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RESEARCH ARTICLE

A Novel Approach Toward Complex Pythagorean Fuzzy Sets and Their Applications in Visualization Technology

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ABSTRACT The theory of complex Pythagorean fuzzy set (CPFS) has been already interpreted. This theory states that in both polar and Cartesian form the degree of membership and degree of non-membership are located in a complex plane's unit disc. However the Cartesian presentation has some drawbacks for instance, this theory can't consider the full belongingness $(1 + \iota 1, 0 + \iota 0)$ of the element because this value is not located in the unit disc of a complex plane but in a unit square of a complex plane. Hence, with the existing Cartesian form of CPFS, the full belongingness of the element can't be described. Further, in polar form, the degrees of membership and non-membership are interpreted by amplitude and phase terms. Where the amplitude acts similar as they act in the Pythagorean fuzzy set (PFS) while phase terms show the periodicity, direction, or position of the element in a set. However, this description of degrees of membership and non-membership is confined to the polar structure and is inappropriate for inclusion in logical operations accompanied by CPFS in Cartesian coordinates. Therefore, in this article, we devise a theory of CPFS in Cartesian coordinates, where both degrees of membership and non-membership are in Cartesian form located in a complex plane's unit disc and containing real and imaginary terms. These terms are fuzzy functions and carry fuzzy information. We also establish a few critical and basic operations for CPFS in Cartesian coordinate and then discuss some aggregation operators (AOs) and a multi-attribute decision-making (MADM) method within CPFS. After that, we employ the established theory in the field of visualization technology to reveal the applicability and requirement of the proposed theory. We interpreted the comparison of the deduced theory with certain prevailing theories to portray the supremacy of this work.

INDEX TERMS Complex Pythagorean fuzzy set in Cartesian coordinate, visualization technology, aggregation operators, MADM.

I. INTRODUCTION

Because crisp sets are binary—an element is either totally in or fully out of the set—they frequently fall short of providing an accurate representation of real-world occurrences. It is challenging to depict circumstances including uncertainty or ill-defined limits because of this constraint.

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Therefore, Zadeh [1] portrayed the theory of fuzzy set (FS) in 1965 which is a more adaptable framework that can make sense of and interpret ambiguous or imprecise data. An FS permits partial membership, in contrast to crisp sets, which have well-defined membership functions that assign entries either wholly within or totally outside the collection. Accordingly, the degree to which an element belongs to the set can be indicated by its degree of membership, which ranges from 0 to 1. Different fields, for example, decision-making, control systems, pattern identification, and artificial intelligence, have tracked down need for the hypothesis of FS. More exact and extensive demonstrating of the veritable world is made conceivable by them, as they give a strong device to adapting to mind boggling and dubious data. Czogala and Drewniak [2] initiated monotonic operations in the notion of FS. Dubois and Prade [3] devised aggregation connectives in FS. Lin et al. [4] diagnosed excavation systems relying on FS and machine learning approaches. Kumar et al. [5] investigated comparative studies in business and management by using the theory of FS. The operations based on the entropy for FS theory were developed by Rudas and Kaynak [6].

In situations when the degree of non-membership and indeterminacy are major factors in decision-making (DM), FS may not be able to fully reflect the range of uncertainty. FS has this limitation because it only takes into account degrees of membership, ignoring degrees of non-membership and the indeterminacy that goes along with them. This problem is addressed by intuitionistic FS (IFS), which was devised by Atanassov and Stoeva. [7]. It makes it possible to depict not just different degrees of membership but also different degrees of non-membership and indeterminacy. In the theory of IFS, the sum of the degrees of membership and non-membership should be located in [0, 1]. Takeuti and Titani [8] devised intuitionistic fuzzy logic. De et al. [9] interpreted certain operations for the model of IFS. Du [10] investigated operations like division and subtraction within IFS. Xu [11] and Xu and Yager [12] devised averaging and geometric AOs for IFS respectively. The group DM and Heronian AOs for IFS were devised by Liu and Chen [13]. If any element has a degree of membership of 0.8 and a degree of non-membership of 0.4, then $0.8+0.4 = 1.2 > 1$. To overcome this, Yager [14] devised the theory of PFS, which is the modification of IFS. In the theory of PFS, the sum of the squares of degrees of membership and non-membership must located in [0, 1]. Peng and Yang [15] deduced some results within PFS and Yager [16] devised various applications and properties of the PFS. Garg [17] investigated confidence levels AOs in the setting of PFS. Akram et al. [18] devised Dombi AOs with PFS. Yager [19] devised a multi-criteria DM (MCDM) approach under PFS.

In 2002, Ramot et al. [20] portrayed complex fuzzy sets (CFS), unlike traditional FS, utilize a degree of membership, which is a combination of a degree of membership in a FS along with a crisp phase value that signifies location within the set. This degree of membership is represented in polar coordinates and carries more information than FS, enabling more efficient reasoning. Ramot et al. [21] discussed complex fuzzy logic and Zhang et al. [22] initiated some operations and equalities for CFS. After that, in 2011, Tamir et al. [23] originated the CFS in the Cartesian coordinate, where the degree of membership is located in the first quadrant of the unit square of the complex plane. Rehman [24] established certain properties and AOs for the Cartesian form of CFS. Mahmood et al. [25] originated a complex fuzzy N-soft set. Alkouri and Salleh [26] devised the notion of complex IFS (CIFS) in polar coordinates, where the degrees of membership and non-membership are located in a complex plane's unit disc. After the establishment of CIFS in polar coordinate, a lot of researchers discussed various theories in CIFS such as Garg and Rani [27] originated information measures, Garg and Rani [28] devised AOs and ranking approach, Rani and Garg [29] devised power AOs and MCDM and Liu and Wang [30] devised MCDM approaches. Ali et al. [31] devised the Cartesian structure of CIFS by changing the range from a complex plane's unit disc to a complex plane's unit square. The degrees of membership and non-membership contain both real and imaginary terms that convey fuzzy information. Fang et al. [32] devised probability AOs for the Cartesian framework of CIFS. Rehman and Mahmood [33] devised a complex intuitionistic fuzzy N-soft set. Ullah et al. [34] interpreted CPFS within the polar framework and after that various researchers utilized this polar form of CPFS and developed various notions such as Hezam et al. [35] devised geometric AOs for CPFS, Akram et al. [36] discussed Dombi AOs and Janani et al. [37] devised Einstein AOs for CPFS. Akram and Naz [38] devised the DM technique in the polar form of CPFS.

A. MOTIVATION AND CONTRIBUTION

Walters et al. [40] devised a theory of CPFS in 2020. This theory states that in both polar and Cartesian forms the degree of membership map from universal set to $\left\{\mathrm{Z}^{\mathbb{M}} : \mathrm{Z}^{\mathbb{M}} \in \mathbb{C}, \left|\mathrm{Z}^{\mathbb{M}}\right| \leq 1\right\}$ and degree of non-membership map from universal set to $\left\{\mathrm{Z}^{\mathbb{N}} : \mathrm{Z}^{\mathbb{N}} \in \mathbb{C}, \left|\mathrm{Z}^{\mathbb{N}}\right| \leq 1\right\}$, that is they are located in a complex plane's unit disc. However, the Cartesian presentation of CPFS contradicts the basic notion of crisp and FS theory as this representation can't interpret the full belongingness of the element to the set. When an element's degree of membership is 1, it fully belongs to the set; when it is 0, it does not entirely belong, according to the theory of crisp sets and FS. Analogously, in the theory of PFS, an element is said to be fully belonging to the set when its degree of membership is 1 and non-membership is 0; in the theory of CFS in Cartesian coordinates, on the other hand, an element is said to be fully belonging to the set when its degree of membership is $1 + \iota 1$. From this, it follows that in the Cartesian form of CPFS, the degree of non-membership should be $0 + \iota 0$ ($\mathrm{Z}^{\mathbb{N}} = 0 + \iota 0$) and the degree of membership for an element to fully belong to a set should be $1 + \iota 1$ ($\mathrm{Z}^{\mathbb{M}} = 1 + \iota 1$). However, the degree of membership in the Walters et al. [40] takes values in the unit disc of a complex plan, which is $\left\{\mathrm{Z}^{\mathbb{M}} : \mathrm{Z}^{\mathbb{M}} \in \mathbb{C}, \left|\mathrm{Z}^{\mathbb{M}}\right| \leq 1\right\}$ and $\left|\mathrm{Z}^{\mathbb{M}}\right| = |1 + \iota 1| = \sqrt{1^2 + 1^2} = \sqrt{2} = 1.4142 > 1$. As to Ullah et al.'s interpreted idea, the degree of membership cannot be $1 + \iota 1$. As a result, the element cannot fully belong to the set in this Cartesian form, which runs counter to the fundamental ideas of crisp and FS theory. Likewise, in the case of non-belonging, the degree of membership should be $0 + \iota 0 \left(\mathrm{Z}^{\mathbb{M}} = 0 + \iota 0\right)$, and the degree of non-membership should be $1 + \iota 1 \left(Z^{\mathbb{N}} = 1 + \iota 1\right)$. Once more, $1 + \iota 1$ cannot be the degree of non-membership.

Additionally, in different DM problems, the expert can assign a value $(0.9 + \iota 0.8, 0.2 + \iota 0.3)$ to an element based on their judgment. However, the magnitude of that number, $\sqrt{(0.9)^2 + (0.8)^2} = 1.204 > 1$, prevents the map for the degree of membership established by Walters et al. [40] from giving this value. Because the value is situated in a complex plane's unit square rather than its unit disc. Moreover, a large number of other numbers are not part of the unit disc. Furthermore, in polar form, the amplitude terms or absolute values of degrees of membership and non-membership are confined inside $[0, 1]$ and the sum of the squares of degrees of membership and non-membership contain in $[0, 1]$ and the phase terms located a position within a set. This interpretation of degrees of membership and non-membership is confined to the polar structure and can't model fuzzy information in Cartesian coordinates. To get over the previously mentioned problems, this script introduces a novel idea of CPFS in a Cartesian coordinate by switching the range from the unit disc of the complex plane to the unit square. In the Cartesian form of CPFS, the degrees of membership and non-membership contain real and imaginary parts that convey fuzzy information and are located in a complex plane's unit square. Applications for CPFS can be found in a number of disciplines, including economics, engineering, and decision sciences, where complex modeling approaches are necessary for the proper representation and analysis of inherently complex uncertainties. We also devise fundamental operational laws for the created CPFS. Furthermore, we deduce averaging and geometric AOs within the Cartesian model of CPFS, these AOs are complex Pythagorean fuzzy (CPF) weighted averaging (CPFWA), CPF ordered weighted averaging (CPFOWA), CPF weighted geometric (CPFWG) and CPF ordered weighted geometric (CPFOWG) operators. Then employing these operators, we initiate a technique of MADM within CPFS for tackling real-life MADM dilemmas that contain CPF information. Finally, in this article, we devise the application of diagnosed theory in the field of visualization technology and then reveal the supremacy of the devised theory by doing a comparison.

B. APPLICATION OF PROPOSED THEORY

The technique of generating virtual instances of computing resources, such as servers, storage devices, networks, or operating systems, is known as virtualization technology. Several virtual environments can operate on a single physical hardware system thanks to the software that creates these virtual instances. This technology makes IT framework the executives more savvy, adaptable, versatile, and asset proficient. Better asset allotment and the executives are made conceivable by virtualization, which confines actual equipment from the working frameworks and applications that suddenly spike in demand for it. Many areas of computing, including software development, cloud computing, data centers, and testing environments, rely on virtualization technology today. Virtualization empowers server combination in server farms, which brings down the expense of equipment, energy use, and space required. It makes it feasible for organizations to rapidly adjust to moving responsibility requests by permitting them to produce and work virtual machines (VMs) on request. Virtualization, which makes it possible to efficiently provide customers with on-demand computer resources, is also the foundation of cloud computing services. Virtualization likewise offers isolated conditions for application testing in programming improvement and testing, ensuring stage similarity and dependability. In light of everything, virtualization innovation changes the IT framework through superior adaptability, smoothed-out administration techniques, and ideal asset use. Kumar and Charu [39] investigated the significance of virtualization technology. Walters et al. [40] provided a comparative investigation of virtualization technologies. Song [41] investigated risks related to virtualization technology. The performance assessment of virtualization technologies was devised by Padala et al. [42]. Employing fuzzy logic, a model for assigning resource recommendations on virtualization technology was devised by Chompoonuch [43].

The most common way of assessing virtualization technology is by deciding their exhibition and capacities for a processing climate. To survey on the off chance that virtualization technologies are reasonable for arrangement, this methodology as a rule incorporates taking a gander at their viability, versatility, and similarity. Evaluators regularly consider factors including generally speaking constancy, responsiveness of the framework, and asset utilization. The goal is to figure out which virtualization technology best suits the remarkable necessities and objectives of the organization to amplify equipment asset use, further develop adaptability, and improve the administration of responsibilities that are virtualized. Moreover, reconciliation with the current IT framework and security contemplations could be very significant in the choice cycle. Since assessing multiple attributes and models immediately is troublesome, the assessment and determination of virtualization technology can be seen as a multi-attribute decision-making (MADM) dilemma. Making informed decisions in MADM requires gauging choices against different elements. While assessing virtualization technology, decision-makers need to consider a few attributes, including performance, ease of management, scalability, compatibility, security, and cost. Contingent upon the special requirements and objectives of the organization, every one of these characteristics might have differing loads and levels of importance. Furthermore, the trade-offs and interdependencies among these characteristics add to the complexity, therefore before making a choice, it is crucial to use MADM approaches to methodically examine and contrast the available virtualization choices. Consequently, in this article, we discuss a case study ‘‘Evaluation and selection of virtualization technologies for cloud-based e-commerce platform. In this case, we select the finest technology for tackling some considered problems of cloud-based e-commerce platforms.

The remaining script is demonstrated: In Section II, the literature review is interpreted. In Section III, first, we devise the already interpreted concept of CPFS and discuss the drawbacks of that concept. Then we deduce a new definition of the CPFS in Cartesian coordinates and its related operations. In Section IV, we anticipate averaging and geometric AOs in the Cartesian framework of CPFS along with connected properties. In Section V, we reveal the application and applicability of the proposed theory. We also initiate a technique of MADM within CPFS and discuss the case study. Section VI of the script contains the comparison of the established theory and Section VII has the conclusion.

II. LITERATURE REVIEW

This section of the script contains certain prevailing concepts such as PFS, CFS, and their associated properties.

The theory of PFS was devised by Yager [14] in 2013, by modifying the notion of IFS.

The PFS $\mathbb{Y}_{PFS}$ over Џ is interpreted by the degree of membership $\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS}}(\boldsymbol{\upsilon}) : \text{Џ} \to [0,1]$ and degree of non-membership $\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS}}(\boldsymbol{\upsilon}) : \text{Џ} \to [0,1]$ for an element $\boldsymbol{\upsilon} \in \text{Џ}$ with the condition that $0 \leq \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS}}((\boldsymbol{\upsilon}))\right)^2 + \left(\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS}}((\boldsymbol{\upsilon}))\right)^2 \leq 1$. Mathematically, a PFS is deduced as

$$\mathbb{Y}_{PFS} = \left\{\left(\boldsymbol{\upsilon}, \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS}}(\boldsymbol{\upsilon}), \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS}}(\boldsymbol{\upsilon})\right)\right) | \boldsymbol{\upsilon} \in \text{Џ}\right\}$$

Moreover,

$$\mathcal{G}^{\mathbb{D}}_{\mathbb{Y}_{PFS}}(\boldsymbol{\upsilon}) = \left(1 - \left(\left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS}}(\boldsymbol{\upsilon})\right)^2 + \left(\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS}}(\boldsymbol{\upsilon})\right)^2\right)\right)^{\frac{1}{2}}$$

is treated as the degree of indeterminacy. The Pythagorean fuzzy number (PFN) will identified as $\mathbb{Y}_{PFS} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS}}\right)$.

In a similar article, Yager also devised operations like complement, union, and intersection for PFS.

Suppose

$$\mathbb{Y}_{PFS-1} = \left\{\left(\boldsymbol{\upsilon}, \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-1}}(\boldsymbol{\upsilon}), \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-1}}(\boldsymbol{\upsilon})\right)\right) | \boldsymbol{\upsilon} \in \text{Џ}\right\}$$

and

$$\mathbb{Y}_{PFS-2} = \left\{\left(\boldsymbol{\upsilon}, \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-2}}(\boldsymbol{\upsilon}), \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-2}}(\boldsymbol{\upsilon})\right)\right) | \boldsymbol{\upsilon} \in \text{Џ}\right\}$$

are two PFS, then

1. $(\mathbb{Y}_{PFS-1})^c = \left\{\left(\boldsymbol{\upsilon}, \left(\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-1}}(\boldsymbol{\upsilon}), \mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-1}}(\boldsymbol{\upsilon})\right)\right) | \boldsymbol{\upsilon} \in \text{Џ}\right\}$
2. $\mathbb{Y}_{PFS-1} \cup \mathbb{Y}_{PFS-2} = \left\{\left(\boldsymbol{\upsilon}, \begin{pmatrix} \max\left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-1}}(\boldsymbol{\upsilon}), \mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-2}}(\boldsymbol{\upsilon})\right), \\ \min\left(\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-1}}(\boldsymbol{\upsilon}), \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-2}}(\boldsymbol{\upsilon})\right) \end{pmatrix}\right) | \boldsymbol{\upsilon} \in \text{Џ}\right\}$
3. $\mathbb{Y}_{PFS-1} \cap \mathbb{Y}_{PFS-2} = \left\{\left(\boldsymbol{\upsilon}, \begin{pmatrix} \min\left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-1}}(\boldsymbol{\upsilon}), \mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-2}}(\boldsymbol{\upsilon})\right), \\ \max\left(\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-1}}(\boldsymbol{\upsilon}), \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-2}}(\boldsymbol{\upsilon})\right) \end{pmatrix}\right) | \boldsymbol{\upsilon} \in \text{Џ}\right\}$

Zhang and Xu [44] devised how one can find the score and accuracy values of PFN.

Suppose $\mathbb{Y}_{PFS} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS}}\right)$ is a PFN, then the score value $\check{S}(\mathbb{Y}_{PFS})$ and accuracy value $\text{Ћ}(\mathbb{Y}_{PFS})$ are expressed as below

$$\check{S}(\mathbb{Y}_{PFS}) = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS}}\right)^2 - \left(\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS}}\right)^2, \quad \check{S}(\mathbb{Y}_{PFS}) \in [-1, 1]$$

$$\text{Ћ}(\mathbb{Y}_{PFS}) = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS}}\right)^2 + \left(\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS}}\right)^2, \quad \text{Ћ}(\mathbb{Y}_{PFS}) \in [0, 1]$$

Yager [14] and Yager and Abbasov [45] interpreted some algebraic operational laws for PFS.

Suppose $\mathbb{Y}_{PFS-1} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-1}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-1}}\right)$ and $\mathbb{Y}_{PFS-2} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-2}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-2}}\right)$ are two PFN and $\varkappa \geq 0$, then

1. $\mathbb{Y}_{PFS-1} \oplus \mathbb{Y}_{PFS-2}$
$$= \begin{pmatrix} \left(\left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-1}}\right)^2 + \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-2}}\right)^2 \right. \\ \left. - \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-1}}\right)^2 \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-2}}\right)^2\right)^{\frac{1}{2}}, \\ \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-1}} \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-2}} \end{pmatrix}$$
2. $\mathbb{Y}_{PFS-1} \otimes \mathbb{Y}_{PFS-2}$
$$= \begin{pmatrix} \mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-1}} \mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-2}}, \\ \left(\left(\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-1}}\right)^2 + \left(\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-2}}\right)^2 \right. \\ \left. - \left(\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-1}}\right)^2 \left(\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-2}}\right)^2\right)^{\frac{1}{2}} \end{pmatrix}$$
3. $\varkappa \mathbb{Y}_{PFS-1}$
$$= \left(\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-1}}\right)^2\right)^{\varkappa}\right)^{\frac{1}{2}}, \left(\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-1}}\right)^{\varkappa}\right)$$
4. $\mathbb{Y}^{\varkappa}_{PFS-1}$
$$= \left(\left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{PFS-1}}\right)^{\varkappa}, \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{PFS-1}}\right)^2\right)^{\varkappa}\right)^2\right)$$

In Cartesian coordinates, the CFS was investigated by Tamir et al. [23] in 2011.

The CFS $\mathbb{Y}_{CFS}$ over Џ is interpreted by the degree of membership $\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CFS}}(\boldsymbol{\upsilon}) : \text{Џ} \to [0,1] + \iota[0,1]$ for an element $\boldsymbol{\upsilon} \in \text{Џ}$, where the degree of membership carries a real term $\mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS}}(\boldsymbol{\upsilon})$ and unreal term $\mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS}}(\boldsymbol{\upsilon})$. Mathematically, a CFS is deduced as

$$\begin{aligned} \mathbb{Y}_{CFS} &= \left\{\left(\boldsymbol{\upsilon}, \mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CFS}}(\boldsymbol{\upsilon})\right) | \boldsymbol{\upsilon} \in \text{Џ}\right\} \\ &= \left\{\left(\boldsymbol{\upsilon}, \mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS}}(\boldsymbol{\upsilon}) + \iota \mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS}}(\boldsymbol{\upsilon})\right) | \boldsymbol{\upsilon} \in \text{Џ}\right\} \end{aligned}$$

$\mathbb{Y}_{CFS} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CFS}}\right) = \left(\mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS}} + \iota \mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS}}\right)$ will treated as a complex fuzzy number (CFN).

Suppose $\mathbb{Y}_{CFS} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CFS}}\right) = \left(\mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS}} + \iota \mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS}}\right)$ is a CFN, then the score value $\check{S}(\mathbb{Y}_{CFS})$ and accuracy value $\text{Ћ}(\mathbb{Y}_{CFS})$ are expressed as below

$$\check{S}(\mathbb{Y}_{CFS}) = \frac{\mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS}} - \mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS}}}{2}, \quad \check{S}(\mathbb{Y}_{CFS}) \in [-1, 1]$$

$$\text{Ћ}(\mathbb{Y}_{CFS}) = \frac{\mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS}} + \mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS}}}{2}, \quad \text{Ћ}(\mathbb{Y}_{CFS}) \in [0, 1]$$

Rehman [24] devised algebraic operational laws for CFNs.

Suppose $\mathbb{Y}_{CFS-1} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CFS-1}}\right) = \left(\mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS-1}} + \iota\mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS-1}}\right)$ and $\mathbb{Y}_{CFS-2} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CFS-2}}\right) = \left(\mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS-2}} + \iota\mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS-2}}\right)$ are two CFNs with $\varkappa \geq 0$, then

1. $\mathbb{Y}_{CFS-1} \oplus \mathbb{Y}_{CFS-2} = \left(\begin{array}{l}\mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS-1}} + \mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS-2}} - \mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS-1}}\mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS-2}} \\ +\iota\left(\mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS-1}} + \mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS-2}} - \mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS-1}}\mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS-2}}\right)\end{array}\right)$
2. $\mathbb{Y}_{CFS-1} \otimes \mathbb{Y}_{CFS-2} = \left(\mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS-1}}\mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS-2}} + \iota\mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS-1}}\mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS-2}}\right)$
3. $\varkappa\mathbb{Y}_{CFS-1} = \left(1 - \left(1 - \mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS-1}}\right)^{\varkappa} + \iota\left(1 - \left(1 - \mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS-1}}\right)^{\varkappa}\right)\right)$
4. $\mathbb{Y}^{\varkappa}_{CFS-1} = \left(\left(\mathcal{G}^{\mathbb{R}}_{\mathbb{Y}_{CFS-1}}\right)^{\varkappa} + \iota\left(\mathcal{G}^{\mathbb{I}}_{\mathbb{Y}_{CFS-1}}\right)^{\varkappa}\right)$.

III. THE CONCEPT OF CPFS IN CARTESIAN COORDINATES

In this part, first, we will devise the already interpreted concept of CPFS and discuss the drawbacks of that concept. Then we will deduce a new definition of the CPFS in Cartesian coordinates and its related operations.

Walters et al. [40] devised the notion of CPFS as follows

Definition 1 [40]: A CPFS $\mathbb{Y}_{CPFS}$ is established as

$$\mathbb{Y}_{CPFS} = \left\{\left(\boldsymbol{\upsilon}, \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}), \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)\right) | \boldsymbol{\upsilon} \in \text{Џ}\right\}$$

Noticed that $\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) : \text{Џ} \to \left\{\mathrm{Z}^{\mathbb{M}} : \mathrm{Z}^{\mathbb{M}} \in \mathbb{C}, \left|\mathrm{Z}^{\mathbb{M}}\right| \leq 1\right\}$ is a degree of membership and $\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) : \text{Џ} \to \left\{\mathrm{Z}^{\mathbb{N}} : \mathrm{Z}^{\mathbb{N}} \in \mathbb{C}, \left|\mathrm{Z}^{\mathbb{N}}\right| \leq 1\right\}$ is a degree of non-membership such that $\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) = \mathrm{Z}^{\mathbb{M}} = \mathrm{Z}^{\mathbb{RM}} + \iota\mathrm{Z}^{\mathbb{IM}}$ and $\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) = \mathrm{Z}^{\mathbb{N}} = \mathrm{Z}^{\mathbb{RN}} + \iota\mathrm{Z}^{\mathbb{IN}}$ with the condition that $0 \leq \left|\mathrm{Z}^{\mathbb{M}}\right|^2 + \left|\mathrm{Z}^{\mathbb{N}}\right|^2 \leq 1$, or $\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) = \mathcal{G}^{\mathbb{MM}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\, e^{\iota 2\pi\left(\mathcal{G}^{\mathbb{PM}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)}$ and $\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) = \mathcal{G}^{\mathbb{MN}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\, e^{\iota 2\pi\left(\mathcal{G}^{\mathbb{PN}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)}$ with the condition that $0 \leq \left(\mathcal{G}^{\mathbb{MM}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2 + \left(\mathcal{G}^{\mathbb{MN}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2 \leq 1$ and $0 \leq \left(\mathcal{G}^{\mathbb{PM}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2 + \left(\mathcal{G}^{\mathbb{PN}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2 \leq 1$. Further, $\mathcal{G}^{H}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) = \mathcal{G}^{\mathbb{MH}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\, e^{\iota 2\pi\left(\mathcal{G}^{\mathbb{PH}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)}$ is identified as the degree of hesitancy, where $\mathcal{G}^{\mathbb{MH}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) = \left(1 - \left(\left|\mathrm{Z}^{\mathbb{M}}\right|^2 + \left|\mathrm{Z}^{\mathbb{N}}\right|^2\right)\right)^{\frac{1}{2}}$ and $\left(\mathcal{G}^{\mathbb{PH}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right) = \left(1 - \left(\left(\mathcal{G}^{\mathbb{PM}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2 + \left(\mathcal{G}^{\mathbb{PN}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2\right)\right)^{\frac{1}{2}}$.

In the theory of crisp set and FS, the element fully belongs to the set when its degree of membership is 1 and does not fully belong when its degree of membership is 0. Similarly, in the theory of PFS, the element fully belongs to the set when its degree of membership is 1 and non-membership is 0 and not fully belong when the degree of membership is 0 and non-membership is 1 and in the theory of CFS interpreted by Tamir et al. [23] the element fully belong to the set when its degree of membership is $1 + \iota 1$. From this, it is obvious that in the Cartesian form of CPFS, the degree of membership for an element to fully belong to a set should be $1 + \iota 1$ which is $\mathrm{Z}^{\mathbb{M}} = 1 + \iota 1$ and the degree of non-membership should be $0 + \iota 0$ that is $\mathrm{Z}^{\mathbb{N}} = 0 + \iota 0$. But in the concept of the Cartesian form of CPFS, initiated by Walters et al. [40], the degree of membership takes values in the unit disc of a complex plan that is $\left\{\mathrm{Z}^{\mathbb{M}} : \mathrm{Z}^{\mathbb{M}} \in \mathbb{C}, \left|\mathrm{Z}^{\mathbb{M}}\right| \leq 1\right\}$ and $\left|\mathrm{Z}^{\mathbb{M}}\right| = |1 + \iota 1| = \sqrt{1^2 + 1^2} = \sqrt{2} = 1.4142 > 1$. Thus according to the interpreted concept of Walters et al. [40], the degree of membership can't be $1+\iota 1$. This means that the element can't fully belong to the set in this Cartesian form which contradicts the basic concept of crisp and FS theory. Similarly, for not belonging the degree of membership should be $0 + \iota 0$ i.e. $\mathrm{Z}^{\mathbb{M}} = 0+\iota 0$ and non-membership should be $1+\iota 1$ i.e. $\mathrm{Z}^{\mathbb{N}} = 1 + \iota 1$. Again the degree of non-membership can't be $1 + \iota 1$.

Moreover, in various DM dilemmas, the expert can give a value $(0.9 + \iota 0.8, 0.2 + \iota 0.3)$ to an element according to their discretion. But the map for the degree of membership defined by Walters et al. [40] can't give this value because the magnitude of that value $\sqrt{(0.9)^2 + (0.8)^2} = 1.204 > 1$ and thus this value is not located in a complex plane's unit disc but located in a unit square of a complex plane. Additionally, many other values do not belong to the unit disc. To overcome above above-discussed drawbacks, in this script, a novel concept of CPFS in a Cartesian coordinate changes the range from the complex plane's unit disc to the unit square. The new interpretation is revealed as

Definition 2: An underneath model

$$\mathbb{Y}_{CPFS} = \left\{\left(\boldsymbol{\upsilon}, \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}), \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)\right) | \boldsymbol{\upsilon} \in \text{Џ}\right\} = \left\{\left(\boldsymbol{\upsilon}, \left(\begin{array}{l}\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}), \\ \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\end{array}\right)\right) | \boldsymbol{\upsilon} \in \text{Џ}\right\}$$

is established as CPFS in the Cartesian framework of complex numbers. Noticed that $\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})$ and $\mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})$ are degrees of membership and non-membership which are located in a complex plane's unit square. Moreover, $\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})$, $\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})$, $\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})$ and $\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) \in [0, 1]$ with the property that $0 \leq \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2 + \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2 \leq 1$ and $0 \leq \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2 + \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2 \leq 1$. The indeterminacy will be deduced by $\mathcal{G}^{\mathbb{D}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) = \left(\mathcal{G}^{\mathbb{RD}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) + \iota\mathcal{G}^{\mathbb{ID}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)$, where $\mathcal{G}^{\mathbb{RD}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) = \left(1 - \left(\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2 + \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2\right)\right)^{\frac{1}{2}}$ and $\mathcal{G}^{\mathbb{ID}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon}) = \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2 + \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS}}(\boldsymbol{\upsilon})\right)^2\right)^{\frac{1}{2}}$. For easiness, in this script, the complex Pythagorean fuzzy (CPF) number (CPFN) would anticipated as $\mathbb{Y}_{CPFS} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS}}\right) = \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}} + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS}}\right)$.

Definition 3: Suppose

$$\mathbb{Y}_{PFS-1} = \left\{\left(\boldsymbol{\upsilon}, \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS-1}}(\boldsymbol{\upsilon}), \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS-1}}(\boldsymbol{\upsilon})\right)\right) | \boldsymbol{\upsilon} \in \text{Џ}\right\}$$

$$= \left\{ \left(\upsilon, \begin{pmatrix} \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}(\upsilon) + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}(\upsilon), \\ \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}(\upsilon) + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}(\upsilon) \end{pmatrix} \right) | \upsilon \in \text{Џ} \right\}$$

and

$$\begin{aligned} &\mathbb{Y}_{PFS-2} \\ &= \left\{ \left(\upsilon, \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS-2}}(\upsilon), \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS-2}}(\upsilon) \right) \right) | \upsilon \in \text{Џ} \right\} \\ &= \left\{ \left(\upsilon, \begin{pmatrix} \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}(\upsilon) + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}(\upsilon), \\ \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}}(\upsilon) + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}}(\upsilon) \end{pmatrix} \right) | \upsilon \in \text{Џ} \right\} \end{aligned}$$

are two CPFS, then

1. $$(\mathbb{Y}_{CPFS-1})^c = \left\{ \left(\upsilon, \begin{pmatrix} \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS-1}}(\upsilon), \\ \mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS-1}}(\upsilon) \end{pmatrix} \right) | \upsilon \in \text{Џ} \right\}$$
2. $$\mathbb{Y}_{CPFS-1} \cup \mathbb{Y}_{CPFS-2} = \left\{ \left(\upsilon, \begin{pmatrix} \max\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}(\upsilon), \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}(\upsilon) \right) \\ +\iota \max\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}(\upsilon), \mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}(\upsilon) \right), \\ \min\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}(\upsilon), \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}}(\upsilon) \right) \\ +\iota \min\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}(\upsilon), \mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}}(\upsilon) \right) \end{pmatrix} \right) | \upsilon \in \text{Џ} \right\}$$
3. $$\mathbb{Y}_{CPFS-1} \cap \mathbb{Y}_{CPFS-2} = \left\{ \left(\upsilon, \begin{pmatrix} \min\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}(\upsilon), \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}(\upsilon) \right) \\ +\iota \min\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}(\upsilon), \mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}(\upsilon) \right), \\ \max\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}(\upsilon), \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}}(\upsilon) \right) \\ +\iota \max\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}(\upsilon), \mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}}(\upsilon) \right) \end{pmatrix} \right) | \upsilon \in \text{Џ} \right\}$$

Definition 4: Suppose $\mathbb{Y}_{CPFS} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS}} \right) = \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}} + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS}} \right)$ is a CPFN, then the score value is expressed as below

$$\check{S}(\mathbb{Y}_{CPFS}) = \frac{1}{2} \begin{pmatrix} \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}} \right)^2 - \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}} \right)^2 \\ + \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}} \right)^2 - \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS}} \right)^2 \end{pmatrix}, \quad \check{S}(\mathbb{Y}_{CPFS}) \in [-1, 1]$$

and the accuracy value would be analyzed as

$$\text{Ƕ}(\mathbb{Y}_{CPFS}) = \frac{1}{2} \begin{pmatrix} \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}} \right)^2 + \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}} \right)^2 \\ + \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}} \right)^2 + \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS}} \right)^2 \end{pmatrix}, \quad \text{Ƕ}(\mathbb{Y}_{CPFS}) \in [0, 1]$$

Based on Def (4), we have the underlying result.

Definition 5: Suppose $\mathbb{Y}_{CPFS-1} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS-1}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS-1}} \right) = \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}} + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}} \right)$, and $\mathbb{Y}_{CPFS-2} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS-2}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS-2}} \right) = \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}} + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}} \right)$ are two CPFNs, then

1. If $\check{S}(\mathbb{Y}_{CPFS-1}) < \check{S}(\mathbb{Y}_{CPFS-2})$ then $\mathbb{Y}_{CPFS-1} < \mathbb{Y}_{CPFS-2}$
2. If $\check{S}(\mathbb{Y}_{CPFS-1}) > \check{S}(\mathbb{Y}_{CPFS-2})$ then $\mathbb{Y}_{CPFS-1} > \mathbb{Y}_{CPFS-2}$
3. If $\check{S}(\mathbb{Y}_{CPFS-1}) = \check{S}(\mathbb{Y}_{CPFS-2})$ then we have
   i. If $\text{Ƕ}(\mathbb{Y}_{CPFS-1}) < \text{Ƕ}(\mathbb{Y}_{CPFS-2})$ then $\mathbb{Y}_{CPFS-1} < \mathbb{Y}_{CPFS-2}$
   ii. If $\text{Ƕ}(\mathbb{Y}_{CPFS-1}) > \text{Ƕ}(\mathbb{Y}_{CPFS-2})$ then $\mathbb{Y}_{CPFS-1} > \mathbb{Y}_{CPFS-2}$
   iii. If $\text{Ƕ}(\mathbb{Y}_{CPFS-1}) = \text{Ƕ}(\mathbb{Y}_{CPFS-2})$ then $\mathbb{Y}_{CPFS-1} = \mathbb{Y}_{CPFS-2}$

Definition 6: Suppose $\mathbb{Y}_{CPFS-1} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS-1}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS-1}} \right) = \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}} + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}} \right)$, and $\mathbb{Y}_{CPFS-2} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS-2}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS-2}} \right) = \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}} + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}} \right)$ are two CPFNs and $\varkappa \geq 0$. Then

1. $$\mathbb{Y}_{CPFS-1} \oplus \mathbb{Y}_{CPFS-2} = \begin{pmatrix} \begin{pmatrix} \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}} \right)^2 + \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}} \right)^2 \\ - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}} \right)^2 \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}} \right)^2 \end{pmatrix}^{\frac{1}{2}} \\ +\iota \begin{pmatrix} \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}} \right)^2 + \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}} \right)^2 \\ - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}} \right)^2 \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}} \right)^2 \end{pmatrix}^{\frac{1}{2}}, \\ \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}} \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}} + \iota \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}} \mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}} \right) \end{pmatrix}$$
2. $$\mathbb{Y}_{CPFS-1} \otimes \mathbb{Y}_{CPFS-2} = \begin{pmatrix} \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}} \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}} \mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}, \\ \begin{pmatrix} \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}} \right)^2 + \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}} \right)^2 \\ - \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}} \right)^2 \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}} \right)^2 \end{pmatrix}^{\frac{1}{2}} \\ \iota \begin{pmatrix} \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}} \right)^2 + \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}} \right)^2 \\ - \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}} \right)^2 \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}} \right)^2 \end{pmatrix}^{\frac{1}{2}} \end{pmatrix}$$
3. $$\varkappa\mathbb{Y}_{CPFS-1} = \begin{pmatrix} \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}} \right)^2 \right)^{\varkappa} \right)^{\frac{1}{2}} \\ +\iota \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}} \right)^2 \right)^{\varkappa} \right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}} \right)^{\varkappa} + \iota \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}} \right)^{\varkappa} \end{pmatrix}$$
4. $$\mathbb{Y}^{\varkappa}_{CPFS-1} = \begin{pmatrix} \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}} \right)^{\varkappa} + \iota \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}} \right)^{\varkappa}, \\ \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}} \right)^2 \right)^{\varkappa} \right)^{\frac{1}{2}} \\ +\iota \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}} \right)^2 \right)^{\varkappa} \right)^{\frac{1}{2}} \end{pmatrix}$$

Theorem 1: Suppose $\mathbb{Y}_{CPFS-1} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS-1}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS-1}} \right) = \begin{pmatrix} \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}, \\ \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}} + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}} \end{pmatrix}$, and $\mathbb{Y}_{CPFS-2}$

$= \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS-2}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS-2}}\right) = \begin{pmatrix} \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}, \\ \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}} + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}} \end{pmatrix}$ are two CPFNs and $\varkappa, \varkappa_1, \varkappa_2 \geq 0$, then the underneath hold.

1. $\mathbb{Y}_{CPFS-1} \oplus \mathbb{Y}_{CPFS-2} = \mathbb{Y}_{CPFS-2} \oplus \mathbb{Y}_{CPFS-1}$
2. $\mathbb{Y}_{CPFS-1} \otimes \mathbb{Y}_{CPFS-2} = \mathbb{Y}_{CPFS-2} \otimes \mathbb{Y}_{CPFS-1}$
3. $\varkappa\left(\mathbb{Y}_{CPFS-1} \oplus \mathbb{Y}_{CPFS-2}\right) = \varkappa\mathbb{Y}_{CPFS-1} \oplus \varkappa\mathbb{Y}_{CPFS-2}$
4. $\left(\mathbb{Y}_{CPFS-1} \otimes \mathbb{Y}_{CPFS-2}\right)^{\varkappa} = \mathbb{Y}^{\varkappa}_{CPFS-1} \otimes \mathbb{Y}^{\varkappa}_{CPFS-2}$
5. $\varkappa_1\mathbb{Y}_{CPFS-1} \oplus \varkappa_2\mathbb{Y}_{CPFS-1} = \left(\varkappa_1 + \varkappa_2\right)\mathbb{Y}_{CPFS-1}$
6. $\mathbb{Y}^{\varkappa_1}_{CPFS-1} \otimes \mathbb{Y}^{\varkappa_2}_{CPFS-1} = \mathbb{Y}^{\varkappa_1+\varkappa_2}_{CPFS-1}$
7. $\left(\mathbb{Y}^{\varkappa_1}_{CPFS-1}\right)^{\varkappa_2} = \mathbb{Y}^{\varkappa_1\varkappa_2}_{CPFS-1}$.

Proof:

1. From Def (6)

$$\mathbb{Y}_{CPFS-1} \oplus \mathbb{Y}_{CPFS-2}$$
$$= \begin{pmatrix} \begin{pmatrix} \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2 + \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}\right)^2 \\ -\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}\right)^2 \end{pmatrix}^{\frac{1}{2}} \\ +\iota\begin{pmatrix} \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2 + \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}\right)^2 \\ -\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}\right)^2 \end{pmatrix}^{\frac{1}{2}}, \\ \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}}\right) \end{pmatrix}$$
$$= \begin{pmatrix} \begin{pmatrix} \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}\right)^2 + \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2 \\ -\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2 \end{pmatrix}^{\frac{1}{2}} \\ +\iota\begin{pmatrix} \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}\right)^2 + \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2 \\ -\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2 \end{pmatrix}^{\frac{1}{2}}, \\ \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}}\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}}\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right) \end{pmatrix}$$
$$= \mathbb{Y}_{CPFS-2} \oplus \mathbb{Y}_{CPFS-1}$$

2. Similarly, 2 can prove.
3. By Def (6)

$$\varkappa\mathbb{Y}_{CPFS-1} = \begin{pmatrix} \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa}\right)^{\frac{1}{2}} \\ +\iota\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa}\right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa} \end{pmatrix}$$

$$\varkappa\mathbb{Y}_{CPFS-2} = \begin{pmatrix} \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\varkappa}\right)^{\frac{1}{2}} \\ +\iota\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\varkappa}\right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\varkappa} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\varkappa} \end{pmatrix}^{\varkappa}$$

Now take the right side, as shown at the bottom of the next page.

1. Similar to 3.
2. Since

$$\varkappa_1\mathbb{Y}_{CPFS-1} = \begin{pmatrix} \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1}\right)^{\frac{1}{2}} \\ +\iota\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1}\right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1} \end{pmatrix}$$

$$\varkappa_2\mathbb{Y}_{CPFS-1} = \begin{pmatrix} \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_2}\right)^{\frac{1}{2}} \\ +\iota\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_2}\right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_2} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_2} \end{pmatrix}$$

Let, as shown at the bottom of page 9.

3. Similar to 5.
4. By Def (6)

$$\mathbb{Y}^{\varkappa_1}_{CPFS-1}$$
$$= \begin{pmatrix} \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1} + \iota\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1}, \\ \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1}\right)^{\frac{1}{2}} \\ +\iota\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1}\right)^{\frac{1}{2}} \end{pmatrix}$$

$$\left(\mathbb{Y}^{\varkappa_1}_{CPFS-1}\right)^{\varkappa_2}$$
$$= \begin{pmatrix} \left(\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1}\right)^{\varkappa_2} + \iota\left(\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1}\right)^{\varkappa_2} \\ \left(1 - \left(\left(1 - \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1}\right)^{\varkappa_2}\right)^{\frac{1}{2}} \\ +\iota\left(1 - \left(\left(1 - \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1}\right)^{\varkappa_2}\right)^{\frac{1}{2}} \end{pmatrix}$$
$$= \begin{pmatrix} \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^{\frac{1}{2}} + \iota\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^{\frac{1}{2}} \\ \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1\varkappa_2}\right)^{\frac{1}{2}} \\ +\iota\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1\varkappa_2}\right)^{\frac{1}{2}} \end{pmatrix}$$
$$= \mathbb{Y}^{\varkappa_1\varkappa_2}_{CPFS-1}.$$

IV. COMPLEX PYTHAGOREAN FUZZY AOS

This part of the article contains averaging and geometric AOs in the Cartesian framework of CPFS along with connected properties. The collection of CPFS would be devised as $\mathbb{Y}_{CPFS-\dot{f}} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS-\dot{f}}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right) = \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{f}}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}} + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right), \dot{f} = 1, 2, \ldots, \varrho$ in this article.

Definition 7: Suppose $CPFWA : \left(\mathbb{Y}_{CPFS-\dot{f}}\right)^{\varrho} \to \mathbb{Y}_{CPFS-\dot{f}}$, if

$$CPFWA\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots, \mathbb{Y}_{CPFS-\varrho}\right) = \bigoplus_{\dot{f}=1}^{\varrho} \Xi_{w-\dot{f}}\mathbb{Y}_{CPFS-\dot{f}}$$

then CPFWA is deduced as CPFWA operator over $\mathbb{Y}_{CPFS-\dot{f}}$. Noted that $\Xi_w = \left(\Xi_{w-1}, \Xi_{w-2}, \ldots, \Xi_{w-\varrho}\right)$ is a

$$\begin{aligned}
\varkappa\mathbb{Y}_{CPFS-1}\varkappa \oplus \mathbb{Y}_{CPFS-2} &= \begin{pmatrix} \left(1-\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa}\right)^{\frac{1}{2}} \\ +\iota\left(1-\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa}\right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa} \end{pmatrix} \\
&\oplus \begin{pmatrix} \left(1-\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\varkappa}\right)^{\frac{1}{2}} \\ +\iota\left(1-\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\varkappa}\right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\varkappa} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\varkappa} \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} 1-\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa} + 1-\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\varkappa} \\ -\left(1-\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)\right)^{\varkappa}\left(1-\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\varkappa}\right) \end{pmatrix}^{\frac{1}{2}} \\ +\iota\begin{pmatrix} 1-\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa} + 1-\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\varkappa} \\ -\left(1-\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)\right)^{\varkappa}\left(1-\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\varkappa}\right) \end{pmatrix}^{\frac{1}{2}} \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa}\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\varkappa} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa}\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\varkappa} \end{pmatrix} \\
&= \begin{pmatrix} \left(1-\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa}\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\varkappa}\right)^{\frac{1}{2}} \\ +\iota\left(1-\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa}\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\varkappa}\right)^{\frac{1}{2}} \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\varkappa} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\varkappa} \end{pmatrix} \\
&= \begin{pmatrix} \left(1-\left(1-\begin{pmatrix} \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2 + \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}\right)^2 \\ -\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}\right)^2 \end{pmatrix}\right)\right)^{\varkappa\frac{1}{2}} \\ +\iota\left(1-\left(1-\begin{pmatrix} \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2 + \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}\right)^2 \\ -\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}\right)^2 \end{pmatrix}\right)\right)^{\varkappa\frac{1}{2}} \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\varkappa} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\varkappa} \end{pmatrix} \\
&= \varkappa\left(\mathbb{Y}_{CPFS-1} \oplus \mathbb{Y}_{CPFS-2}\right)
\end{aligned}$$

weight vector with $0 \leq \Xi_{w-\dot{f}} \leq 1$, $\sum_{\dot{f}=1}^{\varrho} \Xi_{w-\dot{f}} = 1$,

Theorem 2: The aggregated result over $\mathbb{Y}_{CPFS-\dot{f}}$ by utilizing CPFWA operator will be analyzed as CPFN and

$$CPFWA\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\varrho}\right) = \begin{pmatrix} \left(1 - \prod_{\dot{f}=1}^{\varrho}\left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}} \\ +\iota\left(1 - \prod_{\dot{f}=1}^{\varrho}\left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}, \\ \prod_{\dot{f}=1}^{\varrho}\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}} \\ +\iota\prod_{\dot{f}=1}^{\varrho}\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}} \end{pmatrix} \quad (1)$$

$$\begin{aligned}
&\varkappa_1\mathbb{Y}_{CPFS-1} \oplus \varkappa_2\mathbb{Y}_{CPFS-1} \\
&= \begin{pmatrix} \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1}\right)^{\frac{1}{2}} \\ +\iota\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1}\right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1} \end{pmatrix} \\
&\oplus \begin{pmatrix} \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_2}\right)^{\frac{1}{2}} \\ +\iota\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_2}\right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_2} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_2} \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} 1 - \left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1} + 1 - \left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_2} \\ -\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1}\right)\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_2}\right) \end{pmatrix}^{\frac{1}{2}} \\ +\iota\begin{pmatrix} 1 - \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1} + 1 - \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_2} \\ -\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1}\right)\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_2}\right) \end{pmatrix}^{\frac{1}{2}} \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1}\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_2} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1}\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_2} \end{pmatrix} \\
&= \begin{pmatrix} \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1}\left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_2}\right)^{\frac{1}{2}} \\ +\iota\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1}\left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_2}\right)^{\frac{1}{2}} \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1+\varkappa_2} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1+\varkappa_2} \end{pmatrix} \\
&= \begin{pmatrix} \left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1+\varkappa_2}\right)^{\frac{1}{2}} \\ +\iota\left(1 - \left(1 - \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\varkappa_1+\varkappa_2}\right)^{\frac{1}{2}} \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1+\varkappa_2} + \iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\varkappa_1+\varkappa_2} \end{pmatrix} \\
&= \left(\varkappa_1 + \varkappa_2\right)\mathbb{Y}_{CPFS-1}
\end{aligned}$$

Proof: Let $\varrho = 2$. Then we have

$$\Xi_{w-1}\mathbb{Y}_{CPFS-1} = \begin{pmatrix} \left(1-\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\Xi_{w-1}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\Xi_{w-1}}\right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\Xi_{w-1}} \\ +\iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\Xi_{w-1}} \end{pmatrix}$$

$$\Xi_{w-2}\mathbb{Y}_{CPFS-2} = \begin{pmatrix} \left(1-\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\Xi_{w-2}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\Xi_{w-2}}\right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\Xi_{w-2}} \\ +\iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\Xi_{w-2}} \end{pmatrix}$$

and

$$\begin{aligned} &CPFWA\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}\right) \\ &= \Xi_{w-1}\mathbb{Y}_{CPFS-1} \oplus \Xi_{w-2}\mathbb{Y}_{CPFS-2} \\ &= \begin{pmatrix} \left(1-\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\Xi_{w-1}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\Xi_{w-1}}\right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\Xi_{w-1}} \\ +\iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^{\Xi_{w-1}} \end{pmatrix} \\ &\oplus \begin{pmatrix} \left(1-\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\Xi_{w-2}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-2}}\right)^2\right)^{\Xi_{w-2}}\right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\Xi_{w-2}} \\ +\iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-2}}\right)^{\Xi_{w-2}} \end{pmatrix} \\ &= \begin{pmatrix} \left(1-\prod_{\dot{f}=1}^{2}\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\prod_{\dot{f}=1}^{2}\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}, \\ \prod_{\dot{f}=1}^{2}\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}} \\ +\iota\prod_{\dot{f}=1}^{2}\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}} \end{pmatrix} \end{aligned}$$

Thus, for $\varrho = 2$, Eq. (1) is satisfied. Let Eq. (1) is valid for $\varrho = \text{ß}$, that is

$$CPFWA\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots, \mathbb{Y}_{CPFS-\text{ß}}\right) = \begin{pmatrix} \left(1-\prod_{\dot{f}=1}^{\text{ß}}\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\prod_{\dot{f}=1}^{\text{ß}}\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}, \\ \prod_{\dot{f}=1}^{\text{ß}}\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}} \\ +\iota\prod_{\dot{f}=1}^{\text{ß}}\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}} \end{pmatrix}$$

Now to reveal that for $\varrho = \text{ß}+1$, Eq. (1) is valid. Since

$$\begin{aligned} &CPFWA\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\text{ß}+1}\right) \\ &= CPFWA\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\text{ß}}\right) \oplus \mathbb{Y}_{CPFS-\text{ß}+1} \\ &= \begin{pmatrix} \left(1-\prod_{\dot{f}=1}^{\text{ß}}\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\prod_{\dot{f}=1}^{\text{ß}}\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}, \\ \prod_{\dot{f}=1}^{\text{ß}}\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}} \\ +\iota\prod_{\dot{f}=1}^{\text{ß}}\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}} \end{pmatrix} \\ &\oplus \begin{pmatrix} \left(1-\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\text{ß}+1}}\right)^2\right)^{\Xi_{w-\text{ß}+1}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\text{ß}+1}}\right)^2\right)^{\Xi_{w-\text{ß}+1}}\right)^{\frac{1}{2}}, \\ \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\text{ß}+1}}\right)^{\Xi_{w-\text{ß}+1}} \\ +\iota\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\text{ß}+1}}\right)^{\Xi_{w-\text{ß}+1}} \end{pmatrix} \\ &= \begin{pmatrix} \left(1-\prod_{\dot{f}=1}^{\text{ß}+1}\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\prod_{\dot{f}=1}^{\text{ß}+1}\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}, \\ \prod_{\dot{f}=1}^{\text{ß}+1}\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}} \\ +\iota\prod_{\dot{f}=1}^{\text{ß}+1}\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}} \end{pmatrix} \\ &= CPFWA\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\text{ß}}, \mathbb{Y}_{CPFS-\text{ß}+1}\right) \end{aligned}$$

This reveals that for $\varrho = \text{ß}+1$, Eq. (1) is valid and hence for all ϱ.

Next, we have a connected property that is idempotency.

Proposition 1: If $\mathbb{Y}_{CPFS-\dot{f}} = \mathbb{Y}_{CPFS}$ i.e., $\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}} = \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}}$, $\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{f}}} = \mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}}$, $\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}} = \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}}$ and

$\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{f}}} = \mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS}}$, $\forall\dot{f}$, then

$$CPFWA\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\varrho}\right) = \mathbb{Y}_{CPFS-\dot{f}}$$

Proof: As $\mathbb{Y}_{CPFS-\dot{f}} = \mathbb{Y}_{CPFS}\forall\dot{f}$ that is $\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}} = \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}}$, $\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{f}}} = \mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}}$, $\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}} = \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}}$ and $\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{f}}} = \mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS}}$, then

$$\begin{aligned}
&CPFWA\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\varrho}\right)\\
&= \left(\begin{array}{l}
\left(1-\prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}\\
+\iota\left(1-\prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}},\\
\prod_{\dot{f}=1}^{\varrho}\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}}\right)^{\Xi_{w-\dot{f}}}\\
+\iota\prod_{\dot{f}=1}^{\varrho}\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS}}\right)^{\Xi_{w-\dot{f}}}
\end{array}\right)\\
&= \left(\begin{array}{l}
\left(1-\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}}\right)^2\right)^{\sum_{\dot{f}=1}^{\varrho}\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}\\
+\iota\left(1-\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}}\right)^2\right)^{\sum_{\dot{f}=1}^{\varrho}\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}},\\
\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}}\right)^{\sum_{\dot{f}=1}^{\varrho}\Xi_{w-\dot{f}}}\\
+\iota\left(GI^{\mathbb{N}}_{\mathbb{Y}_{CPFS}}\right)^{\sum_{\dot{f}=1}^{\varrho}\Xi_{w-\dot{f}}}
\end{array}\right)\\
&= \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}}+\iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}}+\iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS}}\right)\\
&= \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS}}\right) = \mathbb{Y}_{CPFS}
\end{aligned}$$

The following property of the CPFWA operator is monotonicity.

Proposition 2: Take two gatherings of CPFNs $\mathbb{Y}_{CPFS-\dot{f}} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS-\dot{f}}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right) = \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}}+\iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{f}}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}} +\iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)$, and $\mathbb{Y}^{\#}_{CPFS-\dot{f}} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\right) = \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}+\iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}} \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\right)$, $\dot{f} = 1, 2, \ldots, \varrho$. If $\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}} \leq \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}$, $\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{f}}} \leq \mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}$ $\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}} \geq \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}$, and $\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{f}}} \geq \mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}$ $\forall\dot{f}$, then

$$\begin{aligned}
&CPFWA\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\varrho}\right)\\
&\qquad\leq CPFWA\left(\mathbb{Y}^{\#}_{CPFS-1}, \mathbb{Y}^{\#}_{CPFS-2}, \ldots \mathbb{Y}^{\#}_{CPFS-\varrho}\right)
\end{aligned}$$

Proof: As we have that

$$\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}} \leq \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}} \Rightarrow \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2 \leq \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\right)^2$$

$$\begin{aligned}
1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2 &\geq 1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\right)^2\\
&\Rightarrow \left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\\
&\geq \left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\\
&\Rightarrow \prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\\
&\geq \prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\\
&\Rightarrow 1-\prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\\
&\leq 1-\prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\\
&\Rightarrow \left(1-\prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}\\
&\leq \left(1-\prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\left(1-\prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}\\
&\qquad\leq \left(1-\prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\right)^2\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}
\end{aligned}$$

Now, since,

$$\begin{aligned}
\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}} &\geq \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\\
&\Rightarrow \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right) \geq \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\right)\\
&\Rightarrow \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}} \geq \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}}\\
&\Rightarrow \prod_{\dot{f}=1}^{\varrho}\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}} \geq \prod_{\dot{f}=1}^{\varrho}\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}}
\end{aligned}$$

Similarly, we have

$$\prod_{\dot{f}=1}^{\varrho}\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}} \geq \prod_{\dot{f}=1}^{\varrho}\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}^{\#}_{CPFS-\dot{f}}}\right)^{\Xi_{w-\dot{f}}}$$

Consequently,

$$\begin{aligned}
&CPFWA\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\varrho}\right)\\
&\qquad\leq CPFWA\left(\mathbb{Y}^{\#}_{CPFS-1}, \mathbb{Y}^{\#}_{CPFS-2}, \ldots \mathbb{Y}^{\#}_{CPFS-\varrho}\right)
\end{aligned}$$

The following property of the CPFWA operator is Boundedness.

Proposition 3: If

$$\mathbb{Y}_{CPFS}^{-}=\begin{pmatrix}\min\limits_{\dot{f}}\left\{\mathcal{G}_{\mathbb{Y}_{CPFS-\dot{f}}}^{\mathbb{RM}}\right\}+\iota\max\limits_{\dot{f}}\left\{\mathcal{G}_{\mathbb{Y}_{CPFS-\dot{f}}}^{\mathbb{IM}}\right\},\\ \max\limits_{\dot{f}}\left\{\mathcal{G}_{\mathbb{Y}_{CPFS-\dot{f}}}^{\mathbb{RN}}\right\}+\iota\min\limits_{\dot{f}}\left\{\mathcal{G}_{\mathbb{Y}_{CPFS-\dot{f}}}^{\mathbb{IN}}\right\}\end{pmatrix}$$

and

$$\mathbb{Y}_{CPFS}^{+}=\begin{pmatrix}\max\limits_{\dot{f}}\left\{\mathcal{G}_{\mathbb{Y}_{CPFS-\dot{f}}}^{\mathbb{RM}}\right\}+\iota\min\limits_{\dot{f}}\left\{\mathcal{G}_{\mathbb{Y}_{CPFS-\dot{f}}}^{\mathbb{IM}}\right\},\\ \min\limits_{\dot{f}}\left\{\mathcal{G}_{\mathbb{Y}_{CPFS-\dot{f}}}^{\mathbb{RN}}\right\}+\iota\max\limits_{\dot{f}}\left\{\mathcal{G}_{\mathbb{Y}_{CPFS-\dot{f}}}^{\mathbb{IN}}\right\}\end{pmatrix},$$

then we have

$$\mathbb{Y}_{CPFS}^{-}\leq CPFWA\left(\mathbb{Y}_{CPFS-1},\mathbb{Y}_{CPFS-2},\ldots\mathbb{Y}_{CPFS-\varrho}\right)\leq\mathbb{Y}_{CPFS}^{+}$$

Proof: This can be proved by employing propositions (1) and (2).

Definition 8: Suppose $CPFWA:\left(\mathbb{Y}_{CPFS-\dot{f}}\right)^{\varrho}\rightarrow\mathbb{Y}_{CPFS-\dot{f}}$, if

$$CPFOWA\left(\mathbb{Y}_{CPFS-1},\mathbb{Y}_{CPFS-2},\ldots,\mathbb{Y}_{CPFS-\varrho}\right)=\bigoplus_{\dot{f}=1}^{\varrho}\Xi_{w-\dot{f}}\mathbb{Y}_{CPFS-\Delta(\dot{f})}$$

then CPFOWA is deduced as CPFOWA operator over $\mathbb{Y}_{CPFS-\dot{f}}$. Noted that $\Xi_{w}=\left(\Xi_{w-1},\Xi_{w-2},\ldots,\Xi_{w-\varrho}\right)$ is a weight vector with $0\leq\Xi_{w-\dot{f}}\leq1$, $\sum_{\dot{f}=1}^{\varrho}\Xi_{w-\dot{f}}=1$, and $(\Delta(1),\Delta(2),\ldots,\Delta(\varrho))$ is a permutation of $(1,2,\ldots,\varrho)$ such that $\Delta\left(\dot{f}-1\right)\geq\Delta\left(\dot{f}\right)$, for $\dot{f}=2,3,\ldots,\varrho$.

Theorem 3: The aggregated result over $\mathbb{Y}_{CPFS-\dot{f}}$ by utilizing CPFOWA operator will be analyzed as CPFN and

$$CPFOWA\left(\mathbb{Y}_{CPFS-1},\mathbb{Y}_{CPFS-2},\ldots\mathbb{Y}_{CPFS-\varrho}\right)=\begin{pmatrix}\left(1-\prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}_{\mathbb{Y}_{CPFS-\Delta(\dot{f})}}^{\mathbb{RM}}\right)^{2}\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}\\ +\iota\left(1-\prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}_{\mathbb{Y}_{CPFS-\Delta(\dot{f})}}^{\mathbb{IM}}\right)^{2}\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}},\\ \prod_{\dot{f}=1}^{\varrho}\left(\mathcal{G}_{\mathbb{Y}_{CPFS-\Delta(\dot{f})}}^{\mathbb{RN}}\right)^{\Xi_{w-\dot{f}}}\\ +\iota\prod_{\dot{f}=1}^{\varrho}\left(\mathcal{G}_{\mathbb{Y}_{CPFS-\Delta(\dot{f})}}^{\mathbb{IN}}\right)^{\Xi_{w-\dot{f}}}\end{pmatrix}$$

Next, we deduce the CPFWG operator.

Definition 9: Suppose $CPFWG:\left(\mathbb{Y}_{CPFS-\dot{f}}\right)^{\varrho}\rightarrow\mathbb{Y}_{CPFS-\dot{f}}$, if

$$CPFWG\left(\mathbb{Y}_{CPFS-1},\mathbb{Y}_{CPFS-2},\ldots,\mathbb{Y}_{CPFS-\varrho}\right)=\bigotimes_{\dot{f}=1}^{\varrho}\left(\mathbb{Y}_{CPFS-\dot{f}}\right)^{\Xi_{w-\dot{f}}}$$

then CPFWG is deduced as CPFWG operator over $\mathbb{Y}_{CPFS-\dot{f}}$. Noted that $\Xi_{w}=\left(\Xi_{w-1},\Xi_{w-2},\ldots,\Xi_{w-\varrho}\right)$ is a weight vector with $0\leq\Xi_{w-\dot{f}}\leq1$, and $\sum_{\dot{f}=1}^{\varrho}\Xi_{w-\dot{f}}=1$.

Theorem 4: The aggregated result over $\mathbb{Y}_{CPFS-\dot{f}}$ by utilizing CPFWG operator will be analyzed as CPFN and

$$CPFWG\left(\mathbb{Y}_{CPFS-1},\mathbb{Y}_{CPFS-2},\ldots\mathbb{Y}_{CPFS-\varrho}\right)=\begin{pmatrix}\prod_{\dot{f}=1}^{\varrho}\left(\mathcal{G}_{\mathbb{Y}_{CPFS-\dot{f}}}^{\mathbb{RM}}\right)^{\Xi_{w-\dot{f}}}\\ +\iota\prod_{\dot{f}=1}^{\varrho}\left(\mathcal{G}_{\mathbb{Y}_{CPFS-\dot{f}}}^{\mathbb{IM}}\right)^{\Xi_{w-\dot{f}}},\\ \left(1-\prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}_{\mathbb{Y}_{CPFS-\dot{f}}}^{\mathbb{RN}}\right)^{2}\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}\\ +\iota\left(1-\prod_{\dot{f}=1}^{\varrho}\left(1-\left(\mathcal{G}_{\mathbb{Y}_{CPFS-\dot{f}}}^{\mathbb{IN}}\right)^{2}\right)^{\Xi_{w-\dot{f}}}\right)^{\frac{1}{2}}\end{pmatrix}\quad(2)$$

Proof: Let $\varrho=2$. Then we have

$$\left(\mathbb{Y}_{CPFS-1}\right)^{\Xi_{w-1}}=\begin{pmatrix}\left(\mathcal{G}_{\mathbb{Y}_{CPFS-1}}^{\mathbb{RM}}\right)^{\Xi_{w-1}}\\ +\iota\left(\mathcal{G}_{\mathbb{Y}_{CPFS-1}}^{\mathbb{IM}}\right)^{\Xi_{w-1}},\\ \left(1-\left(1-\left(\mathcal{G}_{\mathbb{Y}_{CPFS-1}}^{\mathbb{RN}}\right)^{2}\right)^{\Xi_{w-1}}\right)^{\frac{1}{2}}\\ +\iota\left(1-\left(1-\left(\mathcal{G}_{\mathbb{Y}_{CPFS-1}}^{\mathbb{IN}}\right)^{2}\right)^{\Xi_{w-1}}\right)^{\frac{1}{2}}\end{pmatrix}$$

$$\left(\mathbb{Y}_{CPFS-2}\right)^{\Xi_{w-2}}=\begin{pmatrix}\left(\mathcal{G}_{\mathbb{Y}_{CPFS-1}}^{\mathbb{RM}}\right)^{\Xi_{w-2}}\\ +\iota\left(\mathcal{G}_{\mathbb{Y}_{CPFS-1}}^{\mathbb{IM}}\right)^{\Xi_{w-2}},\\ \left(1-\left(1-\left(\mathcal{G}_{\mathbb{Y}_{CPFS-1}}^{\mathbb{RN}}\right)^{2}\right)^{\Xi_{w-2}}\right)^{\frac{1}{2}}\\ +\iota\left(1-\left(1-\left(\mathcal{G}_{\mathbb{Y}_{CPFS-1}}^{\mathbb{IN}}\right)^{2}\right)^{\Xi_{w-2}}\right)^{\frac{1}{2}}\end{pmatrix}$$

and,

$$CPFWG\left(\mathbb{Y}_{CPFS-1},\mathbb{Y}_{CPFS-2}\right)=\left(\mathbb{Y}_{CPFS-1}\right)^{\Xi_{w-1}}\otimes\left(\mathbb{Y}_{CPFS-2}\right)^{\Xi_{w-2}}=\begin{pmatrix}\left(\mathcal{G}_{\mathbb{Y}_{CPFS-1}}^{\mathbb{RM}}\right)^{\Xi_{w-1}}\\ +\iota\left(\mathcal{G}_{\mathbb{Y}_{CPFS-1}}^{\mathbb{IM}}\right)^{\Xi_{w-1}},\\ \left(1-\left(1-\left(\mathcal{G}_{\mathbb{Y}_{CPFS-1}}^{\mathbb{RN}}\right)^{2}\right)^{\Xi_{w-1}}\right)^{\frac{1}{2}}\\ +\iota\left(1-\left(1-\left(\mathcal{G}_{\mathbb{Y}_{CPFS-1}}^{\mathbb{IN}}\right)^{2}\right)^{\Xi_{w-1}}\right)^{\frac{1}{2}}\end{pmatrix}$$

$$\otimes\begin{pmatrix} \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-1}}\right)^{\Xi_{\mathscr{w}-2}} \\ +\iota\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-1}}\right)^{\Xi_{\mathscr{w}-2}}, \\ \left(1-\left(1-\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\Xi_{\mathscr{w}-2}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\left(1-\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-1}}\right)^2\right)^{\Xi_{\mathscr{w}-2}}\right)^{\frac{1}{2}}\end{pmatrix}$$

$$=\begin{pmatrix} \prod_{\dot{\mathrm{f}}=1}^{2}\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}} \\ +\iota\prod_{\dot{\mathrm{f}}=1}^{2}\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}}, \\ \left(1-\prod_{\dot{\mathrm{f}}=1}^{2}\left(1-\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^2\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\prod_{\dot{\mathrm{f}}=1}^{2}\left(1-\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^2\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}}\right)^{\frac{1}{2}}\end{pmatrix}$$

Thus, for $\varrho = 2$, Eq. (2) is satisfied. Let Eq. (2) is valid for $\varrho = \text{ß}$, that is

$$CPFWG\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\text{ß}}\right)$$
$$=\begin{pmatrix} \prod_{\dot{\mathrm{f}}=1}^{\text{ß}}\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}} \\ +\iota\prod_{\dot{\mathrm{f}}=1}^{\text{ß}}\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}}, \\ \left(1-\prod_{\dot{\mathrm{f}}=1}^{\text{ß}}\left(1-\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^2\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\prod_{\dot{\mathrm{f}}=1}^{\text{ß}}\left(1-\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^2\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}}\right)^{\frac{1}{2}}\end{pmatrix}$$

Now to reveal that for $\varrho = \text{ß} + 1$, Eq. (2) is valid. Since

$$CPFWG\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\text{ß}+1}\right)$$
$$= CPFWG\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\text{ß}}\right) \otimes \mathbb{Y}_{CPFS-\text{ß}+1}$$
$$=\begin{pmatrix} \prod_{\dot{\mathrm{f}}=1}^{\text{ß}}\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}} \\ +\iota\prod_{\dot{\mathrm{f}}=1}^{\text{ß}}\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}}, \\ \left(1-\prod_{\dot{\mathrm{f}}=1}^{\text{ß}}\left(1-\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^2\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\prod_{\dot{\mathrm{f}}=1}^{\text{ß}}\left(1-\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^2\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}}\right)^{\frac{1}{2}}\end{pmatrix}$$

$$\otimes\begin{pmatrix} \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\text{ß}+1}}\right)^{\Xi_{\mathscr{w}-\text{ß}+1}} \\ +\iota\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\text{ß}+1}}\right)^{\Xi_{\mathscr{w}-\text{ß}+1}}, \\ \left(1-\left(1-\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\text{ß}+1}}\right)^2\right)^{\Xi_{\mathscr{w}-\text{ß}+1}}\right)^{2} \\ +\iota\left(1-\left(1-\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\text{ß}+1}}\right)^2\right)^{\Xi_{\mathscr{w}-\text{ß}+1}}\right)^{\frac{1}{2}}\end{pmatrix}$$

$$=\begin{pmatrix} \prod_{\dot{\mathrm{f}}=1}^{\text{ß}+1}\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}} \\ +\iota\prod_{\dot{\mathrm{f}}=1}^{\text{ß}+1}\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}}, \\ \left(1-\prod_{\dot{\mathrm{f}}=1}^{\text{ß}+1}\left(1-\left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^2\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}}\right)^{\frac{1}{2}} \\ +\iota\left(1-\prod_{\dot{\mathrm{f}}=1}^{\text{ß}+1}\left(1-\left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)^2\right)^{\Xi_{\mathscr{w}-\dot{\mathrm{f}}}}\right)^{\frac{1}{2}}\end{pmatrix}$$
$$= CPFWG\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\text{ß}}, \mathbb{Y}_{CPFS-\text{ß}+1}\right)$$

This reveals that for $\varrho = \text{ß} + 1$, Eq. (2) is valid and hence for all ϱ.

Next, we have a connected property that is idempotency.

Proposition 4: If $\mathbb{Y}_{CPFS-\dot{\mathrm{f}}} = \mathbb{Y}_{CPFS}$ i.e., $\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}} = \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS}}$, $\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}} = \mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS}}$, $\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}} = \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS}}$ and $\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}} = \mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS}}$, $\forall\dot{\mathrm{f}}$, then

$$CPFWG\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\varrho}\right) = \mathbb{Y}_{CPFS-\dot{\mathrm{f}}}$$

The following property of the CPFWG operator is monotonicity.

Proposition 5: Take two gatherings of CPFNs $\mathbb{Y}_{CPFS-\dot{\mathrm{f}}} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right) = \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}} + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right)$, and $\mathbb{Y}^{\#}_{CPFS-\dot{\mathrm{f}}} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}^{\#}_{CPFS-\dot{\mathrm{f}}}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}^{\#}_{CPFS-\dot{\mathrm{f}}}}\right) = \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{\mathrm{f}}}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{\mathrm{f}}}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}^{\#}_{CPFS-\dot{\mathrm{f}}}}\ \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}^{\#}_{CPFS-\dot{\mathrm{f}}}}\right)$, $\dot{\mathrm{f}} = 1, 2, \ldots, \varrho$. If $\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}} \leq \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{\mathrm{f}}}}$, $\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}} \leq \mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}^{\#}_{CPFS-\dot{\mathrm{f}}}}$ $\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}} \geq \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}^{\#}_{CPFS-\dot{\mathrm{f}}}}$, and $\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}} \geq \mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}^{\#}_{CPFS-\dot{\mathrm{f}}}}$ $\forall\dot{\mathrm{f}}$, then

$$CPFWG\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\varrho}\right)$$
$$\leq CPFWG\left(\mathbb{Y}^{\#}_{CPFS-1}, \mathbb{Y}^{\#}_{CPFS-2}, \ldots \mathbb{Y}^{\#}_{CPFS-\varrho}\right)$$

The following property of the CPFWA operator is Boundedness.

Proposition 6: If

$$\mathbb{Y}^{-}_{CPFS} = \begin{pmatrix} \min_{\dot{\mathrm{f}}}\left\{\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right\} + \iota\min_{\dot{\mathrm{f}}}\left\{\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right\}, \\ \max_{\dot{\mathrm{f}}}\left\{\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right\} + \iota\max_{\dot{\mathrm{f}}}\left\{\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right\}\end{pmatrix}$$

and

$$\mathbb{Y}^{+}_{CPFS} = \begin{pmatrix} \max_{\dot{\mathrm{f}}}\left\{\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right\} + \iota \max_{\dot{\mathrm{f}}}\left\{\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right\}, \\ \min_{\dot{\mathrm{f}}}\left\{\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right\} + \iota \min_{\dot{\mathrm{f}}}\left\{\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}}\right\} \end{pmatrix},$$

then we have

$$\mathbb{Y}^{-}_{CPFS} \leq CPFWG\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\varrho}\right) \leq \mathbb{Y}^{+}_{CPFS}$$

Definition 10: Suppose $CPFOWG : \left(\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}\right)^{\varrho} \rightarrow \mathbb{Y}_{CPFS-\dot{\mathrm{f}}}$, if

$$CPFOWG\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots, \mathbb{Y}_{CPFS-\varrho}\right) = \bigotimes_{\dot{\mathrm{f}}=1}^{\varrho} \left(\mathbb{Y}_{CPFS-\Delta(\dot{\mathrm{f}})}\right)^{\Xi_{\mathcal{w}-\dot{\mathrm{f}}}}$$

then CPFOWG is deduced as CPFOWG operator over $\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}$. Noted that $\Xi_{\mathcal{w}} = \left(\Xi_{\mathcal{w}-1}, \Xi_{\mathcal{w}-2}, \ldots, \Xi_{\mathcal{w}-\varrho}\right)$ is a weight vector with $0 \leq \Xi_{\mathcal{w}-\dot{\mathrm{f}}} \leq 1$, $\sum_{\dot{\mathrm{f}}=1}^{\varrho} \Xi_{\mathcal{w}-\dot{\mathrm{f}}} = 1$, and $(\Delta(1), \Delta(2), \ldots, \Delta(\varrho))$ is a permutation of $(1, 2, \ldots, \varrho)$ such that $\Delta\left(\dot{\mathrm{f}} - 1\right) \geq \Delta\left(\dot{\mathrm{f}}\right)$, for $\dot{\mathrm{f}} = 2, 3, \ldots, \varrho$.

Theorem 5: The aggregated result over $\mathbb{Y}_{CPFS-\dot{\mathrm{f}}}$ by utilizing CPFOWG operator will be analyzed as CPFN and

$$CPFOWG\left(\mathbb{Y}_{CPFS-1}, \mathbb{Y}_{CPFS-2}, \ldots \mathbb{Y}_{CPFS-\varrho}\right) = \begin{pmatrix} \prod_{\dot{\mathrm{f}}=1}^{\varrho}\left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\Delta(\dot{\mathrm{f}})}}\right)^{\Xi_{\mathcal{w}-\dot{\mathrm{f}}}} \\ +\iota \prod_{\dot{\mathrm{f}}=1}^{\varrho}\left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\Delta(\dot{\mathrm{f}})}}\right)^{\Xi_{\mathcal{w}-\dot{\mathrm{f}}}}, \\ \left(1 - \prod_{\dot{\mathrm{f}}=1}^{\varrho}\left(1 - \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\Delta(\dot{\mathrm{f}})}}\right)^{2}\right)^{\Xi_{\mathcal{w}-\dot{\mathrm{f}}}}\right)^{\frac{1}{2}} \\ +\iota \left(1 - \prod_{\dot{\mathrm{f}}=1}^{\varrho}\left(1 - \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\Delta(\dot{\mathrm{f}})}}\right)^{2}\right)^{\Xi_{\mathcal{w}-\dot{\mathrm{f}}}}\right)^{\frac{1}{2}} \end{pmatrix}$$

V. APPLICATION

Here, we will reveal the application and practicality of our proposed theory by solving a multi-attribute decision-making (MADM) dilemma in the field of virtualization technology. For that first, we devise a technique of MADM within the Cartesian coordinate of CPFS by employing the invented operators and then discuss a case study "evaluation and selection of virtualization technologies for cloud-based e-commerce platform.

A. AN APPROACH OF MADM WITHIN CPFS

Consider $\left\{\mathcal{V}_{\mathcal{v}t-1}, \mathcal{V}_{\mathcal{v}t-2}, \ldots, \mathcal{V}_{\mathcal{v}t-\varrho}\right\}$ as a class of ϱ alternatives and $\left\{\mathcal{B}_{ab-1}, \mathcal{B}_{a\mathfrak{b}-2}, \ldots, \mathcal{B}_{a\mathfrak{b}-\rho}\right\}$ as a class of $\boldsymbol{\rho}$ attributes in a MADM dilemma. In this dilemma, the decision expert has to assess these alternatives by considering these attributes. As the attributes have their significance decision expert will describe their weight $\Xi_{\mathcal{w}} = \left(\Xi_{\mathcal{w}-1}, \Xi_{\mathcal{w}-2}, \ldots, \Xi_{\mathcal{w}-\boldsymbol{\rho}}\right)$ with $0 \leq \Xi_{\mathcal{w}-\check{\mathfrak{g}}} \leq 1$ and $\sum_{\check{\mathfrak{g}}=1}^{\boldsymbol{\rho}} \Xi_{\mathcal{w}-\check{\mathfrak{g}}} = 1$ according to their significance and his/her choice. The assessment values of these alternatives will be in the model of the Cartesian form of CPFNs which is $\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}} = \left(\mathcal{G}^{\mathbb{M}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}}, \mathcal{G}^{\mathbb{N}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}}\right) = \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}} + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}}\right)$, where $\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}}, \mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}}, \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}}, \mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}} \in [0, 1]$ and $0 \leq \left(\mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}}\right)^2 + \left(\mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}}\right)^2 \leq 1$, and $0 \leq \left(\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}}\right)^2 + \left(\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}}\right)^2 \leq 1$, which develops a CPF decision matrix $\mathbb{D}_{CPFS}$. The decision expert can cope with this dilemma by employing the below steps

Step 1: Most of the time, in MADM issues, the attributes are two sorts that are benefit and cost types. To normalize such sort of attributes we have the underneath formula

$$\left(\mathbb{N}_{CPFS}\right)^{\mathcal{N}} = \begin{cases} \begin{pmatrix} \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}}, \\ \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}} + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}} \end{pmatrix} & \textit{for benefit type} \\ \begin{pmatrix} \mathcal{G}^{\mathbb{RN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}} + \iota\mathcal{G}^{\mathbb{IN}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}} \\ \mathcal{G}^{\mathbb{RM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}} + \iota\mathcal{G}^{\mathbb{IM}}_{\mathbb{Y}_{CPFS-\dot{\mathrm{f}}\check{\mathfrak{g}}}} \end{pmatrix} & \textit{for cost type} \end{cases}$$

Step 2: For aggregating the CPF decision matrix, any of the designed operators (CFPWA, CFPOWA, CFPWG, and CFPOWG) would be utilized.

Step 3: For ordering alternatives, the score values of the aggregated outcomes of the alternatives would be deduced. In the case of the same score values of two different alternatives, accuracy values would be deduced.

Step 4: Based on step 3, the alternatives would be ranked.

B. CASE STUDY "EVALUATION AND SELECTION OF VIRTUALIZATION TECHNOLOGIES"

The multidimensional issue that a multinational company has with an e-commerce platform is improving its infrastructure to satisfy the needs of a continually expanding and erratic user base. This is an analysis of the issue:

- Scalability Challenge: Consistently, client traffic on their site varies at various levels, cresting during occasions and exceptional events. To stay aware of these varieties, customary framework finds it challenging to powerfully scale assets, which could bring about lackluster showing, hindered benefits, and displeased shoppers.
- Performance Bottlenecks: Client disturbance and diminished change rates can result from slow site loads, lazy exchange handling, and coldhearted UIs. The framework's failure to distribute and oversee assets productively compounds these performance issues and makes it more challenging for them to give a consistent client experience.

TABLE 1. The assessment values of virtualization technologies which was devised by IT expert.

	$\mathfrak{B}_{ab-1}$	$\mathfrak{B}_{ab-2}$	$\mathfrak{B}_{ab-3}$	$\mathfrak{B}_{ab-4}$
$\mathcal{V}_{vt-1}$	$\begin{pmatrix}0.11+i0.79,\\0.58+i0.04\end{pmatrix}$	$\begin{pmatrix}0.65+i0.81,\\0.72+i0.46\end{pmatrix}$	$\begin{pmatrix}0.30+i0.47,\\0.13+i0.22\end{pmatrix}$	$\begin{pmatrix}0.25+i0.49,\\0.63+i0.75\end{pmatrix}$
$\mathcal{V}_{vt-2}$	$\begin{pmatrix}0.23+i0.39,\\0.68+i0.83\end{pmatrix}$	$\begin{pmatrix}0.88+i0.85,\\0.12+i0.21\end{pmatrix}$	$\begin{pmatrix}0.80+i0.62,\\0.51+i0.64\end{pmatrix}$	$\begin{pmatrix}0.10+i0.31,\\0.90+i0.95\end{pmatrix}$
$\mathcal{V}_{vt-3}$	$\begin{pmatrix}0.43+i0.52,\\0.38+i0.87\end{pmatrix}$	$\begin{pmatrix}0.01+i0.05,\\0.48+i0.34\end{pmatrix}$	$\begin{pmatrix}0.36+i0.27,\\0.69+i0.61\end{pmatrix}$	$\begin{pmatrix}0.03+i0.17,\\0.86+i0.93\end{pmatrix}$
$\mathcal{V}_{vt-4}$	$\begin{pmatrix}0.59+i0.76,\\0.19+i0.45\end{pmatrix}$	$\begin{pmatrix}0.37+i0.28,\\0.78+i0.84\end{pmatrix}$	$\begin{pmatrix}0.09+i0.16,\\0.97+i0.89\end{pmatrix}$	$\begin{pmatrix}0.96+i0.82,\\0.18+i0.29\end{pmatrix}$

TABLE 2. The aggregating values of virtualization technologies after employing devised operators.

Operators	$\mathcal{V}_{vt-1}$	$\mathcal{V}_{vt-2}$	$\mathcal{V}_{vt-3}$	$\mathcal{V}_{vt-4}$
CPFWA	$\begin{pmatrix}0.418+\iota\,0.663,\\0.389+\iota\,0.287\end{pmatrix}$	$\begin{pmatrix}0.718+\iota\,0.646,\\0.426+\iota\,0.554\end{pmatrix}$	$\begin{pmatrix}0.269+\iota\,0.277,\\0.609+\iota\,0.599\end{pmatrix}$	$\begin{pmatrix}0.74+\iota\,0.607,\\0.455+\iota\,0.585\end{pmatrix}$
CPFOWA	$\begin{pmatrix}0.353+\iota\,0.659,\\0.38+\iota\,0.22\end{pmatrix}$	$\begin{pmatrix}0.632+\iota\,0.573,\\0.525+\iota\,0.655\end{pmatrix}$	$\begin{pmatrix}0.269+\iota\,0.277,\\0.609+\iota\,0.599\end{pmatrix}$	$\begin{pmatrix}0.647+\iota\,0.588,\\0.46+\iota\,0.618\end{pmatrix}$
CPFWG	$\begin{pmatrix}0.3+\iota\,0.592,\\0.57+\iota\,0.51\end{pmatrix}$	$\begin{pmatrix}0.388+\iota\,0.521,\\0.688+\iota\,0.791\end{pmatrix}$	$\begin{pmatrix}0.074+\iota\,0.17,\\0.696+\iota\,0.749\end{pmatrix}$	$\begin{pmatrix}0.327+\iota\,0.363,\\0.829+\iota\,0.765\end{pmatrix}$
CPFOWG	$\begin{pmatrix}0.247+\iota\,0.59,\\0.549+\iota\,0.49\end{pmatrix}$	$\begin{pmatrix}0.311+\iota\,0.465,\\0.719+\iota\,0.826\end{pmatrix}$	$\begin{pmatrix}0.074+\iota\,0.17,\\0.696+\iota\,0.749\end{pmatrix}$	$\begin{pmatrix}0.329+\iota\,0.367,\\0.812+\iota\,0.767\end{pmatrix}$

TABLE 3. The score values of virtualization technologies.

Operators	$\check{S}(\mathcal{V}_{vt-1})$	$\check{S}(\mathcal{V}_{vt-2})$	$\check{S}(\mathcal{V}_{vt-3})$	$\check{S}(\mathcal{V}_{vt-4})$
CPFWA	0.19	0.222	−0.291	0.183
CPFOWA	0.183	0.012	−0.291	−0.085
CPFWG	−0.072	−0.339	−0.505	−0.517
CPFOWG	−0.066	−0.443	−0.505	−0.503

- Security Concerns: It is significant to shield private client data, monetary information, and exchange records. Traditional framework setups, then again, could never have areas of strength for the actions expected to prepare for information breaks, unlawful access, and impedance between different stage parts, leaving them open to serious security concerns.
- Costs Inefficiency: Underutilization causes wasteful asset allotment and conceivable performance bottlenecks while overprovisioning brings about unnecessary framework costs. It's always difficult to strike a balance between cost-effectiveness and performance needs, especially as their platform keeps expanding and changing.

Their e-commerce platform is confronted with issues of infrastructure cost-efficiency, security, scalability, and performance. To meet these issues and secure the platform's future growth and success, a thorough assessment of virtualization technologies is needed to choose the best option that can improve performance, reduce infrastructure costs, increase security, and expand resources dynamically. To tackle these issues and for the selection of best virtualization technology for the e-commerce platform the IT expert of the company considered the underneath 4 virtualization technologies

$\mathcal{V}_{vt-1}$ $(Docker)$: Using lightweight containerization to quickly deploy microservices and efficiently use resources, albeit it might not be as isolated as traditional hypervisor-based virtualization.

TABLE 4. The ranking of virtualization technologies.

Operators	Ranking
CPFWA	$\mathcal{V}_{vt-2} > \mathcal{V}_{vt-1} > \mathcal{V}_{vt-4} > \mathcal{V}_{vt-3}$
CPFOWA	$\mathcal{V}_{vt-1} > \mathcal{V}_{vt-2} > \mathcal{V}_{vt-4} > \mathcal{V}_{vt-3}$
CPFWG	$\mathcal{V}_{vt-1} > \mathcal{V}_{vt-2} > \mathcal{V}_{vt-3} > \mathcal{V}_{vt-4}$
CPFOWG	$\mathcal{V}_{vt-1} > \mathcal{V}_{vt-2} > \mathcal{V}_{vt-4} > \mathcal{V}_{vt-3}$

$\mathcal{V}_{vt-2}$ $(KVM(Kernel - based\ Virtual\ Machine))$: Ideal for cases needing high security and resource isolation, this technology leverages hardware virtualization extensions to provide strong isolation and performance near to native.

$\mathcal{V}_{vt-3}$ $(VMware\ vSphere)$: This virtualization platform with mature features and strong management capabilities is appropriate for enterprise-grade deployments, but it may have more administration overhead than other solutions.

$\mathcal{V}_{vt-4}$ $(Microsoft\ Hyper - V)$: This system offers a compromise between performance, isolation, and administration overhead while integrating seamlessly with Windows environments and giving robust support for Microsoft applications.

To access these the IT expert, consider the following 4 attributes

$\mathcal{B}_{ab-1}$ $(Performance)$: Evaluating how well the virtualization technology manages peak loads while preserving high throughput and low latency.

$\mathcal{B}_{ab-2}$ $(Resource\ Utilization)$: To guarantee maximum cost-effectiveness, assess how well the virtualization

TABLE 5. The comparison outcomes of the deduced and existing theories.

Reference	$\check{S}(\mathcal{V}_{vt-1})$	$\check{S}(\mathcal{V}_{vt-2})$	$\check{S}(\mathcal{V}_{vt-3})$	$\check{S}(\mathcal{V}_{vt-4})$
Xu [11]	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ
Xu and Yager [12]	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ
Yager [19]	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ
Garg [17]	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ
Rehman [24]	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ
Feng et al. [32]	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ
Hezam et al. [35]	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ
Akram et al. [36]	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ
Janani et al. [37]	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ	Ẋ↔Ẋ
CPFWA	0.19	0.222	−0.291	0.183
CPFOWA	0.183	0.012	−0.291	−0.085
CPFWG	−0.072	−0.339	−0.505	−0.517
CPFOWG	−0.066	−0.443	−0.505	−0.503

technology makes use of hardware resources like CPU, memory, and storage.

$\mathcal{B}_{ab-3}$ (*Isolation*): To avoid interference and preserve security and stability inside the e-commerce platform, robust isolation between virtual instances must be ensured.

$\mathcal{B}_{ab-4}$(*Management Cost*): Examining how simple it is to build and maintain virtualized instances, as well as the administrative cost involved.

The IT expert assessed the virtualization technologies relying on the considered four attributes and expressed his assessment values in the term of CPFN in Cartesian coordinate, which created a CPF decision matrix as portrayed in Table 1.

The IT expert also provides weight (0.15, 0.26, 0.32, 0.27) to the attribute according to his expertise about the attributes. To get the finest virtualization technology, the MADM technique with CPFS would be used

Step 1: All attributes are considered as benefit sort so no requirement of normalization.

Step 2: For aggregating the CPF decision matrix, the designed operators (CFPWA, CFPOWA, CFPWG, and CFPOWG) have been utilized and the results are devised in Table 2.

Step 3: For ordering virtualization technologies, the score values of the aggregated outcomes of the virtualization technologies have been deduced and portrayed in Table 3.

Step 4: Based on step 3, the virtualization technologies are ranked, which is devised in Table 4.

Following a careful analysis, we determined that by using the CPFWA operator in the MADM procedure $\mathcal{V}_{vt-2}$ that is KVM is the finest virtualization technology for e-commerce platforms and by using the other operators in the DM process $\mathcal{V}_{vt-1}$ that is Docker is the finest virtualization technology for e-commerce platforms.

VI. COMPARISON

Comparison of the invented work with certain work is critical for describing the significance and supremacy of the devised work. Thus, here, we are going to compare the interpreted theory with certain existing theories which are demonstrated below

TABLE 6. The ranking is based on the deduced and existing theories.

Reference	Ranking
Xu [11]	Ẋ↔Ẋ
Xu and Yager [12]	Ẋ↔Ẋ
Yager [19]	Ẋ↔Ẋ
Garg [17]	Ẋ↔Ẋ
Rehman [24]	Ẋ↔Ẋ
Feng et al. [32]	Ẋ↔Ẋ
Hezam et al. [35]	Ẋ↔Ẋ
Akram et al. [36]	Ẋ↔Ẋ
Janani et al. [37]	Ẋ↔Ẋ
CPFWA	$\mathcal{V}_{vt-2} > \mathcal{V}_{vt-1} > \mathcal{V}_{vt-4} > \mathcal{V}_{vt-3}$
CPFOWA	$\mathcal{V}_{vt-1} > \mathcal{V}_{vt-2} > \mathcal{V}_{vt-4} > \mathcal{V}_{vt-3}$
CPFWG	$\mathcal{V}_{vt-1} > \mathcal{V}_{vt-2} > \mathcal{V}_{vt-3} > \mathcal{V}_{vt-4}$
CPFOWG	$\mathcal{V}_{vt-1} > \mathcal{V}_{vt-2} > \mathcal{V}_{vt-4} > \mathcal{V}_{vt-3}$

- ❖ The theory of IF AOs, was established by Xu [11] in 2007 and Xu and Yager [12] in 2006.
- ❖ The theory of an approach to DM in PFS was devised by Yager [19] in 2013.
- ❖ The idea of confidence level AOs and DM approach within PFS was devised by Garg [17] in 2017.
- ❖ The notion of probability complex fuzzy AOs and DM approach was interpreted by Rehman [24] in 2023.
- ❖ The theory of probability CIF AOs and MCDM technique was established by Fang et al. [32] in 2024.
- ❖ The notion of Geometric AOs and multi-criteria group DM technique within the polar structure of CPFS was demonstrated by Hezam et al. [35] in 2023.
- ❖ CPF Dombi AOs and their DM technique within the polar structure of CPFS were originated by Akram et al. [36] in 2021.
- ❖ CPF Einstein AOs and their DM technique within the polar structure of CPFS were originated by Janani et al. [37] in 2022.

To compare these theories with initiated theory, let us reconsider the information in Table 1, which is in the model of the Cartesian form of CPFS. After applying the prevailing and initiated theories the overall result is portrayed in Tables 5 and 6.

From the outcomes of Tables 5 and 6, none of the prevailing theories solved the information of Table 1, and the reason for that is discussed as: The theory of Xu [11] and Xu and Yager [12] failed because it can merely solve the information in the environment of IFS and are not structured for coping with information that has extra fuzzy information. The theories that were developed by Yager [19] and Garg [17] were unsuccessful because these theories are not capable of tackling information along with extra fuzzy information which is 2^{nd} dimension. The probability complex fuzzy AOs cope 2^{nd} dimension (extra fuzzy information) but can't tackle with non-membership and that was the reason for the failure of this theory. The theory of probability CIF AOs failed because there are some assessment values in Table 1, that the sum of their degrees of membership and non-membership is greater than 1, for example, $0.45 + 0.79 > 1$, and complex intuitionistic FS can't model such information. But the invented structure can model such information because in the invented structure the sum of the square of degree of membership and non-membership should be between 0 and 1.

Further, the geometric, Dombi, and Einstein AOs and their related DM approaches failed to solve the information in Table 1, even these operations and DM approaches are in the model of CPFS. However, these notions are in the polar structure of CPFS and can't be overcome with the Cartesian complex data. Thus, the development of the invented operators and MADM technique was the requirement of today's world as there are no such structures and operators exist in the literature. Furthermore, the invented operators and an approach of MADM can reduce to the Cartesian structure of CIFS and CFS, PFS, IFS, and FS.

VII. CONCLUSION

This script provided a fresh idea of CPFS in a Cartesian coordinate by changing the range from the unit disc of the complex plane to the unit square, hence resolving issues with the prevailing theory of CPFS. The degrees of membership and non-membership in the Cartesian version of CPFS are situated in the unit square of a complex plane and comprise real and imaginary components that transmit fuzzy information. For the developed CPFS, we additionally developed basic operating laws. Additionally, we derived geometric and averaging operators (AOs) for the CPFWA, CPFFOWA, CPFWG, and CPFFOWG operators in the Cartesian model of CPFS. Next, using these operators, we implemented a MADM method inside CPFS to address real-world MADM problems that include CPF data. In this article, we have finally developed the application of diagnosed theory in the sphere of visualization technology, and through comparison, we have shown the superiority of the developed theory.

In the future, we would like to work on the practical usage and application of this theory in various industries. Also we would like to expand this concept in various notion such as complex hesitant FS [46], bipolar complex fuzzy set [47] and bipolar complex fuzzy soft set [48] etc.

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