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RESEARCH ARTICLE

A Novel Approach for Fuzzification of Rough Sets Based on Fuzzy Preference Relation: Properties and Application to Medicine Selection Problem

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ABSTRACT Preference analysis is a significant component in decision-making (DM) when selecting an optimal alternative. By comparing any two alternatives pairwise, preference relations (PRs) effectively depict the preference degrees of decision-makers (DMrs). The rough set theory (RST) has been effectively applied to cope with preference analysis by swapping the equivalence relation (Er) with the dominance relation (DR). In this study, we propose new transfer functions to construct alternatives' upward/downward fuzzy preference degree (FPD) for evaluating upward and downward fuzzy PRs (FPRs). Based on these newly proposed transfer functions, we present a novel method for fuzzifying RSs called the *upward α-fuzzified preference rough sets* ($\alpha^{\uparrow}$-FPRSs). The basic properties of the proposed $\alpha^{\uparrow}$-FPRSs are thoroughly studied. Moreover, several uncertainty measures related to $\alpha^{\uparrow}$-FPRSs are presented. Meanwhile, we offered the notion of upward fuzzy β-covering (UFβC) and upward fuzzy β-neighborhood (UFβ-nghd), upward β-neighborhood (Uβ-nghd), and several related properties are explored. Based on UFβ-nghd and Uβ-nghd, we construct two new models of UFβC rough sets (UFβ-CRSs) along with their properties. We formulate a novel technique of multi-attribute DM (MADM). To legitimise the practicality of our proposed model, we provide a real-life example of selecting an appropriate medication to treat a specific disease. Finally, we look into the efficacy of the launched scheme through a comparison study.

INDEX TERMS Rough set, preference analysis, upward fuzzy β-covering, MADM.

I. INTRODUCTION

Decision analysis is a theoretical framework involving the DM process, criteria, types, and methods. Much of the research on DM under uncertain environments assumes a specific kind of individual's behavior and identifies the preferences that indicate the behavior. Their private and professional lives generally drive individuals' behavior. The main difficulty of the DM process in an uncertain environment is determining how to cope with an individual's attitude while achieving the ultimate goal. Yet, due to the complexity, inaccuracy, and unstructured DM issues under uncertainty and the constraints of information and cognition for individual DMrs, it is hard to get a logical and scientific DM with only a single DMr under uncertainty in practice. To achieve an affordable and consistent optimal outcome, specialists from diverse professions with varying specialties are recruited to form a group and solve the DM issues collectively. So far, the concept and theory of group DM (GDM) have been applied to a variety of DM issues that have arisen in management sciences, medical diagnosis, engineering, and the social sciences.

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A. AN OVERVIEW ON THE DEVELOPMENT OF RST

Pawlak RST [33], [34] supplies us with an effective mathematical way to cope with intelligent systems with inadequate and incomplete knowledge. It has been determined to be very effective in a variety of applications. The rough set (RS) technique is based on an Er specifying the indiscernibility relation between items. Although RST has been implemented proficiently across multiple arenas, several shortcomings might restrict its applications. The problem is that the transitivity of that Er is practically difficult to validate, and theoretical arguments contrary to their use mentioned already in [68] and in [69], Alcantud proposed specific tests for determining the consistency of observed behavior with this framework. These shortcomings result in inaccurate information regarding the objects under consideration. Because of this reason, many academics invented more generic RS variants, which provide an effective and flexible means of data analysis. Zhu [58] proposed the idea of generalized RSs based on relations. Ali et al. [1] analyzed several properties of generalized RSs. She et al. [41] employed logical operators in RS theory. Dubois and Prade [14] created the fuzzy RS (FRS) by swapping out the crisp binary relations in the universe with fuzzy relations. The popularity of RS and FRS can be gauged by their applications in diverse domains, such as medical diagnosis, data mining, attribute reduction, pattern recognition, machine learning, feature selection, neural network, DM, conflict analysis, etc.

While in actual DM, it is vital to take into account DRs between objects in a specific order due to the preference structures among conditions and decisions. Therefore, Greco et al. [17], [18], [19] proposed dominance-based RSs (DB-RSs), generalizing the RS theory by DR. Greco et al. [16], [20] proposed fuzzy DB-RSs (FDB-RSs), where a quantified measure of DR depicts the relationship that an object O_1 is preferred to another object O_2, but also gives the knowledge of how much O_1 is preferred to O_2. Chen et al. [7] initiated dominance-based neighborhood RS (DB-NRS). Shabir and Shaheen [40] created a novel procedure for fuzzifying RSs based on α-indiscernibility by employing a fuzzy tolerance relation. Radzikowska and Kerre [37] explored a model of FRSs using fuzzy logical implication and t-norm.

B. AN OVERVIEW ON THE DEVELOPMENT OF COVERING-BASED RSS

Another perspective for the generalization of RSs consists of replacing the partition induced by Er with a more general concept known as covering. Covering-based RSs (CRSs) are one of the most extensively investigated generalizations of RSs that enable scholars to study roughness and uncertainty in a broader context. In 1983, Zakowski [52] was the first to propose a covering-based generalization of Pawlak approximation operators. Recently, a large spectrum of research has been conducted on this paradigm. In 1998, Bonikowski et al. [5] initiated a novel model of CRS using minimal description. Zhu and Wang [60] addressed three different CRS models. Many significant properties of CRSs were also discussed by Chen et al. [8], [9]. By fusing fuzzy sets and CRSs, numerous researchers have extended CRS versions to covering-based fuzzy RSs (CFRS). Deng et al. [12] suggested a fuzzy covering. D'eer et al. [10] established a fuzzy neighborhood based on fuzzy covering. D'eer et al. [11] put forth neighborhood operators for CRSs. By employing fuzzy β-neighborhoods, Ma [28] developed two types of CFRS models, the generalization of CFRS when $\beta = 1$. Also, Yang and Hu [48] investigated various kinds of fuzzy β-covering based on RSs. Zhu [59] investigated topological approaches to CRSs. Yang [47] pioneered fuzzy CRSs on two different universes with application in DM. Zhang and Dai [54] redefined FRS models in fuzzy β-covering approximation spaces. Atef et al. [2] established several variants of covering-based $(\mathscr{I}, \mathscr{T})$-FRSs with applications in DM. Zhu and Wang [67] interpreted the reduction and axiomization of covering based generalized RSs.

C. A BRIEF REVISIT ON RS THEORY BASED ON FPRs

In certain circumstances, the DMrs are frequently asked to give their preferences over alternatives, like teaching evaluation, university ranking, credit approval, and stock risk estimation. In these circumstances, PRs play a vital role in communicating the preferences of DMrs. Orlovsky [30] proposes FPR to express DMrs' opinions over a group of alternatives. The FPR not only reveals that one alternative is preferred over another, but it also quantifies the degree of preference. Xu [45] proposed intuitionistic PRs with their application in group DM. Wang et al. [44] developed a novel three-way MADM method with fuzzy complementary PRs based on additive consistency. Pawlak's RS theory and FRS cannot analyze the data with preference structures. Subsequently, Hu et al. [22] pioneered a technique to compute FPRs from samples specified by numerical criteria and established the method to attribute reduction of fuzzy preference-based RSs (FPB-RSs). They built upward/downward FPRs of alternatives using the well-known Logsig sigmoid transfer function. Yang et al. [49] established quantitative DB-NRSs (QDB-NRSs) via FPRs. Du and Hu [13] presented a dominance-based rough fuzzy set. Herrera-Viedma et al. [21] discussed sevral issues regarding the consistency of FPRs. Han et al. [63] devised a three-way group consensus mechanism based on probabilistic linguistic PRs with acceptable inconsistency. Guo et al. [64] integrated a large-scale GDM methodology combining three-way clustering and regret theory under FPRs.

Qian et al. [35], [36] generalized Pawlak's RS model to a multi-granulation RS model by employing more than one ER, which offers a novel granular computing perspective. Pan et al. [32] developed a multi-granulation fuzzy preference RS paradigm for the ordinal decision system. In this model, they reported that the FPD using the Logsig sigmoid function is not additively consistent. Therefore, they formulate a new function for constructing upward/downward FPRs. Unfortunately, the Pan et al. [32] method cannot form upward/downward FPRs on a domain with similar values of alternatives over some criteria (see Example 1). To overcome this deficiency, we propose novel transfer functions to construct the upward/downward FPRs of alternatives in this work. Based upon these newly proposed transfer functions, we present a novel method for fuzzification of RSs known as $\alpha^{\uparrow}$-FPRSs to solve the preference analysis problem.

D. A BRIEF REVISIT OF MADM

Generally, the DM is a typical procedure in the real world, which can be defined as a procedure for evaluating alternatives based on the available data under the given scenario. With the advancement of science, DM has expanded from a single attribute to multiple attributes. MADM usually identifies the optimal alternative or arranges all alternatives based on different attributes. There are several classical MADM techniques in the literature. To address the MADM problem, Hwang and Yoon [23] initiated the TOPSIS approach. Krohling and Campanharo [26] pioneered fuzzy TOPSIS for GDM. Xu and Da [65] established the ordered weighted geometric averaging operators with MADM application. Llamazares [66] devised a generalized TODIM method. Zolghadr-Asli et al. [61] review of 20-year applications of MADM. Behret [3] developed a group DM with intuitionistic FPRs. Zhan et al. [6] devised a three-way behavioral DM using hesitant fuzzy information systems. Zhu et al. [15] offered a probabilistic linguistic three-way approach using regret theory and a fuzzy c-means clustering algorithm. Bentkowska et al. [4] put forward a technique of DM with an interval-valued FPR and admissible orders. Zhang et al. [55] successfully applied the fuzzy covering-based $(\mathcal{I}, \mathcal{T})$-FRS model with the MADM problem of company recruitment. A covering-based variable precision $(\mathcal{I}, \mathcal{T})$-FRSs model was introduced by Jiang et al. [24]. Covering-based multi-granulation (I, T)-FRS models are reported by Zhan et al. [53] with their applicability in MADM. Zhang et al. [56] employed the TOPSIS approach in the framework of fuzzy covering approximation space. Jiang et al. [25] suggested a MADM approach to medical diagnosis based on covering-based variable precision FRSs. Zhang et al. [57] put forth covering-based generalized intuitionistic FRSs with applications to MADM. Abdelhafeez et al. [62] proposed a rank and analysis of several solutions for healthcare waste to attain cost-effectiveness and sustainability employing the neutrosophic MADM scheme.

E. MOTIVATION AND AIM OF THIS ARTICLE

In the light of the literature survey, the critical research motivations, knowledge gaps, novelty, and aim of this study can be summed as follows:

1. As previously stated, the transfer functions used by Pan et al. [32] to calculate the FPD for creating upward/downward FPRs are not additively consistent. As a result, we offer novel transfer functions that are additively consistent to create the upward/downward FPD.
2. Furthermore, fewer attempts have been made to explore the structures of RSs via FPR. In the current literature, scholars applied the FPR to create fuzzy approximations. However, even with the help of FPR, the researchers could not determine the crisp approximations. Naturally, the question arises whether we can obtain crisp approximations using FPR. The certifiable answer to this query has driven the current authors to the creation of $\alpha^{\uparrow}$-FPRSs. Furthermore, the $\alpha^{\uparrow}$-FPRSs approximations act as a link between FPR and crisp set.
3. CFRS theory plays a significant role in dealing with ambiguous and uncertain information. Although various CRS models have been constructed within the framework of FSs, there has been no prior investigation into the development of CRSs rooted in UFβC, UFβ-nghd, and Uβ-nghd. Therefore, this article intends to introduce the idea of UFβC, UFβ-nghd, and Uβ-nghd. Moreover, two types of CRS approaches using UFβ-nghd and Uβ-nghd are established.
4. In the process of medical diagnostic, how to choose a suitable medicine from some medicines with the same efficacy values to cure a particular disease has turned into a typical problem for doctors and patients. Generally, in a clinical setting, it is challenging for doctors to quantify the efficacy value of medicine precisely. Therefore, the medicine selection problem can be depicted as a MADM problem in a finite fuzzy covering approximation space. This article primarily addresses a method to choose an optimal medicine among various medicines to cure a particular disease. We can use a MADM technique to rank all medicines according to their efficacy value relative to each symptom and selects the best treatment plan.

F. MAIN OBJECTIVES

Under the contributions of the above investigations, the primary objectives of this research comprise the following primary objectives:

1. To formulate the novel transfer functions for evaluating the FPD of alternatives.
2. In light of the benefits of the newly proposed transfer functions, efforts are made to invent the $\alpha^{\uparrow}$-FPRS model.
3. To study numerous uncertainty measures in the framework of the $\alpha^{\uparrow}$-FPRS model.
4. To devise the concept of UFβC and UFβ-nghd, Uβ-nghd.
5. To construct two innovative CRS models based on UFβ-nghd and Uβ-nghd.
6. To establish a novel MADM strategy based on the developed UFβ-CRS model.

7. To demonstrate the realistic usage of the invented MCDM approach through concrete examples in a medicine selection problem.
8. To demonstrate the developed work's superiority, performance, and validity through a comparative analysis between the designed approach and some prevailing techniques.

G. FRAMEWORK OF THIS STUDY

The script has been structured in the following way.

1. Section II provides an outline of a few fundamental concepts that are crucial for understanding our recommended study.
2. In section III, we formulate new transfer functions to compute upward/downward FPRs, which are additive consistent.
3. Section IV presents a novel framework of fuzzification of RSs via upward FPR known as $\alpha^{\uparrow}$-FPRSs. Then, we discussed some of the structural properties of $\alpha^{\uparrow}$-FPRSs in detail.
4. Some significant uncertainty measures of $\alpha^{\uparrow}$-FPRSs along with properties are presented in Section V.
5. Section VI defines the ideas of UFβC, UFβ-nghd, and Uβ-nghd. Two new RS models based on UFβ-nghd and Uβ-nghd are also established.
6. In Section VII, based on the upward fuzzy β-covering rough sets model, a novel MADM approach to the medicine selection problem is established.
7. Section VIII emphasises comparing various prevailing strategies with our recommended method.
8. Lastly, some concluding remarks are drawn in section IX.

II. PRELIMINARIES

This segment briefly reviews several basic notions related to RSs, fuzzy relations, and FPRS.

A. ROUGH SETS (RSs)

In the RS theory [33], Er is critical to articulating data uncertainty. This Er splits the universe into classes, which are usually stated as information granules. Therefore, in RS theory, we must deal with clusters of objects instead of a single item.

Definition 1 ([33]): Let $\varnothing \neq \mathcal{O}$ be a finite universe and τ be an Er over $\mathcal{O}$. Then $(\tau, \mathfrak{R})$ is termed as *an approximation space* (A_s). Based on A_s, for any subset $\mathcal{S} \subseteq \mathcal{O}$, we can construct the lower and upper approximations of $\mathcal{S}$ as:

$$\left.\begin{aligned} \underline{\mathcal{S}}_{\tau} &= \left\{\flat \in \mathcal{O} \mid [\flat]_{\tau} \subseteq \mathcal{S}\right\}, \\ \overline{\mathcal{S}}^{\tau} &= \left\{\flat \in \mathcal{O} \mid [\flat]_{\tau} \cap \mathcal{S} \neq \varnothing\right\}, \end{aligned}\right\} \tag{1}$$

where,

$$[\flat]_{\tau} = \left\{\mathfrak{q} \in \mathcal{O} \mid (\flat, \mathfrak{q}) \in \tau\right\}. \tag{2}$$

$\mathcal{S} \subseteq \mathcal{O}$ is called *definable* in a given A_s if $\underline{\mathcal{S}}_{\tau} = \overline{\mathcal{S}}^{\tau}$; otherwise, it is called *RS*. Moreover, the regions listed below:

(i) $\mathbb{P}os(\mathcal{S}) = \underline{\mathcal{S}}_{\tau}$,
(ii) $\mathbb{B}nd(\mathcal{S}) = \overline{\mathcal{S}}^{\tau} - \underline{\mathcal{S}}_{\tau}$,
(iii) $\mathbb{N}eg(\mathcal{S}) = \left(\overline{\mathcal{S}}^{\tau}\right)^{c}$,

are called *the positive, boundary, and negative regions* of $\mathcal{S}$, respectively. The Positive region contains the definite elements, the boundary region has doubtful elements, and the negative region contains the definite non-elements of $\mathcal{S}$ subject to the given information.

B. SOME IDEAS RELATED TO FUZZY RELATIONS AND FUZZY PREFERENCE RELATIONS

Definition 2 ([51]): A fuzzy set (FS) ξ on $\mathcal{O}$ is a function from $\mathcal{O}$ to [0, 1], i.e., $\xi : \mathcal{O} \longrightarrow [0, 1]$. The value $\xi(\flat)$ of ξ at $\flat \in \mathcal{O}$ signifies the membership degree of $\flat$ in ξ.
The collection of all FSs over $\mathcal{O}$ is symbolized by $\mathcal{F}(\mathcal{O})$.

Definition 3 ([51]): Let $\xi_1, \xi_2 \in \mathcal{F}(\mathcal{O})$. Then for all $\flat \in \mathcal{O}$, we have

(i) $\xi_1 \leq \xi_2 \Longleftrightarrow \xi_1(\flat) \leq \xi_2(\flat)$;
(ii) $(\xi_1 \cap \xi_2)(\flat) = \xi_1(\flat) \wedge \xi_2(\flat)$;
(iii) $(\xi_1 \cup \xi_2)(\flat) = \xi_1(\flat) \vee \xi_2(\flat)$;
(iv) $\xi_1^{c}(\flat) = 1 - \xi_1(\flat)$.

Definition 4 ([40]): A FS $\mu \in \mathcal{F}(\mathcal{O} \times \mathcal{O})$ is called a fuzzy relation (FR) over $\mathcal{O}$, i.e., $\mu : \mathcal{O} \times \mathcal{O} \longrightarrow [0, 1]$.

If μ is a FR over $\mathcal{O}$, then

(i) μ is called *reflexive FR* if $\forall\, \mathfrak{p} \in \mathcal{O}$, $\mu(\mathfrak{p}, \mathfrak{p}) = 1$.
(ii) μ is said to be *symmetric FR* if $\forall\, \mathfrak{p}, \mathfrak{q} \in \mathcal{O}$, $\mu(\mathfrak{p}, \mathfrak{q}) = \mu(\mathfrak{q}, \mathfrak{p})$.
(iii) μ is called *transitive FR* if $\forall\, \mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in \mathcal{O}$, $\mu(\mathfrak{p}, \mathfrak{r}) \geq \bigvee_{\mathfrak{q} \in \mathcal{O}} \mu(\mathfrak{p}, \mathfrak{q}) \wedge \mu(\mathfrak{q}, \mathfrak{r})$.
(iv) A FR is called a *fuzzy Er* if it is a reflexive, symmetric and transitive FR.

Definition 5 ([21]): A FPR $\widehat{\mathfrak{R}}$ is a FS over $\mathcal{O} \times \mathcal{O}$, which is characterized by a membership function $\mu_{\widehat{\mathfrak{R}}} : \mathcal{O} \times \mathcal{O} \longrightarrow [0, 1]$. For $\mathcal{O} = \{\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n\}$, we can describe it through an $n \times n$ matrix as follows:

$$\widehat{\mathfrak{R}} = \left(\theta_{ij}\right)_{n \times n} = \begin{array}{c c} & \begin{array}{cccc} \mathfrak{x}_1 & \mathfrak{x}_2 & \cdots & \mathfrak{x}_n \end{array} \\ \begin{array}{c} \mathfrak{x}_1 \\ \mathfrak{x}_2 \\ \vdots \\ \mathfrak{x}_n \end{array} & \begin{bmatrix} \theta_{11} & \theta_{12} & \cdots & \theta_{1n} \\ \theta_{21} & \theta_{22} & \cdots & \theta_{2n} \\ \vdots & & \ddots & \vdots \\ \theta_{n1} & \theta_{n2} & \cdots & \theta_{nn} \end{bmatrix} \end{array},$$

where θ_{ij} denotes the preference degree of alternative $\mathfrak{x}_i$ over alternative $\mathfrak{x}_j$, $\theta_{ij} \in [0, 1]$, $\theta_{ij} + \theta_{ji} = 1, \forall i, j = 1, 2, \cdots, n$. Specifically,

- $\theta_{ij} = 0.5$ demonstrates indifference between $\mathfrak{x}_i$ and $\mathfrak{x}_j$;
- $\theta_{ij} > 0.5$ reveals that $\mathfrak{x}_i$ is superior than $\mathfrak{x}_j$;
- $\theta_{ij} < 0.5$ indicates that $\mathfrak{x}_j$ is superior than $\mathfrak{x}_i$;
- $\theta_{ij} = 1$ shows that $\mathfrak{x}_i$ is absolutely superior than $\mathfrak{x}_j$;
- $\theta_{ij} = 0$ means $\mathfrak{x}_j$ is absolutely superior than $\mathfrak{x}_i$.

In Definition 5, the FPR is taken into account, θ_{ij} simply reflects the preference degree $\mathfrak{x}_i$ prior to $\mathfrak{x}_j$. However, in actual situations, we need to show the preference degree of $\mathfrak{x}_i$ is poor than $\mathfrak{x}_j$. To accommodate both scenarios, we titled the FPR in Definition 5 as upward FPR (UFPR) and named the other FPR as downward FPR (DFPR). The UFPR is expressed as $\widehat{\mathfrak{R}}^{\uparrow} = \left(\theta_{ij}^{\uparrow}\right)_{n\times n}$ and DFPR as $\widehat{\mathfrak{R}}^{\downarrow} = \left(\theta_{ij}^{\downarrow}\right)_{n\times n}$. Generally, $\theta_{ij}^{\uparrow} + \theta_{ij}^{\downarrow} = 1$. Thus, for the DFPR:

- $\theta_{ij}^{\downarrow} = 0.5$ demonstrates indifference between $\mathfrak{x}_i$ and $\mathfrak{x}_j$;
- $\theta_{ij}^{\downarrow} > 0.5$ demonstrates that $\mathfrak{x}_i$ is poor than $\mathfrak{x}_j$;
- $\theta_{ij}^{\downarrow} < 0.5$ indicates that $\mathfrak{x}_j$ is poor than $\mathfrak{x}_i$;
- $\theta_{ij}^{\downarrow} = 1$ shows that $\mathfrak{x}_i$ is absolutely poor than $\mathfrak{x}_j$;
- $\theta_{ij}^{\downarrow} = 0$ reveals $\mathfrak{x}_j$ is absolutely poor than $\mathfrak{x}_i$.

Definition 6: A FPR $\widehat{\mathfrak{R}} = (\theta_{ij})_{n\times n}$ is termed as *an additive consistent*, if $\theta_{ij} = \theta_{ik} - \theta_{jk} + 0.5, \forall i, j, k \in \{1, 2, \cdots, n\}$.

Hu et al. [22] employed the Logsig sigmoid transfer function $\frac{1}{1+e^{k\left(f(\mathfrak{x}_i,a)-f(\mathfrak{x}_j,a)\right)}}$ to calculate the FPD of $\mathfrak{x}_i$ to $\mathfrak{x}_j$ as:

$$\theta_{ij}^{\uparrow} = \frac{1}{1+e^{-k\left(f(\mathfrak{x}_i,a)-f(\mathfrak{x}_j,a)\right)}}, \tag{3}$$

$$\theta_{ij}^{\downarrow} = \frac{1}{1+e^{k\left(f(\mathfrak{x}_i,a)-f(\mathfrak{x}_j,a)\right)}}, \tag{4}$$

where $k > 0$.

According to Pan et al. [32], the FPD using the Logsig sigmoid transfer function is not additively consistent. Therefore, they propose different transfer functions to calculate the FPD of $\mathfrak{x}_i$ to $\mathfrak{x}_j$ as:

$$\theta_{ij}^{\uparrow} = 0.5 \times \left(\frac{f(\mathfrak{x}_i,a) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i,a)}{\bigvee_{i=1}^{n} f(\mathfrak{x}_i,a) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i,a)} - \frac{f(\mathfrak{x}_j,a) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i,a)}{\bigvee_{i=1}^{n} f(\mathfrak{x}_i,a) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i,a)} + 1\right) \tag{5}$$

$$\theta_{ij}^{\downarrow} = 0.5 \times \left(\frac{f(\mathfrak{x}_j,a) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i,a)}{\bigvee_{i=1}^{n} f(\mathfrak{x}_i,a) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i,a)} - \frac{f(\mathfrak{x}_i,a) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i,a)}{\bigvee_{i=1}^{n} f(\mathfrak{x}_i,a) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i,a)} + 1\right) \tag{6}$$

where "$\bigvee$" and "$\bigwedge$" are the maximum and minimum value of $f(\mathfrak{x}_i, a) \in [0, 1]$, respectively.

Definition 7: Let $\widehat{\mathfrak{R}}$ be a FPR over $\mathcal{O} \times \mathcal{O}$. Then $\wp = \left(\mathcal{O}, \widehat{\mathfrak{R}}\right)$ is termed as *fuzzy preference approximation space (FPA$_s$)*.

III. PROPOSED TRANSFER FUNCTIONS TO COMPUTE UPWARD/DOWNWARD FPR

We can see that if the values of alternatives on any criterion are different then the transfer functions (V) and (VI) of Pan et al. [32] works, but they do not work when the values of the alternatives on specific criteria are identical. Moreover, in this article, we highlighted that the transfer functions for calculating the FPD of [32] for the formation of UFPR and DFPR are not additively consistent, which is verified in the subsequent example.

TABLE 1. Fuzzy information system.

$f(\mathfrak{x}_i, a)$	a_1	a_2	a_3
$\mathfrak{x}_1$	0.8	0.3	0.2
$\mathfrak{x}_2$	0.3	0.3	0.7
$\mathfrak{x}_3$	0.2	0.3	0.8
$\mathfrak{x}_4$	0.6	0.3	0.6
$\mathfrak{x}_5$	0.4	0.3	0.7
$\mathfrak{x}_6$	0.2	0.3	0.3
$\mathfrak{x}_7$	0.3	0.3	0.3
$\mathfrak{x}_8$	0.3	0.3	0.4
$\mathfrak{x}_9$	0.3	0.3	0.2

Example 1: Table 1 depicts a fuzzified information system for the evaluation of credit card applicants. Let $\mathcal{O} = \{\mathfrak{x}_i : i = 1, 2, \cdots, 9\}$ be the universe of nine applicants and $\mathcal{C} = \{a_1, a_2, a_3\}$ be the set of criteria, where $a_1 =$ *high salery*, and $a_2 =$ *young age*, $a_3 =$ *good education*. Based on criterion a_1, construct the upward FPD of $\mathfrak{x}_i$ to $\mathfrak{x}_j (i, j = 1, 2, \cdots, 9)$ by using Eq. (5), we obtain as shown in the equation at the bottom of the next page.

But based on criterion a_2 and using Eq. (5), we get:

$$\begin{aligned}\theta_{11}^{\uparrow} &= 0.5 \times \left(\frac{f(\mathfrak{x}_1,a_2) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i,a_2)}{\bigvee_{i=1}^{n} f(\mathfrak{x}_i,a_2) - \bigwedge_{i=1}^{n} f(x_i,a_2)} - \frac{f(\mathfrak{x}_1,a_2) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i,a_2)}{\bigvee_{i=1}^{n} f(\mathfrak{x}_i,a_2) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i,a_2)} + 1\right)\\ &= 0.5 \times \left(\frac{0.3-0.3}{0.3-0.3} - \frac{0.3-0.3}{0.3-0.3} + 1\right)\\ &= 0.5 \times \left(\frac{0}{0} - \frac{0}{0} + 1\right) = ?\end{aligned}$$

We can observe that $\theta_{ij}^{\uparrow}$ values do not exist $\forall i, j = 1, 2, \cdots, 9$. Therefore, in the light of Pan et al. [32] method we cannot calculate $\widehat{\mathfrak{R}}_{a_2}^{\uparrow}(\mathfrak{x}_i, \mathfrak{x}_j)$. To get rid of this deficiency, we propose new transfer functions to construct the upward/downward FPD of $\mathfrak{x}_i$ to $\mathfrak{x}_j$ as follows:

$$\theta_{ij}^{\uparrow} = 0.5 \times \left(\frac{f(\mathfrak{x}_i,a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j,a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j,a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j,a)} - \frac{f(\mathfrak{x}_j,a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j,a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j,a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j,a)} + 1\right), \tag{7}$$

$$\theta_{ij}^{\downarrow} = 0.5 \times \left(\frac{f(\mathfrak{x}_j,a) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i,a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j,a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j,a)} - \frac{f(\mathfrak{x}_i,a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j,a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j,a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j,a)} + 1\right), \tag{8}$$

where $f : \mathcal{O} \times \mathcal{C} \longrightarrow [0, 1]$ is a fuzzy membership function and $f(\mathfrak{x}_i, a) \in [0, 1]$.

Based on criterion a_1, using Eqs (7) and (8) to compute the upward/downward FPD of $\mathfrak{x}_i$ to $\mathfrak{x}_j (i, j = 1, 2, \cdots, 9)$, we obtain:

$$\widehat{\mathfrak{R}}^{\uparrow}_{a_1}(\mathfrak{x}_i, \mathfrak{x}_j) = \begin{pmatrix} 0.50 & 0.75 & 0.80 & 0.60 & 0.70 & 0.80 & 0.75 & 0.75 & 0.75 \\ 0.25 & 0.50 & 0.55 & 0.35 & 0.45 & 0.55 & 0.50 & 0.50 & 0.50 \\ 0.20 & 0.45 & 0.50 & 0.30 & 0.40 & 0.50 & 0.45 & 0.45 & 0.45 \\ 0.40 & 0.65 & 0.70 & 0.50 & 0.60 & 0.70 & 0.65 & 0.65 & 0.65 \\ 0.30 & 0.55 & 0.60 & 0.40 & 0.50 & 0.60 & 0.55 & 0.55 & 0.55 \\ 0.20 & 0.45 & 0.50 & 0.30 & 0.40 & 0.50 & 0.45 & 0.45 & 0.45 \\ 0.25 & 0.50 & 0.55 & 0.35 & 0.45 & 0.55 & 0.50 & 0.50 & 0.50 \\ 0.25 & 0.50 & 0.55 & 0.35 & 0.45 & 0.55 & 0.50 & 0.50 & 0.50 \\ 0.25 & 0.50 & 0.55 & 0.35 & 0.45 & 0.55 & 0.50 & 0.50 & 0.50 \end{pmatrix}, \tag{9}$$

$$\widehat{\mathfrak{R}}^{\downarrow}_{a_1}(\mathfrak{x}_i, \mathfrak{x}_j) = \begin{pmatrix} 0.50 & 0.25 & 0.20 & 0.40 & 0.30 & 0.20 & 0.25 & 0.25 & 0.25 \\ 0.75 & 0.50 & 0.45 & 0.65 & 0.55 & 0.45 & 0.50 & 0.50 & 0.50 \\ 0.80 & 0.55 & 0.50 & 0.70 & 0.60 & 0.50 & 0.55 & 0.55 & 0.55 \\ 0.60 & 0.35 & 0.30 & 0.50 & 0.40 & 0.30 & 0.35 & 0.35 & 0.35 \\ 0.70 & 0.45 & 0.40 & 0.60 & 0.50 & 0.40 & 0.45 & 0.45 & 0.45 \\ 0.80 & 0.55 & 0.50 & 0.70 & 0.60 & 0.50 & 0.55 & 0.55 & 0.55 \\ 0.75 & 0.50 & 0.45 & 0.65 & 0.55 & 0.45 & 0.50 & 0.50 & 0.50 \\ 0.75 & 0.50 & 0.45 & 0.65 & 0.55 & 0.45 & 0.50 & 0.50 & 0.50 \\ 0.75 & 0.50 & 0.45 & 0.65 & 0.55 & 0.45 & 0.50 & 0.50 & 0.50 \end{pmatrix}. \tag{10}$$

Based on criterion a_2, using Eqs. (7) and (8) to compute the upward/downward FPD of $\mathfrak{x}_i$ to $\mathfrak{x}_j (i, j = 1, 2, \cdots, 9)$, we obtain:

$$\widehat{\mathfrak{R}}^{\uparrow}_{a_2}(\mathfrak{x}_i, \mathfrak{x}_j) = \begin{pmatrix} 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \end{pmatrix}, \tag{11}$$

$$\widehat{\mathfrak{R}}^{\downarrow}_{a_2}(\mathfrak{x}_i, \mathfrak{x}_j) = \begin{pmatrix} 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 & 0.50 \end{pmatrix}. \tag{12}$$

Based on criterion a_3, using Eqs (7) and (8) to compute the upward/downward FPD of $\mathfrak{x}_i$ to $\mathfrak{x}_j (i, j = 1, 2, \cdots, 9)$, we get:

$$\widehat{\mathfrak{R}}^{\uparrow}_{a_3}(\mathfrak{x}_i, \mathfrak{x}_j) = \begin{pmatrix} 0.50 & 0.25 & 0.20 & 0.30 & 0.25 & 0.45 & 0.45 & 0.40 & 0.50 \\ 0.75 & 0.50 & 0.45 & 0.55 & 0.50 & 0.70 & 0.70 & 0.65 & 0.75 \\ 0.80 & 0.55 & 0.50 & 0.60 & 0.55 & 0.75 & 0.75 & 0.70 & 0.80 \\ 0.70 & 0.45 & 0.40 & 0.50 & 0.45 & 0.65 & 0.65 & 0.60 & 0.70 \\ 0.75 & 0.50 & 0.45 & 0.55 & 0.50 & 0.70 & 0.70 & 0.65 & 0.75 \\ 0.55 & 0.30 & 0.25 & 0.35 & 0.30 & 0.50 & 0.50 & 0.45 & 0.55 \\ 0.55 & 0.30 & 0.25 & 0.35 & 0.30 & 0.50 & 0.50 & 0.45 & 0.55 \\ 0.60 & 0.35 & 0.30 & 0.40 & 0.35 & 0.55 & 0.55 & 0.50 & 0.60 \\ 0.50 & 0.25 & 0.20 & 0.30 & 0.25 & 0.45 & 0.45 & 0.40 & 0.50 \end{pmatrix}, \tag{13}$$

$$\widehat{\mathfrak{R}}^{\downarrow}_{a_3}(\mathfrak{x}_i, \mathfrak{x}_j) = \begin{pmatrix} 0.50 & 0.75 & 0.80 & 0.70 & 0.75 & 0.55 & 0.55 & 0.60 & 0.50 \\ 0.25 & 0.50 & 0.55 & 0.45 & 0.50 & 0.30 & 0.30 & 0.35 & 0.25 \\ 0.20 & 0.45 & 0.50 & 0.40 & 0.45 & 0.25 & 0.25 & 0.30 & 0.20 \\ 0.30 & 0.55 & 0.60 & 0.50 & 0.55 & 0.35 & 0.35 & 0.40 & 0.30 \\ 0.25 & 0.50 & 0.55 & 0.45 & 0.50 & 0.30 & 0.30 & 0.35 & 0.25 \\ 0.45 & 0.70 & 0.75 & 0.65 & 0.70 & 0.50 & 0.50 & 0.55 & 0.45 \\ 0.45 & 0.70 & 0.75 & 0.65 & 0.70 & 0.50 & 0.50 & 0.55 & 0.45 \\ 0.40 & 0.65 & 0.70 & 0.60 & 0.65 & 0.45 & 0.45 & 0.50 & 0.40 \\ 0.50 & 0.75 & 0.80 & 0.70 & 0.75 & 0.55 & 0.55 & 0.60 & 0.50 \end{pmatrix}. \tag{14}$$

Based on the above discussion, the subsequent result is obvious.

Proposition 1: $\left(\widehat{\mathfrak{R}}^{\uparrow}_{a}(x_i, x_j)\right)^T = \widehat{\mathfrak{R}}^{\downarrow}_{a}(x_i, x_j)$.

Proof: Straightforward.

$$\widehat{\mathfrak{R}}^{\uparrow}_{a_1}(\mathfrak{x}_i, \mathfrak{x}_j) = \begin{pmatrix} 0.5000 & 0.9167 & 1 & 0.6667 & 0.8333 & 1 & 0.9167 & 0.9167 & 0.9167 \\ 0.0833 & 0.5000 & 0.5833 & 0.2500 & 0.4167 & 0.5833 & 0.5000 & 0.5000 & 0.5000 \\ 0 & 0.4167 & 0.5000 & 0.1667 & 0.3333 & 0.5000 & 0.4167 & 0.4167 & 0.4167 \\ 0.3333 & 0.7500 & 0.8333 & 0.5000 & 0.6667 & 0.8333 & 0.7500 & 0.7500 & 0.7500 \\ 0.1667 & 0.5833 & 0.6667 & 0.3333 & 0.5000 & 0.6667 & 0.5833 & 0.5833 & 0.5833 \\ 0 & 0.4167 & 0.5000 & 0.1667 & 0.3333 & 0.5000 & 0.4167 & 0.4167 & 0.4167 \\ 0.0833 & 0.5000 & 0.5888 & 0.2500 & 0.4167 & 0.5833 & 0.5000 & 0.5000 & 0.5000 \\ 0.0833 & 0.5000 & 0.5888 & 0.2500 & 0.4167 & 0.5833 & 0.5000 & 0.5000 & 0.5000 \\ 0.0833 & 0.5000 & 0.5888 & 0.2500 & 0.4167 & 0.5833 & 0.5000 & 0.5000 & 0.5000 \end{pmatrix}.$$

According to newly constructed transfer functions (7) and (8), we offer the subsequent definition.

Definition 8: The upward and downward fuzzy preference classes $[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}$ and $[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\downarrow}}$ of $\mathfrak{x}_i$ generated by upward and downward additive FRRs $\widehat{\mathfrak{R}}^{\uparrow} = \left(\theta_{ij}^{\uparrow}\right)_{n\times n}$ and $\widehat{\mathfrak{R}}^{\downarrow} = \left(\theta_{ij}^{\downarrow}\right)_{n\times n}$ are respectively defined as follows:

$$[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}} = \frac{\theta_{i1}^{\uparrow}}{\mathfrak{x}_1} + \frac{\theta_{i2}^{\uparrow}}{\mathfrak{x}_2} + \cdots + \frac{\theta_{in}^{\uparrow}}{\mathfrak{x}_n},$$
$$[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\downarrow}} = \frac{\theta_{i1}^{\downarrow}}{\mathfrak{x}_1} + \frac{\theta_{i2}^{\downarrow}}{\mathfrak{x}_2} + \cdots + \frac{\theta_{in}^{\downarrow}}{\mathfrak{x}_n},$$

where "+" denotes the union operation. Obviously, $[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}$ and $[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\downarrow}}$ are the fuzzy information granules containing $\mathfrak{x}_i$.

The upward and downward additive FPRs form a family of fuzzy information granules from the universe, which constitute the upward and downward fuzzy preference granular structures, respectively given by:

$$\mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow}) = \left\{[\mathfrak{x}_1]_{\widehat{\mathfrak{R}}^{\uparrow}}, [\mathfrak{x}_2]_{\widehat{\mathfrak{R}}^{\uparrow}}, \cdots, [\mathfrak{x}_n]_{\widehat{\mathfrak{R}}^{\uparrow}}\right\},$$
$$\mathscr{P}(\widehat{\mathfrak{R}}^{\downarrow}) = \left\{[\mathfrak{x}_1]_{\widehat{\mathfrak{R}}^{\downarrow}}, [\mathfrak{x}_2]_{\widehat{\mathfrak{R}}^{\downarrow}}, \cdots, [\mathfrak{x}_n]_{\widehat{\mathfrak{R}}^{\downarrow}}\right\}.$$

The subsequent result reveals that our newly constructed transfer functions to evaluate the upward/downward FPD are additive consistent.

Theorem 1: The constructed transfer functions provided in Eqs. (7) and (8) to compute the upward/downward FPD are additive consistent.

Proof: First of all, we prove the required result for the upward FPD in the following three cases:

Case 1:

$$\begin{aligned}\theta_{ii}^{\uparrow} &= 0.5 \times \left(\frac{f(\mathfrak{x}_i, a) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i, a)}{\bigvee_{i=1}^{n} f(\mathfrak{x}_i, a) + \bigwedge_{i=1}^{n} f(\mathfrak{x}_i, a)}\right.\\ &\quad \left. - \frac{f(\mathfrak{x}_i, a) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i, a)}{\bigvee_{i=1}^{n} f(\mathfrak{x}_i, a) + \bigwedge_{i=1}^{n} f(\mathfrak{x}_i, a)} + 1\right)\\ &= 0.5 \times (0 + 1)\\ &= 0.5.\end{aligned}$$

Case 2: as shown in the equation at the bottom of the next page.

Case 3: as shown in the equation at the bottom of the next page.

Analogously, we can prove that the downward FPD given in Eq. (8) is additive consistent.

In the subsequent result, we investigate the characteristics of 0.5− reflexivity, 0.5− symmetry, and 0.5− transitivity of the UFPR based on additive consistency.

Proposition 2: Let $\widehat{\mathfrak{R}}^{\uparrow} = \left(\theta_{ij}^{\uparrow}\right)_{n\times n}$ be an UFPR based on additive consistency on criteria a. Then for each $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{O}$, the subsequent properties are satisfied:

1) 0.5− Reflexivity: $\widehat{\mathfrak{R}}_a^{\uparrow}(\mathfrak{x}, \mathfrak{x}) = 0.5$.
2) 0.5− Symmetry: $\widehat{\mathfrak{R}}_a^{\uparrow}(\mathfrak{x}, \mathfrak{y}) = 0.5 \Longleftrightarrow \widehat{\mathfrak{R}}_a^{\uparrow}(\mathfrak{y}, \mathfrak{x}) = 0.5$.
3) 0.5− Transitivity: $\widehat{\mathfrak{R}}_a^{\uparrow}(\mathfrak{x}, \mathfrak{y}) \geq 0.5, \widehat{\mathfrak{R}}_a^{\uparrow}(\mathfrak{y}, \mathfrak{z}) \geq 0.5 \Longrightarrow \widehat{\mathfrak{R}}_a^{\uparrow}(\mathfrak{x}, \mathfrak{z}) \geq 0.5$.

Proof: It can directly derive by Theorem 1.

IV. UPWARD α-FUZZIFIED ROUGH APPROXIMATIONS UNDER FPR

This segment offers an innovative concept of upward α-fuzzified rough approximations based on upward-FPA_s. We will adopt the transfer function (7) to calculate the upward FPD and introduce $\alpha^{\uparrow}$-FPRSs and their related fundamental properties with some examples.

Definition 9: Let $\wp^{\uparrow} = \left(\mathcal{O}, \widehat{\mathfrak{R}}^{\uparrow}\right)$ be an upward-FPA_s, where $\varnothing \neq \mathcal{O}$ is a finite universe and $\widehat{\mathfrak{R}}$ is a UFPR characterized by its membership function $\mu_{\widehat{\mathfrak{R}}^{\uparrow}} : \mathcal{O} \times \mathcal{O} \longrightarrow [0, 1]$. For any $\alpha \in [0.5, 1)$, the upward α-fuzzified preference lower and upper approximations for any $\mathcal{S} \subseteq \mathcal{O}$ are defined as:

$$\left.\begin{aligned}\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} &= \left\{\mathfrak{x}_i \in \mathcal{O} : \theta_{ij}^{\uparrow} < 1 - \alpha \text{ for all } \mathfrak{x}_j \in \mathcal{S}^c\right\},\\ \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} &= \left\{\mathfrak{x}_i \in \mathcal{O} : \theta_{ij}^{\uparrow} \geq 1 - \alpha \text{ for some } \mathfrak{x}_j \in \mathcal{S}\right\}.\end{aligned}\right\} \tag{15}$$

If $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \neq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$, then $\mathcal{S}$ is titled as an $\alpha^{\uparrow}$-FPRS w.r.t upward-FPA_s; otherwise, it is called $\alpha^{\uparrow}$-fuzzified preference definable.

The information regarding the objects of $\mathcal{O}$ portrayed by the above-described operators are as follows:

- $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$ indicates a crisp set that contains objects $\mathfrak{x}_i \in \mathcal{O}$ equivalent to all objects $\mathfrak{x}_j \in \mathcal{S}^c$ with upward FPD less than to a certain $\alpha \in [0.5, 1)$.
- $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$ indicates a crisp set that contains objects $\mathfrak{x}_i \in \mathcal{O}$ equivalent to at least one object $\mathfrak{x}_j \in \mathcal{S}$ with upward FPD greater than or equal to a certain $\alpha \in [0.5, 1)$.

The corresponding positive, boundary, and negative regions of $\mathcal{S} \subseteq \mathcal{O}$ for $\alpha \in [0.5, 1)$ are characterized as follows:

(i) $\mathbb{P}OS_{\widehat{R}^{\uparrow}}(\mathcal{S}) = \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$,

(ii) $\mathbb{B}ND_{\widehat{R}^{\uparrow}}(\mathcal{S}) = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} - \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$,

(iii) $\mathbb{N}EG_{\widehat{R}^{\uparrow}}(\mathcal{S}) = \left(\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)^c$.

Here, we will provide an example to comprehend the idea upward α-fuzzified rough approximations of $\mathcal{S} \subseteq \mathcal{O}$.

Example 2: Let $\widehat{\mathfrak{R}}^{\uparrow}$ be an UFPR over $\mathcal{O}$ given in (13), where $\mathcal{O} = \{\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_9\}$. If we take $\alpha = 0.6$ and $\mathcal{S} = \{\mathfrak{x}_1, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\} \subseteq \mathcal{O}$. Then the upward α-fuzzified preference lower and upper approximations for $\mathcal{S}$ are calculated by using Definition 9 as:

$$\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \{\mathfrak{x}_1, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_9\},$$
$$\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \mathcal{O}.$$

Consequently, $\mathcal{S}$ is an $\alpha^{\uparrow}$-FPRS, since $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \neq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$. Further by direct computation, we have:

$$\mathbb{P}OS_{\widehat{R}^{\uparrow}}(\mathcal{S}) = \{\mathfrak{x}_1, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_9\},$$
$$\mathbb{B}ND_{\widehat{R}^{\uparrow}}(\mathcal{S}) = \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_5, \mathfrak{x}_8\},$$
$$\mathbb{N}EG_{\widehat{R}^{\uparrow}}(\mathcal{S}) = \varnothing.$$

Proposition 3: Let $\wp^{\uparrow} = (\mathcal{O}, \widehat{\mathfrak{R}}^{\uparrow})$ be an upward-FPA_s, $\mathcal{S} \subseteq \mathcal{O}$ and $\alpha_1, \alpha_2 \in [0.5, 1)$ be such that $\alpha_1 \leq \alpha_2$. Then

1) $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha_2} \subseteq \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha_1}$;
2) $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha_1} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha_2}$.

Proof:

1) For any $\mathfrak{x}_i \in \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha_2}$, we have $\theta_{ij}^{\uparrow} < 1 - \alpha_2$ for all $\mathfrak{x}_j \in \mathcal{S}^c$. As $\alpha_1 \leq \alpha_2$, so $1 - \alpha_2 \leq 1 - \alpha_1$. Thus, $\theta_{ij}^{\uparrow} < 1 - \alpha_1$ for all $\mathfrak{x}_j \in \mathcal{S}^c$. This implies that $\mathfrak{x}_i \in \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha_1}$. Thus, $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha_2} \subseteq \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha_1}$.

$$
\begin{aligned}
\theta_{ij}^{\uparrow} + \theta_{ji}^{\uparrow} &= 0.5 \times \Bigg(\frac{f(\mathfrak{x}_i, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} \\
&\quad - \frac{f(\mathfrak{x}_j, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} + 1 \Bigg) \\
&\quad + 0.5 \times \left(\frac{f(\mathfrak{x}_j, a) - \bigwedge_{i=1}^{n} f(\mathfrak{x}_i, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} - \frac{f(\mathfrak{x}_i, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} + 1 \right) \\
&= 0.5 \times \left(\begin{array}{l} \dfrac{f(\mathfrak{x}_i, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} - \dfrac{f(\mathfrak{x}_j, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} + 1 \\[2ex] \dfrac{f(\mathfrak{x}_j, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} - \dfrac{f(\mathfrak{x}_i, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} + 1 \end{array} \right) \\
&= 0.5 \times (1 + 1) \\
&= 1.
\end{aligned}
$$

$$
\begin{aligned}
\theta_{ij}^{\uparrow} + \theta_{jk}^{\uparrow} &= 0.5 \times \left(\frac{f(\mathfrak{x}_i, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} - \frac{f(\mathfrak{x}_j, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} + 1 \right) \\
&\quad + 0.5 \times \left(\frac{f(\mathfrak{x}_j, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} - \frac{f(\mathfrak{x}_k, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} + 1 \right) \\
&= 0.5 \times \left(\begin{array}{l} \dfrac{f(\mathfrak{x}_i, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} - \dfrac{f(\mathfrak{x}_j, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} + 1 \\[2ex] \dfrac{f(\mathfrak{x}_j, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} - \dfrac{f(\mathfrak{x}_k, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} + 1 \end{array} \right) \\
&= 0.5 \times \left(\frac{f(\mathfrak{x}_i, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} - \frac{f(\mathfrak{x}_k, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} + 1 + 1 \right) \\
&= 0.5 \times \left(\frac{f(\mathfrak{x}_i, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} - \frac{f(\mathfrak{x}_k, a) - \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)}{\bigvee_{j=1}^{n} f(\mathfrak{x}_j, a) + \bigwedge_{j=1}^{n} f(\mathfrak{x}_j, a)} + 1 \right) + 0.5 \\
&= \theta_{ik}^{\uparrow} + 0.5.
\end{aligned}
$$

2) If $\mathfrak{x}_i \in \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha_1}$, then $\theta_{ij}^{\uparrow} \geq 1-\alpha_1$ for some $\mathfrak{x}_j \in \mathcal{S}$. But since $\alpha_1 \leq \alpha_2$, so $1-\alpha_1 \geq 1-\alpha_2$. Therefore, $\theta_{ij}^{\uparrow} \geq 1-\alpha_2$ for some $\mathfrak{x}_j \in \mathcal{S}$. This reveals that $\mathfrak{x}_i \in \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha_2}$. Hence, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha_1} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha_2}$.

Theorem 2: Let $\wp^{\uparrow} = \left(\mathcal{O}, \widehat{\mathfrak{R}}^{\uparrow}\right)$ be an upward-FPA_s, $\alpha \in [0.5, 1)$ and $\mathcal{S}, \mathcal{T} \subseteq \mathcal{O}$. Then

1) $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \subseteq \mathcal{S} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$;
2) $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\varnothing)}_{\alpha} = \varnothing = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\varnothing)}_{\alpha}$;
3) $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{O})}_{\alpha} = \mathcal{O} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{O})}_{\alpha}$;
4) $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}^c)}_{\alpha} = \left(\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)^c$;
5) $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}^c)}_{\alpha} = \left(\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)^c$;
6) $\mathcal{S} \subseteq \mathcal{T} \Longrightarrow \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \subseteq \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$;
7) $\mathcal{S} \subseteq \mathcal{T} \Longrightarrow \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$;
8) $\widehat{\mathfrak{R}}_1^{\uparrow} \subseteq \widehat{\mathfrak{R}}_2^{\uparrow} \Longrightarrow \underline{\widehat{\mathfrak{R}_2}^{\uparrow}(\mathcal{S})}_{\alpha} \subseteq \underline{\widehat{\mathfrak{R}_1}^{\uparrow}(\mathcal{S})}_{\alpha}$;
9) $\widehat{\mathfrak{R}}_1^{\uparrow} \subseteq \widehat{\mathfrak{R}}_2^{\uparrow} \Longrightarrow \overline{\widehat{\mathfrak{R}_1}^{\uparrow}(\mathcal{S})}_{\alpha} \subseteq \overline{\widehat{\mathfrak{R}_2}^{\uparrow}(\mathcal{S})}_{\alpha}$;
10) $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha} \supseteq \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cup \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$;
11) $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cup \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$;
12) $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cap \mathcal{T})}_{\alpha} = \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cap \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$;
13) $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cap \mathcal{T})}_{\alpha} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cap \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$.

Proof: (1)-(3) straightforward.

(4) For any $\mathfrak{x}_i \in \mathcal{O}$,

$$\begin{aligned}
\mathfrak{x}_i \in \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}^c)}_{\alpha} &\Longleftrightarrow \theta_{ij}^{\uparrow} < 1-\alpha \text{ for all } \mathfrak{x}_j \in (\mathcal{S}^c)^c = \mathcal{S} \\
&\Longleftrightarrow \theta_{ij}^{\uparrow} \ngeq 1-\alpha \text{ for any } \mathfrak{x}_j \in \mathcal{S} \\
&\Longleftrightarrow \mathfrak{x}_i \notin \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \\
&\Longleftrightarrow \mathfrak{x}_i \in \left(\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)^c.
\end{aligned}$$

Hence, $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}^c)}_{\alpha} = \left(\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)^c$.

(5) For any $\mathfrak{x}_i \in \mathcal{O}$,

$$\begin{aligned}
\mathfrak{x}_i \in \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}^c)}_{\alpha} &\Longleftrightarrow \theta_{ij}^{\uparrow} \geq 1-\alpha \text{ for some } \mathfrak{x}_j \in \mathcal{S}^c \\
&\Longleftrightarrow \theta_{ij}^{\uparrow} \nless 1-\alpha \text{ for all } \mathfrak{x}_j \in \mathcal{S}^c \\
&\Longleftrightarrow \mathfrak{x}_i \notin \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \\
&\Longleftrightarrow \mathfrak{x}_i \in \left(\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)^c.
\end{aligned}$$

Hence, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}^c)}_{\alpha} = \left(\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)^c$.

(6) Let $\mathfrak{x}_i \in \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$. Then $\theta_{ij}^{\uparrow} < 1-\alpha$ for all $\mathfrak{x}_j \in \mathcal{S}^c$. But since $\mathcal{S} \subseteq \mathcal{T}$ so, $\mathcal{T}^c \subseteq \mathcal{S}^c$. Thus in particular, $\theta_{ij}^{\uparrow} < 1-\alpha$ for all $\mathfrak{x}_j \in \mathcal{T}^c$. Therefore, $\mathfrak{x}_i \in \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$ indicating that $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \subseteq \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$.

(7) If $\mathfrak{x}_i \in \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$. Then $\theta_{ij}^{\uparrow} \geq 1-\alpha$ for some $\mathfrak{x}_j \in \mathcal{S}$. But since $\mathcal{S} \subseteq \mathcal{T}$ so, $\theta_{ij}^{\uparrow} \geq 1-\alpha$ for some $\mathfrak{x}_j \in \mathcal{S} \subseteq \mathcal{T}$ which demonstrates that $\mathfrak{x}_i \in \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$. Thus, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$.

(8) If $\mathfrak{x}_i \in \underline{\widehat{\mathfrak{R}_2}^{\uparrow}(\mathcal{S})}_{\alpha}$. Then $\theta_{2ij}^{\uparrow} < 1-\alpha$ for all $\mathfrak{x}_j \in \mathcal{S}^c$. But since $\widehat{\mathfrak{R}}_1^{\uparrow} \subseteq \widehat{\mathfrak{R}}_2^{\uparrow}$ implies $\theta_{1ij}^{\uparrow} \leq \theta_{2ij}^{\uparrow}$. Hence, $\theta_{1ij}^{\uparrow} < 1-\alpha$ for all $\mathfrak{x}_j \in \mathcal{S}^c$. Therefore, $\mathfrak{x}_i \in \underline{\widehat{\mathfrak{R}_1}^{\uparrow}(\mathcal{S})}_{\alpha}$ showing that $\underline{\widehat{\mathfrak{R}_2}^{\uparrow}(\mathcal{S})}_{\alpha} \subseteq \underline{\widehat{\mathfrak{R}_1}^{\uparrow}(\mathcal{S})}_{\alpha}$.

(9) Analogous to the proof of (8).

(10) As $\mathcal{S} \cup \mathcal{T} \supseteq \mathcal{S}$ and $\mathcal{S} \cup \mathcal{T} \supseteq \mathcal{T}$. So according to part (6), we can write $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha} \supseteq \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$ and $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha} \supseteq \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$. Thus, $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha} \supseteq \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cup \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$.

(11) Since $\mathcal{S} \subseteq \mathcal{S} \cup \mathcal{T}$ and $\mathcal{T} \subseteq \mathcal{S} \cup \mathcal{T}$. So through part (7), $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha}$ and $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha}$. Therefore, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cup \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha}$. Conversely, for any $\mathfrak{x}_i \in \mathcal{O}$,

$$\begin{aligned}
&\mathfrak{x}_i \in \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha} \\
&\Longrightarrow \theta_{ij}^{\uparrow} \geq 1-\alpha \text{ for some } \mathfrak{x}_j \in (\mathcal{S} \cup \mathcal{T}) \\
&\Longrightarrow \theta_{ij}^{\uparrow} \geq 1-\alpha \text{ for some } \mathfrak{x}_j \in \mathcal{S} \text{ or } \theta_{ij}^{\uparrow} \\
&\geq 1-\alpha \text{ for some } \mathfrak{x}_j \in \mathcal{T} \\
&\Longrightarrow \mathfrak{x}_i \in \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \text{ or } \mathfrak{x}_i \in \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha} \\
&\Longrightarrow \mathfrak{x}_i \in \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cup \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha} \\
&\Longrightarrow \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cup \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}.
\end{aligned}$$

Hence, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cup \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$.

(12) According to part (6) and using the fact that $\mathcal{S} \cap \mathcal{T} \subseteq \mathcal{S}$, $\mathcal{S} \cap \mathcal{T} \subseteq \mathcal{T}$ we can get, $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cap \mathcal{T})}_{\alpha} \subseteq \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$ and $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cap \mathcal{T})}_{\alpha} \subseteq \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$. Thus, $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cap \mathcal{T})}_{\alpha} \subseteq \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cap \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$. Conversely, for any $\mathfrak{x}_i \in \mathcal{O}$,

$$\begin{aligned}
&\mathfrak{x}_i \in \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cap \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha} \\
&\Longrightarrow \mathfrak{x}_i \in \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \text{ and } \mathfrak{x}_i \in \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha} \\
&\Longrightarrow \theta_{ij}^{\uparrow} < 1-\alpha \text{ for all } \mathfrak{x}_j \in \mathcal{S}^c \text{ and } \theta_{ij}^{\uparrow} < 1 \\
&\quad -\alpha \text{ for all } \mathfrak{x}_j \in \mathcal{T}^c \\
&\Longrightarrow \theta_{ik}^{\uparrow} < 1-\alpha \text{ for all } \mathfrak{x}_k \in \mathcal{S}^c \cup \mathcal{T}^c = (\mathcal{S} \cap \mathcal{T})^c \\
&\Longrightarrow \mathfrak{x}_i \in \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cap \mathcal{T})}_{\alpha} \\
&\Longrightarrow \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cap \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha} \subseteq \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cap \mathcal{T})}_{\alpha}.
\end{aligned}$$

Hence, $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cap \mathcal{T})}_{\alpha} = \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cap \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$.

(13) By using part (6) and using the fact that $\mathcal{S} \cap \mathcal{T} \subseteq \mathcal{S}$, $\mathcal{S} \cap \mathcal{T} \subseteq \mathcal{T}$ we have, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cap \mathcal{T})}_{\alpha} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$ and $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cap \mathcal{T})}_{\alpha} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$. Therefore, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cap \mathcal{T})}_{\alpha} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cap \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$.

The containments in parts (10) and (13) of the above theorem may hold strictly, which can be verified by the subsequent illustration.

Example 3: Let $\widehat{\mathfrak{R}}^{\uparrow}$ be an UFPR over $\mathcal{O}$ given in (9), where $\mathcal{O} = \{\mathfrak{x}_i : i = 1, 2 \cdots, 9\}$. If we take $\alpha = 0.5$ and $\mathcal{S}, \mathcal{T} \subseteq \mathcal{O}$ such that $\mathcal{S} = \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$, $\mathcal{T} = \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$, then $\mathcal{S} \cup \mathcal{T} = \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$. By direct computation, we get:

$$\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\},$$
$$\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha} = \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\},$$
$$\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha} = \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}.$$

Clearly, $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha} = \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\} \supset \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\} = \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cup \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$, which suggests that the inclusion in part (10) of Theorem 2 might be strict.

Now, if $\mathcal{S}_1 = \{\mathfrak{x}_1, \mathfrak{x}_2\}$, $\mathcal{T}_1 = \{\mathfrak{x}_1, \mathfrak{x}_3\}$, then $\mathcal{S}_1 \cap \mathcal{T}_1 = \{\mathfrak{x}_1\}$. For $\alpha = 0.5$, we obtained:

$$\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}_1)}_{\alpha} = \mathcal{O},$$
$$\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T}_1)}_{\alpha} = \mathcal{O},$$
$$\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}_1 \cap \mathcal{T}_1)}_{\alpha} = \{\mathfrak{x}_1\}.$$

We can observe that, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}_1 \cap \mathcal{T}_1)}_{\alpha} = \{x_1\} \subset \mathcal{O} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}_1)}_{\alpha} \cap \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T}_1)}_{\alpha}$, showing that the containment in part (13) of Theorem 2 may hold strictly.

Remark 1: In RS theory, both the lower and upper approximations of any $\mathcal{S} \subseteq \mathcal{O}$ are definable, that is, $\underline{\underline{\mathfrak{R}(\underline{\mathfrak{R}(\mathcal{S})})}} = \underline{\mathfrak{R}(\mathcal{S})} = \overline{\mathfrak{R}(\underline{\mathfrak{R}(\mathcal{S})})}$ and $\underline{\mathfrak{R}(\overline{\mathfrak{R}(\mathcal{S})})} = \overline{\mathfrak{R}(\mathcal{S})} = \overline{\mathfrak{R}(\overline{\mathfrak{R}(\mathcal{S})})}$. But in the environment of $\alpha^{\uparrow}$-FPRSs, the upward α-fuzzified preference lower and upper approximations of $\mathcal{S} \subseteq \mathcal{O}$ are hardly definable. Generally they still $\alpha^{\uparrow}$-FPRSs, that is, $\underline{\widehat{\mathfrak{R}}^{\uparrow}\left(\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)}_{\alpha} = \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \neq \overline{\widehat{\mathfrak{R}}^{\uparrow}\left(\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)}_{\alpha}$ and $\underline{\widehat{\mathfrak{R}}^{\uparrow}\left(\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)}_{\alpha} \neq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}\left(\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)}_{\alpha}$. To justify this fact, here we elaborate the following example.

Example 4: Let $\widehat{\mathfrak{R}}^{\uparrow}$ be an UFPR over $\mathcal{O}$ given in (9). If we take $\alpha = 0.5$ and $\mathcal{S} = \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\} \subseteq \mathcal{O}$, then by routine computation, we get:

$$\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\},$$
$$\underline{\widehat{\mathfrak{R}}^{\uparrow}\left(\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)}_{\alpha} = \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\},$$
$$\overline{\widehat{\mathfrak{R}}^{\uparrow}\left(\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)}_{\alpha} = \mathcal{O}.$$

Clearly, we can observed that $\underline{\widehat{\mathfrak{R}}^{\uparrow}\left(\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)}_{\alpha} = \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \neq \overline{\widehat{\mathfrak{R}}^{\uparrow}\left(\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right)}_{\alpha}$.

Now, if we consider $\mathcal{S}_1 = \{\mathfrak{x}_1, \mathfrak{x}_4\} \subseteq \mathcal{O}$ and $\alpha = 0.5$, then:

$$\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}_1)}_{\alpha} = \{\mathfrak{x}_1, \mathfrak{x}_4\},$$
$$\underline{\widehat{\mathfrak{R}}^{\uparrow}\left(\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}_1)}_{\alpha}\right)}_{\alpha} = \varnothing,$$
$$\overline{\widehat{\mathfrak{R}}^{\uparrow}\left(\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}_1)}_{\alpha}\right)}_{\alpha} = \{\mathfrak{x}_1, \mathfrak{x}_4\}.$$

We can see that $\underline{\widehat{\mathfrak{R}}^{\uparrow}\left(\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}_1)}_{\alpha}\right)}_{\alpha} \neq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}_1)}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}\left(\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}_1)}_{\alpha}\right)}_{\alpha}$.

Definition 10: Let $\wp^{\uparrow} = \left(\mathcal{O}, \widehat{\mathfrak{R}}^{\uparrow}\right)$ be an upward-FPA_s and $\alpha \in [0.5, 1)$. Then for any $\mathcal{S}, \mathcal{T} \subseteq \mathcal{O}$, we define

1) $\mathcal{S} \cong \mathcal{T}$ if and only if $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$;
2) $\mathcal{S} \overline{\eqsim} \mathcal{T}$ if and only if $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$;
3) $\mathcal{S} \approx \mathcal{T}$ if and only if $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$ and $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$.

In the above definition, we give the new relationships between sets based upon the upward α-fuzzified preference lower and upper approximations.

Proposition 4: The relations $\cong$, $\overline{\eqsim}$ and $\approx$ are Ers.

Proof: Obvious.

Theorem 3: Assume that $\wp^{\uparrow} = \left(\mathcal{O}, \widehat{\mathfrak{R}}^{\uparrow}\right)$ be an upward-FPA_s and $\alpha \in [0.5, 1)$. Then for any $\mathcal{S}, \mathcal{T}, \mathcal{S}_1, \mathcal{T}_1 \subseteq \mathcal{O}$, the following assertions hold:

1) $\mathcal{S} \overline{\eqsim} \mathcal{T}$ if and only if $\mathcal{S} \overline{\eqsim} (\mathcal{S} \cup \mathcal{T})$ and $(\mathcal{S} \cup \mathcal{T}) \overline{\eqsim} \mathcal{T}$;
2) $\mathcal{S} \overline{\eqsim} \mathcal{S}_1$ and $\mathcal{T} \overline{\eqsim} \mathcal{T}_1$ implies $(\mathcal{S} \cup \mathcal{T}) \overline{\eqsim} (\mathcal{S}_1 \cup \mathcal{T}_1)$;
3) $\mathcal{S} \overline{\eqsim} \mathcal{T}$ implies $(\mathcal{S} \cup \mathcal{T}^c) \overline{\eqsim} \mathcal{O}$;
4) $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{T} \overline{\eqsim} \varnothing$ implies $\mathcal{S} \overline{\eqsim} \varnothing$;
5) $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{S} \overline{\eqsim} \mathcal{O}$ implies $\mathcal{T} \overline{\eqsim} \mathcal{O}$.

Proof:

(1) Let $\mathcal{S} \cong \mathcal{T}$. Then $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$. From part (11) of Theorem 2, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cup \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$. Therefore, we get $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$. So $\mathcal{S} \overline{\eqsim} (\mathcal{S} \cup \mathcal{T})$ and $(\mathcal{S} \cup \mathcal{T}) \overline{\eqsim} \mathcal{T}$.
Conversely, let $\mathcal{S} \overline{\eqsim} (\mathcal{S} \cup \mathcal{T})$ and $(\mathcal{S} \cup \mathcal{T}) \overline{\eqsim} \mathcal{T}$. Then by transitivity of $\overline{\eqsim}$, it follows that $\mathcal{S} \overline{\eqsim} \mathcal{T}$.

(2) Let $\mathcal{S} \overline{\eqsim} \mathcal{S}_1$ and $\mathcal{T} \overline{\eqsim} \mathcal{T}_1$, then $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}_1)}_{\alpha}$ and $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T}_1)}_{\alpha}$. Thus $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cup \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}_1)}_{\alpha} \cup \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T}_1)}_{\alpha}$. Therefore, by part (11) of Theorem 2, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}_1 \cup \mathcal{T}_1)}_{\alpha}$. Hence, $(\mathcal{S} \cup \mathcal{T}) \overline{\eqsim} (\mathcal{S}_1 \cup \mathcal{T}_1)$.

(3) Let $\mathcal{S} \cong \mathcal{T}$. Then $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$. Through part (11) of Theorem 2, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S} \cup \mathcal{T}^c)}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cup \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T}^c)}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha} \cup \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T}^c)}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T} \cup \mathcal{T}^c)}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{O})}_{\alpha}$. Hence, $(\mathcal{S} \cup \mathcal{T}^c) \overline{\eqsim} \mathcal{O}$.

(4) Let $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{T} \overline{\eqsim} \varnothing$. Then from Definition 10, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\varnothing)}_{\alpha}$. Therefore, according to parts (2) and (7) of Theorem 2, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\varnothing)}_{\alpha} = \varnothing$. Thus, $\mathcal{S} \overline{\eqsim} \varnothing$.

(5) Suppose that $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{S} \overline{\eqsim} \mathcal{O}$. Then in the light of Definition 10, $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{O})}_{\alpha}$. Therefore, according to parts (3) and (7) of Theorem 2, $\mathcal{O} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{O})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \subseteq \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{T})}_{\alpha}$. Thus, $\mathcal{T} \overline{\eqsim} \mathcal{O}$.

Theorem 4: Let $\wp^{\uparrow} = \left(\mathcal{O}, \widehat{\mathfrak{R}}^{\uparrow}\right)$ be an upward-FPA_s and $\alpha \in [0.5, 1)$. Then for any $\mathcal{S}, \mathcal{T}, \mathcal{S}_1, \mathcal{T}_1 \subseteq \mathcal{O}$, the following assertions hold:

1) $\mathcal{S} \cong \mathcal{T}$ if and only if $\mathcal{S} \cong (\mathcal{S} \cap \mathcal{T})$ and $(\mathcal{S} \cap \mathcal{T}) \cong \mathcal{T}$;
2) $\mathcal{S} \cong \mathcal{S}_1$ and $\mathcal{T} \cong \mathcal{T}_1$ implies $(\mathcal{S} \cap \mathcal{T}) \cong (\mathcal{S}_1 \cap \mathcal{T}_1)$;
3) $\mathcal{S} \cong \mathcal{T}$ implies $(\mathcal{S} \cap \mathcal{T}^c) \cong \varnothing$;
4) $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{T} \cong \varnothing$ implies $\mathcal{S} \cong \varnothing$;
5) $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{S} \cong \mathcal{O}$ implies $\mathcal{T} \cong \mathcal{O}$.

Proof: Straightforward according to Theorems 2 and 3.

Theorem 5: Suppose that $\wp^{\uparrow} = \left(\mathcal{O}, \widehat{\mathfrak{R}}^{\uparrow}\right)$ be an upward-FPA_s and $\alpha \in [0.5, 1)$. Then for any $\mathcal{S}, \mathcal{T}, \mathcal{S}_1, \mathcal{T}_1 \subseteq \mathcal{O}$, the following assertions hold:

1) $\mathcal{S} \approx \mathcal{T}$ if and only if $\mathcal{S} \eqsim (\mathcal{S} \cup \mathcal{T}) \eqsim \mathcal{T}$ and $\mathcal{S} \backsimeq (\mathcal{S} \cap \mathcal{T}) \backsimeq \mathcal{T}$;
2) $\mathcal{S} \approx \mathcal{S}_1$ and $\mathcal{T} \approx \mathcal{T}_1$ implies $(\mathcal{S} \cup \mathcal{T}) \eqsim (\mathcal{S}_1 \cup \mathcal{T}_1)$ and $(\mathcal{S} \cap \mathcal{T}) \backsimeq (\mathcal{S}_1 \cap \mathcal{T}_1)$;
3) $\mathcal{S} \approx \mathcal{T}$ implies $(\mathcal{S} \cup \mathcal{T}^c) \eqsim \mathcal{O}$ and $(\mathcal{S} \cap \mathcal{T}^c) \backsimeq \varnothing$;
4) $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{T} \approx \varnothing$ implies $\mathcal{S} \approx \varnothing$;
5) $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{S} \approx \mathcal{O}$ implies $\mathcal{T} \approx \mathcal{O}$.

Proof: Direct consequence of Theorems 3 and 4.

V. UNCERTAINTY MEASURES ASSOCIATED WITH $\alpha^{\uparrow}$-FPRSs

In this section, we provide several measures to quantify the uncertainty of $\alpha^{\uparrow}$-FPRSs.

Definition 11: Let $\wp^{\uparrow} = \left(\mathcal{O}, \widehat{\mathfrak{R}}^{\uparrow}\right)$ be an upward-FPA_s and $\alpha \in [0.5, 1)$. Then *the measure of precision* $\rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$ of $\mathcal{S}$ under $\alpha^{\uparrow}$-FPRS is defined as:

$$\rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}) = \frac{\left|\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right|}{\left|\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right|}, \tag{16}$$

where $\varnothing \neq \mathcal{S} \subseteq \mathcal{O}$ and $|\bullet|$ denote the set's cardinality. The corresponding *rough degree* $\gamma^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$ of $\mathcal{S}$ under $\alpha^{\uparrow}$-FPRS is defined as:

$$\gamma^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}) = 1 - \rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}) = 1 - \frac{\left|\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right|}{\left|\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right|}. \tag{17}$$

Obviously, $\rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}), \gamma^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}) \in [0, 1]$ for any $\mathcal{S} \subseteq \mathcal{O}$ and $\alpha \in [0.5, 1)$.

The approximate precision can be viewed using the famous Marczewski-Steinhaus metric (MS-metric). The MS-metric measures the distance between two sets, $\mathcal{S}_1$ and $\mathcal{S}_2$, which is given as:

$$\delta(\mathcal{S}_1, \mathcal{S}_2) = \frac{\left|\mathcal{S}_1 \Delta \mathcal{S}_2\right|}{\left|\mathcal{S}_1 \cup \mathcal{S}_2\right|} = 1 - \frac{\left|\mathcal{S}_1 \cap \mathcal{S}_2\right|}{\left|\mathcal{S}_1 \cup \mathcal{S}_2\right|}, \tag{18}$$

where $\mathcal{S}_1 \Delta \mathcal{S}_2 = (\mathcal{S}_1 \cup \mathcal{S}_2) - (\mathcal{S}_1 \cap \mathcal{S}_2)$ indicates the symmetric difference between $\mathcal{S}_1$ and $\mathcal{S}_2$.

From Eq. (18), we observe that:

- $\delta(\mathcal{S}_1, \mathcal{S}_2)$ has a maximum value of 1 when $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint.
- $\delta(\mathcal{S}_1, \mathcal{S}_2)$ has a minimum value of 0 when $\mathcal{S}_1 = \mathcal{S}_2$.

By using the MS-metric to the upward α-fuzzified preference lower and upper approximations, we obtained:

$$\begin{aligned}\delta\left(\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}, \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right) &= 1 - \frac{\left|\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cap \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right|}{\left|\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \cup \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right|} \\ &= 1 - \frac{\left|\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right|}{\left|\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right|} \\ &= 1 - \rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}). \end{aligned} \tag{19}$$

Hence, the measure of precision can be interpreted as an inverse of MS-metric when applied to upward α-fuzzified preference lower and upper approximations. In simple words, the distance between the upward α-fuzzified preference lower and upper approximations determine the measure of precision of the $\alpha^{\uparrow}$-FPRS approximations.

Proposition 5: Assume that $\wp^{\uparrow} = \left(\mathcal{O}, \widehat{\mathfrak{R}}^{\uparrow}\right)$ be an upward-FPA_s and $\alpha \in [0.5, 1)$. Then *the measure of precision* $\rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$ of $\varnothing \neq \mathcal{S} \subseteq \mathcal{O}$ own the following properties:

1) $\rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}) = 1 \Longleftrightarrow \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$,
2) $\rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}) = 0 \Longleftrightarrow \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \varnothing$,
3) For a fixed $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$, $\rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$ strictly monotonically increases with $\left|\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right|$,
4) For a fixed $\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} \neq \varnothing$, $\rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$ strictly monotonically decreases with $\left|\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right|$,
5) $\alpha_1 \leq \alpha_2 \Longrightarrow \rho^{\alpha_1}_{\widehat{R}^{\uparrow}}(\mathcal{S}) \leq \rho^{\alpha_2}_{\widehat{R}^{\uparrow}}(\mathcal{S})$.

Proof: Straightforward.

Definition 12: Let $\wp^{\uparrow} = \left(\mathcal{O}, \widehat{\mathfrak{R}}^{\uparrow}\right)$ be an upward-FPA_s and $\alpha \in [0.5, 1)$. Then *the measure of quality* $\mathcal{Q}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$ of $\varnothing \neq \mathcal{S} \subseteq \mathcal{O}$ under $\alpha^{\uparrow}$-FPRS is defined as:

$$\mathcal{Q}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}) = \frac{\left|\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right|}{\left|\mathcal{S}\right|}. \tag{20}$$

It may be noted that $\mathcal{Q}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$ requires entire information of $\mathcal{S}$; whereas $\rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$ does not.

Proposition 6: For any $\mathcal{S} \subseteq \mathcal{O}$ and $\alpha \in [0.5, 1)$, $\mathcal{Q}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}) \geq \rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$.

Proof: Straightforward.

Proposition 7: Let $\wp^{\uparrow} = \left(\mathcal{O}, \widehat{\mathfrak{R}}^{\uparrow}\right)$ be an upward-FPA_s and $\alpha \in [0.5, 1)$. Then *the measure of quality* $\mathcal{Q}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$ of $\varnothing \neq \mathcal{S} \subseteq \mathcal{O}$ own the following properties:

1) $\mathcal{Q}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}) = 1 \Longleftrightarrow \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \mathcal{S}$,
2) $\mathcal{Q}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}) = 0 \Longleftrightarrow \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \varnothing$,
3) $\alpha_1 \leq \alpha_2 \Longrightarrow \mathcal{Q}^{\alpha_1}_{\widehat{R}^{\uparrow}}(\mathcal{S}) \leq \mathcal{Q}^{\alpha_2}_{\widehat{R}^{\uparrow}}(\mathcal{S})$.

Proof: Straightforward.

Definition 13: Let $\wp^{\uparrow} = \left(\mathcal{O}, \widehat{\mathfrak{R}}^{\uparrow}\right)$ be an upward-FPA_s and $\alpha \in [0.5, 1)$. Then *the measure of completeness of knowledge* $\mathcal{C}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$ of $\varnothing \neq \mathcal{S} \subseteq \mathcal{O}$ under $\alpha^{\uparrow}$-FPRS is defined as:

$$\mathcal{C}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}) = \frac{\left|\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}\right| + \left|\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S}^c)}_{\alpha}\right|}{\left|\mathcal{O}\right|}. \tag{21}$$

Proposition 8: Let $\wp^{\uparrow} = \left(\mathcal{O}, \widehat{\mathfrak{R}}^{\uparrow}\right)$ be an upward-FPA_s and $\alpha \in [0.5, 1)$. Then *the measure of quality* $\mathcal{C}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$ of $\varnothing \neq \mathcal{S} \subseteq \mathcal{O}$ own the following properties:

1) $\mathcal{C}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}) = 1 \Longleftrightarrow \mathcal{S} = \varnothing$ or $\mathcal{S} = \mathcal{O}$,
2) $\mathcal{C}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S}) = 0 \Longleftrightarrow \underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \varnothing$ and $\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha} = \mathcal{O}$.
3) $\alpha_1 \leq \alpha_2 \Longrightarrow \mathcal{C}^{\alpha_1}_{\widehat{R}^{\uparrow}}(\mathcal{S}) \leq \mathcal{C}^{\alpha_2}_{\widehat{R}^{\uparrow}}(\mathcal{S})$.

Proof: Straightforward.

TABLE 2. Uncertainty measures of $\mathcal{S} \subseteq \mathcal{O}$ for different values of $\alpha \in [0.5, 1)$.

$\alpha \in [0.5, 1)$	$\underline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$	$\overline{\widehat{\mathfrak{R}}^{\uparrow}(\mathcal{S})}_{\alpha}$	$\rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$	$\gamma^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$	$\mathcal{Q}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$	$\mathcal{C}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$
$\alpha = 0.50$	$\{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$	$\mathcal{O}$	0.888	0.112	1	0.888
$\alpha = 0.55$	$\{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$	$\mathcal{O}$	0.888	0.112	1	0.888
$\alpha = 0.60$	$\{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$	$\mathcal{O}$	0.777	0.223	0.875	0.777
$\alpha = 0.65$	$\{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$	$\mathcal{O}$	0.777	0.223	0.875	0.777
$\alpha = 0.70$	$\{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\}$	$\mathcal{O}$	0.667	0.333	0.750	0.667
$\alpha = 0.75$	$\{\mathfrak{x}_3, \mathfrak{x}_6\}$	$\mathcal{O}$	0.222	0.778	0.250	0.222
$\alpha = 0.80$	$\varnothing$	$\mathcal{O}$	0	1	0	0
$\alpha = 0.85$	$\varnothing$	$\mathcal{O}$	0	1	0	0
$\alpha = 0.90$	$\varnothing$	$\mathcal{O}$	0	1	0	0
$\alpha = 0.95$	$\varnothing$	$\mathcal{O}$	0	1	0	0

Here, we give an example to understand the concept of the measure of precision, rough degree, measure of quality, and measure of completeness of knowledge.

Example 5: If we take the UFPR $\widehat{\mathfrak{R}}^{\uparrow}$ over $\mathcal{O}$ given in (9) and $\mathcal{S} = \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_5, \mathfrak{x}_6, \mathfrak{x}_7, \mathfrak{x}_8, \mathfrak{x}_9\} \subseteq \mathcal{O}$, then for different values of $\alpha \in [0.5, 1)$, the upward α-fuzzified preference lower and upper approximations for $\mathcal{S}$, and the corresponding values of $\rho^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$, $\gamma^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$, $\mathcal{Q}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$, and $\mathcal{C}^{\alpha}_{\widehat{R}^{\uparrow}}(\mathcal{S})$ are listed in Table 2.

VI. UPWARD FUZZY β-COVERING ROUGH SETS (UFβ-CRSs)

In this segment, we initially describe the concept of UFβ-nghd and Uβ-nghd in upward fuzzy covering approximation space, and some relative properties are studied. After that, we will present two RS models based on UFβ-nghd and Uβ-nghd.

Definition 14: Let $\varnothing \neq \mathcal{O}$ be a finite universe and $\mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})$ be an upward fuzzy preference granular structures. Then for every $\beta \in (0, 1]$, we call $\mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})$ an upward fuzzy β-covering of $\mathcal{O}$, if $\left(\bigcup_{i=1}^{n}[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}\right)(\mathfrak{x}) \geq \beta$ for all $\mathfrak{x} \in \mathcal{O}$. Moreover, $\left(\mathcal{O}, \mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})\right)$ is said to be an upward fuzzy β-covering approximation space (UFβ-CAS).

Definition 15: Assume that $\left(\mathcal{O}, \mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})\right)$ be an UFβ-CAS. For each $\mathfrak{x} \in \mathcal{O}$, we define the UF$\beta$-nghd $\aleph^{\uparrow}_{(\mathfrak{x},\beta)}$ of $\mathfrak{x}$ as follows:

$$\aleph^{\uparrow}_{(\mathfrak{x},\beta)} = \bigcap \left\{[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}} \in \mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow}) : [\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}(\mathfrak{x}) \geq \beta\right\}. \quad (22)$$

Example 6: If we consider the UFPR $\widehat{\mathfrak{R}}^{\uparrow}$ over $\mathcal{O}$ provided in (9), then the upward fuzzy preference classes $[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}$ for $i = 1, 2 \cdots, 9$ are exhibited Table 3. Clearly, we can see that $\mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow}) = \left\{[\mathfrak{x}_1]_{\widehat{\mathfrak{R}}^{\uparrow}}, [\mathfrak{x}_2]_{\widehat{\mathfrak{R}}^{\uparrow}}, \cdots, [\mathfrak{x}_9]_{\widehat{\mathfrak{R}}^{\uparrow}}\right\}$ is an upward fuzzy β-covering of $\mathcal{O}$ $(0 < \beta \leq 0.50)$. Let $\beta = 0.40$. Then,

$$\aleph^{\uparrow}_{(\mathfrak{x}_1,0.40)} = [\mathfrak{x}_1]_{\widehat{\mathfrak{R}}^{\uparrow}} \cap [\mathfrak{x}_4]_{\widehat{\mathfrak{R}}^{\uparrow}},$$
$$\aleph^{\uparrow}_{(\mathfrak{x}_2,0.40)} = [\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}(i = 1, 2 \cdots, 9),$$
$$\aleph^{\uparrow}_{(\mathfrak{x}_3,0.40)} = [\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}(i = 1, 2 \cdots, 9),$$
$$\aleph^{\uparrow}_{(\mathfrak{x}_4,0.40)} = [\mathfrak{x}_1]_{\widehat{\mathfrak{R}}^{\uparrow}} \cap [\mathfrak{x}_4]_{\widehat{\mathfrak{R}}^{\uparrow}} \cap [\mathfrak{x}_5]_{\widehat{\mathfrak{R}}^{\uparrow}},$$
$$\aleph^{\uparrow}_{(\mathfrak{x}_5,0.40)} = [\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}(i = 1, 2 \cdots, 9),$$
$$\aleph^{\uparrow}_{(\mathfrak{x}_6,0.40)} = [\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}(i = 1, 2 \cdots, 9),$$

TABLE 3. The upward fuzzy preference classes $[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}$.

	$[\mathfrak{x}_1]_{\widehat{\mathfrak{R}}^{\uparrow}}$	$[\mathfrak{x}_2]_{\widehat{\mathfrak{R}}^{\uparrow}}$	$[\mathfrak{x}_3]_{\widehat{\mathfrak{R}}^{\uparrow}}$	$[\mathfrak{x}_4]_{\widehat{\mathfrak{R}}^{\uparrow}}$	$[\mathfrak{x}_5]_{\widehat{\mathfrak{R}}^{\uparrow}}$	$[\mathfrak{x}_6]_{\widehat{\mathfrak{R}}^{\uparrow}}$	$[\mathfrak{x}_7]_{\widehat{\mathfrak{R}}^{\uparrow}}$	$[\mathfrak{x}_8]_{\widehat{\mathfrak{R}}^{\uparrow}}$	$[\mathfrak{x}_9]_{\widehat{\mathfrak{R}}^{\uparrow}}$
$\mathfrak{x}_1$	0.50	0.25	0.20	0.40	0.30	0.20	0.25	0.25	0.25
$\mathfrak{x}_2$	0.75	0.50	0.45	0.65	0.55	0.45	0.50	0.50	0.50
$\mathfrak{x}_3$	0.80	0.55	0.50	0.70	0.60	0.50	0.55	0.55	0.55
$\mathfrak{x}_4$	0.60	0.35	0.30	0.50	0.40	0.30	0.35	0.35	0.35
$\mathfrak{x}_5$	0.70	0.45	0.40	0.60	0.50	0.40	0.45	0.45	0.45
$\mathfrak{x}_6$	0.80	0.55	0.50	0.70	0.60	0.50	0.55	0.55	0.55
$\mathfrak{x}_7$	0.75	0.50	0.45	0.65	0.55	0.45	0.50	0.50	0.50
$\mathfrak{x}_8$	0.75	0.50	0.45	0.65	0.55	0.45	0.50	0.50	0.50
$\mathfrak{x}_9$	0.75	0.50	0.45	0.65	0.55	0.45	0.50	0.50	0.50

TABLE 4. The UFβ-nghd $\aleph^{\uparrow}_{(\mathfrak{x}_i,0.40)}$.

	$\aleph^{\uparrow}_{(\mathfrak{x}_1,0.40)}$	$\aleph^{\uparrow}_{(\mathfrak{x}_2,0.40)}$	$\aleph^{\uparrow}_{(\mathfrak{x}_3,0.40)}$	$\aleph^{\uparrow}_{(\mathfrak{x}_4,0.40)}$	$\aleph^{\uparrow}_{(\mathfrak{x}_5,0.40)}$	$\aleph^{\uparrow}_{(\mathfrak{x}_6,0.40))}$	$\aleph^{\uparrow}_{(\mathfrak{x}_7,0.40)}$	$\aleph^{\uparrow}_{(\mathfrak{x}_8,0.40)}$	$\aleph^{\uparrow}_{(\mathfrak{x}_9,0.40)}$
$\mathfrak{x}_1$	0.40	0.20	0.20	0.30	0.20	0.20	0.20	0.20	0.20
$\mathfrak{x}_2$	0.65	0.45	0.45	0.55	0.45	0.45	0.45	0.45	0.45
$\mathfrak{x}_3$	0.70	0.50	0.50	0.60	0.50	0.50	0.50	0.50	0.50
$\mathfrak{x}_4$	0.50	0.30	0.30	0.40	0.30	0.30	0.30	0.30	0.30
$\mathfrak{x}_5$	0.60	0.40	0.40	0.50	0.40	0.40	0.40	0.40	0.40
$\mathfrak{x}_6$	0.70	0.50	0.50	0.60	0.50	0.50	0.50	0.50	0.50
$\mathfrak{x}_7$	0.65	0.45	0.45	0.55	0.45	0.45	0.45	0.45	0.45
$\mathfrak{x}_8$	0.65	0.45	0.45	0.55	0.45	0.45	0.45	0.45	0.45
$\mathfrak{x}_9$	0.65	0.45	0.45	0.55	0.45	0.45	0.45	0.45	0.45

$$\aleph^{\uparrow}_{(\mathfrak{x}_7,0.40)} = [\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}(i = 1, 2 \cdots, 9),$$
$$\aleph^{\uparrow}_{(\mathfrak{x}_8,0.40)} = [\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}(i = 1, 2 \cdots, 9),$$
$$\aleph^{\uparrow}_{(\mathfrak{x}_9,0.40)} = [\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}(i = 1, 2 \cdots, 9),$$

These elements $\aleph^{\uparrow}_{(\mathfrak{x}_i,0.40)}$ $(i = 1, 2 \cdots, 9)$ are listed in Table IV:

Proposition 9: Suppose that $\left(\mathcal{O}, \mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})\right)$ be an UFβ-CAS. Then for each $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathcal{O}$, the subsequent statements hold true:

1) $\aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{x}) \geq \beta$.
2) If $\aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y}) \geq \beta$ and $\aleph^{\uparrow}_{(\mathfrak{y},\beta)}(\mathfrak{z}) \geq \beta$, then $\aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{z}) \geq \beta$.

Proof: Straightforward.

Proposition 10: Let $\beta_1, \beta_2 \in (0, 1]$ be such that $\beta_1 \leq \beta_2$. Then $\aleph^{\uparrow}_{(\mathfrak{x},\beta_1)} \subseteq \aleph^{\uparrow}_{(\mathfrak{x},\beta_2)}$ for all $\mathfrak{x} \in \mathcal{O}$.

Proof: For each $\mathfrak{x} \in \mathcal{O}$, $\beta_1 \leq \beta_2$ gives that $\left\{[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}} : [\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}(\mathfrak{x}) \geq \beta_1\right\} \supseteq \left\{[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}} : [\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}(\mathfrak{x}) \geq \beta_2\right\}$. Thus $\aleph^{\uparrow}_{(\mathfrak{x},\beta_1)} = \bigcap\left\{[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}} : [\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}(\mathfrak{x}) \geq \beta_1\right\} \subseteq \bigcap\left\{[\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}} : [\mathfrak{x}_i]_{\widehat{\mathfrak{R}}^{\uparrow}}(\mathfrak{x}) \geq \beta_2\right\} = \aleph^{\uparrow}_{(\mathfrak{x},\ \beta_2)}$ for all $\mathfrak{x} \in \mathcal{O}$.

Definition 16: Assume that $\left(\mathcal{O}, \mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})\right)$ be an UFβ-CAS with $\mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})$ being an upward fuzzy β-covering of $\mathcal{O}$ for some $\beta \in (0, 1]$. Then for each $\mathfrak{x} \in \mathcal{O}$, we define the U$\beta$-nghd of $\mathfrak{x}$ as:

$$\mathscr{N}^{\uparrow}_{(\mathfrak{x},\beta)} = \left\{\mathfrak{y} \in \mathcal{O} : \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y}) \geq \beta\right\}. \quad (23)$$

Example 7: Suppose that $\left(\mathcal{O}, \mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})\right)$ be an UFβ-CAS in Example 6 with $\beta = 0.40$, then we get

$$\mathscr{N}^{\uparrow}_{(\mathfrak{x}_1,\beta)} = \{\mathfrak{x}_1\},$$
$$\mathscr{N}^{\uparrow}_{(\mathfrak{x}_2,\beta)} = \mathscr{N}^{\uparrow}_{(\mathfrak{x}_3,\beta)} = \mathscr{N}^{\uparrow}_{(\mathfrak{x}_5,\beta)} = \mathscr{N}^{\uparrow}_{(\mathfrak{x}_6,\beta)}$$
$$= \mathscr{N}^{\uparrow}_{(\mathfrak{x}_7,\beta)} = \mathscr{N}^{\uparrow}_{(\mathfrak{x}_8,\beta)} = \mathscr{N}^{\uparrow}_{(\mathfrak{x}_9,\beta)} = \{\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_9\},$$
$$\mathscr{N}^{\uparrow}_{(\mathfrak{x}_4,\beta)} = \{\mathfrak{x}_1, \mathfrak{x}_4\}.$$

Proposition 11: $\mathfrak{x} \in \mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{x})$ for each $\mathfrak{x} \in \mathcal{O}$.

Proof: According to part (1) of Proposition 9, it implies that $\mathfrak{x} \in \left\{\mathfrak{y} \in \mathcal{O} : \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y}) \geq \beta\right\} = \mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)}$ for each $\mathfrak{x} \in \mathcal{O}$.

Proposition 12: If $\mathfrak{x} \in \mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)}$, then $\mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)} \subseteq \mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)}$ for each $\mathfrak{x}, \mathfrak{y} \in \mathcal{O}$.

Proof: For all $\mathfrak{z} \in \mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)}$, we have $\aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{z}) \geq \beta$. Also, $\mathfrak{x} \in \mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)}$ implies $\aleph^{\uparrow}_{(\mathfrak{y},\beta)}(\mathfrak{x}) \geq \beta$. From part (2) of Proposition 9, there is $\aleph^{\uparrow}_{(\mathfrak{y},\beta)}(\mathfrak{z}) \geq \beta$, and thus $\mathfrak{z} \in \mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)}$. Hence, $\mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)} \subseteq \mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)}$.

Proposition 13: $\mathfrak{x} \in \mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)}$ and $\mathfrak{y} \in \mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)}$ if and only if $\mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)} = \mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)}$ for each $\mathfrak{x}, \mathfrak{y} \in \mathcal{O}$.

Proof: Let $\mathfrak{x} \in \mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)}$ and $\mathfrak{y} \in \mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)}$ for $\mathfrak{x}, \mathfrak{y} \in \mathcal{O}$. Then by Proposition 12, $\mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)} \subseteq \mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)}$ and $\mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)} \subseteq \mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)}$. Thus, $\mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)} = \mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)}$.

Conversely, if $\mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)} = \mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)}$, then we have $\mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)} \subseteq \mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)}$ and $\mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)} \subseteq \mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)}$. Hence we can get that $\mathfrak{x} \in \mathcal{N}^{\uparrow}_{(\mathfrak{y},\beta)}$ and $\mathfrak{y} \in \mathcal{N}^{\uparrow}_{(\mathfrak{x},\beta)}$.

Based on the above discussion, here we will propose two new covering RS models by using UFβ-nghd and Uβ-nghd. In one model, the lower and upper approximation operators of each FS are characterized in the fuzzy environment, while in other model, the lower and upper approximation operators of each crisp set are illustrated in the fuzzy context.

A. AN UFβ-CRS MODEL FOR FUZZY SETS

Definition 17: Let $\left(\mathcal{O}, \mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})\right)$ be an UFβ-CAS with $\mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})$ being an upward fuzzy β-covering of $\mathcal{O}$ for some $\beta \in (0, 1]$. For each fuzzy subset $\mathcal{S} \in \mathcal{F}(\mathcal{O})$, the lower approximation $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S})$ and upper approximation $\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S})$ of $\mathcal{S}$ are defined as:

$$\left.\begin{aligned} \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S})(\mathfrak{x}) &= \bigwedge_{\mathfrak{y}\in\mathcal{O}} \left[\left(1-\aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y})\right) \vee \mathcal{S}(\mathfrak{y})\right], \mathfrak{x} \in \mathcal{O}, \\ \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S})(\mathfrak{x}) &= \bigvee_{\mathfrak{y}\in\mathcal{O}} \left[\aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y}) \wedge \mathcal{S}(\mathfrak{y})\right], \mathfrak{x} \in \mathcal{O}. \end{aligned}\right\} \tag{24}$$

If $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}) \neq \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S})$, then $\mathcal{S}$ an UFβ-CRS.

Example 8: Let $\left(\mathcal{O}, \mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})\right)$ be an UFβ-CAS in Example 6. Then for

$$\mathcal{S} = \frac{0.6}{\mathfrak{x}_1} + \frac{0.4}{\mathfrak{x}_2} + \frac{0.3}{\mathfrak{x}_3} + \frac{0.5}{\mathfrak{x}_4} + \frac{0.7}{\mathfrak{x}_5} + \frac{0.4}{\mathfrak{x}_6} + \frac{0.8}{\mathfrak{x}_7} + \frac{0.7}{\mathfrak{x}_8} + \frac{0.2}{\mathfrak{x}_9}$$

and $\beta = 0.40$, we have

$$\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}) = \frac{0.3}{\mathfrak{x}_1} + \frac{0.5}{\mathfrak{x}_2} + \frac{0.5}{\mathfrak{x}_3} + \frac{0.4}{\mathfrak{x}_4} + \frac{0.5}{\mathfrak{x}_5} + \frac{0.5}{\mathfrak{x}_6} + \frac{0.5}{\mathfrak{x}_7} + \frac{0.5}{\mathfrak{x}_8} + \frac{0.5}{\mathfrak{x}_9},$$

$$\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S}) = \frac{0.65}{\mathfrak{x}_1} + \frac{0.45}{\mathfrak{x}_2} + \frac{0.45}{\mathfrak{x}_3} + \frac{0.55}{\mathfrak{x}_4} + \frac{0.45}{\mathfrak{x}_5} + \frac{0.45}{\mathfrak{x}_6} + \frac{0.45}{\mathfrak{x}_7} + \frac{0.45}{\mathfrak{x}_8} + \frac{0.45}{\mathfrak{x}_9}.$$

Proposition 14: Let $\left(\mathcal{O}, \mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})\right)$ be an UFβ-CAS with $\mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})$ being an upward fuzzy β-covering of $\mathcal{O}$ for some $\beta \in (0, 1]$. For each fuzzy subset $\mathcal{S}, \mathcal{T} \in \mathcal{F}(\mathcal{O})$, the following assertions hold true:

1) $\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S}^c) = \left(\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S})\right)^c$;
2) $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}^c) = \left(\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S})\right)^c$;
3) $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{O}) = \mathcal{O}$;
4) $\widetilde{apr}^{\uparrow}_{\beta}(\varnothing) = \varnothing$;
5) $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S} \cap \mathcal{T}) = \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}) \cap \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{T})$;
6) $\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S} \cup \mathcal{T}) = \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S}) \cup \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{T})$;
7) $\mathcal{S} \subseteq \mathcal{T} \Longrightarrow \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}) \subseteq \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{T})$;
8) $\mathcal{S} \subseteq \mathcal{T} \Longrightarrow \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S}) \subseteq \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{T})$;
9) $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S} \cup \mathcal{T}) \supseteq \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}) \cup \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{T})$;
10) $\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S} \cap \mathcal{T}) \subseteq \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S}) \cap \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{T})$;
11) If $1 - \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{x}) \leq \mathcal{S}(\mathfrak{x}) \leq \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{x})$ for all $\mathfrak{x} \in \mathcal{O}$, then $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}) \subset \mathcal{S} \subseteq \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S})$.

Proof:

(1) Since,

$$\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S}^c) = \bigvee_{\mathfrak{y}\in\mathcal{O}} \left[\aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y}) \wedge \mathcal{S}^c(\mathfrak{y})\right],$$

where $\mathcal{S}^c(\mathfrak{y}) = 1 - \mathcal{S}(\mathfrak{y})$

$$\begin{aligned} &= 1 - \bigwedge_{\mathfrak{y}\in\mathcal{O}} \left[\left(1-\aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y})\right) \vee \mathcal{S}(\mathfrak{y})\right] \\ &= 1 - \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}) \\ &= \left(\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S})\right)^c. \end{aligned}$$

Hence, $\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S}^c) = \left(\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S})\right)^c$.

(2) Using $\mathcal{S}$ instead of $\mathcal{S}^c$ in part (1), we get $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}^c) = \left(\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S})\right)^c$.

(3) Since $\mathcal{O}(\mathfrak{x}) = 1$ for every $\mathfrak{x} \in \mathcal{O}$. Thus,

$$\begin{aligned} \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{O})(\mathfrak{x}) &= \bigwedge_{\mathfrak{y}\in\mathcal{O}} \left[\left(1-\aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y})\right) \vee \mathcal{O}(\mathfrak{y})\right] \\ &= 1 = \mathcal{O}(\mathfrak{x}). \end{aligned}$$

Thus $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{O}) = \mathcal{O}$.

(4) As $\varnothing(\mathfrak{x}) = 0$ for every $\mathfrak{x} \in \mathcal{O}$. So,

$$\widetilde{apr}^{\uparrow}_{\beta}(\varnothing)(\mathfrak{x}) = \bigvee_{\mathfrak{y}\in\mathcal{O}} \left[\aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y}) \wedge \varnothing(\mathfrak{y})\right] = 0 = \varnothing(\mathfrak{x}).$$

Hence, $\widetilde{apr}^{\uparrow}_{\beta}(\varnothing) = \varnothing$.

(5) Since

$$\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S} \cap \mathcal{T})(\mathfrak{x}) = \bigwedge_{\mathfrak{y} \in \mathcal{O}} \left[\left(1 - \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y})\right) \vee (\mathcal{S} \cap \mathcal{T})(\mathfrak{y})\right]$$
$$= \bigwedge_{\mathfrak{y} \in \mathcal{O}} \left[\left((1 - \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y})) \vee (\mathcal{S})(\mathfrak{y})\right) \wedge \left((1 - \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y})) \vee (\mathcal{T})(\mathfrak{y})\right)\right]$$
$$= \left(\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}) \cap \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{T})\right)(\mathfrak{x}).$$

Hence, $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S} \cap \mathcal{T}) = \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}) \cap \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{T})$.

(6) Analogous to the proof of (5).

(7) If $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{S}(\mathfrak{x}) \le \mathcal{T}(\mathfrak{x})$ for all $\mathfrak{x} \in \mathcal{O}$. Thus

$$\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S})(\mathfrak{x}) = \bigwedge_{\mathfrak{y} \in \mathcal{O}} \left[\left(1 - \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y})\right) \vee \mathcal{S}(\mathfrak{y})\right]$$
$$\le \bigwedge_{\mathfrak{y} \in \mathcal{O}} \left[\left(1 - \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y})\right) \vee \mathcal{T}(\mathfrak{y})\right]$$
$$= \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{T})(\mathfrak{x}).$$

Hence, $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}) \subseteq \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{T})$.

(8) Analogous to the proof of (7).

(9) Using the fact $\mathcal{S}, \mathcal{T} \subseteq \mathcal{S} \cup \mathcal{T}$ and part (7), we get $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}) \subseteq \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S} \cup \mathcal{T})$ and $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{T}) \subseteq \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S} \cup \mathcal{T})$. Thus $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}) \cup \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{T}) \subseteq \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S} \cup \mathcal{T})$.

(10) Using the fact $\mathcal{S} \cap \mathcal{T} \subseteq \mathcal{S}, \mathcal{T}$ and part (8), we have $\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S} \cap \mathcal{T}) \subseteq \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S})$ and $\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S} \cap \mathcal{T}) \subseteq \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{T})$. Hence $\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S} \cap \mathcal{T}) \subseteq \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S}) \cap \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{T})$.

(11) If $1 - \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{x}) \le \mathcal{S}(\mathfrak{x}) \le \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{x})$ for all $\mathfrak{x} \in \mathcal{O}$, then

$$\mathcal{S}(\mathfrak{x}) = \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{x}) \wedge \mathcal{S}(\mathfrak{x}) \le \bigvee_{\mathfrak{y} \in \mathcal{O}} \left[\aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y}) \wedge \mathcal{S}(\mathfrak{y})\right]$$
$$= \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S})(\mathfrak{x}).$$
$$\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S})(\mathfrak{x}) = \bigwedge_{\mathfrak{y} \in \mathcal{O}} \left[\left(1 - \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{y})\right) \vee \mathcal{S}(\mathfrak{y})\right]$$
$$\le \left[\left(1 - \aleph^{\uparrow}_{(\mathfrak{x},\beta)}(\mathfrak{x})\right) \vee \mathcal{S}(\mathfrak{x})\right] = \mathcal{S}(\mathfrak{x}).$$

Hence, $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{S}) \subset \mathcal{S} \subseteq \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{S})$.

B. AN UFβ-CRS MODEL FOR CRISP SETS

Definition 18: Let $\left(\mathcal{O}, \mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})\right)$ be an UFβ-CAS with $\mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})$ being an upward fuzzy β-covering of $\mathcal{O}$ for some $\beta \in (0, 1]$. For $\mathcal{S} \subseteq \mathcal{O}$, we describe the upward β-lower approximation $\underline{\mathcal{S}}^{\uparrow}_{\beta}$ and upward β-upper approximation $\overline{\mathcal{S}}^{\uparrow}_{\beta}$ of $\mathcal{S}$ as:

$$\left.\begin{aligned} \underline{\mathcal{S}}^{\uparrow}_{\beta} &= \{\mathfrak{x} \in \mathcal{O} : \mathscr{N}^{\uparrow}_{(\mathfrak{x},\beta)} \subseteq \mathcal{S}\}, \\ \overline{\mathcal{S}}^{\uparrow}_{\beta} &= \{\mathfrak{x} \in \mathcal{O} : \mathscr{N}^{\uparrow}_{(\mathfrak{x},\beta)} \cap \mathcal{S} \neq \varnothing\}. \end{aligned}\right\} \tag{25}$$

Moreover, if $\underline{\mathcal{S}}^{\uparrow}_{\beta} \neq \overline{\mathcal{S}}^{\uparrow}_{\beta}$, $\mathcal{S}$ is called an Uβ-CRS. Otherwise, $\mathcal{S}$ is called Uβ-nghd definable. The boundary and negative regions in Uβ-CRS environment are listed as follows:

(i) $\mathbf{POS}_{\beta\uparrow}(\mathcal{S}) = \underline{\mathcal{S}}^{\uparrow}_{\beta}$,

(ii) $\mathbf{B}ND_{\beta\uparrow}(\mathcal{S}) = \overline{\mathcal{S}}^{\uparrow}_{\beta} - \underline{\mathcal{S}}^{\uparrow}_{\beta}$,

(iii) $\mathbf{N}EG_{\beta\uparrow}(\mathcal{S}) = \left(\overline{\mathcal{S}}^{\uparrow}_{\beta}\right)^{c}$.

Example 9: If we revisit Example 7 and take $\mathcal{S} = \{\mathfrak{x}_1, \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_5\}$. Then $\underline{\mathcal{S}}^{\uparrow}_{\beta} = \{\mathfrak{x}_1, \mathfrak{x}_4\}$ and $\overline{\mathcal{S}}^{\uparrow}_{\beta} = \{\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_9\}$. Moreover, $\mathbf{POS}_{\beta\uparrow}(\mathcal{S}) = \{\mathfrak{x}_1, \mathfrak{x}_4\}$, $\mathbf{B}ND_{\beta\uparrow}(\mathcal{S}) = \{\mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_5, \cdots, \mathfrak{x}_9\}$ and $\mathbf{N}EG_{\beta\uparrow}(\mathcal{S}) = \varnothing$.

In the light of Definition 17, we can obtain the subsequent result.

Theorem 6: Let $\left(\mathcal{O}, \mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})\right)$ be an UFβ-CAS with $\mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})$ being an upward fuzzy β-covering of $\mathcal{O}$ for some $\beta \in (0, 1]$. Then for each $\mathcal{S}, \mathcal{T} \subseteq \mathcal{O}$, the following assertions hold true:

1) $\underline{\mathcal{S}}^{\uparrow}_{\beta} \subseteq \mathcal{S} \subseteq \overline{\mathcal{S}}^{\uparrow}_{\beta}$;
2) $\underline{(\mathcal{S}^c)}^{\uparrow}_{\beta} = \left(\overline{\mathcal{S}}^{\uparrow}_{\beta}\right)^{c}$;
3) $\overline{(\mathcal{S}^c)}^{\uparrow}_{\beta} = \left(\underline{\mathcal{S}}^{\uparrow}_{\beta}\right)^{c}$;
4) $\underline{\mathcal{O}}^{\uparrow}_{\beta} = \mathcal{O}$;
5) $\overline{\varnothing}^{\uparrow}_{\beta} = \varnothing$;
6) $\underline{(\mathcal{S} \cap \mathcal{T})}^{\uparrow}_{\beta} = \underline{\mathcal{S}}^{\uparrow}_{\beta} \cap \underline{\mathcal{T}}^{\uparrow}_{\beta}$;
7) $\overline{(\mathcal{S} \cup \mathcal{T})}^{\uparrow}_{\beta} = \overline{\mathcal{S}}^{\uparrow}_{\beta} \cup \overline{\mathcal{T}}^{\uparrow}_{\beta}$;
8) $\mathcal{S} \subseteq \mathcal{T} \Longrightarrow \underline{\mathcal{S}}^{\uparrow}_{\beta} \subseteq \underline{\mathcal{T}}^{\uparrow}_{\beta}$;
9) $\mathcal{S} \subseteq \mathcal{T} \Longrightarrow \overline{\mathcal{S}}^{\uparrow}_{\beta} \subseteq \overline{\mathcal{T}}^{\uparrow}_{\beta}$;
10) $\underline{(\mathcal{S} \cup \mathcal{T})}^{\uparrow}_{\beta} \supseteq \underline{\mathcal{S}}^{\uparrow}_{\beta} \cup \underline{\mathcal{T}}^{\uparrow}_{\beta}$;
11) $\overline{(\mathcal{S} \cap \mathcal{T})}^{\uparrow}_{\beta} \subseteq \overline{\mathcal{S}}^{\uparrow}_{\beta} \cap \overline{\mathcal{T}}^{\uparrow}_{\beta}$.

Proof: Straightforward.

VII. MADM APPROACH BASED ON UFβ-CRSs

In this section, by utilizing the notion of UFβ-CRSs, a general scheme is given for DM of the medicine selection.

Biomedical research is connected to large-scale, developing, and diversified data from various domains. As available resources become increasingly diverse, there is a growing need for multidisciplinary coordination among biomedical researchers to address challenging research problems. As a result, biomedical research has evolved into an interdisciplinary field. FS theory [51] provides a perfectly predictable solution on a certain level due to the complexity of biological systems and the limitations of actual mathematical theories in specific cases. On the other hand, the RS theory [33] has been revealed to be a powerful strategy for tackling many classification and DM problems.

In numerous aspects, medical diagnosis is the field of DM that lacks adequate data. According to the medical perspective, the attribute value is generally vague. We can acquire that incompleteness and uncertainty are inborn characteristics of medical practice. In other words, some medical information is commonly expressed in vague terms.

In actual practice, medical professionals analyze patients and determine best course of action or which is the best medicine. That is to say, medical DM is the pattern of different types of DM in which principles, procedures, and knowledge are approximate. As a result, practical strategies that are employed to tackle the traditional DM problems can be applied to medical DM issues. Also, RS theory is very appropriate for this inaccuracy in the medical sector.

A. DESCRIPTION OF MADM PROBLEM

In clinical practice, medical experts often integrate different medicines to cure certain disease D. Assume that $\mathcal{O} = \{\mathfrak{m}_i : i = 1, 2, \cdots, n\}$ be an assembling of n medicines, and $\mathcal{C} = \{a_1, a_2, \cdots, a_k\}$ be a collection of k most common symptoms/criteria (for instance, fever, fatigue, throat pain, cough, etc.) of a specific disease. The weight vector of all criteria $\mathcal{W} = (\varpi_1, \varpi_2, \cdots, \varpi_k)$, where $0 \leq \varpi_j \leq 1$ and $\sum_{j=1}^{k} \varpi_j = 1$. According to the medical perspective, the attribute value is generally vague. Therefore, the efficacy value of medicines w.r.t. symptoms can be viewed as an FS. Let $\mathcal{E}$ denote a finite set of the domain for the information function $f(\mathfrak{m}_i, a) \in [0, 1]$. In this article, we assume that $f(\mathfrak{m}_i, a)$ stands for the recommendation degree of medicine $\mathfrak{m}_i$ by the medical expert. $[\mathfrak{m}_i]^{a}_{\widehat{\mathfrak{R}}^{\uparrow}}(\mathfrak{m}_j)$ stands for medicine $\mathfrak{m}_j$'s efficacy value for the symptom a_i $(i = 1, 2, \cdots, k,$ $j = 1, 2, \cdots, n)$. For a critical value β, let for each medicine $\mathfrak{m}_i \in \mathcal{O}$, there is at least one symptom $a_k \in \mathcal{C}$ such that efficacy value of the medicine $\mathfrak{m}_j$ for the symptom a_i is not less than β, and $\mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow})$ is an upward fuzzy β-covering of $\mathcal{O}$. Then the UFβ-nghd $\aleph^{\uparrow a}_{(\mathfrak{m}_j,\beta)}$ of $\mathfrak{m}_j$ w.r.t. criteria a is a FS given as:

$$\aleph^{\uparrow a_1}_{(\mathfrak{m}_j,\beta)}(\mathfrak{m}_t) = \left[\bigcap \left\{[\mathfrak{m}_i]^{a}_{\widehat{\mathfrak{R}}^{\uparrow}} : [\mathfrak{m}_i]^{a}_{\widehat{\mathfrak{R}}^{\uparrow}}(\mathfrak{m}_j) \geq \beta\right\}\right](\mathfrak{m}_t);$$
$$t = 1, 2, \cdots, n, \tag{26}$$

which represents the minimum of all efficacy values for every medicine $\mathfrak{m}_t$ used to heal the symptoms. If a FS $\mathcal{S}$ represents the capability of all medicines in $\mathcal{O}$ to combat the disease D, since the inaccuracy of $\mathcal{S}$, then we can take its approximate estimation to the lower and upper approximations of $\mathcal{S}$.

We signify the fuzzified information system $(\mathcal{O}, \mathcal{C}, \mathcal{W}, \mathcal{E})$.

B. DM ALGORITHM

To select the best medicine among the available ones, here we offer a DM algorithm in the framework of UFβ-CRSs. The relevant steps are outlined as follows:

Input: Fuzzified information system $(\mathcal{O}, \mathcal{C}, \mathcal{W}, \mathcal{E})$.

Step 1: Determine $\widehat{\mathfrak{R}}^{\uparrow}_{a_t}; t = 1, 2, \cdots, k$ via transfer function given in Eq.(7).

Step 2: Evaluate $[\mathfrak{m}_i]^{a_t}_{\widehat{\mathfrak{R}}^{\uparrow}}$ of $\mathfrak{m}_j$ w.r.t. a_t.

Step 3: Construct $\aleph^{\uparrow a_t}_{(\mathfrak{m}_j,\beta)}$ of $\mathfrak{m}_j$ w.r.t. a_t.

Step 4: Apply the fuzzy TOPSIS approach to find positive ideal solution $\mathcal{I}^{a_t}_{+}$ and negative ideal solution $\mathcal{I}^{a_t}_{-}$ as:

$$\mathcal{I}^{a_t}_{+} = \max_{1\leq j\leq n} \left\{\aleph^{\uparrow a_t}_{(\mathfrak{m}_i,\beta)}(\mathfrak{m}_j) : i = 1, 2, \cdots, n\right\}, \tag{27}$$

$$\mathcal{I}^{a_t}_{-} = \min_{1\leq j\leq n} \left\{\aleph^{\uparrow a_t}_{(\mathfrak{m}_i,\beta)}(\mathfrak{m}_j) : i = 1, 2, \cdots, n\right\}, \tag{28}$$

where $t = 1, 2, \cdots, k$.

Step 5: By using Definition 17, calculate $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_t}_{+})$, $\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_t}_{+})$, $\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_t}_{-})$ and $\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_t}_{-})$.

Step 6: Determine the ranking index $\varrho_t(\mathfrak{m}_j)$, where

$$\varrho_t(\mathfrak{m}_j) = \sqrt{\begin{aligned}&\left[\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_t}_{+})(\mathfrak{m}_j) - \underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_t}_{-})(\mathfrak{m}_j)\right]^2 \\ &+ \left[\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_t}_{+})(\mathfrak{m}_j) - \widetilde{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_t}_{-})(\mathfrak{m}_j)\right]^2\end{aligned}} \tag{29}$$

Step 7: Compute optimal index $\partial(\mathfrak{m}_j)$, where

$$\partial(\mathfrak{m}_j) = \sum_{t=1}^{k} \varpi_t \varrho_t(\mathfrak{m}_j). \tag{30}$$

Step 8: Rank the medicines w.r.t. $\partial(\mathfrak{m}_j)$.

Output: A ranking outcome of all medicines.

Flowchart manifestation of the devised MADM strategy is provided in Figure 1.

C. EXAMPLE ANALYSIS

In accordance with the problem description in Subsection VII-A, we provide an example of solving a medicine selection issue to confirm the applicability of the recommended approach.

Example 10: Assume that $\mathcal{O} = \{\mathfrak{m}_i : i = 1, 2, \cdots, 9\}$ be the collection of 9 medicines prescribed to combat a disease D and $\mathcal{C} = \{a_1, a_2, a_3\}$ be the set of most common symptoms (criteria) of the disease. The weight vector of all symptoms $\mathcal{W} = (0.30279, 0.32236, 0.37485)$. The evaluation of 9 medicines based on three symptoms are depicted in Table 5.

For the selection of the most suitable medicine, the calculation procedures are shown as follows:

Step 1: Based on criteria a_1, a_2, a_3 and using Eq. (7) to evaluate the upward FPD of $\mathfrak{m}_i$ to $\mathfrak{m}_j (i, j = 1, 2, \cdots, 9)$, we obtain (31)–(33), as shown at the bottom of page 17.

Step 2: The upward fuzzy preference classes $[\mathfrak{m}_i]^{a_1}_{\widehat{\mathfrak{R}}^{\uparrow}}$, $[\mathfrak{m}_i]^{a_2}_{\widehat{\mathfrak{R}}^{\uparrow}}$ and $[\mathfrak{m}_i]^{a_3}_{\widehat{\mathfrak{R}}^{\uparrow}}$ are given in Tables 6, 7, and 8. From Tables 6, 7, and 8, we can see that $\mathscr{P}(\widehat{\mathfrak{R}}^{\uparrow}_{a_k}) = \left\{[\mathfrak{m}_i]^{a_k}_{\widehat{\mathfrak{R}}^{\uparrow}} : i = 1, 2, \cdots, 9, k = 1, 2, 3\right\}$ are upward fuzzy β-coverings of $\mathcal{O}$ $(0 < \beta \leq 0.50)$.

Step 3: Let $\beta = 0.50$ be the critical value. Then the elements $\aleph^{\uparrow a_k}_{(\mathfrak{m}_i,\beta)}$ $(i = 1, 2 \cdots, 9, k = 1, 2, 3)$ are listed in Tables 9, 10 and 11.

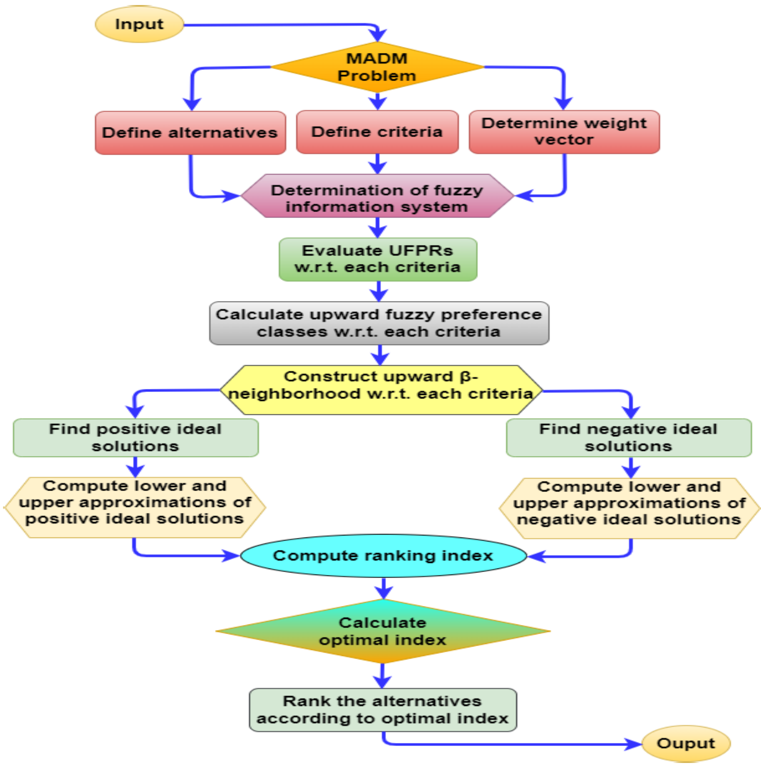

FIGURE 1. Flowchart of the suggested MADM algorithm.

Step 4: In the light of Eqs. (27) and (28), the positive and negative ideal solutions w.r.t. a_t are given as follows:

$$\mathcal{I}_{+}^{a_1} = \frac{0.8000}{\mathfrak{m}_1}+\frac{0.5500}{\mathfrak{m}_2}+\frac{0.5000}{\mathfrak{m}_3}+\frac{0.7000}{\mathfrak{m}_4}+\frac{0.6000}{\mathfrak{m}_5}+\frac{0.5000}{\mathfrak{m}_6}+\frac{0.5500}{\mathfrak{m}_7}+\frac{0.5500}{\mathfrak{m}_8}+\frac{0.5500}{\mathfrak{m}_9},$$

$$\mathcal{I}_{-}^{a_1} = \frac{0.5000}{\mathfrak{m}_1}+\frac{0.2500}{\mathfrak{m}_2}+\frac{0.2000}{\mathfrak{m}_3}+\frac{0.4000}{\mathfrak{m}_4}+\frac{0.3000}{\mathfrak{m}_5}+\frac{0.2000}{\mathfrak{m}_6}+\frac{0.2500}{\mathfrak{m}_7}+\frac{0.2500}{\mathfrak{m}_8}+\frac{0.2500}{\mathfrak{m}_9},$$

$$\mathcal{I}_{+}^{a_2} = \frac{0.5000}{\mathfrak{m}_1}+\frac{0.8333}{\mathfrak{m}_2}+\frac{0.5000}{\mathfrak{m}_3}+\frac{0.6667}{\mathfrak{m}_4}+\frac{0.7500}{\mathfrak{m}_5}+\frac{0.6667}{\mathfrak{m}_6}+\frac{0.6667}{\mathfrak{m}_7}+\frac{0.7500}{\mathfrak{m}_8}+\frac{0.5833}{\mathfrak{m}_9},$$

$$\mathcal{I}_{-}^{a_2} = \frac{0.1667}{\mathfrak{m}_1}+\frac{0.5000}{\mathfrak{m}_2}+\frac{0.1667}{\mathfrak{m}_3}+\frac{0.3333}{\mathfrak{m}_4}+\frac{0.4167}{\mathfrak{m}_5}+\frac{0.3333}{\mathfrak{m}_6}+\frac{0.3333}{\mathfrak{m}_7}+\frac{0.4167}{\mathfrak{m}_8}+\frac{0.2500}{\mathfrak{m}_9},$$

$$\mathcal{I}_{+}^{a_3} = \frac{0.5000}{\mathfrak{m}_1}+\frac{0.5000}{\mathfrak{m}_2}+\frac{0.7500}{\mathfrak{m}_3}+\frac{0.6875}{\mathfrak{m}_4}+\frac{0.5625}{\mathfrak{m}_5}+\frac{0.6875}{\mathfrak{m}_6}+\frac{0.7500}{\mathfrak{m}_7}+\frac{0.5625}{\mathfrak{m}_8}+\frac{0.6250}{\mathfrak{m}_9},$$

$$\mathcal{I}_{-}^{a_3} = \frac{0.2500}{\mathfrak{m}_1}+\frac{0.2500}{\mathfrak{m}_2}+\frac{0.5000}{\mathfrak{m}_3}+\frac{0.4375}{\mathfrak{m}_4}+\frac{0.3125}{\mathfrak{m}_5}+\frac{0.4375}{\mathfrak{m}_6}+\frac{0.5000}{\mathfrak{m}_7}+\frac{0.3125}{\mathfrak{m}_8}+\frac{0.3750}{\mathfrak{m}_9}.$$

TABLE 5. MADM Table.

$f(\mathfrak{m}_i, a)$	a_1	a_2	a_3
$\mathfrak{m}_1$	0.8	0.1	0.2
$\mathfrak{m}_2$	0.3	0.5	0.2
$\mathfrak{m}_3$	0.2	0.1	0.6
$\mathfrak{m}_4$	0.6	0.3	0.5
$\mathfrak{m}_5$	0.4	0.4	0.3
$\mathfrak{m}_6$	0.2	0.3	0.5
$\mathfrak{m}_7$	0.3	0.3	0.6
$\mathfrak{m}_8$	0.3	0.4	0.3
$\mathfrak{m}_9$	0.3	0.2	0.4

Step 5: In the light of Definition 17, we have the following approximations:

$$\underset{\sim}{apr}_{\beta}^{\uparrow}(\mathcal{I}_{+}^{a_1}) = \frac{0.5000}{\mathfrak{m}_1}+\frac{0.5000}{\mathfrak{m}_2}+\frac{0.5000}{\mathfrak{m}_3}+\frac{0.5000}{\mathfrak{m}_4}+\frac{0.5000}{\mathfrak{m}_5}+\frac{0.5000}{\mathfrak{m}_6}+\frac{0.5000}{\mathfrak{m}_7}+\frac{0.5000}{\mathfrak{m}_8}+\frac{0.5000}{\mathfrak{m}_9},$$

TABLE 6. The upward fuzzy preference classes $[\mathfrak{m}_i]^{a_1}_{\widehat{\mathfrak{R}}\uparrow}$.

	$[\mathfrak{m}_1]^{a_1}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_2]^{a_1}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_3]^{a_1}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_4]^{a_1}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_5]^{a_1}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_6]^{a_1}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_7]^{a_1}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_8]^{a_1}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_9]^{a_1}_{\widehat{\mathfrak{R}}\uparrow}$
$\mathfrak{m}_1$	0.5000	0.2500	0.2000	0.4000	0.3000	0.2000	0.2500	0.2500	0.2500
$\mathfrak{m}_2$	0.7500	0.5000	0.4500	0.6500	0.5500	0.4500	0.5000	0.5000	0.5000
$\mathfrak{m}_3$	0.8000	0.5500	0.5000	0.7000	0.6000	0.5000	0.5500	0.5500	0.5500
$\mathfrak{m}_4$	0.6000	0.3500	0.3000	0.5000	0.4000	0.3000	0.3500	0.3500	0.3500
$\mathfrak{m}_5$	0.7000	0.4500	0.4000	0.6000	0.5000	0.4000	0.4500	0.4500	0.4500
$\mathfrak{m}_6$	0.8000	0.5500	0.5000	0.7000	0.6000	0.5000	0.5500	0.5500	0.5500
$\mathfrak{m}_7$	0.7500	0.5000	0.4500	0.6500	0.5500	0.4500	0.5000	0.5000	0.5000
$\mathfrak{m}_8$	0.7500	0.5000	0.4500	0.6500	0.5500	0.4500	0.5000	0.5000	0.5000
$\mathfrak{m}_9$	0.7500	0.5000	0.4500	0.6500	0.5500	0.4500	0.5000	0.5000	0.5000

TABLE 7. The upward fuzzy preference classes $[\mathfrak{m}_i]^{a_2}_{\widehat{\mathfrak{R}}\uparrow}$.

	$[\mathfrak{m}_1]^{a_2}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_2]^{a_2}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_3]^{a_2}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_4]^{a_2}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_5]^{a_2}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_6]^{a_2}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_7]^{a_2}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_8]^{a_2}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_9]^{a_2}_{\widehat{\mathfrak{R}}\uparrow}$
$\mathfrak{m}_1$	0.5000	0.8333	0.5000	0.6667	0.7500	0.6667	0.6667	0.7500	0.5833
$\mathfrak{m}_2$	0.1667	0.5000	0.1667	0.3333	0.4167	0.3333	0.3333	0.4167	0.2500
$\mathfrak{m}_3$	0.5000	0.8333	0.5000	0.6667	0.7500	0.6667	0.6667	0.7500	0.5833
$\mathfrak{m}_4$	0.3333	0.6667	0.3333	0.5000	0.5833	0.5000	0.5000	0.5833	0.4167
$\mathfrak{m}_5$	0.2500	0.5833	0.2500	0.4167	0.5000	0.4167	0.4167	0.5000	0.3333
$\mathfrak{m}_6$	0.3333	0.6667	0.3333	0.5000	0.5833	0.5000	0.5000	0.5833	0.4167
$\mathfrak{m}_7$	0.3333	0.6667	0.3333	0.5000	0.5833	0.5000	0.5000	0.5833	0.4167
$\mathfrak{m}_8$	0.2500	0.5833	0.2500	0.4167	0.5000	0.4167	0.4167	0.5000	0.3333
$\mathfrak{m}_9$	0.4167	0.7500	0.4167	0.5833	0.6667	0.5833	0.5833	0.6667	0.5000

$$\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_1}_{+}) = \frac{0.6000}{\mathfrak{m}_1} + \frac{0.5000}{\mathfrak{m}_2} + \frac{0.5000}{\mathfrak{m}_3} + \frac{0.6000}{\mathfrak{m}_4} + \frac{0.5500}{\mathfrak{m}_5} + \frac{0.5000}{\mathfrak{m}_6} + \frac{0.5000}{\mathfrak{m}_7} + \frac{0.5000}{\mathfrak{m}_8} + \frac{0.5000}{\mathfrak{m}_9},$$

TABLE 8. The upward fuzzy preference classes $[\mathfrak{m}_i]^{a_3}_{\widehat{\mathfrak{R}}\uparrow}$.

	$[\mathfrak{m}_1]^{a_3}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_3]^{a_3}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_3]^{a_3}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_4]^{a_3}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_5]^{a_3}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_6]^{a_3}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_7]^{a_3}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_8]^{a_3}_{\widehat{\mathfrak{R}}\uparrow}$	$[\mathfrak{m}_9]^{a_3}_{\widehat{\mathfrak{R}}\uparrow}$
$\mathfrak{m}_1$	0.5000	0.5000	0.7500	0.6875	0.6525	0.6875	0.7500	0.5625	0.2650
$\mathfrak{m}_2$	0.5000	0.5000	0.7500	0.6875	0.6525	0.6875	0.7500	0.5625	0.2650
$\mathfrak{m}_3$	0.2500	0.2500	0.5000	0.4375	0.3125	0.4375	0.5000	0.3125	0.3750
$\mathfrak{m}_4$	0.3125	0.3125	0.5625	0.5000	0.3750	0.5000	0.5625	0.3750	0.4375
$\mathfrak{m}_5$	0.4375	0.4375	0.6875	0.6250	0.5000	0.6250	0.6875	0.5000	0.6525
$\mathfrak{m}_6$	0.3125	0.3125	0.5625	0.5000	0.3750	0.5000	0.5625	0.3750	0.4375
$\mathfrak{m}_7$	0.2500	0.2500	0.5000	0.4375	0.3125	0.4375	0.5000	0.3125	0.3750
$\mathfrak{m}_8$	0.4375	0.4375	0.6875	0.6250	0.5000	0.6250	0.6875	0.5000	0.6525
$\mathfrak{m}_9$	0.3750	0.3750	0.6250	0.6525	0.4375	0.6525	0.6250	0.4375	0.5000

TABLE 9. The UFβ-nghd $\aleph^{\uparrow a_1}_{(\mathfrak{m}_i,0.50)}$.

	$\aleph^{\uparrow a_1}_{(\mathfrak{m}_1,0.50)}$	$\aleph^{\uparrow a_1}_{(\mathfrak{m}_2,0.50)}$	$\aleph^{\uparrow a_1}_{(\mathfrak{m}_3,0.50)}$	$\aleph^{\uparrow a_1}_{(\mathfrak{m}_4,0.50)}$	$\aleph^{\uparrow a_1}_{(\mathfrak{m}_5,0.50)}$	$\aleph^{\uparrow a_1}_{(\mathfrak{m}_6,0.50)}$	$\aleph^{\uparrow a_1}_{(\mathfrak{m}_7,0.50)}$	$\aleph^{\uparrow a_1}_{(\mathfrak{m}_8,0.50)}$	$\aleph^{\uparrow a_1}_{(\mathfrak{m}_9,0.50)}$
$\mathfrak{m}_1$	0.5000	0.2500	0.2000	0.4000	0.3000	0.2000	0.2500	0.2500	0.2500
$\mathfrak{m}_2$	0.7500	0.5000	0.4500	0.6500	0.5500	0.4500	0.5000	0.5000	0.5000
$\mathfrak{m}_3$	0.8000	0.5500	0.5000	0.7000	0.6000	0.5000	0.5500	0.5500	0.5500
$\mathfrak{m}_4$	0.6000	0.3500	0.3000	0.5000	0.4000	0.3000	0.3500	0.3500	0.3500
$\mathfrak{m}_5$	0.7000	0.4500	0.4000	0.6000	0.5000	0.4000	0.4500	0.4500	0.4500
$\mathfrak{m}_6$	0.8000	0.5500	0.5000	0.7000	0.6000	0.5000	0.5500	0.5500	0.5500
$\mathfrak{m}_7$	0.7500	0.5000	0.4500	0.6500	0.5500	0.4500	0.5000	0.5000	0.5000
$\mathfrak{m}_8$	0.7500	0.5000	0.4500	0.6500	0.5500	0.4500	0.5000	0.5000	0.5000
$\mathfrak{m}_9$	0.7500	0.5000	0.4500	0.6500	0.5000	0.4500	0.5000	0.5000	0.5000

$$\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_1}_{-}) = \frac{0.2000}{\mathfrak{m}_1} + \frac{0.4500}{\mathfrak{m}_2} + \frac{0.5000}{\mathfrak{m}_3} + \frac{0.3000}{\mathfrak{m}_4} + \frac{0.4000}{\mathfrak{m}_5} + \frac{0.5000}{\mathfrak{m}_6} + \frac{0.4500}{\mathfrak{m}_7} + \frac{0.4500}{\mathfrak{m}_8} + \frac{0.4500}{\mathfrak{m}_9},$$

$$\widehat{\mathfrak{R}}^{\uparrow}_{a_1}(\mathfrak{m}_i,\mathfrak{m}_j) = \begin{pmatrix} 0.5000 & 0.7500 & 0.8000 & 0.6000 & 0.7000 & 0.8000 & 0.7500 & 0.7500 & 0.7500 \\ 0.2500 & 0.5000 & 0.5500 & 0.3500 & 0.4500 & 0.5500 & 0.5000 & 0.5000 & 0.5000 \\ 0.2000 & 0.4500 & 0.5000 & 0.3000 & 0.4000 & 0.5000 & 0.4500 & 0.4500 & 0.4500 \\ 0.4000 & 0.6500 & 0.7000 & 0.5000 & 0.6000 & 0.7000 & 0.6500 & 0.6500 & 0.6500 \\ 0.3000 & 0.5500 & 0.6000 & 0.4000 & 0.5000 & 0.6000 & 0.5500 & 0.5500 & 0.5500 \\ 0.2000 & 0.4500 & 0.5000 & 0.3000 & 0.4000 & 0.5000 & 0.4500 & 0.4500 & 0.4500 \\ 0.2500 & 0.5000 & 0.5500 & 0.3500 & 0.4500 & 0.5500 & 0.5000 & 0.5000 & 0.5000 \\ 0.2500 & 0.5000 & 0.5500 & 0.3500 & 0.4500 & 0.5500 & 0.5000 & 0.5000 & 0.5000 \\ 0.2500 & 0.5000 & 0.5500 & 0.3500 & 0.4500 & 0.5500 & 0.5000 & 0.5000 & 0.5000 \end{pmatrix}, \tag{31}$$

$$\widehat{\mathfrak{R}}^{\uparrow}_{a_2}(\mathfrak{m}_i,\mathfrak{m}_j) = \begin{pmatrix} 0.5000 & 0.1667 & 0.5000 & 0.3333 & 0.2500 & 0.3333 & 0.3333 & 0.2500 & 0.4167 \\ 0.8333 & 0.5000 & 0.8333 & 0.6667 & 0.5833 & 0.6667 & 0.6667 & 0.5833 & 0.7500 \\ 0.5000 & 0.1667 & 0.5000 & 0.3333 & 0.2500 & 0.3333 & 0.3333 & 0.2500 & 0.4167 \\ 0.6667 & 0.3333 & 0.6667 & 0.5000 & 0.4167 & 0.5000 & 0.5000 & 0.4167 & 0.5833 \\ 0.7500 & 0.4167 & 0.7500 & 0.5833 & 0.5000 & 0.5833 & 0.5833 & 0.5000 & 0.6667 \\ 0.6667 & 0.3333 & 0.6667 & 0.5000 & 0.4167 & 0.5000 & 0.5000 & 0.4167 & 0.5833 \\ 0.6667 & 0.3333 & 0.6667 & 0.5000 & 0.4167 & 0.5000 & 0.5000 & 0.4167 & 0.5833 \\ 0.7500 & 0.4167 & 0.7500 & 0.5833 & 0.5000 & 0.5833 & 0.5833 & 0.5000 & 0.6667 \\ 0.5833 & 0.2500 & 0.5833 & 0.4167 & 0.3333 & 0.4167 & 0.4167 & 0.3333 & 0.5000 \end{pmatrix}, \tag{32}$$

$$\widehat{\mathfrak{R}}^{\uparrow}_{a_3}(\mathfrak{m}_i,\mathfrak{m}_j) = \begin{pmatrix} 0.5000 & 0.5000 & 0.2500 & 0.3125 & 0.4375 & 0.3125 & 0.2500 & 0.4375 & 0.3750 \\ 0.5000 & 0.5000 & 0.2500 & 0.3125 & 0.4375 & 0.3125 & 0.2500 & 0.4375 & 0.3750 \\ 0.7500 & 0.7500 & 0.5000 & 0.5625 & 0.6875 & 0.5625 & 0.5000 & 0.6875 & 0.625 \\ 0.6875 & 0.6875 & 0.4375 & 0.5000 & 0.6250 & 0.5000 & 0.4375 & 0.6250 & 0.5625 \\ 0.5625 & 0.5625 & 0.3125 & 0.3750 & 0.5000 & 0.3750 & 0.3125 & 0.5000 & 0.4375 \\ 0.6875 & 0.6875 & 0.4375 & 0.5000 & 0.6250 & 0.5000 & 0.4375 & 0.6250 & 0.5625 \\ 0.7500 & 0.7500 & 0.5000 & 0.5625 & 0.6875 & 0.5625 & 0.5000 & 0.6875 & 0.6250 \\ 0.5625 & 0.5625 & 0.3125 & 0.3750 & 0.5000 & 0.3750 & 0.3125 & 0.5000 & 0.4375 \\ 0.6250 & 0.6250 & 0.375 & 0.4375 & 0.5625 & 0.4375 & 0.3750 & 0.5625 & 0.5000 \end{pmatrix}. \tag{33}$$

TABLE 10. The UFβ-nghd $\aleph^{\uparrow a_2}_{(\mathfrak{m}_j,0.50)}$.

	$\aleph^{\uparrow a_2}_{(\mathfrak{m}_1,0.50)}$	$\aleph^{\uparrow a_2}_{(\mathfrak{m}_2,0.50)}$	$\aleph^{\uparrow a_2}_{(\mathfrak{m}_3,0.50)}$	$\aleph^{\uparrow a_2}_{(\mathfrak{m}_4,0.50)}$	$\aleph^{\uparrow a_2}_{(\mathfrak{m}_5,0.50)}$	$\aleph^{\uparrow a_2}_{(\mathfrak{m}_6,0.50)}$	$\aleph^{\uparrow a_2}_{(\mathfrak{m}_7,0.50)}$	$\aleph^{\uparrow a_2}_{(\mathfrak{m}_8,0.50)}$	$\aleph^{\uparrow a_2}_{(\mathfrak{m}_9,0.50)}$
$\mathfrak{m}_1$	0.5000	0.8333	0.5000	0.6667	0.7500	0.6667	0.6667	0.7500	0.5833
$\mathfrak{m}_2$	0.1667	0.5000	0.1667	0.3333	0.4167	0.3333	0.3333	0.4167	0.2500
$\mathfrak{m}_3$	0.5000	0.8333	0.5000	0.6667	0.7500	0.6667	0.6667	0.7500	0.5833
$\mathfrak{m}_4$	0.3333	0.6667	0.3333	0.5000	0.5833	0.5000	0.5000	0.5833	0.4167
$\mathfrak{m}_5$	0.2500	0.5833	0.2500	0.4167	0.5000	0.4167	0.4167	0.5000	0.3333
$\mathfrak{m}_6$	0.3333	0.6667	0.3333	0.5000	0.5833	0.5000	0.5000	0.5833	0.4167
$\mathfrak{m}_7$	0.3333	0.6667	0.3333	0.5000	0.5833	0.5000	0.5000	0.5833	0.4167
$\mathfrak{m}_8$	0.2500	0.5833	0.2500	0.4167	0.5000	0.4167	0.4167	0.5000	0.3333
$\mathfrak{m}_9$	0.4167	0.7500	0.4167	0.5833	0.6667	0.5833	0.5833	0.6667	0.5000

TABLE 11. The UFβ-nghd $\aleph^{\uparrow a_3}_{(\mathfrak{m}_j,0.50)}$.

	$\aleph^{\uparrow a_3}_{(\mathfrak{m}_1,0.50)}$	$\aleph^{\uparrow a_3}_{(\mathfrak{m}_2,0.50)}$	$\aleph^{\uparrow a_3}_{(\mathfrak{m}_3,0.50)}$	$\aleph^{\uparrow a_3}_{(\mathfrak{m}_4,0.50)}$	$\aleph^{\uparrow a_3}_{(\mathfrak{m}_5,0.50)}$	$\aleph^{\uparrow a_3}_{(\mathfrak{m}_6,0.50)}$	$\aleph^{\uparrow a_3}_{(\mathfrak{m}_7,0.50)}$	$\aleph^{\uparrow a_3}_{(\mathfrak{m}_8,0.50)}$	$\aleph^{\uparrow a_3}_{(\mathfrak{m}_9,0.50)}$
$\mathfrak{m}_1$	0.5000	0.5000	0.7500	0.6875	0.5625	0.6875	0.7500	0.5625	0.6250
$\mathfrak{m}_2$	0.5000	0.5000	0.7500	0.6875	0.5625	0.6875	0.7500	0.5625	0.6250
$\mathfrak{m}_3$	0.2500	0.2500	0.5000	0.4375	0.3125	0.4375	0.5000	0.3125	0.3750
$\mathfrak{m}_4$	0.3125	0.3125	0.5625	0.5000	0.3750	0.5000	0.5625	0.3750	0.4375
$\mathfrak{m}_5$	0.4375	0.4375	0.6875	0.6250	0.5000	0.6250	0.6875	0.5000	0.5625
$\mathfrak{m}_6$	0.3125	0.3125	0.5625	0.5000	0.3750	0.5000	0.5625	0.3750	0.4375
$\mathfrak{m}_7$	0.2500	0.2500	0.5000	0.4375	0.3125	0.4375	0.5000	0.3125	0.3750
$\mathfrak{m}_8$	0.4375	0.4375	0.6875	0.6250	0.5000	0.6250	0.6875	0.5000	0.5625
$\mathfrak{m}_9$	0.3750	0.3750	0.6250	0.6525	0.4375	0.6525	0.6250	0.4375	0.5000

$$\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_1}_{-}) = \frac{0.5000}{\mathfrak{m}_1} + \frac{0.3500}{\mathfrak{m}_2} + \frac{0.3000}{\mathfrak{m}_3} + \frac{0.4000}{\mathfrak{m}_4} + \frac{0.4000}{\mathfrak{m}_5} + \frac{0.3000}{\mathfrak{m}_6} + \frac{0.3500}{\mathfrak{m}_7} + \frac{0.3500}{\mathfrak{m}_8} + \frac{0.3500}{\mathfrak{m}_9},$$

$$\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_2}_{+}) = \frac{0.5000}{\mathfrak{m}_1} + \frac{0.5000}{\mathfrak{m}_2} + \frac{0.5000}{\mathfrak{m}_3} + \frac{0.5000}{\mathfrak{m}_4} + \frac{0.5000}{\mathfrak{m}_5} + \frac{0.5000}{\mathfrak{m}_6} + \frac{0.5000}{\mathfrak{m}_7} + \frac{0.5000}{\mathfrak{m}_8} + \frac{0.5000}{\mathfrak{m}_9},$$

$$\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_2}_{+}) = \frac{0.5000}{\mathfrak{m}_1} + \frac{0.6667}{\mathfrak{m}_2} + \frac{0.5000}{\mathfrak{m}_3} + \frac{0.5833}{\mathfrak{m}_4} + \frac{0.5833}{\mathfrak{m}_5} + \frac{0.5833}{\mathfrak{m}_6} + \frac{0.5833}{\mathfrak{m}_7} + \frac{0.5833}{\mathfrak{m}_8} + \frac{0.5000}{\mathfrak{m}_9},$$

$$\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_2}_{-}) = \frac{0.5000}{\mathfrak{m}_1} + \frac{0.1667}{\mathfrak{m}_2} + \frac{0.5000}{\mathfrak{m}_3} + \frac{0.3333}{\mathfrak{m}_4} + \frac{0.2500}{\mathfrak{m}_5} + \frac{0.3333}{\mathfrak{m}_6} + \frac{0.3333}{\mathfrak{m}_7} + \frac{0.2500}{\mathfrak{m}_8} + \frac{0.4167}{\mathfrak{m}_9},$$

$$\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_2}_{-}) = \frac{0.3333}{\mathfrak{m}_1} + \frac{0.5000}{\mathfrak{m}_2} + \frac{0.3333}{\mathfrak{m}_3} + \frac{0.4167}{\mathfrak{m}_4} + \frac{0.4167}{\mathfrak{m}_5} + \frac{0.4167}{\mathfrak{m}_6} + \frac{0.4167}{\mathfrak{m}_7} + \frac{0.4167}{\mathfrak{m}_8} + \frac{0.3333}{\mathfrak{m}_9},$$

$$\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_3}_{+}) = \frac{0.5000}{\mathfrak{m}_1} + \frac{0.5000}{\mathfrak{m}_2} + \frac{0.5000}{\mathfrak{m}_3} + \frac{0.5000}{\mathfrak{m}_4} + \frac{0.5000}{\mathfrak{m}_5} + \frac{0.5000}{\mathfrak{m}_6} + \frac{0.5000}{\mathfrak{m}_7} + \frac{0.5000}{\mathfrak{m}_8} + \frac{0.5000}{\mathfrak{m}_9},$$

$$\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_3}_{+}) = \frac{0.5000}{\mathfrak{m}_1} + \frac{0.5000}{\mathfrak{m}_2} + \frac{0.6250}{\mathfrak{m}_3} + \frac{0.5625}{\mathfrak{m}_4} + \frac{0.5000}{\mathfrak{m}_5} + \frac{0.5625}{\mathfrak{m}_6} + \frac{0.6250}{\mathfrak{m}_7} + \frac{0.5000}{\mathfrak{m}_8} + \frac{0.5625}{\mathfrak{m}_9},$$

$$\underset{\sim}{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_3}_{-}) = \frac{0.5000}{\mathfrak{m}_1} + \frac{0.5000}{\mathfrak{m}_2} + \frac{0.2500}{\mathfrak{m}_3} + \frac{0.3125}{\mathfrak{m}_4} + \frac{0.4375}{\mathfrak{m}_5} + \frac{0.3125}{\mathfrak{m}_6} + \frac{0.2500}{\mathfrak{m}_7} + \frac{0.4375}{\mathfrak{m}_8} + \frac{0.3750}{\mathfrak{m}_9},$$

$$\widetilde{apr}^{\uparrow}_{\beta}(\mathcal{I}^{a_3}_{-}) = \frac{0.3750}{\mathfrak{m}_1} + \frac{0.3750}{\mathfrak{m}_2} + \frac{0.5000}{\mathfrak{m}_3} + \frac{0.4375}{\mathfrak{m}_4} + \frac{0.3750}{\mathfrak{m}_5} + \frac{0.4375}{\mathfrak{m}_6} + \frac{0.5000}{\mathfrak{m}_7} + \frac{0.3750}{\mathfrak{m}_8} + \frac{0.4375}{\mathfrak{m}_9}.$$

Step 6: Based on formula (29), the ranking index can be calculated as follows:

$$\varrho_1(\mathfrak{m}_j) = \frac{0.3162}{\mathfrak{m}_1} + \frac{0.1581}{\mathfrak{m}_2} + \frac{0.2000}{\mathfrak{m}_3} + \frac{0.2828}{\mathfrak{m}_4} + \frac{0.1803}{\mathfrak{m}_5} + \frac{0.2000}{\mathfrak{m}_6} + \frac{0.1581}{\mathfrak{m}_7} + \frac{0.1581}{\mathfrak{m}_8} + \frac{0.1581}{\mathfrak{m}_9},$$

$$\varrho_2(\mathfrak{m}_j) = \frac{0.1667}{\mathfrak{m}_1} + \frac{0.3727}{\mathfrak{m}_2} + \frac{0.1667}{\mathfrak{m}_3} + \frac{0.2357}{\mathfrak{m}_4} + \frac{0.3004}{\mathfrak{m}_5} + \frac{0.2357}{\mathfrak{m}_6} + \frac{0.2357}{\mathfrak{m}_7} + \frac{0.3004}{\mathfrak{m}_8} + \frac{0.1864}{\mathfrak{m}_9},$$

$$\varrho_3(\mathfrak{m}_j) = \frac{0.1250}{\mathfrak{m}_1} + \frac{0.1250}{\mathfrak{m}_2} + \frac{0.2795}{\mathfrak{m}_3} + \frac{0.2253}{\mathfrak{m}_4} + \frac{0.1398}{\mathfrak{m}_5} + \frac{0.2253}{\mathfrak{m}_6} + \frac{0.2795}{\mathfrak{m}_7} + \frac{0.1398}{\mathfrak{m}_8} + \frac{0.1768}{\mathfrak{m}_9}.$$

Step 7: Since the weight of each criterion is specified as $\varpi_1 = 0.30279, \varpi_2 = 0.32236, \varpi_3 = 0.37485$. So according to formula (30), the optimal index can be calculated as follows:

$$\partial(\mathfrak{m}_j) = \frac{0.1963}{\mathfrak{m}_1} + \frac{0.2149}{\mathfrak{m}_2} + \frac{0.2191}{\mathfrak{m}_3} + \frac{0.2461}{\mathfrak{m}_4} + \frac{0.2038}{\mathfrak{m}_5} + \frac{0.2210}{\mathfrak{m}_6} + \frac{0.2286}{\mathfrak{m}_7} + \frac{0.1971}{\mathfrak{m}_8} + \frac{0.1742}{\mathfrak{m}_9}.$$

Step 8: The ranking of the medicines is:

$$\mathfrak{m}_4 \succeq \mathfrak{m}_7 \succeq \mathfrak{m}_6 \succeq \mathfrak{m}_3 \succeq \mathfrak{m}_2 \succeq \mathfrak{m}_5 \succeq \mathfrak{m}_8 \succeq \mathfrak{m}_1 \succeq \mathfrak{m}_9.$$

From the ranking outcome, we deduce that $\mathfrak{m}_4$ is the most suitable medicine for the treatment of disease D. The graphical portrayal of the ranking of the medicines is shown in Figure 2.

TABLE 12. Comparison with the existing literature for $\beta = 0.5$.

Methods	Ranking
Ma [28]	Failed to handle
Yang and Hu [48]	Failed to handle
Our proposed method	$\mathfrak{m}_4 \succeq \mathfrak{m}_7 \succeq \mathfrak{m}_6 \succeq \mathfrak{m}_3 \succeq \mathfrak{m}_2 \succeq \mathfrak{m}_5 \succeq \mathfrak{m}_8 \succeq \mathfrak{m}_1 \succeq \mathfrak{m}_9$

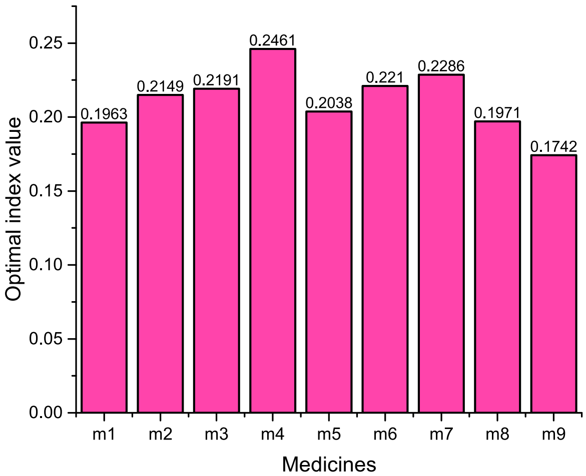

FIGURE 2. Ranking of medicines.

VIII. COMPARISON ANALYSIS AND DISCUSSION

In this segment, we conduct a comparative study from quantitative and qualitative features with various prevailing methods to highlight the efficacy and supremacy of the devised DM scheme and the particular comparison procedure is as follows.

A. QUANTITATIVE COMPARISON

This part is devoted to a comparison study with some existing methods to verify the superiority of our suggested scheme. This section provides a comparison among the models of Yang and Hu [48] and Ma [28] with our suggested MADM approach. In the view of Example 10 of the previous section, we see that the preceding models are unable make a decision in certain cases, for instance, when $\beta = 0.5$. At the same time, our proposed approach can easily accommodate this situation. This indicates that our suggested method is superior to the approaches in [28], [48].

A comparative study among WA method [46], OWA method [46] and the TOPSIS method [23] together with our postulated strategy in the context of Example 10 is given in Table 13. Additionally, the ranking outcomes are plotted graphically in FIGURE 3.

In the light of the data displayed in TABLE 13 revealed that the ranking order of the alternatives exhibits some variations. Nonetheless, it is notable that $\mathfrak{m}_4$ maintains its position as an optimal alternative. Therefore, according to the above discussion, the results acquired by the designed method are highly reliable.

B. QUALITATIVE COMPARISON

In this segment, we examine the characteristics of the developed approach and the launched studies in Zadeh [51], Atef et al. [2], Deng et al. [12], Greco et al. [16], Krohling and Campanharo [26], Pan et al. [32], Shabir and Shaheen [40], and Zhu [58] from the qualitative viewpoint and the comparison outcomes are displayed in TABLE 14. We carry out qualitative comparison from four perspectives: membership degree (MD), depict preference analysis, the roughness of an information system (IS), and covering characteristics to showcase its superiority. According to TABLE 14, it becomes evident that the postulated appraoch possesses all specified qualities, but the mentioned strategies do not have all of them.

C. ADVANTAGES

In summary, the merits of the designed method based on UFβ-CRSs are outlined as follows:

1. In the study of MADM problems with fuzzy information, there are many DM approaches based on FR. However, not all MADM problems can be characterized by a FR. Because of this, we set forth the method to resolve MADM problems with fuzzy information based on UFβ-CRSs.
2. From TABLE 12, we can easily see that the studies of Yang and Hu [48] and Ma [28] cannot rank all the alternatives when $\beta = 0.5$. However, our postulated scheme can provide good sorting outcomes. This means that our proposed methodology is reasonable and feasible.
3. By TABLE 13, we can find that although the specific ranking of the objects by different schemes has minor variations, the optimal choice remains consistent. This phenomenon underscores the rationality of our recommended method.

D. LIMITATIONS

Although the developed methodology has numerous benefits, it is essential to admit its shortcomings:

1. The reported method mainly relies on the appropriate choice of the parameter β. The sensitivity of this parameter might be a shortcoming. In practice, various inputs of parameter β might generate different ranking outcomes, making the devised model less reliable or biased results.
2. In our developed scheme, determining fuzzy approximations for FSs necessitates significant computational

TABLE 13. Comparison of different methods.

Methods	Ranking of alternatives
WA method [46]	$\mathfrak{m}_4 \succeq \mathfrak{m}_7 \succeq \mathfrak{m}_3 \succeq \mathfrak{m}_5 \succeq \mathfrak{m}_1 \succeq \mathfrak{m}_6 \succeq \mathfrak{m}_8 \succeq \mathfrak{m}_2 \succeq \mathfrak{m}_9$
OWA method [46]	$\mathfrak{m}_4 \succeq \mathfrak{m}_7 \succeq \mathfrak{m}_5 \succeq \mathfrak{m}_1 \succeq \mathfrak{m}_6 \succeq \mathfrak{m}_8 \succeq \mathfrak{m}_2 \succeq \mathfrak{m}_3 \succeq \mathfrak{m}_9$
TOPSIS method [23]	$\mathfrak{m}_4 \succeq \mathfrak{m}_7 \succeq \mathfrak{m}_5 \succeq \mathfrak{m}_2 \succeq \mathfrak{m}_1 \succeq \mathfrak{m}_8 \succeq \mathfrak{m}_6 \succeq \mathfrak{m}_3 \succeq \mathfrak{m}_9$
Our proposed method	$\mathfrak{m}_4 \succeq \mathfrak{m}_7 \succeq \mathfrak{m}_6 \succeq \mathfrak{m}_3 \succeq \mathfrak{m}_2 \succeq \mathfrak{m}_5 \succeq \mathfrak{m}_8 \succeq \mathfrak{m}_1 \succeq \mathfrak{m}_9$

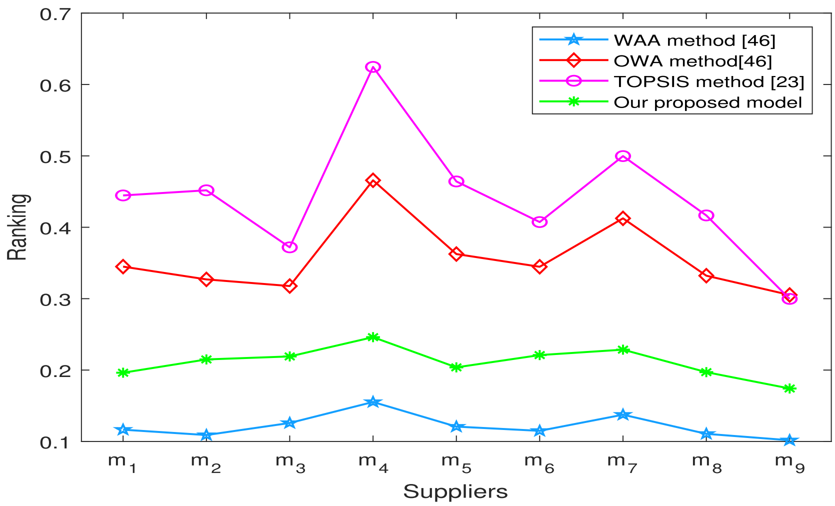

FIGURE 3. Graphical representation of ranking of medicines using different methods.

TABLE 14. Characteristics comparison of different methods with developed approach.

Approaches	Characteristics			
	Handle MD	Depict preference analysis	Roughness of an IS	Covering
Zadeh [51]	Yes	No	No	No
Atef et al. [2]	Yes	No	Yes	Yes
Deng et al. [12]	Yes	No	Yes	Yes
Greco et al. [16]	Yes	No	Yes	No
Krohling and Campanharo [26]	Yes	No	No	No
Pan et al. [32]	Yes	Yes	Yes	No
Shabir and Shaheen [40]	Yes	No	Yes	No
Zhu [58]	Yes	No	Yes	No
Proposed method	Yes	Yes	Yes	Yes

resources, which can increase the DM process's computational complexity.

IX. CONCLUSION AND FUTURE RESEARCH DIRECTIONS

Preference analysis is a substantial tool in decision analysis. The RST was effectively expanded to deal with preference analysis by switching Er with DR. DR cannot capture the fuzziness presented in the criteria. In this article, we pointed out the transfer functions proposed by Pan et al. [32] for the calculation of upward/downward FPRs are not additive consistent through a concrete example. Therefore, we have proposed new transfer functions to determine the upward/downward FPD of alternatives from the fuzzified information system. Based on UFPR, we presented concepts of $\alpha^{\uparrow}$-FPRSs and UFβ-CRSs models.

In general, this article has the following main contributions:

- We have formulated two novel transfer functions to compute upward/downward FPRs of alternatives, which are additively consistent.

- We have introduced the notion of $\alpha^{\uparrow}$-FPRSs by using UFPR. Their respective fundamental structural properties have been investigated in detail.
- Based upon $\alpha^{\uparrow}$-FPRSs, we introduced several uncertainty measures, like the measure of precision, rough degree, the measure of quality, and the measure of completeness of knowledge along with their properties.
- Meanwhile, we established the concept of UFβ-CAS, UFβ-nghd, and Uβ-nghd. Two novel types of RS models using UFβ-nghd and Uβ-nghd are also constructed, along with their structural properties.
- To indicate the application of the postulated strategy with fuzzy information, we have created an innovative scheme to address MADM issues using UFβ-CRSs. The procedure and an algorithm of our devised method have been presented. A practical case study has been provided to illustrate the significance of the proposed scheme.
- At last, a comprehensive comparison has been made with several prevailing approaches to scrutinizing the effectiveness, validity, reliability, and advantages of the suggested technique.

The recommended approach exhibits a broad spectrum of potential applications. The following research directions will deserve our future studies:

- In the future, we will combine our proposed technique with other MADM problems to cope with medical diagnosis problems.
- We will develop attribute reduction based on the proposed UFβ-CRSs.
- By using fuzzy logical implication $\mathcal{I}$ and t-norm $\mathcal{T}$, we will generalize our proposed UFβ-CRSs to upward $(\mathcal{I}, \mathcal{T})$- fuzzy β-covering RS.
- In the context of multi-granulation, we will extend the idea of UFβ-CRSs to multi-granulation UFβ-CRSs model and multi-granulation $(\mathcal{I}, \mathcal{T})$- fuzzy β-covering RS model.
- Also, we will investigate the topological properties of UFβ-CRSs.
- We will also look into the potential hybridization of the invented methodology to improve the accuracy of results and implement these methods to real-world issues with large data sets.

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