IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received 2 March 2024, accepted 15 March 2024, date of publication 25 March 2024, date of current version 29 March 2024.

Digital Object Identifier 10.1109/ACCESS.2024.3380025

RESEARCH ARTICLE

Development of an Investment Sector Selector Using a TOPSIS Method Based on Novel Distances and Similarity Measures for Picture Fuzzy Hypersoft Sets

MUHAMMAD IMRAN HARL[1], MUHAMMAD SAEED[1], MUHAMMAD HARIS SAEED[2], SANAA AHMED BAJRI[3], ALHANOUF ALBURAIKAN[4], AND HAMIDEN ABD EL-WAHED KHALIFA[4,5]

[1]Department of Mathematics, University of Management and Technology, Lahore, Punjab 54000, Pakistan
[2]Department of Chemistry, University of Management and Technology, Lahore, Punjab 54000, Pakistan
[3]Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
[4]Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia
[5]Department of Operations and Management Research, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza 12613, Egypt

Corresponding author: Sanaa Ahmed Bajri (sabajri@pnu.edu.sa)

This work was supported by Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia, through the Researchers Supporting Project PNURSP2024R527.

ABSTRACT The selection of an optimal investment sector is of critical importance not only for individual financial success but also to drive economic development. The allocation of capital into sectors with high potential for growth, innovation, and job creation is key. In addressing the complexity of decision-making scenarios associated with investment sector exploration, we introduce a novel data structure known as Picture fuzzy hypersoft set ($\mathbb{PF}_{\mathbb{HSS}s}$). This specialized approach within computational intelligence and decision-making aims to categorize data into various attributes and sub-attributes, considering the significant role of neutrality. The study stems from the need for a comprehensive framework ($\mathbb{PFHSS}s$) that can effectively handle intricate decision-making scenarios involving attributes, subattributes, and nuanced factors such as neutrality. Traditional tools such as TOPSIS and its extensions of fuzzy sets, while robust in Multiple Criteria Decision Making (MCDM), may face challenges in modeling and analyzing decision-making information within a $\mathbb{PFHSS}s$ environment. The rationale behind this study lies in enhancing the accuracy and efficiency of decision-making processes when dealing with complex, fuzzy, and multi-criteria data. By introducing newly proposed distances and similarity measures tailored to $\mathbb{PFHSS}s$, and constructing a $\mathbb{PFHSS}s$-TOPSIS method, we aim to address the limitations faced by existing models in the $\mathbb{PF}_{\mathbb{HSS}s}$ environment. The application of Hamming distance-based similarity measures further distinguishes our method by determining the weights assigned to each decision maker. The proposed $\mathbb{PF}_{\mathbb{HSS}s}$-TOPSIS method is practically applied in designing an optimal investment sector exploration tool for investors. This method has the potential to establish a crucial connection between alternatives and attributes, providing value across various fields and industries. The research emphasizes bridging the gap in decision-making scenarios where alternatives and attributes need to be effectively connected and analyzed, thereby contributing to the advancement of decision-making processes in complex domains.

INDEX TERMS Soft set, hypersoft set, picture fuzzy hypersoft set, similarity measure, TOPSIS, algorithms, decision making, optimization.

I. INTRODUCTION

The choice of the most suitable investment sector is crucial, not only for individual financial success but also

The associate editor coordinating the review of this manuscript and approving it for publication was Francisco J. Garcia-Penalvo.

for its pivotal role in driving economic development by directing capital towards sectors with the highest potential for growth, innovation, and job creation. In the complex world of investment, decision-makers often rely on a blend of quantitative data and qualitative insights driven by human intuition. However, harnessing these human-intuitionistic data effectively requires tools that can handle the inherent uncertainty and imprecision in decision-making. Here, fuzzy set theory comes into play. Fuzzy set theory provides a mathematical framework for dealing with vague or imprecise information, allowing for a more nuanced and realistic representation of investor preferences, risk perceptions, and market sentiment. By incorporating fuzzy set theory into the analysis, the Investment Sector Selector application can better capture the subtleties of decision-making, ensuring that it considers the intricacies of individual investor's preferences and beliefs when recommending optimal investment strategies. This approach enhances the application's ability to provide well-informed, balanced, and personalized investment sector recommendations, ultimately contributing to more effective economic development strategies.

Zadeh [1] introduced the concept of fuzzy sets (FS), which was a groundbreaking contribution to dealing with uncertainty and vagueness in various fields. Atanassov and Stoeva [2] introduced intuitionistic fuzzy sets (IFS). IFS differs from FS by introducing the concept of a non-membership degree, which quantifies the degree to which an element does not belong to a set. This complements the degree of membership, which quantifies the degree of belongingness. IFS is considered a natural progression from FS and aims to overcome some of the drawbacks of fuzzy set theory. It provides a broader framework for managing information that is fuzzy, uncertain, or incomplete. The introduction of picture fuzzy sets (PFS) by Cuong [3] addresses the limitation of IFS in handling inconsistent information, particularly in scenarios such as voting questions. PFS is designed to accommodate such situations by incorporating three distinct functions: positive membership function, neutral membership function, and negative membership function. By incorporating these three functions, the PFS provides a more nuanced representation of information, especially in cases where people may have varying degrees of support or opposition to a particular option. This is particularly relevant in voting scenarios where individuals can "vote for," "abstain," "vote against," or "refuse to vote." PFS allows for a finer-grained analysis of opinions and attitudes, capturing not only the traditional membership and non-membership degrees but also neutrality. He and Wang [4] proposed a novel information transformation mechanism to facilitate the conversion of unstructured data to picture fuzzy numbers (PFN). This mechanism serves as a bridge between raw unstructured data and the structured representation offered by PFNs, enabling a more effective analysis and interpretation of the data. Wang and Yu [5] contributes to the development of methods and techniques to handle PF information in continuous dynamic emergency decision making scenarios. Luo and Li [6] study on the construction method of similarity measures on Picture Fuzzy Sets (PFS), the focus lies on developing a framework to assess similarity between PFS. By incorporating PFSs into the MRN (Multi-Resolution Network) approach, Zhang et al. [7] have explored a novel methodology that effectively leverages the advantages of MG-PRSs and three way decisions within the FL framework. This comprehensive investigation aims to enhance the efficiency and effectiveness of MRN models in dealing with complex and uncertain data. Molodtsov [8] introduced the concept of soft set (SS) theory as a way to address unpredictability in a parametric manner. This theory aims to overcome restrictions and challenges in dealing with uncertain or unpredictable data. Maji et al. [9] introduced the concept of a fuzzy soft set (FSS), which is hybrid model that combines the characteristics of both FS and SS. The motivation behind introducing this hybrid model is to account for the inherent vagueness and imprecision in real-world data, especially in situations where attributes cannot be accurately characterized using only crisp values of 0 and 1.

Maji et al. [10] introduced the concept of an intuitionistic fuzzy soft set (IFSS), which is a mathematical framework that combines elements from both IFS and SS. In this hybrid model, each element is characterized not only by its membership degree and non-membership degree but also by a hesitation degree, just like in IFS. Additionally, the flexibility of SS is retained, allowing for elements to belong to a set with varying degrees of confidence or uncertainty. The picture fuzzy soft set (PFSS), as defined by Yang et al. [11], represents a hybrid model that combines elements of both PFS and SS. PFSS allows for a parametrization point of view when dealing with uncertainties in a picture fuzzy environment, each element in the universal set can be characterized or described in terms of various attributes, and these attributes introduce different levels of uncertainty, positivity, neutrality, and negativity. The limitations of previous theories in coping with overall inconsistency and inaccuracy in data, particularly when dealing with characteristics of a group of parameters that contain additional subattributes. To address these limitations, Smarandache [12] introduced the concept of hypersoft sets (HSS) by building upon the SS framework. The key innovation in the development of hypersoft sets is the introduction of a multi-decision function, instead of associating just one set of values with each parameter, HSS enable the association of multiple sets of values, reflecting different possible assignments for each attribute or decision.

Saeed et al. [13] conducted a study to explore and establish the fundamental principles and concepts of hypersoft set theory. This likely includes defining the core concepts, properties, and characteristics of HSS. Abbas et al. [14] defined the basic operations that can be performed on HSS. These operations are essential for manipulating HSS and performing various mathematical and computational tasks within the HSS framework. These basic operations likely include set union, intersection, complement, and other fundamental set operations adapted to the context of HSS. Yolcu and Ozturk [15] introduced the concept of fuzzy hypersoft sets (FHSS), which extend traditional HSS to handle fuzzy and uncertain information. These FHSS have found wide applications in various domains where decision-making involves uncertain or imprecise data. The framework allows for a flexible representation of attributes, subattributes, and their relationships, enhancing decision support systems. Yolcu et al. [16] also led to the development of intuitionistic fuzzy hypersoft sets (IFHSS), which further expand the framework to accommodate intuitionistic fuzzy information. This comprehensive framework is capable of handling not only fuzzy and uncertain data but also incomplete and vague information. Saeed and Harl [17] introduced the notion of picture fuzzy hypersoft sets ($\mathbb{PF}_{\mathbb{HSS}_s}$), which is used to handle attributes and subattributes, making them particularly valuable when dealing with inconsistent information. Building on the concept of $\mathbb{PF}_{\mathbb{HSS}_s}$, Saeed et al. [18] introduced the idea of picture fuzzy hypersoft graphs (PFHSG).

These graphs provide a fresh perspective on risk analysis, particularly in product sales, by visually representing the contributing variables and their interactions. They enhance the transparency and interpretability of risk assessment, aiding decision-makers in understanding complex systems. Harl et al. [19] introduced lattice-ordered picture fuzzy hypersoft (LOPFHSS). This structure is particularly beneficial for complex decision-making situations where attributes need to be considered at a sub-attribute level, uncertainty is present, and there is a need to establish ordering or preference among parameters. This approach provides a more flexible and structured way to handle such decision-making problems, offering potential solutions to challenges in real-world applications. The concept of a bipolar picture fuzzy hypersoft set (BPFHSS), as introduced by Harl et al. [20], addresses the importance of considering inconsistent, bipolar, and multiple sub-attribute information in decision-making processes.

Similarity measures play a crucial role in dealing with uncertain data, especially in the context of SS, FSS, IFS, IFSS, PFSS and HSS. These measures are essential for quantifying the degree of similarity or resemblance between two sets or elements characterized by uncertainty.

Majumdar and Samanta [21] initiated the study of similarity measures of SS and introduced new similarity measures for FSS, which were based on distance metrics, set-theoretic approaches, and matching functions. Liu et al. [22] proposed similarity measures and entropy measures for FSS and investigated their properties. Feng and Zheng [23] conducted research on novel similarity measures for FSS, focusing on distance-based measures. Ghose et al. [24] introduced a weighted similarity measure using intuitionistic fuzzy soft sets for the classification of biomarkers. Aggarwal et al. [25] introduced the PFSS similarity measure. These similarity measures contribute to the toolbox of methods available for working with PFSSs, making them more versatile and applicable in various domains where uncertainty and imprecision are inherent. Rahman et al. [26] proposed a similarity measure between two possibility intuitionistic fuzzy hypersoft. Saqlain et al. [27] developed distance and similarity measures for IFHSS with the help of aggregate operators. The introduction of similarity measures for neutrosophic hypersoft sets (NHSSs) by Saqlain et al. [28] is crucial, as these sets are designed to handle data with additional elements of indeterminacy, making them suitable for scenarios where uncertainty and incomplete information are prevalent. Jafar et al. [29] proposed trigonometric similarity measures specifically designed for NHSSs, applying these trigonometric similarity measures to the context of renewable energy source selection.

Hwang et al. [30] originally developed the TOPSIS method, which is a popular multicriteria decision analysis technique used for ranking alternatives based on their similarity to the ideal solution. Chen et al. [31], [32] further studied and possibly extended the TOPSIS method, likely exploring variations or applications of the technique in specific contexts. Boran et al. [33] introduced an adaptation of TOPSIS that incorporates IFS. IFS allow for representing both membership and non-membership degrees of elements, making the TOPSIS approach more suitable for handling uncertainty and vagueness in decision-making problems. Chen et al. [34] developed a proportional interval T2 hesitant fuzzy TOPSIS approach. This likely involves incorporating hesitant fuzzy sets that allow decision-makers to express their uncertainty or hesitancy about the membership of elements in a set. The Hamacher aggregation operators and andness optimization models might be used to aggregate preferences and make decisions. Eraslan and Karaaslan [35] introduced the fuzzy soft TOPSIS method as a multi-criteria decision-making technique. Fuzzy soft sets are a generalization of traditional fuzzy sets and can capture more complex decision scenarios where uncertainty and soft information play a role. The approach proposed by Mehmood et al. [36], involving the use of lattice-ordered t-bipolar soft sets in the TOPSIS method, is a novel and innovative method for solving MCDM problems.

The extension of the TOPSIS method under IFHSS environment, based on correlation coefficients and aggregation operators, as introduced by Zulqarnain et al. [37]. The selection of an effective hand sanitizer to reduce the effects of COVID-19 is a critical decision in the current global health crisis. An extension of the TOPSIS technique based on correlation coefficients under the NHSS presents by Samad et al. [38] an innovative approach to this problem.

Kalaichelvi and Malini [39] conducted a study that applied fuzzy soft sets to investment decision-making problems. This research likely involved collecting data from female employees working in government and private sector companies in Coimbatore, Tamil Nadu, India. The notion of "period" (daily, weekly, monthly, or annually) is crucial for investment decision-making problems. This suggests that the timing of investments, such as how often decisions are made or how long investments are held, plays an important role in the decision-making process. Different periods may have varying degrees of impact on investment choices. özgür and Nihal [40] introduced a new method to incorporate the notion of period into investment decision-making using soft set and matrix theories. Mukherjee and Das [41] has introduced a decision-making procedure that focuses on solving investment decisions. This procedure uses fuzzy fuzzy soft sets and opinion weighting vectors to handle uncertainty, imprecision, and subjective preferences in the investment decision-making process. Khan et al. [42] applied picture fuzzy soft robust VIKOR Method and its associated algorithmic procedures to solve investment problems. Zhang et al. [43] used the matrix version of complex fuzzy hypersoft sets(CFHSS) to construct a decision-support system for the evaluation of real estate residential projects. Zhao et al. [44] proposed the possibility single-valued neutrosophic hypersoft set (psv-NHSS) framework for evaluating investment projects. Khan et al. [45] developed the "q-Rung Orthopair Fuzzy Hypersoft sets" aggregation operator to address a decision-making problem related to real estate investment.

A. MOTIVATION

With that, $\mathbb{PF}_{\mathbb{HSS}_s}$ is a powerful mathematical framework for handling attributes and subattributes, especially in scenarios where information may be inconsistent or imprecise. Here are some key features of $\mathbb{PF}_{\mathbb{HSS}_s}$ that make them valuable in addressing challenges:

(i) $\mathbb{PF}_{\mathbb{HSS}_s}$ excel in situations where the available information related to attributes and subattributes is inconsistent. In real-world decision scenarios, it's common to encounter data inconsistencies, conflicts, or variations. $\mathbb{PF}_{\mathbb{HSS}_s}$ provide a robust tool for managing and making sense of this inconsistent information.

(ii) $\mathbb{PF}_{\mathbb{HSS}_s}$ allow the representation of multiple points of view or perspectives associated with attributes and subattributes. Decision-makers can model and analyze different sources of information within a unified framework. This multi-view representation is valuable when dealing with diverse and sometimes conflicting data sources.

(iii) $\mathbb{PF}_{\mathbb{HSS}_s}$-based decision models enhance transparency by visualizing the relationships and inconsistencies between attributes and subattributes. This transparency helps in the understanding and interpretation of complex decision scenarios. Decision-makers can see how different attributes and subattributes interact and contribute to the overall decision.

(iv) $\mathbb{PF}_{\mathbb{HSS}_s}$ consider the membership degree (MD), non-membership degree (NMD), and abstinence degree (AD) values for attributes and subattributes. This comprehensive approach allows for a more nuanced representation of the uncertainty and vagueness associated with each element.

(v) $\mathbb{PF}_{\mathbb{HSS}_s}$ are capable of handling multiargument data, meaning that they can capture and model complex relationships involving multiple attributes and subattributes simultaneously. This makes them suitable for decision-making in multidimensional environments.

(vi) $\mathbb{PF}_{\mathbb{HSS}_s}$ are versatile and can be applied to various domains and problem types, including decision support, risk analysis, and multi-criteria decision-making (MCDM).

While traditional TOPSIS and its fuzzy set extensions are powerful tools for MCDM, they may face challenges when dealing with complex decision-making information in a $\mathbb{PF}_{\mathbb{HSS}_s}$ environment. The development of a TOPSIS method based on new distances and similarity measures for $\mathbb{PF}_{\mathbb{HSS}_s}$ and matrices represents a valuable contribution to decision science. This approach enhances decision accuracy, streamlines complex decision processes, and accommodates multi-criteria considerations and fuzziness. By leveraging Hamming distance-based similarity measures, it provides a systematic and mathematically grounded framework for assessing similarities in complex and uncertain data, ultimately improving the quality of decision making across various domains.

B. PAPER LAYOUT

The structure and content of the study appear to be organized as follows: The section II provides foundational knowledge by introducing important concepts. The section III introduces and presents distance and similarity measures designed for $\mathbb{PF}_{\mathbb{HSS}_s}$. The section IV outlines the construction of $\mathbb{PF}_{\mathbb{HSS}_s}$-TOPSIS based on the proposed distance and similarity measure. The section V demonstrates the practical application of the developed $\mathbb{PF}_{\mathbb{HSS}_s}$-TOPSIS algorithm in the context of designing an investment sector selector. The section VI is dedicated to a comparative analysis. The final section VII provides concluding remarks and discusses potential directions for future research. It summarizes the findings of the study and suggests areas where further investigation and improvement may be needed

II. PRELIMINARIES

This section serves as a foundation by reviewing the fundamental definitions of various concepts. It covers FS, IFS, PFS, HSS and $\mathbb{PF}_{\mathbb{HSS}_s}$. This is likely to provide readers with the necessary background knowledge for the subsequent sections.

Definition 1 [2]: An IFS on universe of discourse $\mathbb{Q} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \ldots, \mathbf{q}_\alpha\}$ is defined as:

$$\mathrm{Z} = \{\langle m^{\mathrm{Z}^\kappa}_{(\mathbf{q}_i)}, n^{\mathrm{Z}^\kappa}_{(\mathbf{q}_i)}\rangle |\ q_i \in \mathbb{Q}\}$$

where $m^{\mathrm{Z}^\kappa}_{(\mathbf{q}_i)} : \mathbb{Q} \longrightarrow [0, 1]$ denotes membership degree of q_i in Z, $n^{\mathrm{Z}^\kappa}_{(\mathbf{q}_i)} : \mathbb{Q} \longrightarrow [0, 1]$ is non membership degree of q_i in Z. $0 \leq m^{\mathrm{Z}^\kappa}_{(\mathbf{q}_i)} + n^{\mathrm{Z}^\kappa}_{(\mathbf{q}_i)} \leq 1$, $h^{\mathrm{Z}^\kappa}_{(\mathbf{q}_i)} = 1 - m^{\mathrm{Z}^\kappa}_{(\mathbf{q}_i)} - n^{\mathrm{Z}^\kappa}_{(\mathbf{q}_i)}$ represent hesitancy degree of q_i in Z, $\forall\, q_i \in \mathbb{Q}$, $0 \leq h^{\mathrm{Z}^\kappa}_{(\mathbf{q}_i)} \leq 1$.

Definition 2 [3]: A PFS on universe of discourse Let $\mathbb{Q} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \ldots, \mathbf{q}_\alpha\}$ is defined as::

$$\mathrm{V} = \{\langle m^{\mathrm{V}^\kappa}_{(\mathbf{q}_i)}, t^{\mathrm{V}^\kappa}_{(\mathbf{q}_i)}, n^{\mathrm{V}^\kappa}_{(\mathbf{q}_i)}\rangle |\ q_i \in \mathbb{Q}\}$$

where $m^{\mathrm{Z}^\kappa}_{(\mathbf{q}_i)} : \mathbb{Q} \longrightarrow [0, 1]$ denotes membership degree of q_i in Z, $t^{\mathrm{V}^\kappa}_{(\mathbf{q}_i)} : \mathbb{Q} \longrightarrow [0, 1]$ is neutral degree of q_i in Z, $n^{\mathrm{V}^\kappa}_{(\mathbf{q}_i)} : \mathbb{Q} \longrightarrow [0, 1]$ is non membership degree of q_i in Z. $0 \leq m^{\mathrm{V}^\kappa}_{(\mathbf{q}_i)} + t^{\mathrm{V}^\kappa}_{(\mathbf{q}_i)} + n^{\mathrm{V}^\kappa}_{(\mathbf{q}_i)} \leq 1$, $r^{\mathrm{V}^\kappa}_{(\mathbf{q}_i)} = 1 - m^{\mathrm{V}^\kappa}_{(\mathbf{q}_i)} - t^{\mathrm{V}^\kappa}_{(\mathbf{q}_i)} - n^{\mathrm{V}^\kappa}_{(\mathbf{q}_i)}$ represent refusal degree of q_i in V, $\forall\, q_i \in \mathbb{Q}$, $0 \leq r^{\mathrm{V}^\kappa}_{(\mathbf{q}_i)} \leq 1$.

Definition 3 [8]: Let $\mathbb{Q}$ be a universe of discourse, $\mathcal{P}(\mathbb{Q})$ the power set of $\mathbb{Q}$ and h^κ_i a set of attributes. Then, the pair $(\mathbb{F}, h^\kappa_i)$, where $\mathbb{F} : h^\kappa_i \longrightarrow \mathcal{P}(\mathbb{Q})$ is called a SS over $\mathbb{Q}$.

Definition 4 [11]: Let h_i^{κ} is a parametric set. Consider a function $\mathbb{F} : h_i^{\kappa} \longrightarrow \mathcal{P}(PF(\mathbb{Q}))$, $\mathcal{P}(PF(\mathbb{Q}))$ is power set of PFS over $\mathbb{Q}$ then pair $(\mathbb{F}, h_i^{\kappa})$ is representation of PFSS.

Definition 5 [13]: Let $\mathbb{Q}$ be a universe of discourse, $\mathcal{P}(\mathbb{Q})$ the power set of $\mathbb{Q}$. Let $h_i^{\kappa} = \{h_1^{\kappa}, h_2^{\kappa}, h_3^{\kappa}, h_4^{\kappa}, \ldots\ldots\ldots, h_\beta^{\kappa}\}$ be β disjoint parameters set whose corresponding attribute values are $H_1^{\kappa}, H_2^{\kappa}, H_3^{\kappa}, H_4^{\kappa}, \ldots\ldots\ldots, H_\beta^{\kappa}$. Suppose $H^{\kappa} = H_1^{\kappa} \times H_2^{\kappa} \times H_3^{\kappa} \times H_4^{\kappa} \times \ldots\ldots \times H_\beta^{\kappa}$ with $H_s^{\kappa} \cap H_t^{\kappa} = \emptyset$, $t \neq s$, and t, s $\in \{1, 2, \ldots., \beta\}$. The pair $(\mathbb{F}, H^{\kappa})$, where $\mathbb{F}$: $H^{\kappa} \to \mathcal{P}(\mathbb{Q})$ is called a HSS over $\mathbb{Q}$.

Definition 6 [14]: Let $\mathbb{Q}$ be a universe of discourse, $\mathcal{P}(PF(\mathbb{Q}))$ the power set of PFS over $\mathbb{Q}$. Let $h_i^{\kappa} = \{h_1^{\kappa}, h_2^{\kappa}, h_3^{\kappa}, h_4^{\kappa}, \ldots\ldots\ldots, h_\beta^{\kappa}\}$ be β disjoint parameters set whose corresponding attribute values are $H_1^{\kappa}, H_2^{\kappa}, H_3^{\kappa}, H_4^{\kappa}, \ldots\ldots\ldots, H_\beta^{\kappa}$. Suppose $H^{\kappa} = H_1^{\kappa} \times H_2^{\kappa} \times H_3^{\kappa} \times H_4^{\kappa} \times \ldots\ldots \times H_\beta^{\kappa}$ with $H_s^{\kappa} \cap H_t^{\kappa} = \emptyset$, $t \neq s$, and t, s $\in \{1, 2, \ldots., \beta\}$. The pair $(\mathbb{F}, H^{\kappa})$, where $\mathbb{F}$: $H^{\kappa} \to \mathcal{P}(\mathbb{Q})$ is called a $\mathbb{PF}_{\mathbb{HSS}_s}$ over $\mathbb{Q}$. It is represented as follows;

$$(\mathbb{F}, H^{\kappa}) = \mathbb{F}(w_{\varsigma s}^{\kappa}) = \{\langle m_{(\mathbf{q}_i)}^{\mathrm{V}^{\kappa}}, t_{(\mathbf{q}_i)}^{\mathrm{V}^{\kappa}}, n_{(\mathbf{q}_i)}^{\mathrm{V}^{\kappa}}\rangle \mid \; q_i \in \mathbb{Q}\} \;\forall\; w_{\varsigma s}^{\kappa} \in H^{\kappa}$$

III. DEVELOPMENT OF DISTANCE AND SIMILARITY METRICS FOR $\mathbb{PF}_{\mathbb{HSS}_s}$

This section introduces and presents extended distance and similarity measures designed for PFHSS. These measures are likely developed to facilitate more nuanced and accurate comparisons within the PFHSS framework.

Definition 7: Let $\mathbb{Q} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \ldots., \mathbf{q}_\alpha\}$ be the set of alternative and $\mathbf{h}_1^{\kappa}, \mathbf{h}_2^{\kappa}, \ldots.., \mathbf{h}_\beta^{\kappa}$ be the sets of attributes and their corresponding attributive values are respectively the set $\mathbf{H}_{\varsigma 1}^{\kappa}, \mathbf{H}_{\varsigma 2}^{\kappa}, \mathbf{H}_{\varsigma 3}^{\kappa}, \ldots\ldots, \mathbf{H}_{\varsigma\beta}^{\kappa}$. Suppose $\mathbf{H}_{\varsigma}^{\kappa} = \mathbf{H}_{\varsigma 1}^{\kappa} \times \mathbf{H}_{\varsigma 2}^{\kappa} \times \mathbf{H}_{\varsigma 3}^{\kappa} \times \ldots\ldots \times \mathbf{H}_{\varsigma\beta}^{\kappa}$, with $\mathbf{H}_{\varsigma t}^{\kappa} \cap \mathbf{H}_{\varsigma s}^{\kappa} = \emptyset$, $t \neq s$, and t, s $\in \{1, 2, \ldots., n\}$. The pair $(\mathbf{M}_{ij}^{\kappa}, \mathbf{H}_{\varsigma}^{\kappa})$, where $\mathbf{M}_{ij}^{\kappa}$: $\mathbf{H}_{\varsigma}^{\kappa} \to \mathcal{P}(PF(\mathbb{Q}))$ is called a picture fuzzy hypersoft matrix ($\mathbb{PF}_{\mathbb{HSM}}$) of order $\alpha \times \beta$ over $\mathbb{Q}$. Then

$$(\mathbf{M}_{ij}^{\kappa}, \mathbf{H}_{\varsigma}^{\kappa}) = \mathbf{M}_{ij}^{\kappa}(v_j) = \{\langle m_{\mathbf{M}_{ij}^{\kappa}}(\mathbf{q}_i), t_{\mathbf{M}_{ij}^{\kappa}}(\mathbf{q}_i), n_{\mathbf{M}_{ij}^{\kappa}}(\mathbf{q}_i)\rangle \mid q_i \in \mathbb{Q}\} \;\; \forall \;\; v_j \in H_{\varsigma}^{\kappa}.$$

where $i = 1, 2, 3, \ldots.., \alpha; j = 1, 2, 3, \ldots.., \beta$.

The $\mathbb{PF}_{\mathbb{HSM}}$ can be represented as follows

$$[\mathbf{M}_{ij}^{\kappa}(v_j)]_{\alpha\times\beta} = \begin{bmatrix} m_{(\mathbf{q}_1)}^{\mathbf{M}_{11}^{\kappa}}, t_{(\mathbf{q}_1)}^{\mathbf{M}_{11}^{\kappa}}, n_{(\mathbf{q}_1)}^{\mathbf{M}_{11}^{\kappa}} & \ldots\ldots & m_{(\mathbf{q}_1)}^{\mathbf{M}_{1\beta}^{\kappa}}, t_{(\mathbf{q}_1)}^{\mathbf{M}_{1\beta}^{\kappa}}, n_{(\mathbf{q}_1)}^{\mathbf{M}_{1\beta}^{\kappa}} \\ m_{(\mathbf{q}_2)}^{\mathbf{M}_{21}^{\kappa}}, t_{(\mathbf{q}_2)}^{\mathbf{M}_{21}^{\kappa}}, n_{(\mathbf{q}_2)}^{\mathbf{M}_{21}^{\kappa}} & \ldots\ldots & m_{(\mathbf{q}_2)}^{\mathbf{M}_{2\beta}^{\kappa}}, t_{(\mathbf{q}_2)}^{\mathbf{M}_{2\beta}^{\kappa}}, n_{(\mathbf{q}_2)}^{\mathbf{M}_{2\beta}^{\kappa}} \\ \vdots & \vdots & \vdots \\ m_{(\mathbf{q}_\alpha)}^{\mathbf{M}_{\alpha 1}^{\kappa}}, t_{(\mathbf{q}_\alpha)}^{\mathbf{M}_{\alpha 1}^{\kappa}}, n_{(\mathbf{q}_\alpha)}^{\mathbf{M}_{\alpha 1}^{\kappa}} & \ldots\ldots & m_{(\mathbf{q}_\alpha)}^{\mathbf{M}_{\alpha\beta}^{\kappa}}, t_{(\mathbf{q}_\alpha)}^{\mathbf{M}_{\alpha\beta}^{\kappa}}, n_{(\mathbf{q}_\alpha)}^{\mathbf{M}_{\alpha\beta}^{\kappa}} \end{bmatrix}$$

Definition 8: Let $(\mathbf{M}_1^{\kappa}, H_{\varsigma}^{\kappa})$ and $(\mathbf{M}_2^{\kappa}, H_{\varsigma}^{\kappa})$ be two $\mathbb{PF}_{\mathbb{HSS}_s}$ over $\mathbb{Q}$ then the normalized Hamming distance $\mathbf{D}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}_1^{\kappa}, \mathbf{M}_2^{\kappa})$ is defined as:

$$\mathbf{D}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}_1^{\kappa}, \mathbf{M}_2^{\kappa}) = \frac{1}{3n}\sum_{j=1}^{n}\left\{\begin{array}{l} |[m_{(\mathbf{q}_i)}^{\mathbf{M}_1^{\kappa}}]_{v_j} - [m_{(\mathbf{q}_i)}^{\mathbf{M}_2^{\kappa}}]_{v_j}| + \\ |[t_{(\mathbf{q}_i)}^{\mathbf{M}_1^{\kappa}}]_{v_j} - [t_{(\mathbf{q}_i)}^{\mathbf{M}_2^{\kappa}}]_{v_j}| + \\ |[n_{(\mathbf{q}_i)}^{\mathbf{M}_1^{\kappa}}]_{v_j} - [n_{(\mathbf{q}_i)}^{\mathbf{M}_2^{\kappa}}]_{v_j}| \end{array}\right\}.$$

where $i, j = 1, 2, 3, \ldots.., n$.

Definition 9: Let $(\mathbf{M}_1^{\kappa}, H_{\varsigma}^{\kappa})$ and $(\mathbf{M}_2^{\kappa}, H_{\varsigma}^{\kappa})$ be two $\mathbb{PF}_{\mathbb{HSS}_s}$ over $\mathbb{Q}$ then the Normalized Euclidean distance $\mathbf{D}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}_1^{\kappa}, \mathbf{M}_2^{\kappa})$ is defined, as shown in the equation at the bottom of the next page.

Definition 10: Let $(\mathbf{M}_1^{\kappa}, H_{\varsigma}^{\kappa})$ and $(\mathbf{M}_2^{\kappa}, H_{\varsigma}^{\kappa})$ be two $\mathbb{PF}_{\mathbb{HSS}_s}$ over $\mathbb{Q}$ then the Generalized Weighted distance $\mathbf{D}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}_1^{\kappa}, \mathbf{M}_2^{\kappa})_\lambda$ is defined as:

$$\mathbf{D}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}_1^{\kappa}, \mathbf{M}_2^{\kappa})_\lambda = \left[\frac{1}{3n}\sum_{j=1}^{n} r_j \left\{\begin{array}{l} |[m_{(\mathbf{q}_i)}^{\mathbf{M}_1^{\kappa}}]_{v_j} - [m_{(\mathbf{q}_i)}^{\mathbf{M}_2^{\kappa}}]_{v_j}|^\lambda \\ +|[t_{(\mathbf{q}_i)}^{\mathbf{M}_1^{\kappa}}]_{v_j} - [t_{(\mathbf{q}_i)}^{\mathbf{M}_2^{\kappa}}]_{v_j}|^\lambda + \\ |[n_{(\mathbf{q}_i)}^{\mathbf{M}_1^{\kappa}}]_{v_j} - [n_{(\mathbf{q}_i)}^{\mathbf{M}_2^{\kappa}}]_{v_j}|^\lambda \end{array}\right\}\right]^{\frac{1}{\lambda}}$$

Definition 11: Let $(\mathbf{M}_{1(ij)}^{\kappa}, H_{\varsigma}^{\kappa})$ and $(\mathbf{M}_{2(ij)}^{\kappa}, H_{\varsigma}^{\kappa})$ be two picture fuzzy hypersoft matrix $\mathbb{PF}_{\mathbb{HSM}}$ of order $\alpha \times \beta$ over $\mathbb{Q}$ then the Normalized Hamming distance $\mathbf{D}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}_{1(ij)}^{\kappa}, \mathbf{M}_{2(ij)}^{\kappa})$ is defined as:

$$\mathbf{D}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}_{1(ij)}^{\kappa}, \mathbf{M}_{2(ij)}^{\kappa}) = \frac{1}{3\alpha\beta}\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}\left\{\begin{array}{l} |[m_{(\mathbf{q}_i)}^{\mathbf{M}_{1(ij)}^{\kappa}}]_{v_j} - [m_{(\mathbf{q}_i)}^{\mathbf{M}_{2(ij)}^{\kappa}}]_{v_j}| + \\ |[t_{(\mathbf{q}_i)}^{\mathbf{M}_{1(ij)}^{\kappa}}]_{v_j} - [t_{(\mathbf{q}_i)}^{\mathbf{M}_{2(ij)}^{\kappa}}]_{v_j}| + \\ |[n_{(\mathbf{q}_i)}^{\mathbf{M}_{1(ij)}^{\kappa}}]_{v_j} - [n_{(\mathbf{q}_i)}^{\mathbf{M}_{2(ij)}^{\kappa}}]_{v_j}| \end{array}\right\}$$

Definition 12: Let $(\mathbf{M}_{1(ij)}^{\kappa}, H_{\varsigma}^{\kappa})$ and $(\mathbf{M}_{2(ij)}^{\kappa}, H_{\varsigma}^{\kappa})$ be two $\mathbb{PF}_{\mathbb{HSM}}$ over $\mathbb{Q}$ then the Normalized Euclidean distance $\mathbf{D}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}_{1(ij)}^{\kappa}, \mathbf{M}_{2(ij)}^{\kappa})$ is defined, as shown in the equation at the bottom of the next page.

Definition 13: Let $(\mathbf{M}_{1(ij)}^{\kappa}, H_{\varsigma}^{\kappa})$ and $(\mathbf{M}_{2(ij)}^{\kappa}, H_{\varsigma}^{\kappa})$ be two $\mathbb{PF}_{\mathbb{HSM}}$ over $\mathbb{Q}$ then the Generalized Weighted distance $\mathbf{D}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}_{1(ij)}^{\kappa}, \mathbf{M}_{2(ij)}^{\kappa})_\lambda$ is defined as:

$$\mathbf{D}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}_{1(ij)}^{\kappa}, \mathbf{M}_{2(ij)}^{\kappa})_\lambda = \left[\frac{1}{3n}\sum_{j=1}^{n} r_j \left\{\begin{array}{l} |[m_{(\mathbf{q}_i)}^{\mathbf{M}_{1(ij)}^{\kappa}}]_{v_j} - [m_{(\mathbf{q}_i)}^{\mathbf{M}_{2(ij)}^{\kappa}}]_{v_j}|^\lambda + \\ |[t_{(\mathbf{q}_i)}^{\mathbf{M}_{1(ij)}^{\kappa}}]_{v_j} - [t_{(\mathbf{q}_i)}^{\mathbf{M}_{2(ij)}^{\kappa}}]_{v_j}|^\lambda + \\ |[n_{(\mathbf{q}_i)}^{\mathbf{M}_{1(ij)}^{\kappa}}]_{v_j} - [n_{(\mathbf{q}_i)}^{\mathbf{M}_{2(ij)}^{\kappa}}]_{v_j}|^\lambda \end{array}\right\}\right]^{\frac{1}{\lambda}}$$

Definition 14: Let $(\mathbf{M}_1^{\kappa}, H_{\varsigma}^{\kappa})$ and $(\mathbf{M}_2^{\kappa}, H_{\varsigma}^{\kappa})$ be two $\mathbb{PF}_{\mathbb{HSS}_s}$ over $\mathbb{Q}$ then the similarity measure $\mathbf{S}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}_1^{\kappa}, \mathbf{M}_2^{\kappa})$ is defined as:

$$\mathbf{S}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}_1^{\kappa}, \mathbf{M}_2^{\kappa}) = 1 - \frac{1}{3n}\sum_{j=1}^{n}\left\{\begin{array}{l} |[m_{(\mathbf{q}_i)}^{\mathbf{M}_1^{\kappa}}]_{v_j} - [m_{(\mathbf{q}_i)}^{\mathbf{M}_2^{\kappa}}]_{v_j}| + \\ |[t_{(\mathbf{q}_i)}^{\mathbf{M}_1^{\kappa}}]_{v_j} - [t_{(\mathbf{q}_i)}^{\mathbf{M}_2^{\kappa}}]_{v_j}| + \\ |[n_{(\mathbf{q}_i)}^{\mathbf{M}_1^{\kappa}}]_{v_j} - [n_{(\mathbf{q}_i)}^{\mathbf{M}_2^{\kappa}}]_{v_j}| \end{array}\right\}$$

Definition 15: Let $(\mathbf{M}_1^{\kappa}, H_{\varsigma}^{\kappa})$ and $(\mathbf{M}_2^{\kappa}, H_{\varsigma}^{\kappa})$ be two $\mathbb{PF}_{\mathbb{HSS}_s}$ over $\mathbb{Q}$ then the Generalized Weighted similarity measure $\mathbf{S}^{\mathbb{PF}}_{\mathbb{HS}}(\mathbf{M}^{\kappa}_1, \mathbf{M}^{\kappa}_2)_\lambda$ is defined as:

$$\mathbf{S}^{\mathbb{PF}}_{\mathbb{HS}}(\mathbf{M}^{\kappa}_1, \mathbf{M}^{\kappa}_2)_\lambda = 1 - \left[\frac{1}{3n}\sum_{j=1}^{n} r_j \left\{\begin{array}{l} |[m^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [m^{\mathbf{M}^{\kappa}_{2(ij)}}_{(\mathbf{q}_i)}]_{v_j}|^\lambda + \\ |[t^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [t^{\mathbf{M}^{\kappa}_{2(ij)}}_{(\mathbf{q}_i)}]_{v_j}|^\lambda + \\ |[n^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [n^{\mathbf{M}^{\kappa}_{2(ij)}}_{(\mathbf{q}_i)}]_{v_j}|^\lambda \end{array}\right\}\right]^{\frac{1}{\lambda}}$$

Definition 16: Let $(\mathbf{M}^{\kappa}_{1(ij)}, H^{\kappa}_{\varsigma})$ and $(\mathbf{M}^{\kappa}_{2(ij)}, H^{\kappa}_{\varsigma})$ be two $\mathbb{PF}_{\mathbb{HSM}}$ of order $\alpha \times \beta$ over $\mathbb{Q}$ then the similarity measure $\mathbf{S}^{\mathbb{PF}}_{\mathbb{HS}}(\mathbf{M}^{\kappa}_{1(ij)}, \mathbf{M}^{\kappa}_{2(ij)})$ is defined as:

$$\mathbf{S}^{\mathbb{PF}}_{\mathbb{HS}}(\mathbf{M}^{\kappa}_{1(ij)}, \mathbf{M}^{\kappa}_{2(ij)}) = 1 - \frac{1}{3\alpha\beta}\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} \left\{\begin{array}{l} |[m^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [m^{\mathbf{M}^{\kappa}_{2(ij)}}_{(\mathbf{q}_i)}]_{v_j}| + \\ |[t^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [t^{\mathbf{M}^{\kappa}_{2(ij)}}_{(\mathbf{q}_i)}]_{v_j}| + \\ |[n^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [n^{\mathbf{M}^{\kappa}_{2(ij)}}_{(\mathbf{q}_i)}]_{v_j}| \end{array}\right\}$$

Definition 17: Let $(\mathbf{M}^{\kappa}_{1(ij)}, H^{\kappa}_{\varsigma})$ and $(\mathbf{M}^{\kappa}_{2(ij)}, H^{\kappa}_{\varsigma})$ be two $\mathbb{PF}_{\mathbb{HSM}}$ over $\mathbb{Q}$ then the Generalized Weighted similarity measure $\mathbf{S}^{\mathbb{PF}}_{\mathbb{HS}}(\mathbf{M}^{\kappa}_{1(ij)}, \mathbf{M}^{\kappa}_{2(ij)})_\lambda$ is defined as:

$$\mathbf{S}^{\mathbb{PF}}_{\mathbb{HS}}(\mathbf{M}^{\kappa}_{1(ij)}, \mathbf{M}^{\kappa}_{2(ij)})_\lambda = 1 - \left[\frac{1}{3\alpha\beta}\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} r_j \left\{\begin{array}{l} |[m^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [m^{\mathbf{M}^{\kappa}_{2(ij)}}_{(\mathbf{q}_i)}]_{v_j}|^\lambda + \\ |[t^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [t^{\mathbf{M}^{\kappa}_{2(ij)}}_{(\mathbf{q}_i)}]_{v_j}|^\lambda + \\ |[n^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [n^{\mathbf{M}^{\kappa}_{2(ij)}}_{(\mathbf{q}_i)}]_{v_j}|^\lambda \end{array}\right\}\right]^{\frac{1}{\lambda}}$$

IV. DEVELOPMENT OF $\mathbb{PF}_{\mathbb{HSS}_S}$-TOPSIS FOR MCDM USING DISTANCE AND SIMILARITY MEASURES

In this section we introducing a multi attribute decision-making problem that based on $\mathbb{PF}_{\mathbb{HSS}_S}$.

$\mathbb{Q}^{\kappa}_i = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \ldots, \mathbf{q}_\alpha\}$ be the set of alternative and $\mathbf{h}^{\kappa}_1, \mathbf{h}^{\kappa}_2, \ldots, \mathbf{h}^{\kappa}_\beta$ be the sets of attributes and their corresponding attributive values are respectively the set $\mathbf{H}^{\kappa}_1, \mathbf{H}^{\kappa}_2, \mathbf{H}^{\kappa}_3, \ldots, \mathbf{H}^{\kappa}_\beta$. Let r^{κ}_j be the weight of attributes where $0 \leq r^{\kappa}_j \leq 1$ and $\sum_j^\beta r^{\kappa}_j = 1$. Suppose that $\mathbf{D}^{\mathbf{M}^{\kappa}}_v = (\mathbf{D}^{\mathbf{M}^{\kappa}}{}_1, \mathbf{D}^{\mathbf{M}^{\kappa}}{}_2, \ldots \mathbf{D}^{\mathbf{M}^{\kappa}}{}_t)$ be the set of decision makers and Υ^{κ}_v be the weight of decision-makers with $0 \leq \Upsilon^{\kappa}_v \leq 1$ and $\sum_v^t \Upsilon^{\kappa}_v = 1$. Let $[\mathbf{M}^{\kappa}_{v(ij)}]_{\alpha\times\beta}$ be a decision matrix $\mathbf{M}^{\kappa}_{v(ij)} = \{[m^{\mathbf{M}^{\kappa}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}, [t^{\mathbf{M}^{\kappa}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}, [n^{\mathbf{M}^{\kappa}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}\}$ $i = 1, 2, 3, \ldots, \alpha; j = 1, 2, 3, \ldots, \beta; v = 1, 2, 3, \ldots, t$. $\{[m^{\mathbf{M}^{\kappa}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}, [t^{\mathbf{M}^{\kappa}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}, [n^{\mathbf{M}^{\kappa}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}\} \in [0, 1]$, $0 \leq [m^{\mathbf{M}^{\kappa}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j} + [t^{\mathbf{M}^{\kappa}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j} + [n^{\mathbf{M}^{\kappa}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j} \leq 1$.

The decision-making process for choosing an alternative can be acquired using the steps below.

Step 1: Let $[\mathbf{M}^{\kappa}_{v(ij)}]_{\alpha\times\beta}$ be the decision matrix based on the decision makers information given as follows By using $[\mathbf{M}^{\kappa}_{v(ij)}]_{\alpha\times\beta}$ the ideal matrix $\mathbf{M}^{\kappa^\bullet}_{ij}$ can be find as

$$[\mathbf{M}^{\kappa^\bullet}_{ij}]_{\alpha\times\beta} = \{[m^\bullet_{\mathbf{M}^{\kappa}_{ij}}(\mathbf{q}_i)]_{v_j}, t^\bullet\mathbf{M}^{\kappa}_{ij}(\mathbf{q}_i)]_{v_j}, n^\bullet_{\mathbf{M}^{\kappa}_{ij}}(\mathbf{q}_i)]_{v_j}\} = \left\{\begin{array}{l} (1 - \prod_{v=1}^{t}(1 - [m^{\mathbf{M}^{\kappa}_{v(ij}}_{(\mathbf{q}_i)}]_{v_j})^{\frac{1}{t}}), \\ \prod_{v=1}^{t}([t^{\mathbf{M}^{\kappa}_{v(ij}}_{(\mathbf{q}_i)}]_{v_j})^{\frac{1}{t}}, \\ \prod_{v=1}^{t}([n^{\mathbf{M}^{\kappa}_{v(ij}}_{(\mathbf{q}_i)}]_{v_j})^{\frac{1}{t}} \end{array}\right\} \quad (1)$$

The ideal matrix can be represented as follows

$$[\mathbf{M}^{\kappa^\bullet}_{ij}]_{\alpha\times\beta} = \begin{bmatrix} m^{\mathbf{M}^{\kappa^\bullet}_{11}}_{(\mathbf{q}_1)}, t^{\mathbf{M}^{\kappa^\bullet}_{11}}_{(\mathbf{q}_1)}, n^{\mathbf{M}^{\kappa^\bullet}_{11}}_{(\mathbf{q}_1)} & \cdots\cdots & m^{\mathbf{M}^{\kappa^\bullet}_{1\beta}}_{(\mathbf{q}_1)}, t^{\mathbf{M}^{\kappa^\bullet}_{1\beta}}_{(\mathbf{q}_1)}, n^{\mathbf{M}^{\kappa^\bullet}_{1\beta}}_{(\mathbf{q}_1)} \\ m^{\mathbf{M}^{\kappa^\bullet}_{21}}_{(\mathbf{q}_2)}, t^{\mathbf{M}^{\kappa^\bullet}_{21}}_{(\mathbf{q}_2)}, n^{\mathbf{M}^{\kappa^\bullet}_{21}}_{(\mathbf{q}_2)} & \cdots\cdots & m^{\mathbf{M}^{\kappa^\bullet}_{2\beta}}_{(\mathbf{q}_2)}, t^{\mathbf{M}^{\kappa^\bullet}_{2\beta}}_{(\mathbf{q}_2)}, n^{\mathbf{M}^{\kappa^\bullet}_{2\beta}}_{(\mathbf{q}_2)} \\ \vdots & \vdots & \vdots \\ m^{\mathbf{M}^{\kappa^\bullet}_{\alpha 1}}_{(\mathbf{q}_\alpha)}, t^{\mathbf{M}^{\kappa^\bullet}_{\alpha 1}}_{(\mathbf{q}_\alpha)}, n^{\mathbf{M}^{\kappa^\bullet}_{\alpha 1}}_{(\mathbf{q}_\alpha)} & \cdots\cdots & m^{\mathbf{M}^{\kappa^\bullet}_{\alpha\beta}}_{(\mathbf{q}_\alpha)}, t^{\mathbf{M}^{\kappa^\bullet}_{\alpha\beta}}_{(\mathbf{q}_\alpha)}, n^{\mathbf{M}^{\kappa^\bullet}_{\alpha\beta}}_{(\mathbf{q}_\alpha)} \end{bmatrix} \quad (2)$$

Step 2: The similarity between each decision matrix and the ideal matrix is initially determined to calculate the weights of the decision-makers as

$$\mathbf{S}^{\mathbb{PF}}_{\mathbb{HS}}(\mathbf{M}^{\kappa}_{v(ij)}, \mathbf{M}^{\kappa^\bullet}_{(ij)}) = 1 - \frac{1}{3\alpha\beta}\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} \left\{\begin{array}{l} |[m^{\mathbf{M}^{\kappa}_{v(ij)}}_{(q_i)}]_{v_j} - [m^{\mathbf{M}^{\kappa^\bullet}_{v(ij)}}_{(q_i)}]_{v_j}| + \\ |[t^{\mathbf{M}^{\kappa}_{v(ij)}}_{(q_i)}]_{v_j} - [t^{\mathbf{M}^{\kappa^\bullet}_{v(ij)}}_{(q_i)}]_{v_j}| + \\ |[n^{\mathbf{M}^{\kappa}_{v(ij)}}_{(q_i)}]_{v_j} - [n^{\mathbf{M}^{\kappa^\bullet}_{v(ij)}}_{(q_i)}]_{v_j}| \end{array}\right\} \quad (3)$$

Using the information provided above, we now determine the weight Υ^{κ}_v of t decision-makers.

$$\Upsilon^{\kappa}_v = \frac{\mathbf{S}^{\mathbb{PF}}_{\mathbb{HS}}(\mathbf{M}^{\kappa}_{v(ij)}, \mathbf{M}^{\kappa^\bullet}_{v(ij)})}{\sum_{v=1}^{t}\mathbf{S}^{\mathbb{PF}}_{\mathbb{HS}}(\mathbf{M}^{\kappa}_{v(ij)}, \mathbf{M}^{\kappa^\bullet}_{v(ij)})} \quad (4)$$

with $0 \leq \Upsilon^{\kappa}_v \leq 1$ and $\sum_v^t \Upsilon^{\kappa}_v = 1$.

$$\mathbf{D}^{\mathbb{PF}}_{\mathbb{HS}}(\mathbf{M}^{\kappa}_1, \mathbf{M}^{\kappa}_2) = \sqrt{\frac{\sum_{j=1}^{n}\{|[m^{\mathbf{M}^{\kappa}_1}_{(q_i)}]_{v_j} - [m^{\mathbf{M}^{\kappa}_2}_{(q_i)}]_{v_j}|^2 + |[t^{\mathbf{M}^{\kappa}_1}_{(q_i)}]_{v_j} - [t^{\mathbf{M}^{\kappa}_2}_{(q_i)}]_{v_j}|^2 + |[n^{\mathbf{M}^{\kappa}_1}_{(q_i)}]_{v_j} - [n^{\mathbf{M}^{\kappa}_2}_{(q_i)}]_{v_j}|^2\}}{3n}}$$

$$\mathbf{D}^{\mathbb{PF}}_{\mathbb{HS}}(\mathbf{M}^{\kappa}_{1(ij)}, \mathbf{M}^{\kappa}_{2(ij)}) = \sqrt{\frac{\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}\{|[m^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [m^{\mathbf{M}^{\kappa}_{2(ij)}}_{(\mathbf{q}_i)}]_{v_j}|^2 + |[t^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [t^{\mathbf{M}^{\kappa}_{2(ij)}}_{(\mathbf{q}_i)}]_{v_j}|^2 + |[n^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [n^{\mathbf{M}^{\kappa}_{2(ij)}}_{(\mathbf{q}_i)}]_{v_j}|^2\}}{3\alpha\beta}}$$

An aggregated picture fuzzy hypersoft decision matrix is obtained as

$$\mathbf{M}^{\kappa^{ag}}_{\wp(ij)} = \{[m^{\mathbf{M}^{\kappa^{ag}}_{\wp ij}}_{(\mathbf{q}_i)}]_{v_j}, t^{\mathbf{M}^{\kappa^{ag}}_{\wp ij}}_{(\mathbf{q}_i)}]_{v_j}, n^{\mathbf{M}^{\kappa^{ag}}_{\wp ij}}_{(\mathbf{q}_i)}]_{v_j}\} = \left\{\begin{array}{l} (1-\prod_{v=1}^{t}(1-[m^{\mathbf{M}^{\kappa}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j})^{\Upsilon^{\kappa}_{v}}), \\ \prod_{v=1}^{t}([t^{\mathbf{M}^{\kappa}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j})^{\Upsilon^{\kappa}_{v}}, \\ \prod_{v=1}^{t}([n^{\mathbf{M}^{\kappa}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j})^{\Upsilon^{\kappa}_{v}} \end{array}\right\} \tag{5}$$

An aggregate matrix can be represented as follows

$$[\mathbf{M}^{\kappa^{ag}}_{ij}]_{\alpha\times\beta} = \begin{bmatrix} m^{\mathbf{M}^{\kappa^{ag}}_{11}}_{(\mathbf{q}_1)}, t^{\mathbf{M}^{\kappa^{ag}}_{11}}_{(\mathbf{q}_1)}, n^{\mathbf{M}^{\kappa^{ag}}_{11}}_{(\mathbf{q}_1)} & \ldots\ldots\ldots & m^{\mathbf{M}^{\kappa^{ag}}_{1\beta}}_{(\mathbf{q}_1)}, t^{\mathbf{M}^{\kappa^{ag}}_{1\beta}}_{(\mathbf{q}_1)}, n^{\mathbf{M}^{\kappa^{ag}}_{1\beta}}_{(\mathbf{q}_1)} \\ m^{\mathbf{M}^{\kappa^{ag}}_{21}}_{(\mathbf{q}_2)}, t^{\mathbf{M}^{\kappa^{ag}}_{21}}_{(\mathbf{q}_2)}, n^{\mathbf{M}^{\kappa^{ag}}_{21}}_{(\mathbf{q}_2)} & \ldots\ldots\ldots & m^{\mathbf{M}^{\kappa^{ag}}_{2\beta}}_{(\mathbf{q}_2)}, t^{\mathbf{M}^{\kappa^{ag}}_{2\beta}}_{(\mathbf{q}_2)}, n^{\mathbf{M}^{\kappa^{ag}}_{2\beta}}_{(\mathbf{q}_2)} \\ \vdots & \vdots & \vdots \\ m^{\mathbf{M}^{\kappa^{ag}}_{\alpha 1}}_{(\mathbf{q}_\alpha)}, t^{\mathbf{M}^{\kappa^{ag}}_{\alpha 1}}_{(\mathbf{q}_\alpha)}, n^{\mathbf{M}^{\kappa^{ag}}_{\alpha 1}}_{(\mathbf{q}_\alpha)} & \ldots\ldots\ldots & m^{\mathbf{M}^{\kappa^{ag}}_{\alpha\beta}}_{(\mathbf{q}_\alpha)}, t^{\mathbf{M}^{\kappa^{ag}}_{\alpha\beta}}_{(\mathbf{q}_\alpha)}, n^{\mathbf{M}^{\kappa^{ag}}_{\alpha\beta}}_{(\mathbf{q}_\alpha)} \end{bmatrix} \tag{6}$$

Step 3 : During the decision-making process, decision-makers could believe that not all attributes are equally important. Each decision-maker can then form their own opinion about the relative importance of certain attributes. The opinions of all the decision-makers regarding the significance of each attribute must be combined in order to obtain the collective judgement of the attributes that have been chosen. For this purpose, to calculate the weight of attributes based on the importance assigned by each decision-maker is given as

$$[\mathbf{M}^{\kappa}_{v(ij)}]_{1\times\beta} = \{m^{\mathbf{M}^{\kappa^*}_{v(ij)}}_{(\mathbf{q}_i)}, t^{\mathbf{M}^{\kappa^*}_{v(ij)}}_{(\mathbf{q}_i)}, n^{\mathbf{M}^{\kappa^*}_{v(ij)}}_{(\mathbf{q}_i)}\} = \left[m^{\mathbf{M}^{\kappa^*}_{v(11)}}_{(\mathbf{q}_1)}, t^{\mathbf{M}^{\kappa^*}_{v(11)}}_{(\mathbf{q}_1)}, n^{\mathbf{M}^{\kappa^*}_{v(11)}}_{(\mathbf{q}_1)} \ldots m^{\mathbf{M}^{\kappa^*}_{v(1\beta)}}_{(\mathbf{q}_1)}, t^{\mathbf{M}^{\kappa^*}_{v(1\beta)}}_{(\mathbf{q}_1)}, n^{\mathbf{M}^{\kappa^*}_{v(1\beta)}}_{(\mathbf{q}_1)} \right]$$

The weight $r^{\kappa^\diamond}_j$ of attributes $\mathbf{H}^{\kappa}_j$,

$$r^{\kappa^\diamond}_j = \{[m^{\mathbf{M}^{\kappa^\diamond}_{v(ij)}}]_{v_j}, [t^{\mathbf{M}^{\kappa^\diamond}_{v(ij)}}]_{v_j}, [n^{\mathbf{M}^{\kappa^\diamond}_{v(ij)}}]_{v_j}\} = \left\{\begin{array}{l} (1-\prod_{v=1}^{t}(1-[m^{\mathbf{M}^{\kappa^*}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j})^{\Upsilon^{\kappa}_{v}}), \\ \prod_{v=1}^{t}([t^{\mathbf{M}^{\kappa^*}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j})^{\Upsilon^{\kappa}_{v}}, \\ \prod_{v=1}^{t}([n^{\mathbf{M}^{\kappa^*}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j})^{\Upsilon^{\kappa}_{v}} \end{array}\right\} \tag{7}$$

Step 4 After establishing the weights for individual attributes, we apply them to each row of the aggregated decision matrix.

$$[\mathbf{M}^{\kappa^\star}_{v(ij)}]_{\alpha\times\beta} = \{[m^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}, [t^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}, n^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}\} = \left\{\begin{array}{l} ([m^{\mathbf{M}^{\kappa^{ag}}_{\wp ij}}_{(\mathbf{q}_i)}]_{v_j}).([m^{\mathbf{M}^{\kappa^\diamond}_{v(ij)}}]_{v_j}), \\ ([t^{\mathbf{M}^{\kappa^{ag}}_{\wp ij}}_{(\mathbf{q}_i)}]_{v_j} + [t^{\mathbf{M}^{\kappa^\diamond}_{v(ij)}}]_{v_j} - [t^{\mathbf{M}^{\kappa^{ag}}_{\wp ij}}_{(\mathbf{q}_i)}]_{v_j}.[t^{\mathbf{M}^{\kappa^\diamond}_{v(ij)}}]_{v_j}), \\ ([n^{\mathbf{M}^{\kappa^{ag}}_{\wp ij}}_{(\mathbf{q}_i)}]_{v_j} + [n^{\mathbf{M}^{\kappa^\diamond}_{v(ij)}}]_{v_j} - [n^{\mathbf{M}^{\kappa^{ag}}_{\wp ij}}_{(\mathbf{q}_i)}]_{v_j}.[n^{\mathbf{M}^{\kappa^\diamond}_{v(ij)}}]_{v_j}) \end{array}\right\} \tag{8}$$

Step 5 The $\mathbb{PF}_{\mathbb{HSS}s}$ positive ideal solution is given as

$$\mathbf{M}^{\kappa^{\star+}}_j = \{[m^{\mathbf{M}^{\kappa^{\star+}}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}, [t^{\mathbf{M}^{\kappa^{\star+}}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}, n^{\mathbf{M}^{\kappa^{\star+}}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}\} = \{max([m^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}), min([t^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}), min([n^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j})\} \tag{9}$$

The $\mathbb{PF}_{\mathbb{HSS}s}$ negative ideal solution is given as

$$\mathbf{M}^{\kappa^{\star-}}_j = \{[m^{\mathbf{M}^{\kappa^{\star-}}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}, [t^{\mathbf{M}^{\kappa^{\star-}}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}, n^{\mathbf{M}^{\kappa^{\star-}}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}\} = \{\min([m^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}), \min([t^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}), \max([n^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j})\} \tag{10}$$

Step 6 : We should now calculate the normalised Hamming distance with respect to the positive ideal solution and the alternatives using

$$\mathbf{D}^j(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star+}}_j) = \frac{1}{3\beta}\sum_{j=1}^{\beta}\left\{\begin{array}{l} |[m^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [[m^{\mathbf{M}^{\kappa^{\star+}}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}|+ \\ |[t^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [t^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}|+ \\ |[n^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [n^{\mathbf{M}^{\kappa^{\star+}}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}| \end{array}\right\} \tag{11}$$

We should now calculate the normalised Hamming distance with respect to the negative ideal solution and the alternatives using

$$\mathbf{D}^j(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star-}}_j) = \frac{1}{3\beta}\sum_{j=1}^{\beta}\left\{\begin{array}{l} |[m^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [[m^{\mathbf{M}^{\kappa^{\star-}}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}|+ \\ |[t^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [t^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}|+ \\ |[n^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [n^{\mathbf{M}^{\kappa^{\star-}}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j}| \end{array}\right\} \tag{12}$$

Step 7: Ranking the alternatives is achieved using the relative closeness index, which is derived using

$$RK^j = \frac{\mathbf{D}^j(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star-}}_j)}{max[\mathbf{D}^j(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star-}}_j)]} - \frac{\mathbf{D}^j(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star+}}_j)}{min[\mathbf{D}^j(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star+}}_j)]} \tag{13}$$

According to the relative closeness index, the set of chosen alternatives are sorted in descending order. The pseudocode based on the described procedural steps is given below:

Pseudo code of the proposed algorithm:

- Step 1: Start
- Step 2: GetIdealMatrix (decisionMatrices)
  - Calculate the average of individual decision matrices
  - Return the ideal matrix
- Step 3: InputWeights (decisionMakers)
  - Input the weights of decision-makers in MCDM
  - Determine the similarity measure between each decision matrix and the ideal matrix
- Step 4: AddAttributeWeights (decisionMakers)
  - Add the weight of attributes based on the importance assigned by each decision-maker

- Step 5: AddAggregatedMatrix (decisionMatrices, attributeWeights)
  - Add the aggregated decision matrix
- Step 6: CalculatePositiveNegativeIdeals (aggregatedMatrix)
  - Calculate positive ideal solution and negative ideal solution
- Step 7: CalculateNormalizedHammingDistance (alternatives, positiveIdeal, negativeIdeal)
  - Calculate the normalized Hamming distance with respect to the positive ideal solution, negative ideal solution, and the alternatives
- Step 8: RankAlternatives (normalizedHammingDistances)
  - Rank the alternatives using the relative closeness index

  Step 9: Stop

V. DESIGNING AN INVESTMENT SECTOR SELECTOR FOR SELECTION OF MOST SUITABLE SECTOR FOR THEIR INVESTMENT

In today's dynamic and ever-evolving financial landscape, making informed investment decisions has never been more critical. The investment sector you choose can significantly impact the success of your portfolio, and the factors affecting these decisions are becoming increasingly complex. This is where the ''Investment Sector Selector'' application becomes a necessary and invaluable tool for investors. Traditional investment strategies are often based on historical performance and a narrow set of criteria. However, in an era characterized by rapid technological advancements, global market volatility, and a growing emphasis on ethical and sustainable investing, the need for a more comprehensive, data-driven, and adaptable approach has emerged. The ''Investment Sector Selector'' application answers this need by harnessing the power of multi-criteria decision making (MCDM). It enables investors to evaluate, compare, and rank various sectors based on a range of factors that truly matter to them, going beyond mere past returns.The relationship between attributes and alternatives is essential for determining the optimum course of action in investment challenges and decision making scenarios. Investing is a crucial part of building wealth and achieving financial goals, but it comes with its share of challenges and risks. Understanding these challenges is essential for making informed investment decisions. Now, consider $\mathbb{Q}_i^\kappa = \{q_1 = \text{Technology}, q_2 = \text{Healthcare}, q_3 = \text{Real Estate}, q_4 = \text{Finance}\}$ be a set of alternative possible sectors in which an investment can be made. In investment problems, attributes are characteristics or features associated with various investment options or assets. These attributes are essential for investors and financial analysts to evaluate and compare different investment opportunities. Let $H_i^\kappa = \{\mathbf{h}_1^\kappa = \text{Return on Investment (ROI)}, \mathbf{h}_2^\kappa = \text{Risk Tolerance}, \mathbf{h}_3^\kappa = \text{Market Trends}, \mathbf{h}_4^\kappa = \text{Financial Health}\}$ be a attribute set whose corresponding sub attribute values are $\{\mathbf{H}_1^\kappa, \mathbf{H}_2^\kappa, \mathbf{H}_3^\kappa, \mathbf{H}_4^\kappa\}$ respectively. Here,

$\mathbf{H}_1^\kappa = \{g_{11} = \text{Short-Term ROI}, g_{12} = \text{Long-Term ROI}, g_{13} = \text{Dividend Yield}, g_{14} = \text{Historical Performance}\}$

$\mathbf{H}_2^\kappa = \{g_{21} = \text{Beta Value}, g_{22} = \text{Credit Risk}, g_{23} = \text{Volatility}, g_{24} = \text{Market Correlation}\}$

$\mathbf{H}_3^\kappa = \{g_{31} = \text{Economic Indicators}, g_{32} = \text{Consumer Behavior}, g_{33} = \text{Technological Advancements}, g_{34} = \text{Competitive Landscape}\}$

$\mathbf{H}_4^\kappa = \{g_{41} = \text{Profitability}, g_{42} = \text{Debt-to-Equity Ratio}, g_{43} = \text{Cash Flow}, g_{44} = \text{Liquidity}\}$

Based on the attributes mentioned above, $H_i^\kappa = \mathbf{H}_1^\kappa \times \mathbf{H}_2^\kappa \times \mathbf{H}_3^\kappa \times \mathbf{H}_4^\kappa$, there are two hundred and fifty six possible outcomes but for but due to computational barrires and sake of brevity, four of the possible outcomes are addressed below:

$$H_\varsigma^\kappa = \left\{ \begin{array}{ll} \nu_{\varsigma 1}^\kappa = (g_{11}, g_{21}, g_{32}, g_{44}), & \nu_{\varsigma 2}^\kappa = (g_{14}, g_{24}, g_{33}, g_{43}), \\ \nu_{\varsigma 3}^\kappa = (g_{12}, g_{23}, g_{31}, g_{41}). & \nu_{\varsigma 4}^\kappa = (g_{13}, g_{22}, g_{32}, g_{42}) \end{array} \right\}$$

In the context of decision-making in investment problems, observer (decision makers) plays a supervisory role in the decision-making process. Observers are not directly involved in making investment decisions but may have an interest in or influence over the decisions being made. Their involvement can contribute to more responsible and well-informed investment decision-making. These observers in this scenario and market experts and finacial guru's that have studied the economy and are well-aware of the developments taking place around the glove. Now, the use of picture fuzzy hypersoft matrices $\mathbb{PF}_{\mathbb{HSM}}$ in investment decision-making suggests a more advanced and sophisticated approach to gathering and presenting opinions from these expert decision-makers. The structure allows for the incorporation of these opinions in the decision mkaing process. These opinions are then organized into $\mathbb{PF}_{\mathbb{HSS}_s}$. In this matrix

(i) Columns represent the alternatives.
(ii) Rows represent attributes being evaluated.
(iii) Each cell of the matrix contains a picture fuzzy hypersoft number, which represents the degree of preference or assessment of the corresponding decision-maker for a specific attribute.

Step: 01 Finding the ideal matrix by averaging individual decision matrices $\{\mathbf{M}_1^\kappa, \mathbf{M}_2^\kappa, \mathbf{M}_3^\kappa$ and $\mathbf{M}_4^\kappa$ are given on next page$\}$ using the formula (1):

$$[\mathbf{M}_{ij}^{\kappa^\bullet}]_{\alpha\times\beta} = \{[m_{(\mathbf{q}_i)}^{\mathbf{M}_{ij}^{\kappa^\bullet}}]_{\nu_j}, t_{(\mathbf{q}_i)}^{\mathbf{M}_{ij}^{\kappa^\bullet}}]_{\nu_j}, n_{(\mathbf{q}_i)}^{\mathbf{M}_{ij}^{\kappa^\bullet}}]_{\nu_j}\} = \left\{ \begin{array}{l} (1 - \prod_{\nu=1}^{t}(1 - [m_{(\mathbf{q}_i)}^{\mathbf{M}_{\nu(ij}^{\kappa}}]_{\nu_j})^{\frac{1}{t}}), \\ \prod_{\nu=1}^{t}([t_{(\mathbf{q}_i)}^{\mathbf{M}_{\nu(ij}^{\kappa}}]_{\nu_j})^{\frac{1}{t}}, \\ \prod_{\nu=1}^{t}([n_{(\mathbf{q}_i)}^{\mathbf{M}_{\nu(ij}^{\kappa}}]_{\nu_j})^{\frac{1}{t}} \end{array} \right\}$$

To make things easier for the reader, one computation is presented.

$$\mathbf{M}_{(11)}^{\kappa^\bullet} = \{[m_{(\mathbf{q}_1)}^{\mathbf{M}_{11}^{\kappa^\bullet}}]_{\nu_1}, t_{(\mathbf{q}_1)}^{\mathbf{M}_{11}^{\kappa^\bullet}}]_{\nu_1}, n_{(\mathbf{q}_1)}^{\mathbf{M}_{11}^{\kappa^\bullet}}]_{\nu_1}\} = \left\{ \begin{array}{l} (1 - \prod_{\nu=1}^{4}(1 - [m^{\mathbf{M}_{\nu(ij)}^{\kappa}}(\mathbf{q}_1)]_{\nu_1})^{\frac{1}{4}}, \\ \prod_{\nu=1}^{4}([t^{\mathbf{M}_{\nu(ij)}^{\kappa}}(\mathbf{q}_1)]_{\nu_1})^{\frac{1}{4}}, \\ \prod_{\nu=1}^{4}([n^{\mathbf{M}_{\nu(ij)}^{\kappa}}(\mathbf{q}_1)]_{\nu_1})^{\frac{1}{4}}) \end{array} \right\}$$

$$= \left\{\begin{array}{l}(1-[(1-0.5)^{\frac{1}{4}}(1-0.9)^{\frac{1}{4}}(1-0.3)^{\frac{1}{4}}(1-0.4)^{\frac{1}{4}}]),\\ [(0.2)^{\frac{1}{4}}(0.0)^{\frac{1}{4}}(0.3)^{\frac{1}{4}}(0.2)^{\frac{1}{4}}],\\ [(0.1)^{\frac{1}{4}}(0.1)^{\frac{1}{4}}(0.4)^{\frac{1}{4}}(0.2)^{\frac{1}{4}}]\end{array}\right\}$$

$$= \left(0.6193, 0.0000, 0.1682\right)$$

The ideal matrix $[\mathbf{M}^{\kappa\bullet}_{(ij)}]_{4\times 4}$ (14), as shown at the bottom of the page.

Step : 02 Determining the weights of decision-makers in multi-criteria decision-making (MCDM) often involves calculating the similarity measure between each decision matrix and the ideal matrix by using the formula (3),

$$\mathbf{S}^{\mathbb{PF}}_{\mathbb{HS}}(\mathbf{M}^{\kappa}_{v(ij)}, \mathbf{M}^{\kappa\bullet}_{(ij)})$$

$$= 1 - \frac{1}{3\alpha\beta}\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}\left\{\begin{array}{l} |[m^{\mathbf{M}^{\kappa}_{v(ij)}}_{(q_i)}]_{v_j} - [m^{\mathbf{M}^{\kappa\bullet}_{ij}}_{(\mathbf{q}_i)}]_{v_j}|+\\ |[t^{\mathbf{M}^{\kappa}_{v(ij)}}_{(q_i)}]_{v_j} - [m^{\mathbf{M}^{\kappa\bullet}_{ij}}_{(\mathbf{q}_i)}]_{v_j}|+\\ |[n^{\mathbf{M}^{\kappa}_{v(ij)}}_{(q_i)}]_{v_j} - [m^{\mathbf{M}^{\kappa\bullet}_{ij}}_{(\mathbf{q}_i)}]_{v_j}| \end{array}\right\}$$

For the convenience of the reader, one computation is presented.

$$\mathbf{S}^{\mathbb{PF}}_{\mathbb{HS}}(\mathbf{M}^{\kappa}_{1(ij)}, \mathbf{M}^{\kappa\bullet}_{(ij)})$$

$$= 1 - \frac{1}{4(4)}\sum_{i=1}^{4}\sum_{j=1}^{4}\left\{\begin{array}{l} |[m^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [m^{\mathbf{M}^{\kappa\bullet}_{(ij)}}_{(\mathbf{q}_i)}]_{v_j}|+\\ |[t^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [t^{\mathbf{M}^{\kappa\bullet}_{(ij)}}_{(\mathbf{q}_i)}]_{v_j}|+\\ |[n^{\mathbf{M}^{\kappa}_{1(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [n^{\mathbf{M}^{\kappa\bullet}_{(ij)}}_{(\mathbf{q}_i)}]_{v_j}| \end{array}\right\}$$

$$= \frac{1}{4(4)}\sum_{i=1}^{4}\left[\begin{array}{l}
\left\{\begin{array}{l} |[m^{\mathbf{M}^{\kappa}_{1(i1)}}_{(\mathbf{q}_i)}]_{v_1} - [m^{\mathbf{M}^{\kappa\bullet}_{(i1)}}_{(\mathbf{q}_i)}]_{v_1}|+\\ |[t^{\mathbf{M}^{\kappa}_{1(i1)}}_{(\mathbf{q}_i)}]_{v_1} - [t^{\mathbf{M}^{\kappa\bullet}_{(i1)}}_{(\mathbf{q}_i)}]_{v_1}|+\\ |[n^{\mathbf{M}^{\kappa}_{1(i1)}}_{(\mathbf{q}_i)}]_{v_1} - [n^{\mathbf{M}^{\kappa\bullet}_{(i1)}}_{(\mathbf{q}_i)}]_{v_1}| \end{array}\right\}+\\
\left\{\begin{array}{l} |[m^{\mathbf{M}^{\kappa}_{1(i2)}}_{(\mathbf{q}_i)}]_{v_2} - [m^{\mathbf{M}^{\kappa\bullet}_{(i2)}}_{(\mathbf{q}_i)}]_{v_2}|+\\ |[t^{\mathbf{M}^{\kappa}_{1(i2)}}_{(\mathbf{q}_i)}]_{v_2} - [t^{\mathbf{M}^{\kappa\bullet}_{(i2)}}_{(\mathbf{q}_i)}]_{v_2}|+\\ |[n^{\mathbf{M}^{\kappa}_{1(i2)}}_{(\mathbf{q}_i)}]_{v_2} - [n^{\mathbf{M}^{\kappa\bullet}_{(i2)}}_{(\mathbf{q}_i)}]_{v_2}| \end{array}\right\}+\\
\left\{\begin{array}{l} |[m^{\mathbf{M}^{\kappa}_{1(i3)}}_{(\mathbf{q}_i)}]_{v_3} - [m^{\mathbf{M}^{\kappa\bullet}_{(i3)}}_{(\mathbf{q}_i)}]_{v_3}|+\\ |[t^{\mathbf{M}^{\kappa}_{1(i3)}}_{(\mathbf{q}_i)}]_{v_3} - [t^{\mathbf{M}^{\kappa\bullet}_{(i3)}}_{(\mathbf{q}_i)}]_{v_3}|+\\ |[n^{\mathbf{M}^{\kappa}_{1(i3)}}_{(\mathbf{q}_i)}]_{v_3} - [n^{\mathbf{M}^{\kappa\bullet}_{(i3)}}_{(\mathbf{q}_i)}]_{v_3}| \end{array}\right\}+\\
\left\{\begin{array}{l} |[m^{\mathbf{M}^{\kappa}_{1(i4)}}_{(\mathbf{q}_i)}]_{v_4} - [m^{\mathbf{M}^{\kappa\bullet}_{(i4)}}_{(\mathbf{q}_i)}]_{v_4}|+\\ |[t^{\mathbf{M}^{\kappa}_{1(i4)}}_{(\mathbf{q}_i)}]_{v_4} - [t^{\mathbf{M}^{\kappa\bullet}_{(i4)}}_{(\mathbf{q}_i)}]_{v_4}|+\\ |[n^{\mathbf{M}^{\kappa}_{1(i4)}}_{(\mathbf{q}_i)}]_{v_4} - [n^{\mathbf{M}^{\kappa\bullet}_{(i4)}}_{(\mathbf{q}_i)}]_{v_4}| \end{array}\right\}
\end{array}\right]$$

$$[\mathbf{M}^{\kappa}_{1}]_{4\times 4} = \left[\begin{array}{ccccc} & q_1 & q_2 & q_3 & q_4\\ v_1 & (0.5, 0.2, 0.1) & (0.5, 0.3, 0.1) & (0.3, 0.5, 0.1) & (0.4, 0.3, 0.2)\\ v_2 & (0.7, 0.1, 0.1) & (0.2, 0.5, 0.2) & (0.6, 0.1, 0.2) & (0.4, 0.1, 0.3)\\ v_3 & (0.5, 0.2, 0.2) & (0.6, 0.2, 0.1) & (0.3, 0.4, 0.1) & (0.5, 0.2, 0.2)\\ v_4 & (0.4, 0.4, 0.1) & (0.6, 0.2, 0.1) & (0.5, 0.3, 0.2) & (0.5, 0.3, 0.1)\end{array}\right]$$

$$[\mathbf{M}^{\kappa}_{2}]_{4\times 4} = \left[\begin{array}{ccccc} & q_1 & q_2 & q_3 & q_4\\ v_1 & (0.9, 0.0, 0.1) & (0.2, 0.1, 0.7) & (0.6, 0.3, 0.1) & (0.7, 0.1, 0.1)\\ v_2 & (0.3, 0.3, 0.3) & (0.4, 0.2, 0.4) & (0.5, 0.3, 0.1) & (0.1, 0.1, 0.8)\\ v_3 & (0.6, 0.1, 0.1) & (0.7, 0.1, 0.2) & (0.5, 0.2, 0.1) & (0.4, 0.1, 0.4)\\ v_4 & (0.1, 0.4, 0.2) & (0.2, 0.2, 0.2) & (0.5, 0.1, 0.1) & (0.4, 0.1, 0.2)\end{array}\right]$$

$$[\mathbf{M}^{\kappa}_{3}]_{4\times 4} = \left[\begin{array}{ccccc} & q_1 & q_2 & q_3 & q_4\\ v_1 & (0.3, 0.3, 0.4) & (0.6, 0.2, 0.1) & (0.6, 0.1, 0.3) & (0.3, 0.5, 0.2)\\ v_2 & (0.7, 0.2, 0.1) & (0.3, 0.3, 0.2) & (0.4, 0.1, 0.3) & (0.5, 0.3, 0.1)\\ v_3 & (0.6, 0.2, 0.2) & (0.8, 0.1, 0.1) & (0.2, 0.6, 0.2)) & (0.2, 0.3, 0.4)\\ v_4 & (0.4, 0.1, 0.2) & (0.4, 0.1, 0.4) & (0.5, 0.1, 0.2) & (0.3, 0.3, 0.3)\end{array}\right]$$

$$[\mathbf{M}^{\kappa}_{4}]_{4\times 4} = \left[\begin{array}{ccccc} & q_1 & q_2 & q_3 & q_4\\ v_1 & (0.4, 0.2, 0.2) & (0.5, 0.2, 0.1) & (0.6, 0.3, 0.1) & (0.5, 0.3, 0.2)\\ v_2 & (0.8, 0.1, 0.1) & (0.3, 0.4, 0.2) & (0.4, 0.4, 0.2) & (0.7, 0.3, 0.0)\\ v_3 & (0.3, 0.3, 0.4) & (0.5, 0.2, 0.2) & (0.6, 0.2, 0.1) & (0.5, 0.2, 0.2)\\ v_4 & (0.8, 0.0, 0.1) & (0.4, 0.2, 0.4) & (0.5, 0.1, 0.2) & (0.6, 0.3, 0.1)\end{array}\right]$$

$$[\mathbf{M}^{\kappa\bullet}_{(ij)}]_{4\times 4} = \begin{bmatrix}(0.619, 0.000, 0.168) & (0.468, 0.186, 0.162) & (0.539, 0.259, 0.131) & (0.499, 0.259, 0.168)\\ (0.665, 0.156, 0.131) & (0.303, 0.331, 0.237) & (0.482, 0.186, 0.186) & (0.466, 0.173, 0.000)\\ (0.513, 0.186, 0.200) & (0.669, 0.168, 0.141) & (0.421, 0.313, 0.118) & (0.411, 0.186, 0.282)\\ (0.495, 0.000, 0.141) & (0.417, 0.168, 0.237) & (0.500, 0.131, 0.168) & (0.461, 0.228, 0.156)\end{bmatrix} \quad (14)$$

$$
\begin{aligned}
= 1 - \frac{1}{4(4)}\Bigg[&\left\{ \begin{array}{l}
|[m^{\mathbf{M}^{\kappa}_{1(11)}}_{(\mathbf{q}_1)}]_{v_1} - [m^{\mathbf{M}^{\kappa}_{(11)}}_{(\mathbf{q}_1)}]_{v_1}|+ \\
|[m^{\mathbf{M}^{\kappa}_{1(21)}}_{(\mathbf{q}_2)}]_{v_1} - [m^{\mathbf{M}^{\kappa}_{(21)}}_{(\mathbf{q}_2)}]_{v_1}|+ \\
|[m^{\mathbf{M}^{\kappa}_{1(31)}}_{(\mathbf{q}_3)}]_{v_1} - [m^{\mathbf{M}^{\kappa}_{(31)}}_{(\mathbf{q}_3)}]_{v_1}|+ \\
|[m^{\mathbf{M}^{\kappa}_{1(41)}}_{(\mathbf{q}_4)}]_{v_1} - [m^{\mathbf{M}^{\kappa}_{(41)}}_{(\mathbf{q}_4)}]_{v_1}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(11)}}_{(\mathbf{q}_1)}]_{v_1} - [t^{\mathbf{M}^{\kappa}_{(11)}}_{(\mathbf{q}_1)}]_{v_1}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(21)}}_{(\mathbf{q}_2)}]_{v_1} - [t^{\mathbf{M}^{\kappa}_{(21)}}_{(\mathbf{q}_2)}]_{v_1}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(31)}}_{(\mathbf{q}_3)}]_{v_1} - [t^{\mathbf{M}^{\kappa}_{(31)}}_{(\mathbf{q}_3)}]_{v_1}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(41)}}_{(\mathbf{q}_4)}]_{v_1} - [t^{\mathbf{M}^{\kappa}_{(41)}}_{(\mathbf{q}_4)}]_{v_1}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(11)}}_{(\mathbf{q}_1)}]_{v_1} - [n^{\mathbf{M}^{\kappa}_{(11)}}_{(\mathbf{q}_1)}]_{v_1}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(21)}}_{(\mathbf{q}_2)}]_{v_1} - [n^{\mathbf{M}^{\kappa}_{(21)}}_{(\mathbf{q}_2)}]_{v_1}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(31)}}_{(\mathbf{q}_3)}]_{v_1} - [n^{\mathbf{M}^{\kappa}_{(31)}}_{(\mathbf{q}_3)}]_{v_1}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(41)}}_{(\mathbf{q}_4)}]_{v_1} - [n^{\mathbf{M}^{\kappa}_{(41)}}_{(\mathbf{q}_4)}]_{v_1}|
\end{array} \right\} \\
+ &\left\{ \begin{array}{l}
|[m^{\mathbf{M}^{\kappa}_{1(12)}}_{(\mathbf{q}_1)}]_{v_2} - [m^{\mathbf{M}^{\kappa}_{(12)}}_{(\mathbf{q}_1)}]_{v_2}|+ \\
|[m^{\mathbf{M}^{\kappa}_{1(22)}}_{(\mathbf{q}_2)}]_{v_2} - [m^{\mathbf{M}^{\kappa}_{(22)}}_{(\mathbf{q}_2)}]_{v_2}|+ \\
|[m^{\mathbf{M}^{\kappa}_{1(32)}}_{(\mathbf{q}_3)}]_{v_2} - [m^{\mathbf{M}^{\kappa}_{(32)}}_{(\mathbf{q}_3)}]_{v_2}|+ \\
|[m^{\mathbf{M}^{\kappa}_{1(42)}}_{(\mathbf{q}_4)}]_{v_2} - [m^{\mathbf{M}^{\kappa}_{(42)}}_{(\mathbf{q}_4)}]_{v_2}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(12)}}_{(\mathbf{q}_1)}]_{v_2} - [t^{\mathbf{M}^{\kappa}_{(12)}}_{(\mathbf{q}_1)}]_{v_2}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(22)}}_{(\mathbf{q}_2)}]_{v_2} - [t^{\mathbf{M}^{\kappa}_{(22)}}_{(\mathbf{q}_2)}]_{v_2}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(32)}}_{(\mathbf{q}_3)}]_{v_2} - [t^{\mathbf{M}^{\kappa}_{(32)}}_{(\mathbf{q}_3)}]_{v_2}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(42)}}_{(\mathbf{q}_4)}]_{v_2} - [t^{\mathbf{M}^{\kappa}_{(42)}}_{(\mathbf{q}_4)}]_{v_2}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(12)}}_{(\mathbf{q}_1)}]_{v_2} - [n^{\mathbf{M}^{\kappa}_{(12)}}_{(\mathbf{q}_1)}]_{v_2}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(22)}}_{(\mathbf{q}_2)}]_{v_2} - [n^{\mathbf{M}^{\kappa}_{(22)}}_{(\mathbf{q}_2)}]_{v_2}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(32)}}_{(\mathbf{q}_3)}]_{v_2} - [n^{\mathbf{M}^{\kappa}_{(32)}}_{(\mathbf{q}_3)}]_{v_2}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(42)}}_{(\mathbf{q}_4)}]_{v_2} - [n^{\mathbf{M}^{\kappa}_{(42)}}_{(\mathbf{q}_4)}]_{v_2}|
\end{array} \right\} \\
+ &\left\{ \begin{array}{l}
|[m^{\mathbf{M}^{\kappa}_{1(13)}}_{(\mathbf{q}_1)}]_{v_3} - [m^{\mathbf{M}^{\kappa}_{(13)}}_{(\mathbf{q}_1)}]_{v_3}|+ \\
|[m^{\mathbf{M}^{\kappa}_{1}(23)}_{(\mathbf{q}_2)}]_{v_3} - [m^{\mathbf{M}^{\kappa}_{(23)}}_{(\mathbf{q}_2)}]_{v_3}|+ \\
|[m^{\mathbf{M}^{\kappa}_{1(33)}}_{(\mathbf{q}_3)}]_{v_3} - [m^{\mathbf{M}^{\kappa}_{(33)}}_{(\mathbf{q}_3)}]_{v_3}|+ \\
|[m^{\mathbf{M}^{\kappa}_{1(43)}}_{(\mathbf{q}_4)}]_{v_3} - [m^{\mathbf{M}^{\kappa}_{(43)}}_{(\mathbf{q}_4)}]_{v_3}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(13)}}_{(\mathbf{q}_1)}]_{v_3} - [t^{\mathbf{M}^{\kappa}_{(13)}}_{(\mathbf{q}_1)}]_{v_3}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1}(23)}_{(\mathbf{q}_2)}]_{v_3} - [t^{\mathbf{M}^{\kappa}_{(23)}}_{(\mathbf{q}_2)}]_{v_3}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(33)}}_{(\mathbf{q}_3)}]_{v_3} - [t^{\mathbf{M}^{\kappa}_{(33)}}_{(\mathbf{q}_3)}]_{v_3}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(43)}}_{(\mathbf{q}_4)}]_{v_3} - [t^{\mathbf{M}^{\kappa}_{(43)}}_{(\mathbf{q}_4)}]_{v_3}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(13)}}_{(\mathbf{q}_1)}]_{v_3} - [n^{\mathbf{M}^{\kappa}_{(13)}}_{(\mathbf{q}_1)}]_{v_3}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1}(23)}_{(\mathbf{q}_2)}]_{v_3} - [n^{\mathbf{M}^{\kappa}_{(23)}}_{(\mathbf{q}_2)}]_{v_3}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(33)}}_{(\mathbf{q}_3)}]_{v_3} - [n^{\mathbf{M}^{\kappa}_{(33)}}_{(\mathbf{q}_3)}]_{v_3}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(43)}}_{(\mathbf{q}_4)}]_{v_3} - [n^{\mathbf{M}^{\kappa}_{(43)}}_{(\mathbf{q}_4)}]_{v_3}|
\end{array} \right\} \\
+ &\left\{ \begin{array}{l}
|[m^{\mathbf{M}^{\kappa}_{1(14)}}_{(\mathbf{q}_1)}]_{v_4} - [m^{\mathbf{M}^{\kappa}_{(14)}}_{(\mathbf{q}_1)}]_{v_4}|+ \\
|[m^{\mathbf{M}^{\kappa}_{1(24)}}_{(\mathbf{q}_2)}]_{v_4} - [m^{\mathbf{M}^{\kappa}_{(24)}}_{(\mathbf{q}_2)}]_{v_4}|+ \\
|[m^{\mathbf{M}^{\kappa}_{1(34)}}_{(\mathbf{q}_3)}]_{v_4} - [m^{\mathbf{M}^{\kappa}_{(34)}}_{(\mathbf{q}_3)}]_{v_4}|+ \\
|[m^{\mathbf{M}^{\kappa}_{1(44)}}_{(\mathbf{q}_4)}]_{v_4} - [m^{\mathbf{M}^{\kappa}_{(44)}}_{(\mathbf{q}_4)}]_{v_4}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(14)}}_{(\mathbf{q}_1)}]_{v_4} - [t^{\mathbf{M}^{\kappa}_{(14)}}_{(\mathbf{q}_1)}]_{v_4}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(24)}}_{(\mathbf{q}_2)}]_{v_4} - [t^{\mathbf{M}^{\kappa}_{(24)}}_{(\mathbf{q}_2)}]_{v_4}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(34)}}_{(\mathbf{q}_3)}]_{v_4} - [t^{\mathbf{M}^{\kappa}_{(34)}}_{(\mathbf{q}_3)}]_{v_4}|+ \\
|[t^{\mathbf{M}^{\kappa}_{1(44)}}_{(\mathbf{q}_4)}]_{v_4} - [t^{\mathbf{M}^{\kappa}_{(44)}}_{(\mathbf{q}_4)}]_{v_4}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(14)}}_{(\mathbf{q}_1)}]_{v_4} - [n^{\mathbf{M}^{\kappa}_{(14)}}_{(\mathbf{q}_1)}]_{v_4}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(24)}}_{(\mathbf{q}_2)}]_{v_4} - [n^{\mathbf{M}^{\kappa}_{(24)}}_{(\mathbf{q}_2)}]_{v_4}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(34)}}_{(\mathbf{q}_3)}]_{v_4} - [n^{\mathbf{M}^{\kappa}_{(34)}}_{(\mathbf{q}_3)}]_{v_4}|+ \\
|[n^{\mathbf{M}^{\kappa}_{1(44)}}_{(\mathbf{q}_4)}]_{v_4} - [n^{\mathbf{M}^{\kappa}_{(44)}}_{(\mathbf{q}_4)}]_{v_4}|
\end{array} \right\} \Bigg]
\end{aligned}
$$

$$= 1 - \frac{1}{4(4)} \left[\begin{Bmatrix} |0.5-0.619|+|0.5-0.468|+ \\ |0.3-0.539|+|0.4-0.499|+ \\ |0.2-0.000|+|0.3-0.186|+ \\ |0.5-0.259|+|0.3-0.259|+ \\ |0.1-0.168|+|0.1-0.162|+ \\ |0.1-0.131|+|0.2-0.168| \end{Bmatrix} + \begin{Bmatrix} |0.7-0.665|+|0.2-0.303|+ \\ |0.6-0.482|+|0.4-0.466|+ \\ |0.1-0.156|+|0.5-0.331|+ \\ |0.1-0.186|+|0.1-0.173|+ \\ |0.1-0.131|+|0.2-0.237|+ \\ |0.2-0.186|+|0.3-0.000| \end{Bmatrix} + \begin{Bmatrix} |0.5-0.513|+|0.6-0.669|+ \\ |0.3-0.421|+|0.5-0.411|+ \\ |0.2-0.186|+|0.2-0.168|+ \\ |0.4-0.313|+|0.2-0.118|+ \\ |0.2-0.200|+|0.1-0.141|+ \\ |0.1-0.118|+|0.2-0.282| \end{Bmatrix} + \begin{Bmatrix} |0.4-0.495|+|0.6-0.417|+ \\ |0.5-0.500|+|0.5-0.461|+ \\ |0.4-0.000|+|0.2-0.168|+ \\ |0.3-0.131|+|0.3-0.228|+ \\ |0.1-0.141|+|0.1-0.237|+ \\ |0.2-0.168|+|0.1-0.156| \end{Bmatrix} \right]$$

$$\mathbf{S}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}^{\kappa}_{1(ij)}, \mathbf{M}^{\kappa^\bullet}_{(ij)}) = 0.732, \quad \mathbf{S}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}^{\kappa}_{2(ij)}, \mathbf{M}^{\kappa^\bullet}_{(ij)}) = 0.131,$$
$$\mathbf{S}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}^{\kappa}_{3(ij)}, \mathbf{M}^{\kappa^\bullet}_{(ij)}) = 0.087 \quad \mathbf{S}_{\mathbb{HS}}^{\mathbb{PF}}(\mathbf{M}^{\kappa}_{4(ij)}, \mathbf{M}^{\kappa^\bullet}_{(ij)}) = 0.068$$

we now determine the weight Υ_v^κ for (v = 1, 2, 3, 4) of each decision-makers by using formula (4).

$$\Upsilon_1^\kappa == \frac{0.7327}{0.732+0.131+0.087+0.068} = 0.718$$
$$\Upsilon_2^\kappa == \frac{0.131}{0.732+0.131+0.087+0.068} = 0.128$$
$$\Upsilon_3^\kappa = \frac{0.087}{0.732+0.131+0.087+0.068} = 0.085$$
$$\Upsilon_4^\kappa = \frac{0.068}{0.732+0.131+0.087+0.068} = 0.067$$

The construction of an aggregated picture fuzzy hypersoft decision matrix is a key step in obtaining a group decision MCDM using picture fuzzy hypersoft numbers. The aggregated picture fuzzy hypersoft decision matrix is obtained by using formula (5)

$$\mathbf{M}^{\kappa^{ag}}_{\wp(ij)} = \{[m_{(\mathbf{q}_i)}^{\mathbf{M}^{\kappa^{ag}}_{\wp ij}}]_{v_j}, t_{(\mathbf{q}_i)}^{\mathbf{M}^{\kappa^{ag}}_{\wp ij}}]_{v_j}, n_{(\mathbf{q}_i)}^{\mathbf{M}^{\kappa^{ag}}_{\wp ij}}]_{v_j}\}$$
$$= \begin{Bmatrix} (1-\prod_{v=1}^{t}(1-[m_{(\mathbf{q}_i)}^{\mathbf{M}^{\kappa}_{v(ij)}}]_{v_j})^{\Upsilon_v^\kappa}), \\ \prod_{v=1}^{t}([t_{(\mathbf{q}_i)}^{\mathbf{M}^{\kappa}_{v(ij)}}]_{v_j})^{\Upsilon_v^\kappa}, \\ \prod_{v=1}^{t}([n_{(\mathbf{q}_i)}^{\mathbf{M}^{\kappa}_{v(ij)}}]_{v_j})^{\Upsilon_v^\kappa} \end{Bmatrix}$$

For the reader's convenience, one computation is presented.

$$\{[m_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa^{ag}}_{\wp 11}}]_{v_1}, t_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa^{ag}}_{\wp 11}}]_{v_1}, n_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa^{ag}}_{\wp 11}}]_{v_1}\}$$
$$= \begin{pmatrix} 1-((1-[m_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa}_{1(11)}}]_{v_1})^{\Upsilon_1^\kappa} \\ (1-[m_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa}_{2(11)}}]_{v_1})^{\Upsilon_2^\kappa} \\ (1-[m_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa}_{3(11)}}]_{v_1})^{\Upsilon_3^\kappa} \\ (1-[m_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa}_{4(11)}}]_{v_1})^{\Upsilon_4^\kappa}), \\ ([t_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa}_{1(11)}}]_{v_1})^{\Upsilon_1^\kappa}([t_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa}_{2(11)}}]_{v_1})^{\Upsilon_2^\kappa} \\ ([t_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa}_{3(11)}}]_{v_1})^{\Upsilon_3^\kappa}([t_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa}_{4(11)}}]_{v_1})^{\Upsilon_4^\kappa}, \\ ([n_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa}_{1(11)}}]_{v_1})^{\Upsilon_1^\kappa}([n_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa}_{2(11)}}]_{v_1})^{\Upsilon_2^\kappa} \\ ([n_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa}_{(3(11)}}]_{v_1})^{\Upsilon_3^\kappa}([n_{(\mathbf{q}_1)}^{\mathbf{M}^{\kappa}_{4(11)}}]_{v_1})^{\Upsilon_4^\kappa} \end{pmatrix}$$
$$= \begin{pmatrix} 1-((1-0.5)^{0.718} \\ (1-0.9)^{0.128}(1-0.3)^{0.085}(1-0.4)^{0.067}), \\ (0.2)^{0.718}(0.0)^{0.128}(0.3)^{0.085}(0.2)^{0.067}, \\ (0.1)^{0.718}(0.1)^{0.128}(0.4)^{0.085}(0.2)^{0.067} \end{pmatrix}$$

An aggregate matrix (15), as shown at the bottom of the next page.

Step 3: During the decision-making process, decision-makers could believe that not all attributes are equally important. Each decision-maker can then form their own opinion about the relative importance of certain attributes. The opinions of all the decision-makers regarding the significance of each attribute must be combined in order to obtain the collective judgement of the attributes that have been chosen. For this purpose, to calculate the weight of attributes based on the importance assigned by each decision-maker is given as shown in the equation at the bottom of the next page.

The weight $r_j^{\diamond\kappa}$ of attributes $\mathbf{H}_j^\kappa$ can be calculated by (7),

$$r_j^{\diamond\kappa} = \{[m^{\diamond}_{(\mathbf{q}_i)}]_{v_j}, [t^{\diamond}_{(\mathbf{q}_i)}]_{v_j}, [n^{\diamond}_{(\mathbf{q}_i)}]_{v_j}\}$$
$$= \begin{Bmatrix} (1-\prod_{v=1}^{t}(1-[m_{(\mathbf{q}_i)}^{\mathbf{M}^{\kappa^*}_{v(ij)}}]_{v_j})^{\Upsilon_v^\kappa}), \\ \prod_{v=1}^{t}([t_{(\mathbf{q}_i)}^{\mathbf{M}^{\kappa^*}_{v(ij)}}]_{v_j})^{\Upsilon_v^\kappa}, \\ \prod_{v=1}^{t}([n_{(\mathbf{q}_i)}^{\mathbf{M}^{\kappa^*}_{v(ij)}}]_{v_j})^{\Upsilon_v^\kappa} \end{Bmatrix}$$

For the reader's convenience, one computation is presented.

$$r_1^{\diamond\kappa} = \begin{pmatrix} 1-((1-[m^{\mathbf{M}^{\kappa^*}_{1(11)}}(\mathbf{q}_1)]_{v_1})^{\Upsilon_1^\kappa} \\ (1-[m^{\mathbf{M}^{\kappa^*}_{2(11)}}(\mathbf{q}_1)]_{v_1})^{\Upsilon_2^\kappa} \\ (1-[m^{\mathbf{M}^{\kappa^*}_{3(11)}}(\mathbf{q}_1)]_{v_1})^{\Upsilon_3^\kappa} \\ (1-[m^{\mathbf{M}^{\kappa^*}_{4(11)}}(\mathbf{q}_1)]_{v_1})^{\Upsilon_4^\kappa}), \\ ([t^{\mathbf{M}^{\kappa^*}_{1(11)}}(\mathbf{q}_1)]_{v_1})^{\Upsilon_1^\kappa}([t^{\mathbf{M}^{\kappa^*}_{2(11)}}(\mathbf{q}_1)]_{v_1})^{\Upsilon_2^\kappa} \\ ([t^{\mathbf{M}^{\kappa^*}_{3(11)}}(\mathbf{q}_1)]_{v_1})^{\Upsilon_3^\kappa}([t^{\mathbf{M}^{\kappa^*}_{4(11)}}(\mathbf{q}_1)]_{v_1})^{\Upsilon_4^\kappa}, \\ ([n^{\mathbf{M}^{\kappa^*}_{1(11)}}(\mathbf{q}_1)]_{v_1})^{\Upsilon_1^\kappa}([n^{\mathbf{M}^{\kappa^*}_{2(11)}}(\mathbf{q}_1)]_{v_1})^{\Upsilon_2^\kappa} \\ ([n^{\mathbf{M}^{\kappa^*}_{(3(11)}}(\mathbf{q}_1)]_{v_1})^{\Upsilon_3^\kappa}([n^{\mathbf{M}^{\kappa^*}_{4(11)}}(\mathbf{q}_1)]_{v_1})^{\Upsilon_4^\kappa} \end{pmatrix}$$
$$= \begin{pmatrix} 1-((1-0.7)^{0.718}(1-0.6)^{0.128} \\ (1-0.5)^{0.085}(1-0.1)^{0.067}), \\ (0.2)^{0.718}(0.1)^{0.128}(0.2)^{0.085}(0.2)^{0.067}, \\ (0.1)^{0.718}(0.1)^{0.128}(0.1)^{0.085}(0.7)^{0.067} \end{pmatrix}$$

$$= (0.649, 0.182, 0.114)$$
$$r_2^{\diamond\kappa} = (0.541, 0.198, 0.215)$$
$$r_3^{\diamond\kappa} = (0.740, 0.129, 0.115)$$
$$r_4^{\diamond\kappa} = (0.252, 0.240, 0.396)$$

Step 4: After establishing the weights for individual attributes, we apply them to each row of the aggregated decision matrix with the help of formula (8).

$$[\mathbf{M}_{(ij)}^{\kappa^\star}]_{\alpha\times\beta} = \{[m_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^\star}}]_{v_j}, [t_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^\star}}]_{v_j}, n_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^\star}}]_{v_j}\} = \begin{bmatrix} (([m_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^{ag}}}]_{v_j}).([m_{(\mathbf{q}_i)}^{\diamond}]_{v_j}), \\ ([t_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^{ag}}}]_{v_j} + [t_{(\mathbf{q}_i)}^{\diamond}]_{v_j} - [t_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^{ag}}}]_{v_j}.[t_{(\mathbf{q}_i)}^{\diamond}]_{v_j}), \\ ([n_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^{ag}}}]_{v_j} + [n_{(\mathbf{q}_i)}^{\diamond}]_{v_j} - [n_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^{ag}}}]_{v_j}.[n_{(\mathbf{q}_i)}^{\diamond}]_{v_j})) \end{bmatrix}$$

The matrix $[\mathbf{M}_{(ij)}^{\kappa^\star}]_{\alpha\times\beta}$ (16), as shown at the bottom of the page.

Step 5: The $\mathbb{PF}_{\mathbb{HSS}s}$ positive ideal solution calculated by using the formula (9)

$$\mathbf{M}_j^{\kappa^{\star+}} = \{[m_{(\mathbf{q}_i)}^{\mathbf{M}_j^{\kappa^{\star+}}}]_{v_j}, [t_{(\mathbf{q}_i)}^{\mathbf{M}_j^{\kappa^{\star+}}}]_{v_j}, n_{(\mathbf{q}_i)}^{\mathbf{M}_j^{\kappa^{\star+}}}]_{v_j}\}$$
$$= \{\max([m_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^{\star+}}}]_{v_j}), \min([t_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^{\star+}}}]_{v_j}), \min([n_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^{\star+}}}]_{v_j})\}$$
$$= \begin{bmatrix} (0.379, 0.182, 0.218) & (0.467, 0.279, 0.216) \\ (0.304, 0.299, 0.209) & (0.346, 0.293, 0.215) \end{bmatrix}$$

The $\mathbb{PF}_{\mathbb{HSS}s}$ negative ideal solution calculated by (10

$$\mathbf{M}_j^{\kappa^{\star-}} = \{[m_{(\mathbf{q}_i)}^{\mathbf{M}_j^{\kappa^{\star-}}}]_{v_j}, [t_{(\mathbf{q}_i)}^{\mathbf{M}_j^{\kappa^{\star+}}}]_{v_j}, n_{(\mathbf{q}_i)}^{\mathbf{M}_j^{\kappa^{\star-}}}]_{v_j}\}$$
$$= \{\min([m_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^{\star-}}}]_{v_j}), \min([t_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^{\star-}}}]_{v_j}), \max([n_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^{\star-}}}]_{v_j})\}$$
$$= \begin{bmatrix} (0.104, 0.182, 0.466) & (0.132, 0.279, 0.478) \\ (0.126, 0.299, 0.507) & (0.121, 0.293, 0.469) \end{bmatrix}$$

Step 6: We should now calculate the normalised Hamming distance with respect to the positive ideal solution and the alternatives using formula (11)

$$\mathbf{D}^j(\mathbf{M}_{(ij)}^{\kappa^\star}, \mathbf{M}_j^{\kappa^{\star+}}) = \frac{1}{3\beta}\sum_{j=1}^{\beta}\{|[m_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^\star}}]_{v_j} - [m_{(\mathbf{q}_i)}^{\mathbf{M}_j^{\kappa^{\star+}}}]_{v_j}| + |[t_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^\star}}]_{v_j} - [t_{(\mathbf{q}_i)}^{\mathbf{M}_j^{\kappa^{\star+}}}]_{v_j}| + |[n_{(\mathbf{q}_i)}^{\mathbf{M}_{v(ij)}^{\kappa^\star}}]_{v_j} - [n_{(\mathbf{q}_i)}^{\mathbf{M}_j^{\kappa^{\star+}}}]_{v_j}|\}$$

$$\mathbf{D}^1(\mathbf{M}_{(ij)}^{\kappa^\star}, \mathbf{M}_j^{\kappa^{\star+}}) = \frac{1}{12}\begin{pmatrix} (|0.374-0.379| + |0.182-0.182|+ \\ |0.218-0.218| + |0.311-0.467|+ \\ |0.382-0.279| + |0.227-0.216|+ \\ |0.262-0.304| + |0.504-0.299|+ \\ |0.211-0.209| + |0.292-0.346|+ \\ |0.405-0.293| + |0.276-0.215|) \end{pmatrix}$$

$$\mathbf{D}^1(\mathbf{M}_{(ij)}^{\kappa^\star}, \mathbf{M}_j^{\kappa^{\star+}}) = 0.062$$

$$[\mathbf{M}_{1(ij)}^{\kappa^*}]_{1\times4} = \{m_{(\mathbf{q}_i)}^{\mathbf{M}_{1(ij)}^{\kappa^*}}, t_{(\mathbf{q}_i)}^{\mathbf{M}_{1(ij)}^{\kappa^*}}, n_{(\mathbf{q}_i)}^{\mathbf{M}_{1(ij)}^{\kappa^*}}\} = \begin{bmatrix} (0.7, 0.2, 0.1) & (0.6, 0.2, 0.2) & (0.8, 0.1, 0.1) & (0.2, 0.3, 0.5) \end{bmatrix}$$
$$[\mathbf{M}_{2(ij)}^{\kappa^*}]_{1\times4} = \{m_{(\mathbf{q}_i)}^{\mathbf{M}_{2(ij)}^{\kappa^*}}, t_{(\mathbf{q}_i)}^{\mathbf{M}_{2(ij)}^{\kappa^*}}, n_{(\mathbf{q}_i)}^{\mathbf{M}_{2(ij)}^{\kappa^*}}\} = \begin{bmatrix} (0.6, 0.1, 0.1) & (0.2, 0.3, 0.4) & (0.5, 0.2, 0.3) & (0.2, 0.2, 0.2) \end{bmatrix}$$
$$[\mathbf{M}_{3(ij)}^{\kappa^*}]_{1\times4} = \{m_{(\mathbf{q}_i)}^{\mathbf{M}_{3(ij)}^{\kappa^*}}, t_{(\mathbf{q}_i)}^{\mathbf{M}_{3(ij)}^{\kappa^*}}, n_{(\mathbf{q}_i)}^{\mathbf{M}_{3(ij)}^{\kappa^*}}\} = \begin{bmatrix} (0.5, 0.2, 0.1) & (0.5, 0.1, 0.1) & (0.4, 0.4, 0.1) & (0.6, 0.1, 0.2) \end{bmatrix}$$
$$[\mathbf{M}_{4(ij)}^{\kappa^*}]_{1\times4} = \{m_{(\mathbf{q}_i)}^{\mathbf{M}_{4(ij)}^{\kappa^*}}, t_{(\mathbf{q}_i)}^{\mathbf{M}_{4(ij)}^{\kappa^*}}, n_{(\mathbf{q}_i)}^{\mathbf{M}_{4(ij)}^{\kappa^*}}\} = \begin{bmatrix} (0.1, 0.2, 0.7) & (0.4, 0.2, 0.4) & (0.6, 0.2, 0.1) & (0.3, 0.1, 0.3) \end{bmatrix}$$

$$[\mathbf{M}_{ij}^{\kappa^{ag}}]_{\alpha\times\beta} = \begin{bmatrix} (0.576, 0.000, 0.118) & (0.478, 0.244, 0.128) & (0.404, 0.394, 0.109) & (0.450, 0.272, 0.182) \\ (0.674, 0.122, 0.115) & (0.244, 0.419, 0.218) & (0.562, 0.126, 0.189) & (0.406, 0.118, 0.000) \\ (0.512, 0.188, 0.191) & (0.631, 0.172, 0.114) & (0.347, 0.361, 0.106) & (0.467, 0.189, 0.232) \\ (0.413, 0.000, 0.116) & (0.525, 0.182, 0.135) & (0.500, 0.220, 0.182) & (0.481, 0.260, 0.120) \end{bmatrix} \quad (15)$$

$$[\mathbf{M}_{(ij)}^{\kappa^\star}]_{4\times4} = \begin{bmatrix} (0.374, 0.182, 0.218) & (0.311, 0.382, 0.227) & (0.262, 0.504, 0.211) & (0.292, 0.405, 0.276) \\ (0.365, 0.296, 0.306) & (0.132, 0.534, 0.387) & (0.304, 0.299, 0.364) & (0.220, 0.293, 0.215) \\ (0.379, 0.292, 0.284) & (0.467, 0.279, 0.216) & (0.257, 0.443, 0.209) & (0.346, 0.294, 0.320) \\ (0.104, 0.240, 0.466) & (0.132, 0.379, 0.4784) & (0.126, 0.407, 0.507) & (0.121, 0.438, 0.469) \end{bmatrix} \quad (16)$$

TABLE 1. A comprehensive comparison of the proposed structure with hybrid fuzzy structures reported in literature.

Set Structure	Ref	MD	NMD	AD	Multi-Arguments	TOPSIS	TOPSIS Based on Distance/Similarity Measure
IFS	Boran [33]	✓	✓	×	×	✓	×
PFS	Sindu [46]	✓	✓	✓	×	✓	✓
FSS	Selim [35]	✓	×	×	×	✓	×
IFSS	Garg [47]	✓	✓	×	×	✓	×
FHSS	Ahsan [48]	✓	×	×	✓	✓	✓
IFHSS	Rehman, Saqlain [26], [27]	✓	✓	×	✓	✓	✓
$\mathbb{PF}_{\mathbb{HSS}_s}$ (proposed)	Proposed Approach	✓	✓	✓	✓	✓	✓

Similarly

$$\mathbf{D}^2(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star+}}_j) = 0.104$$
$$\mathbf{D}^3(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star+}}_j) = 0.039$$
$$\mathbf{D}^4(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star+}}_j) = 0.207$$

The normalised Hamming distance with respect to the positive ideal solution and the alternatives can be calculated by using formula (12)

$$\mathbf{D}^j(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star-}}_j) = \frac{1}{3\beta}\sum_{j=1}^{\beta}\{|[m^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [m^{\mathbf{M}^{\kappa^{\star-}}_j}_{(\mathbf{q}_i)}]_{v_j}| + |[t^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [t^{\mathbf{M}^{\kappa^{\star-}}_j}_{(\mathbf{q}_i)}]_{v_j}| + |[n^{\mathbf{M}^{\kappa^\star}_{v(ij)}}_{(\mathbf{q}_i)}]_{v_j} - [n^{\mathbf{M}^{\kappa^{\star-}}_j}_{(\mathbf{q}_i)}]_{v_j}|\}$$

$$\mathbf{D}^1(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star-}}_j) = \frac{1}{12}\begin{pmatrix} |0.374-0.104|+|0.182-0.182|+|0.218-0.466| \\ +|0.311-0.132|+|0.382-0.279|+|0.227-0.478| \\ +|0.262-0.126|+|0.504-0.299|+|0.211-0.507| \\ +|0.292-0.121|+|0.405-0.293|+|0.276-0.469| \end{pmatrix}$$

$$\mathbf{D}^1(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star-}}_j) = 0.180$$

Similarly

$$\mathbf{D}^2(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star-}}_j) = 0.129$$
$$\mathbf{D}^3(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star-}}_j) = 0.175$$
$$\mathbf{D}^4(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star-}}_j) = 0.034$$

Step 7: Ranking the alternatives is achieved using the relative closeness index, which is derived using (13)

$$RK^j = \frac{\mathbf{D}^j(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star-}}_j)}{max[\mathbf{D}^j(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star-}}_j)]} - \frac{\mathbf{D}^j(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star+}}_j)}{min[\mathbf{D}^j(\mathbf{M}^{\kappa^\star}_{(ij)}, \mathbf{M}^{\kappa^{\star+}}_j)]}$$

$$RK^1 = -0.588$$
$$RK^2 = -1.942$$
$$RK^3 = -0.024$$
$$RK^4 = -5.066$$

According to the relative closeness index, the set of chosen alternatives are sorted in descending order.

$$RK^3 \geq RK^1 \geq RK^2 \geq RK^4$$

RK^3 corresponds to $\mathbf{q}_3$, so $\mathbf{q}_3$ is the best choice. Based on this ranking, real estate has been identified as the best alternative for designing an Investment Sector Selector for selecting the most suitable sector for investment.

VI. COMPARATIVE ANALYSIS

The inclusion of MD, NMD, and AD values in $\mathbb{PF}_{\mathbb{HSS}_s}$ allows for handling uncertainty and vagueness in data. This is particularly valuable in real-world scenarios where data may not be precise but instead exists along a spectrum of membership and non-membership. The attributes and criteria considered in decision scenarios are often interconnected, meaning that changes or variations in one attribute may affect others. This interdependency makes it difficult to isolate and compare individual attributes in isolation. In some cases, the relationships between attributes and criteria may be complex and nonlinear. Factors that influence the decision may not be linearly related, making direct comparisons less meaningful. To address these challenges, we used TOPSIS that involve distance and similarity measures to compare alternatives. The $\mathbb{PF}_{\mathbb{HSS}_s}$ TOPSIS based on similarity measures is especially beneficial when dealing with complex decision scenarios where attributes are further divided into subcategories, and each attribute and subattribute has MD, NMD, and AD values. The superiority of the $\mathbb{PF}_{\mathbb{HSS}_s}$ TOPSIS technique can be attributed to several factors that make it a powerful approach for multi-criteria decision-making in complex and uncertain environments. Here are some of the advantages and strengths of $\mathbb{PF}_{\mathbb{HSS}_s}$ TOPSIS:

(i) By describing the degrees of membership, non-membership, and neutral values for attributes and subattributes, $\mathbb{PF}_{\mathbb{HSS}_s}$ allows for a more comprehensive analysis. Decision-makers can account for various shades of relevance, irrelevance, and neutrality, which is essential when dealing with complex, multifaceted attributes.

TABLE 2. Numerical analysis with existing structures.

Set Structure	RK^1	RK^2	RK^3	RK^4	Ranking
FHSS	0.000,	-1.990,	-0.7899,	-1.73	$RK^1 \geq RK^3 \geq RK^4 \geq RK^2$
IFHSS	- 3.088	-1.942	-2.204	-2.066	$RK^2 \geq RK^4 \geq RK^3 \geq RK^1$
$\mathbb{PF}_{\mathbb{HSS}_s}$ (proposed)	- 0.588	-1.942	-0.024	-5.066	$RK^3 \geq RK^1 \geq RK^2 \geq RK^4$

(ii) By considering the importance of attributive values by multiplying each attribute's value by its weight before calculating the overall similarity or distance, decision-makers can make more informed comparisons, especially when assessing the similarity or dissimilarity between sets is a critical aspect of the decision-making process.

(iii) The method provides an objective ranking of alternatives based on their similarity to an ideal solution, taking into account the specific values of MD, NMD, and AD for attributes and subattributes. This objectivity is valuable in promoting transparency and fairness in decision-making.

(iv) $\mathbb{PF}_{\mathbb{HSS}_s}$ TOPSIS is not limited to specific domains. It can be applied to a wide range of fields, including finance, healthcare, engineering, and more, where complex decisions need to be made based on multi-dimensional data.

A comparison made in Table 1. Also a numerical analysis of the devolped $\mathbb{PF}_{\mathbb{HSS}_s}$ TOPSIS with some existing structures from the Hypersoft set is provided in Table 2. The $\mathbb{PF}_{\mathbb{HSS}_s}$ TOPSIS has benefits because neutrality is involved, which is helpful in providing the best optimal for decision-making. From these tables likely compares the newly proposed $\mathbb{PF}_{\mathbb{HSS}_s}$-TOPSIS approach with existing methods or approaches used in similar decision-making scenarios. The purpose is to evaluate how the proposed method stacks up against other approaches.

VII. CONCLUSION

$\mathbb{PF}_{\mathbb{HSS}_s}$ are especially valuable when dealing with incomplete, inconsistent, and multi-argument data, which are common in real-world scenarios. By accommodating membership, non-membership, and abstention degrees, $\mathbb{PF}_{\mathbb{HSS}_s}$ offer a comprehensive representation of information, allowing for a more accurate analysis and decision-making process. In this paper we introduced new distance and similarity measures specifically designed for $\mathbb{PF}_{\mathbb{HSS}_s}$. These measures are crucial for quantifying the relationships and similarities between alternatives in a decision-making context. The TOPSIS technique is extended to $\mathbb{PF}_{\mathbb{HSS}_s}$ TOPSIS, allowing it to handle decision problems involving $\mathbb{PF}_{\mathbb{HSS}_s}$ information. In this paper we utilizes Hamming distance to calculate the dissimilarity of alternatives from the positive ideal and negative ideal solutions. After calculating the similarity and dissimilarity of each alternative to the ideal and negative ideal solutions using the Hamming distance for PFHSSs, a relative closeness index is computed. This index quantifies how close each alternative is to the ideal solution relative to all other alternatives. By applying the $\mathbb{PF}_{\mathbb{HSS}_s}$-TOPSIS technique to investment decision-making involves a crucial step of representing data using $\mathbb{PF}_{\mathbb{HSS}_s}$. This representation captures the inherent uncertainty and vagueness associated with the data used in investment analysis.

REFERENCES

[1] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, 1965.
[2] K. T. Atanassov and S. Stoeva, "Intuitionistic fuzzy sets," *Fuzzy sets Syst.*, vol. 20, no. 1, pp. 87–96, 1986.
[3] B. C. Cuong, "Picture fuzzy sets," *JCC*, vol. 30, no. 4, p. 409, Dec. 2014.
[4] S. He and Y. Wang, "Evaluating new energy vehicles by picture fuzzy sets based on sentiment analysis from online reviews," *Artif. Intell. Rev.*, vol. 56, no. 3, pp. 2171–2192, Mar. 2023.
[5] F. Wang and X. Yu, "Time-varying multi-period emergency decision-making based on dissimilarity measure and definite integral in the picture fuzzy context," *Expert Syst. Appl.*, vol. 227, Oct. 2023, Art. no. 120232.
[6] M. Luo and W. Li, "Some new similarity measures on picture fuzzy sets and their applications," *Soft Comput.*, vol. 27, no. 10, pp. 6049–6067, May 2023.
[7] C. Zhang, H. Hou, A. K. Sangaiah, D. Li, F. Cao, and B. Wang, "Efficient mobile robot navigation based on federated learning and three-way decisions," in *Proc. Int. Conf. Neural Inf. Process.* Cham, Switzerland: Springer, 2023, pp. 408–422.
[8] D. Molodtsov, "Soft set theory—First results," *Comput. Math. Appl.*, vol. 37, nos. 4–5, pp. 19–31, Feb. 1999.
[9] P. K. Maji, A. Roy, and R. Biswas, "Fuzzy soft sets," *J. Fuzzy Math.*, vol. 9, pp. 589–602, 2001.
[10] P. K. Maji, R. Biswas, and A. R. Roy, "Intuitionistic fuzzy soft sets," *J. Fuzzy Math.*, vol. 9, no. 3, pp. 677–692, 2001.
[11] Y. Yang, C. Liang, S. Ji, and T. Liu, "Adjustable soft discernibility matrix based on picture fuzzy soft sets and its applications in decision making," *J. Intell. Fuzzy Syst.*, vol. 29, no. 4, pp. 1711–1722, Oct. 2015.
[12] F. Smarandache, "Extension of soft set to hypersoft set, and then to plithogenic hypersoft set," *Neutrosophic Sets Syst.*, vol. 22, pp. 168–170, Jan. 2018.
[13] M. Saeed, A. U. Rahman, M. Ahsan, and F. Smarandache, "An inclusive study on fundamentals of hypersoft set," in *Theory and Application of Hypersoft Set*, vol. 1. Brussels, Belgium: Pons Publication House, 2021, pp. 1–23.
[14] M. Abbas, G. Murtaza, and F. Smarandache, *Basic Operations on Hypersoft Sets and Hypersoft Point*. Gallup, USA: Univ. of New Mexico, 2020.
[15] A. Yolcu and T. Y. Ozturk, "Fuzzy hypersoft sets and it's application to decision-making," in *Theory and Application of Hypersoft Set*, vol. 50, 2021.
[16] A. Yolcu, F. Smarandache, and T. Y. Öztürk, "Intuitionistic fuzzy hypersoft sets," *Commun. Fac. Sci. Univ. Ankara A1 Math. Statist.*, vol. 70, no. 1, pp. 443–455, Jun. 2021.
[17] M. Saeed and M. I. Harl, "Fundamentals of picture fuzzy hypersoft set with application," *Neutrosophic Sets Syst.*, vol. 53, no. 1, p. 24, 2023.
[18] M. Saeed, M. I. Harl, M. H. Saeed, and I. Mekawy, "Theoretical framework for a decision support system for micro-enterprise supermarket investment risk assessment using novel picture fuzzy hypersoft graph," *PLoS ONE*, vol. 18, no. 3, Mar. 2023, Art. no. e0273642.
[19] M. I. Harl, M. Saeed, M. H. Saeed, T. Alballa, and H. A. E.-W. Khalifa, "Human intuitionistic data-based employee performance evaluation with similarity measure using lattice ordered picture fuzzy hypersoft sets," *IEEE Access*, vol. 11, pp. 105642–105654, 2023.

[20] M. I. Harl, M. Saeed, M. H. Saeed, T. Alharbi, and T. Alballa, "Bipolar picture fuzzy hypersoft set-based performance analysis of abrasive textiles for enhanced quality control," *Heliyon*, vol. 9, no. 9, Sep. 2023, Art. no. e19821.
[21] P. Majumdar and S. Samanta, "On similarity measures of fuzzy soft sets," *Int. J. Advance Soft Comput. Appl.*, vol. 3, no. 2, pp. 1–8, 2011.
[22] Z. Liu, K. Qin, and Z. Pei, "Similarity measure and entropy of fuzzy soft sets," *Sci. World J.*, vol. 2014, pp. 1–10, Jun. 2014.
[23] Q. Feng and W. Zheng, "New similarity measures of fuzzy soft sets based on distance measures," *Ann. Fuzzy Math. Informat.*, vol. 7, no. 4, pp. 669–686, 2014.
[24] S. K. Ghosh, A. Ghosh, and S. Bhattacharyya, "Recognition of cancer mediating biomarkers using rough approximations enabled intuitionistic fuzzy soft sets based similarity measure," *Appl. Soft Comput.*, vol. 124, Jul. 2022, Art. no. 109052.
[25] S. Aggarwal, H. Dhumras, and R. K. Bajaj, "On banking site selection decision making problem utilizing similarity measures of picture fuzzy soft sets," in *Proc. 5th Int. Conf. Multimedia, Signal Process. Commun. Technol. (IMPACT)*, Nov. 2022, pp. 1–5.
[26] A. U. Rahman, M. Saeed, H. A. E.-W. Khalifa, and W. A. Afifi, "Decision making algorithmic techniques based on aggregation operations and similarity measures of possibility intuitionistic fuzzy hypersoft sets," *AIMS Math.*, vol. 7, no. 3, pp. 3866–3895, 2022.
[27] M. Saqlain, M. Riaz, R. Imran, and F. Jarad, "Distance and similarity measures of intuitionistic fuzzy hypersoft sets with application: Evaluation of air pollution in cities based on air quality index," *AIMS Math.*, vol. 8, no. 3, pp. 6880–6899, 2023.
[28] M. Saqlain, M. Riaz, M. A. Saleem, and M.-S. Yang, "Distance and similarity measures for neutrosophic HyperSoft set (NHSS) with construction of NHSS-TOPSIS and applications," *IEEE Access*, vol. 9, pp. 30803–30816, 2021.
[29] M. N. Jafar, M. Saeed, M. Saqlain, and M.-S. Yang, "Trigonometric similarity measures for neutrosophic hypersoft sets with application to renewable energy source selection," *IEEE Access*, vol. 9, pp. 129178–129187, 2021.
[30] C. L. Hwang and K. Yoon, "Methods for multiple attribute decision making," in *Multiple Attribute Decision Making* (Lecture Notes in Economics and Mathematical Systems), vol. 186. Berlin, Germany: Springer, 1981, doi: 10.1007/978-3-642-48318-9_3.
[31] C.-T. Chen, C.-T. Lin, and S.-F. Huang, "A fuzzy approach for supplier evaluation and selection in supply chain management," *Int. J. Prod. Econ.*, vol. 102, no. 2, pp. 289–301, Aug. 2006.
[32] Z.-S. Chen, M. Li, W.-T. Kong, and K.-S. Chin, "Evaluation and selection of HazMat transportation alternatives: A PHFLTS- and TOPSIS-integrated multi-perspective approach," *Int. J. Environ. Res. Public Health*, vol. 16, no. 21, p. 4116, Oct. 2019.
[33] F. E. Boran, S. Genç, M. Kurt, and D. Akay, "A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method," *Expert Syst. Appl.*, vol. 36, no. 8, pp. 11363–11368, Oct. 2009.
[34] Z.-S. Chen, Y. Yang, X.-J. Wang, K.-S. Chin, and K.-L. Tsui, "Fostering linguistic decision-making under uncertainty: A proportional interval type-2 hesitant fuzzy TOPSIS approach based on Hamacher aggregation operators and andness optimization models," *Inf. Sci.*, vol. 500, pp. 229–258, Oct. 2019.
[35] S. Eraslan and F. Karaaslan, "A group decision making method based on topsis under fuzzy soft environment," *J. New Theory*, vol. 10, no. 3, pp. 30–40, 2015.
[36] T. Mahmood, K. Hussain, J. Ahmmad, U. U. Rehman, and M. Aslam, "A novel approach toward TOPSIS method based on lattice ordered T-bipolar soft sets and their applications," *IEEE Access*, vol. 10, pp. 69727–69740, 2022.
[37] R. M. Zulqarnain, X. L. Xin, and M. Saeed, "Extension of TOPSIS method under intuitionistic fuzzy hypersoft environment based on correlation coefficient and aggregation operators to solve decision making problem," *AIMS Math.*, vol. 6, no. 3, pp. 2732–2755, 2021.
[38] A. Samad, R. M. Zulqarnain, E. Sermutlu, R. Ali, I. Siddique, F. Jarad, and T. Abdeljawad, "Selection of an effective hand sanitizer to reduce COVID-19 effects and extension of TOPSIS technique based on correlation coefficient under neutrosophic hypersoft set," *Complexity*, vol. 2021, pp. 1–22, May 2021.
[39] A. Kalaichelvi and P. H. Malini, "Application of fuzzy soft sets to investment decision making problem," *Int. J. Math. Sci. Appl.*, vol. 1, no. 3, pp. 1583–1586, 2011.
[40] N. Y. özgür and T. Nihal, "A note on application of fuzzy soft sets to investment decision making problem," *J. New Theory*, vol. 10, no. 7, pp. 1–10, 2015.
[41] A. Mukherjee and A. K. Das, "Application of interval valued intuitionistic fuzzy soft set in investment decision making," in *Proc. 5th Int. Conf. Adv. Comput. Commun. (ICACC)*, Sep. 2015, pp. 61–64.
[42] M. J. Khan, P. Kumam, W. Kumam, and A. N. Al-Kenani, "Picture fuzzy soft robust VIKOR method and its applications in decision-making," *Fuzzy Inf. Eng.*, vol. 13, no. 3, pp. 296–322, Jul. 2021.
[43] H. Zhang, Y. Zhang, A. U. Rahman, and M. Saeed, "An intelligent sv-neutrosophic parameterized MCDM approach to risk evaluation based on complex fuzzy hypersoft set for real estate investments," *Manage. Decis.*, vol. 61, no. 2, pp. 486–505, Mar. 2023.
[44] J. Zhao, B. Li, A. U. Rahman, and M. Saeed, "An intelligent multiple-criteria decision-making approach based on sv-neutrosophic hypersoft set with possibility degree setting for investment selection," *Manage. Decis.*, vol. 61, no. 2, pp. 472–485, Mar. 2023.
[45] S. Khan, M. Gulistan, N. Kausar, D. Pamucar, T.-P. Hong, and H. Abdul Wahab, "Aggregation operators for decision making based on q-Rung orthopair fuzzy hypersoft sets: An application in real estate project," *Comput. Model. Eng. Sci.*, vol. 136, no. 3, pp. 3141–3156, 2023.
[46] M. S. Sindhu, T. Rashid, and A. Kashif, "Modeling of linear programming and extended TOPSIS in decision making problem under the framework of picture fuzzy sets," *PLoS ONE*, vol. 14, no. 8, Aug. 2019, Art. no. e0220957.
[47] H. Garg and R. Arora, "TOPSIS method based on correlation coefficient for solving decision-making problems with intuitionistic fuzzy soft set information," *AIMS Math.*, vol. 5, no. 4, pp. 2944–2966, 2020.
[48] M. Ahsan, M. A. Sarwar, S. A. Lone, S. A. Almutlak, and S. Anwer, "Optimizing new technology implementation through fuzzy hypersoft set: A framework incorporating entropy, similarity measure, and TOPSIS techniques," *IEEE Access*, vol. 11, pp. 80680–80691, 2023.

MUHAMMAD IMRAN HARL was born in Pakistan. He received the M.Phil. degree from the University of Sargodha, Pakistan, in 2013. He is currently pursuing the Ph.D. degree in mathematics with the University of Management and Technology (UMT), Lahore, under the kind supervision of Dr. Muhammad Saeed. He is also a Lecturer in mathematics with the Government Graduate College Jauharabad, Pakistan. He has five research publications under his name. His research interests include algebra, lattice theory, fuzzy mathematics, algebraic hyperstructures, fuzzy algebraic hyperstructures, group theory, soft computing, and neutrosophic sets.

MUHAMMAD SAEED was born in Pakistan, in July 1970. He received the Ph.D. degree in mathematics from Quaid-i-Azam University, Islamabad, Pakistan, in 2012. He taught mathematics at intermediate and degree level with exceptional results. He was involved as a Teacher Trainer for professional development for more than five years. He was the Chairperson of the Department of Mathematics, UMT, Lahore, from 2014 to 2021. Under his dynamics leadership, the Mathematics Department has produced more than 30 Ph.D.'s. He has supervised 30 M.S. students, seven Ph.D.'s, and published more than 150 articles in recognized journals. His research interests include fuzzy mathematics, rough sets, soft set theory, hypersoft set, neutrosophic sets, algebraic and hybrid structures of soft sets and hypersoft sets, multicriteria decision making, optimizations, artificial intelligence, pattern recognition and optimization under convex environments, graph theory in fuzzy-like, soft-like, and hypersoft-like environments, similarly, distance measures and their relevant operators in multipolar hybrid structures. He was awarded the "Best Teacher," in 1999, 2000, 2001, and 2002.

MUHAMMAD HARIS SAEED was born in Pakistan. He received the Intermediate degree in chemistry and in biology from the Government College Township, Lahore, Pakistan. He is currently pursuing the Master of Science degree in chemistry from the University of Management and Technology (UMT), Lahore. During his undergraduate studies with UMT, he was the Basketball Team's Captain and an Assistant Teacher. Currently, he is conducting research in computational chemistry and multi-attribute decision-making with UMT, where he has authored 17 research publications. His research interests include applying multi-criteria decision-making (MCDM) in different aspects of chemistry and utilizing fuzzy and statistical structures for quantitative structure-property relationship (QSPR) analysis.

SANAA AHMED BAJRI received the Ph.D. degree from the Wentworth College, University of York, York, U.K. Since 2012, she has been with the Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia, where she is currently the Director of the Research, Development and Innovation Unit. Her research interests include algebra, especially group theory and semigroup theory.

ALHANOUF ALBURAIKAN received the Ph.D. degree in mathematics from Howard University, Washington, DC, USA. She is currently with the Department of Mathematics, College of Science and Arts, Qassim University, Buraydah, Saudi Arabia. Her research interests include complex analysis, fuzzy mathematics, game theory, and non-linear programming in real and complex spaces.

HAMIDEN ABD EL-WAHED KHALIFA received the Ph.D. degree from the Faculty of Science, Tanta University, Tanta, Gharbia Governorate, Egypt. She was a Full Professor in operations research (mathematical programming). She is currently with the Operations Research Department, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt, as a Professor. She is also with the Department of Mathematics, College of Science and Arts, Qassim University, Buraydah, Saudi Arabia, for research projects. She has published more than 100 publications in SCI journals. Her research interests include game theory, multi-objective programming, fuzzy mathematics, rough sets, and decision making.