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RESEARCH ARTICLE

On the Robustness of a Modified Super-Twisting Algorithm With Prescribed-Time Convergence

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ABSTRACT This paper addresses the robustness of a novel two-stage super-twisting algorithm designed to converge within a prescribed time interval despite disturbances and model uncertainties. Initially, we introduce a method for tuning parameters that guarantees the algorithm's analytic solution will reach the origin precisely at a prescribed instant, assuming an unperturbed scenario. We then enhance this method to maintain prescribed-time convergence, even when faced with unknown bounded disturbances. The algorithm's performance is demonstrated through a numerical simulation of a state estimation problem for a perturbed damped pendulum. The results show that the estimation errors converge robustly to the origin at the prescribed time and remain there afterward.

INDEX TERMS Prescribed-time convergence, robust estimation, super-twisting algorithm.

I. INTRODUCTION

The super-twisting algorithm (STA) [1], [2] has been widely applied to the design of finite-time robust controllers and estimators over the last three decades [3], [4], [5], [6], [7]. Although effective, it is essential to note that we cannot determine a global finite upper bound for the settling time (UBST) when employing the conventional STA. This is because the actual settling time tends towards infinity when the system's initial condition increases without bounds [8]. Recent extensions to this algorithm offer fixed-time stability [9], enabling us to estimate a finite UBST that does not depend on initial conditions [10], [11]. However, this estimate tends to be overly conservative compared to the actual settling time seen in experiments and relates to the system's parameters in a complex way. Moreover, a bounded settling time is insufficient for applications requiring precise timing, such as tactical missile guidance. Hence, this paper explores STA modifications that permit arbitrarily choosing the settling instant.

The pioneering work of Song et al. [12] has demonstrated that it is possible to drive the states of a nonlinear system in the normal form to a desired trajectory at a prescribed instant $t_c \in \mathbb{R}_{>t_0}$ by employing time-varying gains that continuously approach infinity as time approaches t_c. This idea was initially proposed to design regulators and later expanded to state estimation by Holloway and Krstic [13]. However, to avoid encountering a singularity at t_c, the investigated prescribed-time initial value problem is only defined in the finite domain $[t_0, t_c)$ in the aforementioned papers. Furthermore, these approaches are highly sensitive to measurement noise and numerical errors, primarily due to the unbounded growth of their time-varying gains as t approaches t_c. To address this latter limitation, recent papers [14], [15], [16] have investigated using the user-defined instant as a prescribed UBST instead of the convergence instant itself. In doing so, these recent studies ensure that the time-varying gains introduced in [12] and [13] remain finite but render the exact convergence instant unknown, although bounded.

The present paper is a continuation of [17] and proposes a modification of the STA that presents state convergence precisely at a prescribed instant t_c, which is set *a priori* as a system parameter. During the time interval $[t_0, t_c)$, the proposed algorithm uses both time-varying [12], [13] and switching [1], [2] gains to provide robustness and the

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prescribed-time convergence property. At t_c, the algorithm assumes the dynamics of the conventional STA to ensure stability in the infinite time horizon. In contrast to works [1], [2], [10], [11], in which only a conservative bound of the settling time can be estimated, the proposed solution allows the arbitrary prescription of the settling instant. Unlike Song et al. [12] and Holloway and Krstic [13], our approach ensures robust stability in the infinite time domain $[t_0, \infty)$ while also using a time-varying gain only in its first component state equation, which mitigates the sensitivity to measurement noise. Furthermore, our approach differs from [14], [15], and [16] by employing the prescribed instant as the exact settling instant, not as its upper bound. Convergence and robustness proofs for the modified super-twisting algorithm introduced in [17] are also provided. Additionally, for illustration, the proposed method is employed in designing a state estimator for a disturbed damped pendulum. In summary, the main contribution of this paper is the proposal of a novel two-stage STA-like algorithm capable of presenting robust convergence of its states to the origin at a user-defined instant in the presence of bounded disturbances or model uncertainties, along with analytical proofs of the aforementioned properties.

In the remaining text, Section II presents the mathematical preliminaries required to introduce the proposed results and states the paper's primary objective. Section III presents the proposed proofs that indicate the existence of system parameters that guarantee the prescribed-time robust convergence. Section IV presents simulation results and comparisons with a similarly prescribed-time convergent method. Finally, Section V concludes this paper.

A. DEFINITIONS

Consider the system

$$\dot{x} = f(t, x), \quad x(t_0) \triangleq x_0 \in \mathbb{R}^n, \quad t \in \mathbb{R}_{\geq t_0}, \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state vector, t is the time variable, $t_0 \in \mathbb{R}_{\geq 0}$ is the initial instant, and $f : \mathbb{R}_{\geq t_0} \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function such that $f(\cdot, \cdot)$ is jointly continuous in t and x, and $f(t, 0) = 0$, $\forall t \in [t_0, \infty)$. A solution $x(t; t_0, x_0)$ of (1) is viewed in the generalized Filippov sense [18], and is understood as a first-order differentiable function satisfying (1) and the initial condition, *i.e.*, $x(t_0; t_0, x_0) = x_0$.

We present the definition of the stability and convergence concepts required in this paper.

Definition 1 [19]: The equilibrium point $x = 0$ of (1) is said to be *finite-time stable* if it is asymptotically stable, and any solution $x(t; t_0, x_0)$ converges the origin in a finite time interval $t \in [t_0, t_0 + T(t_0, x_0)]$, where

$$T(t_0, x_0) = \inf\{T \geq t_0 : x(t; t_0, x_0) = 0, \ \forall t \geq t_0 + T\} \tag{2}$$

is the settling-time function.

Definition 2: A solution of (1) is said to present *prescribed-time convergence* to its equilibrium point if the system is finite-time stable and its states reach the equilibrium point in a finite time interval upper bounded by $t_0 + t_c$, where t_c can be arbitrarily specified.

II. PROBLEM STATEMENT

The super-twisting algorithm (STA) is given by [1]

$$\dot{\varepsilon}_1 = -\kappa_1 \lfloor \varepsilon_1 \rceil^{1/2} + \varepsilon_2, \tag{3}$$
$$\dot{\varepsilon}_2 = -\kappa_2 \operatorname{sign}(\varepsilon_1) + \delta, \tag{4}$$

where $\boldsymbol{\varepsilon} \triangleq (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2$ is its state vector, $\kappa_1, \kappa_2 \in \mathbb{R}_{>0}$ are scalar parameters, $\lfloor \varepsilon_1 \rceil^{1/2} \triangleq |\varepsilon_1|^{1/2} \operatorname{sign}(\varepsilon_1)$, and $\delta \in \mathbb{R}$ is the disturbance term. Consider that δ is bounded by a known constant L, *i.e.*, $|\delta| \leq L$. Despite of δ, the origin $\varepsilon = 0$ is finite-time stable if the gain condition [20]

$$L < \min\left\{\frac{\kappa_1}{2}, \frac{\kappa_1 \kappa_2}{1 + \kappa_1}\right\} \tag{5}$$

is satisfied.

As the STA (3)–(4) is a finite-time stable system, only a conservative bound of its settling time can be estimated, and the respective estimate is dependent on the system's initial conditions, as expressed in Definition 1. The goal of the present paper is to introduce and analyze a new second-order system analogous to (3)–(4), but endowed with the ability to converge to the origin within a prescribed time interval, despite the presence of the unknown disturbance δ.

III. MAIN RESULTS

To achieve the aforementioned goal, we propose a novel algorithm obtained from a modification of the STA (3)–(4), given by

$$\dot{\varepsilon}_1 = -\sigma(t, \varepsilon_1) + \varepsilon_2, \tag{6}$$
$$\dot{\varepsilon}_2 = -\kappa_2 \operatorname{sign}(\varepsilon_1) + \delta, \tag{7}$$

where

$$\sigma(t, \varepsilon_1) \triangleq \begin{cases} \dfrac{\eta}{t_c - t}\varepsilon_1, & t \in [t_0, t_c), \\ \kappa_1 \lfloor \varepsilon_1 \rceil^{1/2}, & t \in [t_c, \infty), \end{cases} \tag{8}$$

with t_c representing the prescribed convergence instant, and $\eta \in \mathbb{Z}_{\geq 1}$ being a scalar parameter. The algorithm above presents a hybrid structure containing two stages. The first stage comprises the time interval $t \in [t_0, t_c)$, in which the system (6)–(7) combines both a time-varying and a switching gains. In the second stage $t \in [t_c, \infty)$, the proposed algorithm coincides with the conventional STA (3)–(4). Considering the unperturbed case, *i.e.*, $\delta \equiv 0$, and the first stage, the following proposition provides an analytical solution to the initial value problem (IVP) consisting of (6)–(7) and the given initial conditions.

Proposition 1: Consider the IVP consisting of (6)–(7) with $\delta \equiv 0$ and an initial condition $\boldsymbol{\varepsilon}(t_0) = (\varepsilon_1(t_0), \varepsilon_2(t_0)) \in \mathbb{R}^2$, denote its solution in $[t_0, \infty)$ by $\boldsymbol{\varepsilon}(t) := (\varepsilon_1(t), \varepsilon_2(t)) \in \mathbb{R}^2$, and recursively define the zero-crossing instants of $\varepsilon_1(t)$ as

$$t_j \triangleq \inf \mathcal{T}_j, \ j = 1, 2, \ldots, n_r, \tag{9}$$

where n_r is the number of times $\varepsilon_1(t)$ has reset to zero and

$$\mathcal{T}_j \triangleq \left\{ \zeta \in \mathbb{R}_{\geq t_0} : \zeta > t_{j-1} \text{ and } \lim_{t \to \zeta^-} \varepsilon_1(t) = 0 \right\}. \quad (10)$$

On each time interval $t \in [t_{j-1}, t_j)$, the solution to the IVP is recursively given by

$$\varepsilon_1(t) = \frac{t_c - t}{(\eta-2)(\eta-1)} \alpha(t) + \beta(t_{j-1}) \left(\frac{t_c - t}{t_c - t_{j-1}} \right)^\eta, \quad (11)$$

$$\varepsilon_2(t) = \varepsilon_2(t_{j-1}) - \kappa_2 \, \mathrm{s}_{1,j-1} \left(t - t_{j-1} \right), \quad (12)$$

where $\mathrm{s}_{i,j-1} \triangleq \mathrm{sign}(\varepsilon_i(t_{j-1}))$, and

$$\begin{aligned} \alpha(t) \triangleq\ & \mathrm{s}_{1,j-1} \left(\kappa_2 (t_c - t - (\eta-2)(t - t_{j-1})) \right) \\ & + (\eta - 2) \varepsilon_2(t_{j-1}), \end{aligned} \quad (13)$$

$$\begin{aligned} \beta(\tau) \triangleq\ & \varepsilon_1(\tau) - \frac{t_c - \tau}{\eta - 1} \varepsilon_2(\tau) \\ & - \frac{(t_c - \tau)^2}{(\eta-2)(\eta-1)} \kappa_2 \, \mathrm{s}_{1,j-1}. \end{aligned} \quad (14)$$

Proof: By the definition (9), t_j, $j = 1, \ldots, n_r$, are the instants that $\varepsilon_1(t)$ resets to zero and, consequently, $\mathrm{sign}(\varepsilon_1(t))$ switches its sign. Therefore, $\mathrm{sign}(\varepsilon_1(t)) = \mathrm{s}_{1,j-1}\ \forall t \in [t_{j-1}, t_j)$. Hence, the analytic solution of (7) in the interval $t \in [t_{j-1}, t_j)$ is given by

$$\varepsilon_2(t) = \varepsilon_2(t_{j-1}) - \kappa_2 \, \mathrm{s}_{1,j-1} \left(t - t_{j-1} \right). \quad (15)$$

Still considering the time interval $[t_{j-1}, t_j)$, by substituting (15) and (8) into (6), we obtain

$$\dot{\varepsilon}_1 = -\frac{\eta}{t_c - t} \varepsilon_1(t) + \varepsilon_2(t_{j-1}) - \kappa_2 \, \mathrm{s}_{1,j-1} \left(t - t_{j-1} \right), \quad (16)$$

which can be immediately integrated, yielding

$$\varepsilon_1(t) = \frac{t_c - t}{(\eta-2)(\eta-1)} \alpha(t) + \beta(t_{j-1}) \left(\frac{t_c - t}{t_c - t_{j-1}} \right)^\eta, \quad (17)$$

where $\alpha(t)$ and $\beta(.)$ are as defined in (13)–(14). □

Corollary 1: Consider the IVP described in Proposition 1 and assume that $\eta \geq 3$. It holds that $\varepsilon_1(t) \to 0$ as $t \to t_c$.

Proof: From (12), we see that $\varepsilon_2(t)$ will remain bounded in every finite time interval $[t_{j-1}, t_j)$ if its initial condition $\varepsilon_2(t_0)$ and the switching gain κ_2 are finite. Therefore, from (11), we see that for any $\eta \geq 3$, it holds that $\varepsilon_1(t)$ approaches zero as t approaches t_c. □

Corollary 2: Assuming that $\eta \geq 3$, the unperturbed system (6)–(7) is finite-time stable.

Proof: From Corollary 1, it holds that $\boldsymbol{\varepsilon}(t_c) = [0, \varepsilon_2(t_c)]$. Consequently, it is also true that the system (6)–(7) assumes the conventional STA behavior after t_c with finite initial conditions. Therefore, by satisfying (5), the system states will converge to the origin in finite time and remain there afterward. □

The following theorem demonstrates that sufficient conditions exist for the parameters κ_2 and η, which ensure that $\varepsilon_2(t)$ also approaches zero as t approaches t_c.

Theorem 1: Consider the IVP consisting of (6)–(7) with $\delta \equiv 0$, a known initial condition $\boldsymbol{\varepsilon}(t_0) = (\varepsilon_1(t_0), \varepsilon_2(t_0)) \in \mathbb{R}^2$, and $\eta \geq 3$. If the following conditions are satisfied

$$\kappa_2 \left(t_c - t_0 \right) \geq |\varepsilon_2(t_0)|, \quad (18)$$

$$\begin{cases} -\kappa_2 \left(t_c - 2t_1 + t_0 \right) \leq |\varepsilon_2(t_0)| \leq \\ \qquad \kappa_2 \left(t_c - t_0 \right), \ \mathrm{s}_{1,0} = \mathrm{s}_{2,0}, \\ -\kappa_2 \left(t_c - 2t_1 + t_0 \right) \leq -|\varepsilon_2(t_0)| \leq \\ \qquad \kappa_2 \left(t_c - t_0 \right), \ \mathrm{s}_{1,0} \neq \mathrm{s}_{2,0}. \end{cases} \quad (19)$$

then $\varepsilon_2(t)$ converges to zero at the prescribed time t_c.

Proof: Consider a geometric set $\mathcal{E}(\kappa_2)$ defined as

$$\mathcal{E}(\kappa_2) \triangleq \{ (t, \varepsilon_2(t)) : -\kappa_2 \left(t_c - t \right) \leq \varepsilon_2(t) \leq \kappa_2 \left(t_c - t \right) \}, \quad (20)$$

for a given $\kappa_2 < \infty$. This definition directly implies that if the pair $\left(t_{j-1}, \varepsilon_2(t_{j-1}) \right)$ is in $\mathcal{E}(\kappa_2)$ for all $j = 1, \ldots, n_r$, then $\lim_{t \to t_c^-} \varepsilon_2(t) = 0$. Consequently, ensuring the fulfillment of this bounding condition guarantees the convergence of ε_2 to zero at the prescribed time t_c.

From (20) and the known initial condition $\varepsilon_2(t_0)$, we see that choosing κ_2 satisfying

$$\kappa_2 (t_c - t_0) \geq |\varepsilon_2(t_0)|, \quad (21)$$

ensures that $(t_0, \varepsilon_2(t_0)) \in \mathcal{E}(\kappa_2)$.

The next step is to ensure that $(t, \varepsilon_2(t))$ belongs to $\mathcal{E}(\kappa_2)$ at the instant t_1 that $\varepsilon_1(t)$ crosses the time axis for the first time. From (12), this value is given by

$$\varepsilon_2(t_1) = \varepsilon_2(t_0) - \kappa_2 \, \mathrm{s}_{1,0} \left(t_1 - t_0 \right). \quad (22)$$

Considering $t = t_1$ and (20), it must hold that

$$-\kappa_2 (t_c - t_1) \leq \varepsilon_2(t_1) \leq \kappa_2 (t_c - t_1), \quad (23)$$

which can be expanded using (22) to

$$\begin{aligned} & -\kappa_2 \left(t_c - t_1 - \mathrm{s}_{1,0} \left(t_1 - t_0 \right) \right) \\ & \qquad \leq \varepsilon_2(t_0) \leq \kappa_2 \left(t_c - t_1 + \mathrm{s}_{1,0} \left(t_1 - t_0 \right) \right). \end{aligned} \quad (24)$$

Considering all the possible values that $\mathrm{s}_{1,0}$ and $\mathrm{s}_{2,0}$ can assume, we can develop (24) further to obtain

$$\begin{cases} -\kappa_2 \left(t_c - 2t_1 + t_0 \right) \leq |\varepsilon_2(t_0)| \leq \\ \qquad \kappa_2 \left(t_c - t_0 \right), \ \mathrm{s}_{1,0} = \mathrm{s}_{2,0}, \\ -\kappa_2 \left(t_c - 2t_1 + t_0 \right) \leq -|\varepsilon_2(t_0)| \leq \\ \qquad \kappa_2 \left(t_c - t_0 \right), \ \mathrm{s}_{1,0} \neq \mathrm{s}_{2,0}. \end{cases} \quad (25)$$

To compute the boundaries in (25), instant t_1 can be calculated by solving (11) for t, considering $\varepsilon_1(t) = 0$ and adopting parameters η satisfying $\eta \geq 3$, and κ_2 satisfying (5) and (21). Note that by increasing η, $\varepsilon_1(t)$ approaches zero faster in (11), reducing the value of t_1. Additionally, increasing κ_2 steepens the slope of $\varepsilon_2(t)$ in (12), which also reduces t_1. Therefore, (25) can be satisfied by tuning either κ_2 or η.

Given the linear time response of $\varepsilon_2(t)$ and satisfying the conditions in equations (21) and (25), we can conclude that for all t within the interval $[t_0, t_1)$, the pair $(t, \varepsilon_2(t))$ lies within $\mathcal{E}(\kappa_2)$.

Following a similar mathematical procedure for the pair $(t, \varepsilon_2(t))$ at $t = t_2$ yields the inequality

$$\begin{cases} -\kappa_2 (t_c - 2t_2 + t_1) \leq |\varepsilon_2(t_1)| \leq \\ \qquad\qquad \kappa_2 (t_c - t_1), \, s_{1,1} = s_{2,1}, \\ -\kappa_2 (t_c - 2t_2 + t_1) \leq -|\varepsilon_2(t_1)| \leq \\ \qquad\qquad \kappa_2 (t_c - t_1), \, s_{1,1} \neq s_{2,1}, \end{cases} \tag{26}$$

where $s_{1,1} = -s_{1,0}$. The time instant t_2 is similarly determined using (11), with the state vector at t_1 serving as the initial condition. It can be inferred from the fulfillment of the inequalities (25) and (26) that $(t, \varepsilon_2(t)) \in \mathcal{E}(\kappa_2)$ for all $t \in [t_1, t_2)$. Furthermore, it is worth noting that the value of t_2 obtained by solving (11) with the same parameters η and κ_2 that satisfy (25) also satisfies condition (26).

Consider now the generalization of (22) to evaluate $\varepsilon_2(t)$ at every instant that $\varepsilon_1(t)$ crosses the time axis. With $j \geq 1$ and $t_j < t_c$, (23) can be rewritten as

$$-\kappa_2(t_c - t_j) \leq \varepsilon_2(t_j) \leq \kappa_2(t_c - t_j), \tag{27}$$

which analogously yields

$$\begin{cases} -\kappa_2 \left(t_c - 2t_j + t_{j-1}\right) \leq \left|\varepsilon_2(t_{j-1})\right| \leq \\ \qquad\qquad \kappa_2 \left(t_c - t_{j-1}\right), \, s_{1,j-1} = s_{2,j-1}, \\ -\kappa_2 \left(t_c - 2t_j + t_{j-1}\right) \leq -\left|\varepsilon_2(t_{j-1})\right| \leq \\ \qquad\qquad \kappa_2 \left(t_c - t_{j-1}\right), \, s_{1,j-1} \neq s_{2,j-1}, \end{cases} \tag{28}$$

where $s_{1,j} = -s_{1,j-1}$. By induction, we can deduce that if equation (28) holds true, assuming $\left(t_{j-1}, \varepsilon_2(t_{j-1})\right) \in \mathcal{E}(\kappa_2)$, then $(t, \varepsilon_2(t)) \in \mathcal{E}(\kappa_2)$ for all $t \in \left[t_{j-1}, t_j\right)$. Additionally, it is noticeable that equation (28) is satisfied at every time instant t_j, computed from (11), utilizing the state vector at t_{j-1} as the initial condition and the same parameters η and κ_2 that fulfill (25).

Therefore, a convergence rate $\eta \geq 3$ and a switching gain κ_2 that ensure equations (21) and (25) are satisfied, are sufficient to guarantee that $\left(t_{j-1}, \varepsilon_2(t_{j-1})\right) \in \mathcal{E}(\kappa_2)$, $\forall j = 1, 2, \ldots, n_r$. Consequently, $\lim_{t \to t_c^-} \varepsilon_2(t) = 0$, thus concluding the proof. □

The proofs of Corollary 1 and Theorem 1 guarantee the prescribed-time convergence of the complete state vector to the origin of the undisturbed system (6)–(7).

Remark 1: The definition of $\mathcal{E}(\kappa_2)$ makes it evident that its bounding functions are directly influenced by the switching gain κ_2, which, as deduced from (21), relies on the initial value of $\varepsilon_2(t)$. Consequently, since the system parameters depend on its initial conditions, the proposed methodology cannot be classified as fixed-time stable. Furthermore, the initial states of the system are typically unknown in practical scenarios. Nevertheless, it is still feasible to empirically adjust the parameters η and κ_2 to satisfy equations (21) and (25).

Remark 2: Obtaining a convergence proof for the proposed algorithm using conventional Lyapunov methods has proven challenging. This difficulty arises because the method is nonautonomous and discontinuous within the specified interval $[t_0, t_c)$. Motivated by the linear time-response of $\varepsilon_2(t)$, we have explored an alternative approach to establish convergence, which involves employing a bounding region and utilizing the analytic solution of the algorithm in different segments of the prescribed interval.

We present a mathematical analysis in three parts to extend the results of Theorem 1 to disturbed systems. First, we propose formulating two new second-order systems perturbed by constant disturbances of amplitude L and $-L$. Second, we define the parameter conditions that ensure that the states of both systems converge to the origin at the prescribed time t_c. Finally, we demonstrate that the original system will also present prescribed-time convergence if we employ the same parameters that satisfy the conditions obtained in the second part of the analysis.

Consider the IVP consisting of (6)–(7) with a known initial condition $\boldsymbol{\varepsilon}(t_0) = (\varepsilon_1(t_0), \varepsilon_2(t_0)) \in \mathbb{R}^2$ and a bounded disturbance $|\delta| \leq L$, with known bound L. The bounding systems

$$\dot{\check{\varepsilon}}_1 = -\sigma(t, \check{\varepsilon}_1) + \check{\varepsilon}_2, \tag{29}$$

$$\dot{\check{\varepsilon}}_2 = -\kappa_2 \operatorname{sign}\left(\check{\varepsilon}_1\right) - L, \tag{30}$$

and

$$\dot{\hat{\varepsilon}}_1 = -\sigma(t, \hat{\varepsilon}_1) + \hat{\varepsilon}_2, \tag{31}$$

$$\dot{\hat{\varepsilon}}_2 = -\kappa_2 \operatorname{sign}\left(\hat{\varepsilon}_1\right) + L, \tag{32}$$

are obtained by replacing the disturbance $\delta(t)$ by its minimum and maximum values, respectively. Consider also that these bounding systems, as well as the original one (6)–(7), have the same initial conditions and parameters.

The following theorem demonstrates that there exist sufficient conditions for the parameters κ_2 and η which ensure that $\check{\boldsymbol{\varepsilon}}(t)$ and $\hat{\boldsymbol{\varepsilon}}(t)$ converge to the origin as t approaches t_c.

Theorem 2: Consider the IVPs consisting of (29)–(30) with a known initial condition $\check{\boldsymbol{\varepsilon}}(t_0) = \boldsymbol{\varepsilon}(t_0)$, and (31)–(32) with a known initial condition $\hat{\boldsymbol{\varepsilon}}(t_0) = \boldsymbol{\varepsilon}(t_0)$. Consider also a common parameter $\eta \geq 3$ for both IVPs. If the following conditions are satisfied

$$(\kappa_2 - L)(t_c - t_0) \geq |\varepsilon_2(t_0)|, \tag{33}$$

$$\begin{aligned} &-\kappa_2 \left(t_c - \check{t}_1 - \check{s}_{1,0} \left(\check{t}_1 - \check{t}_0\right)\right) + L \left(t_c - \check{t}_0\right) \\ &\qquad \leq \check{\varepsilon}_2(\check{t}_0) \leq \\ &\kappa_2 \left(t_c - \check{t}_1 + \check{s}_{1,0} \left(\check{t}_1 - \check{t}_0\right)\right) - L \left(t_c - 2\check{t}_1 + \check{t}_0\right). \end{aligned} \tag{34}$$

$$\begin{aligned} &-\kappa_2 \left(t_c - \hat{t}_1 - \hat{s}_{1,0} \left(\hat{t}_1 - \hat{t}_0\right)\right) + L \left(t_c - 2\hat{t}_1 + \hat{t}_0\right) \\ &\qquad \leq \hat{\varepsilon}_2(\hat{t}_0) \leq \\ &\kappa_2 \left(t_c - \hat{t}_1 + \hat{s}_{1,0} \left(\hat{t}_1 - \hat{t}_0\right)\right) - L \left(t_c - \hat{t}_0\right), \end{aligned} \tag{35}$$

then both $\check{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\varepsilon}}$ converge to zero at the prescribed time t_c.

Proof: The analytic solution to the IVP (29)–(30), in each interval $t \in [\check{t}_{j-1}, \check{t}_j)$, where $\check{t}_j$ is defined analogously to (9) considering the current system, can be obtained similarly to (11)–(12) as

$$\check{\varepsilon}_1(t) = \frac{t_c - t}{(\eta-2)(\eta-1)}\check{\alpha}(t) + \check{\beta}(\check{t}_{j-1})\left(\frac{t_c - t}{t_c - \check{t}_{j-1}}\right)^{\eta}, \quad (36)$$

$$\check{\varepsilon}_2(t) = \check{\varepsilon}_2(\check{t}_{j-1}) - \left(\kappa_2 \check{s}_{1,\,j-1} + L\right)\left(t - \check{t}_{j-1}\right), \quad (37)$$

where $\check{s}_{i,\,j-1} \triangleq \text{sign}(\check{\varepsilon}_i(\check{t}_{j-1}))$ and

$$\check{\alpha}(t) \triangleq \left(\kappa_2 \check{s}_{1,\,j-1} + L\right)(t_c - t - (\eta-2)(t - \check{t}_{j-1})) + (\eta-2)\check{\varepsilon}_2(\check{t}_{j-1}), \quad (38)$$

$$\check{\beta}(\tau) \triangleq \check{\varepsilon}_1(\tau) - \frac{t_c - \tau}{\eta - 1}\check{\varepsilon}_2(\tau) - \frac{(t_c-\tau)^2}{(\eta-2)(\eta-1)}\left(\kappa_2 \check{s}_{1,\,j-1} + L\right). \quad (39)$$

Since the analysis in the proof of Corollary 1 is also valid for system (29)–(30), it is also true that $\check{\varepsilon}_1(t)$ approaches zero as t approaches t_c.

Similarly to the proof of Theorem 1, let us consider a geometric set $\mathcal{E}^+(\kappa_d)$ defined as

$$\mathcal{E}^+(\kappa_d) \triangleq \{(t, x(t)) \,:\, -\kappa_d\,(t_c - t) \le x(t) \le \kappa_d\,(t_c - t)\}, \quad (40)$$

where $x(t)$ is the evaluated state, and $\kappa_d \triangleq \kappa_2 - L$. Once again, by ensuring that $\left(\check{t}_{j-1}, \check{\varepsilon}_2(\check{t}_{j-1})\right) \in \mathcal{E}^+(\kappa_d)$ for all $j = 1, \ldots, n_r$, we guarantee that $\lim_{t\to t_c^-} \check{\varepsilon}_2(t) = 0$.

From (40) and the known values of $\check{\varepsilon}_2(\check{t}_0)$ and L, by choosing κ_2 satisfying

$$(\kappa_2 - L)(t_c - \check{t}_0) \ge |\check{\varepsilon}_2(\check{t}_0)|, \quad (41)$$

the pair $\left(\check{t}_0, \check{\varepsilon}_2(t_0)\right)$ will belong to $\mathcal{E}^+(\kappa_d)$.

From (37), we can evaluate the value of $\check{\varepsilon}_2(\check{t}_1)$ as

$$\check{\varepsilon}_2(\check{t}_1) = \check{\varepsilon}_2(\check{t}_0) - \left(\kappa_2 \check{s}_{1,0} + L\right)\left(\check{t}_1 - \check{t}_0\right). \quad (42)$$

To ensure that $\left(t, \check{\varepsilon}_2(t)\right) \in \mathcal{E}^+(\kappa_d)$ at $\check{t}_1$, it suffices to use the definition of $\mathcal{E}^+(\kappa_d)$ to obtain

$$-\kappa_d(t_c - \check{t}_1) \le \check{\varepsilon}_2(\check{t}_1) \le \kappa_d(t_c - \check{t}_1), \quad (43)$$

which can be expanded using (42), yielding

$$-\kappa_2\left(t_c - \check{t}_1 - \check{s}_{1,0}\left(\check{t}_1 - \check{t}_0\right)\right) + L\left(t_c - \check{t}_0\right) \le \check{\varepsilon}_2(\check{t}_0) \le \kappa_2\left(t_c - \check{t}_1 + \check{s}_{1,0}\left(\check{t}_1 - \check{t}_0\right)\right) - L\left(t_c - 2\check{t}_1 + \check{t}_0\right). \quad (44)$$

The instant $\check{t}_1$ can be found by solving (36) for t, considering $\check{\varepsilon}_1(t) = 0$ as well as parameters $\eta \ge 3$ and κ_2 satisfying equations (5) and (41). Similar to the undisturbed case, we ensure that $\left(\check{t}_1, \check{\varepsilon}_2(\check{t}_1)\right) \in \mathcal{E}^+(\kappa_d)$ if η and κ_2 are such that (44) holds true. Analogously, it can be shown by induction that these parameters also ensure that

$$-\kappa_d(t_c - \check{t}_j) \le \check{\varepsilon}_2(\check{t}_0) - \sum_{m=1}^{j}\left(\kappa_2 \check{s}_{1,m-1} + L\right)(t_m - t_{m-1}) \le \kappa_d(t_c - \check{t}_j) \quad (45)$$

for every instant $\check{t}_j$ obtained by inverting (36) considering $\check{\varepsilon}_1(t) = 0$. Therefore, there exist parameters η and κ_2 such that $\left(t, \check{\varepsilon}_2(t)\right) \in \mathcal{E}^+(\kappa_d)$, $\forall t \in [t_0, t_c)$.

The prescribed-time convergence of $\hat{\varepsilon}_1(t)$ and existence of parameters that guarantee $\left(t, \hat{\varepsilon}_2(t)\right) \in \mathcal{E}^+(\kappa_d)$, $\forall t \in [t_0, t_c)$ can be analogously deduced. In this case, it suffices to satisfy the following inequality instead of (44)

$$-\kappa_2\left(t_c - \hat{t}_1 - \hat{s}_{1,0}\left(\hat{t}_1 - \hat{t}_0\right)\right) + L\left(t_c - 2\hat{t}_1 + \hat{t}_0\right) \le \hat{\varepsilon}_2(\hat{t}_0) \le \kappa_2\left(t_c - \hat{t}_1 + \hat{s}_{1,0}\left(\hat{t}_1 - \hat{t}_0\right)\right) - L\left(t_c - \hat{t}_0\right), \quad (46)$$

where $\hat{t}_1$ is obtained by inverting

$$\hat{\varepsilon}_1(t) = \frac{t_c - t}{(\eta-2)(\eta-1)}\hat{\alpha}(t) + \hat{\beta}(\hat{t}_0)\left(\frac{t_c - t}{t_c - \hat{t}_0}\right)^{\eta}, \quad (47)$$

considering $\hat{\varepsilon}_1(t) = 0$, and defining $\hat{s}_{1,0} \triangleq \text{sign}(\hat{\varepsilon}_1(\hat{t}_0))$ and

$$\hat{\alpha}(t) \triangleq \left(\kappa_2 \hat{s}_{1,0} - L\right)(t_c - t - (\eta-2)(t - \hat{t}_0)) + (\eta-2)\hat{\varepsilon}_2(\hat{t}_0), \quad (48)$$

$$\hat{\beta}(\hat{t}_0) \triangleq \hat{\varepsilon}_1(\hat{t}_0) - \frac{t_c - \hat{t}_0}{\eta-1}\hat{\varepsilon}_2(\hat{t}_0) - \frac{(t_c - \hat{t}_0)^2}{(\eta-2)(\eta-1)}\left(\kappa_2 \hat{s}_{1,0} - L\right). \quad (49)$$

Hence, if η and κ_2 are chosen to simultaneously satisfy (44) and (46), then the solutions $\check{\varepsilon}_2(t)$ and $\hat{\varepsilon}_2(t)$ will remain in $\mathcal{E}^+(\kappa_d)$. □

Corollary 3: Consider the geometric set $\mathcal{E}^+(\kappa_d)$, and the IVPs (6)–(7), (29)–(30), and (31)–(32). The parameters which ensure the prescribed-time convergence of $\check{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\varepsilon}}$ also ensure the prescribed-time convergence of $\boldsymbol{\varepsilon}$.

Proof: The proof of Theorem 2 readily implies that if the solutions of the IVPs for two systems, subject to constant disturbances whose amplitudes correspond to the maximum and minimum values of an unknown disturbance δ, remain bounded by the geometric set $\mathcal{E}^+(\kappa_d)$, then the solution of the IVP for a system affected by the intermediate disturbance δ will also remain bounded by this set. □

It is possible to obtain functions that delimit a similar geometric set for $(t, \varepsilon_1(t))$ by replacing $\varepsilon_2(t)$ in (6) by the upper and lower line segments that bound $\mathcal{E}^+(\kappa_d)$. Denote this set by $\mathcal{Z}(\kappa_d) \triangleq \{(t, \varepsilon_1(t)) \,:\, -\lambda(t, \varepsilon_1) \le \varepsilon_1(t) \le \lambda(t, \varepsilon_1)\}$, where $\lambda(t, \varepsilon_1)$ is given by

$$\lambda(t, \varepsilon_1) \triangleq \left(|\varepsilon_1(t_0)| - \frac{\kappa_d(t_c - t_0)^2}{\eta - 2}\right)\left(\frac{t_c - t}{t_c - t_0}\right)^{\eta} + \frac{\kappa_d(t_c - t)^2}{\eta - 2}. \quad (50)$$

Thus, if inequalities (44) and (46) are satisfied, then the trajectories of ε_1 and ε_2 are confined in $\mathcal{Z}(\kappa_d)$ and $\mathcal{E}^+(\kappa_d)$, respectively, for any instant $t \in [t_0, t_c)$.

Corollary 4: Consider the σ-function in (8) and the geometric sets $\mathcal{Z}(\kappa_d)$ and $\mathcal{E}^+(\kappa_d)$. It holds that triggering the behavior switch of σ at an instant t slightly earlier than t_c

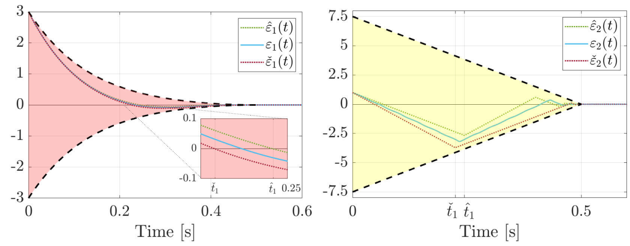

FIGURE 1. State trajectories of the modified STA under different values of disturbance, with the geometric sets $\mathcal{Z}(\kappa_d)$ shaded in red, and $\mathcal{E}^+(\kappa_d)$ shaded in yellow.

TABLE 1. Simulation parameters.

Symbol	Description	Value
t_0	Initial time	0 s
Ts	Integration step	0.0001 s
l	Pendulum length	3 m
m	Pendulum mass	1 kg
g	Gravity acceleration	9.81 m/s^2
b	Friction coefficient	1 Nms
$x_1(t_0)$	Initial angle	0.5 rad
$x_2(t_0)$	Initial angular velocity	0 rad/s
$\hat{x}_1(t_0)$	Initial estimated state	0 rad
$\hat{x}_2(t_0)$	Initial estimated state	0 rad/s

is equivalent to starting a conventional STA with very small initial conditions.

Proof: Consider an instant $t_e = (1 - \Delta)t_c$, where $\Delta \ll 1$ is a user-defined correction factor. With the proper choice of parameters, the time responses of ε_1 and ε_2 will be respectively enveloped by $\mathcal{Z}(\kappa_d)$ and $\mathcal{E}^+(\kappa_d)$ for all $t \in [t_0, t_c)$. Therefore, given that the functions that delimit these sets monotonically converge to the origin as t approaches t_c, as illustrated by the dashed lines in Figure 1, it is evident that any function enclosed by these sets will present a value very close to the origin at an instant close to t_c. From this instant forward, the conventional STA ensures the finite-time convergence of the system states to the origin. □

Example 1: Consider the systems (6)–(7), (29)–(30) and (31)–(32), with initial conditions $\boldsymbol{\varepsilon}(t_0) = \check{\boldsymbol{\varepsilon}}(t_0) = \hat{\boldsymbol{\varepsilon}}(t_0) = (3, 1)$ and $t_0 = \check{t}_0 = \hat{t}_0 = 0$. Consider that the original system is disturbed by $\delta(t) = L\sin(50\pi t)$, with $L = 3$. Figure 1 illustrates the simulated state trajectories of the three systems with parameters $t_c = 0.5$ s, $\eta = 5$, $\kappa_1 = 5$, and $\kappa_2 = 18$, which satisfy equations (5), (41), (44), and (46).

IV. STATE ESTIMATION OF A DISTURBED PENDULUM

In this section, the proposed algorithm is employed in formulating a state estimator for a free damped pendulum affected by a disturbance torque.

A. OBSERVER FORMULATION

The pendulum dynamics are described by

$$\ddot{\theta} = -a\sin(\theta) - \gamma\dot{\theta} + \frac{\nu}{ml^2}, \tag{51}$$

where $\theta \in \mathbb{R}$, and $\dot{\theta} \in \mathbb{R}$ are the pendulum's angular position and velocity, respectively, $a \triangleq g/l$, $\gamma \triangleq b/\left(ml^2\right)$, m is the pendulum mass, l is its length, g is the gravity acceleration, b is the friction coefficient, and $\nu \in [-\rho, \rho]$ is a bounded disturbance torque.

By defining the state vector $\boldsymbol{x} \triangleq (x_1, x_2) \in \mathbb{R}^2$, with $x_1 \triangleq \theta$ and $x_2 \triangleq \dot{\theta}$, and the disturbance input $d \triangleq \nu/\left(ml^2\right) \in \mathbb{R}$, we can rewrite (51) as

$$\dot{x}_1 = x_2, \tag{52}$$

$$\dot{x}_2 = f(\boldsymbol{x}) + d, \tag{53}$$

where $f(\boldsymbol{x}) \triangleq -a\sin(x_1) - \gamma x_2$. Let us assume that d is bounded according to $d \in \left[-\rho/ml^2, \rho/ml^2\right]$.

Denote the state estimate by $\hat{\boldsymbol{x}} \triangleq \left(\hat{x}_1, \hat{x}_2\right) \in \mathbb{R}^2$ and define the estimation error as $\boldsymbol{\varepsilon} \triangleq \boldsymbol{x} - \hat{\boldsymbol{x}}$. By substituting the estimation error definition in (6)–(7), considering the pendulum dynamics in (52)–(53), the modified super-twisting sliding-mode observer (ModSTSMO) is given by

$$\dot{\hat{x}}_1 = \sigma\left(t, \left(x_1 - \hat{x}_1\right)\right) + \hat{x}_2, \tag{54}$$

$$\dot{\hat{x}}_2 = f(\hat{\boldsymbol{x}}) + \kappa_2 \operatorname{sign}(\varepsilon_1). \tag{55}$$

In this case, the disturbance term in (7) is represented by $\delta \equiv \left(f(\boldsymbol{x}) - f(\hat{\boldsymbol{x}})\right) + d$, thus containing both the model uncertainty and the external disturbance. Let us assume that δ is bounded by $\delta \in [-L, L]$, with known $L < \infty$. The conventional super-twisting sliding-mode observer (STSMO) is analogously obtained by substituting the estimation error definition into (3)–(4) instead.

B. SIMULATION RESULTS

The simulation study is conducted in MATLAB using the first-order explicit Euler method. First, we present a numerical verification of the proposed observer's robustness and prescribed-time convergence properties. Next, we present a comparative analysis between our proposed observer and another state-of-the-art prescribed-time convergent observer. Table 1 contains the adopted parameters.

The pendulum starts from rest and is disturbed by $\nu = 0.8\rho\sin(2\pi t) + 0.2\rho\sin(20\pi t)$, where $\rho \triangleq ml^2$, implying that $d \in [-1, 1]$. To account for the bounds of the external disturbance and model uncertainties in δ, we adopt $L = 3$. Consider the observer parameters $\kappa_1 = 6$, $\kappa_2 = 12$, $\eta = 5$, and $t_c = 0.5$ s. Figures 2–3 show the system states, their estimates, and the estimation error, respectively. Verifying that the chosen parameters satisfy the gain criteria (5) and inequality (41) is straightforward. Also, by following the procedure in Theorem 2, inequality (44) is reduced to $-1.04 \leq \check{\varepsilon}_2(t_0) \leq 5.37$, whereas inequality (46) leads to $-1.51 \leq \hat{\varepsilon}_2(t_0) \leq 4.5$. Since $\varepsilon_2(t_0) = \hat{\varepsilon}_2(t_0) = \check{\varepsilon}_2(t_0) = 0.5$ rad, both inequalities hold true.

The previous simulation is repeated with the same parameters, except for the convergence instant, now set to $t_c = 1$ s. The results are shown in Figures 4–5. In all the simulated cases, the prescribed instant t_c is the exact settling instant, and the states remain stable thereafter.

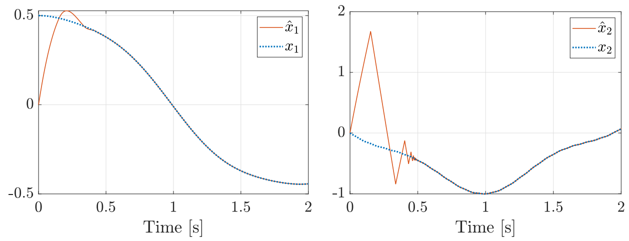

FIGURE 2. **Time response of the system states and its estimates obtained with the proposed observer set with $t_c = 0.5$ s.**

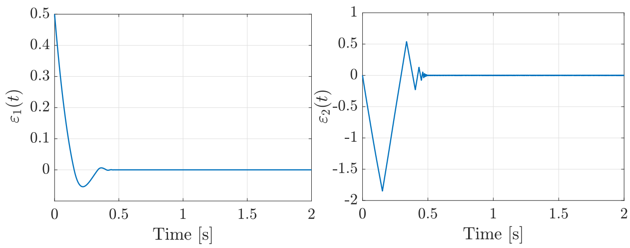

FIGURE 3. **Time response of the state estimation error obtained with the proposed observer set with $t_c = 0.5$ s.**

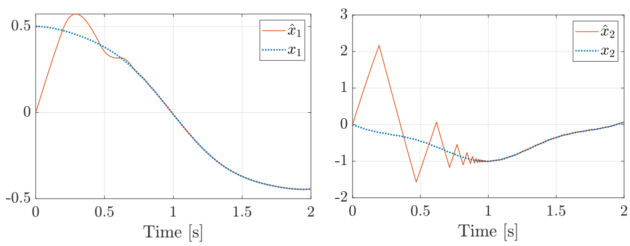

FIGURE 4. **Time response of the system states and its estimates obtained with the proposed observer set with $t_c = 1$ s.**

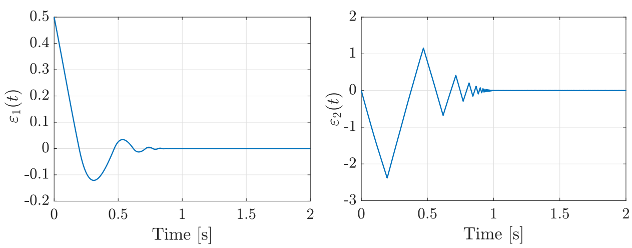

FIGURE 5. **Time response of the state estimation error obtained with the proposed observer set with $t_c = 1$ s.**

Now, the proposed observer is compared to one of the most recently developed methods for second-order estimators with prescribed-time convergence, named here as prescribed-time observer (PTO) [13]. This method uses time-varying gains in both component state equations to drive the system to the origin at precisely the specified instant. As the PTO has been defined only in $[t_0, t_c)$, for fairness of comparison, here we extend it with the conventional STSMO in $t \geq t_c$, yielding

$$\dot{\hat{x}}_1 = g_1(t, \varepsilon_1) + \hat{x}_2, \tag{56}$$

$$\dot{\hat{x}}_2 = g_2(t, \varepsilon_1) + f\left(\hat{\boldsymbol{x}}\right), \tag{57}$$

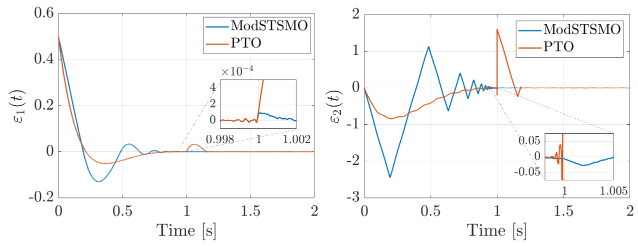

FIGURE 6. **Comparative simulations: time response of the ModSTSMO and PTO estimation errors with $t_c = 1$ s.**

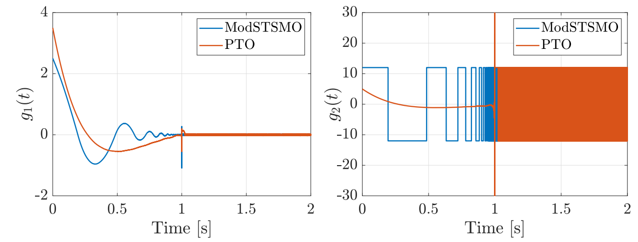

FIGURE 7. **Comparative simulations: time response of the ModSTSMO and PTO injection terms, with $t_c = 1$ s.**

where the injection terms are

$$g_1(t, \varepsilon_1) \triangleq \begin{cases} \left(\ell_1 + 2\dfrac{m+2}{t_c - t}\right)\varepsilon_1, & t \in [t_0, t_c), \\ \kappa_1 \lfloor \varepsilon_1 \rceil^{1/2}, & t \in [t_c, \infty), \end{cases} \tag{58}$$

$$g_2(t, \varepsilon_1) \triangleq \begin{cases} \left(\ell_2 + \ell_1 \dfrac{m+2}{t_c - t} + \dfrac{(m+1)(m+2)}{(t_c - t)^2}\right)\varepsilon_1, \\ \qquad t \in [t_0, t_c), \\ \kappa_2 \operatorname{sign}(\varepsilon_1), \quad t \in [t_c, \infty). \end{cases} \tag{59}$$

The following simulations consider that the angle measurements contain additive zero-mean truncated Gaussian noise with a standard deviation of 10^{-5} on support $\left[-5 \times 10^{-5}, 5 \times 10^{-5}\right]$. Also, to compare the injection terms, we consider that for the proposed method, $g_1(t) = \sigma(t, \varepsilon_1)$ and $g_2(t) = \kappa_2 \operatorname{sign}(\varepsilon_1)$. Considering the parameters $\eta = 5$, $\kappa_1 = 6$, $\kappa_2 = 12$, $t_c = 1$ s, $\ell_1 = 3$, $\ell_2 = 2$, and $m = 1$, we obtain the results shown in Figures 6–7.

The PTO method presents numerical problems near the prescribed instant t_c, as the gains go to infinity while multiplied by a noisy estimation error [13]. Although a similar behavior can be observed in the proposed method, the PTO is shown to be much more susceptible to this effect, as it uses time-varying gains to multiply the estimation error in both observer equations. To mitigate this problem, we adopt an earlier switching instant to the conventional STSMO behavior, as introduced in Corollary (4). With a correction factor $\Delta = 0.005$, we trigger the behavior switch at $t_e = 0.995$ s. Although this approach sacrifices the exactness of the converging instant, Figures 8–9 show that this change eliminates the state divergence caused by the measurement error.

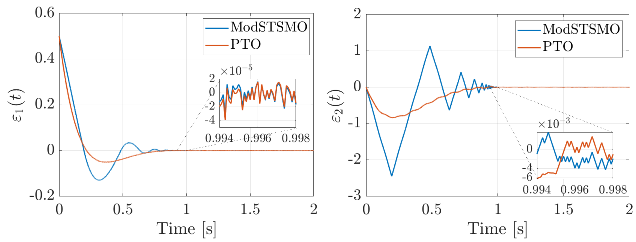

FIGURE 8. Comparative simulations: time response of the ModSTSMO and PTO estimation errors with $t_c = 1$ s, considering an earlier switching instant $t_e = 0.995$ s.

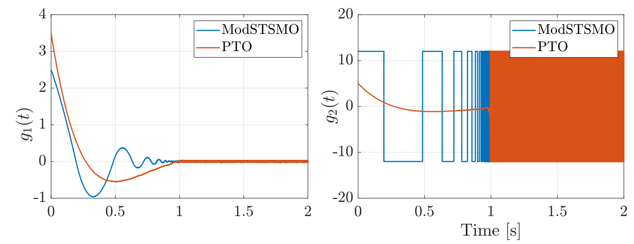

FIGURE 9. Comparative simulations: time response of the ModSTSMO and PTO injection terms with $t_c = 1$ s, considering an earlier switching instant $t_e = 0.995$ s.

V. CONCLUSION

The present paper introduced a modification of the super-twisting algorithm. It showed that, with the appropriate selection of parameters, the transitory behavior of its states remains enveloped by time functions that approach the origin as time approaches the prescribed instant of convergence. Compared to a similar algorithm, our method shows convergence at the prescribed instant, maintaining robust stability thereafter, with a minor sensitivity to inaccurate measurements. In future works, the effectiveness of the proposed method will be evaluated in practical scenarios.

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