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RESEARCH ARTICLE

Finite-Time Annular Domain Stability and Stabilization of Regime-Switching Jump Diffusion System

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ABSTRACT In this paper, the finite-time annular domain stability and stabilization of regime-switching jump diffusion system are discussed. With the help of explicit solution of multi-dimensional regime-switching jump diffusion system, we give the sufficient and necessary conditions for finite-time annular domain stability and stabilization of regime-switching jump diffusion system respectively. Finally, the importance of Markov chain and Poisson jump of our obtained finite-time annular domain stability condition are demonstrated by numerical examples.

INDEX TERMS Finite-time annular domain stability, finite-time annular domain stabilization, regime-switching jump diffusion system.

I. INTRODUCTION

Initially, the researchers mainly study the stability analysis of a deterministic system. Since deterministic systems are susceptible to various random factors, such as epidemics, earthquakes, tsunamis, or terrorist atrocities, stochastic systems as mathematical model are studied. Regime-switching jump diffusion processes can be seen as a jump diffusion process in a stochastic environment, where the evolution of the stochastic environment is modeled by continuous-time Markov chain, or more generally, a continuous-state-dependent switching process with a discrete state space [17]. In order to simulate more systems, many scholars have considered and studied stochastic systems [4], [8], [9], [10], [11], [12], [13], [14], [15], [22], and also studied many properties of the regime-switching jump diffusion system, such as asymptotic stability in probability, finite-time annular domain stability in the sense of expectation and so on.

The finite-time stability of regime-switching jump diffusion system was studied in the 1970s. Since then, this field has gained a great deal of attention and has undergone substantial development. It has been applied into various practical situations, such as the fields of finance [6], biology [7] and engineering.

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Finite-time annular domain stability can be used to solve many problems in engineering practices such as chemical reaction processes, electronic circuit systems [18], and medicine [19]. We investigate finite-time annular domain stability of regime-switching jump diffusion system with Markov chain and Poisson jump. There are many papers about finite-time annular domain stability and the systems in most of the papers involves only one of Markov chain and Poisson jump [1], [2], [5]. There are a few papers containing Markov chain and Poisson jump [3], [18]. All of them use the Lyapunov exponent and linear matrix inequality to give the sufficient condition for finite-time annular domain stability of stochastic systems [1], [2], [3], [5], [18]. With the help of the explicit solution of the regime-switching jump diffusion system, we present the sufficient and necessary condition for finite-time annular domain stability only by the coefficient matrix. This can eliminate the need to find a suitable Lyapunov function, and it greatly simplifies the proof process. Of course, we will encounter difficulties in the process of presenting evidence, such as the expectation about exponential function whose exponent is the random matrix.

The structure of this paper is as follows. Some notations and preliminary results are given in Section II. Section III provides the sufficient and necessary condition of finite-time annular domain stability of regime-switching jump diffusion system. Then, the finite-time annular domain stabilization of regime-switching jump diffusion system is considered in Section IV. Finally, two simulations are discussed to illustrate the superiority of Markov chain and Poisson jump in Section V.

II. NOTATIONS AND PRELIMINARY RESULTS

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $\omega(t) = \big(\omega_1(t), \cdots, \omega_d(t)\big)^\top$ be a d-dimensional standard Brownian motion, $N(\cdot,\cdot)$ be a $\mathscr{F}_t$-adapted Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^n \setminus \{0\}$ and $N(t)$ be a Poisson random process defined on $\mathbb{R}_+ = [0, \infty)$ in the probability space. The corresponding compensated Poisson process of $N(t)$ is $\widetilde{N}(t) = N(t) - \lambda t$ where $0 < \lambda < \infty$ is called the jump intensity of $N(t)$. Let $r(t)$ be a right-continuous Markov chain taking values in $\mathbb{S} = \{1, \dots, m\}$ with generator $\Gamma = (\gamma_{ij})_{z\times z}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here γ_{ij} is the transition rate from i to j and $\gamma_{ij} > 0$ if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$ for $\forall i, j \in \mathbb{S}$. We assume that $\{r(\cdot)\}$, $\{w(\cdot)\}$ and $\{N(\cdot)\}$ are independent.

Throughout this paper, unless otherwise specified, we use the following notations. If A is a matrix or vector, $A^\top$ denotes its transpose. If $A,\ B \in \mathbb{R}^{n\times n}$, $A < B$ means that $B - A$ is a positive definite matrix. $\mathrm{T}r(A)$ denotes the trace of A, $\rho(A)$ means the spectral radius of A, and let $\|\cdot\|$ be the 2-norm of a matrix or vector. Denote by $C^2(\mathbb{R}^n; \mathbb{R}_+)$ the family of all nonnegative functions $V(x)$ which are continuously twice differentiable in x. $I = (e_1, \dots, e_j, \dots, e_n) \in \mathbb{R}^{n\times n}$ is an unit matrix where $j = 1, \dots, n$.

Consider a general regime-switching jump diffusion system

$$\begin{aligned} \mathrm{d}x(t) =& f(x(t), r(t), t)\mathrm{d}t + g(x(t), r(t), t)\mathrm{d}w(t) \\ &+ \int_{\Xi} h(y, x(t-), r(t-), t-)N(\mathrm{d}t, \mathrm{d}y), \end{aligned} \quad \text{(II.1)}$$

on $t \geq 0$ with $x(0)$ being a constant vector, where $f : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n\times d}$, and $h : \Xi \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n$ are Borel-measurable. When $h(y, x(t-), r(t-), t-)$ is independent of y, $\int_\Xi h(y, x(t-), r(t-), t-)N(\mathrm{d}t, \mathrm{d}y)$ can be denoted by $h(x(t-), r(t-), t-)\mathrm{d}N(t)$, and then system (II.1) is written as

$$\begin{aligned} \mathrm{d}x(t) =& f(x(t), r(t), t)\mathrm{d}t + g(x(t), r(t), t)\mathrm{d}w(t) \\ &+ h(x(t-), r(t-), t-)\mathrm{d}N(t),\ t \geq 0. \end{aligned} \quad \text{(II.2)}$$

For a function $V(x) \in C^2(\mathbb{R}^n; \mathbb{R}_+)$, the generalized Itô's formula [14] of $V(x)$ associated with system (II.2) is defined as

$$\begin{aligned} \mathrm{d}V(x) = \Big\{ & V_x(x)f(x, r(t), t) \\ &+ \frac{1}{2}\mathrm{T}r\Big[g^\top(x, r(t), t)V_{xx}(x)g(x, r(t), t)\Big]\Big\}\mathrm{d}t, \\ &+ V_x(x)g(x, r(t), t)\mathrm{d}w(t) \\ &+ \Big[V\big(x + h(x, r(t-), t-)\big) - V(x)\Big]\mathrm{d}N(t). \end{aligned} \quad \text{(II.3)}$$

Next, the definition of finite-time annular domain stability of system (II.2) is given.

Definition 1 [1]: For given positive constants c_3, c_4, T, and a positive definite matrix $R \in \mathbb{R}^{n\times n}$, if there exist positive constants c_1 and c_2, possibly depending on c_3, c_4, T and R such that

$$\begin{aligned} & c_1 \leq \mathbb{E}\big\{x^\top(0)Rx(0)\big\} \leq c_2 \\ & \quad \Rightarrow c_3 < \mathbb{E}\big\{x^\top(t)Rx(t)\big\} < c_4,\ \forall t \in [0,\ T], \end{aligned} \quad \text{(II.4)}$$

then system (II.2) is said to be finite-time annular domain stable with respect to $(c_1, c_2, c_3, c_4, T, R)$.

III. FINITE-TIME ANNULAR DOMAIN STABILITY

In this section, we shall study the finite-time annular domain stability of the homogeneous multi-dimensional regime-switching jump diffusion system

$$\begin{aligned} \mathrm{d}x(t) = F(r(t))x(t)\mathrm{d}t + \sum_{k=1}^{d} G_k x(t)\mathrm{d}w_k(t) \\ + Hx(t-)\mathrm{d}N(t),\ t \geq 0, \end{aligned} \quad \text{(III.1)}$$

with $x(0)$ is a constant vector, $F(1), \cdots, F(m), G_1, \cdots, G_d$, $H \in \mathbb{R}^{n\times n}$ and $r(t) \in \mathbb{S}$.

We will give the sufficient and necessary condition of finite-time solution, and the explicit solution of system (III.1) is given in the following lemma, whose proof will be found in Appendix.

Lemma 2: When the matrices $F(1), \dots, F(m), G_1, \dots, G_d$, H are commutative and $\rho(H) < 1$, the explicit solution of system (III.1) is

$$x(t) = \Phi(t)x_0, \quad \text{(III.2)}$$

where

$$\begin{aligned} \Phi(t) = \exp\Big\{\int_0^t \Big(F(r(s)) - \frac{1}{2}\sum_{k=1}^{d} G_k^2\Big)ds + \sum_{k=1}^{d} G_k w_k(t)\Big\} \\ \cdot \big(I + H\big)^{N(t)}, \end{aligned} \quad \text{(III.3)}$$

with the initial value $x(0) = x_0$ and $r(t) \in \mathbb{S}$. By referring to [20], $\Phi(t)$ is called the fundamental matrix of system (III.1).

Now, we give a sufficient and necessary condition of finite-time annular domain stability for system (III.1).

Theorem 3: When $\rho(H) < 1$, the system (III.1) is finite-time annular domain stable with respect to $(c_1, c_2, c_3, c_4, T, I)$ if and only if $\frac{c_3}{c_1}I < M(t) < \frac{c_4}{c_2}I$ for $t \in [0, T]$, where $0 < \lambda < \infty$ is the jump intensity of $N(t)$, and

$$\begin{aligned} M(t) = e^{-\lambda t} \cdot \exp\Big\{ & \sum_{i\in\mathbb{S}} \pi_i \Big(F^\top(i) - \frac{1}{2}\sum_{k=1}^{d}(G_k^2)^\top + F(i) \\ & - \frac{1}{2}\sum_{k=1}^{d} G_k^2\Big)t + \frac{t}{2}\sum_{k=1}^{d}(G_k^\top + G_k)^2 \\ & + \lambda t\Big[\big(I + H^\top\big)\big(I + H\big)\Big]\Big\}. \end{aligned} \quad \text{(III.4)}$$

Proof: Since $x(t) = \Phi(t)x_0$, we have $x^\top(t)x(t) = x_0^\top\Phi^\top(t)\Phi(t)x_0$, and

$$\mathbb{E}\big(x^\top(t)x(t)\big) = \mathbb{E}\big(x_0^\top\Phi^\top(t)\Phi(t)x_0\big) = x_0^\top\mathbb{E}\big(\Phi^\top(t)\Phi(t)\big)x_0. \tag{III.5}$$

By (III.3), and the independence of $w_1(\cdot), \cdots, w_d(\cdot)$, $r(\cdot)$ and $N(\cdot)$, we obtain

$$\begin{aligned}&\mathbb{E}\Big\{\Phi^\top(t)\Phi(t)\Big\}\\&= \mathbb{E}\Big\{\exp\Big[\int_0^t\Big(F^\top(r(s)) - \frac{1}{2}\sum_{k=1}^d(G_k^2)^\top + F(r(s))\\&\quad - \frac{1}{2}\sum_{k=1}^d G_k^2\Big)\mathrm{d}s\Big]\Big\}\cdot\mathbb{E}\Big\{\exp\Big[\sum_{k=1}^d\Big(G_k^\top + G_k\Big)w_k(t)\Big]\Big\}\\&\quad\cdot\mathbb{E}\Big\{\Big[\big(I+H^\top\big)\big(I+H\big)\Big]^{N(t)}\Big\}.\end{aligned} \tag{III.6}$$

Let

$$Y(t) = \exp\Big\{\int_0^t A(r(s))\mathrm{d}s\Big\}, \tag{III.7}$$

where

$$A(r(s)) = F^\top(r(s)) - \frac{1}{2}\sum_{k=1}^d(G_k^2)^\top + F(r(s)) - \frac{1}{2}\sum_{k=1}^d G_k^2. \tag{III.8}$$

By (III.7), we can obtain

$$Y_j(t) = \exp\Big\{\int_0^t A(r(s))\mathrm{d}s\Big\}e_j, \tag{III.9}$$

where $Y_j(t)$ denotes the jth column of $Y(t)$. Note that (III.9) is an explicit solution of system

$$\mathrm{d}Y_j(t) = A(r(t))Y_j(t)\mathrm{d}t, \tag{III.10}$$

and $Y_j(0) = e_j$. Thus, by the ergodicity of Markov chain [21], we have

$$\begin{aligned}\mathbb{E}Y_j(t) &= \mathbb{E}Y_j(0) + \mathbb{E}\int_0^t A(r(s))Y_j(s)\mathrm{d}s\\&= \mathbb{E}Y_j(0) + \sum_{i\in\mathbb{S}}\pi_i\int_0^t A(i)\mathbb{E}Y_j(s)\mathrm{d}s\\&= \mathbb{E}Y_j(0) + \sum_{i\in\mathbb{S}}\pi_i A(i)\int_0^t\mathbb{E}Y_j(s)\mathrm{d}s,\end{aligned} \tag{III.11}$$

i.e.,

$$\mathrm{d}\Big(\mathbb{E}Y_j(t)\Big) = \sum_{i\in\mathbb{S}}\pi_i A(i)\Big(\mathbb{E}Y_j(t)\Big)\mathrm{d}t, \tag{III.12}$$

then we have

$$\mathbb{E}Y_j(t) = \exp\Big\{\sum_{i\in\mathbb{S}}\pi_i A(i)t\Big\}e_j. \tag{III.13}$$

By using of (III.13) and (III.8),

$$\begin{aligned}&\mathbb{E}\Big\{\exp\Big[\int_0^t\Big(F^\top(r(s)) - \frac{1}{2}\sum_{k=1}^d(G_k^2)^\top + F(r(s))\\&\quad - \frac{1}{2}\sum_{k=1}^d G_k^2\Big)\mathrm{d}s\Big]\Big\}\\&= \exp\Big\{\sum_{i\in\mathbb{S}}\pi_i\Big(F^\top(i) - \frac{1}{2}\sum_{k=1}^d(G_k^2)^\top + F(i)\\&\quad - \frac{1}{2}\sum_{k=1}^d G_k^2\Big)t\Big\},\end{aligned} \tag{III.14}$$

are obtained. Next, let $V(w_k(t)) = \exp\Big\{\sum_{k=1}^d\big(G_k^\top + G_k\big)w_k(t)\Big\}$. Because $G_k^\top + G_k$ is a symmetric matrix, there exists an invertible matrix B_k such that $G_k^\top + G_k = B_k\Lambda_k B_k^{-1}$ where $\Lambda_k = \mathrm{d}iag\big(\lambda_{k_1}, \cdots, \lambda_{k_n}\big)$. Therefore,

$$\begin{aligned}&\exp\Big\{\big(G_k^\top + G_k\big)w_k(t)\Big\}\\&= \exp\Big\{B_k\Lambda_k B_k^{-1}w_k(t)\Big\} = B_k\exp\Big\{\Lambda_k w_k(t)\Big\}B_k^{-1}\\&= B_k\exp\begin{Bmatrix}\lambda_{k_1}w_k(t) & & \\ & \ddots & \\ & & \lambda_{k_n}w_k(t)\end{Bmatrix}B_k^{-1}.\end{aligned} \tag{III.15}$$

Note that

$$\begin{aligned}\mathbb{E}\Big(e^{\lambda_{k_i}w_k(t)}\Big) &= \int_{-\infty}^{+\infty}e^{\lambda_{k_i}s}\frac{1}{\sqrt{2\pi t}}e^{-\frac{s^2}{2t}}\mathrm{d}s\\&= e^{\frac{\lambda_{k_i}^2 t}{2}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{(s-\lambda_{k_i}t)^2}{2t}}\mathrm{d}s\\&= e^{\frac{\lambda_{k_i}^2 t}{2}}.\end{aligned} \tag{III.16}$$

Thus, by (III.15) and the independence of $w_1(t), \ldots, w_d(t)$, we have

$$\begin{aligned}&\mathbb{E}\Big\{\exp\Big[\sum_{k=1}^d\Big(G_k^\top + G_k\Big)w_k(t)\Big]\Big\}\\&= \exp\Big\{\frac{t}{2}\sum_{k=1}^d\big(G_k^\top + G_k\big)^2\Big\}.\end{aligned} \tag{III.17}$$

Since $N(t)$ is a scalar Poisson random process with jump intensity λ and $N(t) \geq 0$, we have

$$\begin{aligned}&\mathbb{E}\Big(\Big[\big(I+H^\top\big)\big(I+H\big)\Big]^{N(t)}\Big)\\&= \sum_{j=0}^{\infty}\Big[\big(I+H^\top\big)\big(I+H\big)\Big]^j\cdot e^{-\lambda t}\cdot\frac{(\lambda t)^j}{j!}\\&= e^{-\lambda t}\cdot\sum_{j=0}^{\infty}\frac{\Big(\lambda t\big[\big(I+H^\top\big)\big(I+H\big)\big]\Big)^j}{j!}\\&= e^{-\lambda t}\cdot e^{\lambda t\big[\big(I+H^\top\big)\big(I+H\big)\big]}.\end{aligned} \tag{III.18}$$

Then, by substituting (III.14), (III.17) and (III.18) into (III.6), we have $\mathbb{E}\big(\Phi^\top(t)\Phi(t)\big) = M(t)$ which is defined as (III.4). This together with (III.5) leads to

$$\mathbb{E}\big(x^\top(t)x(t)\big) = x_0^\top M(t)x_0. \tag{III.19}$$

Sufficiency. When $c_1 \le \mathbb{E}(x_0^\top x_0) \le c_2$, $M(t) > \frac{c_3}{c_1}I$ and $M(t) < \frac{c_4}{c_2}I$ for $\forall t \in [0,T]$, we have

$$\begin{aligned} c_4 &= \frac{c_4}{c_2}\cdot c_2 \ge \frac{c_4}{c_2}x_0^\top x_0 > x_0^\top M(t)x_0 \\ &= \mathbb{E}\big(x^\top(t)x(t)\big) > \frac{c_3}{c_1}x_0^\top x_0 \ge \frac{c_3}{c_1}\cdot c_1 = c_3. \end{aligned}$$

Necessity. The necessity is shown by contradiction. Suppose that there exist times $t_1^*, t_2^* \in [0,T]$ such that $M(t_1^*) \le \frac{c_3}{c_1}I$ or $M(t_2^*) \ge \frac{c_4}{c_2}I$. 1) If $M(t_1^*) \le \frac{c_3}{c_1}I$, by choosing x_0 satisfying $x_0^\top x_0 = c_1$ and using (III.19), we have

$$\mathbb{E}\big(x^\top(t_1^*)x(t_1^*)\big) = x_0^\top M(t_1^*)x_0 \le \frac{c_3}{c_1}x_0^\top x_0 = c_3. \tag{III.20}$$

This contradicts with the definition of finite-time annular domain stability. 2) If $M(t_2^*) \ge \frac{c_4}{c_2}I$, when $x_0^\top x_0 = c_2$, it follows from (III.19) that

$$\mathbb{E}\big(x^\top(t_2^*)x(t_2^*)\big) = x_0^\top M(t_2^*)x_0 \ge \frac{c_4}{c_2}x_0^\top x_0 = c_4, \tag{III.21}$$

which is also a contradiction with the definition of finite-time annular domain stability. Hence, we have $\frac{c_3}{c_1}I < M(t) < \frac{c_4}{c_2}I$ for $\forall t \in [0,T]$. The proof is complete.

Remark 4: Theorem 3 gives the sufficient and necessary condition for finite-time annular domain stability of system (III.1), but the flaw is that $R = I$. From the proof of Theorem 3, it can be seen that if $R = cI$ where c is a constant, we can also obtain the sufficient and necessary condition for finite-time annular domain stability of the system (III.1).

IV. FINITE-TIME ANNULAR DOMAIN STABILIZATION

In this section, we shall study finite-time annular domain stabilization of the linear multi-dimensional stochastic system

$$\begin{aligned} \mathrm{d}x(t) &= \Big(F(r(t))x(t) + B_1(r(t))u_1(t)\Big)\mathrm{d}t \\ &\quad + \sum_{k=1}^{d}\Big(G_k x(t) + B_2 u_2(t)\Big)\mathrm{d}w_k(t) \\ &\quad + \Big(Hx(t-) + B_3 u_3(t)\Big)\mathrm{d}N(t),\ t \ge 0, \end{aligned} \tag{IV.1}$$

where $x(0)$ is a constant vector, $B_1(1), \cdots, B_1(m), B_2, B_3 \in \mathbb{R}^{n\times n}$ and $u_1(t),\ u_2(t),\ u_3(t) \in \mathbb{R}^n$ are control inputs. And for system (IV.1), the state feedback controllers are given as

$$\begin{aligned} &u_1(t) = K_1(r(t))x(t),\ u_2(t) = K_2 x(t), \\ &u_3(t) = K_3 x(t), \end{aligned} \tag{IV.2}$$

where $K_1(1), \cdots, K_1(m), K_2, K_3 \in \mathbb{R}^{n\times n}$ and $r(t) \in \mathbb{S}$. To express the explicit solution of system (IV.1), assume that $F(1) + B_1(1)K_1(1), \cdots, F(m) + B_1(m)K_1(m), G_1 + B_2K_2, \ldots, G_d + B_2K_2, H + B_3K_3$ are commutative. Then, similar to Theorem 3, a sufficient and necessary condition for finite-time annular domain stabilization of system (IV.1) is given as follows.

Theorem 5: When $\rho(H + B_3K_3) < 1$, the system (IV.1) is finite-time annular domain stabilization with respect to $(c_1, c_2, c_3, c_4, T, I)$ if and only if $\frac{c_3}{c_1}I < M(t) < \frac{c_4}{c_2}I$ for $\forall\, t \in [0,T]$, where $0 < \lambda < \infty$ is the jump intensity of $N(t)$, and

$$\begin{aligned} M(t) = e^{-\lambda t}\cdot \exp\Bigg\{ &\sum_{i\in\mathbb{S}}\pi_i\bigg(\Big(F(i) + B_1(i)K_1(i)\Big)^\top \\ &- \frac{1}{2}\sum_{k=1}^{d}((G_k + B_2K_2)^2)^\top \\ &+ \Big(F(i) + B_1(i)K_1(i)\Big) - \frac{1}{2}\sum_{k=1}^{d}(G_k + B_2K_2)^2\bigg)t \\ &+ \frac{t}{2}\sum_{k=1}^{d}\Big((G_k + B_2K_2)^\top + (G_k + B_2K_2)\Big)^2 \\ &+ \lambda t\Big[\Big(I + (H + B_3K_3)^\top\Big)\Big(I + (H + B_3K_3)\Big)\Big]\Bigg\}. \end{aligned} \tag{IV.3}$$

Proof: Substitute (IV.2) into system (IV.1), then system (IV.1) can be changed as

$$\begin{aligned} \mathrm{d}x(t) &= \Big(F(r(t)) + B_1(r(t))K_1(r(t))\Big)x(t)\mathrm{d}t \\ &\quad + \sum_{k=1}^{d}\Big(G_k + B_2K_2\Big)x(t)\mathrm{d}w_k(t) \\ &\quad + \Big(H + B_3K_3\Big)x(t-)\mathrm{d}N(t),\ t \ge 0. \end{aligned} \tag{IV.4}$$

According to Lemma 2 and $\rho(H + B_3K_3) < 1$, the explicit solution of system (IV.4) is $x(t) = \Phi(t)x_0$, where

$$\begin{aligned} \Phi(t) = \exp\Bigg\{ &\int_0^t \bigg(F(r(s)) + B_1(r(s))K_1(r(s)) \\ &- \frac{1}{2}\sum_{k=1}^{d}(G_k + B_2K_2)^2\bigg)\mathrm{d}s \\ &+ \sum_{k=1}^{d}(G_k + B_2K_2)w_k(t)\Bigg\}\Big(I + H + B_3K_3\Big)^{N(t)}, \end{aligned} \tag{IV.5}$$

and $x(0) = x_0$. Then we have

$$\mathbb{E}(x^\top(t)x(t)) = x_0^\top M(t)x_0, \tag{IV.6}$$

where $M(t)$ is defined as (IV.3). According to Theorem 3, system (IV.1) is finite-time annular domain stable with respect to $(c_1, c_2, c_3, c_4, T, I)$ if and only if $\frac{c_3}{c_1}I < M(t) < \frac{c_4}{c_2}I$ for $\forall\, t \in [0,T]$. The proof is complete.

V. NUMERICAL EXAMPLES

In this section, two numerical examples are respectively presented to illustrate the importance of the Markov chain, Poisson jump.

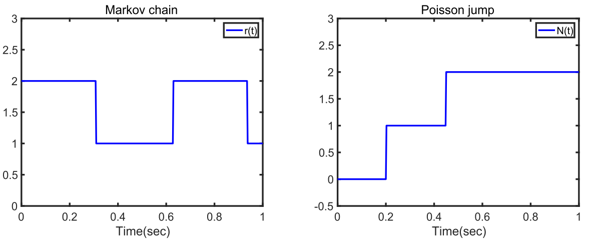

FIGURE 1. **The trajectories of Markov chain and Poisson jump.**

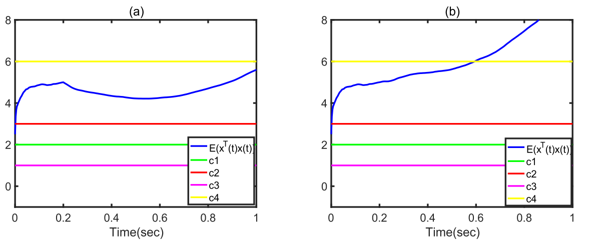

FIGURE 2. **The trajectories of $\mathbb{E}(x^\top(t)x(t))$ for system with Poisson jump (a) and system without Poisson jump (b).**

Example 6: Consider a two-dimensional regime-switching jump diffusion system which is the special case of system (IV.1) where $n = 2$ and $d = 1$, $r(t) \in \mathbb{S} = \{1, 2\}$ with the generator $\Gamma = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$, $u_1(t) = K_1(r(t))x(t)$, $u_2(t) = K_2x(t)$, $u_3(t) = K_3x(t)$ and the coefficient matrices are respectively

$$
\begin{aligned}
F(1) &= \begin{bmatrix} 0.2 & 0.5 \\ 0.3 & 0.4 \end{bmatrix}, \quad F(2) = \begin{bmatrix} 0.17 & 0.26 \\ 0.3 & 0.2 \end{bmatrix}, \\
G &= \begin{bmatrix} 0.17 & 0.15 \\ 0.14 & 0.1 \end{bmatrix}, \quad H = \begin{bmatrix} -0.36 & 0.5 \\ 0.21 & 0.39 \end{bmatrix}, \\
B_1(1) &= \begin{bmatrix} 0.11 & 0.22 \\ 0.07 & 0.15 \end{bmatrix}, \quad B_1(2) = \begin{bmatrix} 0.34 & 0.51 \\ 0.25 & 0.36 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} 0.42 & 0.44 \\ 0.35 & 0.53 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.15 & 0.19 \\ 0.14 & 0.16 \end{bmatrix}, \\
K_1(1) &= \begin{bmatrix} -0.1 & 0.07 \\ 0.15 & -0.05 \end{bmatrix}, \quad K_1(2) = \begin{bmatrix} 0.13 & 0.09 \\ 0.12 & 0.07 \end{bmatrix}, \\
K_2 &= \begin{bmatrix} 0.2 & 0.19 \\ 0.1 & -0.05 \end{bmatrix}, \quad K_3 = \begin{bmatrix} -0.14 & 0.05 \\ 0.12 & -0.03 \end{bmatrix}, \\
x(0) &= \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix}. \qquad \text{(V.1)}
\end{aligned}
$$

In addition, let $c_1 = 2$, $c_2 = 3$, $c_3 = 1$, $c_4 = 6$, $T = 1$, $\lambda = 0.6$. By Theorem 5, we have that system (IV.1) with coefficients (V.1) satisfies $1 = c_3 < \mathbb{E}_1[x^\top(t)x(t)] < 6 = c_4$ when $2 = c_1 \leq \mathbb{E}_1[x^\top(0)x(0)] \leq c_2 = 3$, i.e., the closed-loop system composed of (IV.1) and (V.1) is finite-time annular stable with respect to $(2, 3, 1, 6, 1, I)$.

The simulations results of system (IV.1) with coefficients (V.1) are presented in Figure 1 and Figure 2 (a). In Figure 2 (b), the simulation is about system (IV.1) without Poisson jump, i.e., $B_3 = 0$, $H = 0$ and other coefficients are same as those in (V.1). Figure 2 (b) shows that when there is no Poisson jump, the closed-loop system is not finite-time annular domain stable with respect to $(2, 3, 1, 6, 1, I)$. This implies that Poisson jump plays an active role in the finite-time annular domain stability.

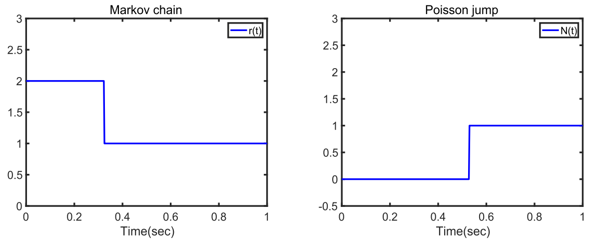

FIGURE 3. **The trajectories of Markov chain and poisson jump.**

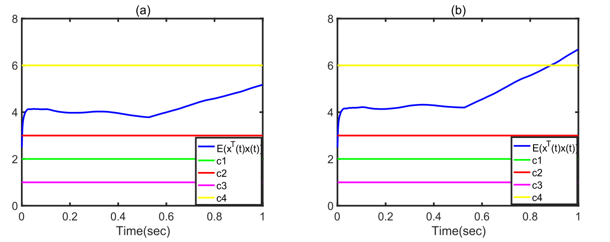

FIGURE 4. **The trajectories of $\mathbb{E}(x^\top(t)x(t))$ for system with Markov chain (a) and system without Markov chain (b).**

Example 7: Consider a two-dimensional regime-switching jump diffusion system which is the special case of system (IV.1) where $n = 2$ and $d = 1$, $r(t) \in \mathbb{S} = \{1, 2\}$ with the generator $\Gamma = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$, the corresponding state feedback controllers $u_1 = K_1(r(t))x(t)$, $u_2 = K_2x(t)$, $u_3 = K_3x(t)$ and the coefficient matrices are respectively

$$
\begin{aligned}
F(1) &= \begin{bmatrix} 0.1 & 0.15 \\ 0.13 & 0.14 \end{bmatrix}, \quad F(2) = \begin{bmatrix} 0.25 & 0.26 \\ 0.23 & 0.32 \end{bmatrix}, \\
G &= \begin{bmatrix} 0.1 & 0.21 \\ 0.12 & 0.15 \end{bmatrix}, \quad H = \begin{bmatrix} 0.36 & 0.3 \\ 0.26 & 0.39 \end{bmatrix}, \\
B_1(1) &= \begin{bmatrix} 0.21 & 0.22 \\ 0.07 & 0.15 \end{bmatrix}, \quad B_1(2) = \begin{bmatrix} 0.34 & 0.51 \\ 0.25 & 0.36 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} 0.22 & 0.14 \\ 0.35 & 0.63 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.25 & 0.16 \\ 0.18 & 0.19 \end{bmatrix}, \\
K_1(1) &= \begin{bmatrix} 0.03 & 0.11 \\ 0.06 & 0.07 \end{bmatrix}, \quad K_1(2) = \begin{bmatrix} 0.13 & 0.09 \\ 0.12 & 0.07 \end{bmatrix}, \\
K_2 &= \begin{bmatrix} 0.04 & 0.1 \\ 0.08 & 0.07 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0.12 & 0.03 \\ 0.04 & 0.07 \end{bmatrix}, \\
F &= \begin{bmatrix} 0.25 & 0.26 \\ 0.23 & 0.32 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.34 & 0.51 \\ 0.25 & 0.36 \end{bmatrix}, \\
K_1 &= \begin{bmatrix} 0.13 & 0.09 \\ 0.12 & 0.07 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix}. \qquad \text{(V.2)}
\end{aligned}
$$

By Theorem 5, we have $1 = c_3 < \mathbb{E}_1[x^\top(t)x(t)] < 5 = c_4$ when $2 = c_1 \leq \mathbb{E}_1[x^\top(0)x(0)] \leq c_2 = 3$, i.e., the closed-loop system composed of (IV.1) with (V.2) is finite-time annular domain stable with respect to $(2, 3, 1, 5, 1, I)$.

The simulations results of system (IV.1) with coefficients (V.2) are presented in Figure 3 and Figure 4 (a). Figure 4 (b) depicts the simulation about system (IV.1) without Markov chain, and F, B_1 and K_1 are constant matrices in (V.2) and other matrices have no change. Figure 4 (b) shows that the closed-loop system with no Markov chain is not finite-time annular domain stable with respect to $(2, 3, 1, 5, 1, I)$. This implies that Markov chain is helpful for finite-time annular domain stability.

VI. CONCLUSION

In this paper, we gave the sufficient and necessary condition of finite-time annular domain stability and stabilization for regime-switching jump diffusion systems. By numerical examples, we showed the importance of Markov chain and Poisson jump for finite-time annular domain stability of stochastic systems.

APPENDIX

PROOF OF LEMMA III.1.

When $\rho\big(H(r(t-))\big) < 1$ for $\forall t \in \mathbb{R}_+$, then it follows from [16] that $\exp\big\{\ln\big(I+H\big)\big\} = I+H$. Let

$$Y(t) = \int_0^t \big(F(r(s)) - \frac{1}{2}\sum_{k=1}^{d} G_k^2\big)\mathrm{d}s + \sum_{k=1}^{d}\int_0^t G_k \mathrm{d}w_k(s) + \int_0^t \ln\big(I+H\big)\mathrm{d}N(s), \tag{A.1}$$

then $\Phi(t) = \exp(Y(t))$. By the generalized Itô's formula, we have

$$\mathrm{d}\Phi(t) = F(r(t))\Phi(t)\mathrm{d}t + \sum_{k=1}^{d} G_k\Phi(t)\mathrm{d}w_k(t) + H\Phi(t-)\mathrm{d}N(t). \tag{A.2}$$

Then,

$$\mathrm{d}\Phi(t)x_0 = F(r(t))\Phi(t)x_0\mathrm{d}t + \sum_{k=1}^{d} G_k\Phi(t)x_0\mathrm{d}w_k(t) + H\Phi(t-)x_0\mathrm{d}N(t). \tag{A.3}$$

Then $\Phi(t)x_0$ is an explicit solution of multi-dimensional linear system (III.1). Because of the uniqueness of solution of system (III.1), the solution $x(t)$ of system (III.1) can be expressed as $x(t) = \Phi(t)x_0$. The proof is complete.

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