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RESEARCH ARTICLE

Synchronization of Multiplex Networks With Stochastic Perturbations via Intermittent Control

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ABSTRACT This paper investigates the synchronization problem for multiplex networks with the stochastic perturbations via intermittent control. In the control schemes, the topologies of different layers of multiplex networks can be different, the control strategy adopts aperiodically intermittent and pinning control methods, and the number of pinned nodes can be different in different layers. Based on the stochastic theory and the Lyapunov stability theory, linear feedback and adaptive controllers are constructed respectively, some sufficient conditions are derived for guaranteeing synchronization of multiplex networks. Furthermore, for the different connectivity of multiplex networks, the conditions for achieving synchronization are discussed. Finally, two numerical simulations are provided to verify the theoretical results.

INDEX TERMS Multiplex networks, synchronization, stochastic perturbations, intermittent control.

I. INTRODUCTION

As an effective tool for studying complex systems, complex networks (CNs) attract extensive attentions from researchers of various fields of science and engineering. Collective dynamic behaviors of CNs such as synchronization of CNs has many potential applications in multi-agent coordinated control and secure communication, and has therefore been widely investigated widely. In general, CNs can not reach synchronization without control, so some control approaches have been proposed to drive CNs to achieve synchronization. However, from a practical perspective, it is very difficult to control all nodes in a large-scale network. Hence, it is highly necessary to develop nodes-based pinning control method for synchronization of CNs. Wang and Chen [1] introduced pinning control method to study synchronization of CNs firstly, since then, the pinning control method has been commonly used in synchronization problem. On the other hand, for reducing the requirements of system hardware and the consumption of communication resources, scholars have proposed some control methods on the discrete signals updated at instant times, such as intermittent control [2], [3], [4], [5], [6], [7], event-triggered control [8], [9], [10], [11], sampled-data control [12], [13], [14] and impulsive control [15], [16], [17].

So far, research of the synchronization of CNs mainly focused on single-layer networks. However, many networks have multiple links in reality, which have more than one layer, and their structure and dynamic behavior are more complicated. For example, in transportation networks, there may be roads, railways, and air routes among cities, in the social networks, people exchange information with each other in many different ways, such as WeChat, QQ, Facebook, E-mail, etc. Such a type of CNs with multi-links have many names, for example, multiplex networks (MNs) [18], [19], [20], multilayer networks [21], [22], [23], multiweighted networks (MWNs) [24], [25], [26], etc. Generally speaking, multilayer networks refer to networks where the node states corresponding to different layers of the network are not the same. And MWNs are networks where there are multiple

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links between nodes, or the weights of the links can be any real number. In recent years, the study of multiplex networks has been a hot topic because of having many applications in some fields. In [27], the public traffic network was took as a MWN, which regarded bus lines as the network nodes, then the synchronization of complex public traffic network with multi-weights was studied. Du et al. [28] proposed a model of a two-layer-coupled public bus and subway traffic, and studied its synchronization problem. The types of synchronization of MNs include intra-layer synchronization [29], [30], inter-layer synchronization [29], [31], complete synchronization [32], [33], and outer synchronization [34], [35], [36] etc. Wherein, the outer synchronization refers to synchronization between different MNs.

Intermittent control strategy implements discontinuous control on systems in the time domain. Compared with the continuous control, this control strategy can reduce control costs. The intermittent control strategy has been widely used in the research of traditional synchronization of CNs [2], [3], [4], [5], [6], [7]. But research on the synchronization of MNs via intermittent control is few. Yi et al [25] studied the synchronization issue of delayed neural networks with multi-weights under aperiodically intermittent pinning control, for the proposed neural network models with the internal delay and coupling delay, derived criteria to guarantee exponential synchronization. Liang et al [35] studied the outer synchronization of multilayer complex networks by intermittent control, designed intermittent controllers in the drive and response configuration, and derived sufficient conditions to achieve the outer synchronization.

In the previous studies of the synchronization of CNs, it was mostly assumed that node dynamics did not involve noise interference. However, in some uncertain environments, the node dynamics are often affected by stochastic perturbations [5], [18], [20], [22]. Thus, it is necessary to take into account the noise effect on the synchronization. Zhao et al. [18] studied the synchronization of MNs with multiple delays and stochastic perturbations, based on the LaSalle-type invariance principle and the Lyapunov stability theory, obtained some pinning synchronization criteria. Jin et al. [20] investigated the adaptive synchronization problem of MNs with stochastic perturbations via pinning control, under the conditions of transmission delay and no delay, derived some pinning criteria for guaranteeing the complete synchronization, respectively. For MNs with stochastic perturbations via pinning control, Zhuang et al. [22] was concerned with the pinning synchronization of delayed multilayer networks with stochastic perturbations, established some sufficient conditions for guaranteeing the synchronization under control input and no control input. Similar to the above MNs model, subsequently Zhuang et al. [34] discussed the pinning synchronization of a kind of drive-response multilayer networks with stochastic perturbations, designed the state-feedback pinning and the adaptive pinning controller, and derived some sufficient conditions to reach the synchronization.

On the basis of the above, this paper studies the synchronization problem of MNs with the stochastic perturbations via aperiodically intermittent and pinning control. The main contributions of the paper can be highlighted in three aspects as follows:

1) To the best of our knowledge, the aperiodically intermittent and pinning methods are firstly proposed to solve the synchronization problem of MNs with stochastic perturbations. Under no perturbation conditions, reference [25] studied the synchronization issue of delayed neural networks with multi-weights under aperiodically intermittent pinning control.

2) For the synchronization problem of MNs with stochastic perturbations in [20] and [22], the synchronization of MNs was investigated by using pinning control. Compared to these two research efforts, this paper adopts aperiodically intermittent and pinning control methods to study the synchronization of MNs with stochastic perturbations, and obtains criteria for guaranteeing the complete synchronization of MNs with state feedback and adaptive controllers.

3) For the different connectivity of MNs, the paper analyzes the relationship between the number of connected branches and the minimum number of pinned nodes in different layers of MNs, discusses some conditions for achieving the synchronization.

The rest of this paper is organized as follows. Some necessary preliminaries, Assumptions, Lemmas, and model description are given in Section II. In Section III, state feedback and adaptive controllers are designed, some sufficient conditions for achieving the synchronization are derived separately. Two numerical examples are presented in Section IV. Finally, conclusions of the paper are drawn.

II. PRELIMINARIES

In the paper, the following notations are adopted. $\mathcal{N}$ is the set of natural numbers, R^n and $R^{n\times n}$ denote n-dimensional real column vectors and n-dimensional real square matrices, respectively. I_n is an identity matrix with order n. R^+ denotes the nonnegative real numbers. The superscript T represents the transpose operation to a corresponding matrix (or vector). $\lambda_{\max}(A)$ represents the maximum eigenvalue of matrix A. $\|\cdot\|$ is the Euclidean norm of a vector in R^n. For all the real-valued functions $V(t, x(t))$ on $R^+ \times R^n$, $C^{2,1}(R^+ \times R^n, R^+)$ stands for the family of $V(t, x(t))$, which is continuously twice differentiable for $x(t) \in R^n$ and once differentiable for $t \in R^+$. For two symmetric matrices P and Q, $P< 0$ means that the matrix P is a negative definite matrix; $P \leq Q$ means that the matrix P - Q is a negative semi-definite matrix.

A. RELATED LEMMAS AND ASSUMPTIONS

Consider the following stochastic differential equation:

$$dx(t) = f(x(t))dt + \varphi(t, x(t))d\omega(t), \tag{1}$$

with initial value $x(t) \in R^n$, $t \geq 0$, $f(x(t)) \in R^n$ is a continuous vector function; $\varphi(t, x(t)) \in R^{n\times n}$ is a noise intensity matrix and $\omega(t) \in R^n$ is a bounded vector-form Weiner process.

Lemma 1 ([5]): For $V(t, x(t)) \in C^{2,1}(R^+ \times R^n, R^+)$, define an operator $\mathcal{L}V(t, x(t))$ from $R^+ \times R^n$ to R^1 by

$$\mathcal{L}V(t,x(t)) = V_t(t,x(t)) + V_x(t,x(t))f(x(t)) + \frac{1}{2}trace(\varphi^T(t,x(t))V_{xx}(t,x(t))\varphi(t,x(t))), \tag{2}$$

where $V_t(t,x(t)) = \partial V(t,x(t))/\partial t$, $V_x(t,x(t)) = (\partial V(t,x(t))/\partial x_1, \partial V(t,x(t))/\partial x_2, \ldots, \partial V(t,x(t))/\partial x_n)$, $V_{xx}(t,x(t)) = (\partial^2 V(t,x(t))/\partial x_i \partial x_j)_{n\times n}$. If $E\int_{t_0}^{t} \mathcal{L}V(t,s(t))ds$ exits, then for all $t \in [t_0, +\infty)$, $t_0 \geq 0$,

$$EV(t) = EV(t_0) + E\int_{t_0}^{t} \mathcal{L}V(t,s(t))ds. \tag{3}$$

Lemma 2 ([5]): For any $X, Y \in R^n$, a positive real number α and $P \in R^{n\times n}$, such that

$$2X^TPY \leq \alpha^{-1}X^TPP^TX + \alpha Y^TY. \tag{4}$$

Lemma 3 ([37]): Let the eigenvalues of matrix A be $\lambda_1, \lambda_2, \ldots, \lambda_n$, the eigenvalues of matrix B be $\mu_1, \mu_2, \ldots, \mu_m$, then eigenvalues of matrix $A \otimes B$ are $\lambda_i\mu_j$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Assumption 1: There exists a $\rho > 0$, for $x, y \in R^n$ and a matrix $\Omega \in R^{n\times n}$, such that $f(\cdot)$ satisfies

$$(x-y)^T(f(x) - f(y)) \leq \rho(x-y)^T\Omega(x-y), \tag{5}$$

where $\Omega = diag(\omega_1, \omega_2, \ldots, \omega_n)$, ω_i is a nonnegative constant, $i = 1, 2, \ldots, n$.

Assumption 2: There exists a $\sigma > 0$, for $x, y \in R^n$, such that $\varphi(\cdot)$ satisfies

$$\begin{aligned} &trace((\varphi(t,x) - \varphi(t,y))^T(\varphi(t,x) - \varphi(t,y))) \\ &\leq 2\sigma(x-y)^T(x-y). \end{aligned} \tag{6}$$

For the aperiodically intermittent control strategy, we will give the following assumption.

Assumption 3: There exits a positive scalar $0 < d_i < T_i < +\infty$, such that

$$\begin{cases} \inf_i(s_i - t_i) = d_i \\ \sup_i(t_{i+1} - t_i) = T_i, \end{cases} \tag{7}$$

where t_i is a time series, T_i is total time width in the i-th time interval, d_i is control width in the i-th time interval. In the paper, Assume that the control ratio (control width d_i to total time width T_i) in any time interval is a real constant $r = d_i/T_i (0 < r < 1)$, $i \in \mathcal{N}$.

B. MODEL DESCRIPTION

Consider a M-layer MN with stochastic perturbations, is described by

$$\begin{aligned} dx_i(t) = [&f(x_i(t)) + \sum_{k=1}^{M} c_k \sum_{j=1}^{N} a_{ij}^{(k)} H^{(k)}(x_j(t) - x_i(t)) \\ &+ u_i(t)]dt + \varphi(x_i(t))d\omega(t), \end{aligned} \tag{8}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^T \in R^n$ is a state vectors of the i-th node of NMs, $i = 1, 2, \ldots, N$, N is the number of nodes in each layer; $f(\cdot) : R^n \to R^n$ is a smooth nonlinear vector field, c_k is the strength of coupling contributed by the k-th layer; $A^{(k)} = (a_{ij}^{(k)})_{N\times N}$ is the outer coupling matrix, if there is a coupling link from node i to node j $(j \neq i)$, then $a_{ij}^{(k)} > 0$, otherwise, $a_{ij}^{(k)} = 0$, which satisfies the diffusion property $\sum_{j=1}^{N} a_{ij}^{(k)} = \sum_{j=1}^{N} a_{ji}^{(k)} = 0$; $H^{(k)} = diag(h_1^{(k)}, h_2^{(k)}, \ldots, h_n^{(k)})$ is an inner coupling matrix in each layer, $h_j^{(k)} > 0$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, M$; $u_i(t)$ is an intermittent pinning controller in i-th node (the pinned node); $\varphi(\cdot) \in R^{n\times n}$ and $\omega(\cdot) \in R^n$ are defined as Eq.(1).

From the properties of the diffusive matrix, Eq. (8) can be rewritten as

$$\begin{aligned} dx_i(t) = [&f(x_i(t)) + \sum_{k=1}^{M} c_k \sum_{j=1}^{N} a_{ij}^{(k)} H^{(k)} x_j(t) \\ &+ u_i(t)]dt + \varphi(x_i(t))d\omega(t). \end{aligned} \tag{9}$$

Defining the desired synchronization state as $x_0(t)$, it satisfies

$$dx_0(t) = f(x_0(t))dt + \varphi(x_0(t))d\omega(t). \tag{10}$$

Error vector is defined as $e_i(t) = x_i(t) - x_0(t)$, $i = 1, 2, \ldots, N$.

Then the system error:

$$\begin{aligned} de_i(t) = [&f(x_i(t)) - f(x_0(t)) + \sum_{k=1}^{M} c_k \sum_{j=1}^{N} a_{ij}^{(k)} H^{(k)} e_j(t) \\ &+ u_i(t)]dt + (\varphi(x_i(t) - \varphi(x_0(t))d\omega(t). \end{aligned} \tag{11}$$

Definition 1: The multiplex network (8) achieves synchronization, if $\lim_{t\to\infty} \|Ee_i(t)\| = 0$, $i = 1, 2, \ldots, N$.

III. MAIN RESULTS

A. THE SYNCHRONIZATION OF MNS WITH A STATE-FEEDBACK INTERMITTENT PINNING CONTROLER

In this section, we study synchronization of MNs under a state-feedback intermittent pinning controller, where different layers of MNs choose to pin different numbers of nodes, and let the number of pinned nodes be l_k in the k-th layer.

A state-feedback intermittent pinning controller in the i-th node is described by

$$u_i(t) = -\sum_{k}^{M} c_k \gamma_{l_i}^{(k)} H^{(k)} e_i(t), \tag{12}$$

where $\gamma_{l_i}^{(k)} > 0$ is a control parameter, when $i \geq l_k, \gamma_{l_i}^{(k)} = 0$. From Eq. (11), one obtains

$$de_i(t) = \begin{cases} [f(x_i(t)) - f(x_0(t)) + \sum_{k=1}^{M} c_k \sum_{j=1}^{N} a_{ij}^{(k)} H^{(k)} e_j(t) \\ - \sum_{k=1}^{M} c_k \gamma_l^{(k)} H^{(k)} e_i(t)]dt + (\varphi(x_i(t) \\ -\varphi(x_0(t))d\omega(t), 1 \leq i \leq \bar{l}, t \in [t_i, s_i) \\ [f(x_i(t)) - f(x_0(t)) + \sum_{k=1}^{M} c_k \sum_{j=1}^{N} a_{ij}^{(k)} H^{(k)} e_j(t) \\ ]dt + (\varphi(x_i(t) - \varphi(x_0(t))d\omega(t), \\ \bar{l} < i, or\ 1 \leq i \leq \bar{l}, t \in [s_i, t_{i+1}) \end{cases} \tag{13}$$

where $\bar{l} = \max\{l_k | k = 1, 2, \ldots, M\}$. Denote $\Gamma^{(k)} = diag(\gamma_1^{(k)}, \gamma_2^{(k)}, \ldots, \gamma_{l_k}^{(k)}, 0, \ldots, 0)_{N\times N}$, $e(t) = [e_1^T(t), e_2^T(t), \ldots, e_N^T(t)]^T$, $F_i(t, e_i(t)) = f(x_i(t)) - f(x_0(t))$, $F(t, e(t)) = [F_1^T(t, e_1(t)), F_2^T(t, e_2(t)), \ldots, F_N^T(t, e_N(t))]^T$, $\Phi_i(t, e_i(t)) = (\varphi(x_i(t) - \varphi(x_0(t))$, and $\Phi(t, e(t)) = [\Phi_1^T(t, e_1(t)), \Phi_2^T(t, e_2(t)), \ldots, \Phi_N^T(t, e_N(t))]^T$.

Then Eq. (13) can be rewritten in the following compact form:

$$de(t) = \begin{cases} [F(t, e(t)) + (\sum_{k=1}^{M} c_k((A^{(k)} - \Gamma^{(k)}) \otimes H^{(k)}))e(t)]dt \\ +\Phi(t, e(t))d\omega(t), \\ 1 \leq i \leq \bar{l}, t \in [t_i, t_i + d_i) \\ [F(t, e(t)) + (\sum_{k=1}^{M} c_k(A^{(k)} \otimes H^{(k)}))e(t)]dt \\ +\Phi(t, e(t))d\omega(t), \bar{l} < i, \\ or 1 \leq i \leq \bar{l}, t \in [t_i + d_i, t_{i+1}) \end{cases} \tag{14}$$

Theorem 1: Under the Assumptions 1-3, the synchronization of MNs with stochastic perturbations via aperiodically intermittent pinning controller (12) can be realized, if $\lim_{j\to\infty} [q_1 \sum_{k=0}^{j} d_k + q_2 \sum_{k=0}^{j} T_k(1 - r)] = -\infty$, where $q_1 = 2\lambda_{\max}[\rho(I_N \otimes \Omega) + \sigma(I_N \otimes I_n) + \sum_{k=1}^{M} c_k((A^{(k)} - \Gamma^{(k)}) \otimes H^{(k)})] < 0$, $q_2 = 2\lambda_{\max}[\rho(I_N \otimes \Omega) + \sigma(I_N \otimes I_n) + \sum_{k=1}^{M} c_k(A^{(k)} \otimes H^{(k)})]$, d_k, T_k and r are defined as above.

Proof: The Lyapunov function is introduced as follows:

$$V(t) = \frac{1}{2} e^T(t)e(t).$$

When $1 \leq i \leq \bar{l}, t \in [t_i, t_i + d_i)$,

$$\begin{aligned} \mathcal{L}V(t) = &e^T(t)F(t, e(t)) + e^T(t)[\sum_{k=1}^{M} c_k((A^{(k)} - \Gamma^{(k)}) \\ &\otimes H^{(k)})]e(t) + \frac{1}{2} trace(\Phi^T(t, e(t))(I_N \otimes I_n) \\ &\Phi(t, e(t))). \end{aligned}$$

From Assumption 1, and Assumption 2, $e^T(t)F(t, e(t)) \leq \rho e^T(t)(I_N \otimes \Omega)e(t)$, and

$$\begin{aligned} &trace(\Phi^T(t, e(t))(I_N \otimes I_n)\Phi(t, e(t))) \\ &= \sum_{i=1}^{N} \Phi_i^T(t, e_i(t))\Phi_i(t, e_i(t)) \leq \sum_{i=1}^{N} 2\sigma e_i^T(t)e_i(t) \\ &= 2\sigma e^T(t)(I_N \otimes I_n)e(t). \end{aligned}$$

One obtains

$$\begin{aligned} \mathcal{L}V(t) \leq &e^T(t)[\rho(I_N \otimes \Omega) + \sigma(I_N \otimes I_n) \\ &+ \sum_{k=1}^{M} c_k((A^{(k)} - \Gamma^{(k)}) \otimes H^{(k)})]e(t) \\ \leq &\lambda_{\max}[\rho(I_N \otimes \Omega) + \sigma(I_N \otimes I_n) \\ &+ \sum_{k=1}^{M} c_k((A^{(k)} - \Gamma^{(k)}) \otimes H^{(k)})]e^T(t)e(t). \end{aligned}$$

Therefore, one gets

$$\mathcal{L}V(t) \leq q_1 V(t). \tag{15}$$

When $\bar{l} < i, or\ 1 \leq i \leq \bar{l}, t \in [t_i + d_i, t_{i+1})$,

$$\begin{aligned} \mathcal{L}V(t) = &e^T(t)F(t, e(t)) + e^T(t)(\sum_{k=1}^{M} c_k A^{(k)} \otimes H^{(k)}))e(t) \\ &+ \frac{1}{2} trace(\Phi^T(t, e(t))(I_N \otimes I_n)\Phi(t, e(t))). \end{aligned}$$

Similarly, from Assumption 1 and Assumption 2,

$$\begin{aligned} \mathcal{L}V(t) = &e^T(t)[F(t, e(t)) + \sum_{k=1}^{M} c_k(A^{(k)} \otimes H^{(k)})]e(t) \\ &+ \frac{1}{2} trace(\Phi^T(t, e(t))\Phi(t, e(t))) \\ \leq &e^T(t)[\rho(I_N \otimes \Omega) + \sigma(I_N \otimes I_n) \\ &+ \sum_{k=1}^{M} c_k A^{(k)} \otimes H^{(k)})]e(t). \end{aligned}$$

Therefore, one gets

$$\mathcal{L}V(t) \leq q_2 V(t). \tag{16}$$

In a word,

$$\mathcal{L}V(t) \leq \begin{cases} q_1 V(t), 1 \leq i \leq \bar{l}, t \in [t_i, t_i + d_i) \\ q_2 V(t), \bar{l} < i, or\ 1 \leq i \leq \bar{l}, t \in [t_i + d_i, t_{i+1}) \end{cases}. \tag{17}$$

According to the principle of calculus,

$$EV(t) \leq \begin{cases} EV(t_j)e^{q_1(t-t_j)}, 1 \leq i \leq \bar{l}, t \in [t_i, t_i + d_i) \\ EV(t_j)e^{q_2(t-(t_j+d_j))}, \bar{l} < i, or\ 1 \leq i \leq \bar{l}, \\ t \in [t_i + d_i, t_{i+1}) \end{cases}. \tag{18}$$

When $t \in [t_0, t_1), EV(t) \leq V(t_0)e^{q_1(t-t_0)}t \in [t_0, t_0 + d_0)$ $EV(t) \leq V(t_0 + d_0)e^{q_2(t-(t_0+d_0))} \leq V(t_0)e^{q_1 d_0 + q_2(t-(t_0+d_0))}, t \in [t_0 + d_0, t_1)$.

By reduction, when $t \in [t_j, t_j + d_j)$,

$$\begin{aligned} EV(t) &\leq EV(t_j)e^{q_1(t-t_j)} \\ &\leq V(t_0)e^{q_1(\sum_{k=0}^{j-1} d_k)+q_2(\sum_{k=0}^{j-1} T_k(1-r))+a_1(t-t_j)}. \\ & t \in [t_j + d_j, t_{j+1}), \\ EV(t) &\leq EV(t_j + d_j)e^{q_2(t-(t_j+d_j))} \\ &\leq V(t_0)e^{q_1(\sum_{k=0}^{j} d_k)+q_2(\sum_{k=0}^{j-1} T_k(1-r))+q_2(t-(t_j+d_j))} \\ &\leq V(t_0)e^{q_1(\sum_{k=0}^{j} d_k)+q_2(\sum_{k=0}^{j} T_k(1-r))}. \end{aligned}$$

When $t = t_{j+1}$,

$$EV(t_{j+1}) \leq V(t_0)e^{q_1(\sum_{k=0}^{j} d_k)+q_2(\sum_{k=0}^{j} T_k(1-r))}.$$

According to the conditions in Theorem 1, one obtain $\lim_{j\to\infty} EV(t_{j+1}) = \lim_{t\to\infty} EV(t) = 0$, which implies $\lim_{t\to\infty} E||x_i(t) - x_0(t)|| = 0$. The proof is completed.

Based on the above analysis, it can be seen that the necessary condition for MNs to achieve synchronization are

$$\rho(I_N \otimes \Omega) + \sigma(I_N \otimes I_n) + \sum_{k=1}^{M} c_k((A^{(k)} - \Gamma^{(k)}) \otimes H^{(k)}) < 0.$$

Below, we will discuss the relationship between network connectivity and the minimum number of pinned nodes.

Corollary 1: As long as any layer of MNs is connected, the synchronization can be achieved by pinning at least one node in that layer.

Now, analyze the maximum eigenvalue of $\sum_{k=1}^{M} c_k((A^{(k)} -\Gamma^{(k)}) \otimes H^{(k)})$. Let's assume that the first layer of the MNs is connected, then $A^{(1)}$ is an irreducible matrix, and $rank(A^{(1)}) = N - 1$. Form reference [38], there is at least one $\gamma_i^{(1)} > 0$, so that $(A^{(1)} - \Gamma^{(1)}) < 0$, From Lemma 3, $c_1((A^{(1)} - \Gamma^{(1)}) \otimes H^{(1)}) < 0$. So $\sum_{k=1}^{M} c_k((A^{(k)} - \Gamma^{(k)}) \otimes H^{(k)}) < c_1((A^{(1)} - \Gamma^{(1)}) \otimes H^{(1)}) < 0$. Obviously, by selecting the appropriate $\gamma_j^{(1)}(j = 1, 2, \ldots, l_1)$ and coupling coefficient c_k, one get

$$\rho(I_N \otimes \Omega) + \sigma(I_N \otimes I_n) + \sum_{k=1}^{M} c_k((A^{(k)} - \Gamma^{(k)}) \otimes H^{(k)}) < 0.$$

Corollary 2: If each layer of the MNs is not connected, let's say, the network in the first layer has m connected branches. Then we can always number the network nodes appropriately so that the coupling matrix of the first layer has the following form: $A^{(1)} = diag(A_1^{(1)}, A_2^{(1)} \ldots, A_m^{(1)})$, where $A_1^{(1)}, A_2^{(2)} \ldots, A_m^{(1)}$ are square matrices with order $N_1, N_2, \ldots, N_m$, and are square irreducible matrices, respectively.

Let $\Gamma^{(1)} = diag(\underbrace{\gamma_1^{(1)}, \ldots, \gamma_{i_1}^{(1)}, 0, \ldots, 0}_{N_1}, \underbrace{\gamma_{i_1+1}^{(1)}, \ldots, \gamma_{i_1+i_2}^{(1)}, 0, \ldots, 0}_{N_2}, \underbrace{\gamma_{i_{m-1}+1}^{(1)}, \ldots, \gamma_{l_1}^{(1)}, 0, \ldots, 0}_{N_m},.$

For $i = 1, 2, \ldots, m$, if there is at least one $\gamma_j^{(1)} > 0, j = 1, 2, \ldots, l_1$, then $\lambda_{\max}(A^{(1)} - \Gamma^{(1)}) < 0$. That is, the number of pinned nodes at each connected branch is greater than or equal to 1, the synchronization of MNs can be achieved.

B. THE SYNCHRONIZATION OF MNS WITH AN ADAPTIVE INTERMITTENT PINNING CONTROLER

In the subsection, by designing an intermittent pinning controller with adaptive control gains, explore the synchronization of two MNs with stochastic perturbations.

For the error equation (11), an adaptive intermittent pinning controller is as follows:

$$u_i(t) = -\sum_{k=1}^{M} c_k \gamma_i^{(k)}(t) H^{(k)} e_i(t), \tag{19}$$

where $\gamma_i^{(k)}(t)$ is an adaptive control gain, and $\dot{\gamma}_i^{(k)}(t) = \theta_k e_i^T(t) H^{(k)} e_i(t), \theta_k > 0$. Denoting $\Gamma^{(k)}(t) = diag(\gamma_1^{(k)}(t), \gamma_2^{(k)}(t), \ldots, \gamma_{l_k}^{(k)}(t), 0, \ldots, 0)_{N\times N}, E^{(k)} = diag(\varepsilon_1^{(k)}, \varepsilon_2^{(k)}, \ldots, \varepsilon_{l_k}^{(k)}, 0, \ldots, 0)_{N\times N}$.

Error system can be written as

$$de(t) = \begin{cases} [F(t, e(t)) + (\sum_{k=1}^{M} c_k(A^{(k)} \otimes H^{(k)}))e(t) \\ -(\sum_{k=1}^{M} c_k(\Gamma^{(k)}(t) \otimes H^{(k)}))e(t)]dt \\ +\Phi(t, e(t))d\omega(t), 1 \leq i \leq \bar{l}, t \in [t_i, t_i + d_i) \\ [F(t, e(t)) + (\sum_{k=1}^{M} c_k(A^{(k)} \otimes H^{(k)}))e(t)]dt \\ +\Phi(t, e(t))d\omega(t), \bar{l} < i, or\ 1 \leq i \leq \bar{l}, \\ t \in [t_i + d_i, t_{i+1}) \end{cases} \tag{20}$$

Theorem 2: Under the Assumptions 1-3, the synchronization of the multiplex networks with stochastic perturbations via an aperiodically adaptive intermittent pinning controller (19) can be realized, if $\lim_{j\to\infty} [\bar{q}_1 \sum_{k=0}^{j} d_k + \bar{q}_2 \sum_{k=0}^{j} T_k(1 - r)] = -\infty$, where $\bar{q}_1 = 2\lambda_{\max}[\rho(I_N \otimes \Omega) + \sigma(I_N \otimes I_n) + \sum_{k=1}^{M} c_k(A^{(k)} - E^{(k)}) \otimes H^{(k)})] < 0$, $\bar{q}_2 = 2\lambda_{\max}[\rho(I_N \otimes \Omega) + \sigma(I_N \otimes I_n) + \sum_{k=1}^{M} c_k(A^{(k)} \otimes H^{(k)})]$, d_k, T_k and r are defined as above.

Proof: The Lyapunov function is introduced as follows:

$$V(t)=\frac{1}{2}e^T(t)e(t)+\sum_{k=1}^{M}\frac{c_k}{2\theta_k}\sum_{i=1}^{l_k}(\gamma_i^{(k)}(t)-\varepsilon_i^{(k)})^2.$$

When $1\le i\le \bar{l},t\in[t_i,t_i+d_i)$

$$\begin{aligned}\mathcal{L}V(t)&=e^T(t)F(t,e(t))+e^T(t)(\sum_{k=1}^{M}c_k(A^{(k)}\otimes H^{(k)})e(t)\\&\quad-e^T(t)(\sum_{k=1}^{M}c_k(\Gamma^{(k)}(t)\otimes H^{(k)})e(t)\\&\quad+\sum_{k=1}^{M}c_k\sum_{i=1}^{l_k}\gamma_i^{(k)}(t)e_i^T(t)H^{(k)}e_i(t)\\&\quad-\sum_{k=1}^{M}c_k\sum_{i=1}^{l_k}\varepsilon_i^{(k)}(t)e_i^T(t)H^{(k)}e_i(t)\\&\quad+\frac{1}{2}trace(\Phi^T(t,e(t))(I_N\otimes I_n)\Phi(t,e(t)))\\&=e^T(t)F(t,e(t))+e^T(t)(\sum_{k=1}^{M}c_k(A^{(k)}\otimes H^{(k)})e(t)\\&\quad-e^T(t)(\sum_{k=1}^{M}c_k(E^{(k)}(t)\otimes H^{(k)})e(t)\\&\quad+\frac{1}{2}trace(\Phi^T(t,e(t))(I_N\otimes I_n)\Phi(t,e(t))).\end{aligned}$$

In view of Assumption 1 and Assumption 2,

$$\begin{aligned}\mathcal{L}V(t)\le e^T(t)[\rho(I_N\otimes\Omega)+\sigma(I_N\otimes I_n)\\+\sum_{k=1}^{M}c_k(A^{(k)}-E^{(k)})\otimes H^{(k)})]e(t).\end{aligned}$$

One can obtain

$$\mathcal{L}V(t)=\bar{q}_1V(t). \tag{21}$$

When $\bar{l}<i, or\ 1\le i\le\bar{l},t\in[t_i+d_i,t_{i+1})$

Similar to derivation in Theorem 1, one has

$$\begin{aligned}\mathcal{L}V(t)&\le e^T(t)[\rho(I_N\otimes\Omega)+\sigma(I_N\otimes I_n)\\&\quad+\sum_{k=1}^{M}c_kA^{(k)}\otimes H^{(k)})]e(t)\\&=\bar{q}_2V(t).\end{aligned}$$

$$\mathcal{L}V(t)\le\begin{cases}\bar{q}_1V(t),\ 1\le i\le\bar{l},t\in[t_i,t_i+d_i)\\ \bar{q}_2V(t),\ \bar{l}<i, or\ 1\le i\le\bar{l},t\in[t_i+d_i,t_{i+1})\end{cases}. \tag{22}$$

It implies the following expression

$$EV(t)\le\begin{cases}EV(t_j)e^{q_1(t-t_j)},\ 1\le i\le\bar{l},t\in[t_i,t_i+d_i)\\ EV(t_j)e^{q_2(t-(t_j+d_j))},\ \bar{l}<i, or\ 1\le i\le\bar{l},\\ t\in[t_i+d_i,t_{i+1})\end{cases} \tag{23}$$

The discussion below is similar to Theorem 1.

IV. SIMULATION EXEMPLES

In this section, two simulation examples with intermittent pinning controllers are given to verify the effectiveness of main results.

A. EXAMPLE WITH A STATE-FEEDBACK INTERMIT-TENT PINNING CONTROLER

Consider a two-layers network with 100 nodes. The first layer of network is constructed by using a Watts-Strogatz small-world network with initial degree $m=4$ and the rewiring probability $p=0.3$. The second layer is constructed as a Barabási–Albert scale-free network with initial nodes $m_0=5$. And for each layer, we use random pinning strategy to generate its coupling matrix.

Taking the following Lorenz system as dynamical system of the ith node:

$$\begin{cases}\dot{x}_{i1}(t)=a(x_{i2}(t)-x_{i1}(t))\\ \dot{x}_{i2}(t)=bx_{i1}(t)-x_{i1}(t)x_{i3}(t)-x_{i2}(t)\\ \dot{x}_{i3}(t)=x_{i1}x_{i2}(t)-cx_{i3}(t)\end{cases},(i=1,2,\ldots,100), \tag{24}$$

where $a=10$, $b=8/3$, $c=28$, then the Lorenz system leads to chaos. According to reference [39], $|x_{i1}(t)|\le 29$, $|x_{i2}(t)|\le 29$, $-1\le x_{i3}(t)\le 57$, $|x_{01}(t)|\le 29$, $|x_{02}(t)|\le 29$, $-1\le x_{03}(t)\le 57$.

$$\begin{aligned}&e_i^T(t)(f(x_i(t))-f(x_0(t)))\\&=-ae_{i1}^2-e_{i2}^2-ce_{i3}^2+(a+b-x_{i3})e_{i1}e_{i2}\\&\quad+x_{i2}e_{i1}e_{i3}\le-ae_{i1}^2-e_{i2}^2-ce_{i3}^2+(a+b+1)\,|e_{i1}e_{i2}|\\&\quad+29\,|e_{i1}e_{i3}|\\&\le(-a+\frac{39\alpha}{2}+\frac{29\beta}{2})e_{i1}^2+(-1+\frac{39}{2\alpha})e_{i2}^2+(-c+\frac{29}{2\beta})e_{i3}^2.\end{aligned}$$

Select $\alpha=0.9810$, $\beta=0.6730$, then $e_i^T(t)(f(x_i(t))-f(x_0(t)))\le 18.89e_i^T(t)e_i(t)$, where $\rho=18.89$, $\Omega=I_3$. That is, $f(\cdot)$ satisfies the condition of Assumption 1. The noise intensity matrix is selected as $\varphi(x_i(t))=1.2x_i(t)$, obviously, $\varphi(\cdot)$ meets Assumption 2, and here $\sigma=0.72$. Select $l_1=l_2=30$, $c_1=10$, $c_2=3$, $h_1^{(k)}=h_2^{(k)}=h_3^{(k)}=3$, $k=1,2$; $\gamma_j^{(1)}=\gamma_j^{(2)}=8$, $j=1,2,\ldots,30$, we can get that $q_1=-48.6$, $q_2=49$. From Theorem 1, for any k, when $q_1d_k+q_2T_k(1-r)<0$. i.e, $r\ge\frac{q_2}{q_2-q_1}=49.8\%$, the two-layers network with stochastic perturbations via the aperiodically intermittent pinning controller can achieve the synchronization. Choose that the control ratio $r=50\%$, the control time intervals $T_1=[0,0.002)$,$T_2=[0.002,0.012)$,$T_3=[0.012,0.032),\ldots$. Initial values of the desired nodes and each node of the network are chosen as $x_0(0)=[2,3,4]^T$,$x_{i1}(0)=2+0.2\times i\times(-1)^i$,$x_{i2}(0)=3+0.2\times i\times(-1)^i$,$x_{i3}(0)=4+0.2\times i\times(-1)^i$, respectively. Stochastic differential equations are solved with step size 0.00001. The following Figure 1 shows evolution trends of three error components, and Figure 2 also draws evolution of error components of the pinned nodes (30 nodes). From Figure 1 and Figure 2, the impact of aperiodically intermittent pinning and disturbances on the synchronization can be seen.

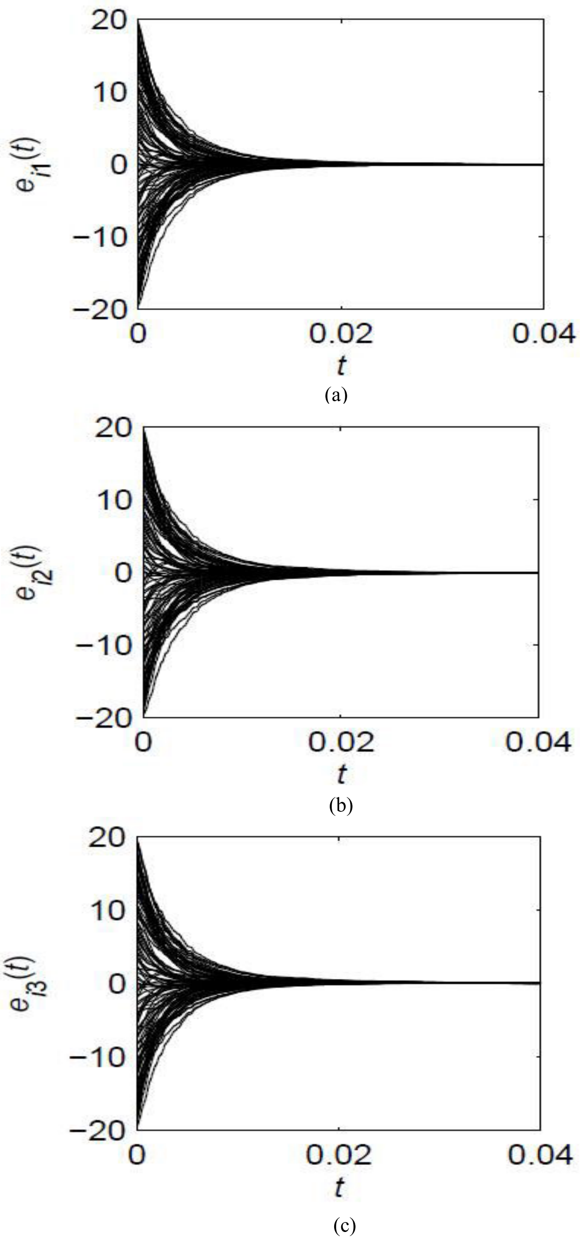

FIGURE 1. **Evolution trends of synchronization errors. (a) Error components $e_{i1}(t)$, (b) Error components $e_{i2}(t)$, (c) Error components $e_{i3}(t)$, $i = 1, 2, \ldots, 100$.**

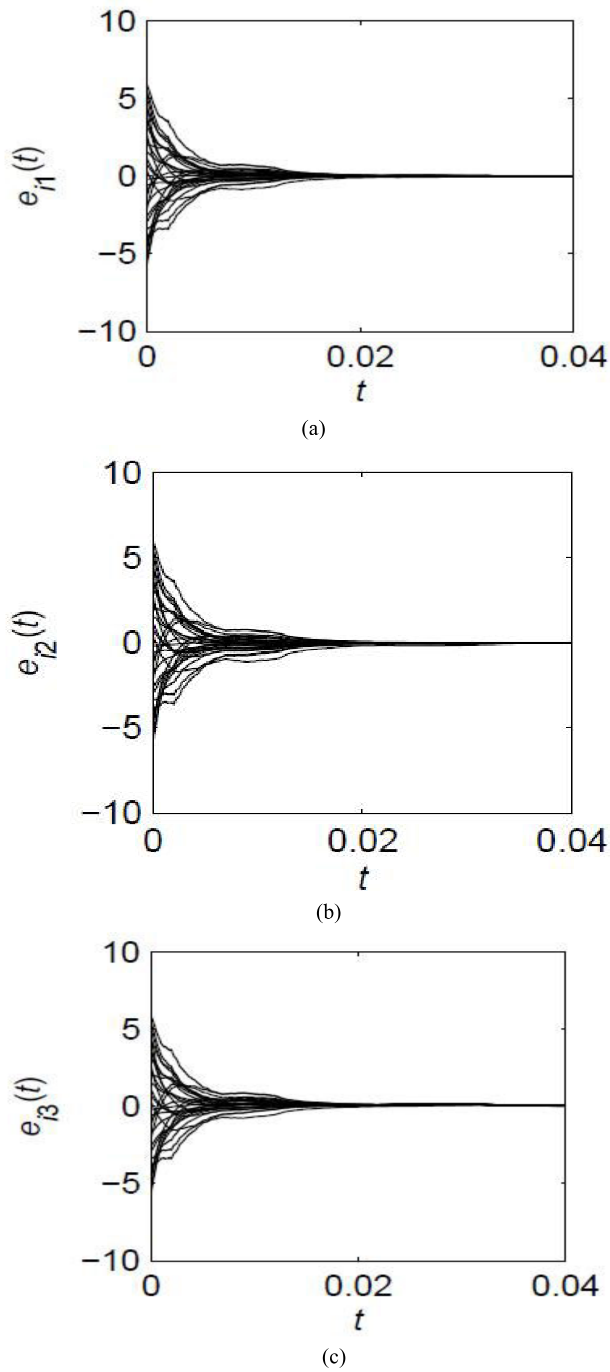

FIGURE 2. **Evolution trends of pinned node errors. Error components $e_{i1}(t)$, (b) Error components $e_{i2}(t)$, (c) Error components $e_{i3}(t)$, $i = 1, 2, \ldots, 30$.**

B. EXAMPLE WITH AN ADAPTIVE INTERMITTENT PINNING CONTROLER

In this example, consider that each layer of a two-layer network with 7 nodes is not connected, whose topology is shown in Figure 3.

Coupling matrices of the two connected sub-networks of the first layer network are $A_1^{(1)} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, $A_2^{(1)} = \begin{pmatrix} -3 & 1 & 1 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 1 & 1 & -3 & 1 & 0 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}$, respectively; Coupling matrices for the two connected sub-networks of the second layer network are $A_1^{(2)} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$, and $A_2^{(2)} = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$, so, $A^{(1)} = \begin{pmatrix} A_1^{(1)} 0 \\ 0 A_2^{(1)} \end{pmatrix}$, and $A^{(2)} = \begin{pmatrix} A_1^{(2)} & 0 \\ 0 & A_2^{(2)} \end{pmatrix}$.

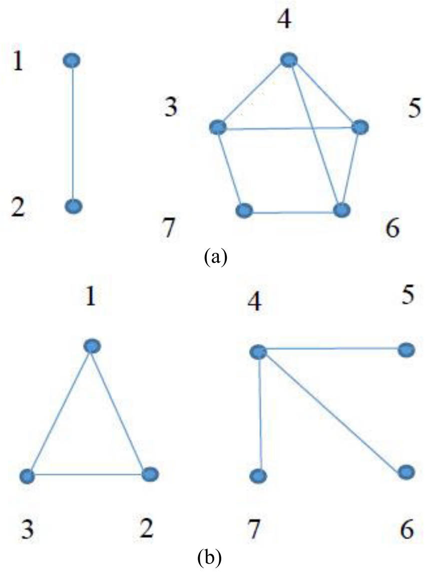

FIGURE 3. The topology of the two-layer network. (a) The topology of the first layer network, (b) The topology of the second layer network.

Dynamic system of the ith node:

$$f(x_i(t)) = \begin{cases} \tanh(x_{i1}(t)) \\ \tanh(x_{i2}(t)) \end{cases}, i = 0, 1, 2, \ldots, 7. \tag{25}$$

First, we prove the dynamical system meets the condition in Assumption 1.

$$\begin{aligned}
&(x_i(t) - x_0(t))^T (f(x_i(t)) - f(x_0(t))) \\
&= (x_{i1}(t) - x_{01}(t), x_{i2}(t) - x_{02}(t)) \\
&\quad \cdot \begin{pmatrix} \tanh x_{i1}(t) - \tanh x_{01}(t) \\ \tanh x_{i2}(t) - \tanh x_{02}(t) \end{pmatrix} \\
&\leq \|(x_{i1}(t) - x_{01}(t), x_{i2}(t) - x_{02}(t))\|_2 \\
&\quad \cdot \left\| \begin{pmatrix} \tanh x_{i1}(t) - \tanh x_{01}(t) \\ \tanh x_{i2}(t) - \tanh x_{02}(t) \end{pmatrix} \right\|_2 \\
&= \sqrt{(x_{i1}(t) - x_{01}(t))^2 + (x_{i2}(t) - x_{02}(t))^2} \\
&\quad . \sqrt{(\tanh x_{i1}(t) - \tanh x_{01}(t))^2 + (\tanh x_{i2}(t) - \tanh x_{02}(t))^2}.
\end{aligned}$$

From the Lagrange Mean Value Theorem, $\tanh x_{i1}(t) - \tanh x_{01}(t) = \tan \dot{h}\theta_1(t)(x_{i1}(t) - x_{01}(t))$, and $\tanh x_{i2}(t) - \tanh x_{02}(t) = \tan \dot{h}\theta_2(t)(x_{i2}(t) - x_{02}(t))$, here $0 < \tan \dot{h}\theta_1(t) \leq [(x_{i1}(t)-x_{01}(t))^2+(x_{i2}(t)-x_{02}(t))^2]$, $\theta_1(t), \theta_2(t) \in R$. Hence,

$$\begin{aligned}
&(x_i(t) - x_0(t))^T (f(x_i(t)) - f(x_0(t))) \\
&\leq \sqrt{(x_{i1}(t) - x_{01}(t))^2 + (x_{i2}(t) - x_{02}(t))^2} \\
&\quad . \sqrt{(x_{i1}(t) - x_{01}(t))^2 + (x_{i2}(t) - x_{02}(t))^2} \\
&\leq [(x_{i1}(t) - x_{01}(t))^2 + (x_{i2}(t) - x_{02}(t))^2] \\
&= (x_i(t) - x_0(t))^T (x_i(t) - x_0(t)).
\end{aligned}$$

So $f(\cdot)$ satisfies the condition of Assumption 1,wherein $\rho = 1$, $\Omega = I_2$. Set $\varphi(x_i(t)) = 0.6x_i(t)$, then $\varphi(\cdot)$ meets Assumption 2, here $\sigma = 0.18$. Set $c_1 = 3$, $c_2 = 1$, $h_1^{(k)} = h_2^{(k)} = 1$, $k = 1, 2$;$\theta_1 = \theta_2 = 1$, $\varepsilon_j^{(1)} = \varepsilon_j^{(2)} = 5$, $j = 1, 3, 4$. Select pinning the 1st, 3rd, and 4th nodes of the first layer, and the second layer does not have a controller (i.e. the number of pinned nodes is 0), then set $\Gamma^{(1)} = diag(1, 0, 1, 1, 0, 0, 0)$, $\Gamma^{(2)} = 0_{7\times7}$,get $q_1 = -3, q_2 = 2.26$, and $r \geq \frac{q_2}{q_2-q_1} = 44\%$ meets the conditions of Theorem 2.

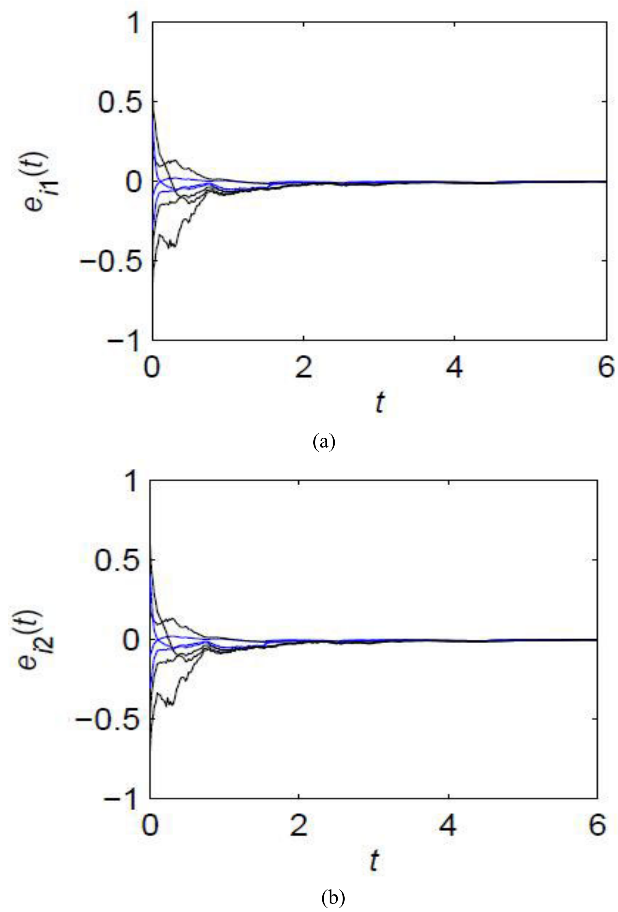

FIGURE 4. Evolution trends of synchronization errors. (a) Error components $e_{i1}(t)$, (b) Error components $e_{i2}(t)$, $i = 1, 2, \ldots, 7$.

Select that $r = 50\%$, the control time intervals are chosen as $T_1 = [0, 1.5), T_2 = [1.5, 2.5), T_3 = [2.5, 4.5), T_4 = [4.5, 5.5), \ldots$. Initial values are chosen as $x_0(0) = [0.1, 0.15]^T$, $x_{i1}(0) = 0.1 + 0.1 \times i \times (-1)^i$, $x_{i2}(0) = 0.15 + 0.1 \times i \times (-1)^i$, respectively. Solve stochastic differential equations with step size 0.02. Evolution trends of synchronization errors are shown in Figure 4.

In Figure 4, the blue curves display the evolution of the pinned nodes $e_{i1}(t)$, $e_{i2}(t)$, $i = 1, 3, 4$, the black curves are the non-pinned nodes $e_{i1}(t)$, $e_{i2}(t)$, $i = 2, 5, 6, 7$. And, aperiodically intermittent pinning and disturbances show the effects on the synchronization curves in Figure 4.

V. CONCLUSION

This paper has proposed new schemes of the synchronization for MNs with the stochastic perturbations in which aperiodically intermittent and pinning control methods are adopted.

In the schemes, the pinned nodes can be different in different layers of MNs. By designing state feedback and adaptive controllers, some sufficient conditions have been given to ensure synchronization. Furtherly, for the different connectivity of MNs, the conditions for synchronization of MNs have been discussed by analyzing the relationship between the number of connected branches and the minimum number of pinned nodes in different layer networks. In the numerical simulation section, two numerical examples have shown effectiveness of the proposed schemes. In the paper, the MNs models do not involve delays. However, some MNs have coupling delays or system delays, which is a further problem to be investigated.

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