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RESEARCH ARTICLE

Identification of Continuous-Time Linear Parameter Varying Systems With Noisy Scheduling Variable Using Local Regression

ARTURO PADILLA [1], HUGUES GARNIER [2], (Member, IEEE), JUAN I. YUZ [3], (Senior Member, IEEE), FENGWEI CHEN [4], AND CARLOS MUÑOZ POBLETE [5], (Member, IEEE)

[1]Department of Mechanical Engineering, Universidad de La Frontera, Temuco 4811230, Chile
[2]Université de Lorraine, CNRS, CRAN, F-54000 Nancy, France
[3]Department of Electronic Engineering, Universidad Técnica Federico Santa María, Valparaíso 2390235, Chile
[4]School of Automation, Chongqing University, Chongqing 400044, China
[5]Department of Electrical Engineering, Universidad de La Frontera, Temuco 4811230, Chile

Corresponding author: Arturo Padilla (arturo.padilla@ufrontera.cl)

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ABSTRACT Some nonlinear systems can be represented through linear parameter varying models. In this work, we address the estimation of continuous-time linear parameter varying models in output error form, using a refined instrumental variable method. A distinguished feature of a linear parameter varying model is that it has parameters that depend on an external signal called the scheduling variable. In this paper, we assume that the scheduling variable is noisy, a condition which is often met in practice, but not frequently considered in the literature. On the other hand, there are applications in which the noise-free version of the scheduling variable is smooth. Under such scenario we can simply filter the scheduling variable before estimating the linear parameter model. Nonetheless, there are cases where special smoothing techniques are required. In this study, we consider one of these special cases, and we use the well-known local regression method as smoothing technique. A numerical example based on a Monte Carlo simulation shows the benefits of the proposed approach.

INDEX TERMS Continuous-time system identification, instrumental variable method, linear parameter varying model, local regression, smoothing method.

I. INTRODUCTION

Nature often exhibits a nonlinear behavior, and one way to represent some nonlinear systems is through linear parameter varying (LPV) models. A strong motivation for studying LPV models is that an accurate and at the same time low-complexity LPV model is of paramount importance in designing an efficient LPV controller [1]. Many control system design methods that use the LPV framework have been developed. An interesting survey on LPV control is given in [2]. In this work, we focus on the estimation of continuous-time LPV models in an open-loop setting through a data-driven approach.

Approaches that have been developed to estimate linear time-invariant (LTI) models have been extended to LPV representations. For instance, that is the case with the methods available to estimate the classical discrete-time (DT) LTI models in input-output (IO) form. One of these approaches that has been extended to the LPV case, is the prediction error minimization method. An overview of the prediction error

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minimization approach applied to the LPV IO framework is presented in [1]. Recently, a MATLAB toolbox for identifying LPV IO models, using the prediction error minimization approach, has been developed [3]. One of the classical DT LTI models is the output error (OE) representation, where it is assumed that the output measurement noise is white. The estimation of DT LPV-OE representations using the prediction error minimization approach has been considered in [4] and [5].

The dynamics of an LPV process depend on an independent external signal called scheduling variable, that can be measured, and is usually denoted by $p(t)$. LPV models have coefficients that depend on $p(t)$. If we assume that the scheduling variable is not available, the coefficients depend instead on time t, that is, we have to deal with a linear time-varying (LTV) system identification problem. The identification of LTV systems is typically done using recursive estimation method, where the forgetting factor or Kalman filter methods are considered to track the time-varying parameters [6]. Different studies have considered the estimation of LTV-OE models, that can be in DT [7] or continuous time (CT) [8].

In this work we focus in particular on CT LPV-OE models. In the case of LTI models, CT OE representations can be estimated using the simplified refined instrumental variable approach for CT models (SRIVC) [9], [10]. In [11], the authors developed the LPV-SRIVC method, which is the extension of SRIVC to CT LPV models. More recently, for the CT LTI case, the consistency and efficiency of the SRIVC approach has been analyzed in [12] and [13], respectively. An important contribution in these two studies is that they present the consistency and efficiency analysis of SRIVC taking into account the intersample behavior of the signals. In this article, we extend the LPV-SRIVC approach to a particular LPV system identification problem that is explained next.

The estimation of LPV models can be done assuming that the scheduling variable is noise-free. However, this assumption is not usually met in practice, since the scheduling variable is often related to a measured signal. The estimation of LPV models considering a noisy scheduling variable yields an identification problem in an errors-in-variables setting [14]. In the literature, this problem has already been addressed, both in open-loop [15] and closed-loop [16] settings. However, an important limitation of these methods is that the coefficients of the LPV model are constrained to polynomials in terms of the scheduling variable. Moreover, all these methods developed so far apply only to DT models. Such DT models can then be transformed into equivalent CT representations; this corresponds to the so-called indirect estimation of CT models. Nonetheless, such transformations are more complicated than in the LTI case [11]. This motivates the direct estimation of CT LPV models through approaches such as LPV-SRIVC.

In this paper, we propose a simple direct estimation method for CT LPV models having a noisy scheduling variable, assuming that the noise-free scheduling variable is smooth (or slowly time-varying). Such a condition is sometimes met in practice [17], [18]. For instance, in [18], the scheduling variable is the absolute speed of a ship, that can be represented using a smooth signal. In the current article, the first step of the proposed method consists in smoothing the scheduling variable.

A simple smoothing or denoising approach consists in filtering the noisy signal in both forward and reverse directions, in order to obtain a signal with zero-phase distortion [19]. Other possibilities include the use of splines or wavelets. In this paper we apply the local regression (LR) method, that has been extensively studied (see e.g. [20], [21], [22], [23]). In the LR method, smoothing of a signal is obtained by local polynomial approximations. The local regression approach has been used in different applications, such as the estimation of kernel density functions [20], frequency response functions [24], [25] and time-varying system [26].

In the LR method, an important hyperparameter is the so-called bandwidth, which defines the window size, i.e., the number of samples that are used for the local model approximation. The bandwidth, which defines the smoothness of the estimation, can be fixed or adaptive. The latter means that the window size varies depending on the location in the data sequence. The performance of the LR method can sometimes be improved by using an adaptive bandwidth. Reasons to use an adaptive bandwidth are: to adapt to the data distribution, to adapt to different levels of noise (heteroscedasticity), and to adapt to changes in the smoothness or curvature of the signals [27].

The contributions of this article are mentioned next:

- We propose a simple direct approach to estimate CT LPV-OE models where the scheduling variable is smooth but corrupted by noise. In the proposed method, the first step is to denoise the scheduling variable using the local regression (LR) approach. Then, the standard LPV-SRIVC method is used to estimate the CT LPV model.
- We assess the LR method in a scenario where it is appropriate to use an adaptive bandwidth, namely, a case where there is an important change in the smoothness of the signal to be denoised. Afterwards we assess the impact of that smoothness process in the estimation of CT LPV-OE models.

The remaining of the paper is organized as follows: the identification problem is formulated in Section II. Next, in Section III, the LPV-SRIVC method for estimating CT LPV-OE models is reviewed. Section IV is focused on the LR method. Firstly, in Section IV-A, we present briefly the proposed estimation method that combines the LPV-SRIVC and LR approaches. In Section IV-B, we show in detail how to smooth the scheduling variable using LR. The LR smoothing technique, using a fixed and adaptive bandwidth, is presented in Sections IV-C-IV-E. The proposed estimation approach and its different versions is summarized in Section IV-F, and it is tested in a numerical example in Section V. Finally, conclusions are drawn in Section VI.

II. PROBLEM FORMULATION

In this section, we formulate the identification problem of interest, namely, the structure of the continuous-time linear parameter varying model and the smooth scheduling variable corrupted by noise.

A. DATA GENERATING SYSTEM

As it is usual in the system identification literature, we assume that the data is generated from a *true system* $\mathcal{S}_o$. This is helpful in order to devise identification methods and understand their properties [6, p. 7 and 250]. Then, let us consider the following data-generating system, which is in a CT LPV-OE form,

$$\mathcal{S}_o \begin{cases} A_o(\mathscr{d}, p_o(t))x(t) = B_o(\mathscr{d}, p_o(t))u(t) \\ y(t_k) = x(t_k) + e_o(t_k) \end{cases} \tag{1}$$

where $\mathscr{d}$ denotes the differentiation operator w.r.t. time, $u(t)$ is the input of the plant, $p_o(t)$ is the noise-free scheduling variable, $x(t)$ is the noise-free output, $e_o(t_k)$ is the output noise and $y(t_k)$ is the noisy output. The sequence $e_o(t_k)$ is assumed to be a DT zero-mean white noise process with Gaussian distribution. Note that $x(t)$ represents a CT signal and $x(t_k)$ the DT counterpart (i.e. a sampled signal). The polynomials A_o and B_o are defined as follows

$$A_o(\mathscr{d}, p_o(t)) = \mathscr{d}^{n_a} + \sum_{i=1}^{n_a} a_i^o(p_o(t))\mathscr{d}^{n_a-i} \tag{2}$$

$$B_o(\mathscr{d}, p_o(t)) = \sum_{j=0}^{n_b} b_j^o(p_o(t))\mathscr{d}^{n_b-j} \tag{3}$$

where n_a, n_b are the polynomial degrees, and a_i^o, b_j^o are real-meromorphic functions with static dependence on p_o. These real-meromorphic functions can be for instance polynomials functions, trigonometric expressions, or rational exponential functions [28, p. 50].

B. MODEL STRUCTURE FOR IDENTIFICATION

While the LPV system depends on the noise-free scheduling variable $p_o(t)$ (see (1)), the LPV model depends on $p(t)$, which is equal to $p_o(t)$ plus some noise sequence. Then, the model is represented as follows

$$\mathcal{M} \begin{cases} A(\mathscr{d}, p(t), \theta)x(t) = B(\mathscr{d}, p(t), \theta)u(t) \\ y(t_k) = x(t_k) + e(t_k) \end{cases} \tag{4}$$

where $e(t_k)$ is assumed to be a zero-mean, normally distributed, DT white noise sequence. The polynomials A and B are given by

$$A(\mathscr{d}, p(t), \theta) = \mathscr{d}^{n_a} + \sum_{i=1}^{n_a} a_i(p(t))\mathscr{d}^{n_a-i} \tag{5}$$

$$B(\mathscr{d}, p(t), \theta) = \sum_{j=0}^{n_b} b_j(p(t))\mathscr{d}^{n_b-j} \tag{6}$$

where $a_i(p(t))$ and $b_j(p(t))$ are parameterized as follows

$$a_i(p(t)) = a_{i,0} + \sum_{l=1}^{n_\alpha} a_{i,l} f_l(p(t)) \tag{7}$$

$$b_j(p(t)) = b_{j,0} + \sum_{l=1}^{n_\beta} b_{j,l} g_l(p(t)) \tag{8}$$

where $\{f\}_{l=1}^{n_\alpha}$ and $\{g\}_{l=1}^{n_\beta}$ are meromorphic functions with static dependence on $p(t)$. The model parameters are stacked together in the parameter vector θ, i.e.

$$\boldsymbol{\theta} = \left[\mathbf{a}_1^T \ \dots \ \mathbf{a}_{n_a}^T \ \mathbf{b}_1^T \ \dots \ \mathbf{b}_{n_b}^T \right] \tag{9}$$

where

$$\mathbf{a}_i = \left[a_{i,0} \ a_{i,1} \ \dots \ a_{i,n_\alpha}\right]^T \tag{10}$$

$$\mathbf{b}_j = \left[b_{j,0} \ b_{j,1} \ \dots \ b_{j,n_\beta}\right]^T. \tag{11}$$

It is assumed that the scheduling variable is corrupted with noise $\gamma_o(t_k)$, i.e.,

$$p(t_k) = p_o(t_k) + \gamma_o(t_k) \tag{12}$$

Then, the following assumptions are made:

A1. The scheduling variable $p(t_k)$ is composed of a noise-free scheduling variable $p_o(t_k)$ and some additive DT white Gaussian noise $\gamma_o(t_k)$, with zero-mean, and unknown variance $\sigma_{\gamma_o}^2$.

A2. $\gamma_o(t_k)$ is independent of output measurement noise $e_o(t_k)$.

A3. The noise-free scheduling variable $p_o(t_k)$ is a smooth function. In our developments, a function is smooth in the sense that the signal is twice differentiable, i.e., the second derivative exists and is finite.

A4. The system belongs to the model set, i.e. $\mathcal{S}_o \in \mathcal{M}$. This means that there exist functions f_l and g_l in (7) and (8), and a parameter vector θ in (9) such that the model structure $\mathcal{M}$ in (4) matches exactly the system $\mathcal{S}_o$ in (1). Note that this assumption is not particularly realistic in practice, but it is quite useful to assess the estimated models [6, p. 250].

Let us define the set of available measurements by $\mathcal{D}_N = \{u(t_k), y(t_k), p(t_k)\}_{k=1}^N$. Then, the identification problem we consider in this paper is to estimate the plant parameter vector θ based on the data set $\mathcal{D}_N$, under the Assumptions A1-A4.

III. THE LPV-SRIVC METHOD FOR LPV-OE IDENTIFICATION

In this section, we briefly present the LPV-SRIVC method for the case in which the scheduling variable is noise free, i.e. $p(t_k) = p_o(t_k)$, and we consider assumption A4 only. For more details about LPV-SRIVC see [11].

In order to apply LPV-SRIVC, we use (5)-(8) to rewrite model (4) as follows

$$\mathcal{M}\begin{cases} \underbrace{\mathscr{d}^{n_a}x(t) + \sum_{i=1}^{n_a} a_{i,0}\mathscr{d}^{n_a-i}x(t)}_{F(\mathscr{d},\boldsymbol{\theta})x(t)} \\ + \sum_{i=1}^{n_a}\sum_{l=1}^{n_\alpha} a_{i,l}\underbrace{f_l(p(t))\mathscr{d}^{n_a-i}x(t)}_{x_{i,l}(t)} \\ = \sum_{j=0}^{n_b}\sum_{l=0}^{n_\beta} b_{j,l}\underbrace{g_l(p(t))\mathscr{d}^{n_b-j}u(t)}_{u_{j,l}(t)} \\ y(t_k) = x(t_k) + e(t_k) \end{cases} \tag{13}$$

where $g_0 = 1$. Note that in (13) we define,

$$F(\mathscr{d}, \boldsymbol{\theta}) = \mathscr{d}^{n_a} + \sum_{i=1}^{n_a} a_{i,0}\mathscr{d}^{n_a-i}. \tag{14}$$

Solving for the variable x in the first equation in (13), and replacing it in the second equation in (13), it is possible to formulate the following model

$$y(t_k) = -\sum_{i=1}^{n_a}\sum_{l=1}^{n_\alpha} \frac{a_{i,l}}{F(\mathscr{d},\boldsymbol{\theta})}x_{i,l}(t_k) + \sum_{j=0}^{n_b}\sum_{l=0}^{n_\beta} \frac{b_{j,l}}{F(\mathscr{d},\boldsymbol{\theta})}u_{j,l}(t_k) + e(t_k) \tag{15}$$

which corresponds to a MISO-LTI representation. Multiplying (15) by $F(\mathscr{d}, \boldsymbol{\theta})$, and solving for the variable $\mathscr{d}^{n_a}y(t_k)$, the model can be written in linear regression from

$$\mathscr{d}^{n_a}y(t_k) = \boldsymbol{\varphi}^T(t_k)\boldsymbol{\theta} + \tilde{e}(t_k) \tag{16}$$

where $x_{i,l}$ and $u_{j,l}$ are defined in (13), $\boldsymbol{\theta}$ is defined in (9),

$$\boldsymbol{\varphi}(t_k) = \big[-\mathscr{d}^{n_a-1}y(t_k)\ \dots\ -y(t_k)\ \ -x_{1,1}(t_k)\ \dots\ -x_{n_a,n_\alpha}(t_k)\ \ u_{0,0}(t_k)\ \dots\ u_{n_b,n_\beta}(t_k)\big]^T \tag{17}$$

and $\tilde{e}(t_k) = F(\mathscr{d}, \boldsymbol{\theta})e(t_k)$. It is important to note that it is not possible to directly estimate the parameters from (16), because both the time-derivatives, and $x_{i,l}$ are not available. The LPV-SRIVC method is an iterative method that has been devised to circumvent these issues. The LPV-SRIVC estimate of the plant model parameter vector at iteration $\tau + 1$ is given by

$$\hat{\boldsymbol{\theta}}^{(\tau+1)} = \left[\frac{1}{N}\sum_{k=1}^{N}\hat{\mathbf{z}}_f(t_k)\hat{\boldsymbol{\varphi}}_{\mathrm{f}}^T(t_k)\right]^{-1}\left[\frac{1}{N}\sum_{k=1}^{N}\hat{\mathbf{z}}_f(t_k)y_{\mathrm{f}}^{(n_a)}(t_k)\right] \tag{18}$$

where

$$y_{\mathrm{f}}^{(n_a)}(t_k) = \mathscr{d}^{n_a}Q_{\mathscr{d}}(\mathscr{d}, \hat{\boldsymbol{\theta}}^{(\tau)})y(t_k) \tag{19}$$

$Q_{\mathscr{d}}$ is a filter that is defined using an estimate of $\boldsymbol{\theta}$ at iteration τ, i.e.

$$Q_{\mathscr{d}}(\mathscr{d}, \hat{\boldsymbol{\theta}}^{(\tau)}) = \frac{1}{F(\mathscr{d}, \hat{\boldsymbol{\theta}}^{(\tau)})} \tag{20}$$

The estimated filtered regressor $\hat{\boldsymbol{\varphi}}_{\mathrm{f}}(t_k)$ is defined as follows

$$\hat{\boldsymbol{\varphi}}_{\mathrm{f}}(t_k) = \Big[-y_{\mathrm{f}}^{(n_a-1)}(t_k)\ \dots\ -y_{\mathrm{f}}(t_k)\ \ -\hat{x}_{1,1}^{\mathrm{f}}(t_k)\ \dots\ -\hat{x}_{n_a,n_\alpha}^{\mathrm{f}}(t_k)\ \ u_{0,0}^{\mathrm{f}}(t_k)\ \dots\ u_{n_b,n_\beta}^{\mathrm{f}}(t_k)\Big]^T \tag{21}$$

where

$$y_{\mathrm{f}}^{(n_a-i)}(t_k) = \mathscr{d}^{n_a-i}Q_{\mathscr{d}}(\mathscr{d}, \hat{\boldsymbol{\theta}}^{(\tau)})y(t_k) \tag{22}$$

with $i = 1, \dots, n_a$,

$$\hat{x}_{i,l}^{\mathrm{f}}(t_k) = Q_{\mathscr{d}}(\mathscr{d}, \hat{\boldsymbol{\theta}}^{(\tau)})\hat{x}_{i,l}(t_k) \tag{23}$$

with $i = 1, \dots, n_a$, $l = 1, \dots, n_\alpha$

$$u_{j,l}^{\mathrm{f}}(t_k) = Q_{\mathscr{d}}(\mathscr{d}, \hat{\boldsymbol{\theta}}^{(\tau)})u_{j,l}(t_k) \tag{24}$$

with $j = 0, \dots, n_b$, $l = 0, \dots, n_\beta$. The estimate $\hat{x}_{i,l}(t_k)$ in (23) is computed using $\hat{x}(t_k)$ obtained from the following auxiliary model

$$A(\mathscr{d}, p(t), \hat{\boldsymbol{\theta}}^{(\tau)})\hat{x}(t) = B(\mathscr{d}, p(t), \hat{\boldsymbol{\theta}}^{(\tau)})u(t). \tag{25}$$

The estimated filtered instrument vector $\hat{\mathbf{z}}_f(t_k)$ is defined as follows

$$\hat{\mathbf{z}}_f(t_k) = \Big[-\hat{x}_{\mathrm{f}}^{(n_a-1)}(t_k)\ \dots\ -\hat{x}_{\mathrm{f}}(t_k)\ \ -\hat{x}_{1,1}^{\mathrm{f}}(t_k)\ \dots\ -\hat{x}_{n_a,n_\alpha}^{\mathrm{f}}(t_k)\ \ u_{0,0}^{\mathrm{f}}(t_k)\ \dots\ u_{n_b,n_\beta}^{\mathrm{f}}(t_k)\Big]^T \tag{26}$$

where $\hat{x}_{\mathrm{f}}(t_k)$ is the filtered version of $\hat{x}(t_k)$, obtained from (25).

The algorithm to compute the LPV-SRIVC estimate defined in (18) is given in [11], where this approach has also been adapted to estimate LPV-Box-Jenkins models.

Regarding hyperparameter selection for LPV-SRIVC, we have to define the initial estimate $\hat{\boldsymbol{\theta}}^{(0)}$. that is automatically obtained using LTI-SRIVC, which does not involve any initialization choices. Note that the hyperparameters of LPV-SRIVC are related to the model structure $\mathcal{M}$, namely n_a, n_b, f_l, g_l, n_α and n_β. However, in this study we assume that $\mathcal{S}_o \in \mathcal{M}$. Therefore, these hyperparameters are a priori chosen and remain fixed.

IV. THE LOCAL REGRESSION METHOD

A. LPV-SRIVC USING A NOISY SCHEDULING VARIABLE

In this section, we discuss how to apply the LPV-SRIVC method when the scheduling variable is noisy. That is, we address the identification problem under assumptions A1 to A4. If the scheduling variable is noisy, it is possible to use a smoothing approach to obtain a reliable estimate of the noise-free scheduling variable $\hat{p}(t_k)$, and then use LPV-SRIVC as usual.

As smoothing approach we use the local regression (LR) method, which is a nonparametric algorithm that consists in fitting local polynomials to segments of data using weighted least-squares. In this study, we apply LR to smooth the scheduling variable $p(t_k)$, assuming that the noise-free scheduling variable $p_o(t_k)$ is smooth (Assumption A3), meaning that its second derivative exists and is finite. As a consequence, the smoothness (or curvature) of $p_o(t_k)$ can be quantified through its second derivative (see Section IV-D). The smoothness assumption allows us to rely on a local polynomial regression to approximate the underlying noise-free scheduling variable. In fact, Weierstrass's theorem [29] states that a continuous function in a finite closed interval can be approximated to any desired accuracy using polynomials, an observation that has been previously made in the literature [30]. Next we present the LR approach in detail. Note that in Section IV-F, we summarize the proposed algorithm and its different versions.

B. SMOOTHING THE SCHEDULING VARIABLE USING LR

We want to obtain an estimated signal $\hat{p}(t_k)$ from the signal measurement $p(t_k)$ using the LR smoothing approach. To that end, let us consider a segment of $p(t_k)$, which is a neighborhood around a certain data point or sample s. We assume that the segment or window has a length $N_l = 2h + 1$, where h is the so-called bandwidth, which we define in terms of number of samples. Then, for the window of size N_l centered at sample s, the local polynomial approximation of the scheduling variable can be written as follows

$$m(t_k, t_s) = \sum_{i=0}^{l_\beta} \beta_i(t_s)[k-s]^i. \tag{27}$$

As an example, in Figure 1, we show an estimated local polynomial approximation $\hat{m}(t_k, t_s)$ centered at sample s, considering a polynomial degree 2, and bandwidth $h = 3$ (i.e., window size $N_l = 7$). Note that for one local approximation we keep only the estimate at s, i.e. $\hat{m}(t_s, t_s)$. Thus, the estimates $\{\hat{p}(t_k)\}_{k=1}^N$ are given by

$$\left[\hat{p}(t_1)\ \hat{p}(t_2)\ \dots\ \hat{p}(t_N)\right]^T \tag{28}$$
$$= \left[\hat{m}(t_1, t_1)\ \hat{m}(t_2, t_2)\ \dots\ \hat{m}(t_N, t_N)\right]^T \tag{29}$$

The parameters $\beta_i(t_s)$ can be gathered in the parameter vector $\boldsymbol{\beta}(t_s) \in \mathbb{R}^{l_\beta+1}$. The weighted least-squares estimate

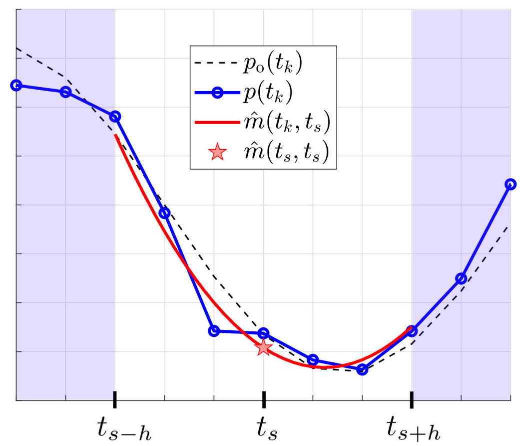

FIGURE 1. Example of an estimated local polynomial approximation $\hat{m}(t_k, t_s)$ centered at s, considering a polynomial degree 2, and bandwidth $h = 3$ (window size $N_l = 7$).

for $\boldsymbol{\beta}(t_s)$ is defined by

$$\hat{\boldsymbol{\beta}}(t_s) = \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^{l_\beta+1}} V_\beta(t_s). \tag{30}$$

$V_\beta(t_s)$ is the cost function,

$$V_\beta(t_s) = \frac{1}{N_l}\sum_{k=s-h}^{s+h} w(r)\varepsilon^2(t_k, \boldsymbol{\beta}(t_s)) \tag{31}$$

where $w(r)$ is a weighting function (also called kernel) with $r = (k-s)/h$, and

$$\varepsilon(t_k, \boldsymbol{\beta}(t_s)) = p(t_k) - m(t_k, t_s). \tag{32}$$

Then, the estimate can be expressed by

$$\hat{\boldsymbol{\beta}}(t_s) = (X^T W X)^{-1} X^T W \mathbf{P} \tag{33}$$

where $W = \text{diag}([w((s-h)/h),\ \dots\ , w((s+h)/h)])$, $\mathbf{P} = [p(t_{s-h})\ \dots p(t_{s+h})]^T$, and X is a Vandermonde type matrix,

$$X = \begin{bmatrix} 1 & (s-h) & \dots & (s-h)^{l_\beta} \\ \vdots & \vdots & & \vdots \\ 1 & (s+h) & \dots & (s+h)^{l_\beta} \end{bmatrix} \tag{34}$$

which can also be written as follows,

$$X = \begin{bmatrix} \mathcal{B}^T(t_{s-h}) \\ \vdots \\ \mathcal{B}^T(t_{s+h}) \end{bmatrix}. \tag{35}$$

Hyperparameters of the LR method are the weight function $w(r)$, the degree of the polynomial l_β, and the bandwidth h. Let us discuss next how to choose them.

Regarding the user choice $w(r)$, different weighting functions can be applied. A simple option is the rectangular weight function ($w(r) = 1$), which yields a 'noisy' estimate, i.e., an estimate with a large variance. This is because all observations within a distance h receive a weight 1 and the rest weight 0, meaning that observations abruptly switch in and out of the smoothing window [31]. Two important requirements for a weighting function are the following [27], [31]: (i) it should have a peak at $r = 0$ (the window center) and decay smoothly to 0 as r increases; (ii) it should go to zero at a finite distance r allowing faster implementations. Although there are different weighting functions that give similar results, and satisfy these requirements, it has been shown that the Epanechnikov function is optimal in a mean square error sense [27]. Thus, we consider in this study the Epanechnikov function [32], which is defined as

$$w(r) = \begin{cases} 0.75(1 - r^2) & \text{for } |r| < 1 \\ 0 & \text{elsewhere.} \end{cases} \tag{36}$$

Regarding the second user choice l_β, on the one hand, there is a risk to run into overfitting using a high-order polynomial. But on the other hand, in comparison to a low-order polynomial, a high-order polynomial will usually yield an estimate with less bias, at the expense of more variance. Thus, the choice of the polynomial degree l_β is a bias-variance trade-off [22]. Usual values for l_β are 1, 2 or 3. In [21] and [32] it is suggested to consider l_β an odd number [21, p. 78] because of the following: when moving from a fit using an even order polynomial to the consecutive odd order polynomial, there is no loss in terms of asymptotic variance. Nonetheless, the odd order polynomial provides less bias. As a consequence, in this study we consider $l_\beta = 3$.

The third hyperparameter, the bandwidth h, defines the smoothness of the estimation. A small h will result in a large variance in the estimates, whereas a large h will lead to a large bias. Thus, as it is the case for the choice of the polynomial degree, the choice of bandwidth has to take into account a bias-variance trade-off [22]. Note that the weighting function (36) is symmetric. Therefore, the window of the local approximation has to contain $N_l = 2h + 1$ samples, where N_l is an odd number.

In LR, the polynomial degree l_β and the bandwidth h are important hyperparameters. Thus, given a certain weight function, different alternatives have been considered to find optimal values for l_β and h, namely: (i) find optimal bandwidth assuming that the polynomial degree is fixed [21], [22], [23], [30], [33], [34]; (ii) find optimal degree assuming that the bandwidth is fixed [30], [34]; and (iii) simultaneously find an optimal bandwidth and degree [34]. In [34], these three alternatives are tested, concluding that alternative (iii) does not provide significant improvements over alternatives (i) or (ii). In our study, we focused on the impact of the bandwidth, i.e., we consider alternative (i) which has also been more frequently considered in the literature.

Two different approaches can be used to find optimal bandwidths. One is the *fixed bandwidth selection* approach, where an optimal and unique bandwidth is obtained for the whole signal. The other is the *adaptive bandwidth selection* approach, where local optimal bandwidths can be obtained. These two bandwidth selection approaches are discussed next.

C. FIXED BANDWIDTH SELECTION

Different methods have been developed for the fixed bandwidth selection problem (see e.g. [21], [22], [23], [33]). Two well-known types of approaches are:

- Classical approaches which are extensions of model selection methods used in parametric statistics, such as cross-validation, Akaike information criterion, or Mallow's C_p.
- Plug-in approaches. These methods are based on the simple idea of 'plugging in' estimates of the unknown quantities that appear in formulas for the asymptotically optimal bandwidth [20, p. 71]. In order to compute such estimates, a preliminary smoothing procedure is done using a so-called pilot bandwidth. Some plug-in approaches rely on Taylor expansions to approximate the mean squared error. However, such approximations are valid for small bandwidth [35]. In addition, plug-in algorithms are complex [36].

Next we present two fixed bandwidth selection approaches that are used in this study. They have been chosen because of their simplicity in comparison to plug-in methods. The first approach has been used as a pilot bandwidth in plug-in methods. The second one is a classical approach.

1) RESIDUAL SQUARES CRITERION

The residual squares criterion (RSC), which is presented in detail in [21] and [32], provides a local estimate of the mean squared error, where the error is the difference $p(t_k) - m(t_k, t_s)$. RSC can be used as a local goodness of fit, and it is defined as follows:

$$\text{RSC}(t_s, h) = \hat{\sigma}^2\{1 + (l_\beta + 1)V\} \tag{37}$$

where $\hat{\sigma}^2$ is the normalized weighted residual sum of squares,

$$\hat{\sigma}^2 = \frac{\sum_{k=s-h}^{s+h} \varepsilon^2(t_k, \boldsymbol{\beta}(t_s))w(r)}{\text{tr}\{W - WX(X^TWX)^{-1}X^TW\}} \tag{38}$$

and V is the first diagonal element of the matrix $(X^TWX)^{-1}(X^TW^2X)(X^TWX)^{-1}$.

For a given signal, an optimal bandwidth can be obtained by solving the following minimization problem

$$\hat{h} = \arg\min_{h \in H} \sum_{s=1}^{N} \text{RSC}(t_s, h) \tag{39}$$

where

$$H \in \left[h_{\min}\ h_{\max}\right]. \tag{40}$$

The RSC bandwidth selector is then $\hat{h}_{\text{RSC}} = \text{adj}\ \hat{h}$, with adj an adjusting constant that depends on l_β and the kernel $w(r)$. As discussed in Section IV-B, in this study we consider $l_\beta = 3$, and the Epanechnikov function. Therefore, the adjusting constant is adj $= 0.8718$. The LR method applied using a fixed bandwidth and RSC is denoted by LR-RSC-F.

Regarding the optimization problem (39), note that for any fixed h, the parameter vector $\boldsymbol{\beta}(t_s)$ in (33) can be readily obtained. Thus, a simple and fast approach presented in [21] and [32] is to numerically search for $\hat{h}$ by evaluating the cost function starting from $h = h_{\min}$, and then successively increase h by a factor $C > 1$ (i.e., $h_j = C^j h_{\min}$, $j = 1, 2, \ldots$), until a (local) minimum is found or until $h > h_{\max}$. Alternatively, a linear search is also possible, iterating the bandwidth increasing 1 sample every time, i.e., $h_j = h_{j-1} + 1$. In this study, we have chosen the second option that gives a more precise result at the expense of more computation time.

The choice of $h_{\min}$ in (40) depends on the polynomial degree l_β. In (33), to ensure that $X^T W X$ is non-singular, we need at least $l_\beta + 1$ samples [34], i.e. $h_{\min} \geq l_\beta/2$. Note, however, that for the Epanechnikov kernel, $w(r = 1) = w(r = -1) = 0$. Thus, $h_{\min} \geq l_\beta/2 + 1$. Regarding $h_{\max}$ in (40), we can consider the maximum possible value, i.e. $h_{\max} = \lfloor (N-1)/2 \rfloor$, where N is the total number of samples, and $\lfloor \cdot \rfloor$ is the floor function.

2) CROSS-VALIDATION

Another local estimate of the mean squared error, which is well-known, is the (local) cross-validation index [22, p. 198],

$$\mathrm{CV}(t_s, h) = \frac{1}{\sum_{k=s-h}^{s+h} w(r)} \sum_{k=s-h}^{s+h} w(r) \frac{(p(t_k) - m(t_k, t_s))^2}{(1 - \mathrm{infl}(s, k))^2} \tag{41}$$

where $\mathrm{infl}(s, k)$ is the influence function defined by

$$\mathrm{infl}(s, k) = \mathcal{B}^T(t_{k-s})(X^T W X)^{-1} \mathcal{B}(t_{k-s}) w(r) \tag{42}$$

with $\mathcal{B}$ given in (35).

Analogous to the RSC optimal bandwidth, a CV optimal bandwidth can be obtained from the following optimization problem:

$$\hat{h}_{\mathrm{CV}} = \arg\min_{h \in H} \sum_{s=1}^{N} \mathrm{CV}(t_s, h). \tag{43}$$

As for the RSC optimization problem (39), for the CV optimization problem (43) we also use $l_\beta = 3$, and the Epanechnikov kernel. The LR method applied using a fixed bandwidth and CV is denoted by LR-CV-F. The optimization problem (43) is solved using a numerical search, in the same way (39) is solved.

Note that the advantage of both indexes RSC and CV is that they do not involve any hyperparameter. For instance, this is not the case for Mallow's C_p [22].

D. ADAPTIVE BANDWIDTH SELECTION

Usually, a suitable smoothing is obtained using a fixed bandwidth. However, if for instance the curvature of the signal is significantly varying, then an adaptive bandwidth should work better. Note that a measure of the smoothness (or curvature) of a signal can be obtained through its second derivative [37, p. 151], [38, p. 212] (see more in the numerical example in Section V). The indexes presented above (RSC and CV) can be used to find an optimal bandwidth for each sample. However, that strategy yields a significant variability in the bandwidth, which deteriorates the smoothness of $\hat{p}(t_k)$. To circumvent that problem, we consider the following two-step approach proposed in [21] and [32]:

1) The whole signal is split up in intervals and an optimal bandwidth is found for each of them. Let us denote by N_{interval} the number of samples of the intervals. Then, it is suggested to split up the signal in $N/(10 \log N)$, with log the natural logarithm. Thus, we use

$$N_{\mathrm{interval}} = \lfloor 10 \log N \rfloor \tag{44}$$

2) The obtained varying bandwidth is considered as a signal $h(t_k)$ to be smoothed. As a filter, it is proposed to simply use a local average over N_{interval} samples.

The LR method applied using an adaptive bandwidth and the two indexes presented above are denoted by LR-RSC-A and LR-CV-A.

This variable bandwidth approach involves only the hyperparameter N_{interval}. Note that, the smaller N_{interval}, the greater the variability (adaptability) of $h(t_k)$ that we can expect. Nonetheless, if N_{interval} is too small, the smoothness of $\hat{p}(t_k)$ is deteriorated (see Section V). On the other hand, in the second step of this approach, a different filter could be used. However that might involve additional hyperparameters.

E. DISCUSSION ABOUT THE ADAPTIVE BANDWIDTH APPROACH

The LR method with fixed bandwidth and rectangular weight function is equivalent to the well-known Savitzky-Golay filter, which corresponds to a FIR filter [38], [39]. Note that when this filter is applied, the filtered data is slid to get a zero-phase response.

In [39], the author provides a plot showing the relationship between the polynomial degree, the bandwidth and the 3 dB cutoff frequency of the corresponding Savitzky-Golay filter. From that analysis, it is clear that the LR method with adaptive bandwidth and/or adaptive polynomial degree is actually an adaptive filter.

As stated in the introduction of the paper, reasons to use an adaptive bandwidth are the following scenarios:

i) to deal with data that is irregularly sampled over time,
ii) to deal with noise in the scheduling variable that may change over time (heteroscedasticity), and
iii) to be able to estimate scheduling variables with smoothness (or curvature, measured by its second order derivative) that may significantly change over time (such as the Doppler function in Section V).

In the numerical example presented in Section V, we have selected scenario (iii) to illustrate the benefits of the adaptive bandwidth approach.

Note, on the one hand, that LR with adaptive bandwidth is suitable for irregularly sampled data (scenario (i)). On the other hand, CT model estimation methods, such as

LPV-SRIVC, can easily handle problems with irregularly sample data [40]. Thus, we can state, that the proposed approach for estimating CT LPV-OE models with noisy scheduling variable, is suitable for the irregularly sampled data case. In this study, for simplicity of exposition, we consider only uniformly sampled data.

F. THE PROPOSED APPROACH AND ITS DIFFERENT VERSIONS

In this section we summarize the proposed method and its different versions for estimating CT LPV-OE models in the particular case when the scheduling variable is noisy. The proposed approach consists of the following two-step procedure:

1) Apply the LR method in order to smooth $p(t_k)$, obtaining an estimate of the noise-free scheduling variable $\hat{p}(t_k)$.
2) Apply the LPV-SRIVC method as usual using the measurements $u(t_k)$ and $y(t_k)$, and the estimate of the noise-free scheduling variable $\hat{p}(t_k)$.

For the smoothing procedure in step 1, the LR method is considered using both a fixed (F), or adaptive (A) bandwidth. In order to define a fixed or adaptive bandwidth we apply two approaches: residual squares criterion (RSC) and cross-validation (CV). This yields the following 4 possible smoothing approaches that are tested in the numerical example in Section V:

1) LR-RSC-F
2) LR-CV-F
3) LR-RSC-A
4) LR-CV-A

V. NUMERICAL EXAMPLE

In this section, a numerical example is presented to show the performance and benefits of the proposed method.

A. DATA GENERATING SYSTEM

The data generating system is described by (1) where

$$a_1^{\text{o}}(p_{\text{o}}(t)) = 2 - 1.5\, p_{\text{o}}(t) + 2p_{\text{o}}^2(t) \tag{45}$$
$$a_2^{\text{o}}(p_{\text{o}}(t)) = 5 + 3\, p_{\text{o}}(t) \tag{46}$$
$$b_1^{\text{o}}(p_{\text{o}}(t)) = 3 + 2\cos(p_{\text{o}}(t)) \tag{47}$$
$$b_2^{\text{o}}(p_{\text{o}}(t)) = 5 - 3\sin(2p_{\text{o}}(t)). \tag{48}$$

The sampling time is $T_s = 0.005$ s and the simulation time $T_f = 15$ s. The input is a white noise with uniform distribution $\mathcal{U}(-1.5, 1.5)$. Regarding the noise-free scheduling variable $p_{\text{o}}(t)$, it is the Doppler function similar to the one in [21, p. 130],

$$p_{\text{o}}(t) = 0.5 + (24/10)\sqrt{t'(1-t')}\sin(2\pi 1.05/(t' + 0.05))$$

where $t' = t/T_f$. The scheduling variable is corrupted by a white noise with Gaussian distribution $\mathcal{N}(0, 0.06^2)$. The signal $e_{\text{o}}(t_k)$ is a white noise with Gaussian distribution $\mathcal{N}(0, 0.01^2)$. This corresponds to the following signal-to-noise-ratios:

$$\text{SNR}_p = 10\log\frac{\sum_{k=1}^{N}[p_{\text{o}}(t_k) - \bar{p}_{\text{o}}]^2}{\sum_{k=1}^{N}\gamma_{\text{o}}^2(t_k)} \approx 21 \text{ dB}$$
$$\text{SNR}_y = 10\log\frac{\sum_{k=1}^{N}[y_{\text{o}}(t_k) - \bar{y}_{\text{o}}]^2}{\sum_{k=1}^{N}e_{\text{o}}^2(t_k)} \approx 23 \text{ dB}$$

where $\bar{p}_{\text{o}}$ and $\bar{y}_{\text{o}}$ are the mean values of the signals $p_{\text{o}}(t_k)$ and $y_{\text{o}}(t_k)$, respectively. Note that this numerical example is similar to a demo available in the CONTSID toolbox [41].

To assess smoothing techniques, usually the mean squared error is used. In this study, we consider a similar indicator which is the fit index [42], [43], that is usually used in system identification to compare measured system output and simulated model output. Then, the fit index for the scheduling variable is given by,

$$\mathcal{F}_p = \left[1 - \sqrt{\frac{\sum_{k=1}^{N}[p_{\text{o}}(t_k) - \hat{p}(t_k)]^2}{\sum_{k=1}^{N}[p_{\text{o}}(t_k) - \bar{p}_{\text{o}}]^2}}\right] \cdot 100\%. \tag{49}$$

Note that the LR method provides estimates of the derivatives of $p_{\text{o}}(t_k)$, namely, the estimate of the ν^{th} derivative is given by $\nu!\hat{\beta}_\nu(t_k)$. As previously mentioned, the second derivative is a measure of the smoothness (or curvature) of a signal. Then, we consider the following index as a measure of the smoothness of $\hat{p}(t_k)$:

$$\mu_s = \frac{1}{N}\sum_{k=1}^{N}|2\hat{\beta}_2(t_k)|. \tag{50}$$

The smoother the signal, the smaller the value of μ_s. We use μ_s to evaluate the hyperparameter N_{interval} in the LR approach.

In LPV-SRIVC, the fit between measured and simulated output is quantified using the fit index $\mathcal{F}_y$, defined analogously to (49). In order to assess the estimated parameters, we can compute the norm of the bias [4]. However, we prefer to compute the relative norm of the bias (RBN), because we believe it is more informative, given that it is a relative measure. RBN is defined as follows,

$$\text{RBN} = \frac{\left\|\boldsymbol{\theta}_{\text{o}} - \bar{\mathbb{E}}(\hat{\boldsymbol{\theta}})\right\|_2}{\|\boldsymbol{\theta}_{\text{o}}\|_2} \tag{51}$$

where $\bar{\mathbb{E}}(\cdot)$ is the mean over the Monte Carlo simulation. In addition we compute the norm of the variance [4],

$$\text{VN} = \left\|\bar{\mathbb{E}}(\hat{\boldsymbol{\theta}} - \bar{\mathbb{E}}(\hat{\boldsymbol{\theta}}))^2\right\|_2. \tag{52}$$

B. SINGLE EXPERIMENT ANALYSIS

In this section, using a single experiment, we analyze the performance of the LR-RSC-A approach, because it involves the choice of an hyperparameter (N_{interval}), and as we will see in Section V-C, it yields the best results. Considering the system described in Section V-A, a single experiment is generated; the corresponding data is shown in Fig. 2.

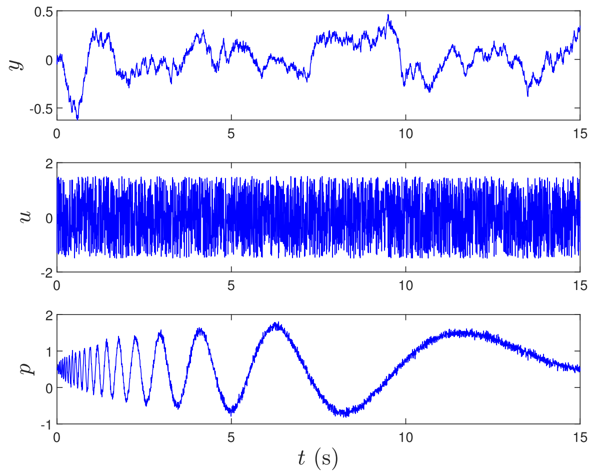

FIGURE 2. Data for the single experiment analysis.

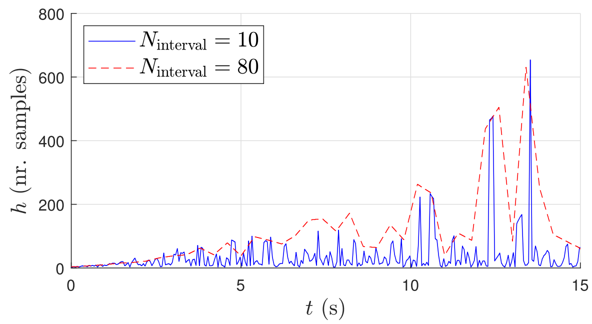

FIGURE 3. Bandwidth h vs time t for two values of N_{interval}.

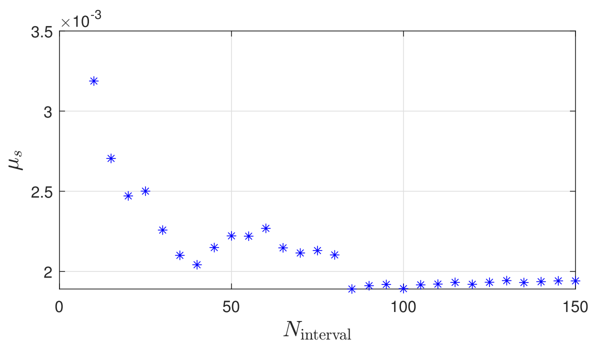

FIGURE 4. Smoothness measure of $\hat{p}(t_k)$ (μ_s) vs N_{interval}.

We explore the use of LR-RSC-A with $h \in H = [3\ 1500]$, and different values of the hyperparameter N_{interval}, with $N_{\text{interval}} \in [10\ \ 150]$. Note that using the suggestion (44), we obtain $N_{\text{interval}} = 80$.

In Fig. 3, for two values of N_{interval}, the bandwidth $h(t_k)$ vs time is shown. We can clearly see that for a smaller N_{interval}, there is more variability in $h(t_k)$. Since the scheduling variable p_o is smooth, also the bandwidth $h(t_k)$ should be smooth. In Fig. 4, the smoothness measure μ_s vs N_{interval} is shown; here we confirm that the smaller the value of N_{interval}, the less smooth the signal $\hat{p}(t_k)$.

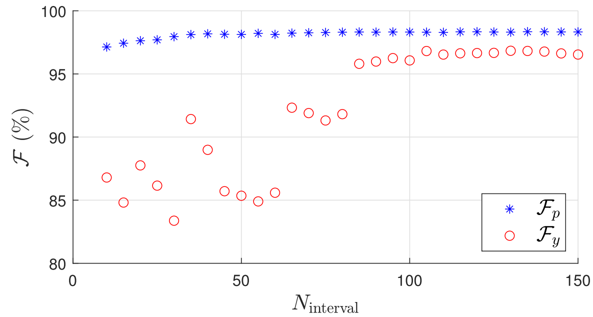

FIGURE 5. Fits $\mathcal{F}_p$ and $\mathcal{F}_y$ for different values of N_{interval}.

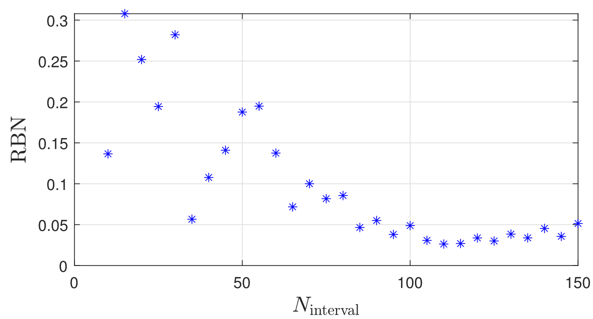

FIGURE 6. Relative norm of the bias (RBN) vs N_{interval}.

TABLE 1. Relative norm of the bias (RBN) and variance of the norm (VN).

Method	RBN	VN
no smoothing	$3.86 \cdot 10^{-1}$	27.36
LR-RSC-F	$6.41 \cdot 10^{-2}$	6.73
LR-CV-F	$7.20 \cdot 10^{-2}$	6.42
LR-RSC-A	$9.09 \cdot 10^{-3}$	0.39
LR-CV-A	$1.35 \cdot 10^{-2}$	0.44

In Fig. 5 we can see how the fit $\mathcal{F}_p$ varies depending on N_{interval}. Considering the smoothed scheduling variables, the LPV-OE system is identified using LPV-SRIVC. The corresponding values for $\mathcal{F}_y$ are shown in Fig. 5. On the other hand, in Fig. 6 the relation between the relative norm of the bias (RBN) and N_{interval} is presented. From these results, and recalling that the smaller N_{interval}, the greater the variability (adaptability) of $h(t_k)$, we can see that an appropriate value for N_{interval} is around 90, which is close to the suggested rule (44), i.e. $N_{\text{interval}} = 80$.

C. MONTE CARLO SIMULATION

In this section, the results of a Monte Carlo simulation with 100 runs are presented in order to evaluate the performance of the proposed methods. The four smoothing approaches presented previously are used for estimating LPV models through LPV-SRIVC. These approaches are also compared with the case when LPV-SRIVC is applied ignoring the noise in the scheduling variable. That case is denoted by 'no smoothing'. For the adaptive bandwidth cases, we use the suggested rule (44), i.e. we set $N_{\text{interval}} = 80$.

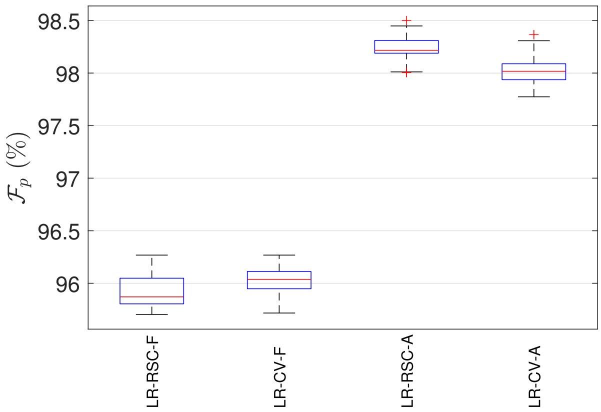

FIGURE 7. **Boxplots of $\mathcal{F}_p$ for the different smoothing methods.**

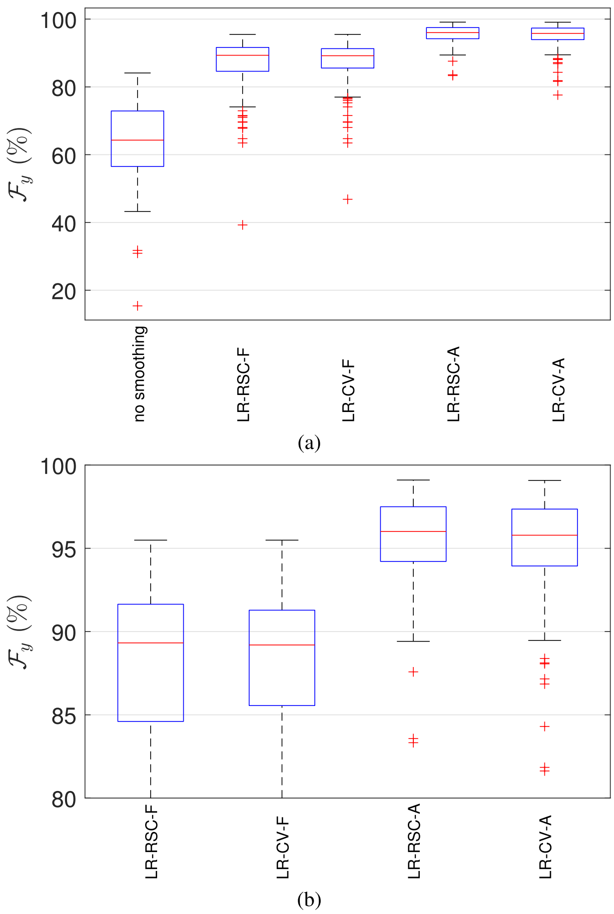

FIGURE 8. **(a) Boxplots of $\mathcal{F}_y$ for the different LPV identification methods. (b) Zoom on the boxes of the boxplots presented in (a) without the case 'no smoothing'.**

In the following results, LPV-SRIVC sometimes fails (it does not converge or yields negative fits); those cases are not considered. When there is no smoothing, LPV-SRIVC fails four times; for LR-RSC-F, LPV-SRIVC fails once; and for LR-CV-F, LPV-SRIVC fails also once. The boxplots of $\mathcal{F}_p$ are shown in Fig. 7, where we see that the best performance is obtained for LR-RSC-A. After computing the smoothed scheduling variables, the LPV-OE models are estimated using LPV-SRIVC. The boxplots of $\mathcal{F}_y$ are shown in Fig. 8(a); a zoom on the boxes of the boxplots without the case 'no smoothing' is presented in Fig. 8(b). We can see that when there is no smoothing, the fits are very

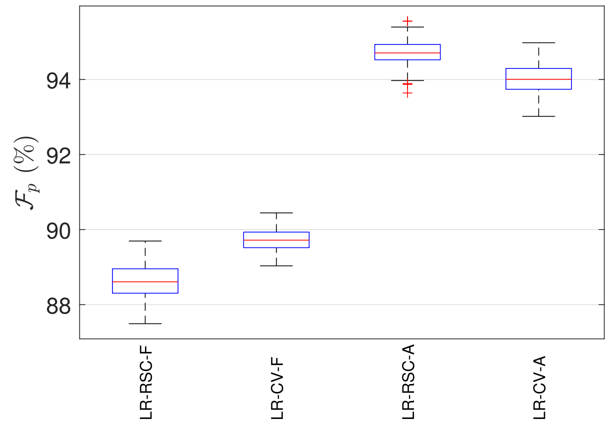

FIGURE 9. **Results of Monte Carlo simulation with larger noise variance in the scheduling variable ($\sigma_{\gamma o}^2 = 0.02^2$). Boxplots of $\mathcal{F}_p$ for the different smoothing methods.**

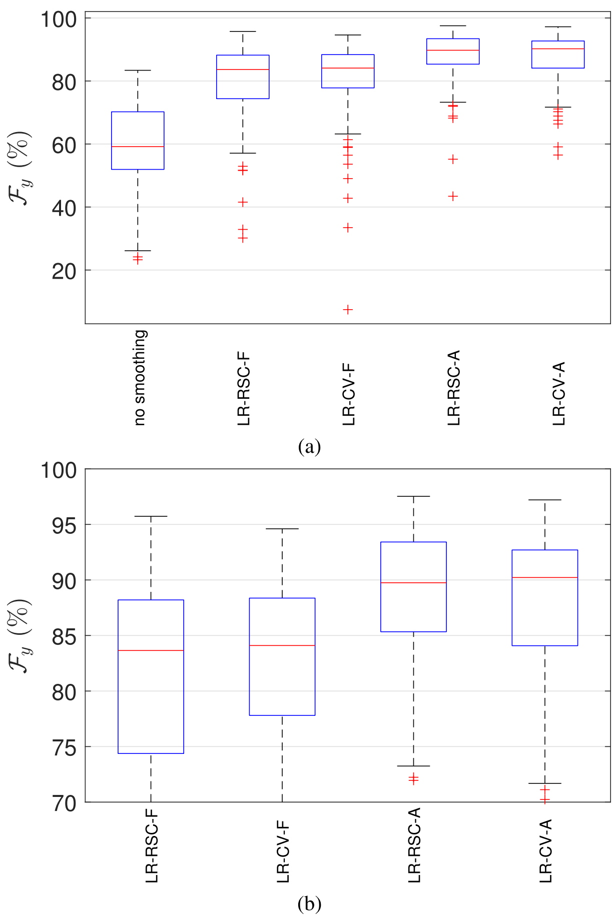

FIGURE 10. **Results of Monte Carlo simulation with larger noise variance in the scheduling variable ($\sigma_{\gamma o}^2 = 0.02^2$). (a) Boxplots of $\mathcal{F}_y$ for the different LPV identification methods. (b) Zoom on the boxes of the boxplots presented in (a) without the case 'no smoothing'.**

TABLE 2. Results of Monte Carlo simulation with larger noise variance in the scheduling variable ($\sigma^2_{\gamma_o} = 0.02^2$). Relative norm of the bias (RBN) and variance of the norm (VN).

Method	RBN	VN
no smoothing	$4.23 \cdot 10^{-1}$	38.54
LR-RSC-F	$7.05 \cdot 10^{-2}$	21.01
LR-CV-F	$1.38 \cdot 10^{-1}$	16.39
LR-RSC-A	$3.20 \cdot 10^{-2}$	2.93
LR-CV-A	$3.84 \cdot 10^{-2}$	3.95

low, and the best outcomes are obtained with LR-RSC-A. Regarding the estimated LPV model parameters, as shown in Table 1, the lowest RBN and VN values are also obtained with LR-RSC-A.

In order to assess the proposed approach with a larger noise variance $\sigma^2_{\gamma_o}$ in the scheduling variable, we repeat the Monte Carlo simulation with 100 runs. Keeping the same noise $e_o(t_k)$, we corrupt $p_o(t_k)$ with a Gaussian noise $\mathcal{N}(0, 0.02^2)$, which yields $\text{SNR}_p \approx 11$ db. In this case, when there is no smoothing, LPV-SRIVC fails five times; for LR-RSC-F, LPV-SRIVC fails six times; for LR-CV-F, LPV-SRIVC fails four times; for LR-RSC-A, LPV-SRIVC fails once; and for LR-CV-A, LPV-SRIVC fails also once. The boxplots of $\mathcal{F}_p$ are shown in Fig. 9, where we see that the best performance is obtained for LR-RSC-A. The boxplots of $\mathcal{F}_y$ are shown in Fig. 10(a); a zoom on the boxes of the boxplots without the case 'no smoothing' is presented in Fig. 10(b), where we can see that the median of LR-CV-A is slightly better than the median of LR-RSC-A. Regarding the estimated LPV model parameters, as shown in Table 2, the lowest RBN and VN values are obtained with LR-RSC-A. Finally, the results show that a larger noise variance $\sigma^2_{\gamma_o}$ deteriorates the performance indexes of the proposed method, but they are still a significant improvement in comparison to the case 'no smoothing'. Moreover, in this case, the adaptive bandwidth approaches also perform better than the fixed bandwidth approaches.

VI. CONCLUSION

The identification of continuous-time LPV models in OE form has been addressed, assuming that a smooth scheduling variable is corrupted by additive white noise. The proposed approach involves the use of the local regression method and the refined instrumental variable approach. An important hyperparameter of the former method is the bandwidth, which can be fixed or adaptive. The latter option is appropriate when: (i) the data is irregularly sampled; (ii) the noise in the scheduling variable changes over time (heteroscedasticity); and (iii) there is a significant variability in the smoothness of the scheduling variable. The latter scenario is considered in this paper, showing through a numerical example the benefits of the LR method for smoothing, and the consequences in the estimation of CT LPV-OE models. The other scenarios can be addressed in a future study, considering both simulated and real data.

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ARTURO PADILLA was born in Talcahuano, Chile. He received the Dr.-Ing. and M.Sc. degrees in mechanical engineering from Universidad de Concepción, Concepcion, Chile, in 2004 and 2005, respectively, and the Ph.D. degree in automatic control from Université de Lorraine, Nancy, France, in 2017. He is currently with Universidad de La Frontera, Temuco, Chile. His research interest includes system identification.

HUGUES GARNIER (Member, IEEE) has been a Professor of control theory and applications with the University of Lorraine, France, since 2003. He was a Visiting Professor with The University of Newcastle, Australia, from 2003 to 2004, and the University of California at San Diego (UCSD), from 2012 to 2013. His current research interests include data-driven modeling for control and predictive maintenance of nonlinear dynamical systems, with a recent focus on robotics and aerospace applications. He has been very active in promoting and developing direct continuous-time model learning approaches and is behind CONTSID, a non-commercial MATLAB toolbox for data-driven continuous-time model learning. He was an editor of two books and was a guest editor for three special issues of international journals.

JUAN I. YUZ (Senior Member, IEEE) received the Ingeniero Civil Electrónico and M.Sc. degrees in electronic engineering from Universidad Técnica Federico Santa María (UTFSM), Valparaíso, Chile, in 2001, and the Ph.D. degree in electrical engineering from The University of Newcastle, Australia, in 2006.

From 2015 to 2019, he was the Director of the Advanced Center for Electrical and Electronic Engineering (AC3E), UTFSM. He is currently a Full Professor with Departamento de Electrónica, UTFSM. He is the coauthor of the book *Sampled-Data Models for Linear and Nonlinear Systems* (Springer, 2014). His research interests include control and identification of sampled-data systems and their applications to power electronics and biomedical systems.

FENGWEI CHEN was born in Chongqing, China. He received the B.Eng. degree in automation and the M.Eng. degree in control theory and control engineering from Wuhan University, Wuhan, China, in 2009 and 2011, respectively, and the Ph.D. degree in automatic control from Université de Lorraine, Nancy, France, in 2014.

From 2015 to 2016, he was a Lecturer with Dalian University of Technology, Dalian, China. From 2017 to 2020, he was an Associate Researcher with Wuhan University, Wuhan, China. Since 2021, he has been with Chongqing University, Chongqing, China, where he is currently an Associate Professor. His research interests include system identification and parameter estimation, with applications to wireless power transfer.

CARLOS MUÑOZ POBLETE (Member, IEEE) received the B.S. degree in electronic engineering from Universidad de La Frontera, Temuco, Chile, in 1990, and the Ph.D. degree in science engineering from Pontificia Universidad Católica de Chile, Santiago, Chile, in 2000.

Currently, he is a Professor of electronic engineering with Universidad de La Frontera. His research interests include feedback control, the IoT, and predictive control.