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RESEARCH ARTICLE

Dynamic Aggregation Operators for Optimal Biometric-Based Attendance Device Selection Under Complex Fermatean Fuzzy Environment

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ABSTRACT Selecting the right Biometric-Based Attendance Device (BBAD) is pivotal for enhancing security, operations, and compliance in today's dynamic environment of identity authentication.Addressing the complexities arising from uncertainty and periodicity, the Complex Fermatean Fuzzy Set (CFFS) theory emerges as adept, encapsulating comprehensive problem specifications. This study introduces two innovative aggregation operators within the CFFS framework: the Complex Fermatean Fuzzy Dynamic Weighted Averaging (CFFDWA) and the Complex Fermatean Fuzzy Dynamic Weighted Geometric (CFFDWG) operators. Some important characteristics of the newly defined operators are established. The shortcomings in the existing score function are rectified along with the introduction of a novel enhanced score function under complex Fermatean fuzzy environment. Moreover, these operators contribute to a systematic framework for handling Multiple Attribute Decision Making (MADM) problems involving complex Fermatean fuzzy information. The article exemplifies their application in resolving a MADM problem, determining the optimal model of BBAD. Finally, to validate the derived methodologies, a thorough comparison study is carried out, demonstrating the superiority of the presented operators against various existing operators.

INDEX TERMS Complex Fermatean fuzzy set, CFFDWA operator, CFFDWG operator, optimization, decision making.

I. INTRODUCTION

Multi-attribute Decision Making (MADM) is vital in modern decision science, integrating preferences to select optimal options from limited alternatives. Kahne's approach involves the assessment of alternatives based on a variety of characteristics, utilizing concise set theory along with logical connectors such as conjunction and disjunction [1]. Supported by aggregation operators, MADM's effectiveness is implemented in numerous disciplines. The limitations of precise sets are, however, highlighted by the unpredictability of human decision making. Fuzzy Set (FS) theory, which Zadeh introduced [2], facilitated ambiguity and imprecision and significantly enhanced the quality of decisions in the social sciences and production management [3], [4], [5]. Utilizing membership functions to manage uncertainty,

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FS involves degree of membership. However, a discrepancy between logical and verbal negations led Atanassov to propose Intuitionistic Fuzzy Sets (IFS) [6]. IFS expands FS by introducing a non-membership function, representing discontentment with human decisions within the [0,1] range. Dual values—the degree of membership and the degree of non-membership—are distinctive features of IFSs. The efficacy of these IFSs has been demonstrated in numerous domains, including medical diagnosis, image segmentation, pattern recognition, and fuzzy time series forecasting [7], [8]. The sum of the membership and non-membership degrees must lie within the unit interval. For example, if we encounter a circumstance where the membership degree is 0.7 and the non-membership degree is 0.6, then $0.7 + 0.6 > 1$ indicates that the IFS is incapable of handling this scenario.

To address the aforementioned deficiencies of IFS, Yager [9], [10] defined the Pythagorean Fuzzy Set (PFS) in which the sum of the squares of membership and non-membership degrees falls within [0,1]. In solving real-world problems, PFS offers more advantages than IFS. Examples include pattern recognition, supplier selection, early warning of industrial accidents, and recommender systems [11], [12], [13], [14]. The PFS system provides a more extensive collection of resources. The case in which the membership degree is 0.7 and the non-membership degree is 0.6, which IFS fails to handle, can be addressed through PFS as $0.7^2 + 0.6^2 \leq 1$. Thus, PFS is a more effective tool for decision-making than IFS. Nonetheless, PFS theory also fails if the membership degree is 0.8 and the non-membership degree is 0.7 because $0.8^2 + 0.7^2 = 1.13 > 1$.

Senapati and Yager [15] proposed the Fermatean Fuzzy Set (FFS) theory, which represents a more comprehensive model compared to the existing theories of IFS and PFS. The FFS theory imposed a condition that the cubic sum of the membership degree and non-membership degree must lie within the unit interval. FFS has the capability to deal with a greater degree of imprecision and ambiguity, hence providing more accurate outcomes within a decision-making framework.

Aggregation operators are tools used to convert n-tuple data into a singular useful form. aggregation operators are widely used in the decision-making process. The definition of geometric aggregation operators based on intuitionistic fuzzy sets is provided in [16]. Xu [17] devised arithmetic aggregation operators for the IFS. Rahman et al. [18], [19], [20] designed various aggregation operators in the framework of PFS. In addition, it is essential to note that researchers acknowledge dynamic aggregation operators as a highly effective method for solving MADM issues. Garg and Akram proposed aggregation operator in FFS. For FFS Rani introduced Einstein aggregation and Aydemir proposed Dombi aggregation operator in [21], [22], [23]. Xu and Yager, WG Wei, S Gumus and Y Liu and J Liu developed the dynamic intuitionistic fuzzy aggregation operator for situations involving intuitionistic fuzzy numbers or interval valued intuitionistic fuzzy numbers to represent attribute decision information in [24], [25], [26], and [27]. Soft aggregation operators on q-rung orthopair fuzzy environment was presented in [28] and [29].

The models discussed above did not apply to two-dimensional problems. Thus, the Complex fuzzy set (CFS) was created by Ramot et al. [30]. It was suggested by the developing relationship between complex and FS theory, with the range of membership function being the complex unit circle. This allows the CFS to accommodate 2-dimensional information, including amplitude and phase components. The amplitude and phase components are both real-valued functions that can accept values from the unit interval to demonstrate the ambiguity of both dimensions. Using the idea of CFS, numerous physical problems have been effectively solved. The present theory is of substantial importance in numerous domains, specifically in the prediction of periodic events and advanced control. These events encompass numerous fuzzy variables that are related in a way that conventional fuzzy operations are unable to sufficiently identify. The notions of Complex Intuitionistic Fuzzy Set (CIFS) and complex cubic intuitionistic fuzzy set were defined in [31] and [32]. Dynamic aggregation operators on CIF was presented in [33]. Frank aggregation operators on CIFS was proposed in [34]. Chinnadurai et al. [35] designed the operations of the complex Interval-valued Pythagorean fuzzy sets. Akram and Naz [36] introduced a novel decision making approach based on Complex Pythagorean Fuzzy Set (CPFS) theory. This set is an extension of CFS and CIFS. The complex fuzzy geometric aggregation operators and complex fuzzy arithmetic aggregation operators were defined in [37] and [38] respectively. In [39], Garg and Rani defined CIF arithmetic and geometric aggregation operators. In the context of the CPFS environment, many operators such as Einstein geometric operators, Dombi operators, prioritized weighted operators, and Yagers aggregation operators have been created [40], [41], [42], [43], [44]. In 2021, Chinnadurai et al. [45] defined the concept of Complex Fermatean Fuzzy Set (CFFS) and also defined aggregation operators in this environment.

The Biometric Behavioral Authentication and Recognition (BBAD) is an essential component of contemporary biometric technology, providing a highly efficient and adaptable method for authenticating and identifying individuals based on their unique behavioral patterns. Amidst the rising worries surrounding security and privacy, BBAD emerges as an invaluable solution. This system captures and evaluates a wide range of behavioral characteristics, such as keystroke dynamics, cursor movement, signature dynamics, and voice patterns. Biometric profiles that are incredibly difficult to copy or fake are made possible by this advanced analysis. We use this phenomenon in the field of dynamic CFF because it is more significant than traditional password-based security measures. It provides a non-intrusive and continuous method of authentication. It operates invisibly in the background, identifying users based on their natural interactions with technology. Additionally, BBAD systems have shown their effectiveness at different times in protecting sensitive data and resources from unauthorized access, fraud, and identity theft. An assortment of biometric techniques are designed to detect and distinguish various physical attributes. The biometric community uses the term "modalities" to denote these various applications. Emerging biometric modalities are discussed in [46]. In [47], models of legal regulation pertaining to biometric identification and authentication utilizing facial recognition technology were delineated with the intention of formulating proposals to enhance the data security of individuals. The biometric technology system, its design, and its performance evaluation were defined in [48], [49], [50], [51], and [52]. These devices ensure that employees are unable to sign in for one another, preventing employee time theft. In view of the aforementioned characteristics of BBAD, Bharat Sanchar Nagar Limited, an agency headquartered in New Delhi, India, has made the strategic decision to implement BBAD across all of its country branches. In order to accomplish this objective, the Bharat Sanchar Nagar Limited authority gathers a group of decision-makers in a conference to evaluate and choose the most efficient model from a set of four alternatives. Decision-makers assign membership and non-membership degrees according to their own choice at three different time periods because, with the passage of time, the version of technology has improved and becomes more efficient and accurate day by day. There are different types of models used for the biometric identification of individuals. In this study, we describe a systematic methodology for the purpose of identifying the optimal alternative using CFF dynamic aggregation operators.

CFFS theory is more capable of handling higher levels of uncertainty compared to IFS, PFS, CIFS, and CPFS. It is more suitable for addressing circumstances that involve greater ambiguity and complexity. For instance, CFFS theory has the ability to cope with two-dimensional data, but IFS, PFS, and FFS cannot handle such data. Moreover, the theories of CIFS and CPFS do not entertain situations where the sum of the squares of membership and non-membership degrees exceeds 1. However, CFFS easily handles such scenarios. This study distinguishes itself from previous research due to its particular focus on utilizing time periods. It is important to design robust, efficient dynamic aggregation operators that can handle decision-making scenarios with evolving ambiguity, imprecision, and vagueness. By facilitating the accumulation of information from various time periods, the proposed research enables an extensive comprehension of the issues by generating a precise depiction. Therefore, it is critical to undertake an investigation that specifically targets the challenges associated with dynamic complex fuzzy MADM. These aspects motivate us to propose dynamic operators in the framework of the CFF environment and present a solution to the MADM problem of selecting the optimal model of BBAD.

Dynamic aggregation operators are employed to handle uncertainties and imprecise information that evolves over time. Current research pertaining to CFF is unable to assess information that is time-dependent. It is also important to note that changes in the significance or value of different data points may not be effectively reflected in the aggregated result without involving the time factor. This lack of sensitivity can lead to inaccurate outcomes, particularly in situations where the importance of data elements varies over time. Dynamic weighted aggregation operators are well suited for situations that necessitate instantaneous decision-making. Their ability to adjust rapidly to changes in the environment or input data makes them highly suitable for time-sensitive decision-making processes. However, in various contexts of decision-making, including dynamic medical diagnostics, multi-period investment decision-making, dynamic assessment of the effectiveness of defense mechanisms, and personnel dynamic evaluation, the essential information pertaining to decisions are frequently acquired at different points in time.

Our primary objective is to devise and evaluate operators designed to the unique characteristics of CFFS. These operators are expressly crafted to enhance adaptability and dynamism in decision-making amidst intricate and unpredictable scenarios. Their incorporation empowers the system to dynamically adjust membership functions and aggregation procedures in response to evolving conditions, fluctuating data, and shifting decision-maker preferences. This flexibility proves invaluable in navigating scenarios with varying levels of uncertainty, rendering conventional static operators comparatively less efficient. Our study aims to unveil these dynamic operators, showcasing their efficacy across diverse contexts, thereby advancing CFFS's utility in addressing complex and dynamic decision environment.

Our research concentrates on the following key objectives of the theoretical framework:

1. Analyze the deficiencies in the current score function and develop an improved score function that strengthens the CFF system. It develops a grading system that is more dependable and precise.
2. Present two novel aggregation operators, namely CFFDWA and CFFDWG operators, developed for decision-making situations that incorporate complex Fermatean fuzzy information and time intervals.
3. Establish a dynamic operational system that is necessary for the execution of CFFSs. This involves the formation of mathematical models that depict the relationships between different complex Fermatean fuzzy numbers.
4. Develop a systematic approach to address the MADM problem involving complex Fermatean fuzzy: data at various time intervals.
5. Utilize the recently proposed approach to determine the optimal model for BBAD in the field of biometric technology: This will include the application of the proposed algorithm in various scenarios.

6. Conduct a comprehensive analysis of CFFDWA and CFFDWG operators, establishing their unique attributes and specific applications to foster a deeper comprehension of their capabilities.

The remaining part of the manuscript is structured as follows: An overview of basic definitions is given in Section II. Section III identifies deficiencies in the existing score function within the CFF environment and introduces a novel score function to address these shortcomings. The Section IV introduces dynamic aggregation operators designed for CFFS and develops their important features. In Section V, the newly defined operators are applied to determine the optimal model of BBAD. Furthermore, a comparative analysis is presented, showcasing the efficacy and practicality of this innovative approach vis-à-vis established techniques. The conclusion is summarized in Section V-A, which also addresses the broader implications of the main findings.

II. PRELIMINARIES

This section provides fundamental definitions which are crucial for comprehending the content presented in this article. Throughout our discourse Γ denotes the universal set.

Definition 1 [30]: A CFS is characterized by a membership function A, which maps each element in the universal set Γ to points within the closed unit disc in the complex plane. Mathematically, it is expressed as $A(w) = \mu_A(w)e^{i2\pi\theta_A(w)}$, where μ_A represents a real-valued function defined on Γ mapping its elements to the closed-unit interval $[0, 1]$ and $e^{i2\pi\theta_A}$ represents a periodic function with a periodic law having a principal period of 2π and $\theta_A(w)$ constrained within the range of 0 to 1.

Definition 2 [31]: A CIFS A defined on a universal set Γ, can be represented as follows: $A = \{(w, \gamma_A(w), q_A(w)) : w \in \Gamma\}$. In this representation, γ_A and q_A are complex-valued functions from Γ to the set $\{w : |w| \le 1\}$.

For $w \in \Gamma$, the membership function $\gamma_A(w)$ is expressed as $\gamma_A(w) = \mu_A(w)e^{i2\pi\theta_A(w)}$ and the nonmembership function $q_A(w)$ is defined as $q_A(w) = \nu_A(w)e^{i2\pi\varphi_A(w)}$. The real-valued functions μ_A, ν_A, θ_A and φ_A are confined to the interval $[0, 1]$, subject to the conditions $0 \le \mu_A(w) + \nu_A(w) \le 1$ and $0 \le \theta_A(w) + \varphi_A(w) \le 1$.

Definition 3 [42]: A CPFS is an object of the form $A = \{(w, \gamma_A(w), q_A(w)) : w \in \Gamma\}$, where the complex-valued functions γ_A and q_A, are complex-valued functions from Γ to the set $\{u : |u| \le 1\}$. For $w \in \Gamma$, the membership function $\gamma_A(w)$ is expressed as $\gamma_A(w) = \mu_A(w)e^{i2\pi\theta_A(w)}$ and the non-membership function $q_A(w)$ is defined as $q_A(w) = \nu_A(w)e^{i2\pi\varphi_A(w)}$. These functions are subject to the conditions $0 \le \mu_A^2(w) + \nu_A^2(w) \le 1$ and $0 \le \theta_A^2(w)+ \varphi_A^2(w) \le 1$. Additionally, the degree of hesitancy functions $\mathcal{H}_A(w) = \hbar(w).e^{i2\pi\alpha(w)}$, where $\hbar(w) = \sqrt{1 - \mu_A^2(w) - \nu_A^2(w)}$ and $\alpha(w) = \sqrt{1 - \theta_A^2(w) - \varphi_A^2(w)}$

Definition 4 [11]: An FFS F in Γ is defined as: $A = \{(w, \mu_A(w), \nu_A(w)) : w \in \Gamma\}$, where $\mu_A, \nu_A : \Gamma \to [0, 1]$ represent the membership degree and non-membership degree functions satisfying $0 \le \mu_A^3(w) + \nu_A^3(w) \le 1$ for all $w \in \Gamma$. Furthermore, the indeterminacy degree of the FFS, denoted as $\varpi_A(w)$, is defined as $\varpi_A(w) = \sqrt[3]{1 - \mu_A^3(w) - \nu_A^3(w)}$

Definition 5 [42]: A CFFS denoted as A is defined as, $A = \{(w, \gamma_A(w), q_A(w)) : w \in \Gamma\}$. Within this framework, γ_A and q_A are complex-valued functions from Γ to the closed unit disc.

For $w \in \Gamma$, the membership function $\gamma_A(w)$ is defined as $\gamma_A(w) = \mu_A(w)e^{i2\pi\theta_A(w)}$ and the nonmembership function $q_A(w)$ is expressed as $q_A(w) = \nu_A(w)e^{i2\pi\varphi_A(w)}$, subject to the conditions $0 \le \mu_A^3(w)+ \nu_A^3(w) \le 1$ and $0 \le \theta_A^3(w) + \varphi_A^3(w) \le 1$.

Moreover, the term $\mathcal{H}_A(w) = \hbar(w) \cdot e^{i2\pi\alpha(w)}$, such that $\hbar(w) = \sqrt[3]{1 - \mu_A^3(w) - \nu_A^3(w)}$ and $\alpha(w) = \sqrt[3]{1 - \theta_A^3(w) - \varphi_A^3(w)}$ define the hesitancy degree of w.

Throughout the remainder of the article, the representation of membership and non-membership degrees of $\sigma \in \Gamma$ is denoted as $\sigma = ((\mu, \theta), (\nu, \varphi))$ and is called Complex Fermatean Fuzzy Number (CFFN).

Definition 6 [42]: Let $\sigma_j = \left((\mu_j, \theta_j), (\nu_j, \varphi_j)\right)$ for $j = 1, 2, 3, \ldots, n$ be a collection of n CFFNs and let $\omega = [\omega_1, \omega_2, \ldots, \omega_n]^T$ represent the weight vector associated with σ_j such that, $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$. Then, a Complex Fermatean Fuzzy Weighted Averaging (CFFWA) operator is a function CFFWA: $\sigma^n \to \sigma$, defined as:

$$CFFWA(\sigma_1, \sigma_2, \ldots, \sigma_n) = \left(\left(\sum_{j=1}^n \omega_j\mu_j, \sum_{j=1}^n \omega_j\theta_j\right), \left(\sum_{j=1}^n \omega_j\nu_j, \sum_{j=1}^n \omega_j\varphi_j\right)\right)$$

Definition 7 [42]: Let $\sigma_j = \left((\mu_j, \theta_j), (\nu_j, \varphi_j)\right)$ for $j = 1, 2, 3, \ldots, n$ be n CFFNs and $\omega = [\omega_1, \omega_2, \ldots, \omega_n]^T$ be the weight vector of σ_j with $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$. Then the Complex Fermatean Fuzzy Weighted Geometric (CFFWG) operator is a function : $\sigma^n \to \sigma$, defined as:

$$CFFWG(\sigma_1, \sigma_2, \ldots, \sigma_n) = \left((\prod_{j=1}^n \mu_j^{\omega_j}, \prod_{j=1}^n \theta_j^{\omega_j}), (\prod_{j=1}^n \nu_j^{\omega_j}, \prod_{j=1}^n \varphi_j^{\omega_j}) \right)$$

Definition 8 [42]: For any CFFN $\sigma = ((\mu, \theta), (\nu, \varphi))$ the score function of σ is defined as

$$S(\sigma) = \frac{1}{2}\left[\left(\mu^3 - \nu^3\right) + \left(\theta^3 - \varphi^3\right)\right], \quad \text{where } S(\sigma)\epsilon[-1, 1]$$

The accuracy function of σ is defined as

$$ac(\sigma) = \frac{1}{2}\left[\left(\mu^3 + \nu^3\right) + \left(\theta^3 + \varphi^3\right)\right], \quad \text{where } ac(\sigma)\epsilon[0, 1]$$

For any two CFFNs σ_1, σ_2, satisfy the following comparison laws

i. If $S(\sigma_1) < S(\sigma_2)$, then $\sigma_1 < \sigma_2$
ii. If $S(\sigma_1) > S(\sigma_2)$, then $\sigma_1 > \sigma_2$
iii. If $S(\sigma_1) = S(\sigma_2)$, then $ac(\sigma_1) < ac(\sigma_2) \Rightarrow \sigma_1 < \sigma_2$, $a c(\sigma_1) < ac(\sigma_2) \Rightarrow \sigma_1 < \sigma_2$ and $ac(\sigma_1) = ac(\sigma_2) \Rightarrow \sigma_1 \sim \sigma_2$

iv. If $S(\sigma_1) = S(\sigma_2)$, then $ac(\sigma_1) < ac(\sigma_2) \Rightarrow \sigma_1 < \sigma_2, a\ c(\sigma_1) < ac(\sigma_2) \Rightarrow \sigma_1 < \sigma_2$ and $ac(\sigma_1) = ac(\sigma_2) \Rightarrow \sigma_1 \sim \sigma_2$

The definitions presented in this section serve as essential building blocks for understanding the subsequent content elucidated within this article.

III. ENHANCEMENT OF THE EXISTING SCORE FUNCTION OF COMPLEX FERMATEAN FUZZY NUMBERS

This section consists of an analysis of the deficiencies present in the score function of CFFNs developed in [42], followed by the proposed improvements.

Example 1: Consider two CFFNs σ_1 and σ_2, defined as follows:
$\sigma_1 = \left(\left((0.5)^{\frac{1}{3}}, (0.3)^{\frac{1}{3}}\right), \left((0.45)^{\frac{1}{3}}, (0.55)^{\frac{1}{3}}\right)\right)$ and $\sigma_2 = \left(\left((0.6)^{\frac{1}{3}}, (0.2)^{\frac{1}{3}}\right), \left((0.35)^{\frac{1}{3}}, (0.65)^{\frac{1}{3}}\right)\right)$.

The application of Definitions 8 to CFFNs σ_1 and σ_2 yields that $S(\sigma_1) = S(\sigma_2) = -0.1$ and $ac(\sigma_1) = ac(\sigma_2) = 0.9$. Property (iii) of Definitions 8 shows that CFFNs σ_1 and σ_2 are incomparable.

The aforementioned example highlights the drawbacks of the current score function. This motivates us to formulate a novel score function, as expounded in the subsequent definition.

Definition 9: For any CFFN denoted as $\sigma = ((\mu, \theta), (\nu, \varphi))$, we introduce the modified score function, defined as $\Omega(\sigma) = \frac{1}{2}\left[\left(\mu^3 - \nu^3\right) + \left(\theta^3 - \varphi^3\right) + \mu^3\nu^3\right]$, where $\Omega(\sigma)\epsilon[-1, 1]$. Notably, the modified score function adheres to the comparison law for any pair of CFFNs, σ_1 and σ_2 : $\Omega(\sigma_1) < \Omega(\sigma_2)$ implies $\sigma_1 < \sigma_2, \Omega(\sigma_1) > \Omega(\sigma_2)$ implies $\sigma_1 > \sigma_2$ and $\Omega(\sigma_1) = \Omega(\sigma_2)$ implies $\sigma_1 \sim \sigma_2$.

Additionally, we define the modified accuracy function for Γ as $\mathfrak{H}(\sigma) = \frac{1}{2}\left[\left(\mu^3 + \nu^3\right) + \left(\theta^3 + \varphi^3\right) - \mu^3\nu^3\right]$, where $\mathfrak{H}(\sigma) \in [0, 1]$. Similarly, the accuracy function also complies with the comparison law for any two CFFNs, σ_1 and σ_2 : $\mathfrak{S}(\sigma_1) < \mathfrak{H}(\sigma_2)$ implies $\sigma_1 < \sigma_2, \mathfrak{H}(\sigma_1) > \mathfrak{H}(\sigma_2)$ implies $\sigma_1 > \sigma_2$ and $\mathfrak{H}(\sigma_1) = \mathfrak{H}(\sigma_2)$ implies $\sigma_1 \sim \sigma_2$.

To demonstrate the efficacy of the proposed score function for CFFNs, we consider the following illustrative example.

Example 2: Consider two CFFNs $\sigma_1 = \left(\left((0.5)^{\frac{1}{3}}, (0.3)^{\frac{1}{3}}\right), \left((0.45)^{\frac{1}{3}}, (0.55)^{\frac{1}{3}}\right)\right)$ and $\sigma_2 = \left(\left((0.6)^{\frac{1}{3}}, (0.2)^{\frac{1}{3}}\right), \left((0.35)^{\frac{1}{3}}, (0.65)^{\frac{1}{3}}\right)\right)$.

The application of Definition 9 to these CFFNs yields that $C(\sigma_1) = 0.0125$ and $C(\sigma_2) = 0.005$. Consequently, in accordance with property 1 of Definition 9, we deduce that $\sigma_1 < \sigma_2$. This finding indicates that σ_2 exhibits superior qualities compared to σ_1.

The efficacy of the proposed score function is demonstrated through Example 2, where the application of Definition 9 successfully evaluates and compares the two complex Fermatean fuzzy numbers (CFFNs), a task that the existing score function was unable to accomplish in Example 1. Example 2 demonstrates its effectiveness in accurately evaluating and distinguishing between CFFNs, indicating improved reliability and accuracy compared to the previous scoring method.

IV. DYNAMIC AGGREGATION OPERATORS ON COMPLEX FERMATEAN FUZZY NUMBERS

In this section, our aim is to develop dynamic aggregation operators in CFF environment.

Definition 10: Let t denote the time variable. A CFF variable is represented as $\sigma_t = ((\mu_t, \theta_t), (\nu_t, \varphi_t))$, where $\mu_t, \theta_t, \nu_t, \varphi_t \in [0, 1]$, subject to the conditions $0 \leq \mu_t^3 + \nu_t^3 \leq 1$ and $0 \leq \theta_t^3 + \varphi_t^3 \leq 1$. For the CFF variable σ_t, if $t = (t_1, t_2, \ldots, t_p)$, then $\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}$ indicate p CFFNs collected at p different periods.

Definition 11: Consider two CFFNs, $\sigma_{t_1} = ((\mu_{t_1}, \theta_{t_1}), (\nu_{t_1}, \varphi_{t_1}))$ and $\sigma_{t_2} = ((\mu_{t_2}, \theta_{t_2}), (\nu_{t_2}, \varphi_{t_2}))$. The essential ordering principal governing their interaction are as follows:

$\sigma_{t_1} \leq \sigma_{t_2}$ if $\mu_{t_1} \leq \mu_{t_2}, \nu_{t_1} \geq \nu_{t_2}$ and $\theta_{t_1} \leq \theta_{t_2}, \varphi_{t_1} \geq \varphi_{t_2}$
$\sigma_{t_1} = \sigma_{t_2}$ if and only if $\sigma_{t_1} \subseteq \sigma_{t_2}$ and $\sigma_{t_2} \subseteq \sigma_{t_1}$

Definition 12: Let $\sigma_t = ((\mu_t, \theta_t), (\nu_t, \varphi_t))$, $\sigma_{t_1} = ((\mu_{t_1}, \theta_{t_1}), (\nu_{t_1}, \varphi_{t_1}))$ and $\sigma_{t_2} = ((\mu_{t_2}, \theta_{t_2}), (\nu_{t_2}, \varphi_{t_2}))$ be three CFFNs, and $\lambda_t > 0$. The dynamic operational laws for these CFFNs are defined as follows:

1. $$\sigma_{t_1} \oplus \sigma_{t_2} = \left(\left(\sqrt[3]{\mu_{t_1}^3 + \mu_{t_2}^3 - \mu_{t_1}^3\mu_{t_2}^3}, \sqrt[3]{\theta_{t_1}^3 + \theta_{t_2}^3 - \theta_{t_1}^3\theta_{t_2}^3}\right), (\nu_{t_1}\nu_{t_2}, \varphi_{t_1}\varphi_{t_2})\right)$$
2. $$\sigma_{t_1} \otimes \sigma_{t_2} = \left(\begin{array}{c}(\mu_{t_1}\mu_{t_2}, \theta_{t_1}\theta_{t_2}), \\ \left(\sqrt[3]{\nu_{t_1}^3 + \nu_{t_2}^3 - \nu_{t_1}^3\nu_{t_2}^3}, \sqrt[3]{\varphi_{t_1}^3 + \varphi_{t_2}^3 - \varphi_{t_1}^3\varphi_{t_2}^3}\right)\end{array}\right)$$
3. $$\lambda_t\sigma_t = \left(\begin{array}{c}\left(\sqrt[3]{1 - \left(1 - \mu_t^3\right)^{\lambda_t}}, \sqrt[3]{1 - \left(1 - \theta_t^3\right)^{\lambda_t}}\right), \\ (\nu_t^{\lambda_t}, \varphi_t^{\lambda_t})\end{array}\right)$$
4. $$\sigma_t^{\lambda_t} = \left(\begin{array}{c}\left(\mu_t^{\lambda_t}, \theta_t^{\lambda_t}\right), (\sqrt[3]{1 - \left(1 - \nu_t^3\right)^{\lambda_t}}, \\ \sqrt[3]{1 - \left(1 - \varphi_t^3\right)^{\lambda_t}})\end{array}\right)$$

In the subsequent definition, we introduce CFFDWA operator.

Definition 13: Consider a collection of CFFNs denoted as $\sigma_{t_k} = ((\mu_{t_k}, \theta_{t_k}), (\nu_{t_k}, \varphi_{t_k}))$, for $k = 1, 2, \ldots, p$, at distinct time periods t_k. Furthermore, let $\lambda_{t_k} = [\lambda_{t_1}, \lambda_{t_2}, \ldots, \lambda_{t_p}]^T$ be the weight vector that is linked to time periods t_k, subject to the conditions $\lambda_{t_k} \in [0, 1]$ and $\sum_{k=1}^{p} \lambda_{t_k} = 1$. The CFFDWA operator, denoted as CFFDWA: $\sigma^p \rightarrow \sigma$ defined as follows:

$$CFFDWA(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}) = \oplus_{k=1}^{p}(\lambda_{t_k}.\sigma_{t_k}) \left(\begin{array}{c}\left(\begin{array}{c}\sqrt[3]{1 - \prod_{k=1}^{p}(1 - \mu_{t_k}^3)^{\lambda_{t_k}}}, \\ \sqrt[3]{1 - \prod_{k=1}^{p}(1 - \theta_{t_k}^3)^{\lambda_{t_k}}}\end{array}\right), \\ \left(\prod_{k=1}^{p}\nu_{t_k}^{\lambda_{t_k}}, \prod_{k=1}^{p}\varphi_{t_k}^{\lambda_{t_k}}\right)\end{array}\right).$$

Theorem 1: Let $\sigma_{t_k} = ((\mu_{t_k}, \theta_{t_k}), (\nu_{t_k}, \varphi_{t_k}))$ for $k = 1, 2, \ldots, p$, represent a collection of CFFNs at p different time periods t_k. Additionally, consider the weight vector with p time period $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2} \ldots, \lambda_{t_p}\right]^T$, where $\lambda_{t_k} \in [0,1]$ and $\sum_{k=1}^{p} \lambda_{t_k} = 1$. The aggregated value of these CFFNs using the CFFDWA operator is itself a CFFN, expressed as:

$$\left(\begin{array}{c} CFFDWA\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}\right) \\ = \left(\sqrt[3]{1-\prod_{k=1}^{p}\left(1-\mu_{t_k}^3\right)^{\lambda_{t_k}}}, \sqrt[3]{1-\prod_{k=1}^{p}\left(1-\theta_{t_k}^3\right)^{\lambda_{t_k}}}\right), \\ \left(\prod_{k=1}^{p} \nu_{t_k}^{\lambda_{t_k}}, \prod_{k=1}^{p} \varphi_{t_k}^{\lambda_{t_k}}\right) \end{array}\right).$$

Proof: The mathematical induction is employed to demonstrate the proof of this result.

Let $p = 2$;

$$CFFDWA\left(\sigma_{t_1}, \sigma_{t_2}\right) = \lambda_{t_1} \cdot \sigma_{t_1} \oplus \lambda_{t_2} \cdot \sigma_{t_2}$$

where;

$$\lambda_{t_1}.\sigma_{t_1} = \left(\begin{array}{c} \left(\sqrt[3]{1-\left(1-\mu_{t_1}^3\right)^{\lambda_{t_1}}}, \sqrt[3]{1-\left(1-\theta_{t_1}^3\right)^{\lambda_{t_1}}}\right), \\ \left(\nu_{t_1}^{\lambda_{t_1}}, \varphi_{t_1}^{\lambda_{t_1}}\right) \end{array}\right)$$

$$\lambda_{t_2}.\sigma_{t_2} = \left(\begin{array}{c} \left(\sqrt[3]{1-\left(1-\mu_{t_2}^3\right)^{\lambda_{t_2}}}, \sqrt[3]{1-\left(1-\theta_{t_2}^3\right)^{\lambda_{t_2}}}\right), \\ \left(\nu_{t_2}^{\lambda_{t_2}}, \varphi_{t_2}^{\lambda_{t_2}}\right) \end{array}\right)$$

Therefore

$$\begin{aligned} &\lambda_{t_1}.\sigma_{t_1} \oplus \lambda_{t_2}.\sigma_{t_2} \\ &= \left(\left(\begin{array}{c} \left(\sqrt[3]{1-\left(1-\mu_{t_1}^3\right)^{\lambda_{t_1}}}, \sqrt[3]{1-\left(1-\theta_{t_1}^3\right)^{\lambda_{t_1}}}\right), \\ \left(\nu_{t_1}^{\lambda_{t_1}}, \varphi_{t_1}^{\lambda_{t_1}}\right) \end{array}\right) \oplus \left(\begin{array}{c} \left(\sqrt[3]{1-\left(1-\mu_{t_2}^3\right)^{\lambda_{t_2}}}, \sqrt[3]{1-\left(1-\theta_{t_2}^3\right)^{\lambda_{t_2}}}\right), \\ \left(\nu_{t_2}^{\lambda_{t_2}}, \varphi_{t_2}^{\lambda_{t_2}}\right) \end{array}\right)\right) \\ &= \left(\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{1-\left(1-\mu_{t_1}^3\right)^{\lambda_{t_1}}\left(1-\mu_{t_2}^3\right)^{\lambda_{t_2}}}, \\ \sqrt[3]{1-\left(1-\theta_{t_1}^3\right)^{\lambda_{t_1}}\left(1-\theta_{t_2}^3\right)^{\lambda_{t_2}}} \end{array}\right), \\ \left(\nu_{t_1}^{\lambda_{t_1}}\nu_{t_2}^{\lambda_{t_2}}, \varphi_{t_1}^{\lambda_{t_1}}\varphi_{t_2}^{\lambda_{t_2}}\right) \end{array}\right) \end{aligned}$$

Consequently,

$$CFFDWA\left(\sigma_{t_1}, \sigma_{t_2}\right) = \left(\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{1-\prod_{k=1}^{2}\left(1-\mu_{t_k}^3\right)^{\lambda_{t_k}}}, \\ \sqrt[3]{1-\prod_{k=1}^{2}\left(1-\theta_{t_k}^3\right)^{\lambda_{t_k}}} \end{array}\right), \\ \left(\prod_{k=1}^{2} \nu_{t_k}^{\lambda_{t_k}}, \prod_{k=1}^{2} \varphi_{t_k}^{\lambda_{t_k}}\right) \end{array}\right)$$

Hence, the base case is proven for $p = 2$.

Suppose that the statement holds for $p = n > 2$, we can express it as follows.

$$\begin{aligned} &CFFDWA\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_n}\right) \\ &= \oplus_{k=1}^{n}\lambda_{t_k}\sigma_{t_k} = \left(\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{1-\prod_{k=1}^{n}\left(1-\mu_{t_k}^3\right)^{\lambda_{t_k}}}, \\ \sqrt[3]{1-\prod_{k=1}^{n}\left(1-\theta_{t_k}^3\right)^{\lambda_{t_k}}} \end{array}\right), \\ \left(\prod_{k=1}^{n} \nu_{t_k}^{\lambda_{t_k}}, \prod_{k=1}^{n} \varphi_{t_k}^{\lambda_{t_k}}\right) \end{array}\right) \end{aligned}$$

Moreover, if $p = n + 1$, then

$$\begin{aligned} &CFFDWA\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_n}, \sigma_{t_{n+1}}\right) \\ &= \lambda_{t_1}\sigma_{t_1} \oplus \lambda_{t_2}\sigma_{t_2} \ldots \oplus \lambda_{t_n}\sigma_{t_n} \oplus \lambda_{t_{n+1}}\sigma_{t_{n+1}} \\ &= \left(\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{1-\prod_{k=1}^{n}\left(1-\mu_{t_k}^3\right)^{\lambda_{t_k}}}, \\ \sqrt[3]{1-\prod_{k=1}^{n}\left(1-\theta_{t_k}^3\right)^{\lambda_{t_k}}} \end{array}\right), \\ \left(\prod_{k=1}^{n} \nu_{t_k}^{\lambda_{t_k}}, \prod_{k=1}^{n} \varphi_{t_k}^{\lambda_{t_k}}\right) \end{array}\right) \\ &\oplus \left(\begin{array}{c} \left(\sqrt[3]{1-\left(1-\mu_{t_{n+1}}^3\right)^{\lambda_{t_{n+1}}}}, \sqrt[3]{1-\left(1-\theta_{t_{n+1}}^3\right)^{\lambda_{t_{n+1}}}}\right), \\ \left(\nu_{t_{n+1}}^{\lambda_{t_{n+1}}}, \varphi_{t_n}^{\lambda_{t_{n+1}}}\right) \end{array}\right) \end{aligned}$$

This means that:

$$\begin{aligned} &CFFDWA\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_{n+1}}\right) \\ &= \left(\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{1-\prod_{k=1}^{n+1}\left(1-\mu_{t_k}^3\right)^{\lambda_{t_k}}}, \\ \sqrt[3]{1-\prod_{k=1}^{n+1}\left(1-\theta_{t_k}^3\right)^{\lambda_{t_k}}} \end{array}\right), \\ \left(\prod_{k=1}^{n+1} \nu_{t_k}^{\lambda_{t_k}}, \prod_{k=1}^{n+1} \varphi_{t_k}^{\lambda_{t_k}}\right) \end{array}\right) \end{aligned}$$

Consequently, we may conclude that the claim is true for any positive integer p.

Example 3: Let us examine CFFNs $\sigma_{t_1} = ((0.9, 0.8), (0.2, 0.5))$, $\sigma_{t_2} = ((0.6, 0.3), (0.5, 0.8))$ and $\sigma_{t_3} = ((0.4, 0.6), (0.6, 0.8))$ each linked to a weight vector $\lambda_{t_k} = (0.2, 0.3, 0.5)^T$ for the respective time intervals t_1, t_2, and t_3. We perform the following calculations: $\prod_{k=1}^{3}(1-\mu_{t_k}^3)^{\lambda_{t_k}} = 0.6926$, $\prod_{k=1}^{3}\left(1-\theta_{t_k}^3\right)^{\lambda_{t_k}} = 0.7608$, $\prod_{k=1}^{3} \nu_{t_k}^{\lambda_{t_k}} = 0.4560$ and $\prod_{k=1}^{3} \varphi_{t_k}^{\lambda_{t_k}} = 0.7282$;

By applying Definition 13, we have:

$$\begin{aligned} &CFFDWA\left(\sigma_{t_2}, \sigma_{t_2}, \sigma_{t_3}\right) \\ &= \oplus_{k=1}^{3}\left(\lambda_{t_k} \cdot \sigma_{t_k}\right) \\ &= ((0.6748, 0.6207), (0.4560, 0.7282)) \end{aligned}$$

Thus, Theorem 1 is validated.

The subsequent theorem investigates the idempotent characteristic of the CFFDWA operator.

Theorem 2: Consider a set of CFFNs $\sigma_{t_k} = \left(\left(\mu_{t_{k'}}, \theta_{t_{k'}}, \left(\nu_{t_{k'}}, \varphi_{t_k}\right)\right)\right.$, where $k = 1, 2, \ldots, p$. If $\sigma_{t_k} = \sigma_{t_j}$ for all k and some $j \in \{1, 2, \ldots, p\}$, where $\sigma_{t_j} = \left(\left(\mu_{t_j}, \theta_{t_j}, \left(\nu_{t_j}, \varphi_{t_j}\right)\right)\right.$ and $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2} \ldots, \lambda_{t_p}\right]^T$ represents the weight vector associated with time periods t_k, such that $\lambda_{t_k} \in [0,1]$ and $\sum_{k=1}^{p} \lambda_{t_k} = 1$, then $CFFDWA\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}\right) = \sigma_{t_j}$.

Proof: Given that $\sigma_{t_k} = \sigma_{t_j}$ for all $k = 1, 2, \ldots, p$ and some $j \in \{1, 2, \ldots, p\}$, we can deduce that $\mu_{t_k} = \mu_{t_j}, \theta_{t_k} = \theta_{t_j}, \nu_{t_k} = \nu_{t_j}$ and $\varphi_{t_k} = \varphi_{t_j}$.

Now, let us delve into the mathematical formulation of CFFDWA:

$$CFFDWA\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}\right)$$

$$= \left(\left(\begin{array}{c} \sqrt[3]{1 - \prod_{k=1}^{p} \left(1 - \mu_{t_k}^3\right)^{\lambda_{t_k}}}, \\ \sqrt[3]{1 - \prod_{k=1}^{p} \left(1 - \theta_{t_k}^3\right)^{\lambda_{t_k}}} \end{array} \right), \\ \left(\prod_{k=1}^{p} \nu_{t_k}^{\lambda_{t_k}}, \prod_{k=1}^{p} \varphi_{t_k}^{\lambda_{t_k}}\right) \right)$$
$$= \left(\begin{array}{c} \left(\sqrt[3]{1-\left(1 - \mu_{t_j}^3\right)^{\sum_{k=1}^{p} \lambda_{t_k}}}, \sqrt[3]{1 - \left(1 - \theta_{t_j}^3\right)^{\sum_{k=1}^{p} \lambda_{t_k}}} \right), \\ \left(\nu_{t_j}^{\sum_{k=1}^{p} \lambda_{t_k}}, \varphi_{t_j}^{\sum_{k=1}^{p} \lambda_{t_k}}\right) \end{array} \right)$$
$$= \left(\left(\sqrt[3]{1 - (1 - \mu_{t_j}^3)}, \sqrt[3]{1 - (1 - \theta_{t_j}^3)} \right), (\nu_{t_j}, \varphi_{t_j}) \right)$$
$$= \left(\left(\sqrt[3]{\mu_{t_j}^3}, \sqrt[3]{\theta_{t_j}^3} \right), (\nu_{t_j}, \varphi_{t_j}) \right)$$
$$= \left((\mu_{t_j}, \theta_{t_j}), (\nu_{t_j}, \varphi_{t_j}) \right) = \left((\mu_{t_j}, \theta_{t_j}), (\nu_{t_j}, \varphi_{t_j}) \right)$$

Consequently,

$$CFFDWA\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}\right) = \sigma_{t_j}.$$

The subsequent theorem investigates the boundedness characteristic of the CFFDWA operator.

Theorem 3: Let $\sigma_t^- = \left((\min_{t_k} \{\mu_{t_k}\}, \min_{t_k} \{\theta_{t_k}\}), (\max_{t_k} \{\nu_{t_k}\}, \max_{t_k} \{\varphi_{t_k}\}) \right)$ and $\sigma_t^+ = \left((\max_{t_k} \{\mu_{t_k}\}, \max_{t_k} \{\theta_{t_k}\}), (\min_{t_k} \{\nu_{t_k}\}, \min_{t_k} \{\varphi_{t_k}\}) \right)$ be the lower and upper bound of the CFFNs $\sigma_{t_k} = \left((\mu_{t_k}, \theta_{t_k}), (\nu_{t_k}, \varphi_{t_k}) \right)$, where $k = 1, 2, 3, \ldots, p$. Let $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2}, \ldots, \lambda_{t_p}\right]^T$ be the weight vector corresponding to t_k, such that $\lambda_{t_k} \in [0, 1]$ and $\sum_{k=1}^{p} \lambda_{t_k} = 1$. Then,

$$\sigma_t^- \leq CFFDWA\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}\right) \leq \sigma_t^+$$

Proof: Consider the result of applying the CFFDWA operator to the collection of CFFNs, denoted as

$$CFFDWA\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}\right) = \left((\mu_t, \theta_t), (\nu_t, \varphi_t) \right).$$

For each μ_{t_k},

$$\min_{t_k} \{\mu_{t_k}\} \leq \mu_{t_k} \leq \max_{t_k} \{\mu_{t_k}\}$$
$$\Rightarrow \min_{t_k} \left\{\mu_{t_k}^3\right\} \leq \mu_{t_k}^3 \leq \max_{t_k} \left\{\mu_{t_k}^3\right\}$$
$$\Rightarrow 1 - \max_{t_k} \left\{\mu_{t_k}^3\right\} \leq 1 - \mu_{t_k}^3 \leq 1 - \min_{t_k} \left\{\mu_{t_k}^3\right\}$$
$$\Rightarrow \prod_{k=1}^{p} \left(1 - \max_{t_k} \left\{\mu_{t_k}^3\right\}\right)^{\lambda_{t_k}} \leq \prod_{k=1}^{p} \left(1 - \mu_{t_k}^3\right)^{\lambda_{t_k}}$$
$$\leq \prod_{k=1}^{p} \left(1 - \min_{t_k} \left\{\mu_{t_k}^3\right\}\right)^{\lambda_{t_k}}$$
$$\Rightarrow \left(1 - \max_{t_k} \left\{\mu_{t_k}^3\right\}\right)^{\sum_{k=1}^{p} \lambda_{t_k}} \leq \prod_{k=1}^{p} \left(1 - \mu_{t_k}^3\right)^{\lambda_{t_k}}$$
$$\leq \left(1 - \min_{t_k} \left\{\mu_{t_k}^3\right\}\right)^{\sum_{k=1}^{p} \lambda_{t_k}}$$
$$\Rightarrow \left(1 - \max_{t_k} \left\{\mu_{t_k}^3\right\}\right)$$
$$\leq \prod_{k=1}^{p} \left(1 - \mu_{t_k}^3\right)^{\lambda_{t_k}} \leq \left(1 - \min_{t_k} \left\{\mu_{t_k}^3\right\}\right)$$
$$\Rightarrow \min_{t_k} \left\{\mu_{t_k}^3\right\} \leq 1 - \prod_{k=1}^{p} \left(1 - \mu_{t_k}^3\right)^{\lambda_{t_k}} \leq \max_{t_k} \left\{\mu_{t_k}^3\right\}$$
$$\Rightarrow \sqrt[3]{\min_{t_k} \left\{\mu_{t_k}^3\right\}} \leq \sqrt[3]{1 - \prod_{k=1}^{p} \left(1 - \mu_{t_k}^3\right)^{\lambda_{t_k}}} \leq \sqrt[3]{\max_{t_k} \left\{\mu_{t_k}^3\right\}}$$

So,

$$\min_{t_k} \{\mu_{t_k}\} \leq \mu_t \leq \max_{t_k} \{\mu_{t_k}\}$$

For each θ_{t_k}, we have

$$\min_{t_k} \{\theta_{t_k}\}$$
$$\leq \theta_{t_k} \leq \max_{t_k} \{\theta_{t_k}\}$$
$$\Rightarrow \min_{t_k} \left\{\theta_{t_k}^3\right\} \leq \theta_{t_k}^3 \leq \max_{t_k} \left\{\theta_{t_k}^3\right\}$$
$$\Rightarrow 1 - \max_{t_k} \left\{\theta_{t_k}^3\right\} \leq 1 - \theta_{t_k}^3 \leq 1 - \min_{t_k} \left\{\theta_{t_k}^3\right\}$$
$$\Rightarrow \prod_{k=1}^{p} \left(1 - \max_{t_k} \left\{\theta_{t_k}^3\right\}\right)^{\lambda_{t_k}} \leq \prod_{k=1}^{p} \left(1 - \theta_{t_k}^3\right)^{\lambda_{t_k}}$$
$$\leq \prod_{k=1}^{p} \left(1 - \min_{t_k} \left\{\theta_{t_k}^3\right\}\right)^{\lambda_{t_k}}$$
$$\Rightarrow \left(1 - \max_{t_k} \left\{\theta_{t_k}^3\right\}\right)^{\sum_{k=1}^{p} \lambda_{t_k}} \leq \prod_{k=1}^{p} \left(1 - \theta_{t_k}^3\right)^{\lambda_{t_k}}$$
$$\leq \left(1 - \min_{t_k} \left\{\theta_{t_k}^3\right\}\right)^{\sum_{k=1}^{p} \lambda_{t_k}}$$
$$\Rightarrow \left(1 - \max_{t_k} \left\{\theta_{t_k}^3\right\}\right)$$
$$\leq \prod_{k=1}^{p} \left(1 - \theta_{t_k}^3\right)^{\lambda_{t_k}} \leq \left(1 - \min_{t_k} \left\{\theta_{t_k}^3\right\}\right)$$
$$\Rightarrow \min_{t_k} \left\{\theta_{t_k}^3\right\} \leq 1 - \prod_{k=1}^{p} \left(1 - \theta_{t_k}^3\right)^{\lambda_{t_k}} \leq \max_{t_k} \left\{\theta_{t_k}^3\right\}$$
$$\Rightarrow \sqrt[3]{\min_{t_k} \{\theta_{t_k}^3\}} \leq \sqrt[3]{1 - \prod_{k=1}^{p} \left(1 - \theta_{t_k}^3\right)^{\lambda_{t_k}}} \leq \sqrt[3]{\max_{t_k} \{\theta_{t_k}^3\}}$$

Therefore,

$$\min_{t_k} \{\theta_{t_k}\} \leq \theta_t \leq \max_{t_k} \{\theta_{t_k}\}$$

Moreover,

$$\min_{t_k} \{\nu_{t_k}\} \leq \nu_{t_k} \leq \max_{t_k} \{\nu_{t_k}\}$$
$$\Rightarrow \prod_{k=1}^{p} \left(\min_{t_k} \{\nu_{t_k}\}\right)^{\lambda_{t_k}}$$
$$\leq \prod_{k=1}^{p} (\nu_{t_k})^{\lambda_{t_k}}$$
$$\leq \prod_{k=1}^{p} \left(\max_{t_k} \{\nu_{t_k}\}\right)^{\lambda_{t_k}}$$
$$\Rightarrow \left(\min_{t_k} \{\nu_{t_k}\}\right)^{\sum_{k=1}^{p} \lambda_{t_k}}$$
$$\leq \prod_{k=1}^{p} (\nu_{t_k})^{\lambda_{t_k}} \leq \left(\max_{t_k} \{\nu_{t_k}\}\right)^{\sum_{k=1}^{p} \lambda_{t_k}}$$
$$\Rightarrow \min_{t_k} \{\nu_{t_k}\} \leq \nu_t \leq \max_{t_k} \{\nu_{t_k}\}$$

Furthermore

$$\begin{aligned}&\min_{t_k}\{\varphi_{t_k}\}\le\varphi_{t_k}\le\max_{t_k}\{\varphi_{t_k}\}\\&\Rightarrow\prod_{k=1}^{p}\left(\min_{t_k}\{\varphi_{t_k}\}\right)^{\lambda_{t_k}}\\&\le\prod_{k=1}^{p}\left(\varphi_{t_k}\right)^{\lambda_{t_k}}\\&\le\prod_{k=1}^{p}\left(\max_{t_k}\{\varphi_{t_k}\}\right)^{\lambda_{t_k}}\\&\Rightarrow\left(\min_{t_k}\{\varphi_{t_k}\}\right)^{\sum_{k=1}^{p}\lambda_{t_k}}\\&\le\prod_{k=1}^{p}\left(\varphi_{t_k}\right)^{\lambda_{t_k}}\le\left(\max_{t_k}\{\varphi_{t_k}\}\right)^{\sum_{k=1}^{p}\lambda_{t_k}}\\&\Rightarrow\min_{t_k}\{\varphi_{t_k}\}\le\varphi_t\le\max_{t_k}\{\varphi_{t_k}\}\end{aligned}$$

Hence by employing Definition 13, we obtain that

$$\sigma_t^-\le CFFDWA\left(\sigma_{t_1},\sigma_{t_2},\ldots,\sigma_{t_p}\right)\le\sigma_t^+$$

The subsequent theorem investigates the monotonic characteristic of the CFFDWA operator.

Theorem 4: Consider sets of two CFFNs, denoted as $\sigma_{t_k}=\left(\left(\mu_{t_k},\theta_{t_k}\right),\left(v_{t_k},\varphi_{t_k}\right)\right)$ and $\sigma'_{t_k}=\left(\left(\mu'_{t_k},\theta'_{t_k}\right),\left(v'_{t_{k'}},\varphi'_{t_k}\right)\right)$, where $k=1,2,3,\ldots,p$. Let $\lambda_{t_k}=\left[\lambda_{t_1},\lambda_{t_2},\ldots,\lambda_{t_p}\right]^T$ represents the weight vector associated with time periods t_k, such that $\lambda_{t_k}\in[0,1]$ and $\sum_{k=1}^{p}\lambda_{t_k}=1$. If for each t_k:

$$\mu_{t_k}\le\mu'_{t_k},\theta_{t_k}\le\theta'_{t_k},\quad\text{and }v_{t_k}\ge v'_{t_k},\varphi_{t_k}\ge\varphi'_{t_k}$$

Then,

$$\begin{aligned}&CFFDWA\left(\sigma_{t_1},\sigma_{t_2},\ldots,\sigma_{t_p}\right)\\&\qquad\le CFFDWA\left(G_{t_1},G_{t_2},\ldots,G_{t_p}\right)\end{aligned}$$

Proof: Based on the provided description σ_{t_k} and σ'_{t_k}, we have $CFFDWA\left(\sigma_{t_1},\sigma_{t_2},\ldots,\sigma_{t_p}\right)=\left(\left(\mu_t,\theta_t\right),\left(v_t,\varphi_t\right)\right)$ and $CFFDWA\left(\sigma'_{t_1},\sigma'_{t_2},\ldots,\sigma'_{t_p}\right)=\left(\left(\mu'_t,\theta'_t\right),\left(v'_t,\varphi'_t\right)\right)$.

Since $\mu_{t_k}\le\mu'_{t_k}$, which implies that $\mu^3_{t_k}\le\mu'^3_{t_k}$, we can deduce that

$$\begin{aligned}&1-\mu^3_{t_k}\\&\ge1-\mu'^3_{t_k}\\&\Rightarrow\prod_{k=1}^{p}\left(1-\mu^3_{t_k}\right)^{\lambda_{t_k}}\ge\prod_{k=1}^{p}\left(1-\mu'^3_{t_k}\right)^{\lambda_{t_k}}\\&\Rightarrow1-\prod_{k=1}^{p}\left(1-\mu^3_{t_k}\right)^{\lambda_{t_k}}\le1-\prod_{k=1}^{p}\left(1-\mu'^3_{t_k}\right)^{\lambda_{t_k}}\\&\Rightarrow\sqrt[3]{1-\prod_{k=1}^{p}\left(1-\mu^3_{t_k}\right)^{\lambda_{t_k}}}\le\sqrt[3]{1-\prod_{k=1}^{p}\left(1-\mu'_{t_k}\right)^{\lambda_{t_k}}}\end{aligned}$$

Hence, we can conclude that

$$\mu_{t_k}\le\mu'_{t_k}\Rightarrow\mu_t\le\mu'_t$$

Similarly, by considering $\theta_{t_k}\le\theta'_{t_k}$ we derive,

$$\sqrt[3]{1-\prod_{k=1}^{p}\left(1-\theta^3_{t_k}\right)^{\lambda_{t_k}}}\le\sqrt[3]{1-\prod_{k=1}^{p}\left(1-\gamma^3_{t_k}\right)^{\lambda_{t_k}}}$$

$$\theta_t\le\gamma_t$$

By adopting the above procedure, we obtain the following inequalities,

$$\begin{aligned}v_t&\ge v'_t\\\varphi_t&\ge\varphi'_t\end{aligned}$$

Therefore, utilizing Definition 13, we obtain that

$$CFFDWA\left(\sigma_{t_1},\sigma_{t_2},\ldots,\sigma_{t_p}\right)\le CFFDWA\left(\sigma'_{t_1},\sigma'_{t_2},\ldots,\sigma'_{t_p}\right)$$

Thus, the monotonicity property is established.

In the ensuing definition, we present a dynamic geometric aggregation operator developed for the CFFNs, namely CFFDWG operator.

Definition 14: Let $\sigma_{t_k}=\left(\left(\mu_{t_k},\theta_{t_k}\right),\left(v_{t_k},\varphi_{t_k}\right)\right)$, for $k=1,2,\ldots,p$, be a collection of CFFNs at p different periods t_k. Moreover, $\lambda_{t_k}=\left[\lambda_{t_1},\lambda_{t_2},\ldots,\lambda_{t_p}\right]^T$, where $\lambda_{t_k}\in[0,1]$, represent the weight vector of the time periods t_k and $\sum_{k=1}^{p}\lambda_{t_k}=1$. A CFFDWG operator is a function $CFFDWG$: $\sigma^p\to\sigma$, defined as:

$$\begin{aligned}&CFFDWG\left(\sigma_{t_1},\sigma_{t_2},\ldots,\sigma_{t_p}\right)=\otimes_{k=1}^{p}\sigma_{t_k}^{\lambda_{t_k}}\\&=\left(\begin{array}{l}\left(\prod_{k=1}^{p}\mu_{t_k}^{\lambda_{t_k}},\prod_{k=1}^{p}\theta_{t_k}^{\lambda_{t_k}}\right),\\\left(\sqrt[3]{1-\prod_{k=1}^{p}\left(1-v^3_{t_k}\right)^{\lambda_{t_k}}},\sqrt[3]{1-\prod_{k=1}^{p}\left(1-\varphi^3_{t_k}\right)^{\lambda_{t_k}}}\right)\end{array}\right),\end{aligned}$$

Theorem 5: Let $\sigma_{t_k}=\left(\left(\mu_{t_k},\theta_{t_k}\right),\left(v_{t_{k'}},\varphi_{t_k}\right)\right)$ for $k=1,2,\ldots,p$, represent a collection of CFFNs at p different time periods t_k. Additionally, consider the weight vector $\lambda_{t_k}=\left[\lambda_{t_1},\lambda_{t_2}\ldots,\lambda_{t_p}\right]^T$, where $\lambda_{t_k}\in[0,1]$ and $\sum_{k=1}^{p}\lambda_{t_k}=1$. The aggregated value of these CFFNs using the CFFDWG operator is itself a CFFN, expressed as:

$$\begin{aligned}&CFFDWG\left(\sigma_{t_1},\sigma_{t_2},\ldots,\sigma_{t_p}\right)\\&=\left(\begin{array}{l}\left(\prod_{k=1}^{p}\mu_{t_k}^{\lambda_{t_k}},\prod_{k=1}^{p}\theta_{t_k}^{\lambda_{t_k}}\right),\\\left(\sqrt[3]{1-\prod_{k=1}^{p}\left(1-v^3_{t_k}\right)^{\lambda_{t_k}}},\sqrt[3]{1-\prod_{k=1}^{p}\left(1-\varphi^3_{t_k}\right)^{\lambda_{t_k}}}\right)\end{array}\right)\end{aligned}$$

Proof: We will use mathematical induction to prove this theorem. Let $p=2$, then

$$CFFDWG\left(\sigma_{t_1},\sigma_{t_2}\right)=\sigma_{t_1}^{\lambda_{t_1}}\otimes\sigma_{t_2}^{\lambda_{t_2}}$$

Exploring the components,

$$\begin{aligned}&\sigma_{t_1}^{\lambda_{t_1}}\\&=\left(\left(\mu_{t_1}^{\lambda_{t_1}},\theta_{t_1}^{\lambda_{t_1}}\right),\left(\sqrt[3]{1-\left(1-v^3_{t_1}\right)^{\lambda_{t_1}}},\sqrt[3]{1-\left(1-\varphi^3_{t_1}\right)^{\lambda_{t_1}}}\right)\right)\\&\sigma_{t_2}^{\lambda_{t_2}}\\&=\left(\left(\mu_{t_2}^{\lambda_{t_2}}\theta_{t_2}^{\lambda_{t_2}}\right),\left(\sqrt[3]{1-\left(1-v^3_{t_2}\right)^{\lambda_{t_2}}},\sqrt[3]{1-\left(1-\varphi^3_{t_2}\right)^{\lambda_{t_2}}}\right)\right)\end{aligned}$$

Then,

$$\sigma_{t_1}^{\lambda_{t_1}}\otimes\sigma_{t_2}^{\lambda_{t_2}}$$

$$= \left(\left(\mu_{t_1}^{\lambda_{t_1}}, \theta_{t_1}^{\lambda_{t_1}} \right), \left(\sqrt[3]{1-(1-\nu_{t_1}^3)^{\lambda_{t_1}}}, \sqrt[3]{1-(1-\varphi_{t_1}^3)^{\lambda_{t_1}}} \right) \right)$$
$$\otimes \left(\left(\mu_{t_2}^{\lambda_{t_2}}, \theta_{t_2}^{\lambda_{t_2}} \right), \left(\sqrt[3]{1-(1-\nu_{t_2}^3)^{\lambda_{t_2}}}, \sqrt[3]{1-(1-\varphi_{t_2}^3)^{\lambda_{t_2}}} \right) \right)$$
$$CFFDWG\left(\sigma_{t_1}, \sigma_{t_2}\right) = \begin{pmatrix} \left(\prod_{k=1}^{2} \mu_{t_k}^{\lambda_{t_k}}, \prod_{k=1}^{2} \theta_{t_k}^{\lambda_{t_k}} \right), \\ \begin{pmatrix} \sqrt[3]{1-\prod_{k=1}^{2}(1-\nu_{t_k}^3)^{\lambda_{t_k}}}, \\ \sqrt[3]{1-\prod_{k=1}^{2}(1-\varphi_{t_k}^3)^{\lambda_{t_k}}} \end{pmatrix} \end{pmatrix}$$

Hence, the assertion holds true for $p = 2$.

Next, we suppose the theory is valid $p = n > 2$, then:

$$CFFDWG\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_n}\right) = \otimes_{k=1}^{n} \sigma_{t_k}^{\lambda_{t_k}} = \begin{pmatrix} \left(\prod_{k=1}^{n} \mu_{t_k}^{\lambda_{t_k}}, \prod_{k=1}^{n} \theta_{t_k}^{\lambda_{t_k}} \right), \\ \begin{pmatrix} \sqrt[3]{1-\prod_{k=1}^{n}(1-\nu_{t_k}^3)^{\lambda_{t_k}}}, \\ \sqrt[3]{1-\prod_{k=1}^{n}(1-\varphi_{t_k}^3)^{\lambda_{t_k}}} \end{pmatrix} \end{pmatrix}$$

Next, if $p = n + 1$, then

$$CFFDWG\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_n}, \sigma_{t_{n+1}}\right) = \sigma_{t_1}^{\lambda_{t_1}} \otimes \sigma_{t_2}^{\lambda_{t_2}} \otimes \ldots \otimes \sigma_{t_n}^{\lambda_{t_n}} \otimes \sigma_{t_{n+1}}^{\lambda_{t_{n+1}}}$$
$$= \begin{pmatrix} \left(\prod_{k=1}^{n} \mu_{t_k}^{\lambda_{t_k}}, \prod_{k=1}^{n} \theta_{t_k}^{\lambda_{t_k}} \right), \\ \begin{pmatrix} \sqrt[3]{1-\prod_{k=1}^{n}(1-\nu_{t_k}^3)^{\lambda_{t_k}}}, \\ \sqrt[3]{1-\prod_{k=1}^{n}(1-\varphi_{t_k}^3)^{\lambda_{t_k}}} \end{pmatrix} \end{pmatrix} \otimes \begin{pmatrix} (\mu_{t_{n+1}}^{\lambda_{t_{n+1}}}, \theta_{t_{n+1}}^{\lambda_{t_{n+1}}}), \\ \left(\sqrt[3]{1-(1-\nu_{t_{n+1}}^3)^{\lambda_{t_{n+1}}}}, \sqrt[3]{1-(1-\varphi_{t_{n+1}}^3)^{\lambda_{t_{n+1}}}} \right) \end{pmatrix}$$
$$CFFDWG\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_{n+1}}\right) = \begin{pmatrix} \left(\prod_{k=1}^{n+1} \mu_{t_k}^{\lambda_{t_k}}, \prod_{k=1}^{n+1} \theta_{t_k}^{\lambda_{t_k}} \right), \\ \begin{pmatrix} \sqrt[3]{1-\prod_{k=1}^{n+1}(1-\nu_{t_k}^3)^{\lambda_{t_k}}}, \\ \sqrt[3]{1-\prod_{k=1}^{n+1}(1-\varphi_{t_k}^3)^{\lambda_{t_k}}} \end{pmatrix} \end{pmatrix}$$

This illustrates that the theorem holds for $p = n + 1$. As a result, we can conclude that statement true for all positive integers p.

Example 4: Consider the CFFNs $\sigma_{t_1} = ((0.5, 0.9), (0.7, 0.3))$, $\sigma_{t_2} = ((0.7, 0.4), (0.5, 0.7))$ and $\sigma_{t_3} = ((0.6, 0.5), (0.7, 0.9))$ and associated weight vector is $\lambda_{t_k} = (0.2, 0.3, 0.5)^T$ of the periods A_t, where $k = 1, 2, 3$. Then,

$$\prod_{k=1}^{3} \mu_{t_k}^{\lambda_{t_k}} = 0.605, \prod_{k=1}^{3} \theta_{t_k}^{\lambda_{t_k}} = 0.525$$
$$\prod_{k=1}^{3} \left(1-\nu_{t_k}^3\right)^{\lambda_{t_k}} = 0.715, \prod_{k=1}^{3} \left(1-\varphi_{t_k}^3\right)^{\lambda_{t_k}} = 0.456.$$

This implies that

$$CFFDWG\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}\right) = \bigotimes_{k=1}^{p} \sigma_{t_k}^{\lambda_{t_k}} = ((0.605, 0.4), (0.658, 0.816)).$$

Thus, Theorem 5 is validated.

The subsequent theorem investigates the idempotent characteristic of the CFFDWG operator.

Theorem 6: Consider a set of CFFNs $\sigma_{t_k} = \left(\left(\mu_{t_k}, \theta_{t_k}, \left(\nu_{t_k}, \varphi_{t_k}\right)\right)\right)$. If $\sigma_{t_k} = \sigma_{t_j} = \left(\left(\mu_{t_j}, \theta_{t_j}, \left(\nu_{t_j}, \varphi_{t_j}\right)\right)\right)$ for all $k = 1, 2, \ldots, p$, and some $j \in \{1, 2, \ldots, p\}$. Let $\lambda_t = \left[\lambda_{t_1}, \lambda_{t_2} \ldots, \lambda_{t_p}\right]^T$ represent the weight vector associated with time period t_k, such that $\lambda_{t_k} \in [0, 1]$ and $\sum_{k=1}^{p} \lambda_{t_k} = 1$, then $CFFDWG\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}\right) = \sigma_{t_j}$.

Proof: Proof can be obtained by using the same approach employed in theorem 2.

The subsequent theorem investigates the boundedness characteristic of the CFFDWG operator.

Theorem 7: Let $\sigma_t^- = \left(\begin{matrix} \left(\min_{t_k}\{\mu_{t_k}\}, \min_{t_k}\{\theta_{t_k}\}\right) \\ , \left(\max_{t_k}\{\nu_{t_k}\}, \max_{t_k}\{\varphi_{t_k}\}\right) \end{matrix} \right)$ and $\sigma_t^+ = \left(\begin{matrix} \left(\max_{t_k}\{\mu_{t_k}\}, \max_{t_k}\{\theta_{t_k}\}\right) \\ , \left(\min_{t_k}\{\nu_{t_k}\}, \min_{t_k}\{\varphi_{t_k}\}\right) \end{matrix} \right)$ be the lower and upper bound of the CFFNs $\sigma_{t_k} = \left(\left(\mu_{t_k}, \theta_{t_k}\right), \left(\nu_{t_k}, \varphi_{t_k}\right)\right)$, where k takes on values from 1 to p. Let $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2}, \ldots, \lambda_{t_p}\right]^T$ be the associated vector of these CFFNs, such that $\lambda_{t_k} \in [0, 1]$, satisfying $\sum_{k=1}^{p} \lambda_{t_k} = 1$. Then,

$$\sigma_t^- \leq CFFDWG\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}\right) \leq \sigma_t^+$$

Proof: Let us apply CFFDWG operator on CFFNs σ_{t_k} for all k.

$$CFFDWG\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}\right) = \left(\left(\mu_t, \theta_t\right), \left(\nu_t, \varphi_t\right)\right).$$

For each μ_{t_k},

$$\min_{t_k}\left\{\mu_{t_k}\right\} \leq \mu_{t_k} \leq \max_{t_k}\left\{\mu_{t_k}\right\}$$
$$\Rightarrow \prod_{k=1}^{p} \left(\min_{t_k}\left\{\mu_{t_k}\right\}\right)^{\lambda_{t_k}} \leq \prod_{k=1}^{p} \left(\mu_{t_k}\right)^{\lambda_{t_k}} \leq \prod_{k=1}^{p} \left(\max_{t_k}\left\{\mu_{t_k}\right\}\right)^{\lambda_{t_k}}$$
$$\Rightarrow \left(\min_{t_k}\left\{\mu_{t_k}\right\}\right)^{\sum_{k=1}^{p}\lambda_{t_k}} \leq \prod_{k=1}^{p} \left(\mu_{t_k}\right)^{\lambda_{t_k}} \leq \left(\max_{t_k}\left\{\mu_{t_k}\right\}\right)^{\sum_{k=1}^{p}\lambda_{t_k}}$$
$$\Rightarrow \min_{t_k}\left\{\mu_{t_k}\right\} \leq \mu_t \leq \max_{t_k}\left\{\mu_{t_k}\right\}$$

For each θ_{t_k},

$$\min_{t_k}\left\{\theta_{t_k}\right\} \leq \theta_{t_k} \leq \max_{t_k}\left\{\theta_{t_k}\right\}$$
$$\Rightarrow \prod_{k=1}^{p} \left(\min_{t_k}\left\{\theta_{t_k}\right\}\right)^{\lambda_{t_k}} \leq \prod_{k=1}^{p} \left(\theta_{t_k}\right)^{\lambda_{t_k}}$$

$$\leq \prod_{k=1}^{p}\left(\max_{t_k}\left\{\theta_{t_k}\right\}\right)^{\lambda_{t_k}}$$

$$\Rightarrow \left(\min_{t_k}\left\{\theta_{t_k}\right\}\right)^{\sum_{k=1}^{p}\lambda_{t_k}}$$

$$\leq \prod_{k=1}^{p}\left(\theta_{t_k}\right)^{\lambda_{t_k}} \leq \left(\max_{t_k}\left\{\theta_{t_k}\right\}\right)^{\sum_{k=1}^{p}\lambda_{t_k}}$$

$$\Rightarrow \min_{t_k}\left\{\theta_{t_k}\right\} \leq \theta_t \leq \max_{t_k}\left\{\theta_{t_k}\right\}$$

Moreover,

$$\min_{t_k}\left\{v_{t_k}\right\}$$

$$\leq v_{t_k} \leq \max_{t_k}\left\{v_{t_k}\right\}$$

$$\Rightarrow \min_{t_k}\left\{v_{t_k}^3\right\} \leq v_{t_k}^3 \leq \max_{t_k}\left\{v_{t_k}^3\right\}$$

$$\Rightarrow 1-\max_{t_k}\left\{v_{t_k}^3\right\} \leq 1-v_{t_k}^3 \leq 1-\min_{t_k}\left\{v_{t_k}^3\right\}$$

$$\Rightarrow \prod_{k=1}^{p}\left(1-\max_{t_k}\left\{v_{t_k}^3\right\}\right)^{\lambda_{t_k}} \leq \prod_{k=1}^{p}\left(1-v_{t_k}^3\right)^{\lambda_{t_k}}$$

$$\leq \prod_{k=1}^{p}\left(1-\min_{t_k}\left\{v_{t_k}^3\right\}\right)^{\lambda_{t_k}}$$

$$\Rightarrow \left(1-\max_{t_k}\left\{v_{t_k}^3\right\}\right)^{\sum_{k=1}^{p}\lambda_{t_k}} \leq \prod_{k=1}^{p}\left(1-v_{t_k}^3\right)^{\lambda_{t_k}}$$

$$\leq \left(1-\min_{t_k}\left\{v_{t_k}^3\right\}\right)^{\Sigma_{k=1}^{p}\lambda_{t_k}}$$

$$\Rightarrow \left(1-\max_{t_k}\left\{v_{t_k}^3\right\}\right)$$

$$\leq \prod_{k=1}^{p}\left(1-v_{t_k}^3\right)^{\lambda_{t_k}} \leq \left(1-\min_{t_k}\left\{v_{t_k}^3\right\}\right)$$

$$\Rightarrow \min_{t_k}\left\{v_{t_k}^3\right\} \leq 1-\prod_{k=1}^{p}\left(1-v_{t_k}^3\right)^{\lambda_{t_k}} \leq \max_{t_k}\left\{v_{t_k}^3\right\}$$

$$\Rightarrow \sqrt[3]{\min_{t_k}\left\{v_{t_k}^3\right\}}$$

$$\leq \sqrt[3]{1-\prod_{k=1}^{3}\left(1-v_{t_k}^3\right)^{\lambda_{t_k}}} \leq \sqrt[3]{\max_{t_k}\left\{v_{t_k}^3\right\}}$$

Therefore,

$$\min_{t_k}\left\{v_{t_k}\right\} \leq v_t \leq \max_{t_k}\left\{v_{t_k}\right\}$$

Furthermore,

$$\min_{t_k}\left\{\varphi_{t_k}\right\}$$

$$\leq \varphi_{t_k} \leq \max_{t_k}\left\{\varphi_{t_k}\right\}$$

$$\Rightarrow \min_{t_k}\left\{\varphi_{t_k}^3\right\} \leq \varphi_{t_k}^3 \leq \max_{t_k}\left\{\varphi_{t_k}^3\right\}$$

$$\Rightarrow 1-\max_{t_k}\left\{\varphi_{t_k}^3\right\} \leq 1-\varphi_{t_k}^3 \leq 1-\min_{t_k}\left\{\varphi_{t_k}^3\right\}$$

$$\Rightarrow \prod_{k=1}^{p}\left(1-\max_{t_k}\left\{\varphi_{t_k}^3\right\}\right)^{\lambda_{t_k}} \leq \prod_{k=1}^{p}\left(1-\varphi_{t_k}^3\right)^{\lambda_{t_k}}$$

$$\leq \prod_{k=1}^{p}\left(1-\min_{t_k}\left\{\varphi_{t_k}^3\right\}\right)^{\lambda_{t_k}}$$

$$\Rightarrow \left(1-\max_{t_k}\left\{\varphi_{t_k}^3\right\}\right)^{\sum_{k=1}^{p}\lambda_{t_k}} \leq \prod_{k=1}^{p}\left(1-\varphi_{t_k}^3\right)^{\lambda_{t_k}}$$

$$\leq \left(1-\min_{t_k}\left\{\varphi_{t_k}^3\right\}\right)^{\Sigma_{k=1}^{p}\lambda_{t_k}}$$

$$\Rightarrow \left(1-\max_{t_k}\left\{\varphi_{t_k}^3\right\}\right)$$

$$\leq \prod_{k=1}^{p}\left(1-\varphi_{t_k}^3\right)^{\lambda_{t_k}}$$

$$\leq \left(1-\min_{t_k}\left\{\varphi_{t_k}^3\right\}\right)$$

$$\Rightarrow \min_{t_k}\left\{\varphi_{t_k}^3\right\} \leq 1-\prod_{k=1}^{p}\left(1-\varphi_{t_k}^3\right)^{\lambda_{t_k}} \leq \max_{t_k}\left\{\varphi_{t_k}^3\right\}$$

$$\Rightarrow \sqrt[3]{\min_{t_k}\left\{\varphi_{t_k}^3\right\}} \leq \sqrt[3]{1-\prod_{k=1}^{p}\left(1-\varphi_{t_k}^3\right)^{\lambda_{t_k}}}$$

$$\leq \sqrt[s]{\max_{t_k}\left\{\varphi_{t_k}^3\right\}}$$

This implies that,

$$\min_{t_k}\left\{\varphi_{t_k}\right\} \leq \varphi_t \leq \max_{t_k}\left\{\varphi_{t_k}\right\}$$

Hence by employing Definition 14, we obtain that

$$\sigma_t^{-} \leq CFFDWG\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}\right) \leq \sigma_t^{+}$$

The subsequent theorem investigates the monotonic characteristic of the CFFDWG operator.

Theorem 8: Consider two sets of CFFNs $\sigma_{t_k} = \left(\left(\mu_{t_k}, \theta_{t_k}\right), \left(v_{t_k}, \varphi_{t_k}\right)\right)$ and $G_{t_k} = \left(\left(\varpi_{t_k}, \gamma_{t_k}\right), \left(\xi_{t_k}, \delta_{t_k}\right)\right)$, where $k = 1, 2, 3, \ldots, p$. Let $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2}, \ldots, \lambda_{t_p}\right]^T$ represent the weight vector associated with time periods t_k, such that $\lambda_{t_k} \in [0, 1]$ and $\sum_{k=1}^{p}\lambda_{t_k} = 1$. If for each t_k,

$$\mu_{t_k} \leq \eta_{t_k}, \theta_{t_k} \leq \gamma_{t_k}, v_{t_k} \geq \xi_{t_k} \text{ and } \varphi_{t_k} \geq \delta_{t_k}$$

Then,

$$CFFDWG\left(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_p}\right) \leq CFFDWG(\left(G_{t_1}, G_{t_2}, \ldots, G_{t_p}\right)$$

Proof: Proof of this theorem is similar to theorem 4.

REMARK: Linguistic and Metaphysical Interpretation of above Theorems: In theorems 1 and 5, it is proved that when we apply DWA and DWG operators to a finite collection of CFFNs, their aggregated value gives a CFFN. Theorems 2 to 6 verify the idempotency property; this property ensures that applying the weighted aggregation operator to the same input CFFS multiple times produces the same result as applying it once. This consistency and stability are essential in decision-making processes, providing predictability and reliability in the aggregate outcome. Theorems 3 and 7 satisfy the monotonicity property; this property ensures that the output of the aggregation operation behaves consistently with changes in the input fuzzy sets' membership and non-membership values. This property also guarantees the preservation of order in the aggregation process. If the input CFFS exhibits a certain order in terms of their membership and non-membership values, the monotonicity property ensures that the order is maintained in the aggregated output. The boundedness property is established by Theorems 4 and 8. This property guarantees that the results generated by the weighted aggregation operator are contained within specific limits. It is important to ensure that the aggregation process produces meaningful and precisely defined outcomes.

V. APPLICATION OF NEWLY DEFINED COMPLEX FERMATEAN FUZZY DYNAMIC WEIGHTED AGGREGATION OPERATORS IN MULTI ATTRIBUTE DECISION MAKING PROBLEM

In this section, we develop a method based on proposed aggregation operators to deal with MADM issues.

- Let $\partial_i = \{\partial_1, \partial_2, \ldots, \partial_m\}$ be a discrete collection of alternatives.
- Suppose that $\rho_j = \{\rho_1, \rho_2, \ldots, \rho_n\}$ denotes the set of attributes and $\omega = [\omega_1, \omega_2, \ldots, \omega_n]^T$ is the associated weighted vector, where $\omega_j \in [0, 1]$ such that $\sum_{j=1}^{n} \omega_j = 1$.
- Let t_k, where $k = 1, 2, \ldots, p$, denote p number of time periods with weight vector $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2}, \ldots, \lambda_{t_p}\right]^T$ such that $\lambda_{t_k} \in [0, 1]$ and $\sum_{k=1}^{p} \lambda_{t_k} = 1$.
- Suppose that $R_{t_k} = \left[r_{ij(t_k)}\right]_{m\times n} = \left(\left(\mu_{ij(t_k)}, \theta_{ij(t_k)}\right), \left(\nu_{ij(t_k)}, \varphi_{ij(t_k)}\right)\right)_{m\times n}$ is the CFF decision matrix at t_k. Here $\left(\mu_{ij(t_k)}, \theta_{ij(t_k)}\right)$ shows how much alternative ∂_i meets the requirements for attribute ρ_j during time interval t_k and $\left(\nu_{ij(t_k)}, \varphi_{ij(t_k)}\right)$ shows how much ∂_i doesn't meet the requirements for attribute ρ_j during time interval t_k. Additionally, $\mu_{ij(t_k)}, \theta_{ij(t_k)}, \nu_{ij(t_k)}, \varphi_{ij(t_k)} \in [0, 1]$ such that $\mu^3_{ij(t_k)} + \nu^3_{ij(t_k)} \leq 1$ and $\theta^3_{ij(t_k)} + \varphi^3_{ij(t_k)} \leq 1$, where i varies from 1 to m and j varies from 1 to n.

By utilizing the aforementioned decision data, we devise an operational approach to rank and choose the most favorable choice.

A. PROCEDURE FOR CFFDWA

Step 1: Apply CFFDWA operator on the matrix R_{t_k}:

$$\begin{aligned} R_{t_k} &= r_{ij(t_k)} = \left(\left(\mu_{ij(t_k)}, \theta_{ij(t_k)}\right), \left(\nu_{ij(t_k)}, \varphi_{ij(t_k)}\right)\right) \\ &= CFFDWA\left(r_{ij(t_1)}, r_{ij(t_2)}, \ldots, r_{ij(t_p)}\right) \\ &= \bigoplus_{k=1}^{p} \lambda_{t_k} F_{t_k} \end{aligned}$$

Apply Theorem 1 to above relationship:

$$\left(\begin{array}{c} r_{ij}\,(t_k) \\ = \left(\sqrt[3]{1-\prod_{k=1}^{p}\left(1-\mu_{t_k}^3\right)^{\lambda_{t_k}}}, \sqrt[3]{1-\prod_{k=1}^{p}\left(1-\theta_{t_k}^3\right)^{\lambda_{t_k}}}\right) \\ \left(\prod_{k=1}^{p} \nu_{t_k}{}^{\lambda_{t_k}} \cdot \prod_{k=1}^{p} \varphi_{t_k}^{\lambda_{t_k}}\right) \end{array}\right)$$

This aggregation combines all CFF decision matrices R_{t_k}, as a collective decision matrix R, where $\lambda_{t_k} = \left[\lambda_{t_1}, \lambda_{t_2}, \ldots, \lambda_{t_p}\right]^T$ denotes the associated weight vector for the time period t_k.

Step 2: Apply CFFWA operator on collective CFF information given in matrix R as follow:

$$\begin{aligned} R = r_i &= \left(\left(\mu_i, \theta_i\right), \left(\nu_i, \varphi_i\right)\right) = CFFWA\left(r_{i1}, r_{i2}, \ldots, r_{in}\right) \\ &= \left(\begin{array}{l} \left(\sqrt[3]{1-\prod_{j=1}^{n}\left(1-\mu_i^3\right)^{\omega_j}}, \sqrt[3]{1-\prod_{j=1}^{n}\left(1-\theta_i^3\right)^{\omega_j}}\right), \\ \left(\prod_{j=1}^{n} \nu_i^{\omega_j}, \prod_{j=1}^{n} \varphi_i^{\omega_j}\right) \end{array}\right) \end{aligned}$$

This action derives the collective overall preference value r_i for the alternative ∂_i, where $\omega_j = [\omega_1, \omega_2, \ldots, \omega_n]^T$ and $\omega_j \in [0, 1]$ are the weight vectors for the attributes.

Step 3: Use Definition 9 to calculate the scores $\Omega\,(r_i)$ for each ∂_i.

Step 4: Rank all the alternatives ∂_i and identify the optimal one.

B. PROCEDURE FOR CFFDWG

Step1. Apply CFFDWG operator on the matrix R_{t_k}:

$$\begin{aligned} R_{t_k} &= r_{ij(t_k)} = \left(\left(\mu_{ij(t_k)}, \theta_{ij(t_k)}\right), \left(\nu_{ij(t_k)}, \varphi_{ij(t_k)}\right)\right) \\ &= CFFDWG\left(r_{ij(t_1)}, r_{ij(t_2)}, \ldots, r_{ij(t_p)}\right) \\ &= \otimes_{k=1}^{p} F_{t_k} \lambda_{t_k} \end{aligned}$$

Apply Theorem 1 to above relationship:

$$r_{ij(t_k)} = \left(\begin{array}{c} \left(\prod_{k=1}^{p} \mu_{t_k}^{\lambda_{t_k}}, \prod_{k=1}^{p} \theta_{t_k}^{\lambda_{t_k}}\right), \\ \left(\begin{array}{c} \sqrt[3]{1-\prod_{k=1}^{p}\left(1-\nu_{t_k}^3\right)^{\lambda_{t_k}}}, \\ \sqrt[3]{1-\prod_{k=1}^{p}\left(1-\varphi_{t_k}^3\right)^{\lambda_{t_k}}} \end{array}\right) \end{array}\right)$$

This aggregation combines all CFF decision matrices R_{t_k}, as a collective decision matrix R.

Step 2: Apply CFFWG operator on collective CFF information given in matrix R as follows:

$$\begin{aligned} r_i &= \left(\left(\mu_i, \theta_i\right), \left(\nu_i, \varphi_i\right)\right) \\ &= CFFWG\left(r_{i1}, r_{i2}, \ldots, r_{in}\right) \\ &= \left(\begin{array}{c} \left(\prod_{j=1}^{n} \mu_i^{\omega_j}, \prod_{i=1}^{m} \theta_i^{\omega_j}\right), \\ \left(\begin{array}{c} \sqrt[3]{1-\prod_{j=1}^{n}\left(1-\nu_i^3\right)^{\omega_j}}, \\ \sqrt[3]{1-\prod_{j=1}^{n}\left(1-\varphi_i^3\right)^{\omega_j}} \end{array}\right)) \end{array}\right). \end{aligned}$$

This procedure yields the aggregated overall preference value r_i, for the alternative ∂_i. Here, $\omega_j = [\omega_1, \omega_2, \ldots, \omega_n]^T$, with $\omega_j \in [0, 1]$ represents the weight vectors corresponding to the attributes.

Step 3: Compute the scores $\Omega\,(r_i)$ of the overall CFF preference values r_i by using Definition 9.

Step 4: Rank all the alternatives ∂_i based on their compatibility with $\Omega\,(r_i)$, and choose the most optimal one.

C. ILLUSTRATIVE EXAMPLE

Increasing technological advances have resulted in numerous enhancements that are quite beneficial to the growth and development of enterprises. Biometric technology is a prominent form of technology that is currently gaining significant

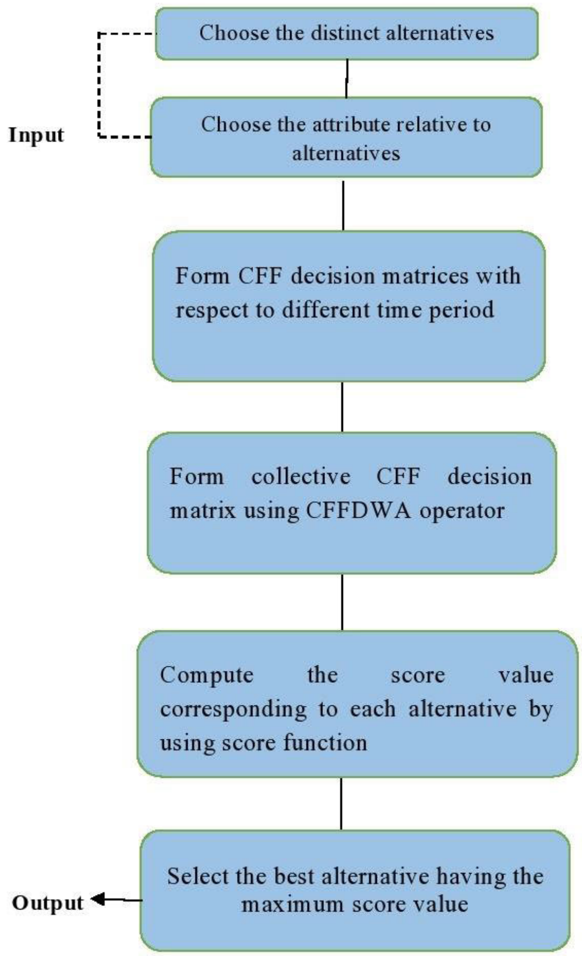

FIGURE 1. Sequential process for selecting the optimal BBAD model using the CFFDWA operator.

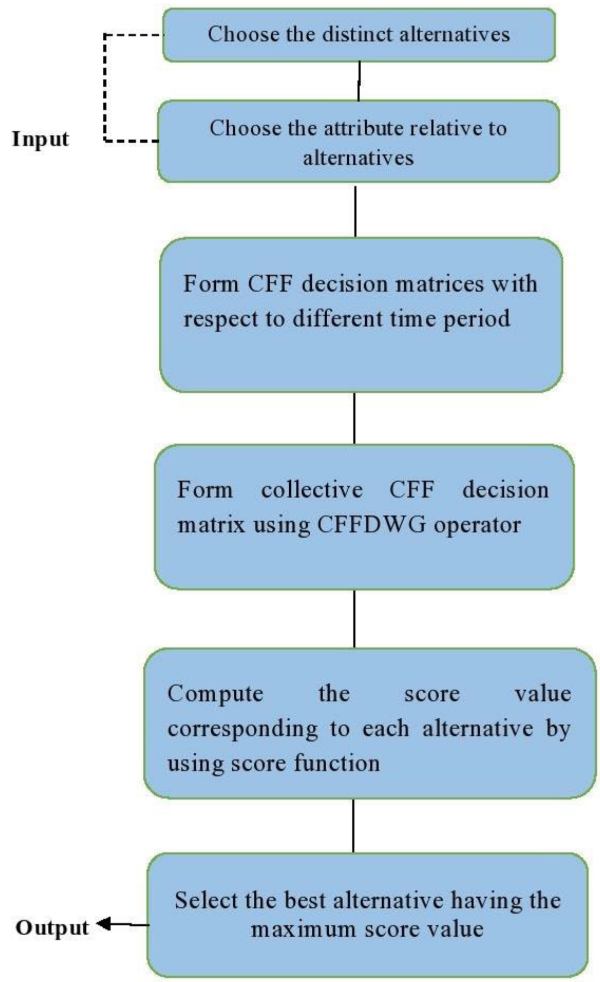

FIGURE 2. Sequential process for selecting the optimal BBAD model using the CFFDWG operator.

popularity. Biometric technologies are used for the purpose of identifying and verifying human traits. Each employee's unique fingerprint, hand shape, face shape, or iris shape is scanned by a biometric-based attendance device, which is gaining widespread recognition. These devices ensure that employees are unable to sign in for one another, preventing employee time theft. In view of the aforementioned characteristics of BBAD, Bharat Sanchar Nagar Limited, an agency headquartered in New Delhi, India, has made the strategic decision to implement BBAD across all of its country branches. In order to accomplish this objective, the Bharat Sanchar Nagar Limited authority gathers a group conference to evaluate and choose the most efficient model from a set of four alternatives. In this study, we describe a systematic methodology for the purpose of identifying the optimal alternative using CFF dynamic aggregation operators.

Let $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ be the set of alternatives to choose the most efficient variant of BBAD.

i. ∂_1 : CP plus
ii. ∂_2 : ESSLX990
iii. ∂_3 : T60
iv. ∂_4 : T 60

The BBAD model is assessed in accordance with four criteria. $\rho_1, \rho_2, \rho_3, \rho_4$.

i. ρ_1: User friendly
ii. ρ_2: Provision for data backup
iii. ρ_3: Battery backup
iv. ρ_4: Employee tracking via GPS.

In the context of CFFNs, we allocate distinct preferences to every alternative. The evaluation of the four potential alternatives $\partial_1, \partial_2, \partial_3, \partial_4$ can be conducted by utilizing the CFF data offered by the decision maker for the four attributes during the specified time intervals t_1, t_2 and t_3. This information is presented in Tables 1, 2 and 3.

Define the weight vector $\lambda_{t_k} = [0.39, 0.28, 0.33]^T$ corresponding to the time intervals t_k, where $k = 1, 2, 3$ and the weight vector $\omega_j = [0.32, 0.24, 0.16, 0.28]^T$ relating to the attributes ρ_j, where $j = 1, 2, 3, 4$.

The following steps addresses the MADM issue discussed previously in relation to the CFFDWA operator.

TABLE 1. Decision matrix R_{t_1}.

	ρ_1	ρ_2	ρ_3	ρ_4
∂_1	$((0.9,0.8),(0.6,0.6))$	$((0.8,0.6),(0.5,0.8))$	$((0.8,0.7),(0.7,0.8))$	$((0.6,0.8),(0.9,0.5))$
∂_2	$((0.6,0.6),(0.8,0.6))$	$((0.5,0.7),(0.5,0.8))$	$((0.6,0.6),(0.9,0.7))$	$((0.7,0.5),(0.4,0.7))$
∂_3	$((0.8,0.4),(0.6,0.8))$	$((0.7,0.6),(0.6,0.7))$	$((0.9,0.6),(0.6,0.5))$	$((0.8,0.4),(0.6,0.9))$
∂_4	$((0.7,0.8),(0.7,0.4))$	$((0.6,0.8),(0.9,0.6))$	$((0.7,0.8),(0.8,0.6))$	$((0.5,0.6),(0.8,0.7))$

TABLE 2. Decision matrix R_{t_1}.

	ρ_1	ρ_2	ρ_3	ρ_4
∂_1	$((0.8,0.7),(0.5,0.5))$	$((0.9,0.5),(0.6,0.7))$	$((0.9,0.6),(0.6,0.8))$	$((0.7,0.8),(0.8,0.5))$
∂_2	$((0.5,0.6),(0.7,0.5))$	$((0.6,0.7),(0.4,0.7))$	$((0.7,0.6),(0.8,0.6))$	$((0.6,0.4),(0.4,0.6))$
∂_3	$((0.7,0.4),(0.5,0.7))$	$((0.8,0.5),(0.6,0.6))$	$((0.8,0.7),(0.7,0.4))$	$((0.7,0.4),(0.7,0.9))$
∂_4	$((0.6,0.8),(0.7,0.3))$	$((0.7,0.7),(0.8,0.3))$	$((0.6,0.8),(0.7,0.6))$	$((0.5,0.7),(0.9,0.7))$

TABLE 3. Decision matrix R_{t_1}.

	ρ_1	ρ_2	ρ_3	ρ_4
∂_1	$((0.8,0.8),(0.5,0.6))$	$((0.9,0.6),(0.5,0.8))$	$((0.8,0.6),(0.5,0.9))$	$((0.6,0.7),(0.9,0.5))$
∂_2	$((0.6,0.7),(0.8,0.5))$	$((0.6,0.6),(0.5,0.8))$	$((0.7,0.5),(0.8,0.6))$	$((0.7,0.4),(0.3,0.7))$
∂_3	$((0.8,0.3),(0.5,0.8))$	$((0.8,0.4),(0.7,0.8))$	$((0.9,0.7),(0.6,0.5))$	$((0.8,0.6),(0.6,0.8))$
∂_4	$((0.7,0.7),(0.6,0.4))$	$((0.7,0.8),(0.9,0.3))$	$((0.8,0.7),(0.7,0.6))$	$((0.6,0.7),(0.5,0.6))$

Step 1: Apply the CFFDWA operator to consolidate all the CFF decision matrices R_{t_k} into a unified CFF decision matrix R displayed in Table 4.

Step 2: Use the CFFWA operator on the values of the matrix R to obtain the overall values of all the alternatives.

$$\partial_1 = ((0.816, 0.723), (0.622, 0.627))$$
$$\partial_2 = ((0.622, 0.662), (0.561, 0.640))$$
$$\partial_3 = ((0.797, 0.513), (0.597, 0.718))$$
$$\partial_4 = ((0.652, 0.749), (0.734, 0.477))$$

Step 3: Determine the ranking of all alternatives by calculating the scores $\Omega(\partial_1)$, $\Omega(\partial_2)$, $\Omega(\partial_3)$ and $\Omega(\partial_4)$, of the CFF values:

$$\Omega(\partial_1) = 0.282$$
$$\Omega(\partial_2) = 0.067$$
$$\Omega(\partial_3) = 0.082$$
$$\Omega(\partial_4) = 0.152$$

Step 4: Since $\Omega(\partial_1) > \Omega(\partial_4) > \Omega(\partial_3) > \Omega(\partial_2)$, as a result, the following is the alternate ranking order:

$$\partial_1 > \partial_4 > \partial_3 > \partial_2$$

Hence CP Plus is the best model of BBAD. Figure 1 illustrates the sequential process for selecting the optimal BBAD model using the CFFDWA operator.

In a similar vein, the above-mentioned MADM problem is resolved as follows within the context of the CFFDWG operator:

Step 1: Apply the *CFFDWG* operator to consolidate all the CFF decision matrices R_{t_k}, in a unified CFF decision matrix R, as delineated in Table 5.

Step 2: Use the CFFWG operator on the values of the matrix R to obtain the overall values of all alternatives ∂_i.

$$\partial_1 = ((0.775, 0.694), (0.716, 0.708))$$
$$\partial_2 = ((0.607, 0.566), (0.716, 0.649))$$
$$\partial_3 = ((0.783, 0.460), (0.610, 0.788))$$
$$\partial_4 = ((0.629, 0.735), (0.787, 0.609))$$

Step 3: Determine the ranking of all alternatives by calculating the scores $\Omega(\partial_1)$, $\Omega(\partial_2)$, $\Omega(\partial_3)$ and $\Omega(\partial_4)$, of the CFF values.

$$\Omega(\partial_1) = 0.123$$
$$\Omega(\partial_2) = -0.077$$
$$\Omega(\partial_3) = -0.015$$
$$\Omega(\partial_4) = 0.027$$

Step 4: Since $\Omega(\partial_1) > \Omega(\partial_4) > \Omega(\partial_3) > \Omega(\partial_2)$, as a result, the following is the alternate ranking order:

$$\partial_1 > \partial_4 > \partial_3 > \partial_2$$

Hence CP Plus is the best model of BBAD. Figure 2 illustrates the sequential process for selecting the optimal BBAD model using the CFFDWG operator.

D. COMPARATIVE ANALYSIS

In this discussion, we conduct a comparative analysis to evaluate the effectiveness of the proposed techniques. The

TABLE 4. Decision matrix R using *CFFDWA*.

	ρ_1	ρ_2	ρ_3	ρ_4
∂_1	$\begin{pmatrix}(0.849,0.777),\\(0.536,0.570)\end{pmatrix}$	$\begin{pmatrix}(0.870,0.576),\\(0.526,0.770)\end{pmatrix}$	$\begin{pmatrix}(0.836,0.644),\\(0.599,0.831)\end{pmatrix}$	$\begin{pmatrix}(0.633,0.772),\\(0.870,0.500)\end{pmatrix}$
∂_2	$\begin{pmatrix}(0.576,0.638),\\(0.770,0.536)\end{pmatrix}$	$\begin{pmatrix}(0.566,0.672),\\(0.469,0.770)\end{pmatrix}$	$\begin{pmatrix}(0.666,0.571),\\(0.837,0.637)\end{pmatrix}$	$\begin{pmatrix}(0.676,0.445),\\(0.363,0.670)\end{pmatrix}$
∂_3	$\begin{pmatrix}(0.777,0.373),\\(0.536,0.770)\end{pmatrix}$	$\begin{pmatrix}(0.767,0.522),\\(0.631,0.700)\end{pmatrix}$	$\begin{pmatrix}(0.879,0.866),\\(0.626,0.469)\end{pmatrix}$	$\begin{pmatrix}(0.777,0.489),\\(0.626,0.865)\end{pmatrix}$
∂_4	$\begin{pmatrix}(0.676,0.772),\\(0.665,0.369)\end{pmatrix}$	$\begin{pmatrix}(0.666,0.777),\\(0.870,0.393)\end{pmatrix}$	$\begin{pmatrix}(0.720,0.772),\\(0.737,0.600)\end{pmatrix}$	$\begin{pmatrix}(0.528,0.666),\\(0.708,0.665)\end{pmatrix}$

TABLE 5. Decision matrix R using *CFFDWG*.

	ρ_1	ρ_2	ρ_3	ρ_4
∂_1	$\begin{pmatrix}(0.837,0.770),\\(0.544,0.576)\end{pmatrix}$	$\begin{pmatrix}(0.859,0.570),\\(0.533,0.777)\end{pmatrix}$	$\begin{pmatrix}(0.826,0.637),\\(0.622,0.879)\end{pmatrix}$	$\begin{pmatrix}(0.626,0.765),\\(0.879,0.500)\end{pmatrix}$
∂_2	$\begin{pmatrix}(0.570,0.631),\\(0.773,0.544)\end{pmatrix}$	$\begin{pmatrix}(0.558,0.665),\\(0.476,0.777)\end{pmatrix}$	$\begin{pmatrix}(0.659,0.564),\\(0.849,0.373)\end{pmatrix}$	$\begin{pmatrix}(0.670,0.436),\\(0.622,0.676)\end{pmatrix}$
∂_3	$\begin{pmatrix}(0.770,0.363),\\(0.544,0.777)\end{pmatrix}$	$\begin{pmatrix}(0.759,0.498),\\(0.638,0.720)\end{pmatrix}$	$\begin{pmatrix}(0.870,0.659),\\(0.633,0.633)\end{pmatrix}$	$\begin{pmatrix}(0.770,0.457),\\(0.635,0.875)\end{pmatrix}$
∂_4	$\begin{pmatrix}(0.670,0.765),\\(0.672,0.377)\end{pmatrix}$	$\begin{pmatrix}(0.659,0.770),\\(0.879,0.472)\end{pmatrix}$	$\begin{pmatrix}(0.700,0.765),\\(0.745,0.792)\end{pmatrix}$	$\begin{pmatrix}(0.531,0.659),\\(0.792,0.672)\end{pmatrix}$

TABLE 6. Aggregated values obtained from existing operators.

	CIFDWA [33]	CIFDWG [33]
∂_1	$((0.81,0.718),(0.622,0.627))$	$((0.774,0.693),(0.692,0.675))$
∂_2	$((0.619,0.590),(0.561,0.639))$	$((0.607,0.565),(0.655,0.664))$
∂_3	$((0.795,0.492),(0.596,0.718))$	$((0.782,0.459),(0.607,0.770))$
∂_4	$((0.647,0.748),(0.733,0.477))$	$((0.629,0.734),(0.778,0.526))$

TABLE 7. Scores and ranking of alternatives using existing and newly designed approaches.

Methods	$\Omega(\partial_1)$	$\Omega(\partial_2)$	$\Omega(\partial_3)$	$\Omega(\partial_4)$	Ranking Order
CIFDWA [33]	0.391	0.178	0.223	0.329	$\partial_1 > \partial_4 > \partial_3 > \partial_2$
CIFDWG [33]	0.317	0.125	0.139	0.148	$\partial_1 > \partial_4 > \partial_3 > \partial_2$
CFFDWA	0.282	0.067	0.082	0.152	$\partial_1 > \partial_4 > \partial_3 > \partial_2$
CFFDWG	0.123	−0.077	−0.015	0.027	$\partial_1 > \partial_4 > \partial_3 > \partial_2$

following key points demonstrate how the proposed aggregation operators are superior to those defined in [15], [22], [23], [24], [25], [26], [27], [28], [36], [37], [38], [39], [40], [41], [42], and [43].

i. The aggregation operators defined in the FF knowledge framework [15], [22], [23], [24] cannot be used to analyze the data presented in Tables 1-3. These operators do not include phase terms, resulting in the loss of a significant amount of data. The enhancement of methodologies presented in this research is facilitated by the advancements made within the dynamic CFF environment. This approach involves various time periods and facilitates a more accurate assessment of the data under consideration.
ii. The techniques proposed in reference [42] are inappropriate for assessing the data contained in Tables 1-3 due to their particular limitations when dealing with time-dependent decision-making challenges. Because of the dynamic CFF environment, the suggested approaches work exceptionally well. They accommodate many time periods and provide for a more accurate assessment of the data being considered.
iii. The aggregation operators created for CIFSs and CPFSs have several limitations. These operators failed to consider the time periods, making them unable to analyze the data in Tables 1-3. Additionally, the absence of time factor results in significant data loss. The suggested strategies are advantageous since they have been developed within the context of dynamic CFF knowledge, using temporal intervals to appropriately assess the data under consideration.
iv. The techniques elucidated in this article have a broader purview than the approaches devised for dynamic IF environments [25], [26], [27], [28]. Membership and nonmembership degrees are less efficient in IF contexts than Figure 1 Sequential process for selecting the optimal BBAD model using the CFFDWA operator in CFF contexts. Consequently, dynamic CFFS can identify and resolve uncertainties through a broader variety of approaches than dynamic IFS.

v. To tackle the MADM problem described earlier, we assess the efficacy and reliability of our proposed operators in comparison to operators in the CIF environment (See Tables 6 and 7).

CFF dynamic aggregation operators, according to the article, provide a more flexible and effective approach to integrating information than CIF aggregation operators. The CFFS has a more comprehensive structure than the CIFS because it satisfies the conditions, $\boldsymbol{\mu}^3+\boldsymbol{\nu}^3 \leq \mathbf{1}$ and $\boldsymbol{\theta}^3+\boldsymbol{\varphi}^3 \leq \mathbf{1}$. Consequently, it offers greater adaptability in resolving decision-making issues that encompass ambiguity.

VI. CONCLUSION

This study investigates MADM using CFF dynamic knowledge. In order to address the complexities of MADM problems, we have devised two novel aggregation operators: the CFFDWA and CFFDWG. A novel scoring system has been developed to prioritize and choose the optimal choice. We have proved several essential characteristics of these operators. Moreover, a systematic procedure to solve MADM problems in the framework of CFF dynamic aggregation operators has been presented. We have also illustrated the realistic efficiency of these operators by proposing techniques to tackle real-world difficulties in multi-attribute decision making. Additionally, this work has provided the best method for choosing the BBAD model in light of the newly described dynamic aggregating operators. The result of this trial shows that CP PLUS is the best model. Finally, a comparative study has been performed to showcase the reliability and effectiveness of the suggested strategies in contrast to current approaches.

A. A LIMITATION OF STUDY

The limitation of the present research emerges when the sum of the cubes representing degrees of membership and non-membership surpasses 1. Furthermore, in the case of decision-making problems involving membership, neutral, and non-membership degrees, the CFFS framework fails.

B. FUTURE WORK

Our future studies will focus on addressing the limits of the current work by exploring recommended approaches for broader contexts, such as complex q-rung fuzzy sets, complex image fuzzy sets, and complex spherical fuzzy sets. Moreover, we will improve the validity and applicability of the suggested techniques by applying them to different dynamic decision-making scenarios such as flexible financial strategies, real-time monitoring of online social media activities, dynamic assessment of military management, secret shortlisting procedures, addressing the energy crisis in developing countries, and resolving timedependent MADM challenges.

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