IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received 5 April 2024, accepted 22 April 2024, date of publication 25 April 2024, date of current version 6 May 2024.

Digital Object Identifier 10.1109/ACCESS.2024.3393555

RESEARCH ARTICLE

New CG-Based Algorithms With Second-Order Curvature Information for NLS Problems and a 4DOF Arm Robot Model

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This work was supported by the Yayasan Universiti Teknologi PETRONAS (YUTP) Grant funded by Universiti Teknologi PETRONAS, Malaysia, and Institute of Autonomous Systems under Grant 015LC0-315.

ABSTRACT This article develops two new algorithms of three-term conjugate gradient (TTCG) coefficients to handle nonlinear least-squares (NLS) problems using the structured secant equation. Motivated by improved conjugacy and sufficient descent conditions, we developed two new formulations of CG coefficients. Furthermore, with these parameters, two search directions are proposed that satisfy the sufficient descent condition, which further enhances the efficiency of the proposed strategies. One key advantage of the proposed techniques is their low memory requirements, rendering them suitable for large-scale nonlinear least squares problems. Moreover, some mild suppositions and a non-monotone line search are used to establish the global convergence properties of the methods. More so, we investigate the robustness and effectiveness of the proposed methods numerically by performing experiments on benchmark test problems, and their performance is compared against existing methods. The outcomes of these experiments indicate that the proposed methods outperform the other techniques regarding the metrics of comparison adopted. Finally, the algorithms are applied to an extension of the model of robotic motion control of four degrees of freedom (4DOF), resulting in positive outcomes for the robot's motion traits.

INDEX TERMS Three-term method, nonlinear least-squares, convergence analysis, robotics, conjugate gradient.

I. INTRODUCTION

A mathematical problem formulated as a nonlinear least-squares (NLS) poses various challenges in domains within science and engineering. This problem encompasses tasks such as data fitting, parameter estimation, robotic motion, experimental design, image restoration, and full waveform inversion, among others (refer to, for example, [1], [2], [3], [4], [5], [6], [7]). Let's examine the general optimization problem without constraints:

$$\min_{x\in\mathbb{R}^n} f(x), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, and $\mathbb{R}^n$ represents the n-dimensional Euclidean space with its norm denoted by $\|\cdot\|$. In this context, the gradient and Hessian of the function (1) at the point x are represented by $g(x) = \nabla f(x)$ and $B = \nabla^2 f(x)$, respectively. CG is frequently utilized among the well-known iterative algorithms employed

The associate editor coordinating the review of this manuscript and approving it for publication was Qiang Li.

to solve the optimization problem (1) [8]. This method updates its sequence of iterations using the formula below,

$$x_{k+1} = x_k + \alpha_k d_k \tag{2}$$

where $k \geq 0$, and $\alpha_k > 0$ represent the step size calculated through a line search technique in the direction of the search vector d_k [9], [10]. For $k = 0$, then $d_k = -g_k$, this is recognized as the steepest descent direction. However, subsequent search directions undergo updates via:

$$d_k = -g_k + \beta_k d_{k-1}. \tag{3}$$

where, the parameter β_k denotes the CG coefficient, the formulas for Hestenes-Stiefel (HS) [11], Dai-Yuan (DY) [12], Polak-Ribiere and Polyak (PRP) [13], [14], and Fletcher-Reeves (FR) [15], four of the well-known traditional two-term CG methods for solving equation (1), are as follows:

$$\beta_k^{HS} = \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}}, \quad \beta_k^{DY} = \frac{\|g_k\|^2}{d_{k-1}^T y_{k-1}}, \tag{4}$$

$$\beta_k^{PRP} = \frac{g_k^T y_{k-1}}{\|g_{k-1}\|^2}, \quad \beta_k^{FR} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2}. \tag{5}$$

where $y_{k-1} = g_k - g_{k-1}$. More recently, additional variations for the parameter β_k have emerged. For example, the (PRP) method was modified, and the formula (RMIL) was proposed in [16],

$$\beta_k^{RMIL} = \frac{g_k^T y_{k-1}}{\|d_{k-1}\|^2}. \tag{6}$$

More so, Dai et al. [17] suggested their formula (DL) employing the new conjugacy condition giving rise to a class of one-parameter CG algorithms, and the β_k being defined as follows:

$$\beta_k^{DL} = \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} - t\frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}, \quad t \geq 0. \tag{7}$$

Many first-order variant approaches have been developed to address NLS problems related to large-scale data sets [18], [19], [20]. Nevertheless, a typical flaw with first-order approaches is the inability of the methods to find a descent direction, potentially leading to stagnation or inaccurate solutions. Recently, numerous research studies have been conducted in the conjugate gradient method, focusing on secant conditions that incorporate second-order information regarding the objective function. For instance, Perry [21] introduced a conjugate gradient method by integrating the secant condition from the quasi-Newton method. Subsequently, Dai et al. [17] proposed two nonlinear CG methods that utilize the secant condition and a novel proposed conjugacy condition. However, within the context of numerical experimentation, one of these methods exhibited spectacular performance compared to the classical HS method. Yabe et al. [22] derived a new conjugacy condition based on the Dai et al. formula and proposed another CG parameter using the modified secant condition proposed by [23] and [24]. Furthermore, Zhou and Zhang [25] proposed a modified version of Dai et al. and Yabe-Takano methods using the MBFGS condition, and in a similar attempt, Ford et al. [26]proposed two CG methods based on the multi-step secant condition. Moreover, for more detail on the secant conditions-based methods, kindly refer to [27], [28], [29], and [30]. Recently, solving NLS problems through utilizing the CG methods has received special attention. Kobayashi et al. [31] extended the nonlinear CG methods by merging the structured secant relation and the Dai et al. [17] scheme to solve NLS problems. However, as highlighted by [32], these methods do not consistently produce a descent search direction. To ameliorate these limitations, some authors have suggested TTCG methods that rely on secant relations, ensuring consistent satisfaction of the sufficient descent criterion. There are quite some methods with three-term CG structures developed recently to solve equation (1). For instance, some studies have proposed a hybrid TTCG algorithm that always satisfies the sufficient descent condition; other studies have introduced a scaled TTCG method that utilizes the DFP update for the inverse Hessian approximation. These TTCG methods have shown excellent applicable characteristics when used on various real-world problems, including COVID-19 modeling, sparse signal restoration, image restoration, portfolio selection, and tomography [33], [34], [35], [36], [37], [38], [39], [40], [41]. Moreover, if the TTCG approaches are compared with conventional two-term CG methods, according to [42]. The TTCG algorithms consistently achieve good numerical performance and possess appropriate theoretical properties such as sufficient descent and trust region properties. Babaie-Kafaki and Ghanbari, as described in [43], introduced a modified version of the three-term HS-DL method where the direction takes on the following structure:

$$d_k = -g_k + \beta_k^{HS} d_{k-1} - t\frac{g_k^T s_{k-1}}{|y_{k-1}^T d_{k-1}|} d_{k-1} + \delta_k^{HS} y_{k-1},$$

$$\delta_k^{HS} = \frac{-g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}}, \quad k \geq 0.$$

and in [44], the authors proposed a hybrid TTCG method that satisfies the sufficient descent condition, and its direction is defined by

$$d_0 = -g_0, \; d_{k+1} = -g_k + \beta_k^{HTHP} d_{k-1} + \kappa_k^{HTHP} r_{k-1},$$

where,

$$\beta_k^{HTHP} := \frac{g_k^T r_{k-1}}{\eta_{k-1}} - \frac{\|r_{k-1}\|^2 g_k^T d_{k-1}}{\eta_{k-1}^2},$$

$$\kappa_k^{HTHP} := c_{k-1}\frac{g_k^T d_{k-1}}{\eta_{k-1}},$$

and $\eta_{k-1} := max\left\{\mu\|d_{k-1}\|\|r_{k-1}\|, d_{k-1}^T r_{k-1}, \|g_{k-1}\|^2\right\}$, $\mu > 0$. where, $c = \frac{g_k^T(r_{k-1}-s_{k-1})}{\|g_k\|^2}$ with $c_k := min\{\bar{c}, max\{0, c\}\}$ and $0 \leq c_k \leq \bar{c} < 1$.

In light of all the developments made in the three-term methods, it is intriguing to devise a more efficient one. However, despite their outstanding features, the TTCG methods for solving the nonlinear least squares problem received little attention. This paper introduces two distinct approaches for determining the coefficients of the TTCG method based on quasi-Newton's secant equation. The first method uses the Dai et al. conjugacy condition, while the second relies on the sufficient descent property. The key contributions of this research can be outlined as follows:

- Propose two new TTCG coefficients using the second-order curvature information of the objective function, developed based on sufficient descent properties and the Dai et al. conjugacy condition.
- The proposed search direction satisfies the criteria of the sufficient descent property.
- With the aid of a non-monotone line search, the proposed coefficients have been shown to achieve global convergence under some mild hypotheses.
- Numerical experiments were carried out to assess the performance of the proposed approaches compared to other methods documented in the literature.
- Finally, the proposed methods were applied to solve the motion-control problem of a four-degree-of-freedom (4DOF) robotic system, demonstrating their practicality.

Moreover, in this work, we adopt Zhang and Hager's [45] non-monotone line-search method to determine the step length α_k. When the direction d_k demonstrates a substantial descent, the step length α_k is determined according to the non-monotone line-search conditions of the Armijo type described below:

$$f(x_k + \alpha_k d_k) \leq Q_k + \delta \alpha_k g_k^T d_k, \tag{8}$$

where,

$$\begin{cases} Q_0 = f(x_0), \\ Q_{k+1} = \dfrac{\eta_k P_k Q_k + f(x_{k+1})}{P_{k+1}}, \\ P_0 = 1, \\ P_{k+1} = \eta_k P_k + 1. \end{cases} \tag{9}$$

Remark 1 [45]: The sequence P_k is situated between $f(x_k)$ and ψ_k, where

$$\psi_k := \frac{1}{k+1} \sum_{i=0}^{k} f(x_i), \ k \geq 0. \tag{10}$$

The remaining sections of this article are arranged as follows: The proposed approach and its algorithms are described in Section II. The global convergence properties of the suggested algorithms are examined in Section III under certain hypotheses using the non-monotone Armijo-type line search. In Section IV, the effectiveness of the proposed algorithms is examined using numerical experiments compared to other methods described in the literature. Finally, Section V applies the new approach to a four-degrees-of-freedom robotics problem.

II. MOTIVATION AND FORMULATION OF THE TTCG COEFFICIENTS

In this section, we begin by considering the three-term gradient method for solving NLS problems as follows:

$$\min_{x \in \mathbb{R}^n} f(x), \quad f(x) = \frac{1}{2} \sum_{i=1}^{m} (R_i(x))^2, \ x \in \mathbb{R}^n \tag{11}$$

Given $R(x)$ as $(R_1(x), R_2(x), \cdots, R_m(x))^T$, where each individual residual $R_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2, \cdots, m$ is a smooth function, we define $J(x) \in \mathbb{R}^{m \times n}$ as the Jacobian matrix associated with the residual function $R(x)$. Additionally, we use $\nabla g(x)$ to represent the gradient of the objective function and $\nabla^2 f(x)$ to represent the Hessian of the same objective function [46]. Notably, the NLS problem expressed in equation (11) exhibits a unique structure for its gradient and Hessian matrix, which can be expressed as follows:

$$g(x) := \sum_{i=1}^{m} R_i(x) \nabla R_i(x) = J(x)^T R(x), \tag{12}$$

$$B(x) := \sum_{i=1}^{m} \nabla R_i(x) \nabla R_i(x)^T + \sum_{i=1}^{m} R_i(x) \nabla^2 R_i(x) \tag{13}$$

$$= J(x)^T J(x) + P(x), \tag{14}$$

where $J(x) = R'(x)$ represents the Jacobian matrix of the residual function, and the matrix $P(x)$ corresponds to the second term mentioned in equation (14). First, we will state as shown in [31], [32], and [47] e.t.c. The structured vector approximation, which is an action of a vector on a matrix, is derived from Taylor series approximations of the Hessian of the objective function defined in (14) such that the following secant equation is fulfilled:

$$B(x)s = \bar{\omega}, \tag{15}$$

here $s = x^+ - x^-$, the difference between current point x^+ and previous point x^- and $\bar{\omega}$ denotes an appropriately structured vector defined as follows:

$$\bar{\omega} = (J^+)^T J^+ s + (J^+ - J^-)^T R(x). \tag{16}$$

where J^+ and J^- denote the current and previous Jacobian matrix of the residual function $R(x)$.

Next, we describe one of the efficient three-term CG (TTCG) methods presented by Zhang et al. [48]; it possesses a decent property. This method utilizes the following search directions,

$$d_k = -g_k + \beta_k^{HS} d_{k-1} - \frac{g_k^T d_{k-1}}{d_{k-1}^T y_{k-1}} y_{k-1}, \ \forall \ k \geq 0. \tag{17}$$

Motivated by their work, We proposed a novel search direction for the three-term parameters, for $n = 1, 2$, defined as follows:

$$d_0 = -g_0, \ d_k = -g_k + \beta_k^{(n)} \bar{\omega}_{k-1} - \beta_k^{(n)} s_{k-1} \ \forall \ k \geq 0. \tag{18}$$

where $s_{k-1} = x_k - x_{k-1}$, and $\beta_k^{(n)}$ is the CG coefficient to be determined through utilization of the idea presented in [49] for the cases when $n = 1, 2$, respectively.

A. DETERMINATION OF β_K^{TTCGC1} VIA THE CONJUGACY CONDITION

In this subsection, we apply the property of the conjugacy condition to derive the first formula for the proposed TTCG coefficient since, for quasi-Newton methods, the search direction d_k can be expressed as

$$d_k = -B_k g_k, \tag{19}$$

where B_k is a symmetric and positive definite matrix that approximates the Hessian $\nabla^2 f(x_k)$, or its inverse, which adheres to the secant equation:

$$B_k \bar{\omega}_{k-1} = s_{k-1},\ \bar{\omega}_{k-1} = J_k^T J_k s_{k-1} + (J_k - J_{k-1})^T R(x_k), \tag{20}$$

where $B_k \approx \nabla^2 f(x_k)^{(-1)}$. Therefore, by multiplying the transpose of the direction d_k as defined in (19) and $\bar{\omega}_{k-1}$, to equation (20) we have

$$d_k^T \bar{\omega}_{k-1} = -(B_k g_k)^T \bar{\omega}_{k-1} = -g_k^T (B_k \bar{\omega}_{k-1}) = -g_k^T s_{k-1}, \tag{21}$$

Hence, equation (21) can be replaced by the relation,

$$d_k^T \bar{\omega}_{k-1} = -\gamma_k g_k^T s_{k-1}, \tag{22}$$

Meanwhile, equation (22) is popularly known as the Dai et al. conjugacy condition (see, for more detail [17]), where γ_k is a scalar. In addition, the original objective function's Hessian matrix is better approximated by the new conjugacy condition. In implementing our proposal for the numerical experiment, we choose $\gamma_k = 0.5\ \forall k$, as suggested by [49].

Now, we incorporate the structured vector, $\bar{\omega}_{k-1}$ into the TTCG direction stated in (18) for $n = 1$ and considering the conjugacy condition in (22), we obtain

$$(-g_k + \beta_k^{(1)} \bar{\omega}_{k-1} - \beta_k^{(1)} s_{k-1})^T \bar{\omega}_{k-1} = -\gamma_k g_k^T s_{k-1}, \tag{23}$$

$$g_k^T \bar{\omega}_{k-1} + \beta_k^{(1)} \bar{\omega}_{k-1}^T \bar{\omega}_{k-1} - \beta_k^{(1)} s_{k-1}^T \bar{\omega}_{k-1} = -\gamma_k g_k^T s_{k-1}, \tag{24}$$

Rearranging the terms in equation (24), we have

$$\beta_k^{(1)} (\bar{\omega}_{k-1}^T \bar{\omega}_{k-1} - s_{k-1}^T \bar{\omega}_{k-1}) = g_k^T (\bar{\omega}_{k-1} - \gamma_k s_{k-1}), \tag{25}$$

$$\beta_k^{(1)} (\bar{\omega}_{k-1} - s_{k-1})^T \bar{\omega}_{k-1} = g_k^T (\bar{\omega}_{k-1} - \gamma_k s_{k-1}), \tag{26}$$

solving for $\beta_k^{(1)}$ from equation (26), we obtain our first coefficient and name it TTCGC1

$$\beta_k^{(TTCGC1)} = \frac{g_k^T (\bar{\omega}_{k-1} - \gamma_k s_{k-1})}{|(\bar{\omega}_{k-1} - s_{k-1})^T \bar{\omega}_{k-1}|},\ \text{if } \bar{\omega}_{k-1} - s_{k-1} \neq 0. \tag{27}$$

B. DETERMINATION OF β_K^{TTCGC2} VIA THE SUFFICIENT DESCENT PROPERTY

This subsection derives the second TTCG coefficient from the sufficient descent property. The sufficient descent condition is stated by:

$$g_k^T d_k = -\vartheta ||g_k||^2,\ \vartheta > 0, \tag{28}$$

by substituting equation (18) for $n = 2$ in equation (28), we have

$$g_k^T (-g_k + \beta_k^{(2)} \bar{\omega}_{k-1} - \beta_k^{(2)} s_{k-1}) = -\vartheta ||g_k||^2, \tag{29}$$

$$g_k^T g_k + \beta_k^{(2)} g_k^T \bar{\omega}_{k-1} - \beta_k^{(2)} g_k^T s_{k-1} = -\vartheta ||g_k||^2, \tag{30}$$

Rearranging the terms in equation (30), we obtain

$$\beta_k^{(2)} g_k^T \bar{\omega}_{k-1} - \beta_k^{(2)} g_k^T s_{k-1} = ||g_k||^2 - \vartheta ||g_k||^2, \tag{31}$$

$$\beta_k^{(2)} g_k^T (\bar{\omega}_{k-1} - s_{k-1}) = ||g_k||^2 (1 - \vartheta), \tag{32}$$

Solving for in equation (32) $\beta_k^{(TTCGC2)}$, we obtain

$$\beta_k^{(TTCGC2)} = \frac{||g_k||^2 (1 - \vartheta)}{|g_k^T (\bar{\omega}_{k-1} - s_{k-1})|}\ \text{if } \bar{\omega}_{k-1} - s_{k-1} \neq 0. \tag{33}$$

Subsequently, we outline the sequential stages of the proposed three-term algorithms used to solve nonlinear least-squares problems as follows:

Algorithm 1 : New TTCG Coefficients (TTCGC)

Inputs: $x_0 \in \mathbb{R}^n$, $\epsilon > 0$, $\rho, \delta, \sigma, \in (0, 1)$, $0 \leq \eta_{min} \leq \eta_{max} \leq 1$, $\mu > 0$, $\gamma = 0.5$, $\vartheta = 0.7$.

Step 1: If $k = 0$, compute g_k, then set $d_k := -g_k$;

Step 2: If $\|g_k\| \leq \epsilon$, then **stop.**

Step 3: Determine ω_{k-1} using (20), β_k^{TTCGC1} or β_k^{TTCGC2} using (27) and (33), respectively.

If $|(\bar{\omega}_{k-1} - s_{k-1})^T \bar{\omega}_{k-1}| < \kappa ||\bar{\omega}_{k-1} - s_{k-1}|| ||\bar{\omega}_{k-1}||$, then set $d_k = -g_k$;

Step 4: Compute the next direction $d_k = -g_k + \beta_k^{(n)} \bar{\omega}_{k-1} - \beta_k^{(n)} s_{k-1}$, with $n = 1$ if β_k^{TTCGC1} is adopted, else $n = 2$, if β_k^{TTCGC2} is used.

Step 5: Compute α_k that satisfy (8).

Step 6: Calculate the next iterate $x_{k+1} = x_k + \alpha_k d_k$, choose $\eta_k \in [\eta_{min}, \eta_{max}]$, and compute P_{k+1} and Q_{k+1} using (9).

Step 7: Set $k := k + 1$ and go to **Step 2.**

Remark 2: It is important to acknowledge that the algorithm outlined above combines two distinct methods. Different algorithms correspond to the choices of either β_k^{TTCGC1} or β_k^{TTCGC2} coefficients. Furthermore, these algorithms have *been designed in MATLAB to compute structured matrices directly as matrix-vector products for each test function, eliminating the need for explicitly creating matrices during the iterative process.*

III. CONVERGENCE RESULTS

Given certain assumptions, this section will present the convergence analysis of the proposed TTCG approaches using equations (27) and (33). For the entirety of this section, it is presumed that $g_k \neq 0$ for all k. To proceed, we adopt the following standard hypothesis regarding the objective function.

H1. The set of points $\ell = \{x \in \mathbb{R}^n \,|\, f(x) \leq f(x_0)\}$ is confined within a bounded region. In other words, there exists a positive constant $\bar{b}$ such that $\|x\| \leq \bar{b}$ for every x within ℓ.

H2. In a certain neighborhood N of ℓ, $R(x)$ is continuously differentiable, and the Jacobian matrix $J(x)$ is Lipschitz continuous, there exists a positive constant a_1 such that

$$\|J(x) - J(y)\| \leq a_1\|x - y\|, \;\; \forall x, y \in N. \tag{34}$$

also, for all $x \in \ell$, there exists a positive constant a_2 such that

$$\|J(x)\| \leq a_2, \;\; \forall x \in \ell. \tag{35}$$

And also $R(x)$ Lipschitz continuous, i.e, there exists a positive constant b_1 such that

$$\|R(x) - R(y)\| \leq b_1\|x - y\|, \;\; \forall x, y \in \ell. \tag{36}$$

Similar to (34), for all $x \in \ell$, there exists a positive constant b_2 such that

$$\|R(x)\| \leq b_2, \;\; \forall x \in \ell. \tag{37}$$

by using (34) and (36), we obtain

$$\begin{aligned}
&\|\nabla f(x) - \nabla f(y)\| \\
&= \|J(x)^T(R(x) - R(y)) + (J(x) - J(y))^T R(y)\| \\
&\leq \|J(x)\|\|R(x) - R(y)\| + \|J(x) - J(y)\|\|R(y)\| \\
&\leq b_1\|J(x)\|\|x - y\| + a_1\|x - y\|\|R(y)\| \\
&\leq (b_1 a_2 + a_1 b_2)\|x - y\|
\end{aligned} \tag{38}$$

Therefore, from (38), it follows that there exists constant $c_1 := (b_1 a_2 + a_1 b_2)$

$$\|\nabla f(x) - \nabla f(y)\| \leq c_1\|x - y\|, \;\; \forall x, y \in \ell. \tag{39}$$

which implies that there exists a positive constant c_2 such that

$$\|\nabla f(x)\| \leq c_2, \;\; \forall x \in \ell. \tag{40}$$

H3. The gradient of the equation (11), represented as $g(x) = J(x)^T R(x)$, exhibits uniform continuity within an open convex set encompassing the level set ℓ.

Next, we present the following lemma:

Lemma 3: Suppose that Hypotheses 1 and 2 hold. Let $\{x_k\}$ and $\{d_k\}$ be generated by Algorithm 1. If the function f is uniformly convex on ℓ, i.e. there exist a constant $\mu > 0$ such that $(\nabla f(x) - \nabla f(y))^T(x - y) \geq \mu\|x - y\|^2$ for any $x, y \in N$, then there exists positive constants c_2 and L such that the subsequent inequalities are valid for every $k \geq 0$.

(a) $\|\bar{\omega}_{k-1}\| \leq L\|s_{k-1}\|$,

(b) $|g_k^T(\bar{\omega}_{k-1} - \gamma_k s_{k-1})| \leq c_2(L + \gamma_k)\|s_{k-1}\|$,

(c) $\frac{1}{|(\bar{\omega}_{k-1} - s_{k-1})^T\bar{\omega}_{k-1}|} \leq \frac{1}{|L^2 - \mu|\|s_{k-1}\|^2}$.

Proof:

(a) From the definition of the structured vector defined in (20), we obtain

$$\begin{aligned}
\|\bar{\omega}_{k-1}\| &= \|J_k^T J_k s_{k-1} + (J_k - J_{k-1})^T R_k\| \\
&\leq \|J_k^T J_k s_{k-1}\| + \|(J_k - J_{k-1})^T R_k\| \\
&\leq \|J_k\|^2\|s_{k-1}\| + \|J_k - J_{k-1}\|\|R_k\| \\
&\leq a_2^2\|s_{k-1}\| + a_1\|x_k - x_{k-1}\|\|R_k\| \\
&\leq a_2^2\|s_{k-1}\| + a_1 b_2\|s_{k-1}\| \\
&= (a_2^2 + a_1 b_2)\|s_{k-1}\|.
\end{aligned}$$

Therefore, by setting $L := a_2^2 + a_1 b_2$, we obtain the inequality (a).

(b) It follows from (27), (34), and inequality in (a) that

$$\begin{aligned}
|g_k^T(\bar{\omega}_{k-1} - \gamma_k s_{k-1})| &\leq \|g_k\|\|\bar{\omega}_{k-1} - \gamma_k s_{k-1}\| \\
&\leq \|g_k\|(\|\bar{\omega}_{k-1}\| + \gamma_k\|s_{k-1}\|) \\
&\leq c_2(L\|s_{k-1}\| + \gamma_k\|s_{k-1}\|) \\
&= c_2(L + \gamma_k)\|s_{k-1}\|.
\end{aligned}$$

Hence, the inequality in (b) holds.

(c) From (a), it follows that $\|\bar{\omega}_{k-1}\| \leq L\|s_{k-1}\|$. Also, from uniform convexity we have $\bar{\omega}_{k-1}^T s_{k-1} \geq \mu\|s_{k-1}\|^2$, for a constant $\mu > 0$.

$$\begin{aligned}
|(\bar{\omega}_{k-1} - s_{k-1})^T\bar{\omega}_{k-1}| &= |\bar{\omega}_{k-1}^T\bar{\omega}_{k-1} - s_{k-1}^T\bar{\omega}_{k-1}| \\
&\geq |\|\bar{\omega}_{k-1}\|^2 - \mu\|s_{k-1}\|^2| \\
&\geq |L^2\|s_{k-1}\|^2 - \mu\|s_{k-1}\|^2| \\
&\geq |L^2 - \mu|\|s_{k-1}\|^2 \\
\frac{1}{|(\bar{\omega}_{k-1} - s_{k-1})^T\bar{\omega}_{k-1}|} &\leq \frac{1}{|L^2 - \mu|\|s_{k-1}\|^2}.
\end{aligned}$$

Therefore, the inequality in (c) also holds. □

Lemma 4: Suppose that the sequence x_k is produced using Algorithm 1, and β_k^{TTCGC1} is defined as per equation (27). The search direction d_k, which is defined by equation (18), exhibits the descent property for all $k \geq 0$, where this property is expressed as $g_k^T d_k \leq -\phi_1\|g_k\|^2$, for every $k \geq 0$ where $\phi_1 > 0$.

Proof: For $k = 0$, from (18) we obtain $g_0^T d_0 \leq -\|g_0\|^2$, this implies that $\phi_1 = 1$. Now, for $k > 0$ and β_k^{TTCGC1} defined by (27), we have the following:

$$\begin{aligned}
&g_k^T d_k \\
&= g_k^T(-g_k + \beta_k^{TTCGC1}\bar{\omega}_{k-1} - \beta_k^{TTCGC1}s_{k-1}),
\end{aligned}$$

$$
\begin{aligned}
&= -\|g_k\|^2 + \beta_k^{TTCGC1} g_k^T \bar{\omega}_{k-1} - \beta_k^{TTCGC1} g_k^T s_{k-1}, \\
&= -\|g_k\|^2 + \frac{g_k^T(\bar{\omega}_{k-1} - \gamma_k s_{k-1})}{|(\bar{\omega}_{k-1} - s_{k-1})^T \bar{\omega}_{k-1}|} g_k^T \bar{\omega}_{k-1} \\
&\quad - \frac{g_k^T(\bar{\omega}_{k-1} - \gamma_k s_{k-1})}{|(\bar{\omega}_{k-1} - s_{k-1})^T \bar{\omega}_{k-1}} g_k^T s_{k-1|}, \\
&\le -\|g_k\|^2 + \frac{\|g_k\|\|(\bar{\omega}_{k-1} - \gamma_k s_{k-1})\|}{|(\bar{\omega}_{k-1} - s_{k-1})^T \bar{\omega}_{k-1}|} \|g_k\|\|\bar{\omega}_{k-1}\| \\
&\quad + \frac{\|g_k\|\|(\bar{\omega}_{k-1} - \gamma_k s_{k-1})\|}{|(\bar{\omega}_{k-1} - s_{k-1})^T \bar{\omega}_{k-1}|} \|g_k\|\|s_{k-1}\|, \\
&\le -\left[1 - \frac{\|(\bar{\omega}_{k-1} - \gamma_k s_{k-1})\|}{|(\bar{\omega}_{k-1} - \gamma_k s_{k-1})^T \bar{\omega}_{k-1}|}(\|\bar{\omega}_{k-1}\| + \|s_{k-1}\|)\right]\|g_k\|^2, \\
&\le -\left[1 + \frac{c_2(L+\gamma_k)\|s_{k-1}\|}{|L^2 - \mu|\|s_{k-1}\|^2}(L\|s_{k-1}\| + \|s_{k-1}\|)\right]\|g_k\|^2, \\
&\le -\left[1 + \frac{c_2(L+\gamma_k)\|s_{k-1}\|}{|L^2 - \mu|\|s_{k-1}\|^2}(L+1)\|s_{k-1}\|\right]\|g_k\|^2 \\
&= -\left[1 + \frac{c_2(L+\gamma_k)(L+1)}{|L^2 - \mu|}\right]\|g_k\|^2.
\end{aligned}
$$

Therefore, by setting $\phi_1 := \left[1 + \frac{c_2(L+\gamma_k)(L+1)}{|L^2-\mu|}\right]$, we have, $g_k^T d_k \le -\phi_1 \|g_k\|^2$, and the proof is completed. □

Lemma 5: Let $\{d_k\}$ be the sequence of directions produced by Algorithm 1, where β_k^{TTCGC1} is defined by (27). Then there exists a constant $\rho_1 > 0$ such $\|d_k\| \le \rho_1$ for all $k \ge 0$.

Proof: For $k = 0$, we have from (18) and β_k^{TTCGC1} defined by (27), we obtain $\|d_0\| = \|g_0\| < \rho_1$, On the other hand, for $k > 0$, we have

$$
\begin{aligned}
&\|d_k\| \\
&= \|g_k + \beta_k^{(TTCGC1)} \bar{\omega}_{k-1} - \beta_k^{(TTCGC1)} s_{k-1}\| \\
&\le \|g_k\| + |\beta_k^{(TTCGC1)}|\|\bar{\omega}_{k-1}\| + |\beta_k^{(TTCGC1)}|\|s_{k-1}\|, \\
&\le \|g_k\| + \frac{\|g_k\|\|(\bar{\omega}_{k-1} - \gamma_k s_{k-1})\|}{|(\bar{\omega}_{k-1} - s_{k-1})^T \bar{\omega}_{k-1}|}\|\bar{\omega}_{k-1}\| + \\
&\frac{\|g_k\|\|(\bar{\omega}_{k-1} - \gamma_k s_{k-1})\|}{|(\bar{\omega}_{k-1} - s_{k-1})^T \bar{\omega}_{k-1}|}\|s_{k-1}\|, \\
&\le \left[1 + \frac{\|(\bar{\omega}_{k-1} - \gamma_k s_{k-1})\|}{|(\bar{\omega}_{k-1} - s_{k-1})^T \bar{\omega}_{k-1}|}\|\bar{\omega}_{k-1}\| \right.\\
&\quad \left. + \frac{\|(\bar{\omega}_{k-1} - \gamma_k s_{k-1})\|}{|(\bar{\omega}_{k-1} - s_{k-1})^T \bar{\omega}_{k-1}|}\|s_{k-1}\|\right]\|g_k\|, \\
&\le \left[1 + \frac{\|(\bar{\omega}_{k-1} - \gamma_k s_{k-1})\|}{|(\bar{\omega}_{k-1} - s_{k-1})^T \bar{\omega}_{k-1}|}(\|\bar{\omega}_{k-1}\| + \|s_{k-1}\|)\right]\|g_k\|, \\
&\le \left[1 + \frac{c_2(L+\gamma_k)\|s_{k-1}\|}{|L^2 - \mu|\|s_{k-1}\|^2}(L\|s_{k-1}\| + \|s_{k-1}\|)\right]\|g_k\|, \\
&= \left[1 + \frac{c_2(L+\gamma_k)\|s_{k-1}\|}{|L^2 - \mu|\|s_{k-1}\|^2}(L+1)\|s_{k-1}\|\right]\|g_k\|.
\end{aligned}
$$

Therefore, if we let $\rho_1 := \left[1 + \frac{c_2(L+\gamma_k)(L+1)}{|L^2-\mu|}\right]$, then we have, $\|d_k\| \le \rho_1$. Hence, the search direction is bounded. □

Lemma 6: Suppose that the sequence x_k is produced using Algorithm 1, and β_k^{TTCGC2} is defined as per equation (33). The search direction d_k, which is defined by equation (18), exhibits the descent property for all $k \ge 0$, where this property state that there exists $\phi_2 > 0$, such that for all $k \ge 0$, we have, $g_k^T d_k \le -\phi_2 \|g_k\|^2$.

Proof: For $k = 0$, we obtain $g_0^T d_0 = -\|g_0\|^2$, this implies that $\omega_2 = 1$. Now, for $k > 0$ and β_k^{TTCGA2} defined by (33), we have the following:

$$
\begin{aligned}
&g_k^T d_k \\
&= g_k^T(-g_k + \beta_k^{TTCGC2} \bar{\omega}_{k-1} - \beta_k^{2TTCGC2} s_{k-1}), \\
&= -\|g_k\|^2 + \beta_k^{TTCGC2} g_k^T \bar{\omega}_{k-1} - \beta_k^{TTCGC2} g_k^T s_{k-1}, \\
&= -\|g_k\|^2 + \frac{||g_k||^2(1-\vartheta)}{|g_k^T(\bar{\omega}_{k-1} - s_{k-1})|} g_k^T \bar{\omega}_{k-1} \\
&\quad - \frac{||g_k||^2(1-\vartheta)}{|g_k^T(\bar{y}_{k-1} - s_{k-1})|} g_k^T s_{k-1}, \\
&\le -\|g_k\|^2 + \frac{||g_k||^2|1-\vartheta|}{|g_k^T(\bar{\omega}_{k-1} - s_{k-1})|}\|g_k\|\|\bar{\omega}_{k-1}\| \\
&\quad + \frac{||g_k||^2|1-\vartheta|}{|g_k^T(\bar{\omega}_{k-1} - s_{k-1})|}\|g_k\|\|s_{k-1}\|, \\
&= -\left[1 - \frac{\|g_k\||1-\vartheta|}{|g_k^T(\bar{\omega}_{k-1} - s_{k-1})|}(\|\bar{\omega}_{k-1}\| + \|s_{k-1}\|)\right]\|g_k\|^2, \\
&\le -\left[1 - \frac{|1-\vartheta|\|g_k\|}{c_2(L+1)\|g_k\|\|s_{k-1}\|}(L\|s_{k-1}\| + \|s_{k-1}\|)\right]\|g_k\|^2, \\
&= -\left[1 - \frac{|1-\vartheta|\|g_k\|}{c_2(L+1)\|g_k\|\|s_{k-1}\|}(L+1)\|s_{k-1}\|\right]\|g_k\|^2, \\
&= -\left[1 - \frac{|1-\vartheta|}{c_2}\right]\|g_k\|^2.
\end{aligned}
$$

Therefore, by setting $\phi_2 := \left[1 - \frac{|1-\vartheta|}{c_2}\right]$, we have, $g_k^T d_k \le -\phi_2 \|g_k\|^2$, and thus, complete the proof. □

Lemma 7: Let $\{d_k\}$ be the sequence of directions produced by Algorithm 1, where β_k^{TTCGC2} is defined by (33), Then there exists a constant $\rho_2 > 0$ such $d_k \le \rho_2$ for all $k \ge 0$.

Proof: For $k = 0$, we have from (18) and β_k^{TTCGC2} defined by (33), we obtain $\|d_0\| = \|g_0\| < \rho_2$, Subsequently, for $k > 0$, we obtain

$$
\begin{aligned}
\|d_k\| &= \|g_k + \beta_k^{TTCGC2} \bar{\omega}_{k-1} - \beta_k^{TTCGC2} s_{k-1}\| \\
&\le \|g_k\| + |\beta_k^{TTCGC2}|\|\bar{\omega}_{k-1}\| - |\beta_k^{TTCGC2}|\|s_{k-1}\|, \\
&= \|g_k\| + \frac{||g_k||^2|1-\vartheta|}{|g_k^T(\bar{\omega}_{k-1} - s_{k-1})|}\|\bar{\omega}_{k-1}\| + \\
&\frac{||g_k||^2|1-\vartheta|}{|g_k^T(\bar{\omega}_{k-1} - s_{k-1})|}\|s_{k-1}\|, \\
&= \left[1 + \frac{||g_k|||1-\vartheta|}{|g_k^T(\bar{\omega}_{k-1} - s_{k-1})|}\|\bar{\omega}_{k-1}\| + \right. \\
&\left. \frac{||g_k|||1-\vartheta|}{|g_k^T(\bar{\omega}_{k-1} - s_{k-1})|}\|s_{k-1}\|\right]\|g_k\|, \\
&= \left[1 + \frac{||g_k|||1-\vartheta|}{|g_k^T(\bar{\omega}_{k-1} - s_{k-1})|}(\|\bar{\omega}_{k-1}\| + \|s_{k-1}\|)\right]\|g_k\|,
\end{aligned}
$$

$$= \left[1 + \frac{|1-\vartheta|\|g_k\|}{c_2(L+1)\|g_k\|\|s_{k-1}\|}(L+1)\|s_{k-1}\|)\right]\|g_k\|,$$
$$= \left[1 + \frac{|1-\vartheta|}{c_2}\right]\|g_k\|,$$

Therefore, if we let $\rho_2 := \left[1 + \frac{|1-\vartheta|}{c_2}\right]$, then we have, $\|d_k\| \leq \rho_2$. Hence, the search direction is bounded. □

Lemma 8: Consider the sequence $\{x_k\}$ generated by Algorithm 1, assuming $g_k \neq 0$, and Hypothesis 3 is satisfied. Furthermore, based on the results from lemmas 3.2 and 3.4, which established that d_k is a sufficient descent direction, it follows that there exists a small positive step length α^* such that

$$f(x+\alpha^* d_k) \leq Q_k + \delta\alpha^* g_k^T, \tag{41}$$

holds, this indicate that α is well-defined.

Proof: In order to demonstrate the existence of a sufficiently small step length α^*, we posit, to establish a contradiction, that

$$f(x_k+\alpha_j d_k) > Q_k + \delta\alpha_j g_k^T d_k,\ \forall\alpha_j \geq 0,\ \text{holds}, \tag{42}$$

such that $\lim_{j\to\infty}\alpha_j = 0$. But, since $f(x) = \frac{1}{2}\|R(x)\|^2$ as stated in (11), this indicates

$$f(x_k) = \frac{1}{2}\|R(x_k)\|^2 \geq 0,\ \forall k, \tag{43}$$

Furthermore, as mentioned in (10), ψ_k lies between P_{k+1} and $f(x_k)$ In other words, it can be represented as a convex combination of both $f(x_k)$ and P_{k+1}. Moreover, $Q_0 = f(x_0)$ at $k = 0$, then we have $\forall k$, $Q_k \geq 0$. Furthermore, by utilizing the constraint that d_k is bounded, equation (8) transforms to $f(x_k+\alpha_j d_k) > Q_k - \delta\alpha_j\eta_j\|g_k\|^2$, therefore, using $\lim_{j\to\infty}\alpha_j = 0$, we have

$$f(x_k) \geq Q_k. \tag{44}$$

On the other hand, $Q_k = \frac{\eta_{k-1}P_{k-1}Q_{k-1}+f(x_k)}{\eta_{k-1}P_{k-1}+1}$, Implying that, Q_k can be expressed convexly as a combination of $f(x_k)$ and Q_{k-1}, consequently, This signifies that $\eta_{k-1} = 0$, since P_{k-1} and Q_{k-1} are both non-zero. Therefore, the non-monotone line search turns monotone that is, equation (8) becomes $f(x_k+\alpha_k d_k) > f(x_k) + \delta\alpha_j g_k^T d_k$, Showing that

$$\frac{f(x_k+\alpha_j d_k) - f(x_k)}{\alpha_j} > \delta g_k^T d_k. \tag{45}$$

by taking the limit as $j \to \infty$ from equation (45) and making use of the assumption of uniform continuity of the gradient, we have $g_k^T d_k \geq \delta g_k^T d_k$. Nevertheless, since $g_k^T d_k < 0$, it holds that $\delta > 1$; this leads to a contradiction, and thus, we conclude the proof. □

The following lemmas are critical for proving Theorem 9. However, their proofs can be found in [45].

Lemma 9: Suppose Hypothesis 1 is valid; if the iterative sequence $\{x_k\}$ is produced by Algorithm 1, then it can be established that $f_k \leq Q_k$ for every value of k.

Lemma 10: Suppose that Hypothesis 2 is valid; if the iterative sequence $\{x_k\}$ is generated by either TTCGC1 or TTCGC2, it can be deduced that

$$\alpha_k \geq \left(\frac{2(1-\delta)}{c_1\zeta}\right)\frac{|g_k^T d_k|}{\|d_k\|^2}. \tag{46}$$

Remark 11: Suppose that $Q_j \leq \psi_j$ for all $0 \leq j < k$, given the initial condition $P_0 = 1$, and considering that $\eta_k \in [0,1]$, then we have

$$P_{j+1} = 1 + \sum_{i=0}^{m}\prod_{m=0}^{i}\eta_{j-m} \leq j+2. \tag{47}$$

Theorem 12: Suppose that equation (11) defines the function $f(x)$, and Hypothesis 1 and 2 are valid. The resulting sequence $\{x_k\}$ produced by either TTCGC1 or TTCGC2 algorithms is encompassed by the level set ℓ, and

$$\lim_{k\to\infty}\inf\|g_k\| = 0, \tag{48}$$

Moreover, if η_{max} is less than 1, then

$$\lim_{k\to\infty}\|g_k\| = 0, \tag{49}$$

Proof: To begin with, we present the fact that

$$f_{k+1} \leq Q_k - \beta\|g_k\|^2, \tag{50}$$

considering the line search in equation (8), we have

$$f_{k+1} \leq Q_k + \delta\alpha_k g_k^T d_k, \tag{51}$$

and the inequality in equation (47), we obtain

$$f_{k+1} \leq Q_k - \left(\frac{2\delta(1-\delta)}{c_1\zeta}\right)\left(\frac{|g_k^T d_k|}{\|d_k\|^2}\right). \tag{52}$$

Based on the sufficient descent property from lemma 3.2. and the bound property from lemma 3.4, we have

$$f_{k+1} \leq Q_k - \left(\frac{2\delta(1-\delta)\phi_1^2}{c_1\zeta\rho_1^2}\right)\|g_k\|^2. \tag{53}$$

where,

$$\beta = \frac{2\delta(1-\delta)\phi_1^2}{c_1\zeta\rho_1^2}. \tag{54}$$

combining the cost update in equation (9) and equation (50), we can obtain

$$Q_k = \frac{\eta_k P_k Q_k + f_{k+1}}{P_{k+1}} \leq \frac{\eta_k P_k Q_k + Q_k - \beta\|g_k\|^2}{P_{k+1}} \tag{55}$$
$$= \frac{Q_k(\eta_k P_{k+1}+1) - \beta\|g_k\|^2}{P_{k+1}}$$
$$= \frac{Q_k P_{k+1} - \beta\|g_k\|^2}{P_{k+1}} = Q_k - \frac{\beta\|g_k\|^2}{P_{k+1}}. \tag{56}$$

As f is bounded from below, and for all k, $f_k \leq Q_k$, we can deduce that Q_k is also bounded from below. Therefore, it follows from (56) that

$$\sum_{k=1}^{\infty}\frac{\|g_k\|^2}{P_{k+1}} < \infty \tag{57}$$

TABLE 1. List of test functions, initial points, and Residual type.

No.	Function name	Initial point	Residual type
F1	Penalty Function 1 [50]	$(3,3,\ldots,3)^T$	Non-Zero
F2	Variably Dimensioned [50]	$(1-\frac{1}{n},1-\frac{2}{n},\ldots,0)^T$	Zero
F3	Trigonometric Function [51]	$(\frac{1}{n},\frac{1}{n},\ldots)^T$	Zero
F4	Discrete Boundary-value [51]	$(\frac{1}{n+1},\frac{1}{n+1}-1,\ldots,\frac{1}{n+1}-1)^T$	Zero
F5	Linear Full Rank [51]	$(1,1,\ldots,1)^T$	Non-zero
F6	Problem 202 [52]	$(2,2,\ldots,2)^T$	Zero
F7	Problem 206 [52]	$(2,2,\ldots,2)^T$	Zero
F8	Problem 212 [52]	$(0.5,0.5,\ldots,0.5)^T$	Zero
F9	Raydan 1 [50]	$(\frac{1}{n},\frac{2}{n},\ldots,1)^T$	Non-zero
F10	Raydan 2 [50]	$(\frac{1}{10n},\frac{2}{10n},\ldots,\frac{1}{10n})^T$	Non-zero
F11	Sine Function 2 [53]	$(1,1,\ldots,1)^T$	Zero
F12	Exponential Function 1 [50]	$(\frac{n}{n-1},\frac{n}{n-1},\ldots,\frac{n}{n-1})^T$	Zero
F13	Exponential Function 2 [50]	$(\frac{1}{n^2},\frac{1}{n^2},\ldots,\frac{1}{n^2})^T$	Zero
F14	Singular Function 2 [50]	$(1,1,\ldots,1)^T$	Zero
F15	Ext. Freudenstein & Roth Function [50]	$(6,3,6,3,\ldots,6,3)^T$	Zero
F16	Ext. Powell Singular Function [50]	$(1.5E-4,\ldots,1.5E-4)^T$	Zero
F17	Function 21 [50]	$(1,1,\ldots,1)^T$	Zero
F18	Broyden Tridiagonal Function [51]	$(-1,-1,\ldots,-1)^T$	Zero
F19	Extended Rosenbrock Function [51]	repmat$([-1:1],[\frac{n}{2},1])$	Zero
F20	Extended Himmelblau Function [54]	$(1,\frac{1}{n},1,\frac{1}{n},\ldots,1,\frac{1}{n})^T$	Zero
F21	Function 27 [50]	$(100,\frac{1}{n^2},\ldots,\frac{1}{n^2})^T$	Zero
F22	Triglog Function [54]	$(1,1,\ldots,1)^T$	Non-zero
F23	Zerojacobian Function [50]	$if \ i=1, \frac{100(n-100)}{n}, \ if \ i \geq 2, \frac{(n-1000)(n-500)}{(60n)^2}$	Zero
F24	Exponential Function [50]	$(0.5,0.5,\ldots,0.5)^T$	Zero
F25	Function 18 [50]	$(0,0,\ldots,0)^T$	Zero
F26	Brown almost linear function [51]	$(0.5,0.5,\ldots,0.5)^T$	Zero

If $\|g_k\|$ were bounded away from 0, the equation (56) would not hold since $P_{k+1} \leq k+2$ by (47). Hence, if $\eta_{max} < 1$, then by (47),

$$P_{k+1} = 1 + \sum_{j=0}^{k}\prod_{i=0}^{j}\eta_{k-i} \leq 1 + \sum_{j=0}^{k}\eta_{max}^{j+1} \leq \sum_{j=0}^{k}\eta_{max}^{j} = \frac{1}{1-\eta_{max}} \tag{58}$$

Therefore, we can infer that (48) directly entails (49). Therefore, the proof is concluded. □

Remark 13: The above proof of theorem 12 is for TTCGC1; the proof for TTCGC2 is similar with slight modifications.

IV. NUMERICAL EXPERIMENTS

This section reports numerical results obtained to solve the NLS problems using the proposed TTCGC1 and TTCGC2 algorithms. We showcase a series of numerical trials to evaluate the effectiveness of these suggested approaches. We investigate 26 benchmark test functions. The list of the test problems, their initial points, and their respective references are reported in Tables 1, we solved each test

TABLE 2. The performance of the TTCGC algorithms vs M1, CGSQN, and SSHS algorithms on problems F1 to F7 are evaluated. The symbol 'F' indicates when an algorithm fails to solve a problem.

METHODS		M1					CGSQN					SSHS					TTCGC1					TTCGC2				
FUNCS	DIM	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F
F1	3000	F	F	F	F	F	3	7	10	0.0119	0.048782	3	7	10	0.1763	0.048782	3	7	10	0.0183	0.032166	20	41	61	0.0359	0.066125
	6000	F	F	F	F	F	3	7	10	0.0299	0.048593	3	7	10	0.0674	0.048593	2	5	7	0.0195	0.11771	15	31	46	0.0609	0.17534
	9000	F	F	F	F	F	3	7	10	0.0388	0.048244	3	7	10	0.0482	0.048244	2	5	7	0.0237	0.11887	12	25	37	0.0612	0.31189
	12000	F	F	F	F	F	2	5	7	0.0199	0.37577	2	5	7	0.0413	0.37577	2	5	7	0.0197	0.11976	10	21	31	0.0817	0.45883
	15000	F	F	F	F	F	2	5	7	0.1534	0.38325	2	5	7	0.0324	0.38325	2	5	7	0.0279	0.12041	8	17	25	0.1064	0.67254
F2	3000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	138	1534	415	0.7882	3.0612E-16	F	F	F	F	F
	6000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	133	1854	400	1.0024	7.727E-18	F	F	F	F	F
	9000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	154	2616	463	2.8126	6.9278E-18	F	F	F	F	F
	12000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	177	3217	532	3.9750	0	F	F	F	F	F
	15000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	183	3878	550	5.2280	4.2379E-19	F	F	F	F	F
F3	3000	F	F	F	F	F	107	1662	322	1.1039	3.4254E-07	50	446	151	0.4861	2.0741E-06	71	1141	214	0.6545	6.4258E-07	79	1283	238	0.7098	1.5304E-07
	6000	F	F	F	F	F	118	1914	355	3.1246	3.9633E-07	60	578	181	1.2966	3.0618E-06	82	1461	247	2.3541	1.0147E-06	93	1696	280	2.6095	0.00000274
	9000	F	F	F	F	F	117	2136	352	4.6771	1.0749E-06	48	253	145	0.8607	6.5403E-06	97	1755	292	4.0113	4.3169E-07	82	1490	247	3.0279	1.9876E-06
	12000	F	F	F	F	F	124	2333	373	6.4502	7.7271E-07	66	765	199	4.1457	2.7111E-06	106	1845	319	4.6306	2.0905E-07	95	1919	286	4.6233	1.0903E-06
	15000	F	F	F	F	F	129	2369	388	7.3917	3.4484E-08	84	1454	253	4.1474	1.5663E-06	98	1855	295	5.9078	3.0588E-06	124	2538	373	7.6103	1.8501E-07
F4	3000	8	22	25	0.0619	1.1407E-08	10	44	31	0.1037	8.6059E-09	7	35	22	0.1225	7.4785E-09	13	59	40	0.0923	4.2093E-09	7	34	22	0.0521	6.0452E-09
	6000	5	14	16	0.0905	3.8254E-09	6	27	19	0.0800	3.9466E-09	5	25	16	0.0753	2.4663E-09	8	35	25	0.0896	1.7865E-09	3	15	10	0.0428	2.7343E-09
	9000	1	5	4	0.0311	2.5098E-09	1	6	4	0.0717	2.5098E-09	1	6	4	0.0377	2.5098E-09	1	6	4	0.0256	2.5098E-09	1	6	4	0.0356	2.5098E-09
	12000	1	5	4	0.0777	1.4115E-09	1	6	4	0.0820	1.4115E-09	1	6	4	0.0350	1.4115E-09	1	6	4	0.0954	1.4115E-09	1	6	4	0.0459	1.4115E-09
	15000	1	5	4	0.0544	9.0322E-10	1	6	4	0.0424	9.0322E-10	1	6	4	0.0711	9.0322E-10	1	6	4	0.1001	9.0322E-10	1	6	4	0.1270	9.0322E-10
F5	3000	2	3	7	0.0082	0.50066	2	5	7	0.0269	0.50067	2	5	7	0.0514	0.50067	2	5	7	0.0073	0.50066	2	5	7	0.0089	0.50066
	6000	2	3	7	0.0129	0.50033	2	5	7	0.0122	0.50033	2	5	7	0.0385	0.50033	2	5	7	0.0092	0.50033	2	5	7	0.0268	0.50033
	9000	2	3	7	0.0162	0.50022	2	5	7	0.0169	0.50022	2	5	7	0.0165	0.50022	2	5	7	0.0133	0.50022	2	5	7	0.0171	0.50022
	12000	2	3	7	0.0244	0.50017	2	5	7	0.0177	0.50017	2	5	7	0.0177	0.50017	2	5	7	0.0191	0.50017	2	5	7	0.0168	0.50017
	15000	2	3	7	0.0262	0.50013	2	5	7	0.0202	0.50013	2	5	7	0.2577	0.50013	2	5	7	0.0251	0.50013	2	5	7	0.0476	0.50013
F6	3000	5	6	16	0.0145	2.8955E-13	5	11	16	0.0367	6.2424E-09	4	9	13	0.0080	2.8221E-13	5	13	16	0.0126	8.4119E-08	4	9	13	0.0080	2.8221E-13
	6000	5	6	16	0.0309	5.791E-13	5	11	16	0.0181	1.2561E-08	4	9	13	0.0222	5.6442E-13	5	13	16	0.0402	1.6824E-07	4	9	13	0.0222	5.6442E-13
	9000	5	6	16	0.0612	8.6865E-13	5	11	16	0.0324	1.8879E-08	4	9	13	0.0261	8.4662E-13	5	13	16	0.0330	2.5236E-07	4	9	13	0.0261	8.4662E-13
	1200	5	6	16	0.0648	1.1582E-12	5	11	16	0.0367	2.5197E-08	4	9	13	0.0331	1.1288E-12	5	13	16	0.0428	3.3648E-07	4	9	13	0.0331	1.1288E-12
	15000	5	6	16	0.0751	1.4477E-12	5	11	16	0.0496	3.1515E-08	4	9	13	0.0484	1.411E-12	5	13	16	0.0593	4.206E-07	4	9	13	0.0484	1.411E-12
F7	3000	5	6	16	0.0272	7.8908E-10	6	14	19	0.0452	3.6379E-13	6	13	19	0.0222	5.232E-09	6	16	19	0.0253	3.2211E-11	6	13	19	0.0222	5.232E-09
	6000	5	6	16	0.1635	1.58E-09	6	14	19	0.0521	4.0788E-13	6	13	19	0.1060	6.6217E-09	6	16	19	0.0534	3.6174E-11	6	13	19	0.1060	6.6217E-09
	9000	5	6	16	0.0651	2.3709E-09	6	14	19	0.0453	4.2414E-13	6	13	19	0.0660	8.5985E-09	6	16	19	0.0633	4.4447E-11	6	13	19	0.0660	8.5985E-09
	12000	5	6	16	0.1092	3.1618E-09	6	14	19	0.2712	4.326E-13	6	13	19	0.0603	1.0993E-08	6	16	19	0.0663	5.3558E-11	6	13	19	0.0603	1.0993E-08
	15000	5	6	16	0.0818	3.9527E-09	6	14	19	0.1790	4.3779E-13	6	13	19	0.1582	1.3836E-08	6	16	19	0.0774	6.2974E-11	6	13	19	0.1582	1.3836E-08

TABLE 3. The performance of the TTCGC algorithms vs M1, CGSQN and SSHS algorithms on problems F8 to F14 are evaluated. The symbol 'F' indicates when an algorithm fails to solve a problem.

METHODS		M1					CGSQN					SSHS					TTCGC1					TTCGC2				
FUNCS	DIM	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F
F8	3000	11	12	34	0.0815	1500	9	19	28	0.0655	1500	10	21	31	0.0577	1500	7	11	22	0.1155	1.8617E-10	10	21	31	0.0577	1500
	6000	11	12	34	0.1493	3000	9	19	28	0.0887	3000	10	21	31	0.0898	3000	7	11	22	0.1234	1.8617E-10	10	21	31	0.0898	3000
	9000	11	12	34	0.2868	4500	9	19	28	0.1223	4500	10	21	31	0.2263	4500	7	11	22	0.2435	1.8617E-10	10	21	31	0.2263	4500
	12000	11	12	34	0.3450	6000	9	19	28	0.2058	6000	10	21	31	0.2282	6000	7	11	22	0.2355	1.8617E-10	10	21	31	0.2282	6000
	15000	11	12	34	0.3449	7500.0001	9	19	28	0.3593	7500	10	21	31	0.2192	7500	7	11	22	0.3259	8.3892E-09	10	21	31	0.2192	7500
F9	3000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	5	5	16	0.0641	2.5168E-08	26	69	79	0.0641	2.5168E-08
	6000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	5	5	16	0.2054	4.1946E-08	26	69	79	0.2054	4.1946E-08
	9000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	7	18	22	0.0152	1.1371E-09	26	69	79	0.0152	1.1371E-09
	12000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	7	18	22	0.0200	3.4113E-09	26	69	79	0.0200	3.4113E-09
	15000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	7	18	22	0.0358	5.6856E-09	26	69	79	0.0358	5.6856E-09
F10	3000	25	36	76	0.0700	1.1639E-06	3	7	10	0.0264	3.5195E-06	5	11	16	0.0405	5.0127E-10	4	9	13	0.0095	2.3796E-08	4	9	13	0.0131	2.6453E-07
	6000	25	36	76	0.1428	2.3278E-06	3	7	10	0.0184	7.0391E-06	5	11	16	0.0224	1.0025E-09	4	9	13	0.0191	4.7592E-08	4	9	13	0.0221	5.2905E-07
	9000	25	36	76	0.3346	3.4916E-06	3	7	10	0.0233	0.000010559	5	11	16	0.0328	1.5038E-09	4	9	13	0.0214	7.1388E-08	4	9	13	0.0407	7.9358E-07
	12000	25	36	76	0.3567	4.6555E-06	3	7	10	0.0274	0.000014078	5	11	16	0.0505	2.0051E-09	4	9	13	0.0317	9.5185E-08	4	9	13	0.0626	1.0581E-06
	15000	25	36	76	0.3068	5.8194E-06	3	7	10	0.0313	0.000017598	5	11	16	0.0494	2.5063E-09	4	9	13	0.0465	1.1898E-07	4	9	13	0.0627	1.3226E-06
F11	3000	1	2	4	0.0061	4.0748E-06	1	3	4	0.0358	4.0748E-06	2	3	4	0.0332	4.0748E-06	1	3	4	0.0073	4.0748E-06	1	3	4	0.0085	4.0748E-06
	6000	1	2	4	0.0084	2.0369E-06	1	3	4	0.0314	2.0369E-06	2	3	4	0.0259	2.0369E-06	1	3	4	0.0119	2.0369E-06	1	3	4	0.0150	2.0369E-06
	9000	1	2	4	0.0143	1.3578E-06	1	3	4	0.0129	1.3578E-06	2	3	4	0.0251	1.3578E-06	1	3	4	0.0087	1.3578E-06	1	3	4	0.0109	1.3578E-06
	12000	1	2	4	0.0050	3.4775E-06	1	3	4	0.0119	3.4775E-06	2	3	4	0.0047	3.4775E-06	1	3	4	0.0047	3.4775E-06	1	3	4	0.0043	3.4775E-06
	15000	1	2	4	0.0033	2.7811E-06	1	3	4	0.0097	2.7811E-06	2	3	4	0.0056	2.7811E-06	1	3	4	0.0041	2.7811E-06	1	3	4	0.0114	2.7811E-06
F12	3000	195	1314	586	1.1487	3.3308E-10	F	F	F	F	F	F	F	F	F	F	1	3	4	0.0056	1.3837E-07	81	1530	244	0.7563	1.6988E-10
	6000	96	724	289	1.0002	2.7983E-10	198	4246	595	3.3135	1.7795E-11	F	F	F	F	F	1	3	4	0.0019	5.8385E-08	111	2314	334	1.9275	5.0598E-11
	9000	115	932	346	2.0758	1.3298E-10	106	2334	319	3.0686	4.4812E-12	F	F	F	F	F	1	3	4	0.0028	5.8385E-08	201	4469	604	5.7471	2.3231E-11
	12000	113	1053	340	2.4950	6.8823E-11	182	4266	547	6.4695	9.5976E-12	F	F	F	F	F	1	3	4	0.0033	5.8385E-08	136	3113	409	4.9286	2.0752E-11
	15000	F	F	F	F	F	146	1094	139	2.0605	2.1397E-11	F	F	F	F	F	1	3	4	0.0068	5.8385E-08	207	4532	622	10.3274	2.027E-11
F13	3000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	83	444	250	0.0926	8.4683E-08	86	1710	259	1.4125	8.4683E-08
	6000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	67	357	202	0.3742	3.3896E-07	84	1914	253	2.7484	3.3896E-07
	9000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	79	421	238	0.5503	1.0065E-06	130	4089	391	9.6327	1.0065E-06
	12000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	37	259	112	0.0936	2.5607E-06	74	1888	223	6.0068	2.5607E-06
	15000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	61	327	184	0.3768	2.2019E-06	188	3942	565	15.2399	2.2019E-06
F14	3000	6	7	19	0.0212	1.9148E-09	4	10	13	0.0353	4.8614E-07	5	12	16	0.0466	3.0562E-06	7	17	22	0.0177	1.3382E-10	5	11	16	0.0146	2.0614E-10
	6000	6	7	19	0.0325	2.2763E-09	4	10	13	0.0280	1.0427E-06	5	12	16	0.0273	5.8312E-06	7	17	22	0.0343	2.135E-10	5	11	16	0.0507	1.455E-10
	9000	6	7	19	0.0579	2.7886E-09	4	10	13	0.0294	1.5992E-06	5	12	16	0.0344	8.6102E-06	7	17	22	0.0491	2.9688E-10	5	11	16	0.0350	1.3009E-10
	12000	6	7	19	0.0665	3.3365E-09	4	10	13	0.0430	2.1558E-06	5	12	16	0.0409	0.00001139	7	17	22	0.0525	3.8109E-10	5	11	16	0.0417	1.2601E-10
	15000	6	7	19	0.1441	3.8983E-09	4	10	13	0.0453	2.7123E-06	5	12	16	0.1379	0.000014171	7	17	22	0.0769	4.6562E-10	5	11	16	0.0945	1.2646E-10

TABLE 4. The performance of the TTCGC algorithms vs M1, CGSQN, and SSHS algorithms on problems F15 to F21 are evaluated. The symbol 'F' indicates when an algorithm fails to solve a problem.

METHODS		M1					CGSQN					SSHS					TTCGC1					TTCGC2				
FUNCS	DIM	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F
F15	3000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	26	69	79	0.0637	2.5168E-08	76	1186	229	0.5239	2.5168E-08
	6000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	26	69	79	0.3639	5.0335E-08	76	1186	229	0.9662	5.0335E-08
	9000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	26	69	79	0.2277	7.5503E-08	76	1185	229	1.3600	7.5503E-08
	12000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	26	69	79	0.2571	1.0067E-07	73	1029	220	1.9727	1.0068E-07
	15000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	26	69	79	0.3489	1.2584E-07	74	1079	223	2.8967	1.2584E-07
F16	3000	12	63	37	0.0946	3.0203E-08	2	11	7	0.0579	3.7048E-12	19	149	58	0.1235	3.4617E-08	5	18	16	0.0252	3.4113E-09	22	167	67	0.1619	1.6931E-08
	6000	12	63	37	0.1482	6.0405E-08	2	11	7	0.0205	7.4097E-12	19	149	58	0.1494	6.9234E-08	5	18	16	0.0632	6.8227E-09	22	167	67	0.2528	3.3863E-08
	9000	12	63	37	0.1630	9.0608E-08	2	11	7	0.0393	1.1115E-11	19	149	58	0.2219	1.0385E-07	5	18	16	0.0487	1.0234E-08	22	167	67	0.3036	5.0794E-08
	12000	12	63	37	0.2782	1.2081E-07	2	11	7	0.0382	1.4819E-11	19	149	58	0.4396	1.3847E-07	5	18	16	0.0540	1.3645E-08	22	167	67	0.3400	6.7725E-08
	15000	12	63	37	0.2129	1.5101E-07	2	11	7	0.0457	1.8524E-11	19	149	58	0.4359	1.7309E-07	5	18	16	0.0626	1.7057E-08	22	167	67	0.5255	8.4656E-08
F17	3000	117	332	352	1.0822	9.1052E-07	63	368	190	0.7045	2.1191E-07	67	432	202	0.5096	2.0292E-07	59	276	178	0.4913	1.5776E-07	61	381	184	0.3926	1.1918E-07
	6000	117	332	352	2.2312	0.000001821	63	368	190	0.8085	4.2382E-07	67	432	202	0.9914	4.0583E-07	59	276	178	0.8029	3.1553E-07	61	381	184	0.8298	2.3836E-07
	9000	114	326	343	2.3716	2.0901E-06	63	368	190	1.9261	6.3573E-07	67	432	202	1.9478	6.0875E-07	59	276	178	0.8985	7.5078E-07	61	381	184	1.2680	3.5754E-07
	12000	114	326	343	2.8493	3.5605E-06	63	368	190	1.3373	8.4764E-07	67	432	202	1.8094	8.1166E-07	59	276	178	1.4494	5.9327E-07	61	381	184	1.3601	4.7672E-07
	15000	114	326	343	3.9204	3.8469E-06	63	368	190	1.9411	1.0595E-06	67	432	202	2.0496	1.0146E-06	59	276	178	1.4754	7.7544E-07	61	381	184	1.7141	5.959E-07
F18	3000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	382	2434	1157	2.3490	0.000000052	101	626	304	1.1802	5.6266E-09
	6000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	398	2542	1195	2.3380	4.9051E-08	103	2347	2128	2.0192	2.3991E-07
	9000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	398	2459	1162	4.8094	6.1015E-08	237	2319	2413	2.2371	2.61E-08
	12000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	402	2567	1178	4.8123	5.1015E-08	285	1782	856	4.2588	2.432E-08
	15000	F	F	F	F	F	F	F	F	F	F	F	F	F	F	F	412	2559	1179	4.8332	5.2015E-08	371	2300	1114	6.3319	7.1891E-08
F19	3000	1	2	4	0.0101	0	1	3	4	0.0527	0	1	3	4	0.0270	0	1	3	4	0.0042	0	1	3	4	0.0061	0
	6000	1	2	4	0.0094	0	1	3	4	0.0177	0	1	3	4	0.0112	0	1	3	4	0.0047	0	1	3	4	0.0054	0
	9000	1	2	4	0.0132	0	1	3	4	0.0342	0	1	3	4	0.0237	0	1	3	4	0.0078	0	1	3	4	0.0103	0
	12000	1	2	4	0.0218	0	1	3	4	0.0977	0	1	3	4	0.0118	0	1	3	4	0.0084	0	1	3	4	0.0608	0
	15000	1	2	4	0.0235	0	1	3	4	0.0433	0	1	3	4	0.0196	0	1	3	4	0.0877	0	1	3	4	0.0298	0
F20	3000	46	132	139	0.1504	7.791E-08	83	469	250	0.2791	1.0222E-07	52	335	157	0.3174	6.3633E-08	67	357	202	0.2436	3.3896E-07	52	309	157	0.3072	5.9964E-08
	6000	46	132	139	0.2651	1.5376E-07	93	536	280	0.7515	2.8244E-07	52	335	157	0.2549	1.2741E-07	83	444	250	0.4511	2.1701E-07	52	309	157	0.8193	1.1993E-07
	9000	46	132	139	0.3852	2.2961E-07	83	485	250	0.9371	4.9459E-07	52	335	157	0.4770	1.9118E-07	85	452	256	0.8718	5.8219E-07	52	309	157	0.4657	1.7989E-07
	12000	46	132	139	0.7930	3.0546E-07	88	510	265	1.3045	2.9312E-08	52	335	157	0.4953	2.5496E-07	76	399	229	0.6840	4.7843E-08	52	309	157	0.6549	2.3986E-07
	15000	46	132	139	0.7073	3.8131E-07	88	520	265	1.1829	8.2899E-08	52	335	157	0.9070	3.1873E-07	89	473	268	0.9817	5.8863E-07	52	309	157	0.6045	2.9982E-07
F21	3000	F	F	F	F	F	46	336	139	0.2731	0.00002064	31	242	94	0.1337	9.5188E-07	61	327	184	0.2680	2.2019E-06	55	279	166	0.2325	8.1566E-07
	6000	F	F	F	F	F	52	304	157	0.5961	1.5093E-06	32	246	97	0.2033	3.786E-07	34	246	103	0.2648	1.0505E-07	52	305	157	0.4332	1.0173E-06
	9000	F	F	F	F	F	55	357	166	1.1232	2.6187E-06	32	244	97	0.4123	0.000033429	43	300	130	0.5784	1.1647E-06	53	283	160	0.8361	8.8258E-07
	12000	F	F	F	F	F	39	277	118	1.8857	5.3587E-07	26	216	79	0.4675	5.8685E-07	42	347	127	0.9517	3.9115E-07	37	283	112	0.5606	0.000010014
	15000	F	F	F	F	F	35	263	106	1.2695	3.0481E-06	31	229	94	0.8068	7.3277E-07	34	218	103	0.5337	1.0149E-06	50	305	151	1.2809	6.2728E-06

TABLE 5. The performance of the TTCGC algorithms vs M1, CGSQN, and SSHS algorithms on problems F22 to F26 are evaluated. The symbol 'F' indicates when an algorithm fails to solve a problem.

METHODS		M1					CGSQN					SSHS					TTCGC1					TTCGC2				
FUNCS	DIM	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F	IT	FE	NG	CPU	V_F
F22	3000	6	7	19	0.0308	1.8969E-09	4	10	13	0.0603	4.3783E-07	5	12	16	0.0460	2.7598E-06	7	17	22	0.0269	5.6722E-11	5	11	16	0.0213	2.0404E-10
	6000	6	7	19	0.0435	2.2646E-09	4	10	13	0.0497	9.9114E-07	5	12	16	0.0339	5.5409E-06	7	17	22	0.0486	1.3978E-10	5	11	16	0.0370	1.4467E-10
	9000	6	7	19	0.0729	2.7786E-09	4	10	13	0.0477	1.5466E-06	5	12	16	0.0506	8.3218E-06	7	17	22	0.0599	2.2409E-10	5	11	16	0.0836	1.2955E-10
	12000	6	7	19	0.1223	3.3273E-09	4	10	13	0.1746	2.1026E-06	5	12	16	0.0634	0.000011103	7	17	22	0.1869	3.0874E-10	5	11	16	0.0658	1.256E-10
	15000	6	7	19	0.1385	3.8896E-09	4	10	13	0.3973	2.6588E-06	5	12	16	0.0723	0.000013884	7	17	22	0.1080	3.9351E-10	5	11	16	0.1077	1.2612E-10
F23	3000	F	F	F	F	F	37	252	112	0.1677	1.3447E-06	34	241	103	0.1112	4.9593E-07	40	266	121	0.1011	6.0373E-06	34	228	103	0.1248	0.000094035
	6000	F	F	F	F	F	43	319	130	0.3829	4.7575E-07	32	227	97	0.1781	8.3514E-07	42	283	127	0.2281	3.8812E-07	57	269	172	0.2442	9.033E-07
	9000	F	F	F	F	F	50	324	151	1.5788	1.6478E-06	29	222	88	0.3334	0.000044508	39	282	118	0.6487	3.9058E-07	56	273	169	0.5383	8.4305E-07
	12000	F	F	F	F	F	47	357	142	0.7846	0.000019352	24	171	73	0.5713	7.029E-07	41	273	124	0.7160	1.7487E-06	57	268	172	0.6178	9.0349E-07
	15000	F	F	F	F	F	41	279	124	0.9896	3.0793E-06	29	230	88	0.5215	0.000046358	42	305	127	0.6489	2.8581E-06	48	262	145	0.5854	7.8383E-07
F24	3000	560	761	1681	2.1513	45000	F	F	F	F	F	26	120	79	0.2191	0.000318	62	223	384	0.6481	867	79	578	238	0.4957	45000
	6000	545	621	1636	3.6663	180000	F	F	F	F	F	19	54	58	0.1244	180000	62	223	384	0.6481	867	88	400	265	0.6481	180000
	9000	545	626	1636	5.1580	405000	F	F	F	F	F	20	58	61	0.1579	405000	158	498	475	1.4069	405000	71	343	214	0.7335	405000
	12000	546	680	1639	6.9658	720000	F	F	F	F	F	25	168	76	0.4152	720000	52	310	157	0.8004	720000	106	961	319	2.3981	720000
	15000	539	611	1618	8.3437	1125000	F	F	F	F	F	26	231	79	0.6274	1125000.001	46	385	139	1.1442	1125000	111	873	334	2.6092	1125000
F25	3000	1	2	2	0.0045	2500	1	2	2	0.004	2500	2	3	3	0.0380	2500	1	2	2	0.0036	2500	1	2	2	0.0060	2500
	6000	1	2	2	0.0027	5000	1	2	2	0.0041	5000	2	3	3	0.0031	5000	1	2	2	0.0025	5000	1	2	2	0.0131	5000
	9000	1	2	2	0.0034	7500	1	2	2	0.0044	7500	2	3	3	0.0034	7500	1	2	2	0.0039	7500	1	2	2	0.0088	7500
	12000	1	2	2	0.0032	10000	1	2	2	0.0073	10000	2	3	3	0.0037	10000	1	2	2	0.0049	10000	1	2	2	0.0384	10000
	15000	1	2	2	0.0071	12500	1	2	2	0.0430	12500	2	3	3	0.0194	12500	1	2	2	0.0049	12500	1	2	2	0.0060	12500
F26	3000	F	F	F	F	F	15	212	46	0.1207	0.5	2	29	7	0.0353	2.4104E-15	21	79	64	0.0511	1.3387E-12	170	4084	511	1.4304	1.4587E-12
	6000	F	F	F	F	F	19	428	58	0.5007	0.5	2	31	7	0.0391	2.2676E-16	23	86	70	0.1759	6.8664E-14	169	4407	508	4.2909	7.1285E-13
	9000	F	F	F	F	F	19	372	58	0.9486	0.5	2	32	7	0.0582	2.7369E-16	23	87	70	0.2090	1.7366E-13	113	3056	340	3.9884	1.8369E-13
	12000	F	F	F	F	F	88	1222	265	2.3783	1.2107E-07	2	33	7	0.0698	9.9658E-16	24	91	73	0.4882	9.5827E-14	173	4870	520	8.4182	3.8914E-13
	15000	F	F	F	F	F	36	671	109	1.3947	0.5	2	33	7	0.1274	6.509E-15	23	88	70	0.3694	2.0454E-13	104	3004	313	5.9008	3.3204E-13

problem with the following five dimensions ranging from 3000, 6000, 9000, 12000, and 15,000. We performed the experiments by evaluating the suggested TTCGC1 and TTCGC2 against Kobayashi et al.'s CGSQN [31], Dehghani et al.'s M1 [32] and Yunus et al.'s SSHS [57] as points of comparison. We carried out the experiment using MATLAB R2022a on a personal computer with an Intel (R) CORE(TM) i7-3537U processor running at 2.00 GHz and having 8 GB of RAM. The algorithms were executed with the specified configuration of parameters:

- TTCGC1 algorithm: $\eta_{min} = 0.1$, $\eta_{max} = 0.85$, $\gamma_k = 0.5$, $\kappa = 10^{-8}$.
- TTCGC2 algorithm: $\epsilon = 10^{-4}$, $\eta_{min} = 0.1$, $\eta_{max} = 0.85$, $\vartheta_k = 7/8$, $\kappa = 10^{-8}$.
- CGSQN algorithm: The parameter settings for this method remained unchanged, as in [31].
- M1 algorithm: The parameter settings for this method remained unchanged, as in [32]
- SSHS algorithm: The parameter settings for this method remained unchanged, as in [57]

Furthermore, throughout the iterative process, we employed a stopping criterion, which is $\|g_k\| \leq 10^{-4}$. We reported that each algorithm had reached a solution when the criterion was met, and we marked any failures with the label "F" in case any of the following conditions occurred:

1) If there are more than a thousand iterations.
2) If the count of function evaluations went beyond 5000.

To evaluate the effectiveness of the proposed algorithms, we utilize the following comparative metrics: (i) Number of iterations, *(IT)* (ii) Number of function evaluations, *(FE)* (iii) Number of gradient evaluations, *(NG)* (iv) CPU time *(CPU)* required to reach the approximate solution, and (vi) Residual Value *(V_F)*. The outcomes of the numerical experiments are presented in Tables 2, 3, 4, and 5, respectively.

Furthermore, to offer a comprehensive perspective on the comparison, we have visually summarized the data presented in Tables 2, 3, 4, and 5 by using the widely recognized Dolan and More [55] performance profile. This can assist in standardizing the comparison of methods. Let's consider a method labeled as n_m and a problem referred to as n_p. In our evaluation, we focus on a performance metric, which can be either the number of iterations, number of function evaluations number of gradient evaluations, CPU time, or the residual value. We denote by $b_{p,m}$ one of the mentioned metrics required by method m to solve the problem. To gauge the performance of a solver s on problem p relative to the best performance achieved by other solvers for the same problem, we utilize the performance ratio $r_{p,s}$, defined as follows:

$$r_{p,s} = \frac{b_{p,s}}{\min\{b_{p,s} : s \in S, p \in P\}}, \tag{59}$$

let M represent the set of solvers and P denote the set of problems. Subsequently, consider $\rho_s(\tau)$ as the probability for a solver $s \in S$ that a performance ratio $r_{p,s}$ falls within a factor of $\tau \in \mathbb{R}^+$ of the best achievable ratio. The formulation for

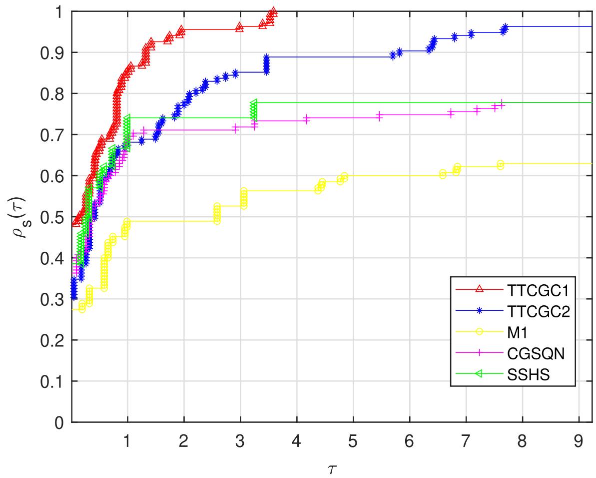

FIGURE 1. Performance assessment relative to the number of iterations.

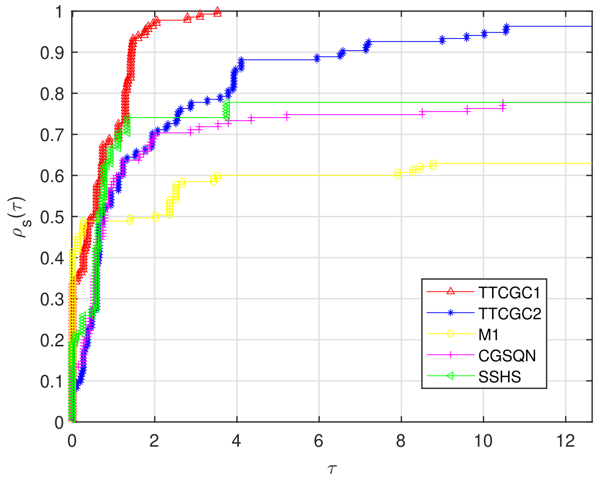

FIGURE 2. Performance assessment relative to the number of functions evaluation.

$\rho_s(\tau)$ is expressed as follows:

$$\rho_s(\tau) = \frac{1}{n_p} size\{p \in P : \log_2(r_{p,s}) \leq \tau\}. \tag{60}$$

According to their guidelines, the solver having the highest probability $\rho_s(\tau)$ is deemed the top-performing solver.

The comparisons were conducted based on the specified metrics depicted in Figures 1 through 5. Upon examination of these figures, it becomes apparent that the proposed TTCGC1 and TTCGC2 algorithms consistently outperform their rivals, M1, CGSQN, and SSHS across all five metrics. In a broader context, Figure 1 within the factor $\tau \geq 1$ indicates that TTCGC1 and TTCGC2 solve approximately 85% and 69% of the test problems with less number of iterations, whereas M1, CGSQN, and SSHS solve nearly 50%, 69%, and 74% of the test problems, respectively. In Figure 2, at $\tau \geq 2$ it is evident that TTCGC1 and TTCGC2 address nearly 97% and 70% of the test problems with the minimum number of function evaluations. In comparison, M1, CGSQN, and SSHS handle approximately 50%, 70%, and 73% of the test problems in terms of the number of function evaluations, respectively. In Figure 3, it is illustrated that TTCGC1 and TTCGC2

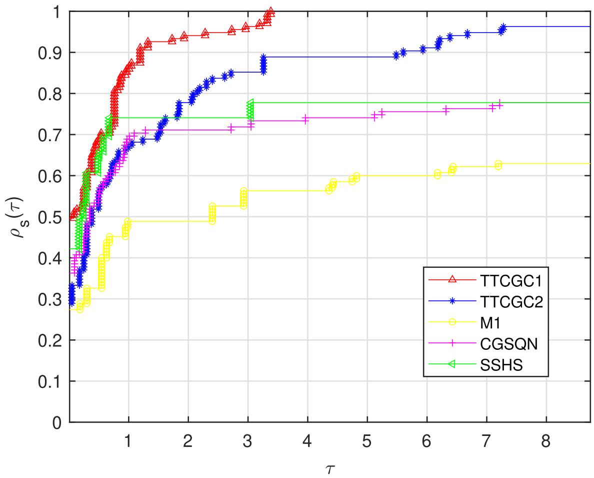

FIGURE 3. Performance assessment relative to the CPU Time.

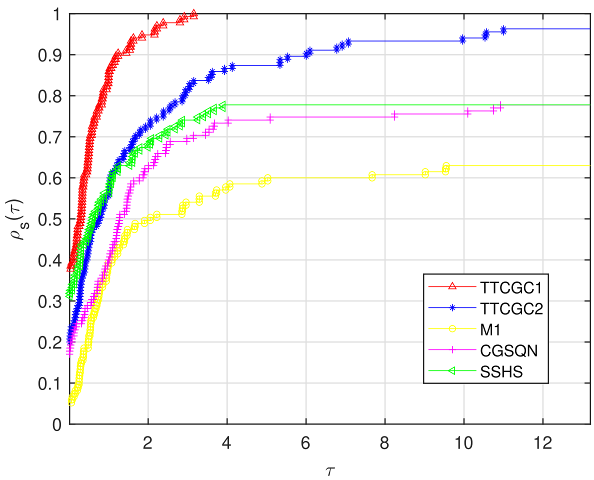

FIGURE 4. Performance assessment relative to the number of gradient evaluations.

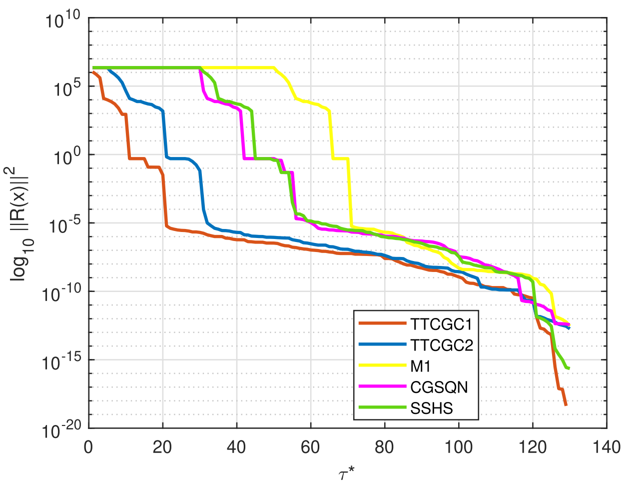

FIGURE 5. Performance assessment relative to the least zero residual errors.

successfully address nearly 95% and 74% of the test problems with a minimal number of gradient evaluations. Conversely, M1, CGSQN, and SSHS manage 50%, 63%, and 70% of the test problems in terms of the number of gradient evaluations,

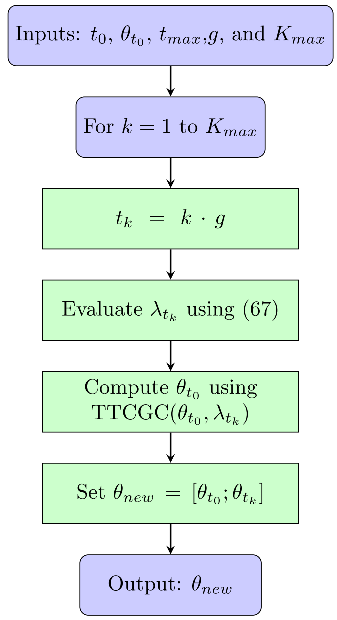

FIGURE 6. Steps for solving Robotic model with 4DOF.

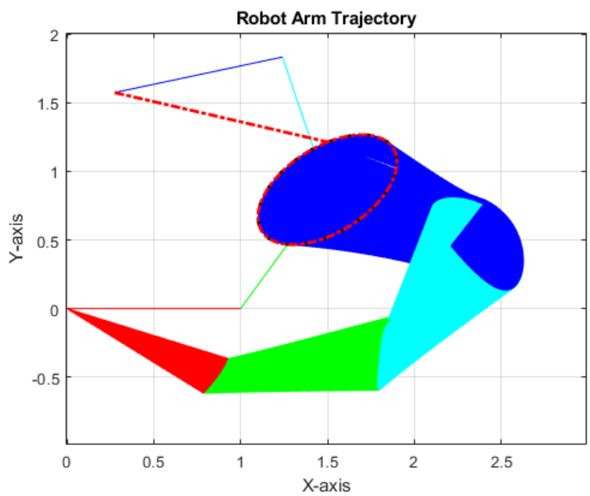

FIGURE 7. Synthesized robot paths of Lissajous curve.

respectively. Moreover, in terms of the CPU time, in Figure 4, in particular at $\tau \geq 2$ TTCGC1 and TTCGC2 demonstrate effectiveness by solving approximately 94% and 79% of the test problems in the shortest amount of time. In contrast, M1, CGSQN, and SSHS address almost 49%, 72%, and 74% of the test problems, respectively. While TTCGC2 demonstrates comparable performance to the SSHS method, it surpasses both M1 and CGSQN across all metrics. The performance of the TTCGC1 algorithms surpasses that of the TTCGC2, M1, CGSQN, and SSHS algorithms, as indicated by Figure 5, it becomes evident that the TTCGC1 methods' convergence offers a notably accurate approximation of the solution with significantly fewer errors, where τ^* represents the count of instances of zero residual problems. Consequently, the proposed approaches are efficient with low memory requirements and may represent a better option.

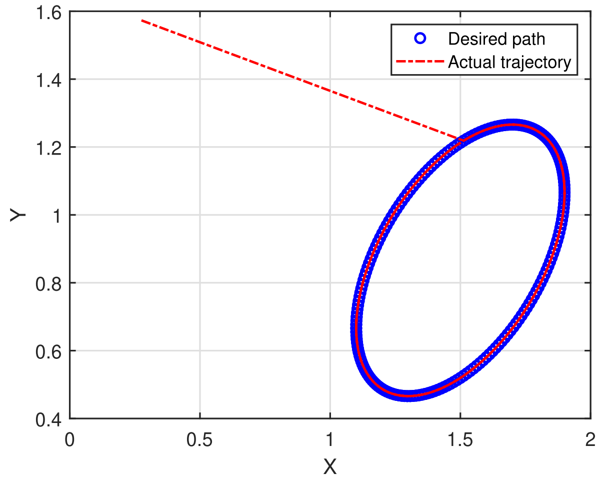

FIGURE 8. End-effector path and intended path of Lissajous curve.

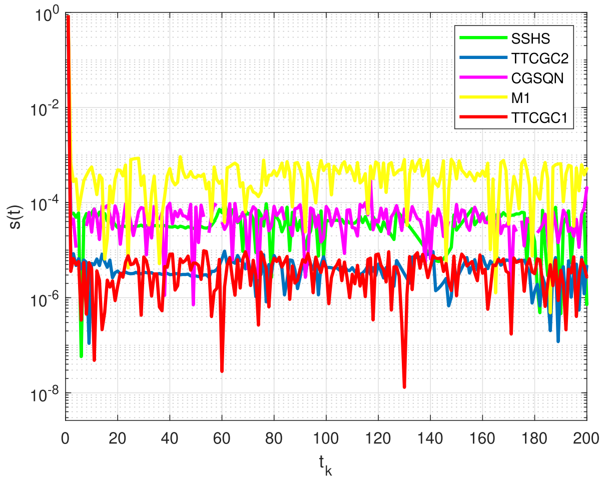

FIGURE 9. Monitoring the residual error of Lissajous curve along x-axis.

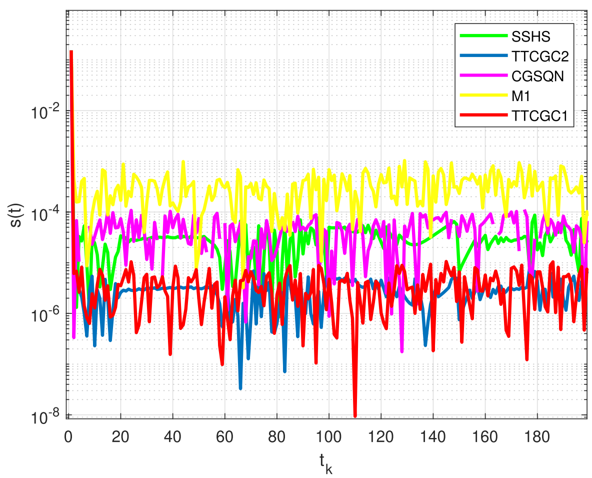

FIGURE 10. Monitoring the residual error of Lissajous curve along y-axis.

V. APPLICATION TO 4DOF MOTION CONTROL OF ROBOT MODEL

This section presents an application of the proposed TTCGC algorithm to address a practical robotic model. The model has been extended from its initial three degrees of freedom (3DOF) state, as discussed in [56] and [57] to encompass four degrees of freedom (4DOF). Furthermore, a 4DOF robot arm can be helpful in various applications such as assembly lines to pick and place parts, painting applications to ensure even coverage, and welding applications to ensure accurate welds [58]. The depiction of the planar four-joint kinematic model and the formulation of the discrete kinematic model equation involving four degrees of freedom are conveyed through the equation below.

$$s(\theta) = \begin{bmatrix} \sum_{j=1}^{4} l_j \cos(\theta_1 + \theta_2 + \ldots + \theta_j) \\ \sum_{j=1}^{4} l_j \sin(\theta_1 + \theta_2 + \ldots + \theta_j) \end{bmatrix}, \tag{61}$$

where $s(\cdot)$ represents the kinematic transformation and positioning of a robot's endpoint or any component of the robot concerning active adjustments in its joints, denoted by $\theta_j \in \mathbb{R}^4$. Each link is designated as l_j (where j takes values $j = 1, 2, \ldots, 4$) and signifies the length of the respective link. Furthermore, within the notion of controlling robotic motion, $s(\theta)$ embodies a vector indicating the position of the end effector. Let us consider $\lambda_{t_k} \in \mathbb{R}^2$ to represent the vector defining the desired path at a specific moment, say t_k. We have devised the subsequent least-squares model. This formulation will be computed within each time segment, denoted as t_k belonging to the interval $[0, t_f]$. We state the optimization problem as follows:

$$\min_{\theta \in \mathbb{R}^4} \frac{1}{2} \|s(\theta) - \lambda_{t_k}\|^2, \tag{62}$$

In which λ_{t_k}, as documented in [56], [57], and [59], represents the desired trajectory at the instance t_k of a Lissajous curve depicted as

$$\lambda_{t_k} = \begin{bmatrix} 1.5 + 0.4 \sin(\frac{\pi t_k}{5}) \\ \frac{\sqrt{3}}{2} + 0.4 \sin(\frac{\pi t_k}{5} + \frac{\pi}{3}) \end{bmatrix}, \tag{63}$$

The structure of (62), as demonstrated earlier, resembles that of (11). This similarity enables the utilization of the TTCGC algorithm to evaluate its solution. The flowchart in Figure 6 outlines the procedure for addressing the challenge of controlling robotic motion. Now, to solve the model and then simulate the outcomes, we make use of the combination at the time moment $t = 0$ denoted as $\theta_{t_0} = [0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}]$, where the length of each link is to $l_j = 1$ for $j = 1, 2, 3, 4$. Additionally, it requires the utmost duration, $t_{max} = 10$ seconds within Algorithm 2.

Upon examination of the illustrated figures that display the results of solving equation (62) using the TTCGC1 algorithm, Figure 8 depicts the robot's end effector model accurately following the desired path. Figure 7 effectively demonstrates the accomplishment of synthesizing robot trajectories for the task. Figures 9 and 10 depict the error rates of residuals, where TTCGC1 and TTCGC2 exhibit the lowest error at approximately 10^{-6}, followed by SSHS with 10^{-5}, CGSQN with 10^{-4}, and M1 with 10^{-3}. The remarkably low residual error rates of TTCGC1 and TTCGC2 in this case underscore the algorithms' good performance, affirming their effectiveness in seeking accurate and reliable solutions for least-square models.

VI. CONCLUSION

In this article, two coefficients for the TTCG method were developed to address nonlinear least-squares problems. These proposed algorithms are constructed based on the structured secant equation and are derived using properties of sufficient descent and the Dai et al. conjugacy condition. We have demonstrated that both algorithms ensure sufficient descent conditions regardless of any specific line search method. Furthermore, by employing the nonmonotone line search technique presented by Zhang and Hager [45], we have established the global convergence of these algorithms, subject to certain conditions. We have also demonstrated these algorithms' numerical efficiency and robustness through benchmark test problems, showing their competitiveness. Additionally, to illustrate the practical applicability of the proposed algorithms, these algorithms are more applicable than the others in addressing a 4DOF robotic motion control problem. Finally, future work in this area will involve the incorporation of the modified structured secant equation as delineated in [32].

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