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RESEARCH ARTICLE

Displacement and Velocity Estimation in Building Structures Using an Algebraic Observer

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ABSTRACT This study addresses the problem of estimating the velocity and displacement of building structures. Initially, the dynamics of a building subject to seismic disturbance are represented using the Euler-Bernoulli beam model. Subsequently, the model is discretized into a finite number of floors through a modal coordinate transformation that produces a set of decoupled dynamics for each story of the building. The Algebraic Observer (AO) takes advantage of this property to ensure the Algebraic Observability Conditions (AOC). This approach allows the representation of velocity and displacement terms as functions of the available acceleration measurements. One notable advantage of the proposed observer is that, due to its design, it does not require tuning of gains to achieve the convergence property. This observer's simplicity of implementation for practical purposes is another noteworthy characteristic. Note that, to minimize the effect of the measurement noise, iterated integrals of the measured acceleration are introduced to mitigate high-frequency noise. In contrast, second-order filters are employed for low-frequency noise. Finally, the experimental demonstration of the effectiveness and accuracy of this estimation method is conducted using a reduced-scale five-story building prototype.

INDEX TERMS Algebraic observer, acceleration measurement, building structure, state estimation.

I. INTRODUCTION

In most applications, measuring the complete state vector of a physical system is either impossible or costly owing to the limited availability of online sensors and their constant maintenance. Indeed, one cannot use as many sensors as signals of interest to characterize the system behavior, especially considering that such signals can be numerous and of various types [1]. Consequently, state estimation has become a crucial task in the field of control system engineering. For instance, in the state feedback control application [2], the state vector is involved to compute the control action based on specified system dynamics. However, there are instances where this information is neither available nor measurable. In such cases, it becomes necessary to reconstruct the system state to apply a designed control strategy.

In the past years, different approaches have been proposed for estimating state variables. For instance, [3] is a pioneering work that introduces a feedback stabilization method to improve the behavior of a hybrid analog-digital integrator circuit to estimate displacement and velocity. Following this line, several numerical integrators have been proposed based on various baseline correction choices [4], [5], [6]. However, previous system knowledge is required to apply these techniques correctly. Nevertheless, the phase gap and bias cannot be eliminated under this scheme. An enhancement of this approach involves the use of state observers, which are

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regarded as auxiliary dynamical systems. Such an estimation problem consists of designing state observers that provide estimations of the real state trajectory, which converge asymptotically/practically to their true values [7]. In this context, two well-known cases related to estimator synthesis are Luenberger [8] and Kalman [9] observers. In the case of linear systems, the absence of singularities explains the design and use of asymptotic observers such as Kalman and Luenberger schemes, which offer arbitrary convergence estimation speeds. Indeed, despite the progress made in this area, there remains much to be accomplished, as evidenced by the extensive literature dedicated to this topic from various perspectives for both nonlinear and linear systems [10]. For instance, sliding-mode (SM) observers [11], [12], interval observers [7], [8], [9], [10], [11], [12], [13], [14], unknown input observers [15], [16], [17], algebraic observers [18]-[19], high-gain observers [20], [21], among others.

In particular, sliding mode control-based observers have been widely used to stabilize the system to estimate states. Some of the most crucial properties include insensitivity to matched perturbations [22], achieving finite-time convergence [23] in the presence of unknown external disturbances [24], and robustness against certain classes of unmatched uncertainties [25]. However, its main disadvantage is the chattering [12], and it usually uses the form of a Luenberger observer. In some cases, the observer's design is complex and presents challenges to implement in practical applications. For example, a decentralized sliding mode control algorithm is proposed in [26] to mitigate the seismic vibrations in offshore jacket platforms induced by earthquake ground motions. According to the traditional state observation problem, interval observers offer upper or lower bounds for the system states; this idea can be easily extended to detect faults in the systems when a residual signal is outside the set of the adaptive thresholds [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], for instance, in [28] the combination of an observer and an interval predictor for a reduced-scale five-story building prototype is implemented to detect the structural damage in the floors. The estimation of unknown inputs [17] represents another application area, as well as system monitoring for damage detection [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], among others. This makes the reconstruction, or observer problem, the heart of a general control problem.

Furthermore, in various studies, accelerometers are the only available instruments for conducting measurements in numerous practical mechanical applications, including civil structures, aerospace, and robotic systems [29]. In the structural vibration control framework, there is a need to develop a state observer to retrieve unavailable signals used in feedback vibration control schemes that reduce the effects of earthquakes and avoid structural damage [30]. Otherwise, these signals are also employed for structural health monitoring tasks. Indeed, sensing displacement and velocity constitutes a significant challenge in the particular case of seismically excited building structures because there is no fixed inertial frame for reference. Moreover, it is important to note that the displacement, velocity, and acceleration measurements are very sensitive to the presence of the disturbance and noise signals [6]. Nowadays, engineers and researchers have made notable studies on different techniques to reduce the effects of external disturbances, addressed by means of robust observer design. In this context, an adaptive observer was designed in [31], based on acceleration measurements allowing simultaneously the estimation of unknown structural parameters and unavailable displacements and velocities. Moreover, a Luenberger observer plus a Kalman filter to estimate the displacements of the structure considering a finite element model and measurement noise were proposed in [32] and later experimentally evaluated in [33]. An extension of this approach is presented in [34], where was developed a robust observer that estimates the states and the seismic event from noisy acceleration measurements. The observer also considers internal and external bounded uncertainties. A challenge faced by observers under this approach is the presence of parametric uncertainty and tuning the observer's gains to converge to the actual values.

One important tool for state estimation is the artificial neural network approach, which has been successfully employed in characterizing structure-unknown nonlinear systems, state estimation, and damage detection. An exhaustive review of these methods can be found in [35]. These algorithms are improved by the convolutional neural network (CNN) algorithm, which belongs to deep learning techniques introduced in [36] for digit recognition. Some applications of state estimation and damage detection can be found respectively in [37] and [38]. On the other hand, machine learning algorithm is another prominent tool for state estimation tasks. For instance, authors in [39] aims to demonstrate how machine learning and data-driven methods can be suitably deployed to improve the visibility, maneuverability, flexibility, profitability, and safety of power generation systems to handle the uncertain challenges at each level. Additionally, authors in [40] present an overview about the advantage using reinforcement learning from expert demonstrations method. The methods are analyzed and classified according to the impact of the demonstrations. However, this kind of method requires a large volume of information for the network training stage; in the absence of this information, the performance of artificial neural network algorithms can be deteriorated. Moreover, due to the extensive data analyzed, algorithms within this framework could consume significant computing resources. Furthermore, in certain instances, data preprocessing is necessary to ensure the reliability of the results.

An interesting alternative for the state estimation task is through algebraic approach that represents states from an arrangement of a finite number of derivatives of the output and input in terms of the available measurements. This approach turns out to be a particularly suitable tool to describe observability and related questions as structural properties of control systems. Under this line, dynamic models are allowed to be implicit and of arbitrary order but restricted to be polynomial in variables and their derivatives [41]. This algorithm is free of some restrictive conditions as the Lipschitz condition does not need the Luenberger structure [18] and algebraic observer always exists whenever the algebraic observability condition is verified [42]. To exemplify applications under this approach we can consult, authors such as Garcia-Rodriguez et. al. [43], who report satisfactory results implementing algebraic observers to estimate signals for trajectory tracking in a perturbed 1-DOF suspension system, whereas comparison between an algebraic and a reduced order observer for online load torque estimation was carried in reference out [44], with application to rectifier direct current motor system. On the other hand, Martínez-Guerra and Flores-Flores [45] presents an algorithm for robustly estimating the COVID-19 pandemic's population by considering undetected individuals. The methodology estimates the susceptible, removed, asymptomatic, and symptomatic populations employing Proportional and Proportional-Integral reduced order under the algebraic approach. A recent improvement related to the estimation of unknown states in buildings can be found in [46]. The observer design is based on a Proportional-Integral estimator. An important difference concerning the predecessor work is that it does not need a coordinate transformation in its design. However, despite the reported satisfactory results, the criteria for selecting gains are specific to each application. Furthermore, observers' performance in some cases also deteriorates in the presence of parametric uncertainty and measurement noise.

It is clear that the body of work based on the methods mentioned above has produced essential results in the field of state estimation and observer design. However, from the authors' perspective, the algebraic approach also offers some promising features for observer design and deserve further research. In this paper, the algebraic observer is inspired by previous studies [47]. The formulation of the proposed observer is based upon the construction of a set of linear time-varying (LTV) differentiation systems; it does not involve a static diffeomorphism, Luenberger form, or Lipschitz constrain. Our primary concern is to develop an efficient, simple, and accessible estimation methodology. Specifically, we focus on the Algebraic Observability Condition (AOC), which in essence, enables us to estimate the unknown displacements and velocities of the building structure using differential polynomials derived from the available acceleration data. One engaging feature is the simplicity of the observer design with significant improvements: i) the algebraic observer is reformulated and applied for a building structure considered like a multi-degree of freedom system, ii) modal transformation is introduced to produce a decoupled system, iii) to illustrate the feasibility of the approach experimental evaluation is carried out. Thus, based on the algebraic observer structure presented here, the main contributions of this work are:

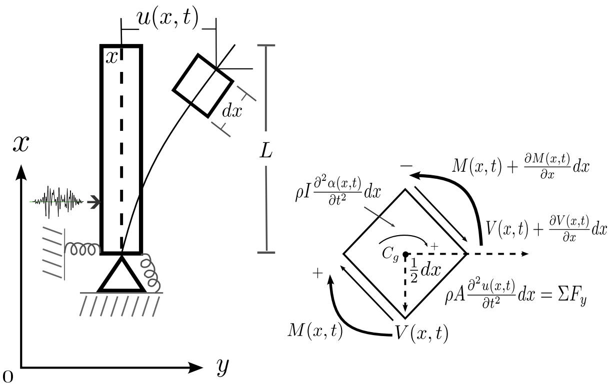


FIGURE 1. Shear beam model.

- The dynamic of multistory building structure subjected to seismic disturbance is modeled using the Euler-Bernoulli beam model.
- We solve the internal state estimation problem, retrieving displacement and velocity signals in building structures based on iterated integrals of the acceleration measurement and the seismic ground motion. A fascinating attribute is that the observers are robust to measurement noise since the estimations exhibit excellent performance and output tracking qualities concerning the model reference. The integral action provides a convenient low-pass filtering on the available signals to reject high-frequency noise signals.
- One engaging feature of this observer is that it can be expanded for multiple estimates using a modal transformation.
- The proposed algebraic observer approach does not require tuning gains to achieve convergence.
- The performance of the proposed observers is experimentally verified through velocity and displacement measurements to compare the estimated position on a reduced-scale five-story building prototype.

The paper structure is organized as follows: Section II presents the mathematical model that describes the structural dynamic. The algebraic observer design applied to a multistory building structure is developed in Section III, whereas the experimental evaluation of the proposed observer is carried out in Section IV. Finally, the conclusions and findings are provided in Section V.

II. DYNAMIC MODEL OF THE BUILDING STRUCTURE

Let building structure subjected to seismic events, with movement constrained to a single axis, hence torsional effects are omitted, see Fig. 1.

Assumption 1: The disturbance signal $\ddot{x}_g(t) \in L_2[0,\infty)$ is bounded and possesses finite energy, i.e.,

$$\|r_2\| = \sqrt{\int_0^\infty \ddot{x}_g(t)^T \ddot{x}_g(t)dt} < \infty \tag{1}$$

The structural response at some time instant is given by using the flexible shear beam model described by the

Euler-Bernoulli equation (2), which adds the effect of shear distortion but not rotary inertial [48], (details can be found in appendix A)

$$m(x)\frac{\partial^2 u(x,t)}{\partial t^2}+c(x)\frac{\partial u(x,t)}{\partial t}+\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right] = f(x,t) \tag{2}$$

whit boundary conditions

$$u(0,t)=u_g,\quad \ddot{u}(0,t)=\ddot{u}_g \tag{3}$$
$$u(1,t)=0,\quad \mu\ddot{u}(1,t)=0 \tag{4}$$

where m and c describe, respectively, the mass and damping coefficient at specific points (x) where the structural behavior is observed, E and I denote the elastic modulus and inertial moment, respectively. Finally, $f(x,t) = -m(x)\ddot{u}_g(t)$, where $\ddot{u}_g(t)$ represents the disturbance generated by the ground acceleration induced by the earthquake.

To address model (2) and determine the solution $u(x,t)$, we employ the decoupling of lateral displacement functionals through the superposition of vibration modes [49].

$$u(x,t)=\sum_{i=1}^{\infty}\phi_i(x)q_i(t) \tag{5}$$

where $\phi_i(x)$ represents the vibration mode, and $q_i(t)$ is the generalized coordinate. Substituting (5) into (2), yields

$$\sum_{i=1}^{\infty}m(x)\phi_i(x)\ddot{q}_i(t)+\sum_{i=1}^{\infty}c(x)\phi_i(x)\dot{q}_i(t) + \sum_{i=1}^{\infty}\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\phi_i(x)}{dx^2}\right]q_i(t) = -\sum_{i=1}^{\infty}m(x)\phi_i(x)\ddot{u}_g(t) \tag{6}$$

Considering the orthogonality condition of the mode shapes,

$$\int \phi_i(x)\phi_j(x)dx = 0,\qquad i\neq j \tag{7}$$

and supposing that

$$EI(x)\left(\frac{d^4\phi_i(x)}{dx^4}\right) = m\omega_i^2\phi_i(x) \tag{8}$$

Substituting (8) into (6), and pre-multiplying both sides of resulting equation by $\phi_i(x)$, yields

$$\begin{aligned}&\int_0^{x(1)}\left[\phi_j(x)m(x)\phi_j(x)\right]\ddot{q}_j(t)\\&\quad+\int_0^{x(1)}\left[\phi_j(x)c(x)\phi_j(x)\right]\dot{q}_j(t)\\&\quad+\int_0^{x(1)}\left[\phi_i(x)m(x)\omega_i^2\phi_i(x)\right]q_j(t)\\&=-\int_0^{x(1)}\left[\phi_j(x)m(x)\phi_j(x)\right]\ddot{u}_g(t)\end{aligned} \tag{9}$$

Assuming that $M_g = \int_0^{x(1)}\phi_j(x)m(x)\phi_j(x)dx$, $C_g = \int_0^{x(1)}\phi_j(x)c(x)\phi_j(x)dx$ and $k_i = m(x)\omega_i^2$, such that, $K_g = \int_0^{x(1)}\phi_j(x)k(x)\phi_j(x)$, hence, the system (9) can be rewritten as

$$M_g\ddot{q}(t)+D_g\dot{q}(t)+K_gq(t) = -M_gl\ddot{u}_g(0,t) \tag{10}$$

Assumption 2: Supposing an equipartitioned spatial discretization with n finite points situated at each story of the building structure and applying the orthogonality condition (7) to matrices M_g, C_g, and K_g in the model (10), yields

$$\begin{aligned}m_{g1}\ddot{q}_1(t)+c_{g1}\dot{q}_1(t)+k_{g1}q_1(t)&=-m_{g1}\ddot{u}_g(t)\\m_{g2}\ddot{q}_2(t)+c_{g2}\dot{q}_2(t)+k_{g2}q_2(t)&=-m_{g2}\ddot{u}_g(t)\\&\vdots\\m_{gn}\ddot{q}_n(t)+c_{gn}\dot{q}_n(t)+k_{gn}q_n(t)&=-m_{gn}\ddot{u}_g(t)\end{aligned} \tag{11}$$

then the matrices and states from the model (10) are defined as follows:

$$\begin{aligned}M_g&=diag\left[m_{g1}\ \ m_{g2}\ \ \ldots\ \ m_{gn}\right]>0\ \in R^{n\times n}\\Cg&=diag\left[c_{g1}\ \ c_{g2}\ \ \ldots\ \ c_{gn}\right]>0\ \in R^{n\times n}\\K_g&=diag\left[k_{g1}\ \ k_{g2}\ \ \ldots\ \ k_{gn}\right]>0\ \in R^{n\times n}\\l&=\left[1\ \ 1\ \ \ldots\ \ 1\right]^T>0\ \in R^{n\times n}\end{aligned} \tag{12}$$

with

$$\begin{aligned}q&=[q_1(t),q_2(t),\ldots,q_n(t)]^T\ \in R^{n\times 1},\\\dot{q}&=[\dot{q}_1(t),\dot{q}_2(t),\ldots,\dot{q}_n(t)]^T\ \in R^{n\times 1},\\\ddot{q}&=[\ddot{q}_1(t),\ddot{q}_2(t),\ldots,\ddot{q}_n(t)]^T\ \in R^{n\times 1}\end{aligned}$$

where M_g, C_g and K_g are, respectively, the modal matrices corresponding to lumped mass, damping and stiffness. These matrices are consistently positive definite. Moreover, q, $\dot{q}$, and $\ddot{q}$ represent the generalized vectors of displacement, velocity, and acceleration, allowing the retrieval of the original coordinates through the expression (5).

Assumption 3: Initially, before earthquake perturbation-meaning $\ddot{x}_g(0) = 0$, the building structure is at rest, that is to say

$$q(0)=0,\quad \dot{q}(0)=0,\quad \ddot{q}(0)=0 \tag{13}$$

Examining the model (10) and incorporating the variable change $v(t) = [v_1(t),\ v_2(t)]^T = [q(t),\ \dot{q}(t)]^T$, thus the system (17) can be represented in the general linear state-space form:

$$\begin{aligned}\dot{v}(t)&=Av(t)+B\ddot{u}_g(t)\\y(t)&=Cv(t)\end{aligned} \tag{14}$$

where

$$A=\begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{I}_{n\times n}\\-K_gM_g^{-1} & -C_gM_g^{-1}\end{bmatrix},\quad B=-\begin{bmatrix}\mathbf{0}_{\mathbf{n}\times\mathbf{1}}\\l\end{bmatrix},$$
$$C=\begin{bmatrix}-K_gM_g^{-1} & -C_gM_g^{-1}\end{bmatrix} \tag{15}$$

Moreover, the system (10) can be rewritten as

$$\ddot{q}(t)+M_g^{-1}C_g\dot{q}(t)+M_g^{-1}K_gq(t) = -l\ddot{u}_g(t) \tag{16}$$

equivalent to

$$\ddot{\mathbf{q}}(t) + 2\Xi\Omega\dot{\mathbf{q}}(t) + \Omega^{\mathbf{2}}\mathbf{q}(t) = -\mathbf{l}\ddot{u}_g(t) \tag{17}$$

with

$$\Xi = diag[\xi_1, \xi_2, \ldots, \xi_n] \in \mathbb{R}^{n\times n}$$
$$\Omega = diag[\omega_1, \omega_2, \ldots, \omega_n] \in \mathbb{R}^{n\times n}$$

here Ξ and Ω are diagonal matrices representing damping and natural frequencies, respectively.

III. ALGEBRAIC OBSERVER DESIGN

In this section, we formulate and analyze the Algebraic Observer (AO) drawing inspiration from [47] and considering the Algebraic Observability Condition (AOC):

Definition 1: The vector values ν satisfy the AOC if it can be represented as a function of the input u and output y of the system along with their sequential derivatives, that is

$$\nu = \mathbf{G}(y, \dot{y}, \cdots, y^{(r_1)}, u, \dot{u}, \cdots, u^{(r_2)}) \tag{18}$$

where $r_1, r_2 \in \mathbb{N}$ and $\mathbf{G}(\cdot)$ is a continuous vector function.

Remark 1: If a system satisfies the AOC, then the observer is feasible and will be called an algebraic observer, where $\gamma(t)$ is a vector containing the states to estimate.

Remark 2: Supposing that the seismic disturbance is bounded, and the building response stays within the elastic range, both the velocity vector $\dot{u}$ and displacements u also remain bounded. This imply that $||\boldsymbol{\nu}_i(\cdot)||$ is upper-bounded by $\psi_i \in \mathbb{R}^+$ such that $\psi_i = \sup_t ||\boldsymbol{\nu}_i(\cdot)||$ with $i \in \{1, 2\}$.

A. FORMULATION OF THE ALGEBRAIC OBSERVER

Let the decoupled dynamic system (10). Note that each linear equation described in (11) can be represented in the following form

$$m_{gi}(\ddot{q}_i(t) + \ddot{u}_g(t)) + c_{gi}\dot{q}_i(t) + k_{gi}q_i(t) = 0$$
$$y_i(t) = \ddot{q}_i(t) \tag{19}$$

with $i = 1, 2, \ldots, n$. Moreover, $y_i(t)$ is the output of the system, which represents the available acceleration.

Remark 3: Considering that acceleration measurements are available, we proceed to estimate the displacement and velocity signals through purely algebraic development.

For simplicity, the model (19) is rewritten as follows

$$z_i(t) = a_i\dot{q}_i(t) + b_i q_i(t)$$
$$y_i(t) = \ddot{q}_i(t) \tag{20}$$

In this context, the auxiliary variable $z_i = (\ddot{q}i + \ddot{u}_g)$ is introduced to represent the absolute acceleration measurement. The parameters $a_i = (-cgi/m_{gi})$ and $b = (-k_{gi}/m_{gi})$ are considered as known constants.

Now, our objective is to estimate the states $q_i(t)$ and $\dot{q}_i(t)$ based on the measurable variables. To achieve this, the AOC condition must be satisfied.

Theorem 1: Since the variables of the system $q_i(t)$ and $\dot{q}_i(t)$ can be estimated and expressed as functions of the known variables $z_i(t)$ and $y_i(t)$, with a finite number of their time derivatives. We can affirm that the conditions outlined in Definition 1, on Algebraic Observability Condition, have been fulfilled and satisfied. Hence, the system (10) is observable, and the development of the algebraic observer is feasible once the coefficient b_i satisfies $b_i \neq 0$.

Proof: To ensure Definition 1, we initially differentiate $z_i(t)$ to time, and upon performing the necessary operations, yields:

$$\dot{q}_i(t) = b_i^{-1}[\dot{z}_i(t) - a_i y_i(t)] \tag{21}$$

Subsequently, $q_i(t)$ is determined from (20), which depends on $\dot{q}_i(t)$. The latter is then substituted with its previously derived expression from (21), enabling the calculation of displacement as follows:

$$q_i(t) = b_i^{-1}[z_i(t) - a_i b_i^{-1}(\dot{z}_i(t) - a_i y_i(t))] \tag{22}$$

■

The algebraic observer formulation to estimate the states $q_i(t)$ and $\dot{q}_i(t)$ involves the application of the following expression:

$$P_i(t)\gamma_i(t) = F_i(t), \tag{23}$$

where $\gamma_i(t)$ represents the estimated vector containing displacement, velocity, the integral of displacement, and other parameter. Vector $\gamma_i(t)$ is defined as follows:

$$\gamma_i(t) = \left(\dot{q}_i(t), q_i(t), \int_0^t q_i(\tau_1)d\tau_1, \int_0^t \tau_1 q_i(\tau_1)d\tau_1\right), \tag{24}$$

and $F_i(t)$ is a vector calculated from the integration of the available measurements, defined as follows:

$$\begin{aligned} F_i(t) &= \begin{bmatrix} F_{1i}(t) & F_{2i}(t) & F_{3i}(t) & F_{4i}(t) \end{bmatrix}^T \\ &= \begin{bmatrix} \int_0^t \tau_1^2 y_i(\tau_1)d\tau_1 \\ \int_0^t \tau_1^3 y_i(\tau_1)d\tau_1 \\ \int_0^t \tau_1 z_i(\tau_1)d\tau_1 \\ \int_0^t \tau_1 z_i(\tau_1)d\tau_1 - \int_0^t \int_0^{\tau_1}(z_i(\tau_2) + a\tau_2 y_i(\tau_2))(\tau_2)d\tau_2 d\tau_1 \end{bmatrix} \end{aligned} \tag{25}$$

and finally, $P_i(t)$ is a matrix constructed with constant parameters a_i and b_i, alongside values of t at each time instant. This matrix is defined as follows:

$$P_i(t) = \begin{bmatrix} t^2 & -2t & 2 & 0 \\ t^3 & -3t^2 & 0 & 6 \\ 0 & a_i t & -a_i & b_i \\ 0 & 0 & -b_i t & 2b_i \end{bmatrix} \tag{26}$$

The matrix $P_i(t)$ must be invertible according to (23). This condition is met when $-b_1^2 t^5 > 0$ as required by the matrix determinant, meaning that $t > 0$ and $b_i \neq 0$, thereby satisfying the AOC. Therefore, we define $t \geq \epsilon > 0$ for a very small arbitrary ϵ. On the other hand, it is important to note that as t approaches infinity, the Euclidean norm $|K_o(t)| \to \infty$, implying that time must also be bounded to a maximum value ($t_{\max}$). Consequently, the estimated states are redefined as:

$$\gamma_{ei}(t)=\left(\dot{q}_{ei}(t), q_{ei}(t), \left[\int_0^t q_{ei}(\tau)d\tau\right]_e, \left[\int_0^t \tau q_{ei}(\tau)d\tau\right]_e\right) \tag{27}$$

and are calculated as:

$$\gamma_{ei}(t)=\begin{cases} arbitrary & for\ t\in[0,\epsilon) \\ P_i^{-1}(t)F_i(t) & for\ t\in[\epsilon, t_{max}] \end{cases} \tag{28}$$

Proof: Let the state algebraic observer (23) constructed by a set of time-varying linear equations. The methodology to obtain these variables is as follows.

The first step involves multiplying the measured output $y(t)$ by t_2 and t_3. The subsequent integration by parts of the resulting expression yields

$$\begin{aligned} F_{1i}(t,y_i) &\triangleq \int_0^t \tau_1^2 y_i(\tau_1)d\tau_1 = \int_0^t \tau_1^2 \ddot{q}_i(\tau_1)d\tau_1 \\ &= t^2\dot{q}_i(t) - 2\int_0^t \tau_1\dot{q}_i(\tau_1)d\tau_1 \\ &= t^2\dot{q}_i(t) - 2tq_i(t) + 2\int_0^t q_i(\tau_1)d\tau_1 \end{aligned} \tag{29}$$

the second expression is:

$$\begin{aligned} F_{2i}(t,y_i) &\triangleq \int_0^t \tau_1^3 y_i(\tau_1)d\tau_1 = \int_0^t \tau_1^3 \ddot{q}_i(\tau_1)d\tau_1 \\ &= t^3\dot{q}_i(t) - 3t^2q_i(t) + 6\int_0^t \tau_1 q_i(\tau_1)d\tau_1 \\ &= t^3\dot{q}_i(t) - 3t^2q_i(t) + 6\int_0^t \tau_1 q_i(\tau_1)d\tau_1 \end{aligned} \tag{30}$$

The third equation is obtained by multiplying the variable $z_i(t)$ from (20) by t and subsequently integrating with respect to time, as follows:

$$\begin{aligned} F_{3i}(t,z_i) &\triangleq \int_0^t \tau_1 z_i(\tau_1)d\tau_1 = \\ &= a_i\left(tq_i(t) - \int_0^t q_i(\tau_1)d\tau_1\right) + b_i\int_0^t \tau_1 q_i(\tau_1)d\tau_1 \end{aligned} \tag{31}$$

On the other hand, to obtain the fourth equation, both sides of equation (20) are differentiated concerning time, producing

$$\dot{z}_i(t) = a_i y_i(t) + b_i\dot{q}_i(t) \tag{32}$$

Multiplying both sides of the expression (20) by t and integrating it twice yields:

$$\begin{aligned} &\int_0^t\int_0^{\tau_1} \tau_2\dot{z}_i(\tau_2)d\tau_2 d\tau_1 - a_i\int_0^t\int_0^{\tau_1} \tau_2 y_i(\tau_2)d\tau_2 d\tau_1 \\ &= b_i\int_0^t\int_0^{\tau_1} \tau_2\dot{q}(\tau_2)d\tau_2 d\tau_1 \end{aligned} \tag{33}$$

Note that (33) depends on unavailable measurement signals $\dot{z}_i(t)$ and $\dot{q}_i(t)$. To address this issue, we rewrite integrals in terms of known variables. Firstly, we eliminate the time derivatives $\dot{z}_i(t)$ and $\dot{q}_i(t)$. Thus, equation (33) is integrated by parts over the time interval $[0, t]$, yielding:

$$\begin{aligned} F_{4i}(t,z_i,y_i) &\triangleq \int_0^t \tau_1 z_i(\tau_1)d\tau_1 \\ &\quad - \int_0^t\int_0^{\tau_1} (z_i(\tau_2) + a_i\tau_2 y_i(\tau_2))(\tau_2)d\tau_2 d\tau_1 \\ &= b_i\int_0^t \tau_1 q_i(\tau_1)d\tau_1 - b_i\int_0^t\int_0^{\tau_1} q_i(\tau_2)d\tau_2 d\tau_1 \end{aligned} \tag{34}$$

Moreover, integrating by part terms into the expression (34), such that

$$\begin{aligned} &\int_0^t \tau_1 q_i(\tau_1)d\tau_1 \\ &\quad = t\int_0^t q_i(\tau_1)d\tau_1 - \int_0^t\int_0^{\tau_1} q_i(\tau_2)d\tau_2 d\tau_1 \\ &\int_0^t\int_0^{\tau_1} q_i(\tau_2)d\tau_2 d\tau_1 \\ &\quad = t\int_0^t q_i(\tau_1)d\tau_1 - \int_0^{\tau_1} \tau_1 q_i(\tau_1)d\tau_1 \end{aligned}$$

and substituting these expressions into (34), the function $F_{4i}(t, z_i, y_i)$ finally can be expressed as:

$$F_{4i}(t,z_i,y_i) = 2b\int_0^t \tau_1 q_i(\tau_1)d\tau_1 - bt\int_0^t q_i(\tau_1)d\tau_1 \tag{35}$$

From equations (29), (30), (31), and (35), four independent equations in four unknown variables are derived. Considering the change of variable $\psi_1(t) = \dot{q}_i(t)$, and iteratively integrating this variable, then $\psi_2(t) = q_i(t)$, $\psi_3(t) = \int_0^t q_i(\tau_1)d\tau_1$, and $\psi_4(t) = \int_0^t \tau_1 q_i(\tau_1)d\tau_1$, representing the desired states, the set of linear equations is then expressed as follows:

$$\begin{aligned} F_{1i}(t,y_i) &= t^2\psi_1(t) - 2t\psi_2(t) + 2\psi_3(t) \\ F_{2i}(t,y_i) &= t^3\psi_1(t) - 3t^2\psi_2(t) + 6\psi_4(t) \\ F_{3i}(t,y_i) &= a_i t\psi_2(t) - a_i\psi_3(t) + b_i\psi_4(t) \\ F_{4i}(t,z_i,y_i) &= 2b_i\psi_4(t) - b_i t\psi_3(t) \end{aligned} \tag{36}$$

such that (36) can be simplified in matrix form (23), where

$$F_i(t) = P_i(t)\gamma_i(t)$$

with

$$\begin{aligned} F_i(t) &= \begin{bmatrix} F_{1i}(t,y_i) & F_{2i}(t,y_i) & F_{3i}(t,y_i) & F_{4i}(t,z_i,y_i) \end{bmatrix}^T \\ P &= \begin{bmatrix} t^2 & -2t & 2 & 0 \\ t^3 & -3t^2 & 0 & 6 \\ 0 & a_i t & -a_i & b_i \\ 0 & 0 & -b_i t & 2b_i \end{bmatrix} \\ \gamma_i &= \begin{bmatrix} \psi_1(t) & \psi_2(t) & \psi_3(t)\psi_4(t) \end{bmatrix}^T \end{aligned}$$

By substituting the original change of variables $\psi_1(t) = \dot{q}_i(t)$, $\psi_2(t) = q_i(t)$, $\psi_3(t) = \int_0^t q_i(\tau_1)d\tau_1$, and $\psi_4(t) = \int_0^t \tau_1 q_i(\tau_1)d\tau_1$ into the vector $\gamma_i(t)$, then the algebraic

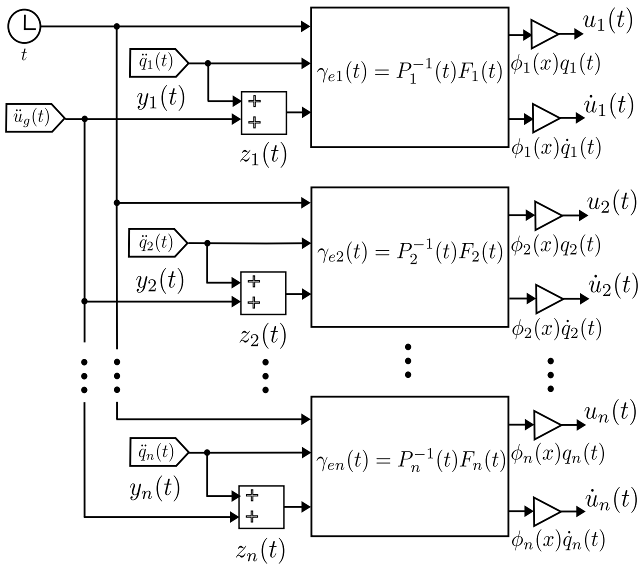


FIGURE 2. Algebraic observer scheme for multi-degree of freedom systems.

observer (23) is retrieved, with the estimated state vector $\gamma_i(t)$ defined by (24),matrix $P_i(t)$ described by (26), and finally, the integration of the available acceleration measurement $F_i(t)$ indicated by (25). ■

Remark 4: For the multidimensional case model (10), with n number of states, the following statement is asserted: Considering the values a_i and b_i as parameters of the algebraic observer, the matrix $P_i(t)$ exhibits an identical structure for each vector $F_i(t)$. This matrix acts as the pathway for measurable signals, producing a set of desired signals represented by the vector $\gamma_{ei}(t)$, where $i = 1, 2, \ldots, n$, as follows:

$$\begin{aligned}\gamma_e(t) &= \left[\gamma_{e1}(t),\ \ \gamma_{e2}(t),\ \ \ldots,\ \ \gamma_{en}(t)\right]\\ &= \left[P_1^{-1}(t)F_1(t),\ \ P_2^{-1}(t)F_2(t),\ \ \ldots,\ \ P_n^{-1}(t)F_n(t)\right]\end{aligned} \tag{37}$$

Finally, the estimated signals are transformed to their original coordinates u by applying the modal transformation (5).

The proposed algebraic observer scheme is summarized in the following Fig. 2. "Note that, for practical implementation, the measurement noise present in acceleration data must be attenuated using filters, taking into account the bandwidth.

IV. EXPERIMENTAL EVALUATION

To illustrate the design methodology and the performance of our observer, we will present the experimental results generated in a realistic framework, using a prototype of a reduced-scale five-story building constructed of aluminum and brass (see Fig. 3).

The prototype has a rectangular base with dimensions of 60 cm×50 cm and a maximum height of 180 cm. The building is divided into five levels with an inter-story separation of 36 cm. Each floor contains four columns, three of which are made from brass with a square cross-section measuring

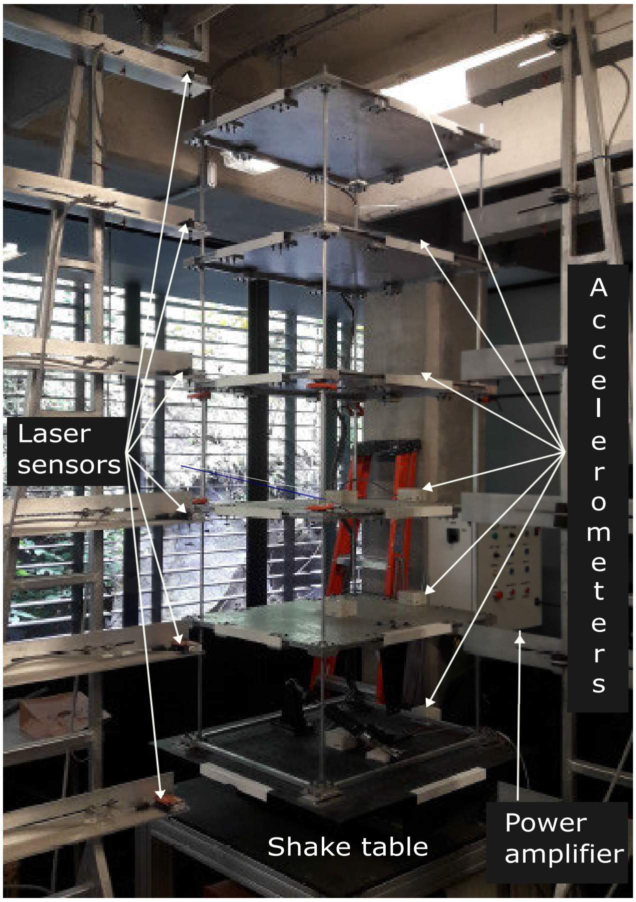


FIGURE 3. Five-story building prototype.

0.635 cm × 0.63 cm, whereas the remaining one is also made from brass. The complete structure is mounted on a shake table, moved by servomotors from the Parker model 406T03LXR. This setup emulates earthquake movements to generate displacement and velocity signals on each floor during experimentation. Moreover, accelerometers, based on MEMS technologies model ADXL203E with a measuring range of ±1.7 g and bandwidth of [0 − 50] Hz, are located at each floor, and one more is located at the base of the prototype. Finally, several OptoNCDT sensors from Micro-Epsilon, specifically the model ILD1302-200 are used to measure the displacement of each floor. It is important to note that this information is only used as reference data to compare the obtained by the proposed algebraic observer. All devices are insulated to an industrial computer utilizing two PCI-6221-M series electronic boards from National Instruments. The workstation operates in Windows XP with Matlab 2011a Real-Time Windows Target toolbox/Simulink using a sample time set to 1 ms.

To gain preliminary insights into the building's behavior, the prototype was initially excited using a chirp signal with a swept frequency ranging from 0.1 Hz to 25 Hz. The frequency response enabled the estimation of the structural bandwidth, corresponding to $f_1 = 1.58$ Hz, $f_2 = 4.76$ Hz, $f_3 = 7.51$ Hz,

$f_4 = 9.83$ Hz, and $f_5 = 11.51$ Hz. Moreover, to attenuate high-frequency noise, real-time filtering is applied to the acceleration measurements using a second-order low-pass filter with a cut-off frequency of 15 Hz, aligning with the dynamic properties of the employed building prototype.

On the other hand, the nominal parameters were identified offline through the Least Square method proposed in [50], requiring the acceleration measurement. Results are presented in the following Table 1.

TABLE 1. Parameters of the reduced scale five-story building prototype.

Floor	m_i [kg]	$\hat{c}_i$ [N·s/m]	$\hat{k}_i$ [N/m]
1	10.78	700.14	19011
2	9.17	278.18	12108
3	9.14	546.84	11966
4	9.12	372.54	11850
5	9.08	375.97	12406

A. ALGEBRAIC OBSERVER IMPLEMENTATION

For experimental analysis, we utilized the North-South component of the Mexico City 1985 earthquake, provided by the Secretaría de Comunicaciones y Transportes (SCT), to perturb the system. The signal was temporally scaled by approximately fivefold and magnified by about two times in magnitude. It exhibits a fundamental frequency of approximately 1.5 Hz, making it well-suited for exciting the vibration modes of the five-story building prototype, as illustrated in Fig. 4.

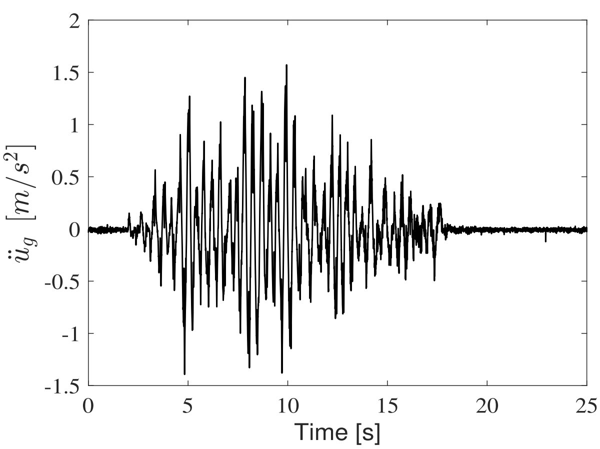


FIGURE 4. Reduce scale of the North-South component of the México city 1985 earthquake.

To implement the proposed algebraic observer (37), the system is initially represented in modal coordinates. To accomplish this task, the parameters in Table 1 are transformed using the equation (5) in matrix form, such that

$$[\phi_{jn}] = \begin{bmatrix} 0.121 & -0.372 & -0.575 & -0.553 & -0.312 \\ 0.299 & -0.621 & -0.250 & 0.443 & 0.539 \\ 0.450 & -0.393 & 0.538 & 0.165 & -0.594 \\ 0.561 & 0.143 & 0.339 & -0.598 & 0.473 \\ 0.616 & 0.549 & -0.449 & 0.337 & -0.187 \end{bmatrix} \tag{38}$$

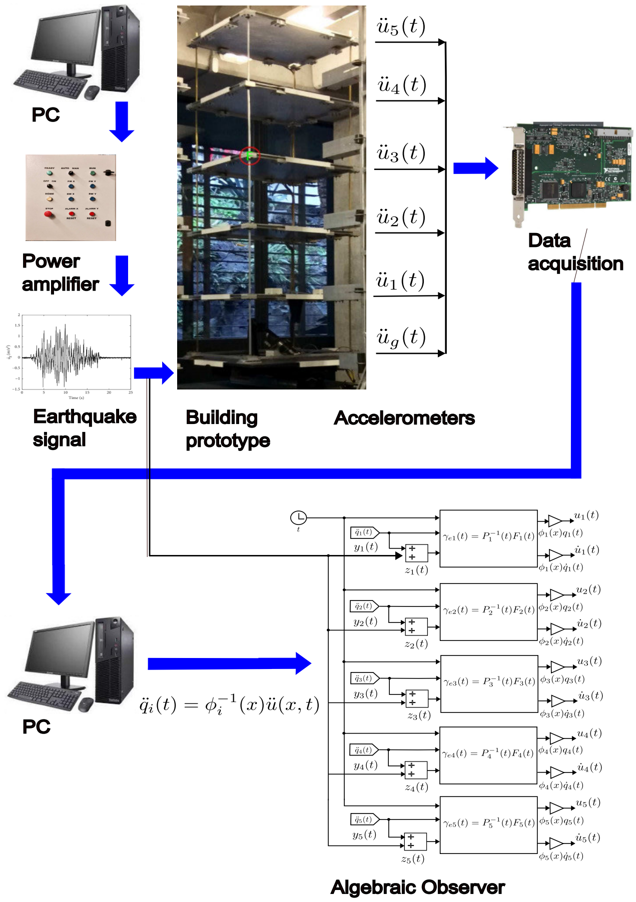


FIGURE 5. Experimental setup.

that produces

$$M_g = diag\begin{bmatrix} 9.1378 & 9.3594 & 9.6701 & 9.6330 & 9.3019 \end{bmatrix} \tag{39}$$

$$C_g = diag\begin{bmatrix} 0.0373 & 0.3114 & 0.8495 & 1.0774 & 1.5604 \end{bmatrix} \times 1.0e+03 \tag{40}$$

$$K_g = diag\begin{bmatrix} 0.1118 & 0.9449 & 2.3188 & 3.6477 & 4.4902 \end{bmatrix} \times 1.0e+04 \tag{41}$$

The transformed system is now represented in the form of equation (20), such that $a_1 = -4.0819$, $b_1 = -122.3489$, $a_2 = -33.2714$, $b_2 = -1009.6$, $a_3 = -87.8481$, $b_3 = -2397.9$, $a_4 = -111.8447$, $b_4 = -3786.7$, $a_5 = -167.7507$ and $b_5 - 4827.2$.
This information is useful for constructing the algebraic observer and thereby recover unavailable measurements. The Fig. 5 illustrates the experimental setup used.

On the other hand, to compare the proposed algebraic observers, a Luenberger observer is introduced by utilizing the following state estimator

$$\begin{aligned} \dot{\hat{v}}(t) &= A\hat{v}(t) + B\ddot{u}_g(t) + L(y_m - \hat{y}_m) \\ \hat{y}_m(t) &= D\hat{v}(t) \end{aligned} \tag{42}$$

where matrices A, B and D have been previously defined in equation (15). Additionally, $L \in R^{2n \times 2n}$ is an appropriate observer gain matrix.

Remark 5: Let the space-state definitions (42) and considering the parameters in Table 1, the observability matrix $\mathcal{O} = (\mathcal{C}^T,\ \mathcal{A}^T\mathcal{C}^T)$ is full range, which has been reported in [51].

Note that for experimental evaluation, the algebraic observer does not require tuning gains considering the aforementioned algebraic design (28); meanwhile, for the Luenberger observer the gains were chosen as $L = (0_{5\times5} \quad -\gamma I_{5\times5})^T$, with $\gamma = 10^5$ following the methodology outlined in [31].

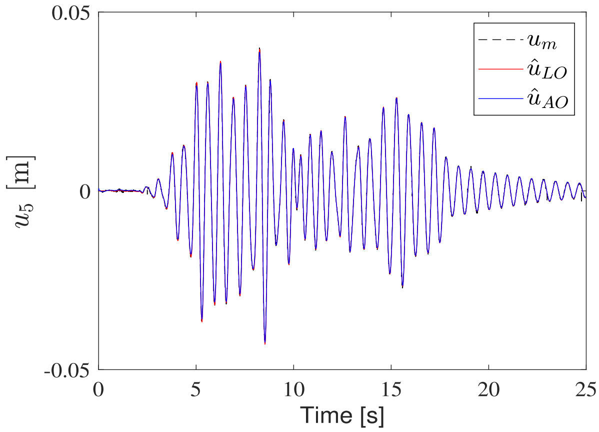
Moreover, the initial conditions for both observers were set to null, i.e., $\nu = 0$. The estimated displacements by the algebraic observer $\hat{u}_{AO}$ and the Luenberger observer $\hat{u}_{LO}$, in comparison with u_m, represent the measured displacement by laser sensors. Figs. 6(a) and 7(a) present the corresponding comparison for the fifth and third floors, respectively. The results demonstrate the proximity of the estimated states to the actual states, with errors consistently decreasing over time, as illustrated in Figs. 6(b) and 7(b), respectively. For instance, the errors are less than 7×10^{-3}, demonstrating the accurate estimation of building displacements by the proposed observers. Note that the fifth and third stories were selected arbitrarily; however, the rest of the floors exhibit a similar behavior in terms of convergence.

Similar results are presented in velocity estimation. However, since velocities cannot be measured directly, the references are generated by filtering displacements $u_i(t)$, $i = 1, 2, \ldots, n$, using the following band-pass filter, such that it attenuates low-frequency noise below 0.1 Hz and high-frequency noise above 15 Hz, respectively.

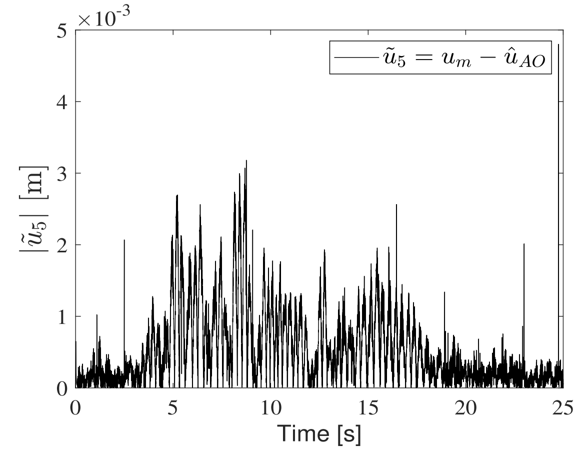
$$f(s) = \frac{100s \times 100}{(s+100)^2}\frac{s^2}{s^2 + 3.77s + 3.56} \tag{43}$$

Figs. 8(a) and 9(a) present a comparison between the signals estimated by the algebraic and Luenberger observers, and the reference signal obtained by filtering the displacement data, corresponding to stories 5 and 3, respectively. Velocity estimation errors on the same floors are presented in Figs. 8(b) and 9(b). In both instances, it is observed that the signal estimated by the algebraic observer converges to values close to the reference values. The estimate is even superior in some cases compared to the results obtained with the Luenberger observer. Indeed, the norm of the estimation error in steady state is approximately 5%, as depicted in Figs.8(b) and 9(b).

Moreover, to assess the performance and accuracy of the proposed algebraic observer in comparison to the Luenberger observer and real measurements, estimations are compared using three different metrics: the root mean square error (RMSE), the correlation coefficient R^2, and the mean absolute error (MAE). These metrics are defined as follows:



(a) Comparison of estimated displacement with respect real measurement and Luenberger observer



(b) Estimation error norm between measured and estimated displacement

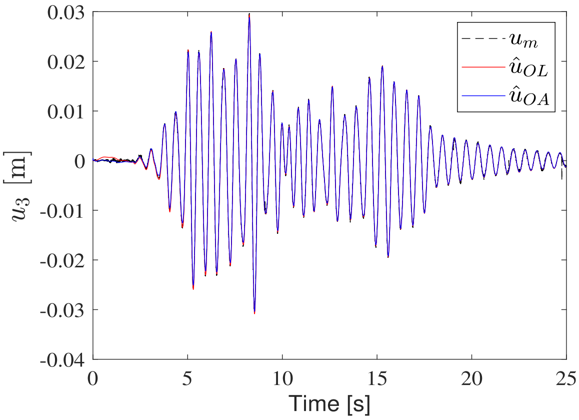
FIGURE 6. **Displacement estimation on the fifth floor.**

$$RMSE = \frac{1}{N}\sum_{i=1}^{N}\sqrt{(\nu_i - \hat{\nu}_i)^2}$$
$$R^2 = 1 - \frac{\frac{1}{N}\sum_{i=1}^{N}(\nu_i - \hat{\nu}_i)^2}{\frac{1}{N}\sum_{i=1}^{N}(\nu_i - \bar{\nu}_i)^2}$$
$$MAE = \frac{1}{N}\sum_{i=1}^{N}|\nu_i - \hat{\nu}_i| \tag{44}$$

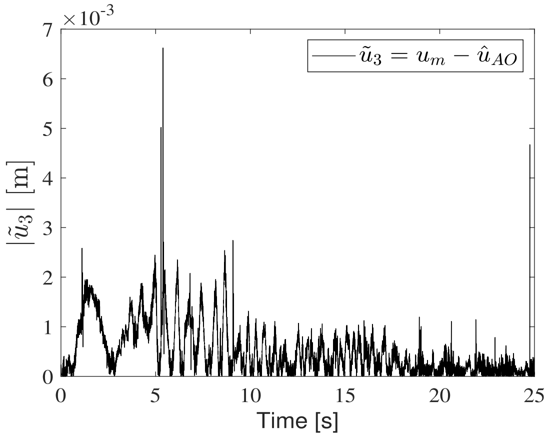
where ν_i and $\hat{\nu}_i$ are the vector of measurement and its estimate, respectively with $i \in \{1, 2\}$.

V. DISCUSSION

This section provides a concise summary of the main findings. Across all experiments, the algebraic observer consistently delivers estimates closer to the experimental values compared to the Luenberger observer, even in scenarios with measurement noise. This highlights the versatility of the proposed observer. The results presented in Tables 2 and 3 demonstrate that the estimated states almost converge to the actual measurements, indicating the

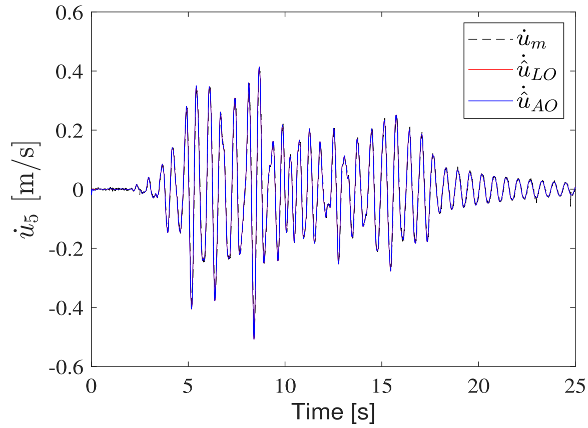


(a) Comparison of estimated displacement with respect real measurement and Luenberger observer

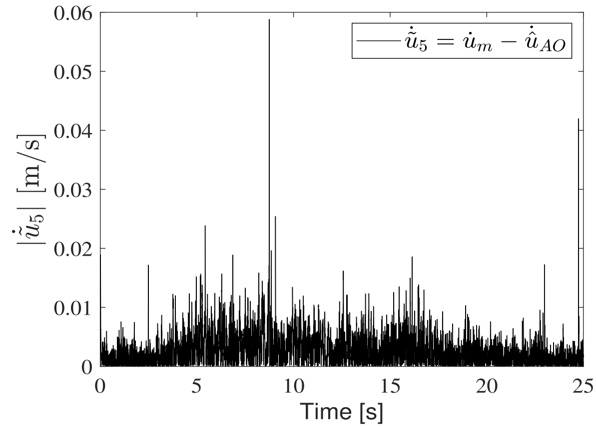


(b) Estimation error norm between measured and estimated displacement

FIGURE 7. Displacement estimation on the third floor.



(a) Comparison of estimated velocity with respect real measurement and Luenberger observer



(b) Estimation error norm between measured and estimated velocity

FIGURE 8. Velocity estimation on the fifth floor.

TABLE 2. Performance metrics for estimated displacement.

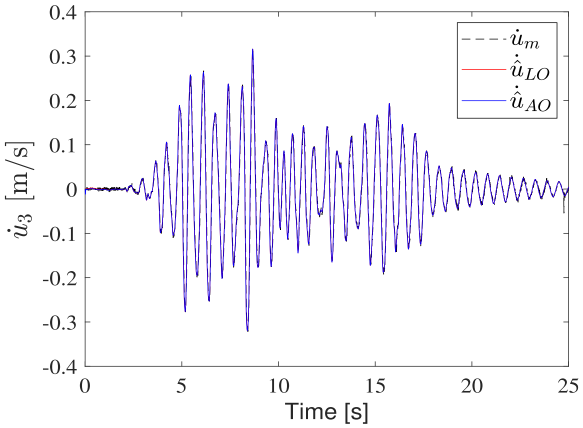
Story	Observer	*RMSE*	R^2	*MAE*
First	Algebraic O.	0.0008	0.8922	5.67×10^{-04}
	Luenberger	0.0008	0.8170	5.52×10^{-04}
Second	Algebraic O.	0.0017	0.9198	0.0012
	Luenberger	0.0023	0.8942	0.0015
Third	Algebraic O.	0.0021	0.9465	0.0015
	Luenberger	0.0027	0.9114	0.0026
Fourth	Algebraic O.	0.0025	0.9510	0.0018
	Luenberger	0.0034	0.9134	0.0023
Fifth	Algebraic O.	0.0028	0.9520	0.0020
	Luenberger	0.0039	0.9058	0.0027

TABLE 3. Performance metrics for estimated velocities.

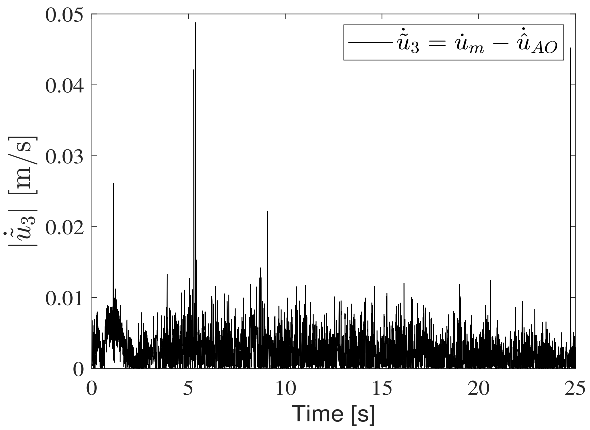
Story	Observer	*RMSE*	R^2	*MAE*
First	Algebraic O.	0.0136	0.7501	0.0096
	Luenberger	0.0123	0.7311	0.0082
Second	Algebraic O.	0.0284	0.8182	0.0203
	Luenberger	0.0234	0.8420	0.0151
Third	Algebraic O.	0.0374	0.8490	0.0268
	Luenberger	0.0337	0.8553	0.0223
Fourth	Algebraic O.	0.0455	0.8543	0.0324
	Luenberger	0.0437	0.8449	0.0281
Fifth	Algebraic O.	0.050	0.8499	0.0360
	Luenberger	0.047	0.8501	0.0308

superior performance of the algebraic observer compared to the Luenberger observer. The experimental assessment using the algebraic observer applied to the reduced-scale five-story building prototype demonstrates an accuracy level of approximately $R^2 = 0.93$ for measured displacement and $R^2 = 0.82$ for estimated velocity. Moreover, the algebraic observer consistently achieved an average R-squared value of 0.82 across all cases, maintaining RMSE and MAE errors below 5×10^{-2}. The steady-state errors were observed to be relatively small, hovering around 6%. Similarly, the Luenberger observer exhibited commendable performance, exhibiting an average R-squared value of 0.82 and error metrics averaging below 3.5×10^{-2}. In light of these results, the algebraic observer stands out as a dependable and practical solution for displacement and velocity estimation in building structures based on acceleration measurements. The performance reaffirms the accuracy of the proposed estimator for displacement and velocity estimation tasks, despite the time-varying behavior.

Although the results are favorable, the authors recognize that more extensive research must be carried out in this direction. For instance, a significant current limitation of the proposed algebraic observer is its design for linear systems or those with integral nonlinearities. However, this limitation can be overcome by employing exact linearization. Moreover, the original formulation of the algebraic observer



(a) Comparison of estimated velocity respect real measurement and Luenberger observer



(b) Estimation error norm between measured and estimated velocity

FIGURE 9. Velocity estimation on the third floor.

is constrained for one degree of freedom system. To extend its applicability to structures like multi-story buildings or other systems with multiple degrees of freedom, this limitation is addressed by employing modal transformations, which allow to the decoupling of the model into a set of uncoupled ordinary differential equations, satisfying the model's form (20).

Regarding experimental testing, a challenge with this approach arises from the lack of precise parameter values for the building prototype. In our study, we identified the structural parameters using a normalized least squares algorithm with a forgetting factor. We compared the estimated with the original response to ensure accuracy. This process ensured that the estimated vibration frequencies matched to the real values.

Additionally, the presence of measurement noise can significantly degrade the performance of the state observer, resulting in less accurate estimates of the state variables and prolonged convergence time. Moreover, significant measurement noise has the potential to induce oscillations in the state estimate. The ultimate effect is mitigated by incorporating iterated integrals into the algebraic observer. The performance was enhanced by filtering the acceleration measurements associated with both the system input and output. The effectiveness of these filters depends on the accurate selection of cutoff frequencies. In our study, we conducted various tests to meticulously choose cutoff frequencies aimed at minimizing phase shifts.

VI. CONCLUSION

The proposed algebraic observer emerges as an alternative for recovering unknown signals, especially considering that, in several practical scenarios, the acceleration signal frequently stands as the only available measurement for various systems. The ease of implementation for practical purposes is a distinct feature of this observer. In contrast to other methodologies proposed in the literature, our approach does not require tuning gains to achieve the convergence property. Note that the majority of methods, including the Luenberger observer, necessitate precise gain-tuning for convergence to the actual states of the system. Experimental results confirm the promising potential of the algebraic observer for real-time applications. Considering their convergence time, they could operate concurrently for vibration control and structural health monitoring, addressing the necessity for displacement and velocity in feedback control, and contributing to the mitigation of vibrations and damage diagnosis in buildings.

However, it is crucial to emphasize that the effectiveness of the algebraic observer relies on accurately identified parameters and attenuating measurement noise. In scenarios characterized by parameter uncertainties and measurement noise, the effectiveness of the proposed algebraic observers may degrade.

APPENDIX A
ANALYSIS OF THE EULER-BERNOULLI BEAM MODEL

Let the shear beam depicted in Fig. 1, with stiffness denoted by $EI(x)$ and a mass per unit length of $m(x)$. Let $f(x, t)$ represent the distribution of force applied to the beam, $u(x, t)$ denote the displacement of the beam, E describe the elastic modulus and $I(x)$ represent the inertial moment.

Consider the free body diagram of a differential element of the beam dx as shown in Fig. 1. The equilibrium equation for the sum of forces is as follows:

$$\frac{\partial V(x,t)}{\partial x}dx - m(x)\frac{\partial^2 u(x,t)}{\partial t^2}dx + f(x,t)dx = 0 \tag{45}$$

which implies that

$$\frac{\partial V(x,t)}{\partial x} - m(x)\frac{\partial^2 u(x,t)}{\partial t^2} + f(x,t) = 0 \tag{46}$$

Now, the equilibrium equation for the sum of moments is

$$\begin{aligned}\left(V(x,t) + \frac{\partial V(x,t)}{\partial x}\right) dx + f(x,t)dx\frac{dx}{2} \\ - m(x)\frac{\partial^2 u(x,t)}{\partial t^2}dx\frac{dx}{2} + M(x,t) \\ + \frac{\partial M(x,t)}{\partial x}dx - M(x,t) = 0\end{aligned} \tag{47}$$

If we neglect higher-order differentials, we have to

$$V(x,t) + \frac{\partial M(x,t)}{\partial x} = 0 \tag{48}$$

If we do not take into account the shear deformations, we have that the curvature is

$$\frac{\partial u(x,t)}{\partial x} = \mathcal{O} \tag{49}$$

Additionally

$$M(x,t) = EI(x)\frac{\partial \mathcal{O}}{\partial x} = EI(x)\frac{\partial^2 u(t,x)}{\partial x^2} \tag{50}$$

Substituting (50) into the moment equilibrium equation (48)

$$V(x,t) = -\frac{\partial M(x,t)}{\partial x} = -\frac{\partial}{\partial x}\left(EI(x)\frac{\partial^2 u(t,x)}{\partial x^2}\right) \tag{51}$$

Moreover, substituting (51) into the force balance equation (46), we obtain

$$-\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 u(t,x)}{\partial x^2}\right) + m(x)\frac{\partial^2 u(x,t)}{\partial^2 t} = f(x,t) \tag{52}$$

Finally, to achieve a response closer to reality, we incorporate the damping term $c(x)\frac{\partial u(x,t)}{\partial t}$ into the model (52), resulting in the complete Euler-Bernoulli beam model as expressed in equation (2).

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