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RESEARCH ARTICLE

Development $p, q, r-$Spherical Fuzzy Einstein Aggregation Operators: Application in Decision-Making in Logo Design

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ABSTRACT $p, q, r-$spherical fuzzy ($p, q, r-$SF) sets are a significant advancement in fuzzy set (FS) theory, providing an effective way to describe hesitation inside a set. This development goes beyond membership degrees (MDs) by incorporating three parameters (p, q and r) that shape and define the spread of FS. The presence of these parameters makes $p, q, r-$SF sets ideal for dealing with complicated decision-making (DM) scenarios in a wide range of applications. In this paper, we proposed a new reliable method for smart DM in logo design projects that uses $p, q, r-$SF Einstein aggregation operators (AOs). First, we present the $p, q, r-$SF Einstein operators, which are accomplished of capturing the relationship and interrelationship of numerous criteria while making decisions on logo design, including originality, relevance, beauty, and usability. Next, we use the proposed framework, a multi-criteria decision analysis tool, to rank the choices. We demonstrate the practical application and efficiency of our method in a scenario in which a team of three designers is entrusted with developing five different logo styles for diverse enterprises. Moreover, we compare proposed procedure with some existing methods, demonstrating its advantages in terms of precision and reliability.

INDEX TERMS $p, q, r-$spherical fuzzy set, Einstein operations, aggregation operators, decision-making, optimization.

I. INTRODUCTION

In the contemporary landscape, decision-making is essential in both personal and professional arenas, with unequaled importance in the dynamic fabric of modern life. Individual decisions concerning schooling, professional routes, and lifestyles have long-term consequences, illustrating empowerment and determining destinies. In the corporate world, strategic decisions affect market dynamics, product innovation, and resource allocation, distinguishing strong businesses from those that are struggling. Governments are confronted with decisions that shape the fates of nations, particularly in the face of global issues such as climate change and epidemics. The digital age amplifies the complexity with an influx of information, necessitating adeptness in data-driven decision-making. Moreover, problem-solving and decision-making are closely related; decision-making serves as a bridge between identifying problems and putting effective solutions in place. Effective decision-making is an art and science that is fundamental to navigating the complex difficulties of the modern world [1], [2], [3], [4], [5].

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Zadeh developed fuzzy sets (FSs) [6], which play an important role in multi-criteria decision making (MCDM) by providing an adaptable basis for dealing with imprecision and ambiguity. FSs make it possible to describe criteria with varying degrees of membership in MCDM choices, which entail numerous conflicting criteria and the need for a nuanced depiction of ambiguity. Fuzzy weighing mechanisms make it easier to give uncertain weights to criteria, while fuzzy aggregation operators, such as weighted sums, allow decision-makers to integrate fuzzy information while accounting for the inherent uncertainty in decision-making [7], [8], [9], [10]. Fuzzy decision-making models, such as fuzzy TOPSIS and fuzzy AHP, use fuzzy sets to rank and choose alternatives according to a variety of criteria [11], [12], [13], [14]. In addition, fuzzy sets are useful in dealing with linguistic factors as they allow for the introduction of qualitative judgments and subjective preferences into decision-making processes. Overall, using fuzzy sets improves MCDM's flexibility and resilience in dealing with the complexities and uncertainties that arise in real-world decision scenarios [15], [16], [17], [18]. FSs represent partial membership using a single degree of membership (MD), whereas Intuitionistic Fuzzy Sets (IFSs) [19] introduce the concept of hesitation, which provides a more elaborate representation of uncertainty by taking into account both membership and non-membership, as well as a degree of hesitation. In IFS, the sum of the degrees of membership (m) and non-membership (n) must be less than or equal to one i.e., $m + n \preccurlyeq 1$. In some real-world scenarios, the conditions for IFSs may not be satisfied. For example, when a decision-maker assesses an item with $m = 0.50$ and $n = 0.70$, it is clear that the total of these values $(0.50 + 0.70)$ is more than 1. As a result, intuitionistic fuzzy (IF) types sets cannot properly handle these specific classes of information. This emphasizes the need to explore alternate methodologies or modifications when dealing with evaluations that fall beyond the established parameters of IFSs in actual decision-making. Yager [20] proposed Pythagorean fuzzy sets (PFSs) as an extension of IFSs to handle situations in which IFS restrictions may not be relevant. PFSs require the square sum of the m and the square of the n to be less than or equal to one i.e., $m^2 + n^2 \preccurlyeq 1$. Furthermore, Senapati and Yager developed Fermatean fuzzy sets (FFSs), which include the requirement that the cube sum of the m and the cube of n be less than or equal to one i.e., $m^3 + n^3 \preccurlyeq 1$. Yager expanded on the concepts of IFSs, PFSs, and FFSs to present $q-$rung orthopair fuzzy sets ($q-$ROFSs). In this generalization, a condition is applied that requires the q^{th} power sum of the m and the q^{th} power of the n to be less than or equal to one, where q is less than or equal to 1 i.e., $m^q + n^q \preccurlyeq 1$. In $q-$ROFSs, there is a single parameter, q, that regulates the influence of both MD and NMD. However, some situations may need the use of various parameter values to control the impact of membership degrees. In such cases, the adaptive nature of $q-$ROFSs may be insufficient to handle the issue. To handle such scenarios, Seikh and Mandal [21] developed $p, q-$Quasirung Orthopair Fuzzy Sets ($p, q-$QOFs), which extended the notion of $q-$ROFSs. In $p, q-$QOFSs, two parameters, p and q, are used to separately control the impact of the degree of MD and NMD. The condition states that the p^{th} power sum of m and the q^{th} power of n must be less than or equal to one ($n^p + n^q \preccurlyeq 1$). This addition provides a more flexible framework, allowing decision-makers to use alternative parameter values for more precise control over the influence MD and NMDs in various decision-making circumstances. FSs sets and their extension are presented in Figure 1.

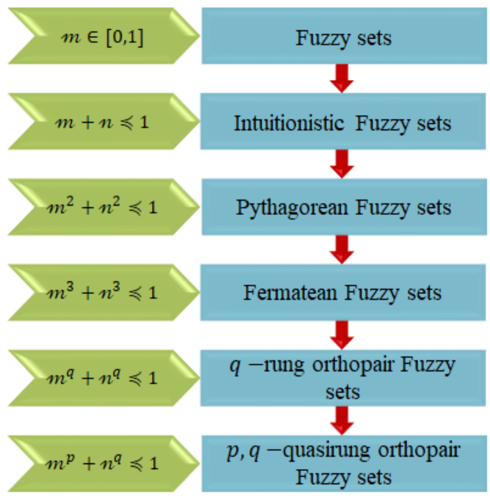

FIGURE 1. **Fuzzy sets and their extensions.**

In the above-mentioned communication, it becomes apparent that all investigations focus on the MD and NDM, ignoring the importance of the natural membership degree (NAMD). However, it should be highlighted that the NAMD plays an important part in appraising an item throughout the decision-making process. Cuong and Kreinovich [22] developed picture fuzzy sets (PiFSs), which incorporated the terms of MD, NMD, and NADM. This system is considered to be more realistic and adaptable than IFSs. The terms of PiFSs are subject to the requirement that the sum of DM, NMD, and NAMD (s) be less than or equal to 1 i.e., $m + s + n \preccurlyeq 1$. Gündoğdu and Kahraman [23] proposed spherical fuzzy sets (SFSs), which are an extension of PiFSs. In SFSs, the square sum of MD, NDM, and NADM must be less than or equal to one i.e., $m^2 + s^2 + n^2 \preccurlyeq 1$. This expansion extends the concept of PiFSs by providing a framework that takes into account a broader number of criteria for a more nuanced depiction of uncertainty in decision-making processes. Mahmood et al. [24] enhanced the concept of spherical fuzzy sets by introducing $T-$spherical fuzzy sets ($T-$SFSs). In $T-$SFSs, the parameter 't' can be adjusted based on decision-makers' information, ensuring that the t^{th} power of MD, NDM, and NAMD are all less than or equal to 1. In $T-$spherical fuzzy environment the decision-makers are bound to use the same value of parameter t for all terms of membership degrees. However, in some situation, the decision makers may need to set different values for MD, NMD, and NAMD. This situation cannot be deal with the $T-$SFSs. Therefore, Rahim et al. [25] proposed $p, q, r-$spherical fuzzy sets ($p, q, r-$SFSs) which the extension of $T-$SFSs. In $p, q, r-$SFSs, three parameters play distinct roles: p governs the influence of MD, q regulates the impact of NMD), and r controls the influence of the natural NAMD within the $p, q, r-$SFSs such that $\mathscr{m}^p + \mathscr{s}^r + \mathscr{n}^q \preccurlyeq 1$ where $p, q \succcurlyeq 1$ and $r = \max(p, q)$. Figure 2 represent the special cases of $p, q, r-$SFSs. Some special cases of $p, q, r-$SFSs are shown in Figure 2.

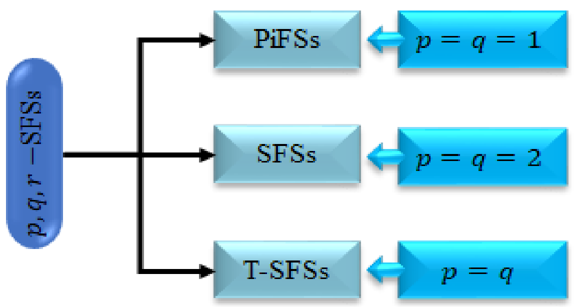

FIGURE 2. $p, q, r-$**SFSs and their special cases.**

A. AGGREGATION OPERATORS

AOs are algorithms for computation that integrate several source data and provide a single outcome. They are essential in a variety of professions, especially quantitative information, computational science, and cognitive studies. These mathematical operators commonly utilize functions such as arithmetic mean, weighted mean, maximum, and minimum, each of which serves a distinct purpose based on the qualities of the aggregated result. AOs are used in decision-making, data analysis, and fuzzy reasoning, and can vary from simple summation and product operators to more complicated approaches like ordered weighted averaging and limited sum operators. They offer a broad toolbox for combining information or preferences from several sources. The choice of an operator is contingent upon the nature of the data and the objectives of the research, highlighting their significance in several computational and decision-support contexts. Scholars have created many AOs that are appropriate for certain circumstances. For example, Wang and Liu [26] presented a collection of Einstein geometric AOs to handle cases when the given inputs take the form of IF values. Khan et al. [27] proposed picture fuzzy Einstein-weighted and Einstein-ordered weighted operators, examined their characteristics and presented a numerical illustration of how to use them in a group decision-making problem. Garg [28] presented a series of Einstein AOs for Pythagorean fuzzy information to handle decision-making problems. Farid et al. [29] proposed Einstein prioritized weighted averaging and geometric operators as a reliable method for tackling Multiple Criteria Group Decision Making (MCGDM) issues. Khan et al. [30] presented a set of aggregation operators (AOs) tailored for spherical fuzzy rough sets, to successfully consolidate information in the context of spherical fuzzy roughness. Readers are encouraged to review the references [31], [32], [33], [34] for a more in-depth understanding of Einstein-based aggregation operators.

B. RELATED WORK

Product development projects involve the process of creating and bringing new products of features to the market. Successful product development is a combination of creativity, market research, technical expertise, and effective project management. Product development projects are complex endeavors that require a myriad of decisions to be made at various stages of the process. From the conceptualization phase to market launch, decision makers navigate uncertainties, ambiguities, and the ever-changing landscape of consumer preferences. Traditional decision-making models often struggle to capture and incorporate the inherent vagueness and imprecision present in the data and information associated with product development. The integration of fuzzy sets into decision making processes offers a promising avenue to address the uncertainties inherent in product development. Büyüközkan and Feyzıoğlu [35] developed fuzzy logic-based decision-making approach for new product development. Furthermore, they also proposed products development projects using artificial intelligence and fuzzy logic [36].

Jung and Seo [37] employed the AHP to evaluate various projects. In the intricate landscape of market competition, numerous fuzzy decision-making methods have been applied to the selection of R & D projects. For instance, Hassanzadeh et al. [38] introduced a fuzzy payoff method for effective assessment of R & D projects. Building upon this, Collan and Luukka [39] extended the fuzzy payoff method, introducing four innovative variants related to fuzzy TOPSIS. Taylan et al. [40] developed a project selection and risk assessment framework that combines fuzzy AHP and fuzzy TOPSIS methods. Wu et al. [41] took a distinct approach, integrating fuzzy MADM with fuzzy multi-objective programming to introduce a comprehensive framework. In a unique contribution, Relich and Pawlowski [42] proposed an innovative model for project portfolio selection, incorporating a fuzzy weighted average approach. Similarly, Karasakal and Aker [43] assessed R & D projects using an extended MADM method integrated with DEA.

C. GAP AND MOTIVATIONS

In order to facilitate decision-making that combines adaptability, robustness in the context of uncertainty, and the ability to accept weighted aggregation, Einstein AOs offer a flexible and useful framework. They are widely used in fields like artificial intelligence and decision science because of their adaptability to a wide range of data types, including FSs and their expansion, as well as the addition of duality.

Because these operators are mathematically simple, they are easy to use and comprehend. Furthermore, their utility in capturing complex relationships and preferences is increased by their capacity to preserve sequence information and perform well with uncertainty. However, the hierarchical structure of $p, q, r-$SF sets is more adjustable and parametric. Unfortunately, there hasn't been much research done on aggregation operators till recently, particularly those built for $p, q, r-$SF datasets. Given the importance of AOs as fundamental mathematical tools in decision-making, this study addresses a need by introducing a set of Einstein AOs for aggregating preferences given as $p, q, r-$SF numbers ($p, q, r-$SFNs). These operators aim to successfully combine multiple preferences and devise methods for making decisions in scenarios involving $p, q, r-$SF sets. Given the importance of aggregation operators as critical mathematical tools in decision-making issues, this study fills a vacuum by providing a set of Einstein aggregation operators designed for aggregating preferences given as $p, q, r-$SFNs. These operators aim to effectively integrate different preferences and develop methods for making decisions in scenarios involving $p, q, r-$SF sets.

This article motivated by three key points:

a) To define Einstein operations for $p, q, r-$spherical fuzzy numbers.
b) To propose some AOs based on these operational laws such as $p, q, r-$spherical fuzzy Einstein weighted averaging ($p, q, r-$SFEWA) and $p, q, r-$spherical fuzzy Einstein weighted geometric ($p, q, r-$SFEWG) operators to aggregate $p, q, r-$spherical fuzzy information.
c) To construct MCDM approach based on the proposed AOs and applied to product development project selection problem.

D. CONTRIBUTIONS OF THE PROPOSED STUDY

The paper aims to enhance the resolution of product development project selection by addressing the evaluation of five different product developments projects as QuantumTech Xperience, EcoHarmony Solutions, StellarGrowth Innovations, Precision Craft Dynamics, EliteVista Creations. The key objectives include formulating Einstein operational laws for $p, q, r-$spherical fuzzy Einstein numbers, including addition, multiplication, and scalar multiplication. The development of $p, q, r-$SFEWA and $p, q, r-$SFEWG operators is proposed to effectively aggregate $p, q, r-$spherical fuzzy information, considering inter$-$attribute relationships. The paper analyzes properties of these AOs and presents some numerical examples of these operators. We constructed a novel MADM approach based on $p, q, r-$SFEWA and $p, q, r-$SFEWG to deal with real-life complex decision-making problems. The application of the proposed approach to product development project selection is proposed to validate its effectiveness and superiority against alternative methods. The proposed $p, q, r-$spherical fuzzy Einstein numbers of algorithms offer completeness and simplicity compared to existing rules. Significantly, the novel MADM method extends beyond product selection, proving versatile for product development teams aiming to enhance their decision-making process and maximize the likelihood of a successful product launch.

E. PAPER OUTLINE

The article is organized as follows:

In Section II, we presented some basic definitions related to the proposed work. In Section III, we defined some operational laws for $p, q, r-$SFSs using Einstein sum and Einstein product. Also, a series of AOs and their properties are presented to aggregate $p, q, r-$SF information. Section IV outlines a novel approach built on the proposed AOs. Demonstrating the efficacy and flexibility of our approach through a practical example, Section V provides empirical evidence. Finally, Section VI, serves as the conclusion of our proposed work.

II. PRELIMINARIES

In this section we have address the fundamental concept.

A. IFSS

Definition 1: [19] Let $\mathfrak{B}$ be a universal set. An IFS $\mathcal{J}$ over an element $\mathscr{b} \in \mathfrak{B}$ can be expressed as follows:

$$\mathcal{J} = \left\{\mathscr{b}, \langle m_{\mathcal{J}}(\mathscr{b}), n_{\mathcal{J}}(\mathscr{b})\rangle | \mathscr{b} \in \mathfrak{B}\right\} \tag{1}$$

where $m_{\mathcal{J}}(\mathscr{b}), n_{\mathcal{J}}(\mathscr{b}) \in [0, 1]$ represent the MD and NMD of an element $\mathscr{b} \in \mathfrak{B}$ such that $m_{\mathcal{J}}(\mathscr{b}) + n_{\mathcal{J}}(\mathscr{b}) \preccurlyeq 1$. The degree of hesitancy between them is calculated as:

$$\Pi_{\mathcal{J}}(\mathscr{b}) = 1 - (m_{\mathcal{J}}(\mathscr{b}) + \Theta_{\mathcal{J}}(\mathscr{b})) \tag{2}$$

Definition 2: [44] Let $\mathcal{J}_1 = \left(m_{\mathcal{J}_1}, n_{\mathcal{J}_1}\right)$, $Q_2 = \left(m_{\mathcal{J}_2}, n_{\mathcal{J}_2}\right)$ and $\mathcal{J} = \left(m_{\mathcal{J}}, n_{\mathcal{J}}\right)$ be any three IFNs. Then, the operational laws between these IFNs are defined as follows:

$\mathcal{J}^C = \left(n_{\mathcal{J}}, m_{\mathcal{J}}\right)$,
$\mathcal{J}_1 \subseteq \mathcal{J}_2$ if and only if $m_{\mathcal{J}_1} \preccurlyeq m_{\mathcal{J}_2}$ and $n_{\mathcal{J}_1} \succcurlyeq n_{\mathcal{J}_2}$,
$\mathcal{J}_1 = \mathcal{J}_2$ if $\mathcal{J}_1 \subseteq \mathcal{J}_2$ and $\mathcal{J}_2 \subseteq \mathcal{J}_1$,
$\mathcal{J}_1 \oplus \mathcal{J}_2 = \left(m_{\mathcal{J}_1} + m_{\mathcal{J}_2} - m_{\mathcal{J}_1} m_{\mathcal{J}_2}, n_{\mathcal{J}_1} n_{\mathcal{J}_2}\right)$,
$\mathcal{J}_1 \otimes \mathcal{J}_2 = \left(m_{\mathcal{J}_1} m_{\mathcal{J}_2}, n_{\mathcal{J}_1} + n_{\mathcal{J}_2} - n_{\mathcal{J}_1} n_{\mathcal{J}_2}\right)$,
$\zeta \mathcal{J} = \left(1 - \left(1 - m_{\mathcal{J}}\right)^{\zeta}, n_{\mathcal{J}}^{\zeta}\right)$,
$\mathcal{J}^{\zeta} = \left(m_{\mathcal{J}}^{\zeta}, 1 - \left(1 - n_{\mathcal{J}}\right)^{\zeta}\right)$.

Where $\mathcal{J}^C$ represent the complement of IFN $\mathcal{J}$ and ζ is any positive real number.

Definition 3: [44] Let $\mathcal{J} = \left(m_{\mathcal{J}}, n_{\mathcal{J}}\right)$ be an IFN. The score function of $\mathcal{J}$ is defined as follows:

$$Sco(\mathcal{J}) = m_{\mathcal{J}} - n_{\mathcal{J}} \tag{3}$$

where $-1 \preccurlyeq Sco(\mathcal{J}) \preccurlyeq 1$.

Definition 4: [44] Let $= \left(m_{\mathcal{J}}, n_{\mathcal{J}}\right)$ be an IFN. The accuracy function of $\mathcal{J}$ is defined as follows:

$$Acc(\mathcal{J}) = m_{\mathcal{J}} + n_{\mathcal{J}} \tag{4}$$

where $0 \preccurlyeq Acc(\mathcal{J}) \preccurlyeq 1$.

Definition 5: [44] Let $\mathcal{J}_1 = \left(m_{\mathcal{J}_1}, n_{\mathcal{J}_1}\right)$ and $\mathcal{J}_2 = \left(m_{\mathcal{J}_2}, n_{\mathcal{J}_2}\right)$ are two IFN, then

If $Sco\left(\mathcal{J}_1\right) \prec Sco\left(J_1\right)$ then $\mathcal{J}_1 \prec \mathcal{J}_2$,
If $Sco\left(\mathcal{J}_1\right) \succ Sco\left(\mathcal{J}_1\right)$ then $\mathcal{J}_1 \succ \mathcal{J}_2$,
If $Sco\left(\mathcal{J}_1\right) = Sco\left(\mathcal{J}_1\right)$and,
If $Acc\left(\mathcal{J}_1\right) \prec Acc\left(\mathcal{J}_1\right)$ then $\mathcal{J}_1 \prec \mathcal{J}_2$,
If $Acc\left(\mathcal{J}_1\right) \succ Acc\left(\mathcal{J}_1\right)$ then $\mathcal{J}_1 \succ \mathcal{J}_2$,
If $Acc\left(\mathcal{J}_1\right) = Acc\left(\mathcal{J}_1\right)$ then $\mathcal{J}_1 \sim \mathcal{J}_2$.

B. PIFSS

Definition 6: [22] For any universal set $\mathfrak{B}$, a PiFS $\mathcal{P}$ can be defined as follows:

$$\mathcal{P} = \left\{b, \langle m_{\mathcal{P}}(b), s_{\mathcal{P}}(b), n_{\mathcal{P}}(b)\rangle | b \in \mathfrak{B}\right\}$$

where $m_{\mathcal{J}}(b), s_{\mathcal{J}}(b)$ and $n_{\mathcal{J}}(x)$, represent positive, neutral and negative membership grade respectively, in element $b \in \mathfrak{B}$ such that $m_{\mathcal{P}}(b), s_{\mathcal{P}}(b), n_{\mathcal{P}}(b) \in [0, 1]$ and $m_{\mathcal{P}}(b) + s_{\mathcal{P}}(b) + n_{\mathcal{P}}(b) \preccurlyeq 1$. The triplet (m, s, n) is called picture fuzzy number (PiFN).

Definition 7: [22] Let $\mathcal{P} = (m, s, n)$ be PiFN then score function of $\mathcal{P}$ can be defined as:

$$Sco(\mathcal{P}) = m - s - n,$$

where $-1 \preccurlyeq Sco(\mathcal{P}) \preccurlyeq 1$. The accuracy function of PiFN $\mathcal{P}$ is defined as follows:

$$Acc(\mathcal{P}) = m + s + n, 0 \preccurlyeq Acc(\mathcal{P}) \preccurlyeq 1.$$

C. SFSS

Definition 8: [23] Let $\mathfrak{B}$ be a universal set, a SFS $\mathcal{S}$ can be symbolized as follows:

$$\mathcal{S} = \{b, \langle m_{\mathcal{S}}(b), s_{\mathcal{S}}(b), n_{\mathcal{S}}(b)\rangle | b \in \mathfrak{B}\} \tag{5}$$

In this context, $m_{\mathcal{S}} : \mathfrak{B} \to [0, 1]$ is the MD, $s_{\mathcal{S}} : \mathfrak{B} \to [0, 1]$ is the NEMD, $n_{\mathcal{S}} : \mathfrak{B} \to [0, 1]$ is the NMD,

which satisfies the condition $m_{\mathcal{S}}(b)^2 + s_{\mathcal{S}}(b)^2 + n_{\mathcal{S}}(b)^2 \leq 1$, for all $b \in \mathfrak{B}$. The triplet is said to be (m, s, n) is called spherical fuzzy number (SFN).

Definition 9: [23] Let $\mathcal{S} = (m, s, n)$ be a SFN then the score function of $\mathcal{S}$ defined as:

$$Sco(\mathcal{S}) = \frac{1 + m^2 - s^2 - n^2}{2} \tag{6}$$

where $-1 \preccurlyeq Sco(\mathcal{S}) \preccurlyeq 1$. The accuracy function of SFN is defined as follows: $Acc(\mathcal{S}) = m^2 + s^2 + n^2$ such that $0 \preccurlyeq Acc(\mathcal{S}) \preccurlyeq 1$.

D. T-SFSS

Definition 10: [24] Let $\mathfrak{B}$ be a universal set, a T-SFN $\mathcal{T}$ can be symbolized as follows:

$$\tau = \left\{b, \langle m_{\mathcal{T}}(b), s_{\mathcal{T}}(b), n_{\mathcal{T}}(b)\rangle | b \in \mathfrak{B}\right\} \tag{7}$$

In this context, $m_{\mathcal{T}} : \mathfrak{B} \to [0, 1]$ is the MD, $s_{\mathcal{T}} : \mathfrak{B} \to [0, 1]$ is the NEMD, $n_{\mathcal{T}} : \mathfrak{B} \to [0, 1]$ is the NMD of τ for an element b and satisfies the condition that $m_{\mathcal{T}}(b)^t, s_{\mathcal{T}}(b)^t, n_{\mathcal{T}}(b)^t \preccurlyeq 1$, for all $b \in \mathfrak{B}$. The triplet is said to be $(m_{\mathcal{T}}, s_{\mathcal{T}}, n_{\mathcal{T}})$ is called T-spherical fuzzy number (T-SFN).

Definition 11: [24] Let $\tau_1 = \left(m_{\mathcal{T}1}, s_{\mathcal{T}1}, n_{\mathcal{T}1}\right)$, $T_2 = \left(m_{\mathcal{T}2}, s_{\mathcal{T}2}, n_{\mathcal{T}2}\right)$ and $\tau = \left(m_{\mathcal{T}}, s_{\mathcal{T}}, n_{\mathcal{T}}\right)$ be any three T-SFN. Then, the operational laws between these T-SFNs are defined as follows:

1. $\mathcal{T}_1 \oplus \mathcal{T}_2 = \left(\begin{array}{l}\sqrt[t]{m_{\mathcal{T}1}^t + m_{\mathcal{T}2}^t - m_{\mathcal{T}1}^t m_{\mathcal{T}2}^t},\\ \sqrt[t]{s_{\mathcal{T}1}^t + s_{\mathcal{T}2}^t - s_{\mathcal{T}1}^t s_{\mathcal{T}2}^t}, n_{\mathcal{T}1} n_{\mathcal{T}2}\end{array}\right)$,
2. $\mathcal{T}_1 \otimes \mathcal{T}_2 = \left(\begin{array}{l}m_{\mathcal{T}1} m_{\mathcal{T}2}, s_{\mathcal{T}1} s_{\mathcal{T}2},\\ \sqrt[t]{n_{\mathcal{T}1}^t + n_{\mathcal{T}2}^t - n_{\mathcal{T}1}^t n_{\mathcal{T}2}^t}\end{array}\right)$,
3. $\zeta \mathcal{T} = \left(\begin{array}{l}\sqrt[t]{1 - \left(1 - m_{\mathcal{T}}^t\right)^\zeta},\\ \sqrt[t]{1 - \left(1 - s_{\mathcal{T}}^t\right)^\zeta}, n_{\mathcal{T}}^\zeta\end{array}\right)$,
4. $\mathcal{T}^\zeta = \left(\begin{array}{l}m_{\mathcal{T}}^\zeta, s_{\mathcal{T}}^\zeta,\\ \sqrt[t]{1 - \left(1 - n_{\mathcal{T}}^t\right)^\zeta},\end{array}\right)$.

Definition 12: [24] Let $\mathcal{T} = \left(m_{\mathcal{T}}, s_{\mathcal{T}}, n_{\mathcal{T}}\right)$ be a T-SFN then score function is defined as:

$$Sc(T) = \frac{1 + m_{\mathcal{S}}^t - s_{\mathcal{S}}^t - n_{\mathcal{S}}^t}{2} \tag{8}$$

In this context $-1 \preccurlyeq Sco(\mathcal{T}) \preccurlyeq 1$. The accuracy function of spherical fuzzy number is defined as follows:

$$Ac(\mathcal{T}) = m_{\mathcal{T}}^t + s_{\mathcal{T}}^t + n_{\mathcal{T}}^t \text{ such that } 0 \preccurlyeq Acc(\mathcal{T}) \preccurlyeq 1.$$

E. $p, q, r-$ SFSS

Definition 13: [25] Let $\mathfrak{B}$ be a universal set, a $p, q, r-$SFS Ψ can be symbolized as follows:

$$\Psi = \{b, \langle m_{\Psi}(b), s_{\Psi}(b), n_{\Psi}(b)\rangle | b \in \mathfrak{B}\} \tag{9}$$

where $m_{\Psi} : \mathfrak{B} \to [0, 1]$ is the MD, $s_{\Psi} : \mathfrak{B} \to [0, 1]$ is the NEMD, $n_{\Psi} : \mathfrak{B} \to [0, 1]$ is the NMD, which satisfies the condition $m_{\Psi}(b)^p + s_{\Psi}(b)^r + n_{\Psi}(b)^q\, 1$, for all $b \in \mathfrak{B}$. The triplet is said to be $(m_{\Psi}, s_{\Psi}, n_{\Psi})$ is called $p, q, r-$spherical fuzzy number ($p, q, r-$SFN). Where p and q are any positive integers such that

- $p = q, p < q$ or $p > q$,
- $r = \max(p, q)$.

Definition 14: [25] Let $\Psi_1 = \left(m_{\Psi_1}, s_{\Psi_1}, n_{\Psi_1}\right)$, $\Psi_2 = \left(m_{\Psi_2}, s_{\Psi_2}, n_{\Psi_2}\right)$ and $\Psi = (m_{\Psi}, s_{\Psi}, n_{\Psi})$ be any three $p, q, r-$SFNs. Then

2. $\Psi_1 \oplus \Psi_2 = \left(\begin{array}{l}\sqrt[p]{\Psi_{\Psi_1}^p + \Psi_{\Psi_2}^p - \Psi_{\Psi_1}^p \Psi_{\Psi_2}^p},\\ \sqrt[r]{\Psi_{\Psi_1}^r + \Psi_{\Psi_2}^r - \Psi_{\Psi_1}^r \Psi_{\Psi_2}^r}, \varphi_{\Psi_1}\varphi_{\Psi_2}\end{array}\right)$,
3. $\Psi_1 \otimes \Psi_2 = \left(\begin{array}{l}\Psi_{\Psi_1}\Psi_{\Psi_2}, \Psi_{\Psi_1}\Psi_{\Psi_2},\\ \sqrt[q]{\varphi_{\Psi_1}^q + \varphi_{\Psi_2}^q - \varphi_{\Psi_1}^q \varphi_{\Psi_2}^q}\end{array}\right)$,
4. $\zeta \Psi = \left(\begin{array}{l}\sqrt[p]{1 - \left(1 - \Psi_{\Psi}^p\right)^\zeta},\\ \sqrt[r]{1 - \left(1 - \Psi_{\Psi}^r\right)^\zeta}, \varphi_{\Psi}^\zeta\end{array}\right)$,

d) $\Psi^\zeta = \left(\Psi_{\Psi}^\zeta, \Psi_{\Psi}^\zeta, \sqrt[q]{1 - \left(1 - \varphi_{\Psi}^q\right)^\zeta},\right)$.

Definition 15: [25] Let $\Psi = (m_\Psi, s_\Psi, n_\Psi)$ be a $p, q, r-$SFN then score function is defined as:

$$Sc(\Psi) = \frac{1 + m_\Psi^p - s_\Psi^r - n_\Psi^q}{2} \tag{10}$$

where $0 \preccurlyeq Sco(\Psi) \preccurlyeq 1$. The accuracy function of $p, q, r-$SFN is defined as follows:

$$Acc(\Psi) = m_\Psi^p + s_\Psi^r + n_\Psi^q (0 \preccurlyeq Acc(\Psi) \preccurlyeq 1) \tag{11}$$

Definition 16: [45] Einstein product θ and sum ϑ are defined as follows:

$$\phi(\theta, \vartheta) = \frac{\theta.\vartheta}{1 + (1-\theta)(1-\vartheta)}, \text{ for all } \theta, \vartheta \in [0, 1]^2$$

$$\varphi(\theta, \vartheta) = \frac{\theta + \vartheta}{1 + \theta\vartheta}, \text{ for all } \theta, \vartheta \in [0, 1]^2$$

III. PROPOSED p, q, r-SPHERICAL FUZZY EINSTEIN OPERATIONS

Definition 17: Let $\Psi_1 = \left(m_{\Psi_1}, s_{\Psi_1}, n_{\Psi_1}\right)$, $\Psi_2 = \left(m_{\Psi_2}, s_{\Psi_2}, n_{\Psi_2}\right)$ and $\Psi = (m_\Psi, s_\Psi, n_\Psi)$ are three p, q, r-SFNs, and $\zeta \succ 0$, then we have Einstein operations for these p, q, r-SFNs are defined as follows:

1. $\Psi_1 \oplus_\varepsilon \Psi_2 = \left(\left\langle \sqrt[p]{\frac{m_{\Psi_1}^p + m_{\Psi_2}^p}{1 + m_{\Psi_1}^p m_{\Psi_2}^p}}, \sqrt[r]{\frac{s_{\Psi_1}^r + s_{\Psi_2}^r}{1 + s_{\Psi_1}^r s_{\Psi_2}^r}}, \frac{n_{\Psi_1} n_{\Psi_2}}{\sqrt[q]{1 + \left(1 - n_{\Psi_1}^q\right)\left(1 - n_{\Psi_2}^q\right)}} \right\rangle \right)$,
2. $\Psi_1 \otimes \Psi_2 = \left(\left\langle \frac{m_{\Psi_1} m_{\Psi_2}}{\sqrt[p]{1 + \left(1 - m_{\Psi_1}^p\right)\left(1 - m_{\Psi_2}^p\right)}}, \frac{s_{\Psi_1} s_{\Psi_2}}{\sqrt[r]{1 + \left(1 - s_{\Psi_1}^r\right)\left(1 - s_{\Psi_2}^r\right)}}, \sqrt[q]{\frac{n_{\Psi_1}^q + n_{\Psi_2}^q}{1 + n_{\Psi_1}^q + n_{\Psi_2}^q}}, \right\rangle \right)$,
3. $\zeta_\varepsilon \Psi = \left(\left\langle \sqrt[p]{\frac{(1+m_\Psi^p)^\zeta - (1-m_\Psi^p)^\zeta}{(1+m_\Psi^p)^\zeta + (1-m_\Psi^p)^\zeta}}, \sqrt[r]{\frac{(1+s_\Psi^r)^\zeta - (1-s_\Psi^r)^\zeta}{(1+s_\Psi^r)^\zeta + (1-s_\Psi^r)^\zeta}}, \frac{\sqrt[q]{2}(n_\Psi)^\zeta}{\sqrt[q]{\left(2 - n_\Psi^q\right)^\zeta + \left(n_\Psi^q\right)^\zeta}} \right\rangle \right)$,
4. $\Psi^\zeta = \left(\left\langle \frac{\sqrt[p]{2}(m_\Psi)^\zeta}{\sqrt[p]{\left(2 - m_\Psi^p\right)^\zeta + \left(m_\Psi^p\right)^\zeta}}, \frac{\sqrt[r]{2}(s_\Psi)^\zeta}{\sqrt[r]{(2 - s_\Psi^r)^\zeta + (s_\Psi^r)^\zeta}}, \sqrt[q]{\frac{(1+n_\Psi^q)^\zeta - (1-n_\Psi^q)^\zeta}{(1+n_\Psi^q)^\zeta + (1-n_\Psi^q)^\zeta}} \right\rangle \right)$.

Definition 18: Let $\Psi_1 = \left(m_{\Psi_1}, s_{\Psi_1}, n_{\Psi_1}\right)$ and $\Psi_2 = \left(m_{\Psi_2}, s_{\Psi_2}, n_{\Psi_2}\right)$ two $p, q, r-$SFNs, then Einstein union and Einstein intersection can be defined as follows:

1. $\Psi_1 \bigcup \Psi_2 = \left(\max\left(m_{\Psi_1}, m_{\Psi_2}\right), \max\left(s_{\Psi_1}, s_{\Psi_2}\right), \min\left(n_{\Psi_1}, n_{\Psi_2}\right) \right)$,
2. $\Psi_1 \bigcap \Psi_2 = \left(\min\left(m_{\Psi_1}, m_{\Psi_2}\right), \min\left(s_{\Psi_1}, s_{\Psi_2}\right), \max\left(n_{\Psi_1}, n_{\Psi_2}\right) \right)$.

Theorem 1: For any three ζ, ζ_1, ζ_3 real numbers . Then for two $p, q, r-$SFNs $\Psi_1 = \left(m_{\Psi_1}, s_{\Psi_1}, n_{\Psi_1}\right)$, and $\Psi_2 = \left(m_{\Psi_2}, s_{\Psi_2}, n_{\Psi_2}\right)$, Einstein sum and product are defined as follows:

1. $\Psi_1 \oplus_E \Psi_2 = \Psi_2 \oplus_E \Psi_1$,
2. $\Psi_1 \otimes_E \Psi_2 = \Psi_2 \otimes_E \Psi_1$,
3. $\zeta(\Psi_1 \oplus_E \Psi_2) = \zeta\Psi_1 \oplus_E \zeta\Psi_2$,
4. $(\Psi_1 \otimes_E \Psi_2)^\zeta = \Psi_1^\zeta \otimes_E \Psi_2^\zeta$,
5. $\zeta_1\Psi \oplus_E \zeta_2\Psi = \Psi(\zeta_1 + \zeta_2)$,
6. $\Psi^{\zeta_1} \otimes_E \Psi^{\zeta_2} = \Psi^{\zeta_1 + \zeta_2}$.

Proof: Let us consider part (1), we have

$$\Psi_1 \oplus \Psi_2 = \left(\left\langle \sqrt[p]{\frac{\Psi_{\Psi_1}^p + \Psi_{\Psi_2}^p}{1 + \Psi_{\Psi_1}^p \Psi_{\Psi_2}^p}}, \sqrt[r]{\frac{\varphi_{\Psi_1}^r + \varphi_{\Psi_2}^r}{1 + \varphi_{\Psi_1}^r \varphi_{\Psi_2}^r}}, \frac{\Theta_{\Psi_1}\Theta_{\Psi_2}}{\sqrt[q]{1 + \left(1 - \Theta_{\Psi_1}^q\right)\left(1 - \Theta_{\Psi_2}^q\right)}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt[p]{\frac{m_{\Psi_2}^p + m_{\Psi_1}^p}{1 + m_{\Psi_2}^p m_{\Psi_1}^p}}, \sqrt[r]{\frac{s_{\Psi_2}^r + s_{\Psi_1}^r}{1 + s_{\Psi_2}^r s_{\Psi_1}^r}}, \frac{n_{\Psi_2} n_{\Psi_1}}{\sqrt[q]{1 + \left(1 - n_{\Psi_2}^q\right)\left(1 - n_{\Psi_1}^q\right)}} \right\rangle \right) = \Psi_2 \oplus \Psi_1.$$

Similarly,

$$\Psi_1 \otimes \Psi_2 = \left(\left\langle \frac{m_{\Psi_1} m_{\Psi_2}}{\sqrt[p]{1 + \left(1 - m_{\Psi_1}^p\right)\left(1 - m_{\Psi_2}^p\right)}}, \frac{s_{\Psi_1} s_{\Psi_2}}{\sqrt[r]{1 + \left(1 - s_{\Psi_1}^r\right)\left(1 - s_{\Psi_2}^r\right)}}, \sqrt[q]{\frac{n_{\Psi_1}^q + n_{\Psi_2}^q}{1 + n_{\Psi_1}^q n_{\Psi_2}^q}}, \right\rangle \right)$$

$$= \left(\left\langle \frac{m_{\Psi_2} m_{\Psi_1}}{\sqrt[p]{1 + \left(1 - m_{\Psi_2}^p\right)\left(1 - m_{\Psi_1}^p\right)}}, \frac{s_{\Psi_2} s_{\Psi_1}}{\sqrt[r]{1 + \left(1 - s_{\Psi_2}^r\right)\left(1 - s_{\Psi_1}^r\right)}}, \sqrt[q]{\frac{n_{\Psi_2}^p + n_{\Psi_1}^p}{1 + n_{\Psi_2}^p n_{\Psi_1}^p}}, \right\rangle \right)$$

$$= \Psi_2 \otimes \Psi_1.$$

By Einstein operational laws, we have

$$\zeta(\Psi_1 \oplus \Psi_2)$$

$$= \zeta \left(\left\langle \sqrt[p]{\frac{a-b}{a+b}}, \sqrt[r]{\frac{e-f}{e+f}}, \frac{\sqrt[q]{2}c}{\sqrt[q]{d+c}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt[p]{\frac{\left(1+\frac{a-b}{a+b}\right)^\zeta - \left(1-\frac{a-b}{a+b}\right)^\zeta}{\left(1+\frac{a-b}{a+b}\right)^\zeta + \left(1-\frac{a-b}{a+b}\right)^\zeta}}, \sqrt[r]{\frac{\left(1+\frac{e-f}{e+f}\right)^\zeta - \left(1-\frac{e-f}{e+f}\right)^\zeta}{\left(1+\frac{e-f}{e+f}\right)^\zeta + \left(1-\frac{e-f}{e+f}\right)^\zeta}}, \frac{\sqrt[q]{2}\left(\frac{\sqrt[q]{2}c}{\sqrt[q]{d+c}}\right)}{\sqrt[q]{\left(\left(2-\frac{2c}{d+c}\right)^\mathfrak{g} + \left(\frac{2c}{d+c}\right)^\mathfrak{g}\right)}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt[p]{\frac{a^\zeta - b^\zeta}{a^\zeta + b^\zeta}}, \sqrt[r]{\frac{e^\zeta - f^\zeta}{e^\zeta + f^\zeta}}, \frac{\sqrt[q]{2}c^\zeta}{\sqrt[q]{\left(d^\zeta + c^\zeta\right)}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt[p]{\frac{\left(1+m_{\Psi_1}^p\right)^\zeta \left(1+m_{\Psi_1}^p\right)^\zeta - \left(1-m_{\Psi_2}^p\right)^\zeta \left(1-m_{\Psi_2}^p\right)^\zeta}{\left(1+m_{\Psi_1}^p\right)^\zeta \left(1+m_{\Psi_1}^p\right)^\zeta + \left(1-m_{\Psi_2}^p\right)^\zeta \left(1-m_{\Psi_2}^p\right)^\zeta}}, \sqrt[r]{\frac{\left(1+s_{\Psi_1}^r\right)^\zeta \left(1+s_{\Psi_1}^r\right)^\zeta - \left(1-s_{\Psi_2}^r\right)^\zeta \left(1-s_{\Psi_2}^r\right)^\zeta}{\left(1+s_{\Psi_1}^r\right)^\zeta \left(1+s_{\Psi_1}^r\right)^\zeta + \left(1-s_{\Psi_2}^r\right)^\zeta \left(1-s_{\Psi_2}^r\right)^\zeta}}, \frac{\sqrt[q]{2} n_{\Psi_1}^\zeta n_{\Psi_2}^\zeta}{\sqrt[q]{\left(2-n_{\Psi_1}^q\right)^\zeta \left(1-n_{\Psi_2}^q\right)^\zeta + n_{\Psi_1}^\zeta n_{\Psi_2}^\zeta}} \right\rangle \right).$$

In other hand, we examine that

$$\zeta\Psi_1 = \begin{pmatrix} \langle \sqrt[p]{\frac{\left(1+m_{\Psi_1}^p\right)^\zeta - \left(1-m_{\Psi_1}^p\right)^\zeta}{\left(1+m_{\Psi_1}^p\right)^\zeta + \left(1-m_{\Psi_1}^p\right)^\zeta}}, \\ \sqrt[r]{\frac{\left(1+s_{\Psi_1}^r\right)^\zeta - \left(1-s_{\Psi_1}^r\right)^\zeta}{\left(1+s_{\Psi_1}^r\right)^\zeta + \left(1-s_{\Psi_1}^r\right)^\zeta}}, \\ \frac{\sqrt[q]{2} n_{\Psi_1}^\zeta}{\sqrt[q]{\left(2-n_{\Psi_1}^q\right)^\zeta + n_{\Psi_1}^\zeta}} \rangle \end{pmatrix}$$

$$= \left(\langle \sqrt[p]{\frac{a_1 - b_1}{a_1 + b_1}}, \sqrt[r]{\frac{e_1 - f_1}{e_1 + f_1}}, \frac{\left(\sqrt[q]{2} c_1\right)}{\sqrt[q]{d_1 + c_1}} \rangle \right)$$

$$\zeta\Psi_2 = \begin{pmatrix} \langle \sqrt[p]{\frac{\left(1+m_{\Psi_2}^p\right)^\zeta - \left(1-m_{\Psi_2}^p\right)^\zeta}{\left(1+m_{\Psi_2}^p\right)^\zeta + \left(1-m_{\Psi_2}^p\right)^\zeta}}, \\ \sqrt[r]{\frac{\left(1+s_{\Psi_2}^r\right)^\zeta - \left(1-s_{\Psi_2}^r\right)^\zeta}{\left(1+s_{\Psi_2}^r\right)^\zeta + \left(1-s_{\Psi_2}^r\right)^\zeta}}, \\ \frac{\sqrt[q]{2} n_{\Psi_2}^\zeta}{\sqrt[q]{\left(2-n_{\Psi_2}^q\right)^\zeta + n_{\Psi_2}^\zeta}} \rangle \end{pmatrix}$$

$$= \left(\langle \sqrt[p]{\frac{a_2 - b_2}{a_2 + b_2}}, \sqrt[r]{\frac{e_2 - f_2}{e_2 + f_2}}, \frac{\left(\sqrt[q]{2} c_2\right)}{\sqrt[q]{d_2 + c_2}} \rangle \right)$$

where $a_1 = \left(1 + m_{\Psi_1}^p\right)^\zeta$, $b_1 = \left(1 - m_{\Psi_1}^p\right)^\zeta$,
$e_1 = \left(1 + s_{\Psi_1}^r\right)^\zeta$, $f_1 = \left(1 - s_{\Psi_1}^r\right)^\zeta$, $c_1 = \left(n_{\Psi_1}\right)^\zeta$,
$d_1 = \left(2 - n_{\Psi_1}\right)^\zeta$, $c_2 = \left(n_{\Psi_2}\right)^\zeta$, $a_2 = \left(1 + m_{\Psi_2}^p\right)^\zeta$,
$b_2 = \left(1 - m_{\Psi_2}^p\right)^\zeta$, $e_2 = \left(1 + s_{\Psi_2}^r\right)^\zeta$, $f_2 = \left(1 - s_{\Psi_2}^r\right)^\zeta$.

$$\zeta\Psi_1 \oplus \zeta\Psi_2$$

$$= \left(\left\langle \sqrt[p]{\frac{a_1 - b_1}{a_1 + b_1}}, \sqrt[r]{\frac{e_1 - f_1}{e_1 + f_1}}, \frac{\left(\sqrt[q]{2} c_1\right)}{\sqrt[q]{d_1 + c_1}} \right\rangle \right) \oplus \left(\langle \sqrt[p]{\frac{a_2 - b_2}{a_2 + b_2}}, \sqrt[r]{\frac{e_2 - f_2}{e_2 + f_2}}, \frac{\left(\sqrt[q]{2} c_2\right)}{\sqrt[q]{d_2 + c_2}} \rangle \right)$$

$$= \begin{pmatrix} \langle \sqrt[p]{\frac{\frac{a_2-b_2}{a_2+b_2} + \frac{a_2-b_2}{a_2+b_2}}{1+\left(\frac{a_1-b_1}{a_1+b_1}\right)\left(\frac{a-b}{a+b}\right)}}, \\ \sqrt[r]{\frac{\frac{e_2-f_2}{e_2+f_2} + \frac{e_2-f_2}{e_2+f_2}}{1+\left(\frac{e_1-f_1}{e_1+f_1}\right)\left(\frac{e-f}{e+f}\right)}}, \\ \frac{2\sqrt[q]{\frac{c_1 c_2}{(d_1+c_1)(d_2+c_2)}}}{\sqrt[q]{\left(1+\left(1-\frac{2c_1}{d_1+c_1}\right)\left(1-\frac{2c_2}{d_2+c_2}\right)\right)}} \rangle \end{pmatrix}$$

$$= \left(\langle \sqrt[p]{\frac{a_1 a_2 - b_1 b_2}{a_1 a_2 + b_1 b_2}}, \sqrt[r]{\frac{e_1 e_2 - f_1 f_2}{e_1 e_2 + f_1 f_2}}, \frac{\sqrt[q]{2} c_1 c_2}{\sqrt[q]{(d_1 d_2 + c_1 c_2)}} \rangle \right)$$

$$= \begin{pmatrix} \langle \sqrt[p]{\frac{\left(1+m_{\Psi_1}^p\right)^\zeta \left(1+m_{\Psi_2}^p\right)^\zeta - \left(1-m_{\Psi_1}^p\right)^\zeta \left(1-m_{\Psi_2}^p\right)^\zeta}{\left(1+m_{\Psi_1}^p\right)^\zeta \left(1+m_{\Psi_2}^p\right)^\zeta + \left(1-m_{\Psi_1}^p\right)^\zeta \left(1-m_{\Psi_2}^p\right)^\zeta}}, \\ \sqrt[r]{\frac{\left(1+s_{\Psi_1}^r\right)^\zeta \left(1+s_{\Psi_2}^r\right)^\zeta - \left(1-s_{\Psi_1}^r\right)^\zeta \left(1-s_{\Psi_2}^r\right)^\zeta}{\left(1+s_{\Psi_1}^r\right)^\zeta \left(1+s_{\Psi_2}^r\right)^\zeta + \left(1-s_{\Psi_1}^r\right)^\zeta \left(1-s_{\Psi_2}^r\right)^\zeta}}, \\ \frac{\sqrt[q]{2} \left(n_{\Psi_1}\right)^\zeta \left(n_{\Psi_2}\right)^\zeta}{\sqrt[q]{\left(\left(2-n_{\Psi_1}\right)^\zeta \left(2-n_{\Psi_2}\right)^\zeta + \left(n_{\Psi_1}\right)^\zeta \left(n_{\Psi_2}\right)^\zeta\right)}} \rangle \end{pmatrix}$$

Hence $\zeta(\Psi_1 \oplus \Psi_2) = \zeta\Psi_1 \oplus \zeta\Psi_2$.

The proof of part (4) is obvious. Let us consider part (5), we have

$$\zeta_1\Psi = \begin{pmatrix} \langle \sqrt[p]{\frac{(1+m_\Psi^p)^{\zeta_1} - (1-m_\Psi^p)^{\zeta_1}}{(1+m_\Psi^p)^{\zeta_1} + (1-m_\Psi^p)^{\zeta_1}}}, \\ \sqrt[r]{\frac{(1+s_\Psi^r)^{\zeta_1} - (1-s_\Psi^r)^{\zeta_1}}{(1+s_\Psi^r)^{\zeta_1} + (1-s_\Psi^r)^{\zeta_1}}}, \\ \frac{\left(\sqrt[q]{2}\right) n_\Psi^{\zeta_1}}{\sqrt[q]{(2-n_\Psi^q)^{\zeta_1} + n_\Psi^{\zeta_1}}} \rangle \end{pmatrix}$$

$$\zeta_2\Psi = \begin{pmatrix} \langle \sqrt[p]{\frac{(1+m_R^p)^{\zeta_2} - (1-m_R^p)^{\zeta_2}}{(1+m_R^p)^{\zeta_2} + (1-m_R^p)^{\zeta_2}}}, \\ \sqrt[r]{\frac{(1+n_R^r)^{\zeta_2} - (1-n_R^r)^{\zeta_2}}{(1+n_R^r)^{\zeta_2} + (1-n_R^r)^{\zeta_2}}}, \\ \frac{\left(\sqrt[q]{2}\right) s_R^{\zeta_2}}{\sqrt[q]{(2-s_R^q)^{\zeta_2} + s_R^{\zeta_2}}} \rangle \end{pmatrix}$$

$$\zeta_1\Psi \oplus \zeta_2\Psi = \begin{pmatrix} \langle \sqrt[p]{\frac{(1+m_\Psi^p)^{\zeta_1} - (1-m_\Psi^p)^{\zeta_1}}{(1+m_\Psi^p)^{\zeta_1} + (1-m_\Psi^p)^{\zeta_1}}}, \\ \sqrt[r]{\frac{(1+s_R^r)^{\zeta_1} - (1-s_R^r)^{\zeta_1}}{(1+s_\Psi^r)^{\zeta_1} + (1-s_\Psi^r)^{\zeta_1}}}, \\ \frac{\left(\sqrt[q]{2}\right) n_\Psi^{\zeta_1}}{\sqrt[q]{(2-n_\Psi^q)^{\zeta_1} + n_\Psi^{\zeta_1}}} \rangle \end{pmatrix} \oplus \begin{pmatrix} \langle \sqrt[p]{\frac{(1+m_\Psi^p)^{\zeta_2} - (1-m_\Psi^p)^{\zeta_2}}{(1+m_\Psi^p)^{\zeta_2} + (1-m_\Psi^p)^{\zeta_2}}}, \\ \sqrt[r]{\frac{(1+s_\Psi^r)^{\zeta_2} - (1-s_\Psi^r)^{\zeta_2}}{(1+s_\Psi^r)^{\zeta_2} + (1-s_\Psi^r)^{\zeta_2}}}, \\ \frac{\sqrt[q]{2} n_\Psi^{\zeta_2}}{\sqrt[q]{(2-n_\Psi^q)^{\zeta_2} + n_\Psi^{\zeta_2}}} \rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle \sqrt[p]{\frac{(1+m_\Psi^p)^{\zeta_1+\zeta_2} - (1-m_\Psi^p)^{\zeta_1+\zeta_2}}{(1+m_\Psi^p)^{\zeta_1+\zeta_2} + (1-m_\Psi^p)^{\zeta_1+\zeta_2}}}, \\ \sqrt[r]{\frac{(1+s_\Psi^r)^{\zeta_1+\zeta_2} - (1-s_\Psi^r)^{\zeta_1+\zeta_2}}{(1+s_\Psi^r)^{\zeta_1+\zeta_2} + (1-s_\Psi^r)^{\zeta_1+\zeta_2}}}, \\ \frac{\sqrt[q]{2} (n_\Psi)^{\zeta_1+\zeta_2}}{\sqrt[q]{((2-s_\Psi)^{\zeta_1+\zeta_2}, +(n_\Psi)^{\zeta_1+\zeta_2})}} \rangle \end{pmatrix}$$

$= (\zeta_1 + \zeta_2)\Psi$. Therefore $\zeta_1\Psi \oplus \zeta_2\Psi = (\zeta_1 + \zeta_2)\Psi$.

Let us consider part (6), we have as shown in the equation at the bottom of the next page.

A. *p, q, r-SF EINSTEIN AVERAGING OPERATORS*

We present a variety of AOs in this section that use the rules from Section III. These operators can effectively combine and simplify information, which facilitates data analysis and decision making.

Definition 19: Let $\Psi_i = \left(m_{\Psi_i}, s_{\Psi_i}, n_{\Psi_i}\right)$ $(i = 1, 2, \ldots k)$ be a set of $p, q, r-$SFNs with their corresponding weight vector ζ_i $(i = 1, 2, \ldots, k)$ such that $\sum_{i=1}^{k} \zeta_i = 1$ and $\zeta_i \in [0, 1]$. Then the operator $p, q, r - SFEWA(\Psi_1, \Psi_2, \ldots, \Psi_k) : \Lambda^k \rightarrow \Lambda$ is defined as

$$p, q, r - SFEWA\left(\Psi_1, \Psi_2, \ldots, \Psi_k\right) = \oplus_{i=1}^{k} \zeta_i \Psi_i \tag{12}$$

Theorem 2: The aggregated value obtained by $p, q, r - SFEWA$ operator of $p, q, r-$SFNs is still a p, q, r-SFNs, and can be expressed as:

$$\begin{aligned} &p, q, r - SFEWA\left(\Psi_1, \Psi_2, \ldots, \Psi_m\right) \\ &= \oplus_{i=1}^{k} \zeta_i \Psi_i \\ &= \left(\begin{array}{c} \langle \sqrt[p]{\frac{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^p\right)^{\zeta_i}-\prod_{i=1}^{k}\left(1-m_{\Psi_i}^p\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^p\right)^{\zeta_i}+\prod_{i=1}^{k}\left(1-m_{\Psi_i}^p\right)^{\zeta_i}}}, \\ \sqrt[r]{\frac{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^r\right)^{\zeta_i}-\prod_{i=1}^{k}\left(1-s_{\Psi_i}^r\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^r\right)^{\zeta_i}+\prod_{i=1}^{k}\left(1-s_{\Psi_i}^r\right)^{\zeta_i}}}, \\ \frac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k}\sqrt[q]{\left(2-n_{\Psi_i}^q\right)^{\zeta_i}+\prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}}\rangle \end{array} \right) \end{aligned} \tag{13}$$

Proof: This theorem is established by using mathematical induction.

Step 1. For $k = 2$, we have

$$\begin{aligned} &p, q, r - SFFWA\left(\Psi_1, \Psi_2\right) \\ &= \oplus_{i=1}^{2} \zeta_i \Psi_i = \zeta_1\Psi_1 \oplus \zeta_2\Psi_2 \end{aligned}$$

$$\zeta_1\Psi_1 = \left(\begin{array}{c} \langle \sqrt[p]{\frac{\left(1+m_{\Psi_1}^p\right)^{\zeta_1}-\left(1-m_{\Psi_1}^p\right)^{\zeta_1}}{\left(1+m_{\Psi_1}^p\right)^{\zeta_1}+\left(1-m_{\Psi_1}^p\right)^{\zeta_1}}}, \\ \sqrt[r]{\frac{\left(1+s_{\Psi_1}^r\right)^{\zeta_1}-\left(1-s_{\Psi_1}^r\right)^{\zeta_1}}{\left(1+s_{\Psi_1}^r\right)^{\zeta_1}+\left(1-s_{\Psi_1}^r\right)^{\zeta_1}}}, \\ \frac{\left(\sqrt[q]{2}\right) n_{\Psi_1}^{\zeta_1}}{\sqrt[q]{\left(2-n_{\Psi_1}^q\right)^{\zeta_1}+n_{\Psi_1}^{\zeta_1}}}\rangle \end{array} \right),$$

$$\zeta_2\Psi_2 = \left(\begin{array}{c} \langle \sqrt[p]{\frac{\left(1+m_{\Psi_2}^p\right)^{\zeta_2}-\left(1-m_{\Psi_2}^p\right)^{\zeta_2}}{\left(1+m_{\Psi_2}^p\right)^{\zeta_2}+\left(1-m_{\Psi_2}^p\right)^{\zeta_2}}}, \\ \sqrt[r]{\frac{\left(1+s_{\Psi_2}^r\right)^{\zeta_2}-\left(1-s_{\Psi_2}^r\right)^{\zeta_2}}{\left(1+s_{\Psi_2}^r\right)^{\zeta_2}+\left(1-s_{\Psi_2}^r\right)^{\zeta_2}}}, \\ \frac{\left(\sqrt[q]{2}\right) n_{\Psi_2}^{\zeta_2}}{\sqrt[q]{\left(2-n_{\Psi_2}^q\right)^{\zeta_2}+n_{\Psi_2}^{\zeta_2}}}\rangle \end{array} \right)$$

$$\begin{aligned} &\zeta_1\Psi_1 \oplus \zeta_2\Psi_2 \\ &\times \left(\begin{array}{c} \langle \sqrt[p]{\frac{\left(1+m_{\Psi_1}^p\right)^{\zeta_1}-\left(1-m_{\Psi_1}^p\right)^{\zeta_1}}{\left(1+m_{\Psi_1}^p\right)^{\zeta_1}+\left(1-m_{\Psi_1}^p\right)^{\zeta_1}}}, \\ \sqrt[r]{\frac{\left(1+s_{\Psi_1}^r\right)^{\zeta_1}-\left(1-s_{\Psi_1}^r\right)^{\zeta_1}}{\left(1+s_{\Psi_1}^r\right)^{\zeta_1}+\left(1-s_{\Psi_1}^r\right)^{\zeta_1}}}, \\ \frac{\left(\sqrt[q]{2}\right) n_{\Psi_1}^{\zeta_1}}{\sqrt[q]{\left(2-n_{\Psi_1}^q\right)^{\zeta_1}+n_{\Psi_1}^{\zeta_1}}}\rangle \end{array} \right) \\ &\oplus \left(\begin{array}{c} \langle \sqrt[p]{\frac{\left(1+m_{\Psi_2}^p\right)^{\zeta_2}-\left(1-m_{\Psi_2}^p\right)^{\zeta_2}}{\left(1+m_{\Psi_2}^p\right)^{\zeta_2}+\left(1-m_{\Psi_2}^p\right)^{\zeta_2}}}, \\ \sqrt[r]{\frac{\left(1+s_{\Psi_2}^r\right)^{\zeta_2}-\left(1-s_{\Psi_2}^r\right)^{\zeta_2}}{\left(1+s_{\Psi_2}^r\right)^{\zeta_2}+\left(1-s_{\Psi_2}^r\right)^{\zeta_2}}}, \\ \frac{\left(\sqrt[q]{2}\right) n_{\Psi_2}^{\zeta_2}}{\sqrt[q]{\left(2-n_{\Psi_2}^q\right)^{\zeta_2}+n_{\Psi_2}^{\zeta_2}}}\rangle \end{array} \right) \\ &= \left(\begin{array}{c} \langle \sqrt[p]{\frac{\left(1+m_{\Psi_1}^p\right)^{\zeta_1}\left(1+m_{\Psi_2}^p\right)^{\zeta_2}-\left(1-m_{\Psi_1}^p\right)^{\zeta_1}\left(1-m_{\Psi_2}^p\right)^{\zeta_2}}{\left(1+m_{\Psi_1}^p\right)^{\zeta_1}\left(1+m_{\Psi_2}^p\right)^{\zeta_2}+\left(1-m_{\Psi_1}^p\right)^{\zeta_1}\left(1-m_{\Psi_2}^p\right)^{\zeta_2}}}, \\ \sqrt[r]{\frac{\left(1+s_{\Psi_1}^r\right)^{\zeta_1}\left(1+s_{\Psi_2}^r\right)^{\zeta_2}-\left(1-s_{\Psi_1}^r\right)^{\zeta_1}\left(1-s_{\Psi_2}^r\right)^{\zeta_2}}{\left(1+s_{\Psi_1}^r\right)^{\zeta_1}\left(1+s_{\Psi_2}^r\right)^{\zeta_2}+\left(1-s_{\Psi_1}^r\right)^{\zeta_1}\left(1-s_{\Psi_2}^r\right)^{\zeta_2}}}, \\ \frac{\left(\sqrt[q]{2}\right) n_{\Psi_1}^{\zeta_1}\left(\sqrt[q]{2}\right) n_{\Psi_2}^{\zeta_2}}{\sqrt[q]{\left(2-n_{\Psi_1}^q\right)^{\zeta_1}+n_{\Psi_1}^{\zeta_1}}\sqrt[q]{\left(2-n_{\Psi_2}^q\right)^{\zeta_2}+n_{\Psi_2}^{\zeta_2}}}\rangle \end{array} \right) \end{aligned}$$

where $\sum_{i=1}^{2} \zeta_i = 1$. Therefore, for $k = 2$, the result is true.

$$\Psi^{\zeta_1} \otimes \Psi^{\zeta_2} = \Psi^{\zeta_1+\zeta_2}.$$

$$\begin{aligned} \Psi^{\zeta_1} \otimes \Psi^{\zeta_2} &= \left(\frac{(m_\Psi)^{\zeta_1}}{\left(\sqrt[p]{1+\left(1-m_\Psi^p\right)}\right)^{\zeta_1}}, \frac{(s_\Psi)^{\zeta_1}}{\left(\sqrt[r]{1+\left(1-s_\Psi^r\right)}\right)^{\zeta_1}}, \sqrt[q]{\frac{\left(n_\Psi^q\right)^{\zeta_1}}{\left(1+n_\Psi^q\right)^{\zeta_1}}} \right) \\ &\otimes \left(\frac{(m_\Psi)^{\zeta_2}}{\left(\sqrt[p]{1+\left(1-m_\Psi^p\right)}\right)^{\zeta_2}}, \frac{(s_\Psi)^{\zeta_2}}{\left(\sqrt[r]{1+\left(1-s_\Psi^r\right)}\right)^{\zeta_2}}, \sqrt[q]{\frac{\left(n_\Psi^q\right)^{\zeta_2}}{\left(1+n_\Psi^q\right)^{\zeta_2}}} \right) \\ &= \left(\begin{array}{c} \frac{(m_\Psi)^{\zeta_1+\zeta_2}}{\left(\sqrt[p]{1+(1-m_\Psi^p)}\right)^{\zeta_1+\zeta_2}}, \\ \frac{(s_\Psi)^{\zeta_1+\zeta_2}}{\left(\sqrt[r]{1+(1-s_\Psi^r)}\right)^{\zeta_1+\zeta_2}}, \sqrt[q]{\frac{\left(n_\Psi^q\right)^{\zeta_1+\zeta_2}}{\left(1+n_\Psi^q\right)^{\zeta_1+\zeta_2}}} \end{array} \right) = \Psi^{\zeta_1+\zeta_2}. \end{aligned}$$

Step 2. Suppose the result is valid for $k = l$ i.e.,

$$p, q, r - SFEWA\left(\Psi_1, \Psi_2, \ldots, \Psi_l\right) = \oplus_{i=1}^{l}\zeta_i\Psi_i = \left(\begin{array}{c} \langle \sqrt[p]{\frac{\prod_{i=1}^{l}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}-\prod_{i=1}^{l}\left(1-m_{\Psi_i}^{p}\right)^{\zeta_i}}{\prod_{i=1}^{l}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}+\prod_{i=1}^{l}\left(1-m_{\Psi_i}^{p}\right)^{\zeta_i}}}, \\ \sqrt[r]{\frac{\prod_{i=1}^{l}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}-\prod_{i=1}^{l}\left(1-s_{\Psi_i}^{r}\right)^{\zeta_i}}{\prod_{i=1}^{l}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}+\prod_{i=1}^{l}\left(1-s_{\Psi_i}^{r}\right)^{\zeta_i}}}, \\ \frac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{l}n_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{l}\sqrt[q]{\left(2-n_{\Psi_i}^{q}\right)^{\zeta_i}+\prod_{i=1}^{l}n_{\Psi_i}^{\zeta_i}}}\rangle \end{array}\right) \tag{14}$$

Step 3. To prove that Equation (13) is true for $k = l+1$ i.e.,

$$p, q, r - SFEWA\left(\Psi_1, \Psi_2, \ldots, \Psi_{l+1}\right) = \oplus_{i=1}^{l+1}\zeta_i\Psi_i = \oplus_{i=1}^{l}\zeta_i\Psi_i \oplus \zeta_{i+1}\Psi_{i+1}$$

$$= \left(\begin{array}{c} \langle \sqrt[p]{\frac{\prod_{i=1}^{l}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}-\prod_{i=1}^{l}\left(1-m_{\Psi_{\mathcal{S}}}^{p}\right)^{\zeta_i}}{\prod_{i=1}^{l}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}+\prod_{i=1}^{l}\left(1-m_{\Psi_i}^{p}\right)^{\zeta_i}}}, \\ \sqrt[r]{\frac{\prod_{i=1}^{l}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}-\prod_{i=1}^{l}\left(1-s_{\Psi_i}^{r}\right)^{\zeta_i}}{\prod_{i=1}^{l}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}+\prod_{i=1}^{l}\left(1-s_{\Psi_i}^{r}\right)^{\zeta_i}}}, \\ \frac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{l}n_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{l}\sqrt[q]{\left(2-n_{\Psi_i}^{q}\right)^{\zeta_i}+\prod_{i=1}^{l}n_{\Psi_i}^{\zeta_i}}}\rangle \end{array}\right)$$

$$\oplus \left(\begin{array}{c} \langle \sqrt[p]{\frac{\left(1+m_{\Psi_{l+1}}^{p}\right)^{\zeta_{l+1}}-\left(1-m_{\Psi_{l+1}}^{p}\right)^{\zeta_{l+1}}}{\left(1+m_{\Psi_{l+1}}^{p}\right)^{\zeta_{l+1}}+\left(1-m_{\Psi_{l+1}}^{p}\right)^{\zeta_{l+1}}}}, \\ \sqrt[r]{\frac{\left(1+s_{\Psi_{l+1}}^{r}\right)^{\zeta_{l+1}}-\left(1-s_{\Psi_{l+1}}^{r}\right)^{\zeta_{l+1}}}{\left(1+s_{\Psi_{l+1}}^{r}\right)^{\zeta_{l+1}}+\left(1-s_{\Psi_{l+1}}^{r}\right)^{\zeta_{l+1}}}}, \\ \frac{\left(\sqrt[q]{2}\right)n_{\Psi_{l+1}}^{\zeta_{l+1}}}{\sqrt[q]{\left(2-n_{\Psi_{l+1}}^{q}\right)^{\zeta_{l+1}}+n_{\Psi_{l+1}}^{\zeta_{l+1}}}}\rangle \end{array}\right)$$

$$= \left(\begin{array}{c} \langle \sqrt[p]{\frac{\prod_{i=1}^{l+1}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}-\prod_{i=1}^{l+1}\left(1-m_{\Psi_{\mathcal{S}}}^{p}\right)^{\zeta_i}}{\prod_{i=1}^{l+1}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}+\prod_{i=1}^{l+1}\left(1-m_{\Psi_i}^{p}\right)^{\zeta_i}}}, \\ \sqrt[r]{\frac{\prod_{i=1}^{l+1}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}-\prod_{i=1}^{l+1}\left(1-s_{\Psi_i}^{r}\right)^{\zeta_i}}{\prod_{i=1}^{l+1}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}+\prod_{i=1}^{l+1}\left(1-s_{\Psi_i}^{r}\right)^{\zeta_i}}}, \\ \frac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{l+1}n_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{l+1}\sqrt[q]{\left(2-n_{\Psi_i}^{q}\right)^{\zeta_i}+\prod_{i=1}^{l+1}n_{\Psi_i}^{\zeta_i}}}\rangle \end{array}\right)$$

where $\sum_{i=1}^{l+1}\zeta_i = 1$.

Thus, Equation (13) hold for $l + 1$. By mathematical induction we conclude that the result is true for all values of k.

Theorem 3: If the p, q, r-SFNs $\Psi_i = \left(m_{\Psi_i}, s_{\Psi_i}, n_{\Psi_i}\right)(i = 1, 2, \ldots, k)$ are identical, i.e., be a $\Psi_i = \Psi$ for all i, where $\Psi = (m_\Psi, s_\Psi, n_\Psi)$, then $p, q, r - SFEWA\left(\Psi_1, \Psi_2, \ldots, \Psi_k\right) = \Psi$.

Proof. As $\Psi_i = \Psi$, for all i, then we obtain

$$p, q, r - SFEWA\left(\Psi_1, \Psi_2, \ldots, \Psi_k\right)$$

$$= \left(\begin{array}{c} \langle \sqrt[p]{\frac{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}-\prod_{i=1}^{k}\left(1-m_{\Psi_i}^{p}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}+\prod_{i=1}^{k}\left(1-m_{\Psi_i}^{p}\right)^{\zeta_i}}}, \\ \sqrt[r]{\frac{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}-\prod_{i=1}^{k}\left(1-s_{\Psi_i}^{r}\right)^{\zeta_i}}{\prod_{\mathrm{k}=1}^{m}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}+\prod_{i=1}^{k}\left(1-s_{\Psi_i}^{r}\right)^{\zeta_i}}}, \\ \frac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{k}n_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k}\sqrt[q]{\left(2-n_{\Psi_i}^{q}\right)^{\zeta_i}+\prod_{i=1}^{k}n_{\Psi_i}^{\zeta_i}}}\rangle \end{array}\right)$$

$$= \left(\begin{array}{c} \langle \sqrt[p]{\frac{\left(1+m_{\Psi_i}^{p}\right)^{\sum_{i=1}^{k}\zeta_i}-\left(1-m_{\Psi_i}^{p}\right)^{\sum_{i=1}^{k}\zeta_i}}{\left(1+m_{\Psi_i}^{p}\right)^{\sum_{i=1}^{k}\zeta_i}+\left(1-m_{\Psi_i}^{p}\right)^{\sum_{i=1}^{k}\zeta_i}}}, \\ \sqrt[r]{\frac{\left(1+s_{\Psi_i}^{r}\right)^{\sum_{i=1}^{k}\zeta_i}-\left(1-s_{\Psi_i}^{r}\right)^{\sum_{i=1}^{k}\zeta_i}}{\left(1+s_{\Psi_i}^{r}\right)^{\sum_{i=1}^{k}\zeta_i}+\left(1-s_{\Psi_i}^{r}\right)^{\sum_{i=1}^{k}\zeta_i}}}, \\ \frac{\left(\sqrt[q]{2}\right)n_{\Psi_i}^{\sum_{i=1}^{k}\zeta_i}}{\sqrt[q]{\left(2-n_{\Psi_i}^{q}\right)^{\sum_{i=1}^{k}\zeta_i}+n_{\Psi_i}^{\sum_{i=1}^{k}\zeta_i}}}\rangle \end{array}\right)$$

$$= \left(\begin{array}{c} \langle \sqrt[p]{\frac{\left(1+m_{\Psi_i}^{p}\right)-\left(1-m_{\Psi_i}^{p}\right)}{\left(1+m_{\Psi_i}^{p}\right)+\left(1-m_{\Psi_i}^{p}\right)}}, \\ \sqrt[r]{\frac{\left(1+s_{\Psi_i}^{r}\right)-\left(1-s_{\Psi_i}^{r}\right)}{\left(1+s_{\Psi_i}^{r}\right)+\left(1-s_{\Psi_i}^{r}\right)}}, \\ \frac{\left(\sqrt[q]{2}\right)n_{\Psi_i}}{\sqrt[q]{\left(2-n_{\Psi_i}^{q}\right)+n_{\Psi_i}}}\rangle \end{array}\right) = \left(m_\Psi, s_\Psi,, n_\Psi\right) = \Psi$$

Therefore, the result can be derived from the information provided.

Theorem 4: Let $\Psi_i = \left(m_{\Psi_i}, s_{\Psi_i}, n_{\Psi_i}\right)(i = 1, 2, \ldots, k)$ be a collection of $p, q, r-$SFNs. If $\Psi^- = \min\{\Psi_1, \Psi_2, \ldots, \Psi_m\}$ and $\Psi^+ = \max\{\Psi_1, \Psi_2, \ldots, \Psi_k\}$, then $\Psi^- p, q, r - SFEWA\left(\Psi_1, \Psi_2, \ldots, \Psi_K\right) \preccurlyeq \Psi^+$.

Proof. Straightforward.

Theorem 5: Let Ω be a positive real number. Then, we have $p, q, r - SFEWA\left(\Omega\Psi_1, \Omega\Psi_2, \ldots, \Omega\Psi_m\right) = \Omega(p, q, r - SFEWA\left(\Psi_1, \Psi_2, \ldots, \Psi_m\right)$

Proof. It is simple to demonstrate.

Theorem 6: Let $\{\Psi_i | i = 1, 2, \ldots, k\}$ and $\left\{\Psi'_i | i = 1, 2, \ldots, k\right\}$ be two sets of $p, q, r-$SFNs, where of $\Psi_i = \left(m_{\Psi_i}, s_{\Psi_i}, n_{\Psi_i}\right)$ and $\Psi'_i = \left(m'_{\Psi_i}, s'_{\Psi_i}, n'_{\Psi_i}\right)$ for $i = 1, 2, \ldots, k$. If $m_{\Psi_i} \preccurlyeq m'_{\Psi_i}$, $s_{\Psi_i} \preccurlyeq s'_{\Psi_i}$ and $n_{\Psi_i} \succcurlyeq n'_{\Psi_i}$ for all i, then

$$p, q, r - SFEWA\left(\Psi_1, \Psi_2, \ldots, \Psi_k\right) \preccurlyeq p, q, r - SFEWA\left(\Psi'_1, \Psi'_2, \ldots, \Psi'_k\right) \tag{15}$$

Proof. Given that $m_{\Psi_i} \preccurlyeq m'_{\Psi_i}$, $s_{\Psi_i} \preccurlyeq s'_{\Psi_i}$ and $n_{\Psi_i} \preccurlyeq n'_{\Psi_i}$ for all $i = 1, 2, \ldots, k$ then we have,

$$\left(\begin{array}{l} \langle \sqrt[p]{\frac{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}-\prod_{i=1}^{k}\left(1-m_{\Psi_i}^{p}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}+\prod_{i=1}^{k}\left(1-m_{\Psi_i}^{p}\right)^{\zeta_i}}}, \\ \sqrt[r]{\frac{\prod_{i=1}^{k}\left(1+s_{\Psi_k}^{r}\right)^{\zeta_i}-\prod_{k=1}^{k}\left(1-s_{\Psi_k}^{r}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}+\prod_{i=1}^{k}\left(1-s_{\Psi_i}^{r}\right)^{\zeta_i}}}, \\ \frac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k}\sqrt[q]{\left(2-n_{\Psi_i}^{q}\right)^{\zeta_i}+\prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}}\rangle \end{array} \right)$$

$$\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}$$

$$\preccurlyeq \left(1-m'_{\Psi_i}\right)^{\zeta_i} \Rightarrow \frac{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}}$$

$$\preccurlyeq \frac{\prod_{i=1}^{k}\left(1-m'_{\Psi_i}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1-m'_{\Psi_i}\right)^{\zeta_i}}$$

$$\Rightarrow \sqrt[p]{\frac{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}}} \preccurlyeq \sqrt[p]{\frac{\prod_{i=1}^{k}\left(1-m'_{\Psi_i}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1-m'_{\Psi_i}\right)^{\zeta_i}}},$$

$$\times \left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}$$

$$\preccurlyeq \left(1-s'_{\Psi_i}\right)^{\zeta_i} \Rightarrow \frac{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}} \frac{\prod_{i=1}^{k}\left(1-s'_{\Psi_i}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1-s'_{\Psi_i}\right)^{\zeta_i}}$$

$$\Rightarrow \sqrt[r]{\frac{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}}} \preccurlyeq \sqrt[r]{\frac{\prod_{i=1}^{k}\left(1-s'_{\Psi_i}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1-s'_{\Psi_i}\right)^{\zeta_i}}}.$$

Similarly, we can prove

$$\frac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k}\sqrt[q]{\left(2-n_{\Psi_i}^{q}\right)^{\zeta_i}+\prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}} \preccurlyeq \frac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{k} n'^{\zeta_i}_{\Psi_i}}{\prod_{i=1}^{k}\sqrt[q]{\left(2-n'^{q}_{\Psi_i}\right)^{\zeta_i}+\prod_{i=1}^{k} n'^{\zeta_i}_{\Psi_i}}}.$$

Thus,

$$\sqrt[p]{\frac{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}}} - \sqrt[r]{\frac{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}}} - \left(\frac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k}\sqrt[q]{\left(2-n_{\Psi_i}^{q}\right)^{\zeta_i}+\prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}}\right)^{q}$$

$$\preccurlyeq \left(\sqrt[p]{\frac{\prod_{i=1}^{k}\left(1-m'_{\Psi_i}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1-m'_{\Psi_i}\right)^{\zeta_i}}}\right)^{p} - \left(\sqrt[r]{\frac{\prod_{i=1}^{k}\left(1-n'_{\Psi_i}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1-s'_{\Psi_i}\right)^{\zeta_i}}}\right)^{r} - \left(\frac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{k} n'^{\zeta_i}_{\Psi_i}}{\prod_{i=1}^{k}\sqrt[q]{\left(2-n'^{q}_{\Psi_i}\right)^{\mathfrak{g}}+\prod_{i=1}^{k} n'^{\zeta_i}_{\Psi_i}}}\right)^{q}$$

Let $\theta = p, q, r - SFEWA\left(\Psi_1, \Psi_2, \ldots, \Psi_k\right)$ and $\theta' = p, q, r - SFEWA\left(\Psi'_1, \Psi'_2, \ldots, \Psi'_k\right)$. Then by Definition 11, we have $\mathcal{S}(\theta) \leq \mathcal{S}\left(\theta'\right)$.

If $Scc\left(\Psi_i\right) \preccurlyeq Scc\left(\Psi'_i\right)$ then we have, $\theta \preccurlyeq \theta'$, i.e., $p, q, r - SFEWA\left(\Psi_1, \Psi_2, \ldots, \Psi_k\right) \preccurlyeq p, q - SFEWA\left(\Psi'_1, \Psi'_2, \ldots, \Psi'_k\right)$.

$Scc\left(\Psi_i\right) = Scc\left(\Psi'_i\right)$ then, we get,

$$\sqrt[p]{\frac{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}}} - \sqrt[r]{\frac{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i}}} - \left(\frac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k}\sqrt[q]{\left(2-n_{\Psi_i}^{q}\right)^{\zeta_i}+\prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}}\right)^{q}$$

$$= \left(\sqrt[p]{\frac{\prod_{i=1}^{k}\left(1-m'_{\Psi_i}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1-m'_{\Psi_i}\right)^{\zeta_i}}}\right)^{p} - \left(\sqrt[r]{\frac{\prod_{i=1}^{k}\left(1-s'_{\Psi_k}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1-s'_{\Psi_k}\right)^{\zeta_i}}}\right)^{r} - \left(\frac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{k} n'^{\zeta_i}_{\Psi_i}}{\prod_{i=1}^{k}\sqrt[q]{\left(2-n'^{q}_{\Psi_i}\right)^{\zeta_i}+\prod_{k=1}^{k} n'^{\zeta_i}_{\Psi_i}}}\right)^{q}$$

then, by condition $\Psi_{\Psi_i} \preccurlyeq \Psi'_{\Psi_i}$, $\varphi_{\Psi_i} \preccurlyeq \varphi'_{\Psi_i}$ and $\Theta_{\Psi_i} \preccurlyeq \Theta'_{\Psi_i}$ for all $i = 1, 2, \ldots, k$ we have

$$\left(\sqrt[p]{\frac{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i}}}\right)^{p} = \left(\sqrt[p]{\frac{\prod_{i=1}^{k}\left(1-m'_{\Psi_i}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1-m'_{\Psi_i}\right)^{\zeta_i}}}\right)^{p},$$

$$\left(\sqrt[r]{\frac{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{p}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{p}\right)^{\zeta_i}}}\right)^{r} = \left(\sqrt[r]{\frac{\prod_{i=1}^{k}\left(1-s'_{\Psi_i}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1-s'_{\Psi_i}\right)^{\zeta_i}}}\right)^{r}$$

and

$$\left(\frac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k}\sqrt[q]{\left(2-n_{\Psi_i}^{q}\right)^{\zeta_i}+\prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}}\right)^{q}$$

$$= \left(\frac{\left(\sqrt[q]{2}\right) \prod_{i=1}^{k} {n'}_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k} \sqrt[q]{\left(2 - {n'}_{\Psi_i}^{q}\right)^{\mathfrak{g}} + \prod_{i=1}^{k} {n'}_{\Psi_i}^{\zeta_i}}} \right)^{q}$$

Therefore, from Equation (10), we have

$$\sqrt[p]{\frac{\prod_{i=1}^{k} \left(1 + m_{\Psi_i}^{p}\right)^{\zeta_i}}{\prod_{i=1}^{k} \left(1 + m_{\Psi_i}^{p}\right)^{\zeta_i}}} - \sqrt[r]{\frac{\prod_{i=1}^{k} \left(1 + s_{\Psi_i}^{r}\right)^{\zeta_i}}{\prod_{i=1}^{k} \left(1 + s_{\Psi_i}^{r}\right)^{\zeta_i}}} - \left(\frac{\left(\sqrt[q]{2}\right) \prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k} \sqrt[q]{\left(2 - n_{\Psi_i}^{q}\right)^{\zeta_i} + \prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}} \right)^{q}$$

$$= \left(\sqrt[p]{\frac{\prod_{i=1}^{k} \left(1 - m'_{\Psi_i}\right)^{\zeta_i}}{\prod_{i=1}^{k} \left(1 - m'_{\Psi_i}\right)^{\zeta_i}}} \right)^{p} + \left(\sqrt[r]{\frac{\prod_{i=1}^{k} \left(1 - s'_{\Psi_i}\right)^{\zeta_i}}{\prod_{i=1}^{k} \left(1 - s'_{\Psi_i}\right)^{\zeta_i}}} \right)^{r} + \left(\frac{\left(\sqrt[q]{2}\right) \prod_{i=1}^{k} {n'}_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k} \sqrt[q]{\left(2 - {n'}_{\Psi_i}^{q}\right)^{\zeta_i} + \prod_{i=1}^{k} {n'}_{\Psi_i}^{\zeta_i}}} \right)^{q}$$

$$= \left(\sqrt[p]{\frac{\prod_{i=1}^{k} \left(1 - m'_{\Psi_i}\right)^{\zeta_i}}{\prod_{i=1}^{k} \left(1 - m'_{\Psi_i}\right)^{\zeta_i}}} \right)^{p} + \left(\sqrt[r]{\frac{\prod_{i=1}^{k} \left(1 - s'_{\Psi_i}\right)^{\zeta_i}}{\prod_{i=1}^{k} \left(1 - s'_{\Psi_i}\right)^{\zeta_i}}} \right)^{r} + \left(\frac{\left(\sqrt[q]{2}\right) \prod_{i=1}^{k} {n'}_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k} \sqrt[q]{\left(2 - {n'}_{\Psi_i}^{q}\right)^{\zeta_i} + \prod_{i=1}^{k} {n'}_{\Psi_i}^{\zeta_i}}} \right)^{q} \quad (16)$$

And hence $p, q, r - SFEWA\left(\Psi_1, \Psi_2, \ldots, \Psi_k\right) p, q, r - SFEWA\left(\Psi'_1, \Psi'_2, \ldots, \Psi'_k\right)$.

B. $p, q, r-$ SF EINSTEIN GEOMETRIC OPERATOR

Definition 20: Let $\Psi_i = \left(m_{\Psi_i}, s_{\Psi_i}, n_{\Psi_i}\right)$ $(i = 1, 2, \ldots, k)$ be a family of p, q, r-SFNs with weight vector $\zeta_i\,(i = 1, 2, \ldots, k)$ such that $\sum_{i=1}^{k} \zeta_i = 1$ and $\zeta_i \in [0, 1]$. Then the geometric operator for $p, q, r-$SF numbers is a mapping $\Lambda^k \to \Lambda$ and can be defined as follows:

$$p, q, r - SFEWG\left(\Psi_1, \Psi_2, \ldots, \Psi_k\right) = \otimes_{i=1}^{k} \left(\Psi_i\right)^{\zeta_i} \quad (17)$$

Theorem 7: The aggregated value of a collection of $p, q, r-$SFNs $\Psi_i = \left(m_{\Psi_i}, s_{\Psi_i}, n_{\Psi_i}\right)$ $(i = 1, 2, \ldots, k)$ utilizing $p, q, r - SFEWG$ operator is still a $p, q-$SFN, and can be defined as follows:

$$p, q, r - SFFWG\left(\Psi_1, \Psi_2, \ldots, \Psi_k\right) = \otimes_{i=1}^{k} \left(\Psi_i\right)^{\zeta_i}$$

$$= \begin{pmatrix} \langle \dfrac{\left(\sqrt[p]{2}\right) \prod_{i=1}^{k} m_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k} \sqrt[p]{\left(2 - m_{\Psi_i}^{p}\right)^{\zeta_i} + \prod_{i=1}^{k} m_{\Psi_i}^{\zeta_i}}}, \\ \langle \dfrac{\left(\sqrt[r]{2}\right) \prod_{i=1}^{k} s_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k} \sqrt[r]{\left(2 - s_{\Psi_i}^{r}\right)^{\zeta_i} + \prod_{i=1}^{k} s_{\Psi_i}^{\zeta_i}}}, \\ \sqrt[q]{\dfrac{\prod_{i=1}^{k} \left(1 + n_{\Psi_i}^{q}\right)^{\zeta_i} - \prod_{i=1}^{k} \left(1 - n_{\Psi_i}^{q}\right)^{\zeta_i}}{\prod_{i=1}^{k} \left(1 + n_{\Psi_i}^{q}\right)^{\zeta_i} + \prod_{i=1}^{k} \left(1 - n_{\Psi_i}^{q}\right)^{\zeta_i}}}, \rangle \end{pmatrix} \quad (18)$$

Proof. Straightforward.

Theorem 8: If the $p, q, r-$SFNs $\Psi_i = \left(m_{\Psi_i}, s_{\Psi_i}, n_{\Psi_i}\right)$ $(i = 1, 2, \ldots, k)$ are identical, i.e., be a $\Psi_i = \Psi$ for all i, where $\Psi = \left(m_{\Psi}, s_{\Psi}, n_{\Psi}\right)$, then $p, q, r - SFEWG\left(\Psi_1, \Psi_2, \ldots, \Psi_k\right) = \Psi$.

Theorem 9: Let $\Psi_i = \left(m_{\Psi_i}, s_{\Psi_i}, n_{\Psi_i}\right)$ $(i = 1, 2, .., k)$ be a collection of $p, q-$SFNs. If $\Psi^- = \min\left\{\Psi_1, \Psi_2, \ldots, \Psi_k\right\}$ and $\Psi^+ = \max\left\{\Psi_1, \Psi_2, \ldots, \Psi_k\right\}$, then $\Psi^- p, q, r - SFEWG\left(\Psi_1, \Psi_2, \ldots, \Psi_m\right) \Psi^+$.

Theorem 10: Let $\{\Psi_i | i = 1, 2, \ldots, k\}$ and $\left\{{\Psi'}_i | i = 1, 2, \ldots, i\right\}$ be two sets of $p, q, r-$SFNs, where of $\Psi_i = \left(m_{\Psi_i}, s_{\Psi_i}, n_{\Psi_i}\right)$ and ${\Psi'}_i = \left({m'}_{\Psi_i}, {s'}_{\Psi_i}, {n'}_{\Psi_i}\right)$ for $i = 1, 2, \ldots, k$. If $m_{\Psi_i} \preccurlyeq {m'}_{\Psi_i}$, $s_{\Psi_i} \preccurlyeq {s'}_{\Psi_i}$ and $n_{\Psi_i} \preccurlyeq {n'}_{\Psi_i}$ for all i, then $p, q, r - SFEWG\left(\Psi_1, \Psi_2, \ldots, \Psi_k\right) \preccurlyeq p, q, r - SFEWG\left(\Psi'_1, \Psi'_2, \ldots, \Psi'_k\right)$.

IV. APPLICATION

To tackle the MADM problem, we can use the proposed AOs, which consider the weight of unknown attributes and attribute values in the form of $p, q, r-$SFNs. Consider the set of distinct alternatives denoted as $\Delta = \{\Delta_1, \Delta_2, \ldots, \Delta_S\}$, which require analysis concerning precise criteria, implied as $\Theta = \{\Theta_1, \Theta_2, \ldots, \Theta_t\}$. The first step is determining the weights for the various attributes using the entropy technique, which compares attributes and measures uncertainty in the decision matrix. This gives each attribute a weight vector. Let the weight vector corresponding to the attribute $\Theta_j (j = 1, 2, \ldots, t)$ be ϖ_j where $\varpi_j \succ 0$ and $\sum_{j=1}^{t} \varpi_j = 1$. The decision matrix is then built using the evaluation values of every alternative with respect to each attribute. The preference values can be represented as $p, q, r-$SFNs. Equation (19) represent the decision matrix which contain the information provided by the decision maker in the form of $p, q, r-$SFNs.

$$\varepsilon = \begin{pmatrix} y_{11} & y_{12} \cdots & y_{1t} \\ y_{21} & y_{22} & \cdots & y_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ y_{s} & y_{s2} \cdots & y_{st} \end{pmatrix} \quad (19)$$

In this matrix, the rows represent the possibilities, while the columns represent the criteria that are considered by the decision-makers. In this context, the term $y_{ij} = \left(m y_{ij}, s y_{ij}, n y_{ij}\right)$ represent a $p, q, r-$SFN such that $\left(m y_{ij}\right)^q + \left(s y_{ij}\right)^r + \left(n y_{ij}\right)^q 1$ for all p, q and $r0$.

A. ALGORITHM

Step 1. Construct a decision matrix E as presented in Equation (19). Where all the data provided by the decision-makers are in the form of $p, q, r-$SFNs.

Step 2. The relative importance of criteria for $p, q, r-$spherical fuzzy information may be determined using a variety of approaches, one of which is the Analytic Hierarchy Process (AHP) [46]. AHP offers a systematic approach to calculating weights based on pairwise comparisons. In this paper we used AHP to determine the weights of criteria.

Step 3. Cost (C) and benefit ($\mathfrak{B}$) considerations are essential to decision-making processes across industries. In business, project management, and government, cost-benefit assessments guide choices by evaluating overall costs against projected benefits, guaranteeing financial viability and optimizing resource allocation. Healthcare decisions are based on health economic evaluations, which balance treatment costs and expected outcomes. Environmental impact assessments balance costs and benefits to guide legislation and promote sustainability. Businesses consider expenses as well as possible market rewards when developing and selling products. Technology adoption decisions weigh early investment against long-term rewards. Risk management performs cost-benefit calculations to evaluate risk mitigation techniques. When a decision matrix includes both cost and benefit criteria, it is important to normalize the information contained inside the matrix using the methodology described in Equation (20). If all attributes in the decision matrix E are of the benefit type, then this step can be skipped to normalize.

$$\mathcal{R} = \begin{cases} \left(m_{y_{ij}}, s_{y_{ij}}, n_{y_{ij}}\right) \text{for} j \in B \\ \left(n_{y_{ij}}, s_{y_{ij}}, m_{y_{ij}}\right) \text{for} j \in C \end{cases} \tag{20}$$

Step 4. Use the $p, q, r - SFEWA$or $p, q, r - SFEWG$ operator to calculate the integrated values of each alternative.

$$p, q, r - SFEWA\left(\Psi_1, \Psi_2, \ldots, \Psi_k\right) = \left(\begin{array}{c} \langle \sqrt[p]{\dfrac{\prod_{i=1}^{k}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i} - \prod_{i=1}^{k}\left(1-m_{\Psi_i}^{p}\right)^{\zeta_i}}{\prod_{i=1}^{m}\left(1+m_{\Psi_i}^{p}\right)^{\zeta_i} + \prod_{i=1}^{k}\left(1-m_{\Psi_i}^{p}\right)^{\zeta_i}}}, \\ \sqrt[r]{\dfrac{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i} - \prod_{i=1}^{k}\left(1-s_{\Psi_i}^{r}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+s_{\Psi_i}^{r}\right)^{\zeta_i} + \prod_{i=1}^{k}\left(1-s_{\Psi_i}^{r}\right)^{\zeta_i}}}, \\ \dfrac{\left(\sqrt[q]{2}\right)\prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k}\sqrt[q]{\left(2-n_{\Psi_i}^{q}\right)^{\zeta_i} + \prod_{i=1}^{k} n_{\Psi_i}^{\zeta_i}}} \rangle \end{array} \right)$$

$$p, q, r - SFEWG\left(\Psi_1, \Psi_2, \ldots, \Psi_k\right) = \left(\begin{array}{c} \langle \dfrac{\left(\sqrt[p]{2}\right)\prod_{i=1}^{k} m_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k}\sqrt[p]{\left(2-m_{\Psi_i}^{p}\right)^{\zeta_i} + \prod_{i=1}^{k} m_{\Psi_i}^{\zeta_i}}}, \\ \dfrac{\left(\sqrt[r]{2}\right)\prod_{i=1}^{k} s_{\Psi_i}^{\zeta_i}}{\prod_{i=1}^{k}\sqrt[r]{\left(2-s_{\Psi_i}^{r}\right)^{\zeta_i} + \prod_{i=1}^{k} s_{\Psi_i}^{\zeta_i}}}, \\ \sqrt[q]{\dfrac{\prod_{i=1}^{k}\left(1+n_{\Psi_i}^{q}\right)^{\zeta_i} - \prod_{i=1}^{k}\left(1-n_{\Psi_i}^{q}\right)^{\zeta_i}}{\prod_{i=1}^{k}\left(1+n_{\Psi_i}^{q}\right)^{\zeta_i} + \prod_{i=1}^{k}\left(1-n_{\Psi_i}^{q}\right)^{\zeta_i}}}, \rangle \end{array} \right)$$

Step 5. Calculate the score value for each aggregated value using Equation (10).

Step 6. Determine the best alternative using the following steps:

i The alternative that possesses the highest score value would be regarded as the best choice.

ii If there are multiple alternatives with the same score values, use Equation (11) to evaluate the accuracy function and compare these alternatives.

iii If multiple alternatives have equal scores and accuracy values, any of them can be selected as the best option.

V. ILLUSTRATIVE EXAMPLE

We are dealing with a case where some experts give their opinion on several product development projects on several attributes. To obtain a complete assessment, we apply the suggested mathematical method to merge the experts' points of view while considering their relative expertise. The flowchart of the proposed model is presented in Figure 3.

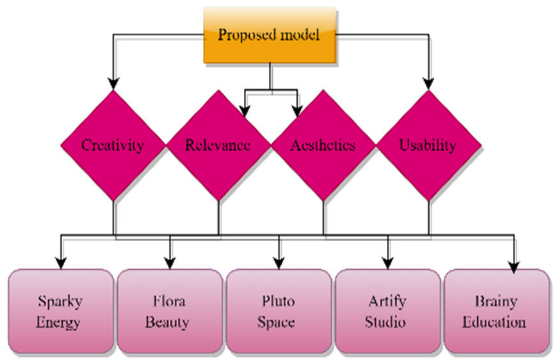

FIGURE 3. Design of the projected MCDM scheme.

A. CASE STUDY

Suppose a scenario where a team of three designers, denoted as $\mathcal{X}^{(1)}$, $\mathcal{X}^{(2)}$, and $\mathcal{X}^{(3)}$, is responsible for forming five different logo designs as follows:

- Sparky Energy (Δ_1)
- Flora Beauty (Δ_2)
- Pluto Space (Δ_2)
- Artify Studio (Δ_2)
- Brainy Education (Δ_2)

These designers are expected to create these logos based on four criteria:

- Creativity (CR) Evaluating the originality and uniqueness of the logo design.
- Relevance (RE) Assessing the suitability and alignment of the logo design with the brand identity and message.
- Aesthetics (AE) Judging the visual appeal and attractiveness of the logo design.
- Usability (US) Measuring the adaptability and versatility of the logo design across different platforms and contexts.

To account for the preferences of each designer, a vector of weights $\omega = (0.25, 0.4, 0.35)$ is assigned to $\mathcal{X}^{(1)}$, $\mathcal{X}^{(2)}$, and $\mathcal{X}^{(3)}$, respectively. The evaluation matrices $R^{(1)}$, $R^{(2)}$, and $R^{(3)}$, generated using experts evaluation method with parameters $p = 2$ and $q = 3$ are presented in Tables 1-3.

The main goal in this scenario is to make a rational decision regarding the best logo design for each brand. To achieve this, a MAGDM framework is proposed. The main objective of MAGDM approach is to enable a comprehensive comparison of the various alternatives based on the criteria important to logo design decision making. These criteria include creativity, relevance, aesthetics, and usability. By following a systematic approach, logo design teams aim to improve their decision-making process and increase the satisfaction of their clients.

TABLE 1. Decision matrix provided expert $\mathcal{R}^{(1)}$.

Δ_i	Θ_1	Θ_2	Θ_3	Θ_4
Δ_1	(0.65,0.55,0.40)	(0.65,0.45,0.25)	(0.55,0.35,0.45)	(0.40,045,0.70)
Δ_2	(0.55,0.65,0.30)	(0.80,0.50,0.15)	(0.65,0.25,0.34)	(0.50,0.70,0.60)
Δ_3	(0.35,0.50,0.75)	(0.65,0.80,0.30)	(0.50,0.60,0.65)	(0.30,0.45,0.65)
Δ_4	(0.70,0.60,0.15)	(0.60,0.70,0.55)	(0.75,0.15,0.30)	(0.50,0.35,0.55)
Δ_5	(0.35,0.44,0.55)	(0.35,0.80,0.65)	(0.35,0.45,0.55)	(0.70,0.150.45)

TABLE 2. Decision matrix provided expert $\mathcal{R}^{(2)}$.

Δ_i	Θ_1	Θ_2	Θ_3	Θ_4
Δ_1	(0.75,0.35,0.55)	(0.70,35,0.50)	(0.60,45,0.35)	(0.55,0.15,0.70)
Δ_2	(0.65,0.15,0.45)	(0.65,0.30,0.55)	(0.55,0.55,0.35)	(0.50,0.35,0.65)
Δ_3	(0.40,0.35,0.65)	(0.45,0.70,0.45)	(0.50,0.65,0.35)	(0.35,0.45,0.65)
Δ_4	(0.35,0.50,0.60)	(0.85,0.10,0.65)	(0.30,0.60,0.35)	(0.65,0.55,0.35)
Δ_5	(0.35,0.75,0.45)	(0.45,0.60,0.75)	(0.45,0.25,0.55)	(0.60,0.35,0.45)

TABLE 3. Decision matrix provided expert $\mathcal{R}^{(3)}$.

Δ_i	Θ_1	Θ_2	Θ_3	Θ_4
Δ_1	(0.70,0.35,0.55)	(0.65,0.45,0.35)	(0.65,0.50,0.45)	(0.70,0.35,0.50)
Δ_2	(0.60,0.15,0.45)	(0.80,0.50,0.25)	(0.55,65,0.35)	(0.65,0.35,0.55)
Δ_3	(0.40,0.35,0.60)	(0.65,0.85,0.30)	(0.30,0.55,0.75)	(0.45,0.70,0.45)
Δ_4	(0.35,0.50,0.65)	(0.60,0.70,0.55)	(0.75,0.60,0.15)	(0.85,0.15,0.65)
Δ_5	(0.55,0.70,0.45)	(0.35,0.80,0.60)	(0.35,0.44,0.55)	(0.45,0.65,0.75)

The data do not need to be standardized since each of the four characteristics belongs to similar categories. This means that the values can be compared directly without the need for normalization or scaling methods. By not standardizing the data, it is easier to analyze and compare the attributes since they are already in the same range and the same units. However, it is essential to mention that if the attributes were not similar in nature, standardization would be necessary to ensure that each attribute has equal weight in the decision-making process. However, in this case, the absence of the need for standardization facilitates the MAGDM process and speeds up the decision-making procedure.

The weight vector for the attributes calculated by AHP is given by: $\mathcal{w} = (0.2515, 0.2017, 0.1996, 0.1627, 0.1845)$. It can be observed that the total of $\mathcal{w}_j (j = 1, 2, 3, 4, 5)$ is equal to one.

B. BY USING $p, q, r-$ SFEWA OPERATOR FOR $p = 2$ AND $q = 3$

Table 4 presents the aggregated values of alternatives by utilizing the $p, q, r-$SFEWA operator and assuming the weights of attributes is (0.2515, 0.2017, 0.1996, 0.1627, 0.1845).

TABLE 4. The aggregated values of experts by using $p, q, r-$SFEWA operator.

Δ_i	Θ_1	Θ_2	Θ_3	Θ_4
Δ_1	(0.5531, 0.4123, 0.4375)	(0.4732, 0.3294, 0.4516)	(0.5142, 0.2916, 0.3686)	(0.5013, 0.3834, 0.4551)
Δ_2	(0.4572, 0.4566, 0.3417)	(0.4523, 0.4520, 0.3946)	(0.4256, 0.4147, 0.3560)	(0.4021, 0.3457, 0.3964)
Δ_3	(0.4641, 0.3368, 0.3280)	(0.4150, 0.3704, 0.4022)	(0.4452, 0.4314, 0.4526)	(0.4388, 0.4024, 0.3423)
Δ_4	(0.5233, 0.4418, 0.4609)	(0.4937, 0.3054, 0.4015)	(0.4521, 0.4678, 0.4322)	(0.4521, 0.3675, 04532)
Δ_5	(0.4890, 0.4532, 0.3883)	(0.4773, 0.4269, 0.3678)	(0.4870, 0.4537, 0.3177)	(0.4436, 0.3902, 0.4110)

Again using $p, q, r-$SFEWA operator to calculate the overall integrating values of each alternative and summarized in Table 5.

TABLE 5. Overall rating values of alternatives.

Alternatives	Aggregated values	$Sco(\Delta_i)$
Δ_1	(0.5130,0.4045,0.3867)	0.5696
Δ_2	(0.4515,0.4032,0.4123)	0.5341
Δ_3	(0.3955,0.4320,0.3809)	0.5103
Δ_4	(0.4617,0.4546,0.4726)	0.5068
Δ_5	(0.4760,0.4058,0.3812)	0.5522

Table 5 makes it clear that the ranking is determined by the score values and goes like this: a1 is clearly the most advantageous choice out of the options that are offered since $\Delta_1 \succ \Delta_5 \succ \Delta_2 \succ \Delta_3 \succ \Delta_4$. This rating highlights option Δ_1'S (Sparky Energy) beneficial position inside the decision framework and emphasizes how much better it is than the other options. The graphical representation of score values is presented in Figure 4.

C. BY USING $p, q, r-$ SFEWG OPERATOR FOR $p = 2$ AND $q = 3$

Using the $p, q, r-$SFEWG operator and to aggregate the rating values of each alternative. We used the attribute weight vector (0.2515, 0.2017, 0.1996, 0.1627, 0.1845). The aggregated values are listed in Table 6.

The assembled data in Table 6 is subjected to another round of aggregation, with the results given simply in Table 7. By applying Equation (11) to calculate the score values of alternatives.

Table 1 analysis shows that option Δ_1 continuously retains the highest score in all scenarios. However, it is remarkable that the score values of alternatives Δ_3 and Δ_4 overlap for

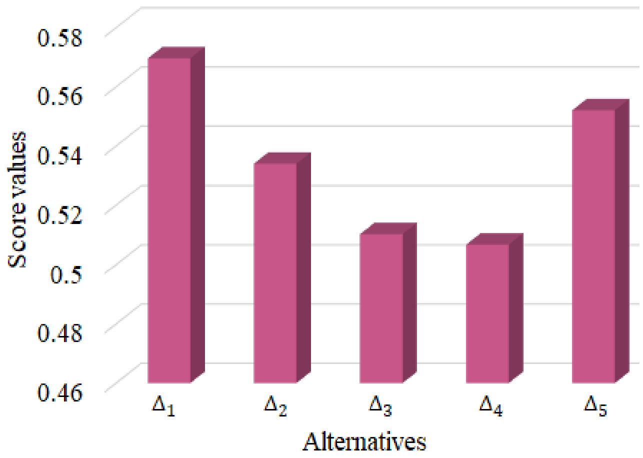

FIGURE 4. Score values of available alternatives.

TABLE 6. The aggregated values of alternative by using $p, q, r-$SFEWG operator.

Δ_i	Θ_1	Θ_2	Θ_3	Θ_4
Δ_1	(0.3914, 0.3723, 0.3145)	(0.3759, 0.36730. 2841)	(0.3533, 0.3069, 0.2966)	(0.3479, 0.3024, 0.2830)
Δ_2	(0.3425, 0.3076, 0.2901)	(0.3245, 0.3168, 0.3194)	(0.3260, 0.3451, 0.3431)	(0.3405, 0.3135, 0.2678)
Δ_3	(0.3523, 0.3456, 0.2880)	(0.2927, 0.3532, 0.2806)	(0.3089, 0.2903, 0.2899)	(0.3455, 0.3006, 0.2902)
Δ_4	(0.3601, 0.3729, 0.3972)	(0.3543, 0.3411, 0.2764)	(0.3246, 0.3453, 0.3014)	(0.3140, 0.3022, 0.3245)
Δ_5	(0.3858, 0.3168, 0.2750)	(0.3728, 0.3462, 0.2788)	(0.36781, 0.3067, 0.2908)	(0.3466, 0.3216, 0.2620)

TABLE 7. Aggregated values and score values of alternatives.

Δ_i	Aggregated values	$Sco(\Delta_i)$
Δ_1	(0.4895,0.3778,0.3576)	0.5700
Δ_2	(0.4239,0.3783,0.3870)	0.5338
Δ_3	(0.3668,0.4096,0.3575)	0.5101
Δ_4	(0.4327,0.4264,0.4473)	0.5101
Δ_5	(0.4354,0.3699,0.3461)	0.5488

$p = 2$, $q = 3$, and $r = 3$. As a result, the accuracy values of these alternatives can be determined using Equation (12), resulting in $Acc(\Delta_3) = 0.2490$ and $Acc(\Delta_4) = 0.2816$. Therefore, the overall ranking order of the available alternatives is $\mathbf{\Delta_1 \succ \Delta_5 \succ \Delta_2 \succ \Delta_4 \succ \Delta_3}$.

D. THE IMPACT OF PARAMETERS p, q AND r OVER SCORE VALUES AND RANKING ORDER

As the proposed AOs retain symmetry with respect to the parameters p, q and r, an investigation of the influence of these factors on the final ranking of alternatives is carried out. This study uses simultaneous variations of p and q, and the associated score values, as well as the resulting ranking order, are shown in Tables 8 and 9. Examining these tables reveals that providing different sets of parameters p and q yields varied score values for the aggregated numbers. Notably, despite these adjustments, the alternate rankings remain intact. Note that for this investigation we used $p, q, r-$SFEWA operator.

TABLE 8. The impact of p over decision results.

p	q	r	Score values				
			Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
3	2	3	0.5518	0.5227	0.5021	0.4934	0.5446
4	2	4	0.5575	0.5243	0.5066	0.4956	0.5463
5	2	5	0.5603	0.5272	0.5097	0.4971	0.5487
6	2	6	0.5621	0.5291	0.5113	0.4985	0.5502
7	2	7	0.5645	0.5305	0.5129	0.4996	0.5509
8	2	8	0.5669	0.5319	0.5146	0.5004	0.5516
9	2	9	0.5683	0.5331	0.5162	0.5011	0.5521
10	2	10	0.5698	0.5338	0.5173	0.5017	0.5524

This feature of the suggested operators is more important in real decision-making situations. For instance, observations show that when the parameters grow, so do the score values of the alternatives, providing decision-makers with a more positive outlook. Therefore, it would be appropriate to give these parameters greater values throughout the aggregation process when decision-makers have an optimistic view. The influence of parameter p is shown in Figure 5.

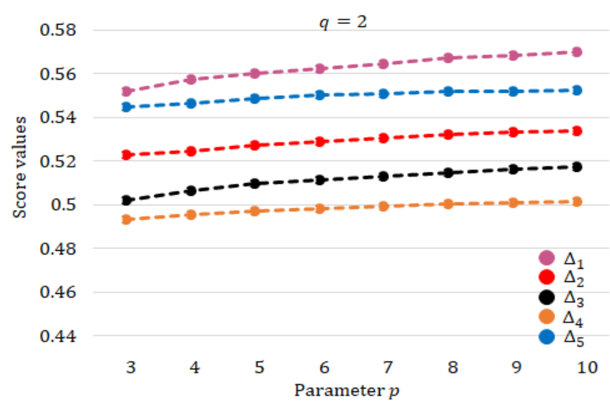

FIGURE 5. The influence of parameter p.

TABLE 9. The impact of q over score values.

q	p	r	Score values				
			Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
3	2	3	0.5696	0.5341	0.5103	0.5068	0.5522
4	2	4	0.5724	0.5360	0.5135	0.5088	0.5536
5	2	5	0.5743	0.5387	0.5168	0.5097	0.5547
6	2	6	0.5766	0.5410	0.5186	0.5108	0.5561
7	2	7	0.5784	0.5429	0.5213	0.5115	0.5572
8	2	8	0.5797	0.5441	0.5226	0.5125	0.5580
9	2	9	0.5807	0.5464	0.5239	0.5133	0.5591
10	2	10	0.5812	0.5476	0.5251	0.5142	0.5602

On the other hand, lower values might be assigned to these criteria when decision-makers have a gloomy perspective, which will reduce the overall rating values. However, it is significant that the optimal option remains stable, suggesting that the results are unbiased and unaffected by the optimism or pessimism of those making the decisions. As such, the ranking results are considered to be trustworthy.

E. VALIDITY TEST

To highlight the adaptability of our proposed approach in various scenarios, we applied evaluation protocols developed by Wang et al. [26] as follows:

Step 1. "Assuming constant relative weights of criteria, substituting lower rated values for suboptimal options should not impact the identification of the optimal alternative. Thus, the top ranked alternative would remain unchanged.

Step 2. The procedure should be transitive.

Step 3. If a specific issue is divided into smaller problems, applying the same decision-making method should maintain the overall order of options as the original ranking."

Validity test using criterion 1. The ranking order obtained by our proposed approach is

$\Delta_1 \succ \Delta_5 \succ \Delta_2 \succ \Delta_3 \succ \Delta_4$. To test criterion 1, we replaced the non-optimal alternative Δ_4 with the worst alternative Δ_4^*, assuming rating values of Δ_4^* to be $(0.20, 0.15, 0.50)$, $(0.10, 0.10, 0.60)$, $(0.15, 0.10, 0.40)$ and $(0.30, 0.15, 0.80)$. Following the observation, our approach yielded aggregated values are $Sco\left(\Delta_4^*\right) = 0.4128$, $Sco(\Delta_1) = 0.5696$, $Sco(\Delta_2) = 0.5341$, $Sco(\Delta_3) = 0.5101$, $Sco(\Delta_3) = 0.5101$ and $Sco(\Delta_5) = 0.5446$ where we used the parametric values $p = 2$ and $q = 3$. The ranking order remained $\Delta_1 \succ \Delta_5 \succ \Delta_2 \succ \Delta_3 \succ \Delta_4$, with the best alternative remaining consistent. Thus, our approach provides consistent findings with respect to criterion 1.

F. COMPARATIVE STUDIES

The suggested MAGDM approach is compared against a number of current operators in this section, including T-spherical fuzzy aggregation operators [47], [48], [49], $p, q, r-$spherical fuzzy aggregation operators [25], [50], and spherical fuzzy aggregation operators [51], [52]. Table 10 combines the ideal score values with the options' order of ranking. This table'ß analysis shows that the best option is consistent with the results of the suggested strategy, demonstrating the method'ß resilience when compared against cutting-edge techniques.

TABLE 10. Comparison.

Methods	Score values					Option
	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	
Garg et al. [47]	0.2349	0.1864	0.1790	0.1548	0.1945	Δ_1
Chen et al. [48]	0.3241	0.2421	0.2280	0.2179	0.2908	Δ_1
Mahnaz et al. [49]	0.2763	0.2355	0.2278	0.2031	0.2534	Δ_1
Rahim et al. [25]	0.5482	0.5147	0.5038	0.4970	0.5326	Δ_1
Akhtar et al. [50]	0.4355	0.3848	0.3716	0.3418	0.4093	Δ_1
Ashraf et al. [51]	0.3147	0.2560	0.2481	0.2357	0.2839	Δ_1
Ashraf et al. [52]	0.4912	0.4544	0.4356	0.4265	0.4873	Δ_1

Thus, the $p, q, r-$spherical fuzzy set framework's suggested generalized Einstein aggregation operators provide a new and adaptable approach to dealing with uncertainty in decision-making issues. Due to their design for the $p, q, r-$SP environment, these operators provide a more flexible and effective means of determining the best choice in real-world scenarios. The graphical view of score values obtained by different existing approaches is presented in Figure 6.

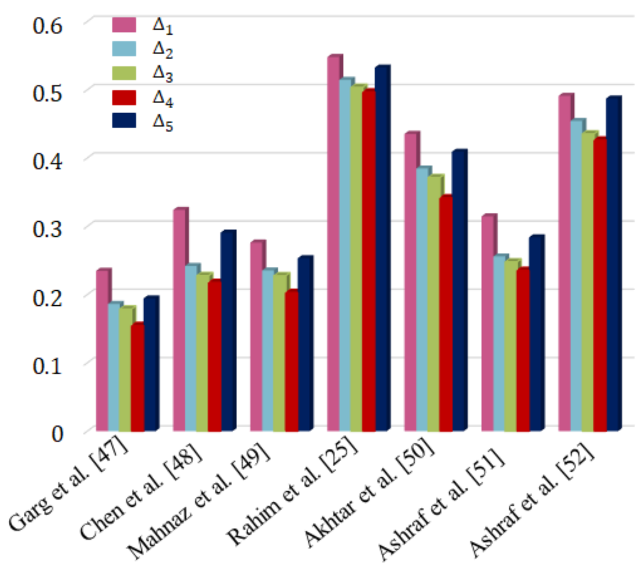

FIGURE 6. Score values of available alternative obtained by existing approaches.

G. ADVANTAGES

In compared to preceding methodologies, the suggested approach to decision-making allows for more flexibility. This is the outcome of its widespread application in a variety of scenarios and issue areas. The approach extends beyond both flexible systems and constraints. This flexibility enables decision-makers to adjust and personalize the decision-making process based on their needs and preferences. The suggested decision-making method clearly allows for more flexibility. By using parameters like p, q, and r, this enhancement is made possible. Decision-makers may customize and modify the decision-making process to suit their own requirements and preferences thanks to these features. Consequently, the proposed technique yields a personalized and adaptable foundation. The widespread use of this framework results in a strong, adaptable framework that can manage a variety of decision-making scenarios. This resilience ensures dependable and accurate outcomes under a range of conditions.

H. LIMITATIONS

- Using $p, q, r-$SF Einstein AOs with parameterized parameters (p, q and r) might complicate procedures for making decisions.
- Implementing $p, q, r-$SF Einstein AOs and algorithms for multi-criteria DM may be time-consuming and

computationally challenging, particularly in substantial scenarios with multiple choices.

VI. CONCLUSION

In this paper, we defined novel $p, q, r-$SF operation laws using Einstein sum and Einstein product, which are more elastic and adjustable than the existing ones. Based on these operational laws a series AOs such as $p, q, r-$SFEWA and $p, q, r-$SFEWG are defined to integrate $p, q, r-$SF information. Furthermore, some properties and special cases of the proposed AOs are discussed in detail. By using these AOs, we constructed a MAGDM approach to deal to real-life DM problems. To illustrate the applicability and effectiveness of the technique we presented a real-life example, where we use the proposed operators to evaluate logo design, which is a vital part of product development. The outcomes of the proposed method are compared with some existing approach to validation. It the end some advantages and limitations of the proposed approach are discussed.

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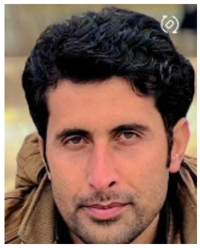

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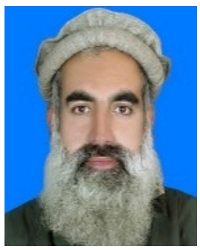

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