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RESEARCH ARTICLE

Selection of Software Development Methodology by Employing a Multi-Criteria Decision-Making Approach Based on Logarithmic Bipolar Complex Fuzzy Aggregation Operators

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ABSTRACT Numerous factors, including project complexity, team size, customer collaboration, and development pace, must be taken into consideration while choosing a software development methodology. This problem is a multi-criteria decision-making dilemma since the many criteria frequently fight with one another and fluctuate in priority. In order to systematically select the best methodology, the multi-criteria decision-making technique assists in quantifying and assessing these factors. This ensures that decisions are made in a fair and informed manner within the context of the challenging software development environment. Thus, in this article, we develop a multi-criteria decision-making approach in the setting of bipolar complex fuzzy information for the prioritization and selection of optimal software development methodology. For this, we first, invent logarithmic operational laws and associated results for the bipolar complex fuzzy set. Then, we invent four aggregation operators by utilizing these logarithmic operational laws under the structure of bipolar complex fuzzy information that is, logarithmic bipolar complex fuzzy weighted averaging, logarithmic bipolar complex fuzzy ordered weighted averaging, logarithmic bipolar complex fuzzy weighted geometric, and logarithmic bipolar complex fuzzy ordered weighted geometric. After that, we solve a multi-criteria decision-making dilemma related to the prioritization and selection of software development methodology by considering the artificial data in the setting of bipolar complex fuzzy information and achieve that "Agile" is the optimal software development methodology among the considered four different software development methodologies, i.e., Waterfall, DevOps, Spiral, and Agile. In the last, we investigate the comparison study of the deduced theory to a few current theories to reveal the importance and supremacy of the constructed theory.

INDEX TERMS Software development methodology, logarithmic operations, bipolar complex fuzzy set, MCDM.

I. INTRODUCTION

When discussing planning, creating, testing, and managing software systems, the term "software development methodology (SDM)" is used. From the inception of the first idea through the ultimate deployment and continuous maintenance, it includes a set of ideas, techniques, and rules that direct the whole software development process. Over time, many approaches have been created to accommodate various project requirements, team sizes, technological difficulties, and organizational demands. For a software project to be completed successfully, each methodology outlines a certain set of tasks, roles, and responsibilities. The capacity of software development methodologies to offer

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structure and predictability to the sometimes intricate and varied process of building software is what makes them so important. Methodologies aid teams in remaining organized, streamlining communication, and efficiently managing resources by giving a defined structure for project management and cooperation. They help to reduce the risk of project failure or delays by helping to define reasonable expectations, estimate project durations, and allocate resources. Furthermore, approaches provide a methodical approach to problem-solving and decision-making, guaranteeing that the software output satisfies the necessary quality standards and functional objectives. Saeed et al. [1] analyzed the approaches to software development. Chandra [2] and Despa [3] investigated the comparison of numerous SDMs.

Software development methodologies also improve team communication and collaboration. By outlining roles and duties, they encourage responsibility and open channels of communication among the team. When teams are dispersed across several locations or when numerous teams work together on the same project, this is very important. Furthermore, methods make it possible for development processes to be continuously improved. Teams may improve their approach and increase efficiency over time by reflecting on previous projects, discovering bottlenecks, and iterating on procedures. Because they offer structure, order, and predictability, software development techniques are essential to the software development lifecycle. They make sure that resources are used efficiently, communication is simplified, and projects are properly managed. The effective delivery of software products that satisfy user expectations and quality standards is made possible by techniques, which promote cooperation, problem-solving, and continual development. Geambasu et al. [4] devised the factors that influence the selection of SDM. Hijazi et al. [5] investigated risk management in various SDMs. A conceptual foundation related to the SDMs was discussed by Gonzalez-Perez and Henderson-Sellers [6]. Blum [7] investigated a taxonomy of SDM. Vavpotic and Bajec [8] deduced the evaluation of technical and social aspects of SDMs.

Crisp sets, a key idea in traditional set theory, have several drawbacks. Their binary membership function, which either includes or excludes items without taking degrees of belonging into account, makes it difficult to handle ambiguity or vagueness in real-world circumstances. Furthermore, crisp sets cannot reflect overlapping or ambiguous borders between sets, which are frequent in complex systems. Zadeh [9] developed the fuzzy set (FS) theory to overcome these constraints by enabling elements to have degrees of belonging ranging from 0 to 1. FS better represents slow transitions and partial belongingness because they can handle ambiguity and vagueness. They are therefore suitable for simulating real-world scenarios where elements may to variable degrees belong to multiple sets. FS offers a more realistic framework for deliberation and reasoning, notably in domains like artificial intelligence, control systems, and linguistics, by permitting the representation of fuzzy or ambiguous notions. Wang and Lin [10] initiated the multi-criteria decision-making (MCDM) technique within the fuzzy structure for the selection of configuration items for software development. Chen [11] discussed risk software development by employing FS. Lee et al. [12] initiated a novel algorithm for employing FS for assessing risk in software development. Hapke et al. [13] discussed a scheduling system for software development under a fuzzy environment. Poulik et al. [14] devised a crossroads ranking relying on the Randic index graph under fuzzy theory. Moreover, Chen et al. [15] discussed linguistic Z-number, Liu [16] assessed the logistics efficiency of agriculture produced under intuitionistic FS, and Das et al. [17] devised ϕ-tolerance competition graphs under picture fuzzy information along with its applications. Zhang [18] devised the notion of the bipolar fuzzy set (BFS) by expanding the notion of FS. BFS better represents human opinion, as BFS interprets both positive and negative aspects (dual aspects) at a time. In the notion of BFS, there is a positive degree of belonging placed in $[0, 1]$ and negative degree of belonging placed in $[-1, 0]$ of each element. Riaz et al. [19] deduced sine trigonometric aggregation operators (AOs) within bipolar fuzzy (BF) information. Jana et al. [20] devised logarithmic AOs in the setting of BFS. Jana et al. [21] deduced Dombi prioritized AOs for the BF set. Akram [22] initiated bipolar fuzzy graphs (BFGs) and Samanta and Pal [23] devised irregular BFGs. Rashmanlou et al. [24] devised a product of BFGs and Akram and Akmal [25] devised an application of BFGs. Poulik and Ghorai [26] devised the notion of connectivity in BF incidence graphs. The Randic index of BFGs was deduced by Poulik et al. [27].

Ramot et al. [28] deduced the notion of the complex fuzzy set (CFS) by expanding the notion of FS (by changing $[0, 1]$ to a complex plane's unit disc). The structure of CFS was invented by Ramot et al. [28] in the model of polar form which represents two-dimensional information. Afterward, in 2011, Tamir et al. [29] deduced a novel approach to CFS in the cartesian form by transforming the complex plane's unit disc to the complex plane's unit square. The arithmetic and geometric AOs for complex fuzzy (CF) information were devised by Bi et al. [30] and Bi et al. [31] respectively. Zhang et al. [32] investigated the properties of operation under CF information. Hu et al. [33] discussed the continuity of CF operations. Rehman [34] established probability AOs in the model of CFS. For tackling two-dimensional information with dual aspects (positive and negative) at a time, Mahmood and Rehman [35] devised the notion of the bipolar complex fuzzy set (BCFS). In the notion of BCFS, there is a positive degree of belonging which is placed in the first quadrant of a complex plane's unit square, and a negative degree of belonging which is placed in 3^{rd} quadrant of a complex plane's unit square of each element. Rehman and Mahmood [36] employed the notion of BCFS in pattern recognition and medical diagnosis. Mahmood and Rehman [37] discussed Dombi AOs within the bipolar complex fuzzy (BCF) set. The Aczel-Alsina AOs and Maclaurin symmetric mean operators in the setting of BCF set were discussed by Mahmood et al. [38] and Mahmood and Rehman [39]. The digital technology implementation under the setting of BCFS was devised by Mahmood and Rehman [40].

It is possible to obtain comparably accurate estimates by using logarithmic operations, which are frequently disregarded in favor of algebraic operations. However, only a small amount of research has been done on the use of logarithmic operations for coping with dual aspects and no research has been done on the use of logarithmic operations for tackling 2nd dimension (extra fuzzy information). Also, there is no research on the utilization of logarithmic operations for tackling dual aspects and extra fuzzy information at the same time. Further, the selection and prioritization of software development methodology is an MCDM dilemma since various criteria are involved. In many situations, the criteria of software development methodologies can have dual aspects (positive and negative aspects) along with extra fuzzy information. For example, the criterion "development speed" has dual aspects "high development speed" and "low development speed" along with extra fuzzy information that is its positive and negative effects. There is no research, which considers the dual aspects and related extra fuzzy information of criteria of the software development methodologies. Inspired by these research gaps, in this script, we deduce an MCDM approach by employing logarithmic AOs within BCFS for the selection and prioritization of software development methodology. Through the deduced MCDM approach, one can cope with any MCDM dilemma, where criteria contain dual aspects and extra fuzzy information. The main contribution of this script is interpreted as We investigate logarithmic operations for BCF information that can fill in the gaps left by algebraic operations while capturing the interrelationships across numerous BCFSs. Based on logarithmic operations for BCF information, we invent AOs such as logarithmic BCF weighted averaging (L-BCFWA), logarithmic BCF ordered weighted averaging (L-BCFOWA), logarithmic BCF weighted geometric (L-BCFWG), and logarithmic BCF ordered weighted geometric (L-BCFOWG) and discuss related axioms. Compared to traditional methods, BCF logarithmic aggregation operators have several advantages: they compress large differences, highlight smaller values, manage diverse ranges well, and incorporate decision-maker attitudes through weighting. These features enable a more flexible and nuanced representation of complex data and preferences in scenarios such as information fusion and decision-making. After that, we construct an approach to MCDM within the structure of BCFS to handle MCDM dilemmas. Employing the invented approach of MCDM, we discuss the case study "Selection of software development methodology". We investigate the comparison study of the deduced theory to a few current theories to reveal the importance and supremacy of the constructed theory.

The rest of the script is managed as: The notion of BCFS, score and accuracy function, and fundamental operations of BCFS are recalled in Section II. In Section III, we explore a few new logarithmic operational laws (LOLs) based on BCFNs and talk about their properties. In Section IV, we invent logarithmic AOs based on the invented LOLs in the setting of BCF information. These AOs are L-BCFWA, L-BCFOWA, L-BCFWG, and L-BCFOWG operators. In Section V, we invent the MCDM approach and then discuss a case study related to the software development methodology. We also have a comparison study in Section V. In Section VI, we interpret the conclusion.

II. PRELIMINARIES

The notion of BCFS, score and accuracy function, and fundamental operations of BCFS are recalled in this section of the article.

Definition 1 [35]: The underneath structure

$$\text{Ꝁ} = \left\{\left(\mathrm{d},\ \Pi_{P-\text{Ꝁ}}(\mathrm{d}),\ \Pi_{N-\text{Ꝁ}}(\mathrm{d})\right) \mid \mathrm{d} \in \mathfrak{D}\right\}$$

designated the structure of BCFS, where $\Pi_{P-\text{Ꝁ}}(\mathrm{d}) = \Pi_{RP-\text{Ꝁ}}(\mathrm{d}) + \iota\, \Pi_{IP-\text{Ꝁ}}(\mathrm{d})$ is a positive degree of belonging and $\Pi_{N-\text{Ꝁ}}(\mathrm{d}) = \Pi_{RN-\text{Ꝁ}}(\mathrm{d}) + \iota\, \Pi_{IN-\text{Ꝁ}}(\mathrm{d})$, is a negative degree of belonging, with $\Pi_{RP-\text{Ꝁ}}(\mathrm{d}), \Pi_{IP-\text{Ꝁ}}(\mathrm{d}) \in [0, 1]$ and $\Pi_{RN-\text{Ꝁ}}(\mathrm{d}), \Pi_{IN-\text{Ꝁ}}(\mathrm{d}) \in [-1, 0]$. The bipolar CF number (BCFN) is revealed as $\text{Ꝁ} = \left(\Pi_{P-\text{Ꝁ}}, \Pi_{N-\text{Ꝁ}}\right) = \left(\Pi_{RP-\text{Ꝁ}} + \iota\Pi_{IP-\text{Ꝁ}},\ \Pi_{RN-\text{Ꝁ}} + \iota\, \Pi_{IN-\text{Ꝁ}}\right)$

Definition 2 [37]: Under a BCFN $\text{Ꝁ} = \left(\Pi_{P-\text{Ꝁ}}, \Pi_{N-\text{Ꝁ}}\right) = \left(\Pi_{RP-\text{Ꝁ}} + \iota\Pi_{IP-\text{Ꝁ}},\ \Pi_{RN-\text{Ꝁ}} + \iota\, \Pi_{IN-\text{Ꝁ}}\right)$

$$\dot{\check{S}}_{\mathbb{SF}}(\text{Ꝁ}) = \frac{1}{4}\left(2+\Pi_{RP-\text{Ꝁ}}+\Pi_{IP-\text{Ꝁ}}+\Pi_{RN-\text{Ꝁ}}+\Pi_{IN-\text{Ꝁ}}\right), \quad \dot{\check{S}}_{\mathbb{SF}} \in [0,\ 1] \tag{1}$$

$$\tilde{H}_{\mathbb{AF}}(\text{Ꝁ}) = \frac{\Pi_{RP-\text{Ꝁ}}+\Pi_{IP-\text{Ꝁ}}-\Pi_{RN-\text{Ꝁ}}-\Pi_{IN-\text{Ꝁ}}}{4}, \quad \tilde{H}_{\mathbb{AF}} \in [0,\ 1] \tag{2}$$

are score and accuracy values of Ꝁ respectively. By employing Eq. (1) and Eq. (2), we have

1. if $\dot{\check{S}}_{\mathbb{SF}}(\text{Ꝁ}_1) < \dot{\check{S}}_{\mathbb{SF}}(\text{Ꝁ}_2)$, then $\text{Ꝁ}_1 < \text{Ꝁ}_2$;
2. if $\dot{\check{S}}_{\mathbb{SF}}(\text{Ꝁ}_1) > \dot{\check{S}}_{\mathbb{SF}}(\text{Ꝁ}_2)$, then $\text{Ꝁ}_1 > \text{Ꝁ}_2$;
3. if $\dot{\check{S}}_{\mathbb{SF}}(\text{Ꝁ}_1) = \dot{\check{S}}_{\mathbb{SF}}(\text{Ꝁ}_2)$, then
   (1) if $\tilde{H}_{\mathbb{AF}}(\text{Ꝁ}_1) < \tilde{H}_{\mathbb{AF}}(\text{Ꝁ}_2)$, then $\text{Ꝁ}_1 < \text{Ꝁ}_2$;
   (2) if $\tilde{H}_{\mathbb{AF}}(\text{Ꝁ}_1) > \tilde{H}_{\mathbb{AF}}(\text{Ꝁ}_2)$, then $\text{Ꝁ}_1 > \text{Ꝁ}_2$;
   (3) if $\tilde{H}_{\mathbb{AF}}(\text{Ꝁ}_1) = \tilde{H}_{\mathbb{AF}}(\text{Ꝁ}_2)$, then $\text{Ꝁ}_1 = \text{Ꝁ}_2$.

Definition 3 [37]: Under two BCFNs $\text{Ꝁ}_1 = \left(\Pi_{P-\text{Ꝁ}_1}, \Pi_{N-\text{Ꝁ}_1}\right) = \left(\Pi_{RP-\text{Ꝁ}_1} + \iota\Pi_{IP-\text{Ꝁ}_1},\ \Pi_{RN-\text{Ꝁ}_1} + \iota\, \Pi_{IN-\text{Ꝁ}_1}\right)$ and $\text{Ꝁ}_2 = \left(\Pi_{P-\text{Ꝁ}_2}, \Pi_{N-\text{Ꝁ}_2}\right) = \left(\Pi_{RP-\text{Ꝁ}_2} + \iota\Pi_{IP-\text{Ꝁ}_2},\ \Pi_{RN-\text{Ꝁ}_2} + \iota\, \Pi_{IN-\text{Ꝁ}_2}\right)$ and $\mathfrak{b} > 0$, we have

1. $$\text{Ꝁ}_1 \oplus \text{Ꝁ}_2 = \begin{pmatrix} \Pi_{RP-\text{Ꝁ}_1} + \Pi_{RP-\text{Ꝁ}_2} - \Pi_{RP-\text{Ꝁ}_1}\Pi_{RP-\text{Ꝁ}_2} \\ +\iota\left(\Pi_{IP-\text{Ꝁ}_1} + \Pi_{RP-\text{Ꝁ}_2} - \Pi_{IP-\text{Ꝁ}_1}\Pi_{IP-\text{Ꝁ}_2}\right), \\ -\left(\Pi_{RN-\text{Ꝁ}_1}\Pi_{RN-\text{Ꝁ}_2}\right) + \iota\left(-\left(\Pi_{IN-\text{Ꝁ}_1}\Pi_{IN-\text{Ꝁ}_2}\right)\right) \end{pmatrix}$$
2. $\text{Ꝁ}_1 \otimes \text{Ꝁ}_2$

$$= \begin{pmatrix} \Pi_{RP-\text{Ꝁ}_1} \Pi_{RP-\text{Ꝁ}_2} + \iota\, \Pi_{IP-\text{Ꝁ}_1} \Pi_{IP-\text{Ꝁ}_2}, \\ \Pi_{RN-\text{Ꝁ}_1} + \Pi_{RN-\text{Ꝁ}_2} \Pi_{RN-\text{Ꝁ}_1} + \Pi_{RN-\text{Ꝁ}_2} \\ +\iota\, \left(\Pi_{IN-\text{Ꝁ}_1} + \Pi_{IN-\text{Ꝁ}_2} \Pi_{IN-\text{Ꝁ}_1} + \Pi_{IN-\text{Ꝁ}_2} \right) \end{pmatrix}$$

3. $\mathfrak{b}\text{Ꝁ}_1$

$$= \begin{pmatrix} 1 - \left(1 - \Pi_{RP-\text{Ꝁ}_1}\right)^{\mathfrak{b}} + \iota \left(1 - \left(1 - \Pi_{IP-\text{Ꝁ}_1}\right)^{\mathfrak{b}}\right), \\ -\left|\Pi_{RN-\text{Ꝁ}_1}\right|^{\mathfrak{b}} + \iota \left(-\left|\Pi_{IN-\text{Ꝁ}_1}\right|^{\mathfrak{b}}\right) \end{pmatrix}$$

4. ${\text{Ꝁ}_1}^{\mathfrak{b}}$

$$= \left(\begin{pmatrix} \left(\Pi_{RP-\text{Ꝁ}_1}\right)^{\mathfrak{b}} + \iota \left(\Pi_{IP-\text{Ꝁ}_1}\right)^{\mathfrak{b}}, \\ -1 + \left(1 + \Pi_{RN-\text{Ꝁ}_1}\right)^{\mathfrak{b}} + \iota \left(-1 + \left(1 + \Pi_{IN-\text{Ꝁ}_1}\right)^{\mathfrak{b}}\right) \end{pmatrix}\right)$$

Theorem 1 [37]: Under two BCFNs, $\text{Ꝁ}_1 = \left(\Pi_{P-\text{Ꝁ}_1}, \Pi_{N-\text{Ꝁ}_1}\right) = \left(\Pi_{RP-\text{Ꝁ}_1} + \iota \Pi_{IP-\text{Ꝁ}_1}, \Pi_{RN-\text{Ꝁ}_1} + \iota\, \Pi_{IN-\text{Ꝁ}_1}\right)$ and $\text{Ꝁ}_2 = \left(\Pi_{P-\text{Ꝁ}_2}, \Pi_{N-\text{Ꝁ}_2}\right) = \left(\Pi_{RP-\text{Ꝁ}_2} + \iota \Pi_{IP-\text{Ꝁ}_2}, \Pi_{RN-\text{Ꝁ}_2} + \iota\, \Pi_{IN-\text{Ꝁ}_2}\right)$, and $\mathfrak{b}, \mathfrak{b}_1, \mathfrak{b}_2 > 0$, then

1. $\text{Ꝁ}_1 \oplus \text{Ꝁ}_2 = \text{Ꝁ}_2 \oplus \text{Ꝁ}_1$
2. $\text{Ꝁ}_1 \otimes \text{Ꝁ}_2 = \text{Ꝁ}_2 \otimes \text{Ꝁ}_1$
3. $\mathfrak{b}\left(\text{Ꝁ}_1 \oplus \text{Ꝁ}_2\right) = \mathfrak{b}\text{Ꝁ}_1 \oplus \mathfrak{b}\text{Ꝁ}_2$
4. $\left(\text{Ꝁ}_1 \otimes \text{Ꝁ}_2\right)^{\mathfrak{b}} = {\text{Ꝁ}_1}^{\mathfrak{b}} \otimes {\text{Ꝁ}_2}^{\mathfrak{b}}$
5. $\mathfrak{b}_1\text{Ꝁ}_1 \oplus \mathfrak{b}_2\text{Ꝁ}_1 = \left(\mathfrak{b}_1 + \mathfrak{b}_2\right)\text{Ꝁ}_1$
6. ${\text{Ꝁ}_1}^{\mathfrak{b}_1} \otimes {\text{Ꝁ}_1}^{\mathfrak{b}_2} = {\text{Ꝁ}_1}^{\mathfrak{b}_1 + \mathfrak{b}_2}$
   $\left({\text{Ꝁ}_2}^{\mathfrak{b}_1}\right)^{\mathfrak{b}_2} = {\text{Ꝁ}_2}^{\mathfrak{b}_1\mathfrak{b}_2}$.

III. BCF LOGARITHMIC OPERATIONAL LAWS OF BCFS AND BCFN

In this segment, we explore a few new logarithmic operational laws (LOLs) based on BCFNs and talk about their properties. The fundamental motivation behind this part is to interpret new logarithmic AOs relying on BCF data. Clearly, $\log_\xi 1$ is not well-defined and $\log_\xi 0$ has no meaning in the real numbers. Thus, here, we consider that $\text{Ꝁ} \neq 0$, where Ꝁ is a BCFN and $\xi \neq 1$ in this manuscript.

Definition 4: Assume a BCFS $\text{Ꝁ} = \{(\mathrm{d}, \Pi_{P-\text{Ꝁ}}(\mathrm{d}), \Pi_{N-\text{Ꝁ}}(\mathrm{d})) \mid \mathrm{d} \in \mathfrak{D}\} = \{(\mathrm{d}, \Pi_{RP-\text{Ꝁ}}(\mathrm{d}) + \iota\, \Pi_{IP-\text{Ꝁ}}(\mathrm{d}), \Pi_{RN-\text{Ꝁ}}(\mathrm{d}) + \iota\, \Pi_{IN-\text{Ꝁ}}(\mathrm{d})) \mid \mathrm{d} \in \mathfrak{D}\}$ over $\mathfrak{D}$, then the LOL of BCFS Ꝁ is deduced as

$$\log_\xi(\text{Ꝁ}) = \left\{ \begin{pmatrix} \mathrm{d},\ 1 - \log_\xi \Pi_{P-\text{Ꝁ}}(\mathrm{d}), \\ -\log_\xi\left(1 + \Pi_{N-\text{Ꝁ}}(\mathrm{d})\right) \end{pmatrix} \mid \mathrm{d} \in \mathfrak{D} \right\}$$
$$= \left\{ \left(\mathrm{d}, \begin{pmatrix} 1 - \log_\xi \Pi_{RP-\text{Ꝁ}}(\mathrm{d}) \\ +\iota\left(1 - \log_\xi \Pi_{IP-\text{Ꝁ}}(\mathrm{d})\right), \\ -\log_\xi\left(1 + \Pi_{RN-\text{Ꝁ}}(\mathrm{d})\right) \\ +\iota\left(-\log_\xi\left(1 + \Pi_{IN-\text{Ꝁ}}\right)(\mathrm{d})\right) \end{pmatrix} \right) \mid \mathrm{d} \in \mathfrak{D} \right\}$$

Noted that $0 < \xi \leq \min(\Pi_{RP-\text{Ꝁ}}(\mathrm{d}), 1 + \Pi_{RN-\text{Ꝁ}}(\mathrm{d}), \Pi_{IP-\text{Ꝁ}}(\mathrm{d}), 1 + \Pi_{IN-\text{Ꝁ}}(\mathrm{d})) \leq 1, \xi \neq 1$. Obviously, $\log_\xi(\text{Ꝁ})$ is again a BCFS. Actually, from the definition of BCFS, we have that the real and unreal parts of a positive degree of belonging hold the underneath

$$\Pi_{RP-\text{Ꝁ}}: \mathfrak{D} \to [0, 1], \text{ i.e., } \forall\, \mathrm{d} \in \mathfrak{D},\ \Pi_{RP-\text{Ꝁ}}(\mathrm{d}) \in [0, 1]$$
$$\Pi_{IP-\text{Ꝁ}}: \mathfrak{D} \to [0, 1], \text{ i.e., } \forall\, \mathrm{d} \in \mathfrak{D},\ \Pi_{IP-\text{Ꝁ}}(\mathrm{d}) \in [0, 1]$$

Similarly, the real and unreal parts of the negative degree of belonging hold the underneath

$$\Pi_{RN-\text{Ꝁ}}: \mathfrak{D} \to [-1, 0], \text{ i.e., } \forall\, \mathrm{d} \in \mathfrak{D},\ \Pi_{RN-\text{Ꝁ}}(\mathrm{d}) \in [-1, 0]$$
$$\Pi_{IN-\text{Ꝁ}}: \mathfrak{D} \to [-1, 0], \text{ i.e., } \forall\, \mathrm{d} \in \mathfrak{D},\ \Pi_{IN-\text{Ꝁ}}(\mathrm{d}) \in [-1, 0]$$

If $0 < \xi \leq \min(\Pi_{RP-\text{Ꝁ}}(\mathrm{d}), 1 + \Pi_{RN-\text{Ꝁ}}(\mathrm{d}), \Pi_{IP-\text{Ꝁ}}(\mathrm{d}), 1 + \Pi_{IN-\text{Ꝁ}}(\mathrm{d})) \leq 1, \xi \neq 1$, then, the real and unreal parts of the positive degree of belonging

$$1 - \log_\xi \Pi_{RP-\text{Ꝁ}}: \mathfrak{D} \to [0, 1], \text{ i.e.,}$$
$$\forall\, \mathrm{d} \in \mathfrak{D},\ 1 - \log_\xi \Pi_{RP-\text{Ꝁ}}(\mathrm{d}) \in [0, 1]$$
$$1 - \log_\xi \Pi_{IP-\text{Ꝁ}}: \mathfrak{D} \to [0, 1], \text{ i.e.,}$$
$$\forall\, \mathrm{d} \in \mathfrak{D},\ 1 - \log_\xi \Pi_{IP-\text{Ꝁ}}(\mathrm{d}) \in [0, 1]$$

and the real and unreal parts of the negative degree of belonging

$$-\log_\xi\left(1 + \Pi_{RN-\text{Ꝁ}}\right): \mathfrak{D} \to [-1, 0], \text{ i.e.,}$$
$$\forall\, \mathrm{d} \in \mathfrak{D},\ -\log_\xi\left(1 + \Pi_{RN-\text{Ꝁ}}(\mathrm{d})\right) \in [-1, 0]$$
$$-\log_\xi\left(1 + \Pi_{IN-\text{Ꝁ}}\right): \mathfrak{D} \to [-1, 0], \text{ i.e.,}$$
$$\forall\, \mathrm{d} \in \mathfrak{D},\ -\log_\xi\left(1 + \Pi_{IN-\text{Ꝁ}}(\mathrm{d})\right) \in [-1, 0]$$

Consequently,

$$\log_\xi(\text{Ꝁ})$$
$$= \left\{\left(\mathrm{d},\ 1 - \log_\xi \Pi_{P-\text{Ꝁ}}(\mathrm{d}),\ -\log_\xi\left(1 + \Pi_{N-\text{Ꝁ}}(\mathrm{d})\right)\right) \mid \mathrm{d} \in \mathfrak{D}\right\}$$
$$= \left\{ \mathrm{d}, \begin{pmatrix} 1 - \log_\xi \Pi_{RP-\text{Ꝁ}}(\mathrm{d}) \\ +\iota\left(1 - \log_\xi \Pi_{IP-\text{Ꝁ}}(\mathrm{d})\right), \\ -\log_\xi\left(1 + \Pi_{RN-\text{Ꝁ}}(\mathrm{d})\right) \\ +\iota\left(-\log_\xi\left(1 + \Pi_{IN-\text{Ꝁ}}\right)(\mathrm{d})\right) \end{pmatrix} \mid \mathrm{d} \in \mathfrak{D} \right\}$$

where, $0 < \xi \leq \min(\Pi_{RP-\text{Ꝁ}}(\mathrm{d}), 1 + \Pi_{RN-\text{Ꝁ}}(\mathrm{d}), \Pi_{IP-\text{Ꝁ}}(\mathrm{d}), 1 + \Pi_{IN-\text{Ꝁ}}(\mathrm{d})) \leq 1, \xi \neq 1$ is a BCFS.

Definition 5: Assume a BCFN $\text{Ꝁ} = \left(\Pi_{P-\text{Ꝁ}}, \Pi_{N-\text{Ꝁ}}\right) = \left(\Pi_{RP-\text{Ꝁ}} + \iota\, \Pi_{IP-\text{Ꝁ}}, \Pi_{RN-\text{Ꝁ}} + \iota\, \Pi_{IN-\text{Ꝁ}}\right)$ over $\mathfrak{D}$, if,

$$\log_\xi(\text{Ꝁ})$$
$$= \begin{cases} \begin{pmatrix} 1 - \log_\xi \Pi_{P-\text{Ꝁ}}, \\ -\log_\xi\left(1 + \Pi_{N-\text{Ꝁ}}\right) \end{pmatrix} = \begin{pmatrix} 1 - \log_\xi \Pi_{RP-\text{Ꝁ}} \\ +\iota\left(1 - \log_\xi \Pi_{IP-\text{Ꝁ}}\right), \\ -\log_\xi\left(1 + \Pi_{RN-\text{Ꝁ}}\right) \\ +\iota\left(-\log_\xi\left(1 + \Pi_{IN-\text{Ꝁ}}\right)\right) \end{pmatrix} \\ \begin{pmatrix} 1 - \log_{\frac{1}{\xi}} \Pi_{P-\text{Ꝁ}}, \\ -\log_{\frac{1}{\xi}}\left(1 + \Pi_{N-\text{Ꝁ}}\right) \end{pmatrix} = \begin{pmatrix} 1 - \log_{\frac{1}{\xi}} \Pi_{RP-\text{Ꝁ}} \\ +\iota\left(1 - \log_{\frac{1}{\xi}} \Pi_{IP-\text{Ꝁ}}\right), \\ -\log_{\frac{1}{\xi}}\left(1 + \Pi_{RN-\text{Ꝁ}}\right) \\ +\iota\left(-\log_{\frac{1}{\xi}}\left(1 + \Pi_{IN-\text{Ꝁ}}\right)\right) \end{pmatrix} \end{cases}$$

with $0 < \xi \leq \min(\Pi_{RP-\text{Ꝁ}}, 1 + \Pi_{RN-\text{Ꝁ}}, \Pi_{IP-\text{Ꝁ}}, 1 + \Pi_{IN-\text{Ꝁ}}) \leq 1, \xi \neq 1$, and $0 < \frac{1}{\xi} \leq \min(\Pi_{RP-\text{Ꝁ}}, 1 + \Pi_{RN-\text{Ꝁ}}, \Pi_{IP-\text{Ꝁ}}, 1 + \Pi_{IN-\text{Ꝁ}}) \leq 1, \xi \neq 1$ then $\log_\xi(\text{Ꝁ})$ would be revealed as a logarithmic operator and the value of $\log_\xi(\text{Ꝁ})$ would be identified as logarithmic BCFN (L-BCFN). Further, $\log_\xi(0) = 0$, where $0 = (0 + \iota\, 0, -0 - \iota\, 0)$ and $\xi > 0$ and $\xi \neq 1$.

Definition 6: For two BCFNs $\text{Ꝁ}_1 = \left(\Pi_{P-\text{Ꝁ}_1}, \Pi_{N-\text{Ꝁ}_1}\right) = \left(\Pi_{RP-\text{Ꝁ}_1} + \iota \Pi_{IP-\text{Ꝁ}_1}, \Pi_{RN-\text{Ꝁ}_1} + \iota\, \Pi_{IN-\text{Ꝁ}_1}\right)$ and $\text{Ꝁ}_2 =$

$(\Pi_{P-\mathbb{K}_2}, \Pi_{N-\mathbb{K}_2}) = (\Pi_{RP-\mathbb{K}_2} + \iota \Pi_{IP-\mathbb{K}_2}, \Pi_{RN-\mathbb{K}_2} + \iota\, \Pi_{IN-\mathbb{K}_2})$ and $\mathfrak{b} > 0$, we interpret the underneath LOLs

1. $\log_\xi \mathbb{K}_1 \oplus \log_\xi \mathbb{K}_2$

$$= \begin{pmatrix} \log_\xi \Pi_{RP-\mathbb{K}_1} + \log_\xi \Pi_{RP-\mathbb{K}_2} - \log_\xi \Pi_{RP-\mathbb{K}_1} \log_\xi \Pi_{RP-\mathbb{K}_2} \\ +\iota\,(\log_\xi \Pi_{IP-\mathbb{K}_1} + \log_\xi \Pi_{IP-\mathbb{K}_2} - \log_\xi \Pi_{IP-\mathbb{K}_1} \log_\xi \Pi_{IP-\mathbb{K}_2}) \\ -((-\log_\xi (1+\Pi_{RN-\mathbb{K}_1})(-\log_\xi (1+\Pi_{RN-\mathbb{K}_2})))) \\ +\iota\,(-((-\log_\xi (1+\Pi_{IN-\mathbb{K}_1})(-\log_\xi (1+\Pi_{IN-\mathbb{K}_2}))))) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \log_\xi \Pi_{RP-\mathbb{K}_1} . \log_\xi \Pi_{RP-\mathbb{K}_2} \\ +\iota\, \left(1 - \log_\xi \Pi_{IP-\mathbb{K}_1} . \log_\xi \Pi_{IP-\mathbb{K}_2}\right) \\ -\left(\left(-\log_\xi \left(1+\Pi_{RN-\mathbb{K}_1}\right)\left(-\log_\xi \left(1+\Pi_{RN-\mathbb{K}_2}\right)\right)\right)\right) \\ +\iota\,(-((-\log_\xi (1+\Pi_{IN-\mathbb{K}_1})(-\log_\xi (1+\Pi_{IN-\mathbb{K}_2}))))) \end{pmatrix};$$

2. $\log_\xi \mathbb{K}_1 \otimes \log_\xi \mathbb{K}_2$

$$= \begin{pmatrix} \left(1 - \log_\xi \Pi_{RP-\mathbb{K}_1}\right)\left(1 - \log_\xi \Pi_{RP-\mathbb{K}_2}\right) \\ +\iota\, \left(1 - \log_\xi \Pi_{IP-\mathbb{K}_1}\right)\left(1 - \log_\xi \Pi_{IP-\mathbb{K}_2}\right), \\ \begin{pmatrix} \log_\xi \left(1+\Pi_{RN-\mathbb{K}_1}\right) + \log_\xi \left(1+\Pi_{RN-\mathbb{K}_2}\right) \\ +\log_\xi \left(1+\Pi_{RN-\mathbb{K}_1}\right) \log_\xi \left(1+\Pi_{RN-\mathbb{K}_2}\right) \end{pmatrix} \\ +\iota \begin{pmatrix} \log_\xi \left(1+\Pi_{IN-\mathbb{K}_1}\right) + \log_\xi \left(1+\Pi_{IN-\mathbb{K}_2}\right) \\ +\log_\xi \left(1+\Pi_{IN-\mathbb{K}_1}\right) \log_\xi \left(1+\Pi_{IN-\mathbb{K}_2}\right) \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \left(1 - \log_\xi \Pi_{RP-\mathbb{K}_1}\right)\left(1 - \log_\xi \Pi_{RP-\mathbb{K}_2}\right) \\ +\iota\, \left(1 - \log_\xi \Pi_{IP-\mathbb{K}_1}\right)\left(1 - \log_\xi \Pi_{IP-\mathbb{K}_2}\right), \\ \begin{pmatrix} -1 + \left(1 - \log_\xi \left(1+\Pi_{RN-\mathbb{K}_1}\right)\right) \\ \left(1 - \log_\xi \left(1+\Pi_{RN-\mathbb{K}_2}\right)\right) \end{pmatrix} \\ +\iota \begin{pmatrix} -1 + \left(1 - \log_\xi \left(1+\Pi_{IN-\mathbb{K}_1}\right)\right) \\ \left(1 - \log_\xi \left(1+\Pi_{IN-2}\right)\right) \end{pmatrix} \end{pmatrix};$$

3. $\mathfrak{b}\log_\xi \mathbb{K}_1$

$$= \begin{pmatrix} 1 - \left(1 - \left(1 - \log_\xi \Pi_{RP-\mathbb{K}_1}\right)\right)^{\mathfrak{b}} \\ +\iota \left(1 - \left(1 - \left(1 - \log_\xi \Pi_{IP-\mathbb{K}_1}\right)\right)^{\mathfrak{b}}\right), \\ -\left|\log_\xi \left(1+\Pi_{RN-\mathbb{K}_1}\right)\right|^{\mathfrak{b}} \\ +\iota\, \left(-\left|\log_\xi \left(1+\Pi_{IN-\mathbb{K}_1}\right)\right|^{\mathfrak{b}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \left(\log_\xi \Pi_{RP-\mathbb{K}_1}\right)^{\mathfrak{b}} \\ +\iota \left(1 - \left(\log_\xi \Pi_{IP-\mathbb{K}_1}\right)^{\mathfrak{b}}\right), \\ -\left|\log_\xi \left(1+\Pi_{RN-\mathbb{K}_1}\right)\right|^{\mathfrak{b}} \\ +\iota\, \left(-\left|\log_\xi \left(1+\Pi_{IN-\mathbb{K}_1}\right)\right|^{\mathfrak{b}}\right) \end{pmatrix};$$

4. $\left(\log_\xi \mathbb{K}_1\right)^{\mathfrak{b}}$

$$= \begin{pmatrix} \left(1 - \log_\xi \Pi_{RP-\mathbb{K}_1}\right)^{\mathfrak{b}} \\ +\iota\left(1 - \log_\xi \Pi_{IP-\mathbb{K}_1}\right)^{\mathfrak{b}}, \\ -1+\left(1 - \log_\xi \left(1+\Pi_{RN-\mathbb{K}_1}\right)^{\mathfrak{b}}\right) \\ +\iota\, \left(-1+\left(1-\log_\xi \left(1+\Pi_{RN-\mathbb{K}_1}\right)\right)^{\mathfrak{b}}\right) \end{pmatrix}$$

The underneath theorem will express the association between exponential operational law and LOL.

Theorem 2: Assume $\mathbb{K} = (\Pi_{P-\mathbb{K}}, \Pi_{N-\mathbb{K}}) = (\Pi_{RP-\mathbb{K}} + \iota\, \Pi_{IP-\mathbb{K}}, \Pi_{RN-\mathbb{K}} + \iota\, \Pi_{IN-\mathbb{K}})$ as a BCFN and if $0 < \xi \leq \min(\Pi_{RP-\mathbb{K}}, 1+\Pi_{RN-\mathbb{K}}, \Pi_{IP-\mathbb{K}}, 1+\Pi_{IN-\mathbb{K}}) \leq 1, \xi \neq 1$, then we have

1. $\xi^{\log_\xi (\mathbb{K})} = \mathbb{K}$
2. $\log_\xi (\xi)^{\mathbb{K}} = \mathbb{K}$

Proof: Omitted.

Theorem 3: Under two BCFNs $\mathbb{K}_{\mathfrak{z}} = (\Pi_{P-\mathbb{K}_{\mathfrak{z}}}, \Pi_{N-\mathbb{K}_{\mathfrak{z}}}) = (\Pi_{RP-\mathbb{K}_{\mathfrak{z}}} + \iota\, \Pi_{IP-\mathbb{K}_{\mathfrak{z}}}, \Pi_{RN-\mathbb{K}_{\mathfrak{z}}} + \iota\, \Pi_{IN-\mathbb{K}_{\mathfrak{z}}}), \mathfrak{z} = 1, 2,$ we have

1. $\log_\xi \mathbb{K}_1 \oplus \log_\xi \mathbb{K}_2 = \log_\xi \mathbb{K}_2 \oplus \log_\xi \mathbb{K}_1$
2. $\log_\xi \mathbb{K}_1 \otimes \log_\xi \mathbb{K}_2 = \log_\xi \mathbb{K}_2 \otimes \log_\xi \mathbb{K}_1$

Noted that $0 < \xi \leq \min(\Pi_{RP-\mathbb{K}}(d), 1+\Pi_{RN-\mathbb{K}}(d), \Pi_{IP-\mathbb{K}}(d), 1+\Pi_{IN-\mathbb{K}}(d)) \leq 1$ and $\xi \neq 1$.

Proof:

1. From Def (6), we have

$$\log_\xi \mathbb{K}_1 \oplus \log_\xi \mathbb{K}_2$$

$$= \begin{pmatrix} 1 - \log_\xi \Pi_{RP-\mathbb{K}_1} \log_\xi \Pi_{RP-\mathbb{K}_2} \\ +\iota\, \left(1 - \log_\xi \Pi_{IP-\mathbb{K}_1} \log_\xi \Pi_{IP-\mathbb{K}_2}\right) \\ -\left(\begin{pmatrix} -\log_\xi \left(1+\Pi_{RN-\mathbb{K}_1}\right) \\ \left(-\log_\xi \left(1+\Pi_{RN-\mathbb{K}_2}\right)\right) \end{pmatrix}\right) \\ +\iota\, \left(-\left(\begin{pmatrix} -\log_\xi \left(1+\Pi_{IN-\mathbb{K}_1}\right) \\ \left(-\log_\xi \left(1+\Pi_{IN-\mathbb{K}_2}\right)\right) \end{pmatrix}\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \log_\xi \Pi_{RP-\mathbb{K}_2} \log_\xi \Pi_{RP-\mathbb{K}_1} \\ +\iota\, \left(1 - \log_\xi \Pi_{IP-\mathbb{K}_2} \log_\xi \Pi_{IP-\mathbb{K}_1}\right) \\ -\left(\begin{pmatrix} -\log_\xi \left(1+\Pi_{RN-\mathbb{K}_2}\right) \\ \left(-\log_\xi \left(1+\Pi_{RN-\mathbb{K}_1}\right)\right) \end{pmatrix}\right) \\ +\iota\, \left(-\left(\begin{pmatrix} -\log_\xi \left(1+\Pi_{IN-\mathbb{K}_2}\right) \\ \left(-\log_\xi \left(1+\Pi_{IN-\mathbb{K}_1}\right)\right) \end{pmatrix}\right)\right) \end{pmatrix}$$

$$= \log_\xi \mathbb{K}_2 \oplus \log_\xi \mathbb{K}_1$$

2. By employing Def (6), we achieve

$$\log_\xi \mathbb{K}_1 \otimes \log_\xi \mathbb{K}_2$$

$$= \begin{pmatrix} \left(1 - \log_\xi \Pi_{RP-\mathbb{K}_1}\right)\left(1 - \log_\xi \Pi_{RP-\mathbb{K}_2}\right) \\ +\iota\, \left(1 - \log_\xi \Pi_{IP-\mathbb{K}_1}\right)\left(1 - \log_\xi \Pi_{IP-\mathbb{K}_2}\right), \\ \begin{pmatrix} -1 + \left(1 - \log_\xi \left(1+\Pi_{RN-\mathbb{K}_1}\right)\right) \\ \left(1 - \log_\xi \left(1+\Pi_{RN-\mathbb{K}_2}\right)\right) \end{pmatrix} \\ +\iota \begin{pmatrix} -1 + \left(1 - \log_\xi \left(1+\Pi_{IN-\mathbb{K}_1}\right)\right) \\ \left(1 - \log_\xi \left(1+\Pi_{IN-2}\right)\right) \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \left(1 - \log_\xi \Pi_{RP-\mathbb{K}_2}\right)\left(1 - \log_\xi \Pi_{RP-\mathbb{K}_1}\right) \\ +\iota\, \left(1 - \log_\xi \Pi_{IP-\mathbb{K}_2}\right)\left(1 - \log_\xi \Pi_{IP-\mathbb{K}_1}\right), \\ \begin{pmatrix} -1 + \left(1 - \log_\xi \left(1+\Pi_{RN-\mathbb{K}_2}\right)\right) \\ \left(1 - \log_\xi \left(1+\Pi_{RN-\mathbb{K}_1}\right)\right) \end{pmatrix} \\ +\iota \begin{pmatrix} -1 + \left(1 - \log_\xi \left(1+\Pi_{IN-\mathbb{K}_2}\right)\right) \\ \left(1 - \log_\xi \left(1+\Pi_{IN-\mathbb{K}_1}\right)\right) \end{pmatrix} \end{pmatrix}$$

$$= \log_\xi \mathbb{K}_2 \otimes \log_\xi \mathbb{K}_1$$

Theorem 4: Under three BCFNs $\mathbb{K}_{\mathfrak{z}} = (\Pi_{P-\mathbb{K}_{\mathfrak{z}}}, \Pi_{N-\mathbb{K}_{\mathfrak{z}}}) = (\Pi_{RP-\mathbb{K}_{\mathfrak{z}}} + \iota\, \Pi_{IP-\mathbb{K}_{\mathfrak{z}}}, \Pi_{RN-\mathbb{K}_{\mathfrak{z}}} + \iota\, \Pi_{IN-\mathbb{K}_{\mathfrak{z}}}), \mathfrak{z} = 1, 2, 3$ we have

1. $\left(\log_\xi \mathbb{K}_1 \oplus \log_\xi \mathbb{K}_2\right) \oplus \log_\xi \mathbb{K}_2 = \log_\xi \mathbb{K}_1 \oplus \left(\log_\xi \mathbb{K}_2 \oplus \log_\xi \mathbb{K}_3\right)$
2. $\left(\log_\xi \mathbb{K}_1 \otimes \log_\xi \mathbb{K}_2\right) \otimes \log_\xi \mathbb{K}_2 = \log_\xi \mathbb{K}_1 \otimes \left(\log_\xi \mathbb{K}_2 \otimes \log_\xi \mathbb{K}_3\right)$

Noted that $0 < \xi \leq \min(\Pi_{RP-\mathbb{K}}(d), 1+\Pi_{RN-\mathbb{K}}(d), \Pi_{IP-\mathbb{K}}(d), 1+\Pi_{IN-\mathbb{K}}(d)) \leq 1$ and $\xi \neq 1$.

Proof: Trivial, so omitted here.

Theorem 5: Under two BCFNs $\text{Ҡ}_{\mathfrak{z}} = \left(\Pi_{P-\text{Ҡ}_{\mathfrak{z}}},\ \Pi_{N-\text{Ҡ}_{\mathfrak{z}}}\right) = \left(\Pi_{RP-\text{Ҡ}_{\mathfrak{z}}} + \iota\,\Pi_{IP-\text{Ҡ}_{\mathfrak{z}}},\ \Pi_{RN-\text{Ҡ}_{\mathfrak{z}}} + \iota\,\Pi_{IN-\text{Ҡ}_{\mathfrak{z}}}\right),\ \mathfrak{z} = 1,\ 2,$ and $\mathfrak{b}_1, \mathfrak{b}_2, > 0$, we have

1. $\mathfrak{b}_1\left(\log_{\xi}\text{Ҡ}_1 \oplus \log_{\xi}\text{Ҡ}_2\right) = \mathfrak{b}_1\log_{\xi}\text{Ҡ}_1 \oplus \mathfrak{b}_1\log_{\xi}\text{Ҡ}_2$
2. $\left(\log_{\xi}\text{Ҡ}_1 \otimes \log_{\xi}\text{Ҡ}_2\right)^{\mathfrak{b}_1} = \left(\log_{\xi}\text{Ҡ}_1\right)^{\mathfrak{b}_1} \otimes \left(\log_{\xi}\text{Ҡ}_2\right)^{\mathfrak{b}_1}$
3. $\mathfrak{b}_1\log_{\xi}\text{Ҡ}_1 \oplus \mathfrak{b}_2\log_{\xi}\text{Ҡ}_1 = (\mathfrak{b}_1 + \mathfrak{b}_2)\log_{\xi}\text{Ҡ}_1$
4. $\left(\log_{\xi}\text{Ҡ}_1\right)^{\mathfrak{b}_1} \otimes \left(\log_{\xi}\text{Ҡ}_1\right)^{\mathfrak{b}_2} = \left(\log_{\xi}\text{Ҡ}_1\right)^{(\mathfrak{b}_1+\mathfrak{b}_2)}$
5. $\left(\left(\log_{\xi}\text{Ҡ}_1\right)^{\mathfrak{b}_1}\right)^{\mathfrak{b}_2} = \left(\log_{\xi}\text{Ҡ}_1\right)^{\mathfrak{b}_1\mathfrak{b}_2}$

Noted that $0 < \xi \leq \min(\Pi_{RP-\text{Ҡ}}(\mathrm{d}),\ 1 + \Pi_{RN-\text{Ҡ}}(\mathrm{d}),\ \Pi_{IP-\text{Ҡ}}(\mathrm{d}),\ 1 + \Pi_{IN-\text{Ҡ}}(\mathrm{d})) \leq 1$ and $\xi \neq 1$.

Proof:

1. By employing Def (6), we have

$$\log_{\xi}\text{Ҡ}_1 \oplus \log_{\xi}\text{Ҡ}_2 = \begin{pmatrix} \begin{pmatrix}\log_{\xi}\Pi_{RP-\text{Ҡ}_1} + \log_{\xi}\Pi_{RP-\text{Ҡ}_2} \\ -\log_{\xi}\Pi_{RP-\text{Ҡ}_1}\log_{\xi}\Pi_{RP-\text{Ҡ}_2}\end{pmatrix} \\ +\iota\begin{pmatrix}\log_{\xi}\Pi_{IP-\text{Ҡ}_1} + \log_{\xi}\Pi_{IP-\text{Ҡ}_2} \\ -\log_{\xi}\Pi_{IP-\text{Ҡ}_1}\log_{\xi}\Pi_{IP-\text{Ҡ}_2}\end{pmatrix} \\ -\left(\begin{pmatrix}-\log_{\xi}\left(1 + \Pi_{RN-\text{Ҡ}_1}\right) \\ \left(-\log_{\xi}\left(1 + \Pi_{RN-\text{Ҡ}_2}\right)\right)\end{pmatrix}\right) \\ +\iota\left(-\left(\begin{pmatrix}-\log_{\xi}\left(1 + \Pi_{IN-\text{Ҡ}_1}\right) \\ \left(-\log_{\xi}\left(1 + \Pi_{IN-\text{Ҡ}_2}\right)\right)\end{pmatrix}\right)\right)\end{pmatrix}$$

$$\begin{pmatrix} 1 - \log_{\xi}\Pi_{RP-\text{Ҡ}_1}\log_{\xi}\Pi_{RP-\text{Ҡ}_2} \\ +\iota\left(1 - \log_{\xi}\Pi_{IP-\text{Ҡ}_1}\log_{\xi}\Pi_{IP-\text{Ҡ}_2}\right) \\ -\left(\begin{pmatrix}-\log_{\xi}\left(1 + \Pi_{RN-\text{Ҡ}_1}\right) \\ \left(-\log_{\xi}\left(1 + \Pi_{RN-\text{Ҡ}_2}\right)\right)\end{pmatrix}\right) \\ +\iota\left(-\left(\begin{pmatrix}-\log_{\xi}\left(1 + \Pi_{IN-\text{Ҡ}_1}\right) \\ \left(-\log_{\xi}\left(1 + \Pi_{IN-\text{Ҡ}_2}\right)\right)\end{pmatrix}\right)\right)\end{pmatrix}$$

$$\mathfrak{b}_1\left(\log_{\xi}\text{Ҡ}_1 \oplus \log_{\xi}\text{Ҡ}_2\right) = \begin{pmatrix} 1 - \left(\log_{\xi}\Pi_{RP-\text{Ҡ}_1}\log_{\xi}\Pi_{RP-\text{Ҡ}_2}\right)^{\mathfrak{b}_1} \\ +\iota\left(1 - \left(\log_{\xi}\Pi_{IP-\text{Ҡ}_1}\log_{\xi}\Pi_{IP-\text{Ҡ}_2}\right)^{\mathfrak{b}_1}\right), \\ -\left|\left(-\log_{\xi}\left(1+\Pi_{RN-\text{Ҡ}_1}\right)\left(-\log_{\xi}\left(1+\Pi_{RN-\text{Ҡ}_2}\right)\right)\right)\right|^{\mathfrak{b}} \\ +\iota\left(-\left|\left(-\log_{\xi}\left(1+\Pi_{IN-\text{Ҡ}_1}\right)\left(-\log_{\xi}\left(1+\Pi_{IN-\text{Ҡ}_2}\right)\right)\right)\right|^{\mathfrak{b}}\right)\end{pmatrix}$$

$$= \mathfrak{b}_1\log_{\xi}\text{Ҡ}_1 \oplus \mathfrak{b}_1\log_{\xi}\text{Ҡ}_2$$

The rest is similar and can be proved by employing Def (6).

IV. LOGARITHMIC AOS FOR BCFNS

Here, we invent logarithmic AOs based on the invented LOLs in the setting of BCF information. These AOs are L-BCFWA, L-BCFOWA, L-BCFWG, and L-BCFOWG operators.

Definition 7: Under the gathering of BCFNs $\text{Ҡ}_{\mathfrak{z}} = (\Pi_{P-\text{Ҡ}_{\mathfrak{z}}},\ \Pi_{N-\text{Ҡ}_{\mathfrak{z}}}) = (\Pi_{RP-\text{Ҡ}_{\mathfrak{z}}} + \iota\,\Pi_{IP-\text{Ҡ}_{\mathfrak{z}}},\ \Pi_{RN-\text{Ҡ}_{\mathfrak{z}}} + \iota\,\Pi_{IN-\text{Ҡ}_{\mathfrak{z}}}),\ \mathfrak{z} = 1,\ 2,\ \ldots,\ \bar{\text{й}}$, with $0 < \xi_{\mathfrak{z}} \leq \min_{\mathfrak{z}}(\Pi_{RP-\text{Ҡ}_{\mathfrak{z}}},\ 1+\Pi_{RN-\text{Ҡ}_{\mathfrak{z}}},\ \Pi_{IP-\text{Ҡ}_{\mathfrak{z}}},\ 1+\Pi_{IN-\text{Ҡ}_{\mathfrak{z}}}) \leq 1$ and $\xi \neq 1$, the L-BCFWA operator is deduced as

$$L - BCFWA\left(\text{Ҡ}_1,\ \text{Ҡ}_2,\ \ldots,\ \text{Ҡ}_{\bar{\text{й}}}\right) = \bigoplus_{\mathfrak{z}=1}^{\bar{\text{й}}} \omega_{\mathrm{v}-\mathfrak{z}}\log_{\xi_{\mathfrak{z}}}\text{Ҡ}_{\mathfrak{z}}$$

Noted that $\omega_{\mathrm{v}} = (\omega_{\mathrm{v}-1},\ \omega_{\mathrm{v}-2},\ \ldots,\ \omega_{\mathrm{v}-\bar{\text{й}}})$ is a weight vector with the axiom that $0 \leq \omega_{\mathrm{v}-\mathfrak{z}} \leq 1$ and $\sum_{\mathfrak{z}=1}^{\bar{\text{й}}}\omega_{\mathrm{v}-\mathfrak{z}} = 1$.

Theorem 6: Under the gathering of BCFNs $\text{Ҡ}_{\mathfrak{z}} = (\Pi_{P-\text{Ҡ}_{\mathfrak{z}}},\ \Pi_{N-\text{Ҡ}_{\mathfrak{z}}}) = (\Pi_{RP-\text{Ҡ}_{\mathfrak{z}}} + \iota\,\Pi_{IP-\text{Ҡ}_{\mathfrak{z}}},\ \Pi_{RN-\text{Ҡ}_{\mathfrak{z}}} + \iota\,\Pi_{IN-\text{Ҡ}_{\mathfrak{z}}}),\ \mathfrak{z} = 1,\ 2,\ \ldots,\ \bar{\text{й}}$, with $0 < \xi_{\mathfrak{z}} \leq \min_{\mathfrak{z}}(\Pi_{RP-\text{Ҡ}_{\mathfrak{z}}},\ 1+\Pi_{RN-\text{Ҡ}_{\mathfrak{z}}},\ \Pi_{IP-\text{Ҡ}_{\mathfrak{z}}},\ 1+\Pi_{IN-\text{Ҡ}_{\mathfrak{z}}}) \leq 1$ and $\xi \neq 1$, the usage of the L-BCFWA operator will deduce again a BCFN and

$$L - BCFWA\left(\text{Ҡ}_1,\ \text{Ҡ}_2,\ \ldots,\ \text{Ҡ}_{\bar{\text{й}}}\right) = \begin{cases} \begin{pmatrix} 1 - \prod_{\mathfrak{z}=1}^{\bar{\text{й}}}\left(\log_{\xi_{\mathfrak{z}}}\Pi_{RP-\text{Ҡ}_{\mathfrak{z}}}\right)^{\omega_{\mathrm{v}-\mathfrak{z}}} \\ +\iota\left(1 - \prod_{\mathfrak{z}=1}^{\bar{\text{й}}}\left(\log_{\xi_{\mathfrak{z}}}\Pi_{IP-\text{Ҡ}_{\mathfrak{z}}}\right)^{\omega_{\mathrm{v}-\mathfrak{z}}}\right), \\ -\prod_{\mathfrak{z}=1}^{\bar{\text{й}}}\left(\left|\log_{\xi_{\mathfrak{z}}}\left(1 + \Pi_{RN-\text{Ҡ}_{\mathfrak{z}}}\right)\right|\right)^{\omega_{\mathrm{v}-\mathfrak{z}}} \\ +\iota\left(-\prod_{\mathfrak{z}=1}^{\bar{\text{й}}}\left(\left|\log_{\xi_{\mathfrak{z}}}\left(1 + \Pi_{IN-\text{Ҡ}_{\mathfrak{z}}}\right)\right|\right)^{\omega_{\mathrm{v}-\mathfrak{z}}}\right)\end{pmatrix} \\ \begin{pmatrix} 1 - \prod_{\mathfrak{z}=1}^{\bar{\text{й}}}\left(\log_{\frac{1}{\xi_{\mathfrak{z}}}}\Pi_{RP-\text{Ҡ}_{\mathfrak{z}}}\right)^{\omega_{\mathrm{v}-\mathfrak{z}}} \\ +\iota\left(1 - \prod_{\mathfrak{z}=1}^{\bar{\text{й}}}\left(\log_{\frac{1}{\xi_{\mathfrak{z}}}}\Pi_{IP-\text{Ҡ}_{\mathfrak{z}}}\right)^{\omega_{\mathrm{v}-\mathfrak{z}}}\right), \\ -\prod_{\mathfrak{z}=1}^{\bar{\text{й}}}\left(\left|\log_{\frac{1}{\xi_{\mathfrak{z}}}}\left(1 + \Pi_{RN-\text{Ҡ}_{\mathfrak{z}}}\right)\right|\right)^{\omega_{\mathrm{v}-\mathfrak{z}}} \\ +\iota\left(-\prod_{\mathfrak{z}=1}^{\bar{\text{й}}}\left(\left|\log_{\frac{1}{\xi_{\mathfrak{z}}}}\left(1 + \Pi_{IN-\text{Ҡ}_{\mathfrak{z}}}\right)\right|\right)^{\omega_{\mathrm{v}-\mathfrak{z}}}\right)\end{pmatrix}\end{cases} \tag{3}$$

where, $0 < \xi_{\mathfrak{z}} \leq \min_{\mathfrak{z}}(\Pi_{RP-\text{Ҡ}_{\mathfrak{z}}},\ 1+\Pi_{RN-\text{Ҡ}_{\mathfrak{z}}},\ \Pi_{IP-\text{Ҡ}_{\mathfrak{z}}},\ 1+\Pi_{IN-\text{Ҡ}_{\mathfrak{z}}}) \leq 1,\ \xi \neq 1$, and $0 < \frac{1}{\xi_{\mathfrak{z}}} \leq \min_{\mathfrak{z}}(\Pi_{RP-\text{Ҡ}_{\mathfrak{z}}},\ 1 + \Pi_{RN-\text{Ҡ}_{\mathfrak{z}}},\ \Pi_{IP-\text{Ҡ}_{\mathfrak{z}}},\ 1 + \Pi_{IN-\text{Ҡ}_{\mathfrak{z}}}) \leq 1,\ \xi \neq 1$.

Proof: This proof would be done by mathematical induction. Assume $\bar{\text{й}} = 2$, and since

$$\omega_{\mathrm{v}-1}\log_{\xi_1}\text{Ҡ}_1 = \begin{pmatrix} 1 - \left(\log_{\xi_1}\Pi_{RP-\text{Ҡ}_1}\right)^{\omega_{\mathrm{v}-1}} \\ +\iota\left(1 - \left(\log_{\xi_1}\Pi_{IP-\text{Ҡ}_1}\right)^{\omega_{\mathrm{v}-1}}\right), \\ -\left|\log_{\xi_1}\left(1 + \Pi_{RN-\text{Ҡ}_1}\right)\right|^{\omega_{\mathrm{v}-1}} \\ +\iota\left(-\left|\log_{\xi_1}\left(1 + \Pi_{IN-\text{Ҡ}_1}\right)\right|^{\omega_{\mathrm{v}-1}}\right)\end{pmatrix}$$

$$\omega_{\mathrm{v}-2}\log_{\xi_2}\text{Ҡ}_2 = \begin{pmatrix} 1 - \left(\log_{\xi_2}\Pi_{RP-\text{Ҡ}_2}\right)^{\omega_{\mathrm{v}-2}} \\ +\iota\left(1 - \left(\log_{\xi_2}\Pi_{IP-\text{Ҡ}_2}\right)^{\omega_{\mathrm{v}-2}}\right), \\ -\left|\log_{\xi_2}\left(1 + \Pi_{RN-\text{Ҡ}_2}\right)\right|^{\omega_{\mathrm{v}-2}} \\ +\iota\left(-\left|\log_{\xi_2}\left(1 + \Pi_{IN-\text{Ҡ}_2}\right)\right|^{\omega_{\mathrm{v}-2}}\right)\end{pmatrix}$$

then,

$$
\begin{aligned}
&\omega_{\psi-1}\log_{\xi_1}\text{Ꝁ}_1 \oplus \omega_{\psi-2}\log_{\xi_2}\text{Ꝁ}_2\\
&=\begin{pmatrix}
\begin{pmatrix}
1-\left(\log_{\xi_1}\Pi_{RP-\text{Ꝁ}_1}\right)^{\omega_{\psi-1}}\\
+1-\left(\log_{\xi_2}\Pi_{RP-\text{Ꝁ}_2}\right)^{\omega_{\psi-2}}\\
-\left(\left(1-\left(\log_{\xi_1}\Pi_{RP-\text{Ꝁ}_1}\right)^{\omega_{\psi-1}}\right)\left(1-\left(\log_{\xi_2}\Pi_{RP-\text{Ꝁ}_2}\right)^{\omega_{\psi-2}}\right)\right)
\end{pmatrix}\\
+\iota\begin{pmatrix}
1-\left(\log_{\xi_1}\Pi_{IP-\text{Ꝁ}_1}\right)^{\omega_{\psi-1}}\\
+1-\left(\log_{\xi_2}\Pi_{IP-\text{Ꝁ}_2}\right)^{\omega_{\psi-2}}\\
-\left(\left(1-\left(\log_{\xi_1}\Pi_{IP-\text{Ꝁ}_1}\right)^{\omega_{\psi-1}}\right)\left(1-\left(\log_{\xi_2}\Pi_{IP-\text{Ꝁ}_2}\right)^{\omega_{\psi-2}}\right)\right)
\end{pmatrix},\\
-\left(\left(-\left|\log_{\xi_1}\left(1+\Pi_{RN-\text{Ꝁ}_1}\right)\right|^{\omega_{\psi-1}}\right)\left(-\left|\log_{\xi_2}\left(1+\Pi_{RN-\text{Ꝁ}_2}\right)\right|^{\omega_{\psi-2}}\right)\right)\\
+\iota\left(-\left(\left(-\left|\log_{\xi_1}\left(1+\Pi_{IN-\text{Ꝁ}_1}\right)\right|^{\omega_{\psi-1}}\right)\left(-\left|\log_{\xi_2}\left(1+\Pi_{IN-\text{Ꝁ}_2}\right)\right|^{\omega_{\psi-2}}\right)\right)\right)
\end{pmatrix}\\
&=\begin{pmatrix}
1-\prod_{\mathfrak{z}=1}^{2}\left(\log_{\xi_\mathfrak{z}}\Pi_{RP-\text{Ꝁ}_\mathfrak{z}}\right)^{\omega_{\psi-\mathfrak{z}}}\\
+\iota\left(1-\prod_{\mathfrak{z}=1}^{2}\left(\log_{\xi_\mathfrak{z}}\Pi_{IP-\text{Ꝁ}_\mathfrak{z}}\right)^{\omega_{\psi-\mathfrak{z}}}\right),\\
-\prod_{\mathfrak{z}=1}^{2}\left|\log_{\xi_\mathfrak{z}}\left(1+\Pi_{RN-\text{Ꝁ}_\mathfrak{z}}\right)\right|^{\omega_{\psi-\mathfrak{z}}}\\
+\iota\left(-\prod_{\mathfrak{z}=1}^{2}\left|\log_{\xi_\mathfrak{z}}\left(1+\Pi_{IN-\text{Ꝁ}_\mathfrak{z}}\right)\right|^{\omega_{\psi-\mathfrak{z}}}\right)
\end{pmatrix}\\
&= L-BCFWA\left(\text{Ꝁ}_1,\ \text{Ꝁ}_2\right)
\end{aligned}
$$

Assume that Eq. (3) is valid for $\bar{\text{й}} = {}^\circ C$, then we have

$$
L-BCFWA\left(\text{Ꝁ}_1,\ \text{Ꝁ}_2,\ \ldots,\ \text{Ꝁ}_{{}^\circ C}\right)
=\begin{pmatrix}
1-\prod_{\mathfrak{z}=1}^{{}^\circ C}\left(\log_{\xi_\mathfrak{z}}\Pi_{RP-\text{Ꝁ}_\mathfrak{z}}\right)^{\omega_{\psi-\mathfrak{z}}}\\
+\iota\left(1-\prod_{\mathfrak{z}=1}^{{}^\circ C}\left(\log_{\xi_\mathfrak{z}}\Pi_{IP-\text{Ꝁ}_\mathfrak{z}}\right)^{\omega_{\psi-\mathfrak{z}}}\right),\\
-\prod_{\mathfrak{z}=1}^{{}^\circ C}\left|\log_{\xi_\mathfrak{z}}\left(1+\Pi_{RN-\text{Ꝁ}_\mathfrak{z}}\right)\right|^{\omega_{\psi-\mathfrak{z}}}\\
+\iota\left(-\prod_{\mathfrak{z}=1}^{{}^\circ C}\left|\log_{\xi_\mathfrak{z}}\left(1+\Pi_{IN-\text{Ꝁ}_\mathfrak{z}}\right)\right|^{\omega_{\psi-\mathfrak{z}}}\right)
\end{pmatrix}
$$

last let $\bar{\text{й}} = {}^\circ C+1$, then

$$
\begin{aligned}
&L-BCFWA\left(\text{Ꝁ}_1,\ \text{Ꝁ}_2,\ \ldots,\ \text{Ꝁ}_{{}^\circ C}\right)\oplus \text{Ꝁ}_{{}^\circ C+1}\\
&=\begin{pmatrix}
1-\prod_{\mathfrak{z}=1}^{{}^\circ C}\left(\log_{\xi_\mathfrak{z}}\Pi_{RP-\text{Ꝁ}_\mathfrak{z}}\right)^{\omega_{\psi-\mathfrak{z}}}\\
+\iota\left(1-\prod_{\mathfrak{z}=1}^{{}^\circ C}\left(\log_{\xi_\mathfrak{z}}\Pi_{IP-\text{Ꝁ}_\mathfrak{z}}\right)^{\omega_{\psi-\mathfrak{z}}}\right),\\
-\prod_{\mathfrak{z}=1}^{{}^\circ C}\left|\log_{\xi_\mathfrak{z}}\left(1+\Pi_{RN-\text{Ꝁ}_\mathfrak{z}}\right)\right|^{\omega_{\psi-\mathfrak{z}}}\\
+\iota\left(-\prod_{\mathfrak{z}=1}^{{}^\circ C}\left|\log_{\xi_\mathfrak{z}}\left(1+\Pi_{IN-\text{Ꝁ}_\mathfrak{z}}\right)\right|^{\omega_{\psi-\mathfrak{z}}}\right)
\end{pmatrix}\\
&\oplus\begin{pmatrix}
1-\left(\log_{\xi_{{}^\circ C+1}}\Pi_{RP-\text{Ꝁ}_{{}^\circ C+1}}\right)^{\omega_{\psi-{}^\circ C+1}}\\
+\iota\left(1-\left(\log_{\xi_{{}^\circ C+1}}\Pi_{IP-\text{Ꝁ}_{{}^\circ C+1}}\right)^{\omega_{\psi-{}^\circ C+1}}\right),\\
-\left|\log_{\xi_{{}^\circ C+1}}\left(1+\Pi_{RN-\text{Ꝁ}_{{}^\circ C+1}}\right)\right|^{\omega_{\psi-{}^\circ C+1}}\\
+\iota\left(-\left|\log_{\xi_{{}^\circ C+1}}\left(1+\Pi_{IN-\text{Ꝁ}_{{}^\circ C+1}}\right)\right|^{\omega_{\psi-{}^\circ C+1}}\right)
\end{pmatrix}\\
&=\begin{pmatrix}
1-\prod_{\mathfrak{z}=1}^{{}^\circ C+1}\left(\log_{\xi_\mathfrak{z}}\Pi_{RP-\text{Ꝁ}_\mathfrak{z}}\right)^{\omega_{\psi-\mathfrak{z}}}\\
+\iota\left(1-\prod_{\mathfrak{z}=1}^{{}^\circ C+1}\left(\log_{\xi_\mathfrak{z}}\Pi_{IP-\text{Ꝁ}_\mathfrak{z}}\right)^{\omega_{\psi-\mathfrak{z}}}\right),\\
-\prod_{\mathfrak{z}=1}^{{}^\circ C+1}\left|\log_{\xi_\mathfrak{z}}\left(1+\Pi_{RN-\text{Ꝁ}_\mathfrak{z}}\right)\right|^{\omega_{\psi-\mathfrak{z}}}\\
+\iota\left(-\prod_{\mathfrak{z}=1}^{{}^\circ C+1}\left|\log_{\xi_\mathfrak{z}}\left(1+\Pi_{IN-\text{Ꝁ}_\mathfrak{z}}\right)\right|^{\omega_{\psi-\mathfrak{z}}}\right)
\end{pmatrix}
\end{aligned}
$$

Therefore, Eq. (3) is valid $\forall\ \bar{\text{й}}$. Likewise, if $0 < \frac{1}{\xi_\mathfrak{z}} \leq \min_{\mathfrak{z}}\left(\Pi_{RP-\text{Ꝁ}_\mathfrak{z}},\ 1+\Pi_{RN-\text{Ꝁ}_\mathfrak{z}},\ \Pi_{IP-\text{Ꝁ}_\mathfrak{z}},\ 1+\Pi_{IN-\text{Ꝁ}_\mathfrak{z}}\right) \leq 1$, $\xi \neq 1$, then we can demonstrate

$$
L-BCFWA\left(\text{Ꝁ}_1,\ \text{Ꝁ}_2,\ \ldots,\ \text{Ꝁ}_{\bar{\text{й}}}\right)
=\begin{pmatrix}
1-\prod_{\mathfrak{z}=1}^{\bar{\text{й}}}\left(\log_{\frac{1}{\xi_\mathfrak{z}}}\Pi_{RP-\text{Ꝁ}_\mathfrak{z}}\right)^{\omega_{\psi-\mathfrak{z}}}\\
+\iota\left(1-\prod_{\mathfrak{z}=1}^{\bar{\text{й}}}\left(\log_{\frac{1}{\xi_\mathfrak{z}}}\Pi_{IP-\text{Ꝁ}_\mathfrak{z}}\right)^{\omega_{\psi-\mathfrak{z}}}\right),\\
-\prod_{\mathfrak{z}=1}^{\bar{\text{й}}}\left(\left|\log_{\frac{1}{\xi_\mathfrak{z}}}\left(1+\Pi_{RN-\text{Ꝁ}_\mathfrak{z}}\right)\right|\right)^{\omega_{\psi-\mathfrak{z}}}\\
+\iota\left(-\prod_{\mathfrak{z}=1}^{\bar{\text{й}}}\left(\left|\log_{\frac{1}{\xi_\mathfrak{z}}}\left(1+\Pi_{IN-\text{Ꝁ}_\mathfrak{z}}\right)\right|\right)^{\omega_{\psi-\mathfrak{z}}}\right)
\end{pmatrix}
$$

Hence proved.

Further, if $\xi_1 = \xi_2 = \ldots = \xi_{\bar{\text{й}}} = \xi$ and $0 < \xi \leq \min_{\mathfrak{z}}\left(\Pi_{RP-\text{Ꝁ}_\mathfrak{z}},\ 1+\Pi_{RN-\text{Ꝁ}_\mathfrak{z}},\ \Pi_{IP-\text{Ꝁ}_\mathfrak{z}},\ 1+\Pi_{IN-\text{Ꝁ}_\mathfrak{z}}\right) \leq 1$ and $\xi \neq 1$, then the invented L-BCFWA operator would be transformed underneath

$$L-BCFWA\left(\text{Ҟ}_1,\ \text{Ҟ}_2,\ \ldots,\ \text{Ҟ}_{\text{ӣ}}\right)=\begin{pmatrix} 1-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\left(\log_{\xi}\Pi_{RP-\text{Ҟ}_{\mathfrak{z}}}\right)^{\omega_{\text{ѵ}-\mathfrak{z}}} \\ +\iota\left(1-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\left(\log_{\xi}\Pi_{IP-\text{Ҟ}_{\mathfrak{z}}}\right)^{\omega_{\text{ѵ}-\mathfrak{z}}}\right), \\ -\prod_{\mathfrak{z}=1}^{\text{ӣ}}\left(\left|\log_{\xi}\left(1+\Pi_{RN-\text{Ҟ}_{\mathfrak{z}}}\right)\right|\right)^{\omega_{\text{ѵ}-\mathfrak{z}}} \\ +\iota\left(-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\left(\left|\log_{\xi}\left(1+\Pi_{IN-\text{Ҟ}_{\mathfrak{z}}}\right)\right|\right)^{\omega_{\text{ѵ}-\mathfrak{z}}}\right)\end{pmatrix}$$

Properties: L-BCFWA operator holds the underneath axioms.

❖ Idempotency: Under the gathering of BCFNs $\text{Ҟ}_{\mathfrak{z}}=\left(\Pi_{P-\text{Ҟ}_{\mathfrak{z}}},\ \Pi_{N-\text{Ҟ}_{\mathfrak{z}}}\right)=\left(\Pi_{RP-\text{Ҟ}_{\mathfrak{z}}}+\iota\,\Pi_{IP-\text{Ҟ}_{\mathfrak{z}}},\ \Pi_{RN-\text{Ҟ}_{\mathfrak{z}}}+\iota\,\Pi_{IN-\text{Ҟ}_{\mathfrak{z}}}\right)$, $\mathfrak{z}=1,\ 2,\ \ldots,\ \text{ӣ}$, then if $\text{Ҟ}_{\mathfrak{z}}=\text{Ҟ}\ \forall\mathfrak{z}$, that is $\Pi_{RP-\text{Ҟ}_{\mathfrak{z}}}=\Pi_{RP-\text{Ҟ}}$, $\Pi_{IP-\text{Ҟ}_{\mathfrak{z}}}=\Pi_{IP-\text{Ҟ}}$, $\Pi_{RN-\text{Ҟ}_{\mathfrak{z}}}=\Pi_{RN-\text{Ҟ}}$ and $\Pi_{IN-\text{Ҟ}_{\mathfrak{z}}}=\Pi_{IN-\text{Ҟ}}\ \forall\,\mathfrak{z}$, then

$$L-BCFWA\left(\text{Ҟ}_1,\ \text{Ҟ}_2,\ \ldots,\ \text{Ҟ}_{\text{ӣ}}\right)=\log_{\xi}\text{Ҟ}$$

Noted that $\xi_1=\xi_2=\ldots=\xi_{\text{ӣ}}=\xi$.

Proof: Since we have

$$L-BCFWA\left(\text{Ҟ}_1,\ \text{Ҟ}_2,\ \ldots,\ \text{Ҟ}_{\text{ӣ}}\right)=\begin{pmatrix} 1-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\left(\log_{\xi}\Pi_{RP-\text{Ҟ}_{\mathfrak{z}}}\right)^{\omega_{\text{ѵ}-\mathfrak{z}}} \\ +\iota\left(1-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\left(\log_{\xi}\Pi_{IP-\text{Ҟ}_{\mathfrak{z}}}\right)^{\omega_{\text{ѵ}-\mathfrak{z}}}\right), \\ -\prod_{\mathfrak{z}=1}^{\text{ӣ}}\left(\left|\log_{\xi}\left(1+\Pi_{RN-\text{Ҟ}_{\mathfrak{z}}}\right)\right|\right)^{\omega_{\text{ѵ}-\mathfrak{z}}} \\ +\iota\left(-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\left(\left|\log_{\xi}\left(1+\Pi_{IN-\text{Ҟ}_{\mathfrak{z}}}\right)\right|\right)^{\omega_{\text{ѵ}-\mathfrak{z}}}\right)\end{pmatrix}$$

and $\Pi_{RP-\text{Ҟ}_{\mathfrak{z}}}=\Pi_{RP-\text{Ҟ}}$, $\Pi_{IP-\text{Ҟ}_{\mathfrak{z}}}=\Pi_{IP-\text{Ҟ}}$, $\Pi_{RN-\text{Ҟ}_{\mathfrak{z}}}=\Pi_{RN-\text{Ҟ}}$ and $\Pi_{IN-\text{Ҟ}_{\mathfrak{z}}}=\Pi_{IN-\text{Ҟ}}\ \forall\,\mathfrak{z}$, then

$$L-BCFWA\left(\text{Ҟ}_1,\ \text{Ҟ}_2,\ \ldots,\ \text{Ҟ}_{\text{ӣ}}\right)=\begin{pmatrix} 1-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\left(\log_{\xi}\Pi_{RP-\text{Ҟ}}\right)^{\omega_{\text{ѵ}-\mathfrak{z}}} \\ +\iota\left(1-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\left(\log_{\xi}\Pi_{IP-\text{Ҟ}}\right)^{\omega_{\text{ѵ}-\mathfrak{z}}}\right), \\ -\prod_{\mathfrak{z}=1}^{\text{ӣ}}\left(\left|\log_{\xi}\left(1+\Pi_{RN-\text{Ҟ}}\right)\right|\right)^{\omega_{\text{ѵ}-\mathfrak{z}}} \\ +\iota\left(-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\left(\left|\log_{\xi}\left(1+\Pi_{IN-\text{Ҟ}}\right)\right|\right)^{\omega_{\text{ѵ}-\mathfrak{z}}}\right)\end{pmatrix}$$

$$=\begin{pmatrix} 1-\left(\log_{\xi}\Pi_{RP-\text{Ҟ}}\right)^{\sum_{\mathfrak{z}=1}^{\text{ӣ}}\omega_{\text{ѵ}-\mathfrak{z}}} \\ +\iota\left(1-\left(\log_{\xi}\Pi_{IP-\text{Ҟ}}\right)^{\sum_{\mathfrak{z}=1}^{\text{ӣ}}\omega_{\text{ѵ}-\mathfrak{z}}}\right), \\ -\left(\left|\log_{\xi}\left(1+\Pi_{RN-\text{Ҟ}}\right)\right|\right)^{\sum_{\mathfrak{z}=1}^{\text{ӣ}}\omega_{\text{ѵ}-\mathfrak{z}}} \\ +\iota\left(-\left(\left|\log_{\xi}\left(1+\Pi_{IN-\text{Ҟ}}\right)\right|\right)^{\sum_{\mathfrak{z}=1}^{\text{ӣ}}\omega_{\text{ѵ}-\mathfrak{z}}}\right)\end{pmatrix}$$

$$=\begin{pmatrix} 1-\log_{\xi}\Pi_{RP-\text{Ҟ}} \\ +\iota\left(1-\log_{\xi}\Pi_{IP-\text{Ҟ}}\right), \\ -\log_{\xi}\left(1+\Pi_{RN-\text{Ҟ}}\right) \\ +\iota\left(-\log_{\xi}\left(1+\Pi_{IN-\text{Ҟ}}\right)\right)\end{pmatrix}=\log_{\xi}\text{Ҟ}.$$

❖ Monotonicity: Under two gatherings of BCFNs $\text{Ҟ}_{\mathfrak{z}}=\left(\Pi_{P-\text{Ҟ}_{\mathfrak{z}}},\ \Pi_{N-\text{Ҟ}_{\mathfrak{z}}}\right)=\left(\Pi_{RP-\text{Ҟ}_{\mathfrak{z}}}+\iota\,\Pi_{IP-\text{Ҟ}_{\mathfrak{z}}},\ \Pi_{RN-\text{Ҟ}_{\mathfrak{z}}}+\iota\,\Pi_{IN-\text{Ҟ}_{\mathfrak{z}}}\right)$, and $\text{Ҟ}^{\star}_{\mathfrak{z}}=\left(\Pi_{P-\text{Ҟ}^{\star}_{\mathfrak{z}}},\ \Pi_{N-\text{Ҟ}^{\star}_{\mathfrak{z}}}\right)=\left(\Pi_{RP-\text{Ҟ}^{\star}_{\mathfrak{z}}}+\iota\,\Pi_{IP-\text{Ҟ}^{\star}_{\mathfrak{z}}},\ \Pi_{RN-\text{Ҟ}^{\star}_{\mathfrak{z}}}+\iota\,\Pi_{IN-\text{Ҟ}^{\star}_{\mathfrak{z}}}\right)$, $\mathfrak{z}=1,\ 2,\ \ldots,\ \text{ӣ}$, if $\Pi_{RP-\text{Ҟ}_{\mathfrak{z}}}\leq\Pi_{RP-\text{Ҟ}^{\star}_{\mathfrak{z}}}$, $\Pi_{IP-\text{Ҟ}_{\mathfrak{z}}}\leq\Pi_{IP-\text{Ҟ}^{\star}_{\mathfrak{z}}}$, $\Pi_{RN-\text{Ҟ}_{\mathfrak{z}}}\leq\Pi_{RN-\text{Ҟ}^{\star}_{\mathfrak{z}}}$ and $\Pi_{IN-\text{Ҟ}_{\mathfrak{z}}}\leq\Pi_{IN-\text{Ҟ}^{\star}_{\mathfrak{z}}}$, then

$$\begin{aligned}&L-BCFWA\left(\text{Ҟ}_1,\ \text{Ҟ}_2,\ \ldots,\ \text{Ҟ}_{\text{ӣ}}\right)\\&\quad\leq L-BCFWA\left(\text{Ҟ}^{\star}_1,\ \text{Ҟ}^{\star}_2,\ \ldots,\ \text{Ҟ}^{\star}_{\text{ӣ}}\right)\end{aligned}$$

Proof: Since, for any $\mathfrak{z}$, we have

$$\begin{aligned}&\Pi_{RP-\text{Ҟ}_{\mathfrak{z}}}\leq\Pi_{RP-\text{Ҟ}^{\star}_{\mathfrak{z}}}\\&\Rightarrow\log_{\xi}\Pi_{RP-\text{Ҟ}_{\mathfrak{z}}}\leq\log_{\xi}\Pi_{RP-\text{Ҟ}^{\star}_{\mathfrak{z}}}\\&\Rightarrow\prod_{\mathfrak{z}=1}^{\text{ӣ}}\log_{\xi}\Pi_{RP-\text{Ҟ}_{\mathfrak{z}}}\leq\prod_{\mathfrak{z}=1}^{\text{ӣ}}\log_{\xi}\Pi_{RP-\text{Ҟ}^{\star}_{\mathfrak{z}}}\\&\Rightarrow 1-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\log_{\xi}\Pi_{RP-\text{Ҟ}_{\mathfrak{z}}}\leq 1-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\log_{\xi}\Pi_{RP-\text{Ҟ}^{\star}_{\mathfrak{z}}}\end{aligned}$$

Similarly, for any $\mathfrak{z}$

$$1-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\log_{\xi}\Pi_{IP-\text{Ҟ}_{\mathfrak{z}}}\leq 1-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\log_{\xi}\Pi_{IP-\text{Ҟ}^{\star}_{\mathfrak{z}}}$$

Next, for any $\mathfrak{z}$, we have that

$$\begin{aligned}&\Pi_{RN-\text{Ҟ}_{\mathfrak{z}}}\leq\Pi_{RN-\text{Ҟ}^{\star}_{\mathfrak{z}}}\\&\Rightarrow 1+\Pi_{RN-\text{Ҟ}_{\mathfrak{z}}}\leq 1+\Pi_{RN-\text{Ҟ}^{\star}_{\mathfrak{z}}}\\&\Rightarrow\log_{\xi}\left(1+\Pi_{RN-\text{Ҟ}_{\mathfrak{z}}}\right)\geq\log_{\xi}\left(1+\Pi_{RN-\text{Ҟ}^{\star}_{\mathfrak{z}}}\right)\\&\Rightarrow\prod_{\mathfrak{z}=1}^{\text{ӣ}}\log_{\xi}\left(1+\Pi_{RN-\text{Ҟ}_{\mathfrak{z}}}\right)\geq\prod_{\mathfrak{z}=1}^{\text{ӣ}}\log_{\xi}\left(1+\Pi_{RN-\text{Ҟ}^{\star}_{\mathfrak{z}}}\right)\\&\Rightarrow-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\log_{\xi}\left(1+\Pi_{RN-\text{Ҟ}_{\mathfrak{z}}}\right)\leq-\prod_{\mathfrak{z}=1}^{\text{ӣ}}\log_{\xi}\left(1+\Pi_{RN-\text{Ҟ}^{\star}_{\mathfrak{z}}}\right)\end{aligned}$$

Similarly, for any $\mathfrak{z}$, we have

$$-\prod_{\mathfrak{z}=1}^{\bar{И}} \log_{\xi}\left(1+\Pi_{IN-\mathcal{K}_{\mathfrak{z}}}\right) \leq -\prod_{\mathfrak{z}=1}^{\bar{И}} \log_{\xi}\left(1+\Pi_{IN-\mathcal{K}^{\star}_{\mathfrak{z}}}\right)$$

By employing the Def of score and accuracy values, we achieve that

$$\begin{aligned} &L-BCFWA\left(\mathcal{K}_1,\ \mathcal{K}_2,\ \ldots,\ \mathcal{K}_{\bar{И}}\right)\\ &\quad \leq L-BCFWA\left(\mathcal{K}^{\star}_1,\ \mathcal{K}^{\star}_2,\ \ldots,\ \mathcal{K}^{\star}_{\bar{И}}\right). \end{aligned}$$

❖ Boundedness: Under the gathering of BCFNs

$$\begin{aligned} \mathcal{K}_{\mathfrak{z}} &= \left(\Pi_{P-\mathcal{K}_{\mathfrak{z}}},\ \Pi_{N-\mathcal{K}_{\mathfrak{z}}}\right)\\ &= \left(\Pi_{RP-\mathcal{K}_{\mathfrak{z}}}+\iota\,\Pi_{IP-\mathcal{K}_{\mathfrak{z}}},\ \Pi_{RN-\mathcal{K}_{\mathfrak{z}}}+\iota\,\Pi_{IN-\mathcal{K}_{\mathfrak{z}}}\right), \end{aligned}$$

$\mathfrak{z}=1,\ 2,\ \ldots,\ \bar{И}$, then if

$$\mathcal{K}^{-}=\begin{pmatrix}\min_{\mathfrak{z}}\left\{\Pi_{RP-\mathcal{K}_{\mathfrak{z}}}\right\}+\iota\,\min_{\mathfrak{z}}\left\{\Pi_{IP-\mathcal{K}_{\mathfrak{z}}}\right\},\\ \max_{\mathfrak{z}}\left\{\Pi_{RN-\mathcal{K}_{\mathfrak{z}}}\right\}+\iota\,\max_{\mathfrak{z}}\left\{\Pi_{IN-\mathcal{K}_{\mathfrak{z}}}\right\}\end{pmatrix},$$

$$\mathcal{K}^{+}=\begin{pmatrix}\max_{\mathfrak{z}}\left\{\Pi_{RP-\mathcal{K}_{\mathfrak{z}}}\right\}+\iota\,\max_{\mathfrak{z}}\left\{\Pi_{IP-\mathcal{K}_{\mathfrak{z}}}\right\},\\ \min_{\mathfrak{z}}\left\{\Pi_{RN-\mathcal{K}_{\mathfrak{z}}}\right\}+\iota\,\min_{\mathfrak{z}}\left\{\Pi_{IN-\mathcal{K}_{\mathfrak{z}}}\right\}\end{pmatrix},\ \text{then}$$

$$\mathcal{K}^{-}\leq L-BCFWA\left(\mathcal{K}_1,\ \mathcal{K}_2,\ \ldots,\ \mathcal{K}_{\bar{И}}\right)\leq \mathcal{K}^{+}$$

Proof: By employing idempotency and monotonicity, boundedness, we achieve that

$$\mathcal{K}^{-}\leq L-BCFWA\left(\mathcal{K}_1,\ \mathcal{K}_2,\ \ldots,\ \mathcal{K}_{\bar{И}}\right)\leq \mathcal{K}^{+}.$$

Definition 8: Under the gathering of BCFNs $\mathcal{K}_{\mathfrak{z}} = (\Pi_{P-\mathcal{K}_{\mathfrak{z}}},\ \Pi_{N-\mathcal{K}_{\mathfrak{z}}}) = (\Pi_{RP-\mathcal{K}_{\mathfrak{z}}}+\iota\,\Pi_{IP-\mathcal{K}_{\mathfrak{z}}},\ \Pi_{RN-\mathcal{K}_{\mathfrak{z}}}+\iota\,\Pi_{IN-\mathcal{K}_{\mathfrak{z}}})$, $\mathfrak{z}=1,\ 2,\ \ldots,\ \bar{И}$, with $0<\xi_{\mathfrak{z}}\leq \min_{\mathfrak{z}}(\Pi_{RP-\mathcal{K}_{\mathfrak{z}}},\ 1+\Pi_{RN-\mathcal{K}_{\mathfrak{z}}},\ \Pi_{IP-\mathcal{K}_{\mathfrak{z}}},\ 1+\Pi_{IN-\mathcal{K}_{\mathfrak{z}}})\leq 1$ and $\xi\neq 1$, the L-BCFOWA operator is deduced as

$$L-BCFOWA\left(\mathcal{K}_1,\ \mathcal{K}_2,\ \ldots,\ \mathcal{K}_{\bar{И}}\right)=\bigoplus_{\mathfrak{z}=1}^{\bar{И}}\omega_{\psi-\mathfrak{z}}\log_{\xi_{\mathfrak{z}}}\mathcal{K}_{\hat{y}(\mathfrak{z})}$$

Noted that $\omega_{\psi}=\left(\omega_{\psi-1},\ \omega_{\psi-2},\ \ldots,\ \omega_{\psi-\bar{И}}\right)$ is a weight vector with the axiom that $0\leq\omega_{\psi-\mathfrak{z}}\leq 1$, $\sum_{\mathfrak{z}=1}^{\bar{И}}\omega_{\psi-\mathfrak{z}}=1$ and $\left(\hat{y}(1),\ \hat{y}(2),\ \ldots,\ \hat{y}(\bar{И})\right)$ is a permutation of $1,\ 2,\ \ldots,\ \bar{И}$ such that for any $\mathfrak{z}$ $\hat{y}(\mathfrak{z}-1)\leq\hat{y}(\mathfrak{z})$.

Theorem 7: Under the gathering of BCFNs $\mathcal{K}_{\mathfrak{z}} = (\Pi_{P-\mathcal{K}_{\mathfrak{z}}},\ \Pi_{N-\mathcal{K}_{\mathfrak{z}}}) = (\Pi_{RP-\mathcal{K}_{\mathfrak{z}}}+\iota\,\Pi_{IP-\mathcal{K}_{\mathfrak{z}}},\ \Pi_{RN-\mathcal{K}_{\mathfrak{z}}}+\iota\,\Pi_{IN-\mathcal{K}_{\mathfrak{z}}})$, $\mathfrak{z}=1,\ 2,\ \ldots,\ \bar{И}$, with $0<\xi_{\mathfrak{z}}\leq \min_{\mathfrak{z}}(\Pi_{RP-\mathcal{K}_{\mathfrak{z}}},\ 1+\Pi_{RN-\mathcal{K}_{\mathfrak{z}}},\ \Pi_{IP-\mathcal{K}_{\mathfrak{z}}},\ 1+\Pi_{IN-\mathcal{K}_{\mathfrak{z}}})\leq 1$ and $\xi\neq 1$, the usage of the L-BCFOWA operator will deduce again a BCFN and

$$L-BCFOWA\left(\mathcal{K}_1,\ \mathcal{K}_2,\ \ldots,\ \mathcal{K}_{\bar{И}}\right)$$

$$=\begin{cases}\begin{pmatrix}1-\prod_{\mathfrak{z}=1}^{\bar{И}}\left(\log_{\xi_{\mathfrak{z}}}\Pi_{RP-\mathcal{K}_{\hat{y}(\mathfrak{z})}}\right)^{\omega_{\psi-\mathfrak{z}}}\\ +\iota\left(1-\prod_{\mathfrak{z}=1}^{\bar{И}}\left(\log_{\xi_{\mathfrak{z}}}\Pi_{IP-\mathcal{K}_{\hat{y}(\mathfrak{z})}}\right)^{\omega_{\psi-\mathfrak{z}}}\right),\\ -\prod_{\mathfrak{z}=1}^{\bar{И}}\left(\left|\log_{\xi_{\mathfrak{z}}}\left(1+\Pi_{RN-\mathcal{K}_{\hat{y}(\mathfrak{z})}}\right)\right|\right)^{\omega_{\psi-\mathfrak{z}}}\\ +\iota\left(-\prod_{\mathfrak{z}=1}^{\bar{И}}\left(\left|\log_{\xi_{\mathfrak{z}}}\left(1+\Pi_{IN-\mathcal{K}_{\hat{y}(\mathfrak{z})}}\right)\right|\right)^{\omega_{\psi-\mathfrak{z}}}\right)\end{pmatrix}\\ \begin{pmatrix}1-\prod_{\mathfrak{z}=1}^{\bar{И}}\left(\log_{\frac{1}{\xi_{\mathfrak{z}}}}\Pi_{RP-\hat{y}(\mathfrak{z})}\right)^{\omega_{\psi-\mathfrak{z}}}\\ +\iota\left(1-\prod_{\mathfrak{z}=1}^{\bar{И}}\left(\log_{\frac{1}{\xi_{\mathfrak{z}}}}\Pi_{IP-\mathcal{K}_{\hat{y}(\mathfrak{z})}}\right)^{\omega_{\psi-\mathfrak{z}}}\right),\\ -\prod_{\mathfrak{z}=1}^{\bar{И}}\left(\left|\log_{\frac{1}{\xi_{\mathfrak{z}}}}\left(1+\Pi_{RN-\mathcal{K}_{\hat{y}(\mathfrak{z})}}\right)\right|\right)^{\omega_{\psi-\mathfrak{z}}}\\ +\iota\left(-\prod_{\mathfrak{z}=1}^{\bar{И}}\left(\left|\log_{\frac{1}{\xi_{\mathfrak{z}}}}\left(1+\Pi_{IN-\mathcal{K}_{\hat{y}(\mathfrak{z})}}\right)\right|\right)^{\omega_{\psi-\mathfrak{z}}}\right)\end{pmatrix}\end{cases}$$

where, $0<\xi_{\mathfrak{z}}\leq\min_{\mathfrak{z}}(\Pi_{RP-\mathcal{K}_{\mathfrak{z}}},\ 1+\Pi_{RN-\mathcal{K}_{\mathfrak{z}}},\ \Pi_{IP-\mathcal{K}_{\mathfrak{z}}},\ 1+\Pi_{IN-\mathcal{K}_{\mathfrak{z}}})\leq 1$, $\xi\neq 1$, and $0<\frac{1}{\xi_{\mathfrak{z}}}\leq\min_{\mathfrak{z}}(\Pi_{RP-\mathcal{K}_{\mathfrak{z}}},\ 1+\Pi_{RN-\mathcal{K}_{\mathfrak{z}}},\ \Pi_{IP-\mathcal{K}_{\mathfrak{z}}},\ 1+\Pi_{IN-\mathcal{K}_{\mathfrak{z}}})\leq 1$, $\xi\neq 1$.

Proof: This proof is identical to the proof of Theorem (2).

Further, if $\xi_1=\xi_2=\ldots=\xi_{\bar{И}}=\xi$ and $0<\xi\leq\min_{\mathfrak{z}}\left(\Pi_{RP-\mathcal{K}_{\mathfrak{z}}},\ 1+\Pi_{RN-\mathcal{K}_{\mathfrak{z}}},\ \Pi_{IP-\mathcal{K}_{\mathfrak{z}}},\ 1+\Pi_{IN-\mathcal{K}_{\mathfrak{z}}}\right)\leq 1$ and $\xi\neq 1$, then the invented L-BCFOWA operator would be transformed underneath

$$L-BCFOWA\left(\mathcal{K}_1,\ \mathcal{K}_2,\ \ldots,\ \mathcal{K}_{\bar{И}}\right)$$

$$=\begin{pmatrix}1-\prod_{\mathfrak{z}=1}^{\bar{И}}\left(\log_{\xi}\Pi_{RP-\mathcal{K}_{\hat{y}(\mathfrak{z})}}\right)^{\omega_{\psi-\mathfrak{z}}}\\ +\iota\left(1-\prod_{\mathfrak{z}=1}^{\bar{И}}\left(\log_{\xi}\Pi_{IP-\mathcal{K}_{\hat{y}(\mathfrak{z})}}\right)^{\omega_{\psi-\mathfrak{z}}}\right),\\ -\prod_{\mathfrak{z}=1}^{\bar{И}}\left(\left|\log_{\xi}\left(1+\Pi_{RN-\mathcal{K}_{\hat{y}(\mathfrak{z})}}\right)\right|\right)^{\omega_{\psi-\mathfrak{z}}}\\ +\iota\left(-\prod_{\mathfrak{z}=1}^{\bar{И}}\left(\left|\log_{\xi}\left(1+\Pi_{IN-\mathcal{K}_{\hat{y}(\mathfrak{z})}}\right)\right|\right)^{\omega_{\psi-\mathfrak{z}}}\right)\end{pmatrix}$$

Properties: L-BCFOWA operator holds the underneath axioms.

❖ Idempotency: Under the gathering of BCFNs $\mathcal{K}_{\mathfrak{z}} = (\Pi_{P-\mathcal{K}_{\mathfrak{z}}},\ \Pi_{N-\mathcal{K}_{\mathfrak{z}}}) = (\Pi_{RP-\mathcal{K}_{\mathfrak{z}}}+\iota\,\Pi_{IP-\mathcal{K}_{\mathfrak{z}}},\ \Pi_{RN-\mathcal{K}_{\mathfrak{z}}}+\iota\,\Pi_{IN-\mathcal{K}_{\mathfrak{z}}})$, $\mathfrak{z}=1,\ 2,\ \ldots,\ \bar{И}$, then if $\mathcal{K}_{\mathfrak{z}}=\mathcal{K}\ \forall\,\mathfrak{z}$, that is $\Pi_{RP-\mathcal{K}_{\mathfrak{z}}}=\Pi_{RP-\mathcal{K}}$, $\Pi_{IP-\mathcal{K}_{\mathfrak{z}}}=\Pi_{IP-\mathcal{K}}$, $\Pi_{RN-\mathcal{K}_{\mathfrak{z}}}=\Pi_{RN-\mathcal{K}}$ and $\Pi_{IN-\mathcal{K}_{\mathfrak{z}}}=\Pi_{IN-\mathcal{K}}\ \forall\,\mathfrak{z}$, then

$$L-BCFOWA\left(\mathcal{K}_1,\ \mathcal{K}_2,\ \ldots,\ \mathcal{K}_{\bar{И}}\right)=\log_{\xi}\mathcal{K}$$

Noted that $\xi_1 = \xi_2 = \ldots = \xi_{\bar{\text{Й}}} = \xi$.

❖ Monotonicity: Under two gatherings of BCFNs $\text{Ҝ}_{\mathfrak{z}} = (\Pi_{P-\text{Ҝ}_{\mathfrak{z}}},\ \Pi_{N-\text{Ҝ}_{\mathfrak{z}}}) = (\Pi_{RP-\text{Ҝ}_{\mathfrak{z}}} + \iota\,\Pi_{IP-\text{Ҝ}_{\mathfrak{z}}},\ \Pi_{RN-\text{Ҝ}_{\mathfrak{z}}} + \iota\,\Pi_{IN-\text{Ҝ}_{\mathfrak{z}}})$, and $\text{Ҝ}^{\star}_{\mathfrak{z}} = (\Pi_{P-\text{Ҝ}^{\star}_{\mathfrak{z}}},\ \Pi_{N-\text{Ҝ}^{\star}_{\mathfrak{z}}}) = (\Pi_{RP-\text{Ҝ}^{\star}_{\mathfrak{z}}} + \iota\,\Pi_{IP-\text{Ҝ}^{\star}_{\mathfrak{z}}},\ \Pi_{RN-\text{Ҝ}^{\star}_{\mathfrak{z}}} + \iota\,\Pi_{IN-\text{Ҝ}^{\star}_{\mathfrak{z}}})$, $\mathfrak{z} = 1, 2, \ldots, \bar{\text{Й}}$, if $\Pi_{RP-\text{Ҝ}_{\mathfrak{z}}} \le \Pi_{RP-\text{Ҝ}^{\star}_{\mathfrak{z}}}$, $\Pi_{IP-\text{Ҝ}_{\mathfrak{z}}} \le \Pi_{IP-\text{Ҝ}^{\star}_{\mathfrak{z}}}$, $\Pi_{RN-\text{Ҝ}_{\mathfrak{z}}} \le \Pi_{RN-\text{Ҝ}^{\star}_{\mathfrak{z}}}$ and $\Pi_{IN-\text{Ҝ}_{\mathfrak{z}}} \le \Pi_{IN-\text{Ҝ}^{\star}_{\mathfrak{z}}}$, then

$$\begin{aligned} &L-BCFOWA\left(\text{Ҝ}_1,\ \text{Ҝ}_2,\ \ldots,\ \text{Ҝ}_{\bar{\text{Й}}}\right) \\ &\quad \le L-BCFOWA\left(\text{Ҝ}^{\star}_1,\ \text{Ҝ}^{\star}_2,\ \ldots,\ \text{Ҝ}^{\star}_{\bar{\text{Й}}}\right) \end{aligned}$$

❖ Boundedness: Under the gathering of BCFNs

$$\begin{aligned} \text{Ҝ}_{\mathfrak{z}} &= \left(\Pi_{P-\text{Ҝ}_{\mathfrak{z}}},\ \Pi_{N-\text{Ҝ}_{\mathfrak{z}}}\right) \\ &= \left(\Pi_{RP-\text{Ҝ}_{\mathfrak{z}}} + \iota\,\Pi_{IP-\text{Ҝ}_{\mathfrak{z}}},\ \Pi_{RN-\text{Ҝ}_{\mathfrak{z}}} + \iota\,\Pi_{IN-\text{Ҝ}_{\mathfrak{z}}}\right), \\ &\mathfrak{z} = 1,\ 2,\ \ldots,\ \bar{\text{Й}},\ \text{then if} \\ \text{Ҝ}^{-} &= \begin{pmatrix} \min\limits_{\mathfrak{z}}\left\{\Pi_{RP-\text{Ҝ}_{\mathfrak{z}}}\right\} + \iota\,\min\limits_{\mathfrak{z}}\left\{\Pi_{IP-\text{Ҝ}_{\mathfrak{z}}}\right\}, \\ \max\limits_{\mathfrak{z}}\left\{\Pi_{RN-\text{Ҝ}_{\mathfrak{z}}}\right\} + \iota\,\max\limits_{\mathfrak{z}}\left\{\Pi_{IN-\text{Ҝ}_{\mathfrak{z}}}\right\} \end{pmatrix},\ \text{and} \\ \text{Ҝ}^{+} &= \begin{pmatrix} \max\limits_{\mathfrak{z}}\left\{\Pi_{RP-\text{Ҝ}_{\mathfrak{z}}}\right\} + \iota\,\max\limits_{\mathfrak{z}}\left\{\Pi_{IP-\text{Ҝ}_{\mathfrak{z}}}\right\}, \\ \min\limits_{\mathfrak{z}}\left\{\Pi_{RN-\text{Ҝ}_{\mathfrak{z}}}\right\} + \iota\,\min\limits_{\mathfrak{z}}\left\{\Pi_{IN-\text{Ҝ}_{\mathfrak{z}}}\right\} \end{pmatrix},\ \text{then} \\ \text{Ҝ}^{-} &\le L-BCFOWA\left(\text{Ҝ}_1,\ \text{Ҝ}_2,\ \ldots,\ \text{Ҝ}_{\bar{\text{Й}}}\right) \le \text{Ҝ}^{+} \end{aligned}$$

Underneath, we investigate the logarithmic BCF geometric AOs.

Definition 9: Under the gathering of BCFNs $\text{Ҝ}_{\mathfrak{z}} = (\Pi_{P-\text{Ҝ}_{\mathfrak{z}}},\ \Pi_{N-\text{Ҝ}_{\mathfrak{z}}}) = (\Pi_{RP-\text{Ҝ}_{\mathfrak{z}}} + \iota\,\Pi_{IP-\text{Ҝ}_{\mathfrak{z}}},\ \Pi_{RN-\text{Ҝ}_{\mathfrak{z}}} + \iota\,\Pi_{IN-\text{Ҝ}_{\mathfrak{z}}})$, $\mathfrak{z} = 1, 2, \ldots, \bar{\text{Й}}$, with $0 < \xi_{\mathfrak{z}} \le \min\limits_{\mathfrak{z}}(\Pi_{RP-\text{Ҝ}_{\mathfrak{z}}},\ 1+\Pi_{RN-\text{Ҝ}_{\mathfrak{z}}},\ \Pi_{IP-\text{Ҝ}_{\mathfrak{z}}},\ 1+\Pi_{IN-\text{Ҝ}_{\mathfrak{z}}}) \le 1$ and $\xi \ne 1$, the L-BCFWG operator is deduced as

$$L-BCFWG\left(\text{Ҝ}_1,\ \text{Ҝ}_2,\ \ldots,\ \text{Ҝ}_{\bar{\text{Й}}}\right) = \bigotimes_{\mathfrak{z}=1}^{\bar{\text{Й}}} \left(\log_{\xi_{\mathfrak{z}}} \text{Ҝ}_{\mathfrak{z}}\right)^{\omega_{\mathbb{v}-\mathfrak{z}}}$$

Noted that $\omega_{\mathbb{v}} = (\omega_{\mathbb{v}-1},\ \omega_{\mathbb{v}-2},\ \ldots,\ \omega_{\mathbb{v}-\bar{\text{Й}}})$ is a weight vector with the axiom that $0 \le \omega_{\mathbb{v}-\mathfrak{z}} \le 1$ and $\sum_{\mathfrak{z}=1}^{\bar{\text{Й}}} \omega_{\mathbb{v}-\mathfrak{z}} = 1$.

Theorem 8: Under the gathering of BCFNs $\text{Ҝ}_{\mathfrak{z}} = (\Pi_{P-\text{Ҝ}_{\mathfrak{z}}},\ \Pi_{N-\text{Ҝ}_{\mathfrak{z}}}) = (\Pi_{RP-\text{Ҝ}_{\mathfrak{z}}} + \iota\,\Pi_{IP-\text{Ҝ}_{\mathfrak{z}}},\ \Pi_{RN-\text{Ҝ}_{\mathfrak{z}}} + \iota\,\Pi_{IN-\text{Ҝ}_{\mathfrak{z}}})$, $\mathfrak{z} = 1, 2, \ldots, \bar{\text{Й}}$, with $0 < \xi_{\mathfrak{z}} \le \min\limits_{\mathfrak{z}}(\Pi_{RP-\text{Ҝ}_{\mathfrak{z}}},\ 1+\Pi_{RN-\text{Ҝ}_{\mathfrak{z}}},\ \Pi_{IP-\text{Ҝ}_{\mathfrak{z}}},\ 1+\Pi_{IN-\text{Ҝ}_{\mathfrak{z}}}) \le 1$ and $\xi \ne 1$, the usage of the L-BCFWG operator will deduce again a BCFN and

$$L-BCFWG\left(\text{Ҝ}_1,\ \text{Ҝ}_2,\ \ldots,\ \text{Ҝ}_{\bar{\text{Й}}}\right)$$

$$= \begin{cases} \begin{pmatrix} \prod\limits_{\mathfrak{z}=1}^{\bar{\text{Й}}}\left(1-\log_{\xi_{\mathfrak{z}}}\Pi_{RP-\text{Ҝ}_{\mathfrak{z}}}\right)^{\omega_{\mathbb{v}-\mathfrak{z}}} \\ +\iota\prod\limits_{\mathfrak{z}=1}^{\bar{\text{Й}}}\left(1-\log_{\xi_{\mathfrak{z}}}\Pi_{IP-\text{Ҝ}_{\mathfrak{z}}}\right)^{\omega_{\mathbb{v}-\mathfrak{z}}}, \\ -1+\prod\limits_{\mathfrak{z}=1}^{\bar{\text{Й}}}\left(1-\log_{\xi_{\mathfrak{z}}}\left(1+\Pi_{RN-\text{Ҝ}_{\mathfrak{z}}}\right)\right)^{\omega_{\mathbb{v}-\mathfrak{z}}} \\ +\iota\left(-1+\prod\limits_{\mathfrak{z}=1}^{\bar{\text{Й}}}\left(1-\log_{\xi_{\mathfrak{z}}}\left(1+\Pi_{IN-\text{Ҝ}_{\mathfrak{z}}}\right)\right)^{\omega_{\mathbb{v}-\mathfrak{z}}}\right) \end{pmatrix} \\ \begin{pmatrix} \prod\limits_{\mathfrak{z}=1}^{\bar{\text{Й}}}\left(1-\log_{\frac{1}{\xi_{\mathfrak{z}}}}\Pi_{RP-\text{Ҝ}_{\mathfrak{z}}}\right)^{\omega_{\mathbb{v}-\mathfrak{z}}} \\ +\iota\prod\limits_{\mathfrak{z}=1}^{\bar{\text{Й}}}\left(1-\log_{\frac{1}{\xi_{\mathfrak{z}}}}\Pi_{IP-\text{Ҝ}_{\mathfrak{z}}}\right)^{\omega_{\mathbb{v}-\mathfrak{z}}}, \\ -1+\prod\limits_{\mathfrak{z}=1}^{\bar{\text{Й}}}\left(1-\log_{\frac{1}{\xi_{\mathfrak{z}}}}\left(1+\Pi_{RN-\text{Ҝ}_{\mathfrak{z}}}\right)\right)^{\omega_{\mathbb{v}-\mathfrak{z}}} \\ +\iota\left(-1+\prod\limits_{\mathfrak{z}=1}^{\bar{\text{Й}}}\left(1-\log_{\frac{1}{\xi_{\mathfrak{z}}}}\left(1+\Pi_{IN-\text{Ҝ}_{\mathfrak{z}}}\right)\right)^{\omega_{\mathbb{v}-\mathfrak{z}}}\right) \end{pmatrix} \end{cases} \tag{4}$$

where, $0 < \xi_{\mathfrak{z}} \le \min\limits_{\mathfrak{z}}(\Pi_{RP-\text{Ҝ}_{\mathfrak{z}}},\ 1+\Pi_{RN-\text{Ҝ}_{\mathfrak{z}}},\ \Pi_{IP-\text{Ҝ}_{\mathfrak{z}}},\ 1+\Pi_{IN-\text{Ҝ}_{\mathfrak{z}}}) \le 1$, $\xi \ne 1$, and $0 < \frac{1}{\xi_{\mathfrak{z}}} \le \min\limits_{\mathfrak{z}}(\Pi_{RP-\text{Ҝ}_{\mathfrak{z}}},\ 1+\Pi_{RN-\text{Ҝ}_{\mathfrak{z}}},\ \Pi_{IP-\text{Ҝ}_{\mathfrak{z}}},\ 1+\Pi_{IN-\text{Ҝ}_{\mathfrak{z}}}) \le 1$, $\xi \ne 1$.

Proof: This proof would be done by mathematical induction. Assume $\bar{\text{Й}} = 2$, and since

$$\begin{aligned} &\left(\log_{\xi_1}\text{Ҝ}_1\right)^{\omega_{\mathbb{v}-1}} \\ &= \begin{pmatrix} \left(1-\log_{\xi_1}\Pi_{RP-\text{Ҝ}_1}\right)^{\omega_{\mathbb{v}-1}} \\ +\iota\left(1-\log_{\xi_1}\Pi_{IP-\text{Ҝ}_1}\right)^{\omega_{\mathbb{v}-1}}, \\ -1+\left(1-\log_{\xi_1}\left(1+\Pi_{RN-\text{Ҝ}_1}\right)^{\omega_{\mathbb{v}-1}}\right) \\ +\iota\left(-1+\left(1-\log_{\xi_1}\left(1+\Pi_{RN-\text{Ҝ}_1}\right)\right)^{\omega_{\mathbb{v}-1}}\right) \end{pmatrix} \\ &\left(\log_{\xi_2}\text{Ҝ}_2\right)^{\omega_{\mathbb{v}-2}} \\ &= \begin{pmatrix} \left(1-\log_{\xi_2}\Pi_{RP-\text{Ҝ}_2}\right)^{\omega_{\mathbb{v}-2}} \\ +\iota\left(1-\log_{\xi_2}\Pi_{IP-\text{Ҝ}_2}\right)^{\omega_{\mathbb{v}-2}}, \\ -1+\left(1-\log_{\xi_2}\left(1+\Pi_{RN-\text{Ҝ}_2}\right)^{\omega_{\mathbb{v}-2}}\right) \\ +\iota\left(-1+\left(1-\log_{\xi_2}\left(1+\Pi_{RN-\text{Ҝ}_2}\right)\right)^{\omega_{\mathbb{v}-2}}\right) \end{pmatrix} \end{aligned}$$

then,

$$\left(\log_{\xi_1}\text{Ҝ}_1\right)^{\omega_{\mathbb{v}-1}} \otimes \left(\log_{\xi_2}\text{Ҝ}_2\right)^{\omega_{\mathbb{v}-2}}$$

$$
= \begin{pmatrix} \left(1-\log_{\xi_1}\Pi_{RP-\text{Ꝁ}_1}\right)^{\omega_{\mathbb{v}-1}} \\ +\iota\left(1-\log_{\xi_1}\Pi_{IP-\text{Ꝁ}_1}\right)^{\omega_{\mathbb{v}-1}}, \\ -1+\left(1-\log_{\xi_1}\left(1+\Pi_{RN-\text{Ꝁ}_1}\right)^{\omega_{\mathbb{v}-1}}\right) \\ +\iota\left(-1+\left(1-\log_{\xi_1}\left(1+\Pi_{RN-\text{Ꝁ}_1}\right)\right)^{\omega_{\mathbb{v}-1}}\right) \end{pmatrix}
\otimes \begin{pmatrix} \left(1-\log_{\xi_2}\Pi_{RP-\text{Ꝁ}_2}\right)^{\omega_{\mathbb{v}-2}} \\ +\iota\left(1-\log_{\xi_2}\Pi_{IP-\text{Ꝁ}_2}\right)^{\omega_{\mathbb{v}-2}}, \\ -1+\left(1-\log_{\xi_2}\left(1+\Pi_{RN-\text{Ꝁ}_2}\right)^{\omega_{\mathbb{v}-2}}\right) \\ +\iota\left(-1+\left(1-\log_{\xi_2}\left(1+\Pi_{RN-\text{Ꝁ}_2}\right)\right)^{\omega_{\mathbb{v}-2}}\right) \end{pmatrix}
$$

$$
= \begin{pmatrix} \left(1-\log_{\xi_1}\Pi_{RP-\text{Ꝁ}_1}\right)^{\omega_{\mathbb{v}-1}}\left(1-\log_{\xi_2}\Pi_{RP-\text{Ꝁ}_2}\right)^{\omega_{\mathbb{v}-2}} \\ +\iota\left(1-\log_{\xi_1}\Pi_{IP-\text{Ꝁ}_1}\right)^{\omega_{\mathbb{v}-1}}\left(1-\log_{\xi_2}\Pi_{IP-\text{Ꝁ}_2}\right)^{\omega_{\mathbb{v}-2}}, \\ \begin{pmatrix} -1+\left(1-\log_{\xi_1}\left(1+\Pi_{RN-\text{Ꝁ}_1}\right)^{\omega_{\mathbb{v}-1}}\right) \\ -1+\left(1-\log_{\xi_2}\left(1+\Pi_{RN-\text{Ꝁ}_2}\right)^{\omega_{\mathbb{v}-2}}\right) \\ +\left(-1+\left(1-\log_{\xi_1}\left(1+\Pi_{RN-\text{Ꝁ}_1}\right)^{\omega_{\mathbb{v}-1}}\right)\right) \\ \left(-1+\left(1-\log_{\xi_2}\left(1+\Pi_{RN-\text{Ꝁ}_2}\right)^{\omega_{\mathbb{v}-2}}\right)\right) \end{pmatrix} \\ +\iota\begin{pmatrix} -1+\left(1-\log_{\xi_1}\left(1+\Pi_{IN-\text{Ꝁ}_1}\right)^{\omega_{\mathbb{v}-1}}\right) \\ -1+\left(1-\log_{\xi_2}\left(1+\Pi_{IN-\text{Ꝁ}_2}\right)^{\omega_{\mathbb{v}-2}}\right) \\ +\left(-1+\left(1-\log_{\xi_1}\left(1+\Pi_{IN-\text{Ꝁ}_1}\right)^{\omega_{\mathbb{v}-1}}\right)\right) \\ \left(-1+\left(1-\log_{\xi_2}\left(1+\Pi_{IN-\text{Ꝁ}_2}\right)^{\omega_{\mathbb{v}-2}}\right)\right) \end{pmatrix} \end{pmatrix}
$$

$$
= \begin{pmatrix} \prod_{\text{ꝣ}=1}^{2}\left(1-\log_{\xi_{\text{ꝣ}}}\Pi_{RP-\text{Ꝁ}_{\text{ꝣ}}}\right)^{\omega_{\mathbb{v}-\text{ꝣ}}} \\ +\iota\prod_{\text{ꝣ}=1}^{2}\left(1-\log_{\xi_{\text{ꝣ}}}\Pi_{IP-\text{Ꝁ}_{\text{ꝣ}}}\right)^{\omega_{\mathbb{v}-\text{ꝣ}}}, \\ -1+\prod_{\text{ꝣ}=1}^{2}\left(1-\log_{\xi_{\text{ꝣ}}}\left(1+\Pi_{RN-\text{Ꝁ}_{\text{ꝣ}}}\right)\right)^{\omega_{\mathbb{v}-\text{ꝣ}}} \\ +\iota\left(-1+\prod_{\text{ꝣ}=1}^{2}\left(1-\log_{\xi_{\text{ꝣ}}}\left(1+\Pi_{IN-\text{Ꝁ}_{\text{ꝣ}}}\right)\right)^{\omega_{\mathbb{v}-\text{ꝣ}}}\right) \end{pmatrix}
$$

$$
= L-BCFWG\left(\text{Ꝁ}_1,\ \text{Ꝁ}_2\right)
$$

Assume that Eq. (4) is valid for $\bar{й} = {}^{\circ}\mathrm{C}$, then we have

$$
L-BCFWG\left(\text{Ꝁ}_1,\ \text{Ꝁ}_2,\ \ldots,\ \text{Ꝁ}_{{}^{\circ}\mathrm{C}}\right)
$$

$$
= \begin{pmatrix} \prod_{\text{ꝣ}=1}^{{}^{\circ}\mathrm{C}}\left(1-\log_{\xi_{\text{ꝣ}}}\Pi_{RP-\text{Ꝁ}_{\text{ꝣ}}}\right)^{\omega_{\mathbb{v}-\text{ꝣ}}} \\ +\iota\prod_{\text{ꝣ}=1}^{{}^{\circ}\mathrm{C}}\left(1-\log_{\xi_{\text{ꝣ}}}\Pi_{IP-\text{Ꝁ}_{\text{ꝣ}}}\right)^{\omega_{\mathbb{v}-\text{ꝣ}}}, \\ -1+\prod_{\text{ꝣ}=1}^{{}^{\circ}\mathrm{C}}\left(1-\log_{\xi_{\text{ꝣ}}}\left(1+\Pi_{RN-\text{Ꝁ}_{\text{ꝣ}}}\right)\right)^{\omega_{\mathbb{v}-\text{ꝣ}}} \\ +\iota\left(-1+\prod_{\text{ꝣ}=1}^{{}^{\circ}\mathrm{C}}\left(1-\log_{\xi_{\text{ꝣ}}}\left(1+\Pi_{IN-\text{Ꝁ}_{\text{ꝣ}}}\right)\right)^{\omega_{\mathbb{v}-\text{ꝣ}}}\right) \end{pmatrix}
$$

last let $\bar{й} = {}^{\circ}\mathrm{C}+1$, then

$$
L-BCFWG\left(\text{Ꝁ}_1,\ \text{Ꝁ}_2,\ \ldots,\ \text{Ꝁ}_{{}^{\circ}\mathrm{C}}\right)\otimes\text{Ꝁ}_{{}^{\circ}\mathrm{C}+1}
$$

$$
= \begin{pmatrix} \prod_{\text{ꝣ}=1}^{{}^{\circ}\mathrm{C}}\left(1-\log_{\xi_{\text{ꝣ}}}\Pi_{RP-\text{Ꝁ}_{\text{ꝣ}}}\right)^{\omega_{\mathbb{v}-\text{ꝣ}}} \\ +\iota\prod_{\text{ꝣ}=1}^{{}^{\circ}\mathrm{C}}\left(1-\log_{\xi_{\text{ꝣ}}}\Pi_{IP-\text{Ꝁ}_{\text{ꝣ}}}\right)^{\omega_{\mathbb{v}-\text{ꝣ}}}, \\ -1+\prod_{\text{ꝣ}=1}^{{}^{\circ}\mathrm{C}}\left(1-\log_{\xi_{\text{ꝣ}}}\left(1+\Pi_{RN-\text{Ꝁ}_{\text{ꝣ}}}\right)\right)^{\omega_{\mathbb{v}-\text{ꝣ}}} \\ +\iota\left(-1+\prod_{\text{ꝣ}=1}^{{}^{\circ}\mathrm{C}}\left(1-\log_{\xi_{\text{ꝣ}}}\left(1+\Pi_{IN-\text{Ꝁ}_{\text{ꝣ}}}\right)\right)^{\omega_{\mathbb{v}-\text{ꝣ}}}\right) \end{pmatrix}
\otimes \begin{pmatrix} \left(1-\log_{\xi_{{}^{\circ}\mathrm{C}+1}}\Pi_{RP-\text{Ꝁ}_{{}^{\circ}\mathrm{C}+1}}\right)^{\omega_{\mathbb{v}-{}^{\circ}\mathrm{C}+1}} \\ +\iota\left(1-\log_{\xi_{{}^{\circ}\mathrm{C}+1}}\Pi_{IP-\text{Ꝁ}_{{}^{\circ}\mathrm{C}+1}}\right)^{\omega_{\mathbb{v}-{}^{\circ}\mathrm{C}+1}}, \\ -1+\left(1-\log_{\xi_{{}^{\circ}\mathrm{C}+1}}\left(1+\Pi_{RN-\text{Ꝁ}_{{}^{\circ}\mathrm{C}+1}}\right)^{\omega_{\mathbb{v}-{}^{\circ}\mathrm{C}+1}}\right) \\ +\iota\left(-1+\left(1-\log_{\xi_{{}^{\circ}\mathrm{C}+1}}\left(1+\Pi_{RN-\text{Ꝁ}_{{}^{\circ}\mathrm{C}+1}}\right)\right)^{\omega_{\mathbb{v}-{}^{\circ}\mathrm{C}+1}}\right) \end{pmatrix}
$$

$$
= \begin{pmatrix} \prod_{\text{ꝣ}=1}^{{}^{\circ}\mathrm{C}+1}\left(1-\log_{\xi_{\text{ꝣ}}}\Pi_{RP-\text{Ꝁ}_{\text{ꝣ}}}\right)^{\omega_{\mathbb{v}-\text{ꝣ}}} \\ +\iota\prod_{\text{ꝣ}=1}^{{}^{\circ}\mathrm{C}+1}\left(1-\log_{\xi_{\text{ꝣ}}}\Pi_{IP-\text{Ꝁ}_{\text{ꝣ}}}\right)^{\omega_{\mathbb{v}-\text{ꝣ}}}, \\ -1+\prod_{\text{ꝣ}=1}^{{}^{\circ}\mathrm{C}+1}\left(1-\log_{\xi_{\text{ꝣ}}}\left(1+\Pi_{RN-\text{Ꝁ}_{\text{ꝣ}}}\right)\right)^{\omega_{\mathbb{v}-\text{ꝣ}}} \\ +\iota\left(-1+\prod_{\text{ꝣ}=1}^{{}^{\circ}\mathrm{C}+1}\left(1-\log_{\xi_{\text{ꝣ}}}\left(1+\Pi_{IN-\text{Ꝁ}_{\text{ꝣ}}}\right)\right)^{\omega_{\mathbb{v}-\text{ꝣ}}}\right) \end{pmatrix}
$$

Therefore, Eq. (4) is valid $\forall\ \bar{й}$. Likewise, if $0 < \frac{1}{\xi_{\text{ꝣ}}} \leq \min_{\text{ꝣ}}\left(\Pi_{RP-\text{Ꝁ}_{\text{ꝣ}}},\ 1+\Pi_{RN-\text{Ꝁ}_{\text{ꝣ}}},\ \Pi_{IP-\text{Ꝁ}_{\text{ꝣ}}},\ 1+\Pi_{IN-\text{Ꝁ}_{\text{ꝣ}}}\right) \leq 1$, $\xi \neq 1$, then we can demonstrate

$$
L-BCFWG\left(\text{Ꝁ}_1,\ \text{Ꝁ}_2,\ \ldots,\ \text{Ꝁ}_{\bar{й}}\right)
$$

$$= \begin{pmatrix} \prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\frac{1}{\xi_{\mathfrak{z}}}} \Pi_{RP-\Bbbk_{\mathfrak{z}}}\right)^{\omega_{v-\mathfrak{z}}} \\ +\iota \prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\frac{1}{\xi_{\mathfrak{z}}}} \Pi_{IP-\Bbbk_{\mathfrak{z}}}\right)^{\omega_{v-\mathfrak{z}}}, \\ -1+\prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\frac{1}{\xi_{\mathfrak{z}}}} \left(1+\Pi_{RN-\Bbbk_{\mathfrak{z}}}\right)\right)^{\omega_{v-\mathfrak{z}}} \\ +\iota \left(-1+\prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\frac{1}{\xi_{\mathfrak{z}}}} \left(1+\Pi_{IN-\Bbbk_{\mathfrak{z}}}\right)\right)^{\omega_{v-\mathfrak{z}}}\right) \end{pmatrix}$$

Hence proved.

Further, if $\xi_1 = \xi_2 = \ldots = \xi_{\bar{\text{й}}} = \xi$ and $0 < \xi \leq \min_{\mathfrak{z}} \left(\Pi_{RP-\Bbbk_{\mathfrak{z}}}, 1+\Pi_{RN-\Bbbk_{\mathfrak{z}}}, \Pi_{IP-\Bbbk_{\mathfrak{z}}}, 1+\Pi_{IN-\Bbbk_{\mathfrak{z}}}\right) \leq 1$ and $\xi \neq 1$, then the invented L-BCFWG operator would be transformed underneath

$$L-BCFWG\left(\Bbbk_1, \Bbbk_2, \ldots, \Bbbk_{\bar{\text{й}}}\right) = \begin{pmatrix} \prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\xi} \Pi_{RP-\Bbbk_{\mathfrak{z}}}\right)^{\omega_{v-\mathfrak{z}}} \\ +\iota \prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\xi} \Pi_{IP-\Bbbk_{\mathfrak{z}}}\right)^{\omega_{v-\mathfrak{z}}}, \\ -1+\prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\xi} \left(1+\Pi_{RN-\Bbbk_{\mathfrak{z}}}\right)\right)^{\omega_{v-\mathfrak{z}}} \\ +\iota \left(-1+\prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\xi} \left(1+\Pi_{IN-\Bbbk_{\mathfrak{z}}}\right)\right)^{\omega_{v-\mathfrak{z}}}\right) \end{pmatrix}$$

Properties: L-BCFWG operator holds the underneath axioms.

❖ Idempotency: Under the gathering of BCFNs $\Bbbk_{\mathfrak{z}} = (\Pi_{P-\Bbbk_{\mathfrak{z}}}, \Pi_{N-\Bbbk_{\mathfrak{z}}}) = (\Pi_{RP-\Bbbk_{\mathfrak{z}}} + \iota\,\Pi_{IP-\Bbbk_{\mathfrak{z}}}, \Pi_{RN-\Bbbk_{\mathfrak{z}}} + \iota\,\Pi_{IN-\Bbbk_{\mathfrak{z}}})$, $\mathfrak{z} = 1, 2, \ldots, \bar{\text{й}}$, then if $\Bbbk_{\mathfrak{z}} = \Bbbk\ \forall\, \mathfrak{z}$, that is $\Pi_{RP-\Bbbk_{\mathfrak{z}}} = \Pi_{RP-\Bbbk}$, $\Pi_{IP-\Bbbk_{\mathfrak{z}}} = \Pi_{IP-\Bbbk}$, $\Pi_{RN-\Bbbk_{\mathfrak{z}}} = \Pi_{RN-\Bbbk}$ and $\Pi_{IN-\Bbbk_{\mathfrak{z}}} = \Pi_{IN-\Bbbk}\ \forall\, \mathfrak{z}$, then

$$L-BCFWG\left(\Bbbk_1, \Bbbk_2, \ldots, \Bbbk_{\bar{\text{й}}}\right) = \log_{\xi}\Bbbk$$

Noted that $\xi_1 = \xi_2 = \ldots = \xi_{\bar{\text{й}}} = \xi$.

❖ Monotonicity: Under two gatherings of BCFNs $\Bbbk_{\mathfrak{z}} = (\Pi_{P-\Bbbk_{\mathfrak{z}}}, \Pi_{N-\Bbbk_{\mathfrak{z}}}) = (\Pi_{RP-\Bbbk_{\mathfrak{z}}} + \iota\,\Pi_{IP-\Bbbk_{\mathfrak{z}}}, \Pi_{RN-\Bbbk_{\mathfrak{z}}} + \iota\,\Pi_{IN-\Bbbk_{\mathfrak{z}}})$, and $\Bbbk^{\star}_{\mathfrak{z}} = (\Pi_{P-\Bbbk^{\star}_{\mathfrak{z}}}, \Pi_{N-\Bbbk^{\star}_{\mathfrak{z}}}) = (\Pi_{RP-\Bbbk^{\star}_{\mathfrak{z}}} + \iota\,\Pi_{IP-\Bbbk^{\star}_{\mathfrak{z}}}, \Pi_{RN-\Bbbk^{\star}_{\mathfrak{z}}} + \iota\,\Pi_{IN-\Bbbk^{\star}_{\mathfrak{z}}})$, $\mathfrak{z} = 1, 2, \ldots, \bar{\text{й}}$, if $\Pi_{RP-\Bbbk_{\mathfrak{z}}} \leq \Pi_{RP-\Bbbk^{\star}_{\mathfrak{z}}}$, $\Pi_{IP-\Bbbk_{\mathfrak{z}}} \leq \Pi_{IP-\Bbbk^{\star}_{\mathfrak{z}}}$, $\Pi_{RN-\Bbbk_{\mathfrak{z}}} \leq \Pi_{RN-\Bbbk^{\star}_{\mathfrak{z}}}$ and $\Pi_{IN-\Bbbk_{\mathfrak{z}}} \leq \Pi_{IN-\Bbbk^{\star}_{\mathfrak{z}}}$, then

$$L-BCFWG\left(\Bbbk_1, \Bbbk_2, \ldots, \Bbbk_{\bar{\text{й}}}\right) \leq L-BCFWG\left(\Bbbk^{\star}_1, \Bbbk^{\star}_2, \ldots, \Bbbk^{\star}_{\bar{\text{й}}}\right)$$

❖ Boundedness: Under the gathering of BCFNs

$$\Bbbk_{\mathfrak{z}} = \left(\Pi_{P-\Bbbk_{\mathfrak{z}}}, \Pi_{N-\Bbbk_{\mathfrak{z}}}\right) = \left(\Pi_{RP-\Bbbk_{\mathfrak{z}}} + \iota\,\Pi_{IP-\Bbbk_{\mathfrak{z}}}, \Pi_{RN-\Bbbk_{\mathfrak{z}}} + \iota\,\Pi_{IN-\Bbbk_{\mathfrak{z}}}\right),$$

$\mathfrak{z} = 1, 2, \ldots, \bar{\text{й}}$, then if

$$\Bbbk^{-} = \begin{pmatrix} \min_{\mathfrak{z}}\left\{\Pi_{RP-\Bbbk_{\mathfrak{z}}}\right\} + \iota \min_{\mathfrak{z}}\left\{\Pi_{IP-\Bbbk_{\mathfrak{z}}}\right\}, \\ \max_{\mathfrak{z}}\left\{\Pi_{RN-\Bbbk_{\mathfrak{z}}}\right\} + \iota \max_{\mathfrak{z}}\left\{\Pi_{IN-\Bbbk_{\mathfrak{z}}}\right\} \end{pmatrix}, \text{ and}$$

$$\Bbbk^{+} = \begin{pmatrix} \max_{\mathfrak{z}}\left\{\Pi_{RP-\Bbbk_{\mathfrak{z}}}\right\} + \iota \max_{\mathfrak{z}}\left\{\Pi_{IP-\Bbbk_{\mathfrak{z}}}\right\}, \\ \min_{\mathfrak{z}}\left\{\Pi_{RN-\Bbbk_{\mathfrak{z}}}\right\} + \iota \min_{\mathfrak{z}}\left\{\Pi_{IN-\Bbbk_{\mathfrak{z}}}\right\} \end{pmatrix}, \text{ then}$$

$$\Bbbk^{-} \leq L-BCFWG\left(\Bbbk_1, \Bbbk_2, \ldots, \Bbbk_{\bar{\text{й}}}\right) \leq \Bbbk^{+}$$

Definition 10: Under the gathering of BCFNs $\Bbbk_{\mathfrak{z}} = (\Pi_{P-\Bbbk_{\mathfrak{z}}}, \Pi_{N-\Bbbk_{\mathfrak{z}}}) = (\Pi_{RP-\Bbbk_{\mathfrak{z}}} + \iota\,\Pi_{IP-\Bbbk_{\mathfrak{z}}}, \Pi_{RN-\Bbbk_{\mathfrak{z}}} + \iota\,\Pi_{IN-\Bbbk_{\mathfrak{z}}})$, $\mathfrak{z} = 1, 2, \ldots, \bar{\text{й}}$, with $0 < \xi_{\mathfrak{z}} \leq \min_{\mathfrak{z}}(\Pi_{RP-\Bbbk_{\mathfrak{z}}}, 1+\Pi_{RN-\Bbbk_{\mathfrak{z}}}, \Pi_{IP-\Bbbk_{\mathfrak{z}}}, 1+\Pi_{IN-\Bbbk_{\mathfrak{z}}}) \leq 1$ and $\xi \neq 1$, the L-BCFOWG operator is deduced as

$$L-BCFOWG\left(\Bbbk_1, \Bbbk_2, \ldots, \Bbbk_{\bar{\text{й}}}\right) = \bigotimes_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(\log_{\xi_{\mathfrak{z}}} \Bbbk_{\hat{y}(\mathfrak{z})}\right)^{\omega_{v-\mathfrak{z}}}$$

Noted that $\omega_{v} = (\omega_{v-1}, \omega_{v-2}, \ldots, \omega_{v-\bar{\text{й}}})$ is a weight vector with the axiom that $0 \leq \omega_{v-\mathfrak{z}} \leq 1$, $\sum_{\mathfrak{z}=1}^{\bar{\text{й}}} \omega_{v-\mathfrak{z}} = 1$ and $\left(\hat{y}(1), \hat{y}(2), \ldots, \hat{y}(\bar{\text{й}})\right)$ is a permutation of $1, 2, .., \bar{\text{й}}$ such that for any $\mathfrak{z}$ $\hat{y}(\mathfrak{z}-1) \leq \hat{y}(\mathfrak{z})$.

Theorem 9: Under the gathering of BCFNs $\Bbbk_{\mathfrak{z}} = \left(\Pi_{P-\Bbbk_{\mathfrak{z}}}, \Pi_{N-\Bbbk_{\mathfrak{z}}}\right) = \left(\Pi_{RP-\Bbbk_{\mathfrak{z}}} + \iota\,\Pi_{IP-\Bbbk_{\mathfrak{z}}}, \Pi_{RN-\Bbbk_{\mathfrak{z}}} + \iota\,\Pi_{IN-\Bbbk_{\mathfrak{z}}}\right)$, $\mathfrak{z} = 1, 2, \ldots, \bar{\text{й}}$, with $0 < \xi_{\mathfrak{z}} \leq \min_{\mathfrak{z}}\left(\Pi_{RP-\Bbbk_{\mathfrak{z}}}, 1+\Pi_{RN-\Bbbk_{\mathfrak{z}}}, \Pi_{IP-\Bbbk_{\mathfrak{z}}}, 1+\Pi_{IN-\Bbbk_{\mathfrak{z}}}\right) \leq 1$ and $\xi \neq 1$, the usage of the L-BCFOWG operator will deduce again a BCFN and

$$L-BCFOWG\left(\Bbbk_1, \Bbbk_2, \ldots, \Bbbk_{\bar{\text{й}}}\right) = \begin{cases} \begin{pmatrix} \prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\xi_{\mathfrak{z}}} \Pi_{RP-\Bbbk_{\hat{y}(\mathfrak{z})}}\right)^{\omega_{v-\mathfrak{z}}} \\ +\iota \prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\xi_{\mathfrak{z}}} \Pi_{IP-\Bbbk_{\hat{y}(\mathfrak{z})}}\right)^{\omega_{v-\mathfrak{z}}}, \\ -1+\prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\xi_{\mathfrak{z}}} \left(1+\Pi_{RN-\Bbbk_{\hat{y}(\mathfrak{z})}}\right)\right)^{\omega_{v-\mathfrak{z}}} \\ +\iota \left(-1+\prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\xi_{\mathfrak{z}}} \left(1+\Pi_{IN-\Bbbk_{\hat{y}(\mathfrak{z})}}\right)\right)^{\omega_{v-\mathfrak{z}}}\right) \end{pmatrix} \\ \begin{pmatrix} \prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\frac{1}{\xi_{\mathfrak{z}}}} \Pi_{RP-\Bbbk_{\hat{y}(\mathfrak{z})}}\right)^{\omega_{v-\mathfrak{z}}} \\ +\iota \prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\frac{1}{\xi_{\mathfrak{z}}}} \Pi_{IP-\Bbbk_{\hat{y}(\mathfrak{z})}}\right)^{\omega_{v-\mathfrak{z}}}, \\ -1+\prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\frac{1}{\xi_{\mathfrak{z}}}} \left(1+\Pi_{RN-\Bbbk_{\hat{y}(\mathfrak{z})}}\right)\right)^{\omega_{v-\mathfrak{z}}} \\ +\iota \left(-1+\prod_{\mathfrak{z}=1}^{\bar{\text{й}}} \left(1-\log_{\frac{1}{\xi_{\mathfrak{z}}}} \left(1+\Pi_{IN-\Bbbk_{\hat{y}(\mathfrak{z})}}\right)\right)^{\omega_{v-\mathfrak{z}}}\right) \end{pmatrix} \end{cases}$$

where, $0 < \xi_{\text{ʒ}} \le \min_{\text{ʒ}}(\Pi_{RP-\text{Ꝁ}_{\text{ʒ}}},\ 1+\Pi_{RN-\text{Ꝁ}_{\text{ʒ}}},\ \Pi_{IP-\text{Ꝁ}_{\text{ʒ}}},\ 1+\Pi_{IN-\text{Ꝁ}_{\text{ʒ}}}) \le 1,\ \xi \neq 1$, and $0 < \frac{1}{\xi_{\text{ʒ}}} \le \min_{\text{ʒ}}(\Pi_{RP-\text{Ꝁ}_{\text{ʒ}}},\ 1+\Pi_{RN-\text{Ꝁ}_{\text{ʒ}}},\ \Pi_{IP-\text{Ꝁ}_{\text{ʒ}}},\ 1+\Pi_{IN-\text{Ꝁ}_{\text{ʒ}}}) \le 1, \xi \neq 1$.

Properties: L-BCFOWG operator holds the underneath axioms.

❖ Idempotency: Under the gathering of BCFNs $\text{Ꝁ}_{\text{ʒ}} = (\Pi_{P-\text{Ꝁ}_{\text{ʒ}}},\ \Pi_{N-\text{Ꝁ}_{\text{ʒ}}}) = (\Pi_{RP-\text{Ꝁ}_{\text{ʒ}}} + \iota\, \Pi_{IP-\text{Ꝁ}_{\text{ʒ}}},\ \Pi_{RN-\text{Ꝁ}_{\text{ʒ}}} + \iota\, \Pi_{IN-\text{Ꝁ}_{\text{ʒ}}})$, $\text{ʒ} = 1,\ 2,\ \ldots,\ \text{ӣ}$, then if $\text{Ꝁ}_{\text{ʒ}} = \text{Ꝁ}\ \forall\, \text{ʒ}$, that is $\Pi_{RP-\text{Ꝁ}_{\text{ʒ}}} = \Pi_{RP-\text{Ꝁ}}$, $\Pi_{IP-\text{Ꝁ}_{\text{ʒ}}} = \Pi_{IP-\text{Ꝁ}}$, $\Pi_{RN-\text{Ꝁ}_{\text{ʒ}}} = \Pi_{RN-\text{Ꝁ}}$ and $\Pi_{IN-\text{Ꝁ}_{\text{ʒ}}} = \Pi_{IN-\text{Ꝁ}}\ \forall\, \text{ʒ}$, then

$$L-BCFOWG\left(\text{Ꝁ}_1,\ \text{Ꝁ}_2,\ \ldots,\ \text{Ꝁ}_{\text{ӣ}}\right) = \log_{\xi}\text{Ꝁ}$$

Noted that $\xi_1 = \xi_2 = \ldots = \xi_{\text{ӣ}} = \xi$.

❖ Monotonicity: Under two gatherings of BCFNs $\text{Ꝁ}_{\text{ʒ}} = (\Pi_{P-\text{Ꝁ}_{\text{ʒ}}},\ \Pi_{N-\text{Ꝁ}_{\text{ʒ}}}) = (\Pi_{RP-\text{Ꝁ}_{\text{ʒ}}} + \iota\, \Pi_{IP-\text{Ꝁ}_{\text{ʒ}}},\ \Pi_{RN-\text{Ꝁ}_{\text{ʒ}}} + \iota\, \Pi_{IN-\text{Ꝁ}_{\text{ʒ}}})$, and $\text{Ꝁ}^{\star}_{\text{ʒ}} = (\Pi_{P-\text{Ꝁ}^{\star}_{\text{ʒ}}},\ \Pi_{N-\text{Ꝁ}^{\star}_{\text{ʒ}}}) = (\Pi_{RP-\text{Ꝁ}^{\star}_{\text{ʒ}}} + \iota\, \Pi_{IP-\text{Ꝁ}^{\star}_{\text{ʒ}}},\ \Pi_{RN-\text{Ꝁ}^{\star}_{\text{ʒ}}} + \iota\, \Pi_{IN-\text{Ꝁ}^{\star}_{\text{ʒ}}})$, $\text{ʒ} = 1,\ 2,\ \ldots,\ \text{ӣ}$, if $\Pi_{RP-\text{Ꝁ}_{\text{ʒ}}} \le \Pi_{RP-\text{Ꝁ}^{\star}_{\text{ʒ}}}$, $\Pi_{IP-\text{Ꝁ}_{\text{ʒ}}} \le \Pi_{IP-\text{Ꝁ}^{\star}_{\text{ʒ}}}$, $\Pi_{RN-\text{Ꝁ}_{\text{ʒ}}} \le \Pi_{RN-\text{Ꝁ}^{\star}_{\text{ʒ}}}$ and $\Pi_{IN-\text{Ꝁ}_{\text{ʒ}}} \le \Pi_{IN-\text{Ꝁ}^{\star}_{\text{ʒ}}}$, then

$$\begin{aligned} &L-BCFOWG\left(\text{Ꝁ}_1,\ \text{Ꝁ}_2,\ \ldots,\ \text{Ꝁ}_{\text{ӣ}}\right) \\ &\quad \le L-BCFOWG\left(\text{Ꝁ}^{\star}_1,\ \text{Ꝁ}^{\star}_2,\ \ldots,\ \text{Ꝁ}^{\star}_{\text{ӣ}}\right) \end{aligned}$$

❖ Boundedness: Under the gathering of BCFNs

$$\begin{aligned} \text{Ꝁ}_{\text{ʒ}} &= \left(\Pi_{P-\text{Ꝁ}_{\text{ʒ}}},\ \Pi_{N-\text{Ꝁ}_{\text{ʒ}}}\right) \\ &= \left(\Pi_{RP-\text{Ꝁ}_{\text{ʒ}}} + \iota\, \Pi_{IP-\text{Ꝁ}_{\text{ʒ}}},\ \Pi_{RN-\text{Ꝁ}_{\text{ʒ}}} + \iota\, \Pi_{IN-\text{Ꝁ}_{\text{ʒ}}}\right), \end{aligned}$$

$\text{ʒ} = 1,\ 2,\ \ldots,\ \text{ӣ}$, then if

$$\text{Ꝁ}^{-} = \begin{pmatrix} \min_{\text{ʒ}}\left\{\Pi_{RP-\text{Ꝁ}_{\text{ʒ}}}\right\} + \iota\ \min_{\text{ʒ}}\left\{\Pi_{IP-\text{Ꝁ}_{\text{ʒ}}}\right\}, \\ \max_{\text{ʒ}}\left\{\Pi_{RN-\text{Ꝁ}_{\text{ʒ}}}\right\} + \iota\ \max_{\text{ʒ}}\left\{\Pi_{IN-\text{Ꝁ}_{\text{ʒ}}}\right\} \end{pmatrix}, \text{ and}$$

$$\text{Ꝁ}^{+} = \begin{pmatrix} \max_{\text{ʒ}}\left\{\Pi_{RP-\text{Ꝁ}_{\text{ʒ}}}\right\} + \iota\ \max_{\text{ʒ}}\left\{\Pi_{IP-\text{Ꝁ}_{\text{ʒ}}}\right\}, \\ \min_{\text{ʒ}}\left\{\Pi_{RN-\text{Ꝁ}_{\text{ʒ}}}\right\} + \iota\ \min_{\text{ʒ}}\left\{\Pi_{IN-\text{Ꝁ}_{\text{ʒ}}}\right\} \end{pmatrix}, \text{ then}$$

$$\text{Ꝁ}^{-} \le L-BCFOWG\left(\text{Ꝁ}_1,\ \text{Ꝁ}_2,\ \ldots,\ \text{Ꝁ}_{\text{ӣ}}\right) \le \text{Ꝁ}^{+}$$

V. APPLICATION (MCDM APPROACH BASED ON LOGARITHM BCF AOS)

In this portion, we invent an MCDM approach relying on the devised logarithm BCF (L-BCF) AOs.

Under the presence of ӣ alternative that is $\mathcal{B}_{\mathbb{AT}} = \left\{\mathcal{B}_{\mathbb{AT}-1},\ \mathcal{B}_{\mathbb{AT}-2}, \ldots,\ \mathcal{B}_{\mathbb{AT}-\text{ӣ}}\right\}$ and Ɱ criteria that are $\mathcal{E}_{\mathbb{AB}} = \left\{\mathcal{E}_{\mathbb{AB}-1},\ \mathcal{E}_{\mathbb{AB}-2},\ \ldots,\ \mathcal{E}_{\mathbb{AB}-\text{Ɱ}}\right\}$ with a weight that is $\omega_{\mathbb{v}} = \left(\omega_{\mathbb{v}-1},\ \omega_{\mathbb{v}-2},\ \ldots,\ \omega_{\mathbb{v}-\text{ӣ}}\right)$ which holds the axiom that $0 \le \omega_{\mathbb{v}-\text{ƒ}} \le 1,\ \sum_{\text{ƒ}=1}^{\text{Ɱ}} \omega_{\mathbb{v}-\text{ƒ}} = 1$. The decision expert assesses the revealed alternatives by keeping in mind the demonstrated criteria and interprets the evaluation arguments under the model of BCFNs $\text{Ꝁ}_{\text{ʒƒ}} = \left(\Pi_{P-\text{Ꝁ}_{\text{ʒƒ}}},\ \Pi_{N-\text{Ꝁ}_{\text{ʒƒ}}}\right) = \left(\Pi_{RP-\text{Ꝁ}_{\text{ʒƒ}}} + \iota\, \Pi_{IP-\text{Ꝁ}_{\text{ʒƒ}}},\ \Pi_{RN-\text{Ꝁ}_{\text{ʒƒ}}} + \iota\Pi_{IN-\text{Ꝁ}_{\text{ʒƒ}}}\right)$, to devise a BCF decision matrix $\mathcal{M}_{\mathbb{DM}}$. For tackling this MCDM dilemma, we have the underneath steps

Step 1: In so many MCDM dilemmas, the attributes can be cost type and benefit type. Thus, the standardization of the BCF decision matrix is required, which will be done by the underneath formula.

$$(\mathcal{M}_{\mathbb{DM}})^{S} = \begin{cases} \begin{pmatrix} \Pi_{RP-\text{Ꝁ}_{\text{ʒƒ}}} \\ +\iota\ \Pi_{IP-\text{Ꝁ}_{\text{ʒƒ}}}, \\ \Pi_{RN-\text{Ꝁ}_{\text{ʒƒ}}} \\ +\iota\Pi_{IN-\text{Ꝁ}_{\text{ʒƒ}}} \end{pmatrix} & \textit{For benefit type} \\ \begin{pmatrix} 1-\Pi_{RP-\text{Ꝁ}_{\text{ʒƒ}}} \\ +\iota\ \left(1-\Pi_{IP-\text{Ꝁ}_{\text{ʒƒ}}}\right), \\ -1-\ \Pi_{RN-\text{Ꝁ}_{\text{ʒƒ}}} \\ +\iota\ \left(-1-\Pi_{IN-\text{Ꝁ}_{\text{ʒƒ}}}\right) \end{pmatrix} & \textit{For cost type} \end{cases}$$

Step 2: Aggregate the standardization BCF decision matrix by employing one of the invented operators i.e., L-BCFWA, L-BCFOWA, L-BCFWG, and L-BCFOWG operators.

Step 3: Achieve the score values of the aggregated outcomes and in case of the same score values of two distinct aggregated outcomes, then achieve the accuracy values.

Step 4: Order the alternatives by employing the score or accuracy values.

5.1. Case Study

To optimize the software development process for a future project, a choice of the best software development methodology is required. The four alternatives that are being considered are

$\mathcal{B}_{\mathbb{AT}-1}$**: Waterfall:** A methodological approach that is linear and sequential and requires completion of each phase (maintenance, design, deployment, implementation, testing, requirements) before moving on to the next. Once a phase is over, accommodating changes can be difficult.

$\mathcal{B}_{\mathbb{AT}-2}$**: DevOps:** DevOps is a set of practices that aims to close the gap between development and IT operations, despite not being a conventional methodology. To produce faster and more reliable software releases, it places a strong emphasis on automation, collaboration, and continuous delivery.

$\mathcal{B}_{\mathbb{AT}-3}$**: Spiral:** Waterfall and iterative development are both used in this methodology. It entails repeatedly iterating on the software to refine and improve it, building on the knowledge gained from the previous cycle.

$\mathcal{B}_{\mathbb{AT}-4}$**: Agile:** A collaborative, adaptable, and iterative strategy that prioritizes client feedback. By dividing the project into smaller iterations or sprints, agile approaches like Scrum and Kanban enable frequent modifications and continual development.

The decision will depend on four criteria that is $\mathcal{E}_{\mathbb{AB}-1}$: Project flexibility, $\mathcal{E}_{\mathbb{AB}-2}$: Development speed, $\mathcal{E}_{\mathbb{AB}-3}$: Communication and collaboration, and $\mathcal{E}_{\mathbb{AB}-4}$: Risk management.

A thorough evaluation will be carried out by comparing the four methodologies to these criteria and giving each criteria weight i.e., $(0.14,\ 0.36,\ 0.24,\ 0.26)$ based on the particulars of the project. The methodology that received the highest overall score will be suggested for the software development project, ensuring that the choice is well-informed and fits

TABLE 1. Evaluation values of software development methodologies.

	$Ɛ_{\mathbb{AB}-1}$	$Ɛ_{\mathbb{AB}-2}$	$Ɛ_{\mathbb{AB}-3}$	$Ɛ_{\mathbb{AB}-4}$
$Ɓ_{\mathbb{AT}-1}$	$\begin{pmatrix}0.33+\iota 0.57,\\-0.23-\iota 0.67\end{pmatrix}$	$\begin{pmatrix}0.42+\iota 0.37,\\-0.63-\iota 0.58\end{pmatrix}$	$\begin{pmatrix}0.73+\iota 0.48,\\-0.18-\iota 0.27\end{pmatrix}$	$\begin{pmatrix}0.64+\iota 0.35,\\-0.47-\iota 0.36\end{pmatrix}$
$Ɓ_{\mathbb{AT}-2}$	$\begin{pmatrix}0.67+\iota 0.39,\\-0.53-\iota 0.35\end{pmatrix}$	$\begin{pmatrix}0.83+\iota 0.38,\\-0.73-\iota 0.59\end{pmatrix}$	$\begin{pmatrix}0.29+\iota 0.83,\\-0.47-\iota 0.37\end{pmatrix}$	$\begin{pmatrix}0.67+\iota 0.25,\\-0.22-\iota 0.43\end{pmatrix}$
$Ɓ_{\mathbb{AT}-3}$	$\begin{pmatrix}0.46+\iota 0.36,\\-0.36-\iota 0.56\end{pmatrix}$	$\begin{pmatrix}0.46+\iota 0.89,\\-0.42-\iota 0.53\end{pmatrix}$	$\begin{pmatrix}0.26+\iota 0.58,\\-0.35-\iota 0.36\end{pmatrix}$	$\begin{pmatrix}0.25+\iota 0.47,\\-0.57-\iota 0.68\end{pmatrix}$
$Ɓ_{\mathbb{AT}-4}$	$\begin{pmatrix}0.68+\iota 0.7,\\-0.42-\iota 0.35\end{pmatrix}$	$\begin{pmatrix}0.64+\iota 0.46,\\-0.58-\iota 0.35\end{pmatrix}$	$\begin{pmatrix}0.68+\iota 0.35,\\-0.58-\iota 0.13\end{pmatrix}$	$\begin{pmatrix}0.46+\iota 0.79,\\-0.57-\iota 0.24\end{pmatrix}$

TABLE 2. The aggregated outcomes of software development methodologies after employing L-BCFWA, L-BCFOWA, L-BCFWG, and L-BCFOWG operators.

Operators	$Ɓ_{\mathbb{AT}-1}$	$Ɓ_{\mathbb{AT}-2}$	$Ɓ_{\mathbb{AT}-3}$	$Ɓ_{\mathbb{AT}-4}$
L-BCFWA	$\begin{pmatrix}0.743+\iota\,0.624,\\-0.216-\iota\,0.257\end{pmatrix}$	$\begin{pmatrix}0.827+\iota\,0.69,\\-0.287-\iota\,0.265\end{pmatrix}$	$\begin{pmatrix}0.552+\iota\,0.839,\\-0.244-\iota\,0.225\end{pmatrix}$	$\begin{pmatrix}0.788+\iota\,0.762,\\-0.35-\iota\,0.127\end{pmatrix}$
L-BCFOWA	$\begin{pmatrix}0.727+\iota\,0.632,\\-0.213-\iota\,0.273\end{pmatrix}$	$\begin{pmatrix}0.833+\iota\,0.623,\\-0.248-\iota\,0.251\end{pmatrix}$	$\begin{pmatrix}0.527+\iota\,0.757,\\-0.229-\iota\,0.218\end{pmatrix}$	$\begin{pmatrix}0.776+\iota\,0.788,\\-0.35-\iota\,0.121\end{pmatrix}$
L-BCFWG	$\begin{pmatrix}0.702+\iota\,0.611,\\-0.372-\iota\,0.298\end{pmatrix}$	$\begin{pmatrix}0.747+\iota\,0.589,\\-0.372-\iota\,0.282\end{pmatrix}$	$\begin{pmatrix}0.519+\iota\,0.764,\\-0.256-\iota\,0.251\end{pmatrix}$	$\begin{pmatrix}0.776+\iota\,0.708,\\-0.356-\iota\,0.141\end{pmatrix}$
L-BCFOWG	$\begin{pmatrix}0.685+\iota\,0.614,\\-0.263-\iota\,0.315\end{pmatrix}$	$\begin{pmatrix}0.783+\iota\,0.543,\\-0.33-\iota\,0.266\end{pmatrix}$	$\begin{pmatrix}0.495+\iota\,0.703,\\-0.355-\iota\,0.134\end{pmatrix}$	$\begin{pmatrix}0.761+\iota\,0.729,\\-0.355-\iota\,0.134\end{pmatrix}$

TABLE 3. The score values of software development methodologies.

Operators	$\dot{S}_{\mathbb{SF}}(Ɓ_{\mathbb{AT}-1})$	$\dot{S}_{\mathbb{SF}}(Ɓ_{\mathbb{AT}-2})$	$\dot{S}_{\mathbb{SF}}(Ɓ_{\mathbb{AT}-3})$	$\dot{S}_{\mathbb{SF}}(Ɓ_{\mathbb{AT}-4})$
L-BCFWA	0.723	0.741	0.731	0.768
L-BCFOWA	0.718	0.739	0.709	0.773
L-BCFWG	0.684	0.67	0.694	0.747
L-BCFOWG	0.68	0.682	0.678	0.75

TABLE 4. The ranking of software development methodologies.

Operators	Ranking
L-BCFWA	$Ɓ_{\mathbb{AT}-4} > Ɓ_{\mathbb{AT}-2} > Ɓ_{\mathbb{AT}-3} > Ɓ_{\mathbb{AT}-1}$
L-BCFOWA	$Ɓ_{\mathbb{AT}-4} > Ɓ_{\mathbb{AT}-2} > Ɓ_{\mathbb{AT}-3} > Ɓ_{\mathbb{AT}-1}$
L-BCFWG	$Ɓ_{\mathbb{AT}-4} > Ɓ_{\mathbb{AT}-2} > Ɓ_{\mathbb{AT}-3} > Ɓ_{\mathbb{AT}-1}$
L-BCFOWG	$Ɓ_{\mathbb{AT}-4} > Ɓ_{\mathbb{AT}-2} > Ɓ_{\mathbb{AT}-3} > Ɓ_{\mathbb{AT}-1}$

with the project's particular needs and limitations.Bottom of FormThe evaluated values are in Table 1.

Step 1: In this MCDM dilemma, all criteria are benefits type so skipping this step.

Step 2: Aggregated BCF decision matrix by employing L-BCFWA, L-BCFOWA, L-BCFWG, and L-BCFOWG operators, and the result is displayed in Table 2.

Step 3: Achieved the score values of each software development methodology and revealed in Table 3.

Step 4: In Table 4, order the software development methodology.

Thus, we got that the $Ɓ_{\mathbb{AT}-4}$ that is "Agile" is the best software development methodology among the considered four methodologies.

VI. COMPARATIVE STUDY

This portion of the article contains a comparative study of the invented theory with a few prevailing theories to reveal the supremacy and dominance of the invented work.

For this purpose, we consider four various articles from the literature whose basic theme is discussed underneath.

- The theory of logarithmic AOs and MADM technique under BF information was deduced by Jana et [20].
- The theory of sine trigonometric AOs and SIR technique under BF information was invented by Riaz et al. [19].
- The notion of arithmetic AOs in the setting of complex fuzzy information was deduced by Bi et [30].
- The notion of geometric AOs in the setting of complex fuzzy information was deduced by Bi et [31].

and also consider the information from the case study demonstrated in Table 1. As the data is in the model of bipolar complex fuzzy information and AOs and methods invented

TABLE 5. The comparison among a few current theories and devised theories.

Reference	$\dot{\acute{S}}_{\mathbb{SF}}(\mathcal{B}_{\mathbb{AT}-1})$	$\dot{\acute{S}}_{\mathbb{SF}}(\mathcal{B}_{\mathbb{AT}-2})$	$\dot{\acute{S}}_{\mathbb{SF}}(\mathcal{B}_{\mathbb{AT}-3})$	$\dot{\acute{S}}_{\mathbb{SF}}(\mathcal{B}_{\mathbb{AT}-4})$	Ranking
Jana et al. [20]	⌧⌧⌧	⌧⌧⌧	⌧⌧⌧	⌧⌧⌧	⌧⌧⌧
Riaz et al. [19]	⌧⌧⌧	⌧⌧⌧	⌧⌧⌧	⌧⌧⌧	⌧⌧⌧
Bi et al. [30]	⌧⌧⌧	⌧⌧⌧	⌧⌧⌧	⌧⌧⌧	⌧⌧⌧
Bi et al. [31]	⌧⌧⌧	⌧⌧⌧	⌧⌧⌧	⌧⌧⌧	⌧⌧⌧
L-BCFWA	0.723	0.741	0.731	0.768	$\mathcal{B}_{\mathbb{AT}-4} > \mathcal{B}_{\mathbb{AT}-2} > \mathcal{B}_{\mathbb{AT}-3} > \mathcal{B}_{\mathbb{AT}-1}$
L-BCFOWA	0.718	0.739	0.709	0.773	$\mathcal{B}_{\mathbb{AT}-4} > \mathcal{B}_{\mathbb{AT}-2} > \mathcal{B}_{\mathbb{AT}-3} > \mathcal{B}_{\mathbb{AT}-1}$
L-BCFWG	0.684	0.67	0.694	0.747	$\mathcal{B}_{\mathbb{AT}-4} > \mathcal{B}_{\mathbb{AT}-2} > \mathcal{B}_{\mathbb{AT}-3} > \mathcal{B}_{\mathbb{AT}-1}$
L-BCFOWG	0.68	0.682	0.678	0.75	$\mathcal{B}_{\mathbb{AT}-4} > \mathcal{B}_{\mathbb{AT}-2} > \mathcal{B}_{\mathbb{AT}-3} > \mathcal{B}_{\mathbb{AT}-1}$

by Jana et al. [20] and Riaz et al. [19] merely tackle the bipolar fuzzy and fuzzy information and can't handle the two-dimensional information. Thus, the theories of Jana et al. [20] and Riaz et al. [19] failed to cope with the data in Table 1. Similarly, the AOs invented by Bi et al. [30] and Bi et al. [31] merely aggregate complex fuzzy and fuzzy information and can't tackle the negative opinion or aspects. Thus, these AOs can't aggregate the data in the structure of BCFS and hence failed to cope with the data in Table 1. The result is interpreted in Table 5.

Further, the invented logarithmic BCF AOs are more generalized and accurate than the logarithmic AOs for bipolar fuzzy information invented by Jana et a. [20] because by removing the unreal part in the invented logarithmic BCF AOs, we would get the logarithmic AOs for bipolar fuzzy information invented by Jana al. [20], so the logarithmic AOs for BFS are the special case of the invented operators. Furthermore, by ignoring the negative degree of belonging the invented logarithmic BCF AOs would convert in the model of the cartesian form of CFS and by removing the negative degree of belonging and unreal part into the positive degree of belonging the deduced logarithmic BCF AOs would transform in the structure of FS.

Moreover, there are various techniques for tackling dilemmas in the literature such as SPOTIS, COMET, SIMUS, RANCOM, etc. Here, we would compare the invented MCDM approach with SPOTIS in the structure of FS was deduced by Shekhovtsov et al. [41], COMET for intuitionistic FS (IFS) was diagnosed by Faizi et al. [42], SIMUS in model of FS was delivered by Stoilova and Munier [43], and RANCOM in single-valued Neutrosophic set (SVNS) was deduced by Rani et al. [44]. These already existing theory can't tackle the information displayed in Table 1, because the theories of FS, IFS, and SVNS can't cope with bipolarity and extra fuzzy information. Thus, for the information in the structure of BCFS, these approaches are not applicable. Merely the invented MCDM can cope with BCF information.

VII. CONCLUSION

In this article, we invented LOLs for BCF information and associated properties. Then by employing these operations, we deduced four various AOs that are L-BCFWA, L-BCFOWA, L-BCFWG, and L-BCFOWG operators. we also investigated the associated axioms of these logarithmic AOs. After that, we devised an approach to MCDM under the setting of BCF information to cope with MCDM dilemmas. Further, in this script, we discussed the selection of software development methodology since, software development methodology is an MCDM dilemma because the selection of the best software development methodology would be based on various criteria which ensures that the decision is made in a fair and informed manner. Thus, we investigate case study related to the selection and prioritization of software development methodology. To demonstrate the significance and superiority of the constructed theory, we last looked into a comparison study of the inferred theory to a few other contemporary theories.

A. LIMITATIONS AND FUTURE DIRECTION

The invented theory can't overcome with information that is in the model of BCF linguistic set [45], BCF soft set [46], picture FS [47], complex picture FS [48], complex bipolar FS [49], bipolar complex spherical FS [50] and other generalization of BCFS. Because of this limitation and the benefits of logarithmic AOs, there is a deliberate plan to use them in a wider range of frameworks, such as BCF linguistic set, BCF soft se, picture FS, bipolar complex spherical FS, etc. By utilizing the natural advantages of logarithmic AOs, this extension aims to improve accuracy and efficiency in a range of computational tasks. Logarithmic AOs are expected to open up new possibilities for analysis and problem-solving, spurring innovation across several fields, when incorporated into these frameworks. These initiatives seek to meet difficult problems and promote improvements in both theoretical and practical research by utilizing the natural adaptability and durability of logarithmic AOs.

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