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THEORY

Graphs Drawn With Some Vertices per Face: Density and Relationships

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ABSTRACT Graph drawing beyond planarity is a research area that has received an increasing attention in the last twenty years, driven by the necessity to mitigate the visual complexity inherent in geometric representations of non-planar graphs. This research area stems from the study of graph layouts with forbidden crossing configurations, a well-established subject in geometric and topological graph theory. In this context, the contribution of this paper is as follows: 1) We introduce a new hierarchy of graph families, called k^+-*real face graphs*; for any integer $k \geq 1$, a graph G is a k^+-real face graph if it admits a drawing Γ in the plane such that the boundary of each face (formed by vertices, crossings, and edges) contains at least k vertices of G ("k^+" stands for k or more); 2) We give tight upper bounds on the edge density of k^+-real face graphs, namely we prove that n-vertex 1^+-real face and 2^+-real face graphs have at most $5n - 10$ and $4n - 8$ edges, respectively. Furthermore, in a constrained scenario in which all vertices must lie on the boundary of the external face, 1^+-real face and 2^+-real face graphs have at most $3n - 6$ and $2.5n - 4$ edges, respectively; 3) We characterize the complete graphs that admit a k^+-real face drawing or an outer k^+-real face drawing for any $k \geq 1$. We also provide a clear picture for the majority of complete bipartite graphs; and 4) We establish relationships between k^+-real face graphs and other prominent beyond-planar graph families; notably, we show that for any $k \geq 1$, the class of k^+-real face graphs is not included in any family of beyond-planar graphs with hereditary property.

INDEX TERMS Beyond-planar graph drawing, edge density, geometric graph theory, graph visualization, inclusion relationships.

I. INTRODUCTION

Graph drawing (also known as graph visualization) is a well-established research area that addresses the problem of automatically computing readable visual representations of graphs and networks, by developing models, algorithms, and systems [2], [3], [4]. Graph drawing finds application in many real-world domains where data are naturally modeled as complex graphs and the user can benefit from visualization paradigms and tools for exploring the data space. Examples of these domains include social sciences, software engineering, biology, finance, and computer networks (see, e.g., [5], [6], [7], [8], [9]). The importance of graph visualization has also been highlighted in the contexts of machine learning,

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knowledge discovery, and explainable AI, which further characterizes the interdisciplinary nature of this research area (see, e.g., [10], [11], [12], [13]).

Within the above scenario, the study of non-planar drawings of graphs with forbidden substructures has a long tradition, stemming from topological and geometric graph theory (see, e.g., [14]). This topic is commonly recognized as *beyond-planar graph drawing*, and it has become increasingly popular in the last two decades. From an application perspective, this growth in interest is motivated by the necessity of visually representing complex relational data (modeled as non-planar graphs), by reducing the negative impact of edge crossings on the readability of a graph visualization. This motivation is further reinforced by some human cognitive experiments designed to assess users' analytical capabilities when specific types of edge crossings are avoided in graph visualizations. See [15], [16], [17], and [18] for surveys or books on the subject.

A *beyond-planar graph family* is a class of non-planar graphs that can be drawn in the plane by avoiding some specific type of edge crossing configurations. For instance, for a given positive integer k, the family of *k-planar graphs* contains all graphs that admit a drawing with no more than k crossings per edge [19], [20], while *k-quasi planar graphs* are those that can be drawn without k mutually (i.e., pairwise) crossing edges [21], [22], [23], [24], [25], [26]. A generalization of k-planar graphs, called *min-k-planar graphs* has been recently proposed, in which for any two crossing edges one of the two must contain at most k crossings [27], [28]. Other prominent examples of beyond-planar graph families are *fan-planar* graphs [29], [30], [31], [32], [33], [34], [35], where an edge is not allowed to cross two independent edges, and *k-gap planar* graphs ($k \geq 1$) [36], [37], [38], where for each pair of crossing edges one of the two edges contains a small gap through which the other edge can pass, and only k gaps per edge are allowed. In other words, in a k-gap planar drawing it is possible to assign each crossing to one of the two edges that form it, in such a way that no more than k crossings are assigned to the same edge. Geometric properties of edge crossings have also been proposed. In particular, *right-angle-crossing graphs* (*RAC graphs* for short) are those graphs that admit a drawing with straight-line edges such that any two crossing edges form angles of 90° at their crossing point [39], [40]; generalizations and variants of RAC drawings consider the possibility of inserting bends along the edges or forbid crossing angles that are below a given threshold (see, e.g., [41], [42], [43], [44]). RAC graphs are among the most investigated families in beyond-planar graph drawing, because there are human cognitive studies showing that a good angular resolution at the crossing points makes a graph layout more readable [45]. Refer to [16] for additional references on the subject.

Given a family $\mathcal{F}$ of beyond-planar graphs, a core problem with a long tradition, which originates from extremal graph theory [46], [47], [48], is establishing the *edge density* for the elements of $\mathcal{F}$, i.e., the maximum number of edges that an n-vertex graph in $\mathcal{F}$ can have with respect to n. Besides its theoretical interest, finding upper bounds on the edge density of graphs in a family $\mathcal{F}$ represents a key factor in the design of recognition algorithms for $\mathcal{F}$, i.e., algorithms that take as input a graph and establish whether this graph belongs to $\mathcal{F}$ (see, e.g., [30], [33], [49], [50], [51]). For example, it is known that n-vertex 1-planar graphs and 2-planar graphs have at most $4n - 8$ edges and $5n - 10$ edges, respectively, and both these bounds are tight, in the sense that there are graphs in these families that can actually achieve them [20], [52]. In the literature, a graph of $\mathcal{F}$ whose number of edges is the maximum possible over its number of vertices is usually called an *optimal* graph of $\mathcal{F}$ (see, e.g., [49], [53], [54], [55], [56], [57]).

A complementary research direction investigates how different beyond-planar graph families relate to each other in terms of inclusion [16]. For instance, the family of simple k-planar graphs is a subset of $(k+1)$-quasi planar graphs for any $k \geq 2$ [58]. Other examples of inclusion relationships involve the families of fan-planar and k-gap planar graphs [33], [37].

A. CONTRIBUTION

In this paper we introduce a new hierarchy of graph families. Namely, for any positive integer k, we define the family of *k^+-real face graphs* as follows. Let Γ be a graph G in the plane, where edge crossing points are regarded as dummy vertices. The drawing Γ divides the plane into topologically connected regions, called *faces* (or *cells*): if Γ is a planar drawing (i.e., if Γ does not have crossings), the boundary of each face consists of vertices (and edges) of G; otherwise, there are some faces whose boundaries contain dummy vertices. We say that G is a *k^+-real face graph* if it admits a drawing such that each face boundary contains at least k real vertices, i.e., k vertices of G ("k^+" stands for "k or more"). Note that, for any $k \geq 1$, the family of $(k + 1)^+$-real face graphs is included in the family of k^+-real face graphs. For example, Figure 1 shows three different drawings of the same graph, namely the complete graph on five vertices. While the leftmost drawing does not belong to any family of k^+-real face drawings (as it contains a face formed by crossing points only), the other two drawings are a 1^+-real face and a 2^+-real face drawing, respectively.

The study of the k^+-real face graph hierarchy has both a theoretical and a practical motivation. From a theoretical perspective, k^+-real face drawings are a generalization of planar drawings whose face sizes are above a desired threshold [59], [60], [61]. Also, finding k^+-real face graphs can be regarded as a generalization to non-planar graphs of the classical guarding planar graph problem [62], where the vertices that cover the face set are the (real) vertices of G. From the practical side, the advantage of computing k^+-real face drawings comes from the observation that faces mostly consisting of crossing points carry out little information about the relational dataset modeled by the graph and make the layout less readable. Indeed, the number of real vertices per face provides a measure of how much the drawing is far from

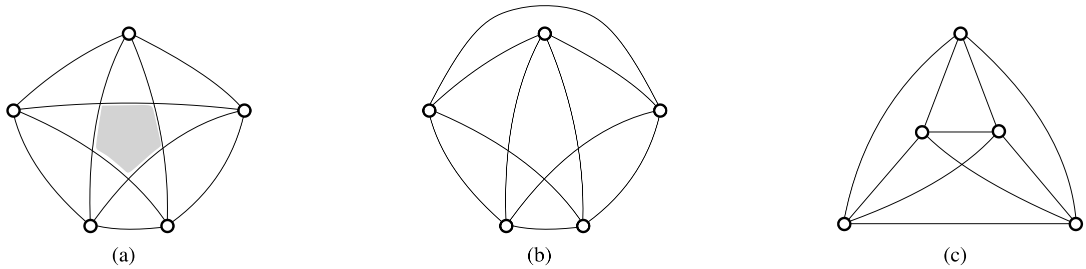


FIGURE 1. Three different drawings of the complete graph K_5: (a) A drawing with a face that has no vertices (in gray); (b) a 1^+-real face drawing; (c) a 2^+-real face drawing.

TABLE 1. Summary of density results in this paper; n, m, and χ denote the number of vertices, edges, and crossings, respectively.

Graph Family	Crossings ($\chi \leq$)	Edges ($m \leq$)	Ref.
k^+-real face graphs ($k \geq 3$)	$\frac{2-k}{k} \cdot m + n - 2$	$\frac{k}{k-2}(n-2)$	Le. 1,Th. 1
2^+-real face graphs	$n-2$	$4n-8$	Le. 1,Th. 2
1^+-real face graphs	$m+n-2$	$5n-10$	Le. 1,Th. 3
outer k^+-real face graphs ($k \geq 3$)	$\frac{2-k}{k} \cdot m + \frac{k-1}{k} \cdot n - 1$	$\frac{k-1}{k-2} \cdot n - \frac{k}{k-2}$	Le. 2,Th. 4
outer 2^+-real face graphs	$\frac{1}{2}n-1$	$2.5n-4$	Le. 2,Th. 5
outer 1^+-real face graphs	$m-1$	$3n-6$	Le. 2,Th. 6

being planar; when possible, faces formed only by crossing points should be avoided.

It is also worth remarking that our study differs from all previous approaches in the field of graph drawing beyond planarity. Indeed, the class of k^+-real face graphs focuses on relating edge crossings to the structure of face boundaries, rather than just considering forbidden crossing configurations along edges. Our results are as follows:

- We provide tight upper bounds on the edge density of k^+-real face graphs, for all values of k, both in the general case and in the constrained scenario in which all vertices of the graph are forced to stay on the external face. The constrained scenario can be seen as a generalization of the study of outerplanar graphs to our graph family, which we call *outer k^+-real face graphs*. We remark that similar constraints have been previously studied for other families of beyond-planar graphs (see, e.g., [30], [33], [51], [63], [64]). Table 1 reports the set of our results on edge density. See also Table 2 for a comparison between our results and previous results in graph drawing beyond planarity.
- We use the edge density results to characterize the complete graphs that admit k^+-real face drawings, for any integer $k \geq 1$. Interestingly, we observe the existence of k^+-real face drawings of complete graphs that match both the maximum value of k and the minimum number of edge crossings, i.e., the so-called *crossing number* of the graphs. We also provide a clear picture for the majority of complete bipartite graphs.
- We establish inclusion relationships between k^+-real face graphs and families of beyond-planar graphs with hereditary property, such as h-planar and h-quasi planar graphs. Notably, we show that, for any positive integer k, the family of k^+-real face graphs is not included in any beyond-planar graph family with hereditary property. However, we prove that this is not always the case if we restrict our attention to optimal graphs.

We remark that very recently, and after the conference version of the present paper was published [1], Kaufmann et al. [65] published a technical report about a new general formula, used to provide alternative proofs for several edge density results in the beyond-planar graph drawing area. In particular, the authors of [65] describe an alternative proof of the upper bounds on the edge density of unconstrained k^+-real face graphs given in our paper. The work in [65] confirms the correctness of our results and reinforces the scientific interest in the graph families introduced in our article.

B. PAPER STRUCTURE

The remainder of the paper is structured as follows: Section II discusses the literature that is mainly related to our contribution; Section III contains basic notation and terminology adopted throughout the paper; Sections IV and V provide the edge-density results of k^+-real face and outer k^+-real face graphs, respectively; Section VI focuses on k^+-real face drawings of complete graphs and complete bipartite graphs; Section VII presents the results about inclusion relationships of k^+-real face graphs and other beyond-planar graph families; Section VIII concludes with final remarks and suggestions about future research directions.

II. RELATED WORK

Our work provides advances on two main lines of research in graph drawing beyond planarity, namely the study of the edge density of specific graph families and the study

TABLE 2. Edge density bounds for some popular beyond-planar graph families; n and m denote the number of vertices and edges of the graphs. The bounds assume simple graphs and drawings.

Graph Family	Edges ($m \leq$)	Tight	Ref.
1-planar	$4n-8$	•	[52]
2-planar	$5n-10$	•	[52]
3-planar	$5.5n-11$	+	[66]
4-planar	$6n-12$	+	[67]
k-planar ($k \geq 5$)	$3.81\sqrt{k}n$	×	[67]
3-quasi planar	$6.5n-20$	•	[22], [65]
4-quasi planar	$72(n-2)$	×	[21]
k-quasi planar ($k \geq 5$)	$c_k n \log n$	∘	[26]
fan-planar	$5n-10$	•	[35], [65]
1-gap planar	$5n-10$	•	[38]
k-gap planar ($k \geq 1$)	$O(\sqrt{k}n)$	×	[38]
0-bend RAC	$4n-10$	•	[40]
1-bend RAC	$5n-10$	•	[65]
2-bend RAC	$10n-19$	+	[65]

of inclusion relationships among different graph families. In the following we briefly survey the literature on these two research lines, and we also discuss other common topics in the field.

A. EDGE DENSITY

The problem of establishing the maximum number of edges that an n-vertex graph in a given beyond-planar graph family can have is extensively studied in the literature. Notably, the majority of beyond-planar graph classes that have been investigated exhibit linear upper bounds on the edge density. Nonetheless, unlike testing graph planarity, deciding whether an input graph belongs to some given beyond-planar graph family is typically an NP-hard problem. While the interested reader can refer to the survey in [16] for the edge density results of a wide range of graph families, we summarize in Table 2 the bounds for some of the most popular graph families. For each result, we report the corresponding reference. Also, the symbol • indicates that the bound is tight; the symbols + and × indicate that the bound is tight up to an additive or to a multiplicative constant, respectively; the symbol ∘ means that the bound may be far from being tight.

The definition of most of the families reported in Table 2 has already been given in Section I. The last three lines in the table refer to straight-line RAC drawings (0-bend RAC), RAC drawings with at most one bend per edge (1)-bend RAC), and RAC drawings with at most two bends per edge (2)-bend RAC). We also recall that every graph admits a RAC drawing with at most three bends per edge, i.e, n-vertex 3-bend RAC drawings can achieve the density of the complete graph K_n [40].

B. INCLUSION RELATIONSHIPS

Table 2 shows that the maximum edge density of different beyond-planar graph families is sometimes the same. In these cases it is natural asking whether the two families coincide. In fact, this is not true most of the times. For example, although both 2-planar graphs and fan-planar graphs can have up to $5n-10$ edges, it is proven that none of these classes is included in the other [34].

There are also cases in which two families are not in any inclusion relationship, even if the maximum edge density of one of the two families is strictly less than the maximum edge density of the other. To this regard, a remarkable example is given by 0-bend RAC graphs (whose density is at most $4n-10$), which are not included in the class of 1-planar graphs (whose density can be higher, namely up to $4n-8$) [68]. As another example, there exist 1-gap planar graphs that are not k-planar, for any given $k \geq 1$ [38]. Recall that k-planar graphs might have much higher density than 1-gap planar graphs when $k \geq 3$ (see Table 2).

A non-trivial inclusion relationship involves the hierarchies of k-planar graphs and k-quasi planar graphs. Namely, by using sophisticated rerouting arguments, Angelini et al. [58] showed that every k-planar drawing can be transformed into a $(k+1)$-quasi planar drawing, for any integer $k \geq 2$. This implies that the family of k-planar graphs is properly included in the family of k-quasi planar graphs. Another interesting result by Bae et al. [38] establishes that all $2k$-planar graphs are k-gap planar, for any $k \geq 1$; this result is proven by exploiting the famous Hall's theorem [69].

Refer to [16] for other relationships between beyond-planar graph families.

C. OTHER TOPICS IN BEYOND-PLANAR GRAPH DRAWING

Although our paper concentrates on edge density and inclusion relationships for k^+-real face graphs, it is worth remarking that other types of problems are studied in the literature in the field of beyond-planar graph drawing. One of them is the already mentioned *recognition problem*, that aims to establish the membership of graphs to specific graph families. Another problem concerns the *stretchability* of topological drawings, that is, establishing whether a certain type of beyond-planar drawing can be transformed into a drawing of the same family using straight-line edges only. While this is always true for planar graphs with fixed embedding, as the famous Fary's theorem proves [70], there are examples of beyond-planar drawings that are not stretchable, or that are stretchable only if we can change the embedding [71], [72], [73], [74].

To conclude this short literature review, we observe that many of the classical problems arising in beyond-planar graph drawing are addressed in constrained scenarios. As mentioned in Section I, some of our density results are about the constrained scenario in which all vertices belong to the external face. Other common constrained settings assume that the vertices are drawn on two parallel lines (e.g., when the graph is bipartite) or that all vertices lie on the same line (called *spine*) and the edges are drawn on a given number of distinct planes that contain this line (called *pages*). See [16] for an extensive discussion of this and other topics on graph drawing beyond planarity.

TABLE 3. Symbols for the cardinality of sets of vertices, edges, and faces.

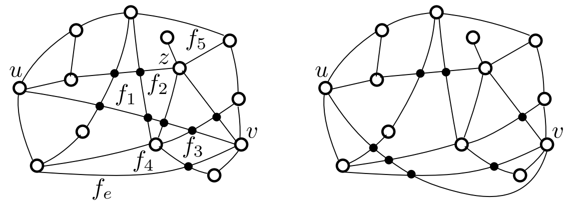
$\lvert V(G)\rvert$	$\lvert E(G)\rvert$	$\lvert V(\Gamma)\rvert$	$\lvert E(\Gamma)\rvert$	$\lvert F(\Gamma)\rvert$	$\lvert V(\Gamma)\setminus V(G)\rvert$
n	m	ν	μ	φ	χ

III. BASIC NOTATION AND TERMINOLOGY

Let G be a graph. We assume that G is simple, that is, it contains neither multiple edges (also called parallel edges) nor self-loops. We also assume, without loss of generality, that G is connected, as otherwise we can just consider each connected component of G independently. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. A *drawing* Γ of G is a geometric representation of G that maps each vertex $v \in V(G)$ to a distinct point of the plane and each edge $uv \in E(G)$ to a simple Jordan arc between the points corresponding to u and v. We assume that Γ is a *simple* drawing (sometimes called *good* drawing in the literature), that is: (*i*) adjacent edges do not intersect, except at their common endpoint; (*ii*) two independent (i.e., non-adjacent) edges intersect at most in one of their interior points, called a *crossing point*; and (*iii*) no three edges intersect at a common crossing point.

Refer to Figure 2 for an illustration of the next definitions. Let Γ be a drawing of G. A *vertex* of Γ is either a point corresponding to a vertex of G, called a *real-vertex*, or a point corresponding to a crossing point, called a *crossing-vertex*. Observe that a crossing-vertex has degree four. We remark that in the literature a plane graph obtained by replacing crossing points with dummy vertices is often referred to as a *planarization* [3]. We denote by $V(\Gamma)$ the set of vertices of Γ. An *edge* of Γ is a curve connecting two vertices of Γ; an edge of Γ whose endpoints are both real-vertices coincides with an edge of G. We denote by $E(\Gamma)$ the set of edges of Γ. Drawing Γ subdivides the plane into topologically connected regions, called *faces* (sometimes called *cells* in the literature). The boundary of each face consists of a circular sequence of vertices and edges of Γ. The set of faces of Γ is denoted by $F(\Gamma)$. Exactly one face in $F(\Gamma)$ corresponds to an infinite region of the plane, called the *external face* (or *outer face*) of Γ (face f_e in Figure 2(a)); the other faces are the *internal faces* of Γ. When the boundary of a face f of Γ contains a vertex v (or an edge e), we also say that f contains v (or e).

From now on, we denote by $n = \lvert V(G)\rvert$ and $m = \lvert E(G)\rvert$ the number of vertices and the number of edges of G, respectively. For a drawing Γ of G, we denote by $\nu = \lvert V(\Gamma)\rvert$, $\mu = \lvert E(\Gamma)\rvert$, and $\varphi = \lvert F(\Gamma)\rvert$ the number of vertices, edges, and faces of Γ, respectively. Also, we denote by $\chi = \lvert V(\Gamma) \setminus V(G)\rvert = \nu - n$ the number of crossing-vertices of Γ. For example in Figure 2(b) the number of crossing-vertices of $\mathbf{\Gamma_2}$ is $\chi = 7$. Table 3 summarized the symbols used to denote the cardinality of the different vertex, edge, and face sets. Also, refer to Table 5 at the end of the paper for a more comprehensive glossary of the symbols used in the technical parts.



(a) Γ_1 (b) Γ_2

FIGURE 2. Two non-planar drawings Γ_1 and Γ_2 of the same graph G having $n = 13$ vertices and $m = 22$ edges. In both drawings the real-vertices are represented by empty circles, while the crossing vertices are represented by black circles. (a) The number of vertices of Γ_1 is $\nu = 21$ (it has 8 crossing-vertices). The number of edges and the number of faces of Γ_1 are $\mu = 38$ and $\varphi = 19$, respectively. Face f_1 is a 0-real face, face f_4 is a 2-real triangle while face f_2 and face f_3 are a 1-real quadrilateral and a 2-real quadrilateral, respectively. Face f_e is the external face. For face f_2 we have that $\delta^r_\Gamma(f_2) = 1$, $\delta^c_\Gamma(f_2) = 3$, and then $\delta_\Gamma(f_2) = 4$. The boundary of face f_5 is not a simple cycle since vertex z is traversed twice, then $\delta_\Gamma(f_5) = 6$ because $\delta^r_\Gamma(f_5) = 5$ and $\delta^c_\Gamma(f_5) = 1$. (b) Drawing Γ_2 is obtained from Γ_1 by rerouting the edge uv of G. Γ_2 is a 1^+-real face drawing and has $\nu = 20$ vertices (with 7 crossing-vertices), $\mu = 33$ edges, and $\varphi = 18$ faces.

A. DEGREE OF VERTICES AND FACES

For a vertex $v \in V(G)$, denote by $\delta_G(v)$ the *degree of v in G*, i.e., the number of edges incident to v. Analogously, for a vertex $v \in V(\Gamma)$, let $\delta_\Gamma(v)$ be the *degree of v in* Γ. For a face $f \in F(\Gamma)$, let $\delta_\Gamma(f)$ denote the *degree of f*, i.e., the number of times we traverse vertices (either real- or crossing-vertices) while walking on the boundary of f clockwise. Each vertex contributes to $\delta_\Gamma(f)$ the number of times we traverse it (possibly more than once if the boundary of f is not a simple cycle); see face f_5 in Figure 2(a) for an example. Also, denote by $\delta^r_\Gamma(f)$ the *real-vertex degree of f*, i.e., the number of times we traverse a real-vertex of Γ while walking on the boundary of f clockwise. Again, each real-vertex contributes to $\delta^r_\Gamma(f)$ the number of times we traverse it. Finally, $\delta^c_\Gamma(f)$ denotes the number of times we traverse a crossing-vertex of Γ while walking on the boundary of f clockwise. Clearly, $\delta_\Gamma(f) = \delta^r_\Gamma(f) + \delta^c_\Gamma(f)$.

B. K^+-FACE DRAWINGS

Given a graph G and a positive integer k, a *k^+-real face drawing* of G is a drawing Γ of G such that the boundary of each face of Γ has at least k real-vertices. If G admits a k^+-real face drawing, we say that G is a *k^+-real face graph*. An *outer k^+-real face drawing* of G is a k^+-real face drawing Γ of G such that all its real-vertices are on the boundary of the external face. If G admits an outer k^+-real face drawing we say that G is an *outer k^+-real face graph*. We say that a face $f \in F(\Gamma)$ is an *h-real face*, where h is a non-negative integer, if $\delta^r_\Gamma(f) = h$. An h-real face of degree d is called an *h-real d-gon*. An h-real 3-gon is also called an *h-real triangle*, and an h-real 4-gon is also called an *h-real quadrilateral*. We say that an edge $e = uv \in E(\Gamma)$ is an *h-real edge* ($h \in \{0, 1, 2\}$) if $\lvert\{u, v\} \cap V(G)\rvert = h$, that is, e contains h real-vertices.

IV. DENSITY OF K^+-REAL FACE GRAPHS

This sections gives tight upper bounds on the number of edges that a k^+-real face graph can have. All our proofs exploit the well-known Euler's formula for planar drawings, which relates the number of edges, vertices, and faces. In our case, the idea is to use several applications of this formula on (non-planar) k^+-real face drawings, where the vertices are both the real-vertices and the crossing-vertices of the drawing, and where the edges of the drawing are portions of edges of the input graph.

We start by proving an upper bound on the number χ of crossing-vertices in a k^+-real face drawing. It will be exploited to prove some of our edge density results.

Lemma 1: For any given k^+-real face drawing drawing Γ of a graph G, the following inequality holds:

$$\chi \leq \frac{2-k}{k} \cdot m + n - 2 \qquad (1)$$

Proof: By hypothesis, each face $f \in F(\Gamma)$ contains at least k real-vertices. Since each real-vertex $v \in V(G)$ can belong to at most $\delta_G(v)$ faces of Γ and since $\sum_{v \in V(G)} \delta_G(v) = 2m$, we have that the number φ of faces of Γ is such that $\varphi \leq \frac{2m}{k}$. Also, the number of edges μ of Γ is such that $\mu = m + 2\chi$. Hence, by Euler's formula applied to Γ, we have $\varphi = \mu + 2 - \nu = m + 2\chi + 2 - n - \chi$, and hence $\varphi = m + \chi + 2 - n$. It follows that $\chi = \varphi - m + n - 2 \leq \frac{2m}{k} - m + n - 2 = \frac{2-k}{k} \cdot m + n - 2$. □

A. K^+-REAL FACE GRAPHS, WITH $K \geq 2$

We first prove the edge density upper bound for the the case $k \geq 3$ (Theorem 1) and then we consider the case $k = 2$ (Theorem 2).

Theorem 1: Let k be an integer such that $k \geq 3$. If G is a k^+-real face graph with n vertices and m edges, then $m \leq \frac{k}{k-2}(n-2)$, and this bound is tight. Also, the k^+-real face drawings of optimal n-vertex k^+-real face graphs are exactly the n-vertex planar drawings in which each face is a simple k-gon.

Proof: Let Γ be any k^+-real face drawing of G. When $k \geq 3$, the term $\frac{2-k}{k}$ is negative and, equivalently, $\frac{k}{k-2}$ is positive. Since the number χ of crossing-vertices of Γ cannot be negative, i.e., $\chi \geq 0$, by Equation (1) of Lemma 1 we have that the number m of edges of G must satisfy the inequality $m \leq \frac{k}{k-2}(n-2)$.

To show that the bound is tight, consider any n-vertex planar graph with a given planar embedding, where each face has exactly k vertices. By Euler's formula, it is immediate to see that such a graph has $m = \frac{k}{k-2}(n-2)$ edges.

Finally, for an n-vertex graph with $m = \frac{k}{k-2}(n-2)$ edges, Equation (1) implies $\chi \leq 0$. It follows that in every k^+-real face drawing Γ of an optimal n-vertex k^+-real face graph G the number χ of crossings equals 0, that is, G is necessarily planar. This implies that each face of Γ has degree k. Indeed, since each face of Γ has at least k vertices, if one of these faces had more than k vertices, then G would have more than $m = \frac{k}{k-2}(n-2)$ edges, which is not possible. □

Theorem 2: If G is a 2^+-real face graph with n vertices and m edges, then $m \leq 4n - 8$, and this bound is tight. Also, the optimal n-vertex 2^+-real face graphs are exactly the optimal 1-planar graphs.

Proof: Let Γ be any 2^+-real face drawing of G. With $k = 2$, from Equation (1) of Lemma 1 we get $\chi \leq n - 2$. Since $\mu \leq 3\nu - 6$, since $\mu = m + 2\chi$, and since $\nu = n + \chi$, we have $m \leq \chi + 3n - 6$, and therefore $m \leq n - 2 + 3n - 6 = 4n - 8$. This proves the claimed upper bound on m.

To see that this bound is tight, consider the family of 1-planar graphs, which admit a drawing Γ with at most one crossing per edge. Each face of Γ contains at least two real-vertices (see also [75]), thus Γ is a 2^+-real face drawing. In particular, for $n = 8$ and for every $n \geq 12$, there exists an optimal 1-planar graph with n vertices and $4n - 8$ edges [20]. To complete the proof, we show that every 2^+-real face drawing Γ of an optimal 2^+-real face graph G is also a 1-planar drawing of G, and hence that G is an optimal 1-planar graph. Suppose by contradiction that this is not true. Since G does not have multiple edges, this implies that there is at least one edge crossed twice in Γ, and hence a face of degree at least four in Γ (namely a face with at least two real-vertices and two crossing-vertices). Since each other face of Γ has at least degree three and since the sum of the degrees of the faces of Γ equals twice the number μ of its edges, we have $2\mu \geq 3(\varphi - 1) + 4 = 3\varphi + 1$. Since $\mu = m + 2\chi$ and $\varphi = m + \chi + 2 - n$, we get $m \leq \chi - 7 + 3n$. From the optimality of Γ, we can plug $m = 4n - 8$ into the previous inequality, and we get $\chi \geq n - 1$, which contradicts Equation (1) of Lemma 1 when $k = 2$. □

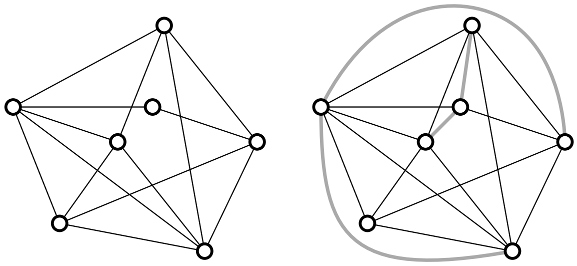
B. 1^+-REAL FACE GRAPHS

To prove an upper bound on the number of edges in 1^+-real face graphs, we exploit a *discharging* technique, which has been successfully used in previous papers on other beyond-planar graph families. See for example [22], [67], and [44]. Intuitively, a discharging technique consists of assigning an initial charge to the faces of the drawing, mainly based on the number of real- and crossing-vertices of their boundaries, and then moving some charges from faces with high capacity to faces with smaller capacities, without increasing the total charge. This idea, combined with Euler's formula, is used to count the maximum number of edges in the graph. More formally, as in [22], we consider a *charging function* ch : $F(\Gamma) \to \mathbb{R}$ such that, for each $f \in F(\Gamma)$, we set:

$$\mathrm{ch}(f) = \delta_\Gamma(f) + \delta_\Gamma^r(f) - 4 = 2\delta_\Gamma^r(f) + \delta_\Gamma^c(f) - 4 \qquad (2)$$

The value ch(f) is called the *initial charge* of f. In practice, each crossing-vertex of f contributes exactly once to the initial charge of f, while each real-vertex contributes twice. By using Euler's formula, it can be easily seen that the following relationship holds (refer to [22] for details):

$$\sum_{f \in F(\Gamma)} \mathrm{ch}(f) = 4n - 8 \qquad (3)$$



(a) (b)

FIGURE 3. (a) A 1^+-real face drawing Γ. (b) An example of augmentation of Γ as described in the proof of Theorem 3; the added edges are gray and thick.

The aim of a discharging technique is to derive from the initial charging function ch a new function ch′ that satisfies the next two properties (see also [22]):

- **C1.** $\text{ch}'(f) \geq \alpha\delta^r_\Gamma(f)$, for some real number $\alpha > 0$;
- **C2.** $\sum_{f\in F(\Gamma)} \text{ch}'(f) \leq \sum_{f\in F(\Gamma)} \text{ch}(f)$

If $\alpha > 0$ is a real number for which a charging function ch′ satisfies **C1** and **C2**, by Equation (3) we have: $4n - 8 = \sum_{f\in F(\Gamma)} \text{ch}(f) \geq \sum_{f\in F(\Gamma)} \text{ch}'(f) \geq \alpha \sum_{f\in F(\Gamma)} \delta^r_\Gamma(f)$. Also, since $\sum_{f\in F(\Gamma)} \delta^r_\Gamma(f) = \sum_{v\in V(G)} \delta_G(v) = 2m$, we get the following:

$$m \leq \frac{2}{\alpha}(n-2) \qquad (4)$$

Thus, Equation (4) can be exploited to prove upper bounds on the edge-density of a graph for specific values of α, whenever we find a charging function ch′ that satisfies **C1** and **C2**. We are now ready to present the main result of this section.

Theorem 3: If G is a 1^+-real face graph with n vertices and m edges, then $m \leq 5n - 10$, and this bound is tight.

Proof: Let Γ be a 1^+-real face drawing of G. We first augment Γ and G with extra edges as follows (refer to Figure 3 for an example). If some face f of Γ contains a pair u and v of real-vertices but does not contain an edge uv on its boundary, then we augment Γ (and G) with an edge uv drawn in the interior of f, in such a way that it does not create any crossing. We then repeat this process until every pair of real-vertices in each face f is connected by an edge on the boundary of f. Note that this augmentation is not unique and may introduce multiple edges in G. However, it does not create any 0-real faces and any faces of degree two in the drawing; also, the drawing remains a 1^+-real face drawing. Denoted by Γ′ the drawing resulting from the edge augmentation on Γ, for each face $f \in F(\Gamma')$ we have that $\delta_{\Gamma'}(f) \geq 3$ and $1 \leq \delta^r_{\Gamma'}(f) \leq 3$. Also, denoted by G' the graph resulting from the augmentation on G, we have $V(G') = V(G)$ and $E(G) \subseteq E(G')$; hence, an upper bound on the number of edges m' of G' is also an upper bound on the number of edges m of G.

Suppose given on Γ′ the initial charging function ch : $F(\Gamma') \to \mathbb{R}$ of Equation (2). If we are able to define a charging function ch′ : $F(\Gamma') \to \mathbb{R}$ that satisfies **C1** and **C2** for $\alpha = \frac{2}{5}$, then by Equation (4) we get $m \leq m' \leq 5n - 10$, and we are done. We show how to define ch′.

For every face $f \in \Gamma'$, we initially set $\text{ch}'(f) = \text{ch}(f) = 2\delta^r_{\Gamma'}(f) + \delta^c_{\Gamma'}(f) - 4$. With this choice and with $\alpha = \frac{2}{5}$, function ch′ satisfies **C2**. Also, **C1** becomes $2\delta^r_{\Gamma'}(f) + \delta^c_{\Gamma'} - 4 \geq \frac{2}{5}\delta^r_{\Gamma'}(f)$, that is, $8\delta^r_{\Gamma'}(f) + 5\delta^c_{\Gamma'}(f) \geq 20$. Hence, since $\delta_{\Gamma'}(f) \geq 3$, **C1** is always satisfied for each face f such that either $\delta^r_{\Gamma'}(f) \geq 2$, or $\delta^r_{\Gamma'}(f) = 1$ and $\delta^c_{\Gamma'}(f) \geq 3$. It follows that, the only faces that do not satisfy **C1** are the 1-real triangles, i.e., each face t for which $\delta^r_{\Gamma'}(t) = 1$ and $\delta^c_{\Gamma'}(t) = 2$. Indeed, for a 1-real triangle t the initial charge equals 0, thus we need to suitably increase the value of $\text{ch}'(t)$.

For each 1-real triangle t, let f be the face incident to the unique 0-real edge of t; see Figure 4(a). Note that it must be $\delta_{\Gamma'}(f) \geq 4$. Indeed, if it were $\delta_{\Gamma'}(f) = 3$ then G would contain two multiple edges (which is impossible because G is simple) or there would be two adjacent edges of G that cross in Γ (which is impossible because Γ is a simple drawing). Also, since Γ′ is a 1^+-real face drawing, we have $\delta^r_{\Gamma'}(f) \geq 1$. We apply a discharging operation, by moving a fraction $\frac{2}{5}$ of charge from f to t across their shared 0-real edge. In this way, we set $\text{ch}'(t) = \frac{2}{5}$ and reduce $\text{ch}'(f)$ by $\frac{2}{5}$. The total charge of Γ′ determined by ch′ does not change (hence **C2** is still satisfied) but now $\text{ch}'(t)$ satisfies **C1**.

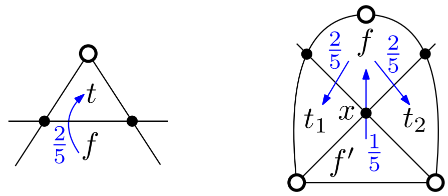
Since for a face f the reduction of $\text{ch}'(f)$ by $\frac{2}{5}$ occurs across a 0-real edge of f, the number of times this happens is at most $\delta^c_{\Gamma'}(f) - 1$. Therefore, after we have applied a discharging operation for each 1-real triangle, the charge $\text{ch}'(f)$ of each face f of degree at least four is such that:

$$\text{ch}'(f) \geq 2\delta^r_{\Gamma'}(f) + \delta^c_{\Gamma'}(f) - 4 - \frac{2}{5}\delta^c_{\Gamma'}(f) + \frac{2}{5} = 2\delta^r_{\Gamma'}(f) + \frac{3}{5}\delta^c_{\Gamma'}(f) - \frac{18}{5}$$

Hence f satisfies **C1** (i.e., $\text{ch}'(f) \geq \frac{2}{5}\delta^r_{\Gamma'}(f)$) if this relation holds:

$$8\delta^r_{\Gamma'}(f) + 3\delta^c_{\Gamma'}(f) \geq 18 \qquad (5)$$

It can be easily verified that the above relation is always satisfied for a face f of degree at least four, except when f is a 1-real quadrilateral (which consists of one real-vertex and 3 crossing-vertices). Indeed, if f is a 1-real quadrilateral it could have moved a fraction $\frac{2}{5}$ of charge towards a 1-real triangle t_1 and a fraction $\frac{2}{5}$ of charge towards another 1-real triangle t_2; see Figure 4(b). Both t_1 and t_2 share a crossing-vertex x with f and with another face f'. In this case $\text{ch}'(f) =$



(a) (b)

FIGURE 4. Illustration for the proof of Theorem 3.

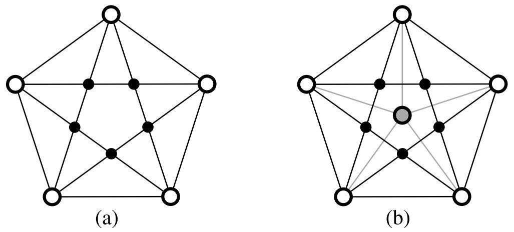


FIGURE 5. (a) A pentagon of an optimal 2-planar drawing with five inner chords that create a 0-real face. (b) To transform the 2-planar drawing into a 1^+-real face drawing, it is sufficient to add a vertex (in gray) inside the 0-real face of each pentagon an connect this vertex to the real-vertices of the pentagon.

$\mathrm{ch}(f) - \frac{4}{5} = 1 - \frac{4}{5} = \frac{1}{5} = \frac{2}{5}\delta^r_{\Gamma'}(f) - \frac{1}{5}$, thus f has a deficit of $\frac{1}{5}$. Observe that the boundary of f' contains two real-vertices adjacent to x, which are connected by an edge due to the edge augmentation initially performed on Γ. Hence, f' is a 2-real triangle and at this point we have $\mathrm{ch}'(f') = \mathrm{ch}(f') = 1 = \frac{2}{5}\delta^r_{\Gamma'}(f') + \frac{1}{5}$. It follows that $\mathrm{ch}'(f')$ has a surplus of $\frac{1}{5}$, and we can move this surplus from f' to f, i.e., we increase $\mathrm{ch}'(f)$ by $\frac{1}{5}$ and decrease $\mathrm{ch}'(f')$ by $\frac{1}{5}$. Since this reduction of $\mathrm{ch}(f')$ can happen at most once for f', both f and f' satisfy C1 at the end of this operation. This completes the proof that $m \leq 5n - 10$.

To show the tightness of the bound, consider the class of n-vertex optimal 2-planar graphs, that is, graphs that admit a drawing with at most two crossings per edge and that have the maximum possible number of edges over all 2-planar graphs with n vertices. Pach and Tóth [20] show that these graphs have $5n - 10$ edges. Also, Bekos et al. [76] show that an optimal 2-planar drawing Γ has a regular structure, namely: The crossing-free edges of Γ form a pentangulation $P(G)$ (i.e., every face is a simple cycle of degree five), and inside each face of $P(G)$ there are five crossing edges; see Figure 5(a). We can transform an optimal 2-planar drawing Γ into a 1^+-real face drawing Γ' by inserting, for each pentagon p of $P(G)$, a real vertex inside the 0-real face of Γ that is contained in the interior of p, and by connecting this vertex to each of the five vertices of p; see Figure 5(b). If $P(G)$ consists of h faces (pentagons), the augmented drawing has $n' = n + h$ vertices and $m' = 5n' - 10$ edges. □

V. DENSITY OF OUTER K^+-REAL FACE GRAPHS

In this section we provide tight upper bounds on the maximum number of edges that an outer k^+-real face graph can have, depending on k. For an outer k^+-real face drawing Γ of a graph G, we denote by $F_{\mathrm{int}}(\Gamma) \subset F(\Gamma)$ the subset of internal faces of Γ. Additionally, φ_{int} denotes the number of internal faces of Γ, that is $\varphi_{\mathrm{int}} = |F_{\mathrm{int}}(\Gamma)|$. Note that $\varphi = \varphi_{\mathrm{int}} + 1$. As for k^+-real face graphs, we first give an upper bound on the number χ of crossing-vertices in an outer k^+-real face drawing, which will be exploited to prove some edge density results in this constrained scenario.

Lemma 2: Let G be a graph and let k be any positive integer. If Γ is an outer k^+-real face drawing of G then the following holds:

$$\chi \leq \frac{2-k}{k} \cdot m + \frac{k-1}{k} \cdot n - 1 \tag{6}$$

Proof: By hypothesis, each face $f \in F_{\mathrm{int}}(\Gamma)$ contains at least k real-vertices. Since each real-vertex $v \in V(G)$ can belong to at most $\delta_G(v) - 1$ internal faces of Γ and since $\sum_{v \in V(G)}(\delta_G(v) - 1) = 2m - n$, we have that the number φ_{int} of internal faces of Γ is such that $\varphi_{\mathrm{int}} \leq \frac{2m-n}{k}$, and therefore $\varphi \leq \frac{2m-n}{k} + 1$. By Euler's formula applied to Γ, we have $\varphi = \mu + 2 - \nu = m + 2\chi + 2 - n - \chi$, and hence $\varphi = m + \chi + 2 - n$. It follows that $\chi = \varphi - m + n - 2 \leq \frac{2m-n}{k} - m + n - 1 = (1 - \frac{1}{k}) \cdot n - (1 - \frac{2}{k}) \cdot m - 1 = \frac{2-k}{k} \cdot m + \frac{k-1}{k} \cdot n - 1$. □

A. OUTER K^+-REAL FACE GRAPHS, WITH $K \geq 2$

As for the unconstrained scenario, we first consider the case $k \geq 3$ and then the case $k = 2$.

Theorem 4: Let k be a positive integer such that $k \geq 3$. If G is an outer k^+-real face graph with n vertices and m edges, then $m \leq \frac{k-1}{k-2} \cdot n - \frac{k}{k-2}$, and this bound is tight. Also, the optimal n-vertex outer k^+-real face drawings are exactly the n-vertex outerplanar drawings whose internal faces are simple k-gons.

Proof: Let Γ be any outer k^+-real face drawing of G, with $k \geq 3$. Since the number χ of crossing-vertices of Γ cannot be negative, by Equation (6) of Lemma 2 we have that $\frac{2-k}{k} \cdot m + \frac{k-1}{k} \cdot n - 1 \geq 0$. Since $\frac{2-k}{k}$ is negative, this implies that $m \leq (\frac{k-1}{k} \cdot n - 1)\frac{k}{k-2}$, and therefore $m \leq \frac{k-1}{k-2} \cdot n - \frac{k}{k-2}$.

To see that the bound is tight, consider the family of n-vertex outerplane graphs (i.e., outerplanar embedded graphs) where each internal face has exactly k vertices. For such an n-vertex graph, we have $2m = k\varphi_{\mathrm{int}} + n$. Also, $\varphi_{\mathrm{int}} + 1 = m + 2 - n$, and hence $\varphi_{\mathrm{int}} = m - n + 1$. Thus, $2m = k(m - n + 1) + n$, which yields $m = \frac{k-1}{k-2} \cdot n - \frac{k}{k-2}$. The final part of the theorem follows from the same considerations as in the proof of Theorem 1. □

Theorem 5: Let G be an outer 2^+-real face graph with n vertices and m edges. We have that $m \leq 2.5n - 4$, and this bound is tight. Also, the n-vertex optimal outer 2^+-real face graphs are exactly the optimal outer-1-planar graphs.

Proof: Let Γ be any outer 2^+-real face drawing of G. By Lemma 2, with $k = 2$, we get $\chi \leq \frac{n}{2} - 1$. If we remove from Γ exactly one edge of G per crossing-vertex, we get an outerplanar graph with $m' = m - \chi$ edges and n vertices. Since a maximal outerplanar graph with n vertices has at most $2n - 3$ edges, we have $m - \chi \leq 2n - 3$, and therefore $m \leq 2n - 3 + \chi \leq 2n - 3 + \frac{n}{2} - 1$, that is, $m \leq \frac{5}{2}n - 4 = 2.5n - 4$. This proves that $2.5n - 4$ is an upper bound on the number of edges of G.

To show that this bound is tight, we describe an infinite family of outer 2^+-real face graphs whose number of edges matches the bound. Refer to Figures 6(a) and 6(b). Let $h \geq 1$ be any positive integer. Construct a graph G_h with $n = 2h + 2$ vertices and $m = 2.5n - 4$ edges as follows. Suppose that in a drawing Γ of G_h the n vertices are distributed along a circumference such that u and v are

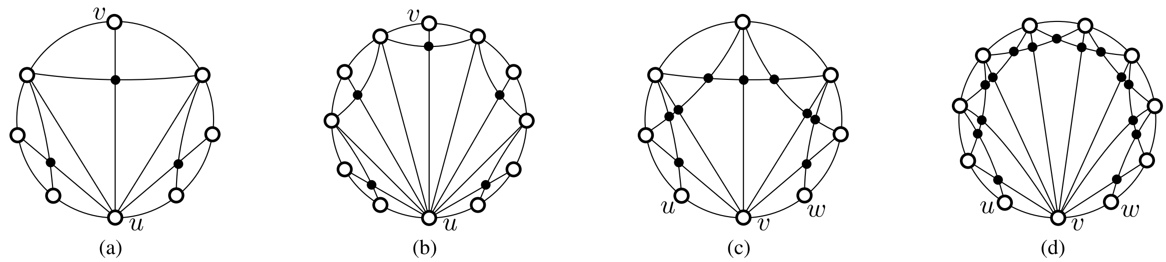


FIGURE 6. **An optimal outer 2^+-real face graph with (a) $n = 8$ vertices, and (b) $n = 12$ vertices. An optimal outer 1^+-real face graph with (c) $n = 8$ vertices, and (d) $n = 11$ vertices.**

the bottom-most and the top-most vertices, respectively; this means that there is a set V_l of h vertices encountered walking from u to v on the circumference clockwise, and another set V_r of h vertices encountered walking from u to v on the circumference counterclockwise. The m edges of G_h are as follows. There is a cycle of n edges that contains all n vertices of G_h; it forms the boundary of the external face of Γ. There are other $n-3$ internal edges that connect u to all vertices of G_h that are not adjacent to u on the external boundary. Finally, there are h edges, each connecting a pair of distinct vertices in $V_l \cup V_r$ that have a topological distance equal to two on the external cycle, but not the two vertices adjacent to u. Hence, $m = n + (n-3) + h = 2.5n - 4$.

We finally prove that every n-vertex optimal outer 2^+-real face drawing Γ is also 1-planar; since optimal outer-1-planar graphs have at most $2.5n - 4$ edges [63], [77], this will imply that optimal outer 2^+-real face graphs are exactly the optimal outer-1-planar graphs. Suppose, by contradiction, that an edge of G crosses twice in Γ. This implies that there is an internal face of Γ of degree at least 4. Therefore $2\mu \geq 3(\varphi-2)+4+n$, which leads to $2\mu \geq 3\varphi-2+n$. Since $\mu = m+2\chi$ and $\varphi = m+\chi+2-n$, we have $\chi \geq m+4-2n$. Since Γ is optimal, then $m = 2.5n - 4$, and hence $\chi \geq 0.5n$, which contradicts Lemma 2 for $k = 2$. □

B. OUTER 1^+-REAL FACE GRAPHS

As for 1^+-real face graphs, we use a discharging technique to prove an upper bound on the number of edges of outer 1^+-real face graphs. An outer 1^+-real face Γ is *edge-maximal* if the drawing obtained by adding to Γ any new edge between two of its real-vertices is no longer outer 1^+-real face. An example of edge-maximal outer 1^+-real face drawing is illustrated in Figure 7(a). However, this graph is not optimal, because for any $n \geq 3$ there exist outer 1^+-real face graphs that contain $3n - 6$ edges (Theorem 6).

The next lemma is fundamental to prove Theorem 6. It shows some important structural properties of edge-maximal outer 1^+-real face drawings. For an illustration of the properties stated in the lemma refer to Theorem 7.

Lemma 3: Let G be an n-vertex outer 1^+-real face graph, with $n \geq 4$, and let Γ be an edge-maximal outer 1^+-real face drawing of G. The following properties hold:

- **a)** The boundary of the external face is a simple cycle without crossing vertices and with exactly n real-vertices.
- **b)** Each internal face of Γ is either a 3-real triangle, or a 2-real d-gon ($d \geq 3$), or a 1-real triangle, or a 1-real quadrilateral.
- **c)** We can map each 1-real triangle to exactly one face of Γ that is either a 2-real d-gon, for $d \geq 4$, or a 1-real quadrilateral, in such a way that: (i) at most $(d-3)$ 1-real triangles are mapped to the same 2-real d-gon; and (ii) at most two 1-real triangles are mapped to the same 1-real quadrilateral.
- **d)** The number of 3-real triangles plus the number of 2-real d-gons is exactly n, and the number of 1-real quadrilaterals is at most $n-4$.

Proof: Let f_0 be the external face of Γ. Suppose by contradiction that the boundary of f_0 contains a crossing-vertex c. Let u be the first real-vertex encountered starting from c and moving on the boundary of f_0 clockwise. Analogously, let v be the first real-vertex encountered starting from c and moving on the boundary of f_0 counterclockwise. Note that Γ does not have an edge uv. Indeed, since we are assuming that c is on the external face, if Γ contained uv, then either Γ would not be simple (which is impossible) or some real-vertices would not be on the external face (which contradicts the fact that Γ is outer 1^+-real face). At this point we can easily augment Γ by adding edge uv in the external face while keeping the drawing outer 1^+-real face, which contradicts the fact that Γ is edge-maximal. This proves that the boundary of f_0 consists of exactly n real-vertices and no crossing-vertices. Also, if the boundary of f_0 were not a simple cycle, we could add a minimal subset of edges in the external face to achieve this property, while keeping all real-vertices on the external boundary. Again, this contradicts the edge-maximality of Γ and completes the proof of Property (a).

We now turn our attention to Property (b). Let f be any internal face of Γ. We first observe that either f has at most two real-vertices or f is a 3-real triangle. Indeed, suppose that f contains (at least) three real-vertices. If f is not a triangular face than there must be two real-vertices u and v in f that are not connected by an edge. Hence, we can augment Γ by adding an edge uv that splits f; the new drawing

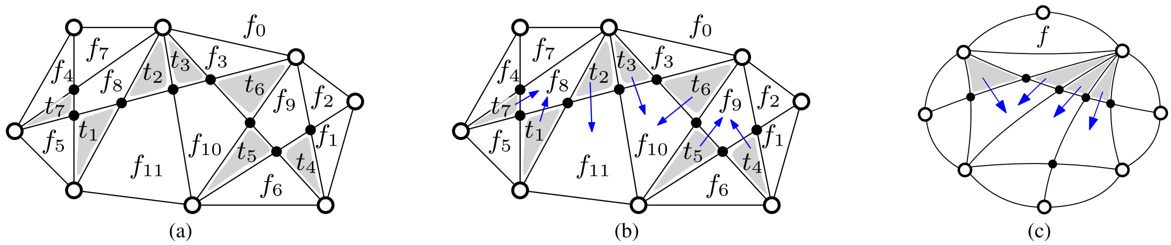


FIGURE 7. (a) An edge-maximal outer 1^+-real face drawing of a graph with 8 vertices and 17 edges. The faces filled in gray represent 1-real triangles. Each face f_i with $i \in \{1, \ldots, 7\}$ is a 2-real triangle. The faces f_8, f_9, f_{10} are 1-real quadrilateral, and f_{11} is a 2-real 4-gon. (b) A mapping of the 1-real triangles to the other faces of the drawing; the mapping, showed by blue arrows, satisfies property (c) of Lemma 3. (c) Another example of an edge-maximal outer 1^+-real face drawing, with a mapping that satisfies property (c) of Lemma 3 is shown.

would still be an outer 1^+-real face drawing of G, which contradicts the edge-maximality of Γ. Therefore, if f contains three real-vertices then it must be a 3-real triangle. Suppose now that f has exactly two real-vertices u and v. In this case, by Property (a), f has an edge uv shared with the external face; also, since f has degree $d \geq 3$, the remaining $d-2$ vertices of f are all crossing-vertices, i.e., f is a 2-real d-gon. Finally consider a face f with exactly one real-vertex v and suppose for contradiction that it is neither a 1-real triangle nor a 1-real quadrilateral. This implies that f has degree at least five, and that there is a 0-real edge e in Γ shared by f and another face f' that contains a real vertex u not adjacent to v. If we augment Γ with an edge uv that crosses e we get a new outer 1^+-real face drawing with one edge more than Γ, a contradiction. This completes the proof of Property (b).

We now focus on Property (c). Let t be any 1-real triangle of Γ, let e be the 0-real edge of t, and let f be the face of Γ sharing e with t. Since G has no multiple edges, f cannot be a 1-real triangle. Also, since the boundary of f contains at least two crossing-vertices, by Property (b), f is neither a 3-real triangle nor a 2-real 3-gon. Hence, by Property (b), f is either a 2-real d-gon, with $d \geq 4$, or a 1-real quadrilateral. We map t on f. In this way, every 1-real triangle is either mapped to exactly one 2-real d-gon, with $d \geq 4$, or to exactly 1-real quadrilateral. Moreover, since each 2-real d-gon f, with $d \geq 4$, has exactly $(d-3)$ 0-real edges on its boundary, at most $(d-3)$ 1-real triangles are mapped to f. Analogously, since each 1-real quadrilateral f has exactly two 0-real edges, at most two 1-real triangles are mapped to f. This completes the proof of Property (c).

It remains to prove Property (d). By Property (a), the boundary of the external face f_0 consists of exactly n 2-real edges. By Property (b), any 2-real edge is incident to an internal face that is either a 3-real triangle or a 2-real d-gon, with $d \geq 3$. Also, since all real-vertices of Γ belong to f_0, each 3-real triangle is adjacent to f_0. If there is no 3-real triangle in Γ, then each edge of f_0 is incident to a 2-real d-gon and, since no two 2-real d-gons share a 2-real edge, there are exactly n 2-real d-gons. Suppose vice versa that there are some 3-real triangles. Each 3-real triangle is incident to two edges of f_0 and to a third 2-real edge shared with a 2-real d-gon. Hence, for each pair of external edges of a 3-real triangle f, we count a pair of internal faces of Γ consisting of f and of the 2-real d-gon adjacent to f. This implies that the number of 3-real triangles plus the number of 2-real d-gons of Γ equals n. Finally, consider the number of 1-real quadrilaterals in an outer 1^+-real face drawing. To form a single 1-real quadrilateral we need at least 5 vertices, and it can be seen that any additional 1-real quadrilateral requires to use at least one more vertex. Hence a drawing with a number q of 1-real quadrilaterals requires at least $n = q + 4$ vertices, that is, $q \leq n - 4$. □

Theorem 6: If G is an outer 1^+-real face graph with n vertices and m edges, then $m \leq 3n - 6$, and this bound is tight.

Proof: To prove the upper bound on the maximum number of edges, it is enough to concentrate on edge-maximal outer 1^+-real face drawings of G. Let Γ be such a drawing. If G has three vertices only, then Γ is a planar 3-cycle and the statement trivially holds. Assume that $n \geq 4$. We exploit a discharging technique like in the proof of Theorem 3. However, in this case, we aim to show the existence of a charging function ch' that satisfies **C1** and **C2** for $\alpha = \frac{2}{3}$. If such a function exists then, by Equation (4), we get $m \leq 3n - 6$. For each face $f \in F(\Gamma)$, initially set $ch'(f) = ch(f)$, where $ch(f)$ is the charging function of Equation (2). Denote by f_0 the external face of Γ. Based on Properties (a) and (b) of Lemma 3, it holds $\delta_\Gamma(f_0) = \delta^r_\Gamma(f_0) = n$ and each internal face of Γ is either a 3-real triangle, or a 2-real d-gon, or a 1-real triangle, or a 1-real quadrilateral. At this point we have:

- $ch'(f_0) = 2\delta^r_\Gamma(f_0) + \delta^c_\Gamma(f_0) - 4 = 2n - 4$; the charge excess of f_0 with respect to $\frac{2}{3}\delta^r_\Gamma(f_0)$ is $\frac{4}{3}n - 4$.
- If f is a 3-real triangle, then $ch'(f) = 2$; the charge excess (and the charge deficit) of f is zero.
- If f is a 2-real d-gon, then $ch'(f) = d-2$; hence, if $d = 3$ (i.e., f is a 2-real triangle) then f has a charge deficit of $\frac{1}{3}$, while if $d \geq 4$ then it has an excess of $d - \frac{10}{3}$.
- If f is a 1-real triangle then $ch(f) = 0$ and f has a charge deficit of $\frac{2}{3}$.
- If f is a 1-real quadrilateral then $ch(f) = 1$ and f has a charge excess of $\frac{1}{3}$.

We now want to modify ch' by moving charges from faces with an excess to faces with a deficit, in such a way that **C1** is satisfied. Based on the aforementioned analysis, the only faces with a deficit are the 2-real triangles (with a deficit of $\frac{1}{3}$) and the 1-real triangles (with a deficit of $\frac{2}{3}$).

Consider a mapping of each 1-real triangle to either a 2-real d-gon (with $d \geq 4$) or to a 1-real quadrilateral that satisfies the conditions (i) and (ii) in Property (c) of Lemma 3. We do the following:

- For every 1-real triangle t mapped to a 2-real d-gon f, we decrease $\mathrm{ch}'(f)$ by $\frac{2}{3}$ and increase $\mathrm{ch}'(t)$ by $\frac{2}{3}$. After this operation, $\mathrm{ch}'(t)$ satisfies **C1**. Also, the charge $\mathrm{ch}'(f)$ is decreased by at most $(d-3)\frac{2}{3}$. Since the excess of f before the operation was equal to $d - \frac{10}{3}$, the new excess after the operation is at least $d - \frac{10}{3} - (d-3)\frac{2}{3} = \frac{1}{3}d - \frac{4}{3}$, which is always non-negative for $d \geq 4$. Hence, $\mathrm{ch}'(f)$ still satisfies **C1**.
- For every 2-real triangle t, we decrease $\mathrm{ch}'(f_0)$ by $\frac{1}{3}$ and increase $\mathrm{ch}'(t)$ by the same quantity. After this operation, $\mathrm{ch}'(t)$ satisfies **C1**. Also, since by Property (d) of Lemma 3. the number of 2-real triangles is at most n, the excess of f_0 after this operation becomes $n-4$. For each 1-real quadrilateral f, we further move one unit of charge from f_0 to f. After this operation, the excess of each 1-real quadrilateral grows from $\frac{1}{3}$ to $\frac{4}{3}$. At the same time, since by Property (d) of Lemma 3. there are at most $(n-4)$ 1-real quadrilaterals, the excess of f_0 either remains positive or becomes 0; hence, $\mathrm{ch}'(f_0)$ still satisfies **C1**.
- Finally, for 1-real triangle t mapped to a 1-real quadrilateral f, we increase $\mathrm{ch}'(t)$ by $\frac{2}{3}$, so that it now satisfies **C1**, and we decrease $\mathrm{ch}'(f)$ by $\frac{2}{3}$. Since there are at most two 1-real triangles mapped to f, the charge of f is decreased by at most $\frac{4}{3}$, and therefore the excess of f either remains positive or it becomes zero, which still satisfies **C1**.

This completes the proof that $m \leq 3n-6$. To prove that the bound is tight, for each $n \geq 3$ we describe how to construct an outer 1^+-real face graph with n vertices and $m = 3n-6$ edges. If $n = 3$ then G is just a triangle, while if $n = 4$ then G is K_4 (which can be drawn with all vertices on the boundary of the external face and a single edge crossing inside). Let $n \geq 5$. The construction of G is as follows; see Figures 6(c) and 6(d). Start with a cycle of n vertices, which defines the boundary of the external face in an outer 1^+-real face drawing Γ of G. Arbitrarily select a vertex v of G and augment Γ (and G) with $n-3$ internal edges that connect v to all vertices that were not already adjacent to it. Finally, let u be the vertex that immediately follows v while walking on the external boundary of Γ clockwise, and let w be the vertex that immediately precedes v by walking on the external boundary of Γ counterclockwise. Augment Γ (and G) with two interlaced chains of edges, each connecting a sequence of pairs of vertices that have topological distance equal to two on the external face but excluding the pair $\{u, w\}$ and any other pair that involves v (which is already connected to any other vertex). If n is odd, each of these two chains consists of $\frac{n-3}{2}$ edges. If n is even, one of the chain consists of $\frac{n-2}{2}$ edges and the other consists of $\frac{n-2}{2} - 1$ edges. It is immediate to check that, with this construction, G has $3n-6$ edges and Γ is an outer 1^+-real face drawing. □

VI. COMPLETE GRAPHS AND COMPLETE BIPARTITE GRAPHS

In this section we study complete graphs and complete bipartite graphs. For the complete graphs we provide a characterization of those that admit k^+-real face drawings, for any $k \geq 1$. Concerning complete bipartite graphs, for most of them we are able to establish whether they belong or not to the different k^+-real face graph families; the question remains open for few instances.

A. COMPLETE GRAPHS

Denote by K_n the complete graph on n vertices. The number of edges of K_n is $m = \frac{n(n-1)}{2}$. From the edge density results summarized in Table 1, the following facts immediately hold:

- K_n with $n \geq 9$ is not a k^+-real face graph, for any $k \geq 1$.
- K_n with $n \geq 7$ is not a k^+-real face graph, for any $k \geq 2$.
- K_n with $n \geq 5$ is not an outer k^+-real face graph, for any $k \geq 1$.
- K_n with $n \geq 4$ is not an outer k^+-real face graphs, for any $k \geq 3$.

On the positive side, we have the following:

- K_3 and K_4 are outer 2^+-real face graphs (trivial).
- K_5 is a 2^+-real face graph (see Figure 1).
- K_6 is a 2^+-real face graph (see Figure 8(a)).
- K_7 is a 1^+-real face graph (see Figure 8(b)).
- K_8 is a 1^+-real face graph (see Figure 8(c)).

It might be interesting to observe that the k^+-real face drawings shown in Figure 8(a), Figure 8(b), and Figure 8(c) correspond to drawings that match the *crossing number* for the graphs K_6, K_7, and K_8, respectively. Recall that the crossing number of a graph G is the minimum number of crossings required by any drawing of G in the plane.

The discussion above implies the following characterization.

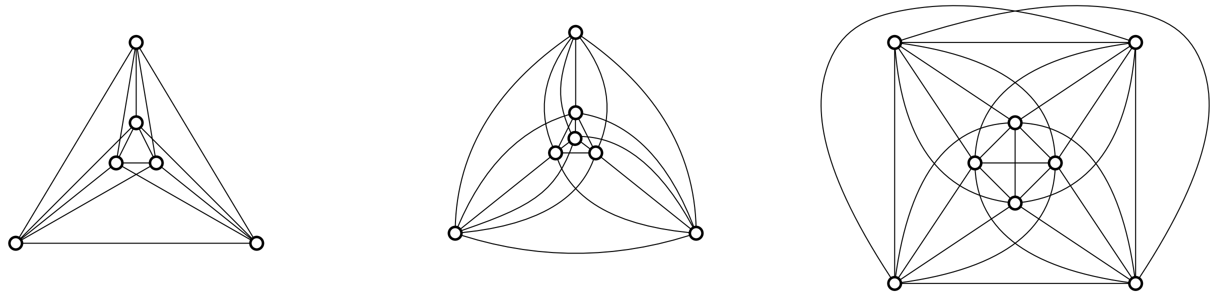
Theorem 7: The complete graphs that admit a k^+-real face drawing or an outer k^+-real face drawing (with $k \geq 1$) are those reported in Table 4.

The following corollary immediately holds from Theorem 7.

Corollary 1: For any $k \geq 1$ and $n \geq 3$, there exists an $O(1)$-time algorithm that decides whether the complete graph K_n is k^+-real face or outer k^+-real face.

B. COMPLETE BIPARTITE GRAPHS

Denote by $K_{a,b}$ the complete bipartite graph on $n = a + b$ vertices, where a is the cardinality of one partition set and b is the cardinality of the other partition set. Recall that $K_{a,b}$ has $m = a \cdot b$ edges. Without loss of generality, in the following we always assume that $b \geq a$. We start with the following.



(a) (b) (c)

FIGURE 8. **(a) 2^+-real face drawing of K_6; (b) 1^+-real face drawing of K_7; (c) 1^+-real face drawing of K_8.**

TABLE 4. **Complete graphs that admit a k^+-real face drawing or an outer k^+-real face drawing.**

Graph	1^+-real face	2^+-real face	k^+-real face ($k \geq 3$)	outer 1^+-real face	outer 2^+-real face
K_3	yes	yes	yes	yes	yes
K_4	yes	yes	yes	yes	yes
K_5	yes	yes	no	no	no
K_6	yes	yes	no	no	no
K_7	yes	no	no	no	no
K_8	yes	no	no	no	no
$K_{\geq 9}$	no	no	no	no	no

Lemma 4: For any $b \geq 2$, the graph $K_{2,b}$ is a k^+-real face graph for $k \leq 4$, whereas it is not k^+-real face for $k \geq 5$.

Proof: The graph $K_{2,b}$ is a planar graph for every $b \geq 2$. Also, in every planar embedding of $K_{2,b}$ all faces have degree four. By Theorem 1, it follows that $K_{2,b}$ is a k^+-real face graph for $k \leq 4$, whereas it is not k^+-real face for $k \geq 5$. □

To study what happens for the complete bipartite graphs $K_{a,b}$ that are not planar, that is, those for which $a \geq 3$ and $b \geq a$, we exploit the upper bounds on the number of crossings summarized in Table 1, rather than their edge density. Namely, the famous Zarankiewicz's conjecture [78] claims that the crossing number of $K_{a,b}$ equals the value of the following function:

$$Z(a, b) = \left\lfloor \frac{a}{2} \right\rfloor \left\lfloor \frac{a-1}{2} \right\rfloor \left\lfloor \frac{b}{2} \right\rfloor \left\lfloor \frac{b-1}{2} \right\rfloor$$

Kleitman [79] showed that the conjecture holds for all values $b \leq 6$ and also proved that the smallest counterexample to Zarankiewicz's conjecture (if any) must occur for odd values of a and b. Furthermore, Woodall [80] used a computer program to verify the correctness of the conjecture for $K_{7,7}$ and $K_{7,9}$. Thus, the smallest unsettled instances of Zarankiewicz's conjecture are $K_{7,11}$ and $K_{9,9}$. Observe that the number of edges of $K_{9,9}$ is higher than the maximum number of edges of any 1^+-real face graph, thus this graph (and every other complete bipartite graph with higher edge density) does not have a k^+-real face drawing, for any $k \geq 1$. About $K_{7,11}$, we cannot use the same argument. However, by Lemma 1 we know that a 1^+-real face drawing of a graph with n vertices and m edges has at most $m + n - 2$ crossings; hence any 1^+-real face drawing of $K_{7,11}$ cannot have more than 93 crossings. Nonetheless, the crossing number of $K_{7,10}$ equals $Z(7, 10) = 225$, and the crossing number of $K_{7,11}$ is larger than the one of $K_{7,10}$. It follows that also $K_{7,11}$ is not 1^+-real face, for any $k \geq 1$. Therefore, we can conclude that for every complete bipartite graph $K_{a,b}$ that might have a chance of being 1^+-real face, we can assume that the crossing number of $K_{a,b}$ equals $Z(a, b)$. We now prove the following.

Lemma 5: For $b \geq 3$ and $k \geq 3$, the graph $K_{3,b}$ is not k^+-real face.

Proof: It is enough to show that the statement holds for $k = 3$. By Lemma 1, a 3^+-real face drawing has at most $-\frac{m}{3} + n - 2$ crossings. Hence, for $b \geq 3$, the number of crossings allowed in a 3^+-real face drawing is at most 1. However, the crossing number of $K_{3,b}$ is $Z(3, b)$, which is always greater than 1 when $b \geq 4$. Hence $K_{3,b}$ is not 3^+-real face if $b \geq 4$.

To complete the proof, we must show that $K_{3,3}$ is not 3^+-real face. Suppose by contradiction that a 3^+-real face drawing Γ of $K_{3,3}$ exists. As observed above, Γ has at most one crossing. Call e one of the two crossing edges of $K_{3,3}$ in Γ, and let Γ' be the drawing obtained from Γ by removing e. Γ' is planar and all its faces have degree four. Also, the two end-vertices of e do not belong to the same face of Γ', as otherwise it would be possible to reinsert e in Γ' without crossings, which is impossible because $K_{3,3}$ is not planar. It follows that e cuts an edge of a degree-4 face in Γ', by generating a triangular face in Γ with only two real-vertices and one crossing-vertex, a contradiction. □

We also observe that for every graph $K_{a,b}$ with $a \geq 4$ (and $b \geq a$) the upper bound of Lemma 1 is strictly less than $Z(a, b)$; thus $K_{a,b}$ is not a 3^+-real face graph if $a \geq 4$. This observation, together with Lemma 4 and Lemma 5, immediately implies the following characterization.

Theorem 8: Let $k \geq 3$. The complete bipartite graph $K_{a,b}$, with $b \geq a$ is a k^+-real face graph if and only if one of these two conditions holds: (*i*) $a \leq 2$, $b \geq 2$, and $k = 3$; (*ii*) $a = 2$ and $k = 4$.

An implication of Theorem 8 is that there are no complete bipartite graphs that are k^+-real face for $k \geq 5$. From Theorem 8 we get the following corollary.

Corollary 2: For any $k \geq 3$ and for any given complete bipartite graph $K_{a,b}$, there exists an $O(1)$-time algorithm that decides whether $K_{a,b}$ is k^+-real face.

We now consider 2^+-real face drawings of non-planar complete bipartite graphs. By Lemma 1, the number of crossings in any 2^+-real face drawing of a graph with n vertices is at most $n - 2$. Still using Zarankiewicz's formula, it is immediate to verify that this upper bound is below the crossing number of $K_{a,b}$ when one of the following cases holds: (*i*) $a = 3$ and $b \geq 7$; (*ii*) $a = 4$ and $b \geq 5$; (*iii*) $a \geq 5$ and $b \geq a$;

On the other hand, each of the few remaining graphs, namely $K_{3,3}$, $K_{3,4}$, $K_{3,5}$, $K_{3,6}$, and $K_{4,4}$, admit a 2^+-real face drawing, as shown in Figure 9. These considerations, together with Lemma 4, imply the following characterization of the complete bipartite graphs that are 2^+-real face.

Theorem 9: A complete bipartite graph $K_{a,b}$, with $b \geq a$, is 2^+-real face if and only if one of the following holds: (*i*) $a = 2$; (*ii*) $a = 3$ and $b \leq 6$; (*iii*) $a = 4$ and $b = 4$.

The next corollary immediately follows.

Corollary 3: Given any complete bipartite graph $K_{a,b}$, there exists an $O(1)$-time algorithm that decides whether $K_{a,b}$ is 2^+-real face.

We finally turn our attention to the study of the complete bipartite graphs that admit a 1^+-real face drawing, in addition to those that also admit a k^+-real face drawing for $k \in \{2, 3, 4\}$. By Lemma 1, the number of crossings in any 1^+-real face drawing of a graph with n vertices and m edges is at most $m + n - 2$. As before, if we compare this upper bound with the crossing number given by Zarankiewicz's formula, we can conclude that the graph $K_{a,b}$ does not have a 1^+-real face drawing in each of the following cases: (*i*) $a = 3$ and $b \geq 19$; (*ii*) $a = 4$ and $b \geq 13$; (*iii*) $a = 5$ and $b \geq 9$; (*iv*) $a = 6$ and $b \geq 7$; (*v*) $a \geq 8$ and $b \geq a$.

On the positive side, we are able to establish that each of the following complete bipartite graphs admits a 1^+-real face drawing (see Figure 10 and Figure 11): $K_{3,7}$, $K_{3,8}$, $K_{3,9}$, $K_{3,10}$, $K_{4,5}$, $K_{4,6}$, $K_{4,7}$, $K_{5,5}$. It is interesting to observe that, as for the complete graphs, also all k^+-real face drawings that we presented for the complete bipartite graphs match the crossing number, that is, they achieve the minimum number of crossings.

We leave it open to establish whether each of the following complete bipartite graphs is 1^+-real face or not: (*i*) $K_{3,b}$ for $b \in [11, 18]$; (*ii*) $K_{4,b}$ for $b \in [8, 12]$, (*iii*) $K_{5,b}$ for $b \in [6, 8]$, and (*iv*) $K_{6,6}$.

We now consider the constrained scenario, that is outer k^+-real face drawings of complete bipartite graphs. We characterize, for all values $k \geq 1$, the complete bipartite graphs that are outer k^+-real face; for each graph listed below, we indicate the minimum k for which it is outer k^+-real face.

- $K_{1,1}$ is outer 2^+-real face (trivial).
- $K_{1,2}$ is outer 3^+-real face (trivial).
- $K_{2,2}$ is outer 4^+-real face (trivial).
- $K_{2,3}$ and $K_{2,4}$ are outer 2^+-real face (see Figures 12(a) and 12(b)); it is easy to verify that any outer k^+-real face drawing of these two graphs has at least one face with only two real-vertices.
- $K_{2,5}$ and $K_{2,6}$ are outer 1^+-real face (see Figures 12(c) and 12(d)); it is easy to verify that any outer k^+-real face drawing of these two graphs has at least one face with one real-vertex.
- $K_{2,b}$ with $b \geq 7$ is not an outer k^+-real face graph for any $k \geq 1$. It can be easily checked that in any drawing of this graph with all faces on the external face, there is at least one 0-real face.
- $K_{a,b}$ with $a, b \geq 3$ is not an outer k^+-real face graph for any $k \geq 1$. It is sufficient to restrict to $K_{3,3}$ and verify that for any of the three non-symmetric circular orderings of black and white vertices along the boundary of the outer face, there is at least one 0-real face (see Figure 13).

Based on the above facts, the following immediately holds.

Corollary 4: For any $k \geq 1$ and for any complete bipartite graph $K_{a,b}$, there exists an $O(1)$-time algorithm that decides whether $K_{a,b}$ is outer k^+-real face.

VII. INCLUSION RELATIONSHIPS

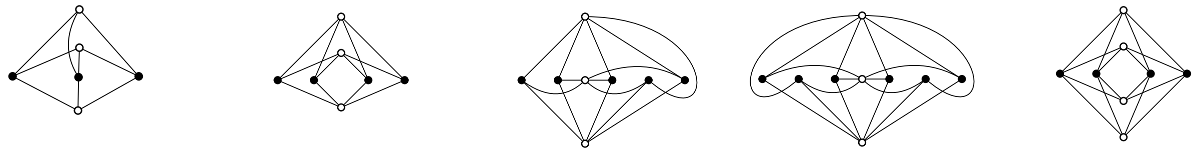
We have observed that, by definition, the family of $(k+1)^+$-real face graphs is properly included in the family of k^+-real face graphs, for any integer $k \geq 1$. For k-planar graphs we have the opposite: each k-planar graph is also $(k+1)$-planar. The results of Section IV yield some inclusion relationships between the families of k-planar graphs and k^+-real face graphs, for $k \in \{1, 2\}$. Namely, Theorem 2 shows that 1-planar graphs are 2^+-real face graphs and that the sets of optimal 1-planar graphs and optimal 2^+-real face graphs coincide. These relationships are summarized in Figure 14.

In the following, we prove a more general result about the relationship between k^+-real face graphs and any other beyond-planar graph family with hereditary property. This result (Theorem 10) implies that, for any fixed positive integer k, there cannot exist a beyond-planar graph family with hereditary property that includes the whole set of k^+-real face graphs, independent of the maximum edge density of the two families. We now formalize this concept.

We say that a family $\mathcal{F}$ of beyond-planar graphs has the *hereditary property* if any subgraph of a graph in $\mathcal{F}$ also belongs to $\mathcal{F}$. Most of the beyond-planar graph families studied in the literature (see, e.g., [16]) have the hereditary property. Conversely, k^+-real face graphs do not always satisfy this property; indeed, removing vertices from a k^+-real face graph G may give rise to a subgraph G' whose drawings necessarily have a face with less than k vertices. However, it is immediate to see that any subgraph of G that has the same vertex set of G is still a k^+-real face graph. We prove the following.

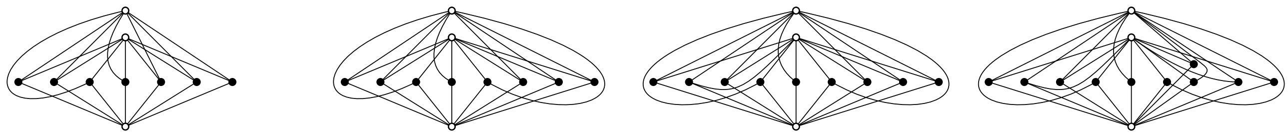
Lemma 6: For any integer $k > 0$ and for any family $\mathcal{F}$ of beyond-planar graphs with hereditary property, there exists a k^+-real face graph that does belong to $\mathcal{F}$.

Proof: Let G be any (connected) graph that does not belong to $\mathcal{F}$, i.e., $G \notin \mathcal{F}$. The idea is to augment G with



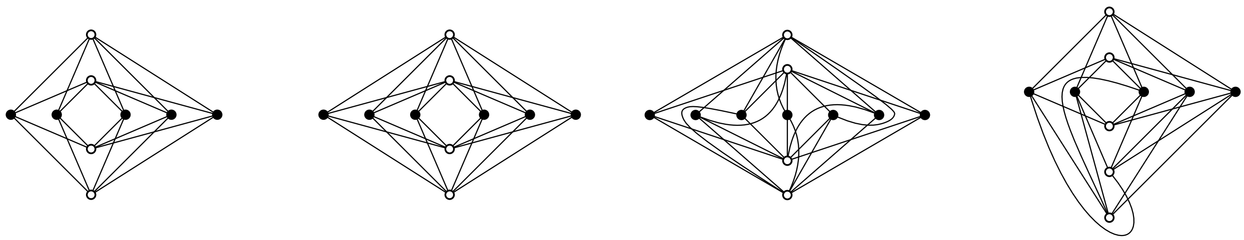
(a) $K_{3,3}$ (b) $K_{3,4}$ (c) $K_{3,5}$ (d) $K_{3,6}$ (e) $K_{4,4}$

FIGURE 9. 2^+-real face drawings of $K_{3,3}$, $K_{3,4}$, $K_{3,5}$, $K_{3,6}$, and $K_{4,4}$.



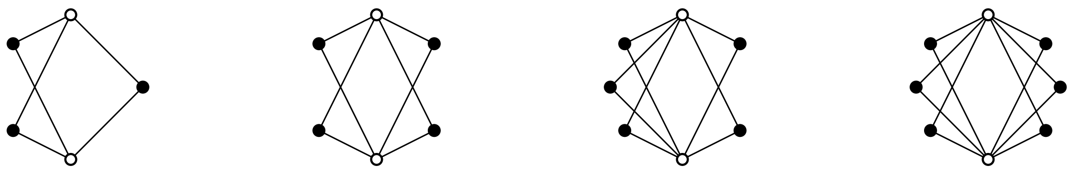
(a) $K_{3,7}$ (b) $K_{3,8}$ (c) $K_{3,9}$ (d) $K_{3,10}$

FIGURE 10. 1^+-real face drawings for $K_{3,7}$, $K_{3,8}$, $K_{3,9}$, and $K_{3,10}$.



(a) $K_{4,5}$ (b) $K_{4,6}$ (c) $K_{4,7}$ (d) $K_{5,5}$

FIGURE 11. 1^+-real face drawings for $K_{4,5}$, $K_{4,6}$, $K_{4,7}$, and $K_{5,5}$.



(a) $K_{2,3}$ (b) $K_{2,4}$ (c) $K_{2,5}$ (d) $K_{2,6}$

FIGURE 12. (a) and (b): outer 2^+-real face drawings for $K_{2,3}$ and $K_{2,4}$, respectively. (c) and (d): outer 1^+-real face drawings for $K_{2,5}$ and $K_{2,6}$, respectively.

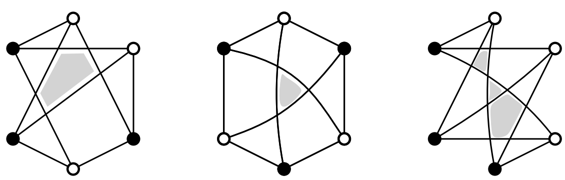


FIGURE 13. The three possible (non-symmetric) outer k^+-real face embeddings for $K_{3,3}$; 0-real faces are colored gray.

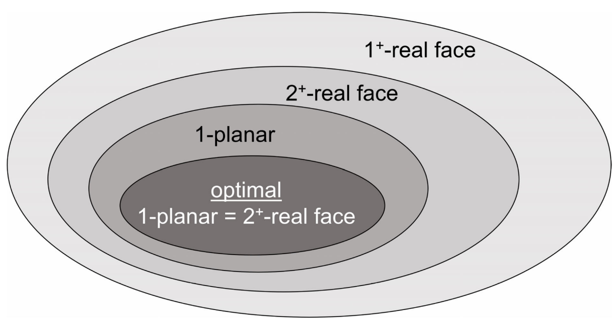


FIGURE 14. Venn diagram describing relationships between k^+-real face graphs and k-planar graphs.

vertices and edges so that the resulting graph G' is a k^+-real face graph. If we are able to do that, then G' cannot belong to $\mathcal{F}$, because otherwise also G would belong to $\mathcal{F}$ (due to the hereditary property of $\mathcal{F}$), which contradicts our initial hypothesis.

To construct G' starts from any drawing Γ of G in the plane. If Γ is already a k^+-real face drawing, we are done, i.e., G' coincides with G. Otherwise, we augment Γ (and hence G) into a new drawing Γ' as explained below; see Figure 15 for an example of this augmentation. We first consider the set of 0-real faces of Γ. If this set is not empty, there must be a 0-real face f that is adjacent to a face f' containing a real-vertex v. Add to Γ a new real-vertex u in the interior of f and connect u to v with an edge that crosses exactly one edge shared by f and f'. After this augmentation, the set of 0-real faces in the new drawing is decreased by one element. Repeat this procedure until there is no more 0-real faces in the drawing. Then, for every face f of Γ with $h < k$ real-vertices (if any), arbitrarily select a real-vertex v of f, and attach to v a chain of $k - h$

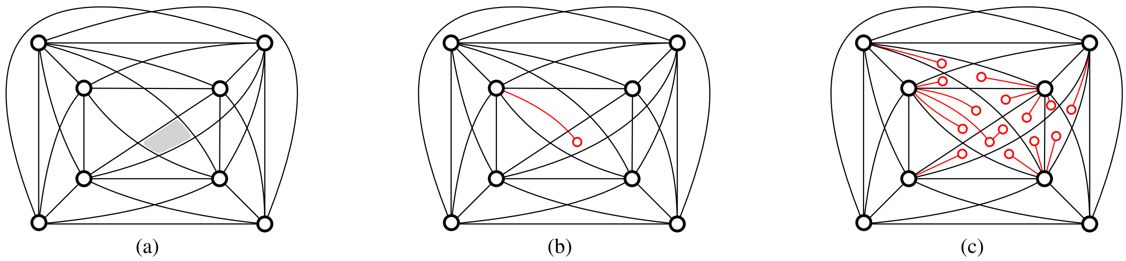


FIGURE 15. Example of the augmentation procedure in the proof of Lemma 6. The example shows how to construct an instance in the family of 2^+-real face graphs, which is not in the family of 1-planar graphs. (a) A drawing of a graph G that is not 1-planar; it has only one 0-real face (filled in gray). (b) An augmentation that removes the 0-real face; the new elements (a vertex and an edge) are in red. (c) A further augmentation that yields a 2^+-real face drawing; for each face f with less than two vertices, we attach an extra real-vertex to an arbitrary real-vertex on the boundary of f.

vertices in the interior of f. This creates a new face f' in place of f, whose boundary has k real-vertices. Once all those faces have been processed, the underlying graph G' of the resulting drawing is a (connected) k^+-real face graph. □

Lemma 6 immediately implies the following.

Theorem 10: For any positive integer k, the family of k^+-real face graphs is not included in any beyond-planar graph family with hereditary property.

We say that two graph families are *incomparable* if none of the two families includes the other. Theorem 10 allows us to easily prove a series of incomparability relationships.

Corollary 5: For any integers $k \geq 1$ and $h \geq 3$, the family of k^+-real face graphs is incomparable with the families of h-planar graphs, min-h-planar graphs, and h-quasi planar graphs.

Proof: Theorem 10 proves the existence of 1^+-real face graphs that are neither h-planar, nor min-h-planar, nor h-quasi planar. On the other hand, since the maximum number of edges of n-vertex h-planar graphs and min-h-planar graphs, for $h \geq 3$, can be greater than $5n - 10$ [20], [27], [28], [66], there exist h-planar graphs and min-h-planar graphs that are not 1^+-real face graphs (and hence that are not k^+-real face graphs, for any $k \geq 2$). Similarly, h-quasi planar graphs, for any $h \geq 3$, can have higher density than 1^+-real face graphs, because 3-quasi planar graphs can have up to $6.5n - 20$ edges [21]. The claimed incomparabilities follow. □

Corollary 6: For any $k \geq 2$, the family of k^+-real face graphs is incomparable with each of the following families: (*i*) 2-planar graphs; (*ii*) min-2-planar graphs; (*iii*) fan-planar graphs; and (*iv*) h-gap planar graphs, for any $h \geq 1$.

Proof: As we have already seen, n-vertex 2-planar graphs can have up to $5n - 10$ edges [52]. Similarly, min-2-planar graphs [28], fan-planar graphs [35], and 1-gap planar graphs [38] with n vertices can have up to $5n - 10$ edges. Since n-vertex 2^+-real face graphs have at most $4n - 8$ edges (Theorem 2), none of the three families above is included in the family of 2^+-real face graphs (and thus in the family of k^+-real face graphs, for any $k \geq 3$). Theorem 10 proves the other direction. □

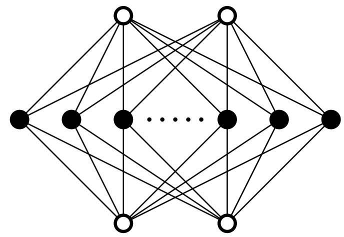


FIGURE 16. A fan-planar drawing of $K_{4,b}$. The two vertex sets of the partition have distinct colors (black and white). If $b \geq 13$, this graph is not 1^+-real face, as its crossing number is too high with respect to the maximum number of crossings allowed in any 1^+-real face drawing.

VIII. CONCLUSION

This paper contributed to enhance the field of graph drawing beyond planarity by introducing the hierarchy of k^+-real face graphs, for $k \geq 1$. To the best of our knowledge, this is the first beyond-planar graph family that concentrates on the properties of face-boundaries, rather than just on the crossing configurations along edges.

For any positive integer k, we provided tight upper bounds on the number of edges that a k^+-real face graph can have, also in a setting in which all vertices must lie on the external boundary of the drawing. The edge density results have also been used to characterize the complete graphs that admit a k^+-real face drawing, for any value $k \geq 1$, and to provide a clear picture for almost all complete bipartite graphs. We finally established an array of incomparability relationships between k^+-real face graphs and several types of popular beyond-planar graph families. These relationships are proved as a consequence of a general result, showing that there cannot exist any beyond-planar graph class with hereditary property that includes all k^+-real face graphs, regardless of the value of k.

A. LIMITATIONS AND OPEN PROBLEMS

The contribution of this paper mainly focused on providing edge density bounds and relationships with previous graph classes in the field of beyond-planar graph drawing. Our combinatorial findings leave some unanswered questions; also, several algorithmic aspects remain mostly unexplored.

TABLE 5. Glossary of the main symbols used in the paper.

Notation	Description
G	Connected graph
Γ	Drawing of a connected graph G
$V(G)$	Set of vertices of a graph G
$E(G)$	Set of edges of a graph G
$V(\Gamma)$	Set of vertices of a drawing Γ
$E(\Gamma)$	Set of edges of a drawing Γ
$F(\Gamma)$	Set of faces of a drawing Γ
$F_{\text{int}}(\Gamma)$	Set of internal faces of a drawing Γ
n	Number of vertices of G, i.e., $n = \lvert V(G)\rvert$
m	Number of edges of G, i.e., $m = \lvert E(G)\rvert$
ν	Number of vertices of Γ, i.e., $\nu = \lvert V(\Gamma)\rvert$
μ	Number of edges of Γ, i.e., $m = \lvert E(\Gamma)\rvert$
φ	Number of faces of Γ, i.e., $\varphi = \lvert F(\Gamma)\rvert$
φ_{int}	Number of internal faces of Γ, i.e., $\varphi = \lvert F_{\text{int}}(\Gamma)\rvert$
χ	Number of crossing-vertices of Γ, i.e., $\chi = \nu - n$
$\delta_G(v)$	Degree of a vertex v in a graph G
$\delta_\Gamma(v)$	Degree of a vertex v in a drawing Γ
$\delta_\Gamma(f)$	Degree of a face f in a drawing Γ
$\delta^r_\Gamma(f)$	Real-vertex degree of a face f in a drawing Γ
$\delta^c_\Gamma(f)$	Crossing-vertex degree of a face f in a drawing Γ

In the following we list some research directions that we consider among the most relevant for future advances.

- **Research direction 1**. As we have seen, both optimal 2-planar graphs and optimal 1^+-real face graphs have density $5n - 10$. Also, Theorem 10 implies that there are 1^+-real face graphs that are not 2-planar. It remains open to establish whether the family of 2-planar graphs and the family of 1^+-real face graphs are incomparable or not. Specifically, can we find a 2-planar graph that is not 1^+-real face? Observe that every 2-planar drawing of an optimal 2-planar graph G is not 1^+-real face, as it contains 0-real faces. However, one cannot exclude in principle that G admits a 1^+-real face drawing that is not 2-planar. In the same direction, one can wonder whether there exist incomparability or inclusion relationships between the family of 1^+-real face graphs and the families of min-2-planar, fan-planar, and 1-gap planar graphs, all containing n-vertex graphs with up to $5n - 10$ edges (see Table 2). In fact, at least for fan-planar graphs, the results that we established in Section VI-B for complete bipartite graphs lead to the following.

  Lemma 7: There exist infinitely many fan-planar graphs that are not 1^+-real face. In other words, the families of 1^+-real face and fan-planar graphs are incomparable.

  Proof: As observed in Section VI-B, each graph in the infinite family of $K_{4,b}$ for $b \geq 13$ is not 1^+-real face. However, all the graphs in this family can be easily drawn as fan-planar graphs, as shown in Figure 16. □
- **Research direction 2**. A second line of research is to establish the complexity of testing whether a graph is k^+-real face or outer k^+-real face for a given k. We remark that this type of decision problems is known to be NP-hard for most of the families in beyond-planar graph drawing, while it is sometimes solvable in polynomial time when we restrict to optimal graphs or other subfamilies (see, e.g., [30], [49], [81]). If the recognition problem turns out to be NP-hard also for the k^+-real face graphs, one can investigate its parameterized complexity. A related work in this direction is [82]. We observe that our results lead to some preliminary consequences about the recognition of optimal k^+-real face graphs when $k > 2$. Namely, by Theorem 1, recognizing optimal k^+-real face graphs, for $k \geq 3$, corresponds to recognizing planar graphs that have an embedding with all faces of degree k. This last problem is polynomial-time solvable for $k \leq 6$ and it is NP-complete for odd $k \geq 7$ and for even $k \geq 10$ [83]. Further, by Theorem 2, recognizing optimal 2^+-real face graphs coincides with the problem of recognizing optimal 1-planar graphs, which is linear-time solvable [49].
- **Research direction 3**. It is interesting to study *geometric* k^+-real face drawings, that is, k^+-real face drawings with straight-line edges. For example, as already done for other beyond-planar graph families [39], [63], [71], [72], [73], [74], [84] one can either establish tight upper bounds on the edge density of graphs that admit geometric k^+-real face drawings or characterize the non-planar embeddings that admit such a drawing. As a preliminary consideration, we observe that the geometric setting for k^+-real face drawings is actually more restrictive than the topological setting. For example, our Theorem 2 proves that every 2^+-real face drawing of graph with $4n - 8$ edges is also 1-planar. On the other hand, geometric 1-planar drawings have at most

$4n - 9$ edges [63]. This implies that also the density of geometric 2^+-real face drawings is at most $4n - 9$.

- **Research direction 4**. As explained in the introduction of this paper, one of the practical motivations behind the study of beyond-planar graph families is to increase the readability of non-planar graph layouts. Nonetheless, user studies that assess the advantage of avoiding specific crossing configurations only exist for a limited number of beyond-planar graph families (see, e.g., [15], [45], [85], [86]). It would be interesting to design human cognitive experiments aimed to assess users' analytical capabilities when k^+-real face drawings are adopted in place of drawings that contain several 0-real faces. Additionally, for the realization of practical visualization systems, it is fundamental the design and experimentation of efficient algorithms that compute k^+-real face drawings or approximations of them.

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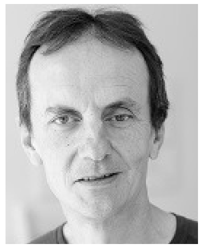

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