IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received 4 February 2024, accepted 20 February 2024, date of publication 26 February 2024, date of current version 11 March 2024.

Digital Object Identifier 10.1109/ACCESS.2024.3369893

RESEARCH ARTICLE

Selection of Cloud Services Provider by Utilizing Multi-Attribute Decision-Making Based on Hesitant Bipolar Complex Fuzzy Dombi Aggregation Operators

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This work was supported by the Deanship of Scientific Research, King Khalid University, Abha, Saudi Arabia, for funding this work through the Research Groups Program, under Grant R.G. P-2/99/44.

ABSTRACT A trustworthy cloud services provider makes sure that data backups, disaster recovery, and compliance with industry standards are all met, allowing businesses to concentrate on their core competencies and innovation while leaving the complexity of IT infrastructure management to the professionals. The selection of the finest cloud services provider is a multi-attribute decision-making problem and thus, in this article, we investigate the selection of the finest cloud services provider by employing a multi-attribute decision-making technique using the invention of hesitant bipolar complex fuzzy set. The hesitant bipolar complex fuzzy sets deal not only with the data involving complex and hesitant information but also the data of dual aspects that are both positive and negative because a lot of real-life situation contains information that has hesitancy, 2^{nd} dimension, and a dual aspect simultaneously. Furthermore, in this manuscript, using the structure of a hesitant bipolar complex fuzzy set we initiate numerous aggregation operators. Under the environment of the proposed methodology, we invent different aggregation operators such as hesitant bipolar complex fuzzy Dombi weighted averaging, hesitant bipolar complex fuzzy Dombi ordered weighted averaging, hesitant bipolar complex fuzzy Dombi hybrid averaging, hesitant bipolar complex fuzzy Dombi weighted geometric, hesitant bipolar complex fuzzy Dombi ordered weighted geometric and hesitant bipolar complex fuzzy Dombi hybrid geometric operators. All these operators are used to solve different situations at different times. When any decision-maker wants to solve hesitation under a complex environment with 2^{nd} dimensions then these operators are very useful. Without proposed operators, we cannot handle the above situations because no doubt different tools for different situations are available in the market but there is no tool available for solving hesitant bipolar complex fuzzy information. Afterward, we devise a technique of multi-attribute decision-making for tackling genuine life decision-making dilemmas under the setting of hesitant bipolar complex fuzzy sets. We investigate a case study associated with cloud service providers by employing the invented technique of decision-making. In the end, we compare our newly suggested work with a few other current theories to show the effectiveness of our work.

INDEX TERMS Cloud services, cloud services provider, hesitant bipolar complex fuzzy set, MADM.

I. INTRODUCTION

In today's technological and commercial environments, cloud services (CS) and cloud service providers are essential. The value of cloud services resides in their capacity to

The associate editor coordinating the review of this manuscript and approving it for publication was Peter Langendoerfer.

provide scalable and adaptable computer resources through the Internet. Through the use of cloud services, people and businesses may access a variety of computer resources, including networking, databases, processing power, storage, and networking, without making significant investments in physical infrastructure. Businesses can develop quickly, roll out applications more quickly, and react to shifting market needs more effectively because of this affordable and on-demand access to resources. Furthermore, CS makes it possible for teams to collaborate and operate remotely while remaining in sync.Kwon and Seo [1] discussed a decision-making (DM) model to indicate a CS using fuzzy AHP. Coppolino et al. [2] proposed FS theory-based comparative evaluation of CS offerings. Moreover, a novel context for viable CS selection as a service (CSSAAS) under a fuzzy environment was given by Hussain et al. [3]. Kumar et al. [4] discuss prioritizing the solution of CS selection using multi-criteria decision-making (MCDM) methods based on fuzzy settings.The success and broad use of cloud computing are greatly influenced by cloud service providers. First and foremost, they provide a wide variety of services and solutions that are tailored to the unique requirements of both organizations and people. Users can select the degree of abstraction and control they desire with these services, which include Infrastructure as a Service (IaaS), Platform as a Service (PaaS), and Software as a Service (SaaS). To guarantee their services' high availability, dependability, and security, CS providers make significant investments in cutting-edge technology and data centers. Security is of utmost importance in the digital era, and reliable cloud service providers have strong security measures in place to protect the data of their clients. To guard against unwanted access and data breaches, they use encryption, access limits, and frequent audits. Furthermore, cloud service providers give alternatives for data backup and disaster recovery, assuring data resilience and ongoing corporate operations. Businesses of all sizes may benefit from top-notch IT skills without having to invest in dedicated IT staff because of their experience handling complicated infrastructures and enormous volumes of data. Businesses may concentrate on innovation and growth while leaving the technical side of IT to professionals by utilizing the services of CS providers. Tanoum et al. [5] give valuable context on selecting cloud computing service providers with fuzzy AHP. The idea of multicriteria evaluation of CS providers using Pythagorean fuzzy TOPSIS was given by Onar et al. [6]. Furthermore, Ghorui et al. [7] discussed the selection of CS providers using the MCDM procedure under IF uncertainty.

The ability to make decisions is crucial in many branches of science and technology. The DM technique gives a valuable framework to many researchers, scientists, and other common people to design their big decisions in many fields of life. We always note that decision is very important in real-life issues. If anyone wants to plan his business, journey, and any other then first he makes a decision and even no one can do any of his work without a decision. Many decision-makers (DMs) perform their task only after the decision and there is very little chance of harm to them. So, viewing all those qualities of decision we can say that DM plays a very important role in everyday life. Fuzzy sets (FSs) theory gives a valuable framework to DMs. Before the invention of FS theory, most DMs used the framework of crisp set theory (CST) to make their decisions. But CST has a very limited framework and cannot give a free hand to DMs. FS theory generalized the CST and gives a void range for choices and cannot make decisions only in terms of binary options 0 and 1. In CST, there are only two possible outcomes for every element it either belongs to the set or it does not. However, FS generalizes the CST and provides a void range for choice in the form of fuzziness. FS theory was developed by Zadeh [8] in 1965. In FS theory each element from the fixed set has membership values and is called the grade of membership (GM). GM must belong to the interval [0, 1]. FS theory gives a massive mathematical framework for all types of decisions. Many DMs utilize the FS theory in different fields of life. Mardani et al. [9] discussed that DM methods depend on fuzzy aggregation operators (AOs). Hadi et al. [10] proposed the idea of MADM problems under Fermatean fuzzy Hamacher AOs. Similarly, many researchers utilize the FS theory and give different ideas in many fields. Merigo and Casanovas [11] discussed the concept of fuzzy generalized hybrid AOs and their use in fuzzy DM. Boukezzoula et al. [12] discussed min and max operators for fuzzy intervals and their potential use in AOs. A hybrid model based on fuzzy AHP and fuzzy WASPAS for construction site selection was invented by Turskis et al. [13]. A novel multi-attribute decision-making (MADM) method based on fuzzy rough sets was proposed by Ye et al. [14]. FS gives a single grade to each element from the fixed set while sometimes DMs want to discuss their opinion in the form of a group of grades or a set of information. In this situation, hesitant FSs (HFS) is a tool that can help DMs. So, HFS provides a useful environment to DMs when they want to discuss their opinion in the form of a set or the form of hesitation. The idea of HFS was given by Torra [15]. After the invention of HFSs, many researchers or scholars realized the importance of HFSs and tried to use them in many fields, some of the fields are discussed below. Zhang [16] proposed the idea of aggregation and discussed HF power AOs and their application to multiple attribute group decision-making (MAGDM). Qin et al. [17] also proposed the Frank AOs and their application to HF MADM. Tang et al. [18] discussed their expertise in the MADM approach based on dual HF Frank AOs. Saha et al. [19] invented the notions of some new hybrid HF-weighted AOs based on Archimedean and Dombi operations for MADM. AOs on HFSs give a single value framework to DMs after the execution. Always AOs converts a set of values into a single value and many researchers utilize this technique for converting a set of information into a single value. Also keeping these concepts in mind Batool et al. [20] proposed the optimization approach with single-valued neutrosophic HF Dombi AOs. Liu et al. [21] invent MADM using HF Dombi–Archimedean weighted AOs.

FSs only can handle information in the form of a single element but as compared to FS, HFSs can solve a set of information FSs and HFSs are only solvable for GM and if a grade of non-membership (GNM) comes in any information, then FS and HFS are failed to solve that kind of information. Many DM problems in real life are related to both grades. To resolve that kind of information or issues Atanassov [22] gave the idea of the intuitionistic fuzzy set (IFS). IFS can solve both grades simultaneously and give a useful result to DMs. After the invention of IFSs DMs felt much freer to make their decisions. Many researchers utilize this concept in different fields of life and have many new ideas some of the ideas are discussed below. Wang and Liu [23] discussed the IF geometric AOs based on Einstein operations. Zhao and Wei [24] proposed some IF Einstein hybrid AOs and their application to MADM. Huang [25] proposed the concept of IF Hamacher AOs and their application to MADM. Similarly, many scholars discussed a lot of aggregation based on IFSs and utilized them in different fields of life. Senapati et al. [26] discussed a useful aggregation of Aczel Alsina (AA) and its application to IF MADM. Garg et al. [27] invent the idea of Choquet integral-based information AOs under the interval-valued IFS and its applications to the DM process. Mahmood and Ali [28] proposed MADM methods based on AA power AOs for managing complex IFS. A prioritized AOs-based approach to MCDM using interval-valued IFSs invented by Chen [29].

The invention of the Bipolar fuzzy set (BFS) was a great achievement in FS theory. BFS gives the right path to DMs. BFS is different from other fuzzy-related sets. BFS is one of the only sets that can explain the positive and negative aspects of any object. When any DM wants to discuss the goodness and badness of any object then he must take the recourse of BFSs. BFSs give the right way for many researchers, and scholars to utilize them in different fields with different applications. Jana et al. [30] introduce the idea of BF Dombi AOs and their uses in the MADM process. BF Dombi prioritized AOs in MADM given by Jana et al. [31]. Moreover, Bipolar neutrosophic DAOs with application in MADM situations proposed by Mahmood et al. [32]. Akram et al. [33] proposed the idea of extensions of Dombi AOs for DM under m-polar fuzzy information. Zhang et al. [34] introduce the Study on risk assessment of pharmaceutical distribution supply chain with BF information. Mandal [35] generalize the idea of Bipolar PFS and its application in MADM. Naz et al. [36] discuss the different ideas on a new MAGDM method with 2-tuple linguistic BF Heronian mean operators. Riaz et al. [37] proposed MCDM based on bipolar picture fuzzy operators and

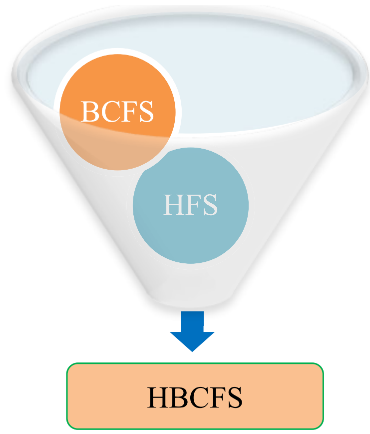

FIGURE 1. Graphical representation of proposed structure.

new distance measures. Wei et al. [38] invented the idea of BF Hamacher AOs in MADM. Furthermore, cubic BF ordered weighted geometric AOs and their application using internal and external cubic BF data given by Riaz and Tehrim [39]. No doubt BFS gives a valuable framework but when information occurs in the form of bipolar hesitation than to handle such types of difficulties Mandal and Ranadive [40] introduce the concept of Hesitant bipolar-valued fuzzy sets (HBVFSs) and bipolar-valued HFS (BVHFSs) and its uses in MAGDM. Liu et al. [41] also contribute to HBFSs and evaluate the idea of the MAGDM framework based on double hierarchy BHF linguistic information and its application to the optimal selection of talents. Awang et al. [42] proposed an HBV neutrosophic set.

Sometimes while making decisions, we have to face such information which is in bipolar complex form. Bipolar complex form means the positive and negative aspects of any objects in the form of complex information or the form of two-dimensional information. So, the BFSs and HBFSs fail within such cases. To handle such type of information the idea of a Bipolar complex fuzzy set (BCFS) was given by Mahmood and Rehman [43] with the title A Novel Approach towards BCFSs and their use in generic similarity metrics. Mahmood and Rehman [44] also proposed the Artificial intelligence deployment in digital technology and its effects based on BCF Schweizer-Sklar power AOs. Additionally, Xu et al. [45] explored the use of the MADM technique based on the BCF Dombi Prioritized AOs to analyze structural systems to design earthquake-resistant structures. Mahmood and Rehman [46] also invented the idea of the MADM technique based on BCF Maclaurin symmetric mean operators. BCF framework is capable of facing two-dimensional information and can easily solve all related problems. But when any DMs face the BCF information in the form of hesitant or in the form of set then there is no tool available yet that can easily solve the set of two-dimensional information. So, this is the motivation for developing the theory of hesitant bipolar complex fuzzy sets (HBCFSs). In the manuscript, we proposed the new concept of HBCFSs and also proposed the Dombi AOs in the environment of HBCFSs. Furthermore, the graphical representation of proposed structure is given in the following figure 1.

This article is divided into sections. In section II, we review some fundamental definitions and procedures for BCFSs and HFSs. In section III we invent our new theory of HBCFSs with their related laws and properties. Also in this section, for comparison of two HBCFNs, we propose score and accuracy functions. Dombi AOs under the framework of HBCFNs are proposed in Section IV. Section V consisted of the MADM technique based on HBCFSs. Moreover, the application of the proposed theory with numerical examples is discussed in section VI. At the end of this paper in section VII, we compare newly presented and established hypotheses. We discussed the conclusion of the whole manuscript in section 8.

II. PRELIMINARIES

In this section, we linked some basic concepts and notions of HFSs and BCFSs. We also show or discuss their related essential operational laws and main characteristics.

Definition 1 ([15]): A HFS hat ($\hat{\mathbb{B}}$ over fixed hat ($\hat{\mathbb{G}}$ is noted as

$$\hat{\mathbb{B}} = \left\{ < \eth,\ \S_{\hat{\mathbb{B}}}(\eth) > \mid \eth \in \hat{\mathbb{G}} \right\} \tag{1}$$

where $\overline{\mathfrak{U}} = \S_{\hat{\mathbb{B}}}(\eth)$ is the set of some values in $[0,\ 1]$, shows the MD of each $\eth \in \hat{\mathbb{G}}$. For easiness $\overline{\mathfrak{U}} = \S_{\hat{\mathbb{B}}}(\eth)$ represents a hesitant fuzzy number (HFN).

Definition 2 ([15]): For three HFNs $\overline{\mathfrak{U}}_1$, $\overline{\mathfrak{U}}_2$ and $\overline{\mathfrak{U}}_3$

1) $\overline{\mathfrak{U}}^c = \cup_{\check{\check{\mathfrak{z}}} \in \overline{\mathfrak{U}}} \left\{ 1 - \check{\check{\mathfrak{z}}} \right\}$
2) $\overline{\mathfrak{U}}_1 \cup \overline{\mathfrak{U}}_2 = \cup_{\check{\check{\mathfrak{z}}}_1 \in \overline{\mathfrak{U}}_1,\ \check{\check{\mathfrak{z}}}_2 \in \overline{\mathfrak{U}}_2} \max \left\{ \check{\check{\mathfrak{z}}}_1,\ \check{\check{\mathfrak{z}}}_2 \right\}$
3) $\overline{\mathfrak{U}}_1 \cap \overline{\mathfrak{U}}_2 = \cup_{\check{\check{\mathfrak{z}}}_1 \in \overline{\mathfrak{U}}_1, \check{\check{\mathfrak{z}}}_2 \in \overline{\mathfrak{U}}_2} \min \left\{ \check{\check{\mathfrak{z}}}_1,\ \check{\check{\mathfrak{z}}}_2 \right\}$

Definition 3 ([16]): Let $\overline{\mathfrak{U}}_1$, $\overline{\mathfrak{U}}_2$ and $\overline{\mathfrak{U}}_3$ be three HFNs and $\lambda > 0$, then

1) $\overline{\mathfrak{U}}^{\lambda} = \cup_{\check{\check{\mathfrak{z}}} \in \overline{\mathfrak{U}}} \left\{ \check{\check{\mathfrak{z}}}^{\lambda} \right\}$
2) $\lambda \overline{\mathfrak{U}} = \cup_{\check{\check{\mathfrak{z}}} \in \overline{\mathfrak{U}}} \left\{ 1 - \left(1 - \check{\check{\mathfrak{z}}}\right)^{\lambda} \right\}$
3) $\overline{\mathfrak{U}}_1 \oplus \overline{\mathfrak{U}}_2 = \cup_{\check{\check{\mathfrak{z}}}_1 \in \overline{\mathfrak{U}}_1,\ \check{\check{\mathfrak{z}}}_2 \in \overline{\mathfrak{U}}_2} \left\{ \check{\check{\mathfrak{z}}}_1 + \check{\check{\mathfrak{z}}}_2 - \check{\check{\mathfrak{z}}}_1 \check{\check{\mathfrak{z}}}_2 \right\}$
4) $\overline{\mathfrak{U}}_1 \otimes \overline{\mathfrak{U}}_2 = \cup_{\check{\check{\mathfrak{z}}}_1 \in \overline{\mathfrak{U}}_1,\ \check{\check{\mathfrak{z}}}_2 \in \overline{\mathfrak{U}}_2} \left\{ \check{\check{\mathfrak{z}}}_1 \check{\check{\mathfrak{z}}}_2 \right\}$

Definition 4 ([43]): A BCFS hat ($\hat{\mathbb{B}}$ over fixed hat ($\hat{\mathbb{G}}$ is noted as

$$\hat{\mathbb{B}} = \left\{ < \eth,\ \left(\S^{+}_{\hat{\mathbb{B}}}(\eth),\ \S^{-}_{\hat{\mathbb{B}}}(\eth) \right) > \mid \eth \in \hat{\mathbb{G}} \right\} \tag{2}$$

where $\S^{+}_{\hat{\mathbb{B}}}(\eth) = \check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{R}}]}_{\hat{\mathbb{B}}}(\eth) + \iota \check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{I}}]}_{\hat{\mathbb{B}}}(\eth)$ denotes the PMG and $\S^{-}_{\hat{\mathbb{B}}}(\eth) = \check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{R}}]}_{\hat{\mathbb{B}}}(\eth) + \iota \check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{I}}]}_{\hat{\mathbb{B}}}(\eth)$ denotes the NMG for each $\eth \in \hat{\mathbb{G}}$. All values of PMG and NMG are from the unit square of a complex plane. Where $\check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{R}}]}_{\hat{\mathbb{B}}}(\eth)$, $\check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{I}}]}_{\hat{\mathbb{B}}}(\eth) \in [0,\ 1]$ and $\check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{R}}]}_{\hat{\mathbb{B}}}(\eth)$, $\check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{I}}]}_{\hat{\mathbb{B}}}(\eth) \in [-1,\ 0]$. For simplicity, the bipolar complex fuzzy number is symbolized by $\overline{\mathfrak{U}} = \left(\S^{+},\ \S^{-}\right) = \left(\check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{R}}]} + \iota \check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{I}}]}, \check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{R}}]} + \iota \check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{I}}]}\right)$.

Definition 5 ([43]): Let $\overline{\mathfrak{U}}, \overline{\mathfrak{U}}_1$ and $\overline{\mathfrak{U}}_2$ be three BCFNs then.

1) $$\overline{\mathfrak{U}}^{c} = \left\{ \begin{array}{c} \left(\left[1 - \check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{R}}]}\right] + \iota \left[1 - \check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{I}}]}\right] \right), \\ \left(\left[-1 - \check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{R}}]}\right] + \iota \left[-1 - \check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{I}}]}\right] \right) \end{array} \right\}$$

2) $\overline{\mathfrak{U}}_1 \cup \overline{\mathfrak{U}}_2$
$$= \left\{ \begin{array}{c} \left(max\left[\check{\check{\mathfrak{z}}}_1^{+[\tilde{\mathbb{R}}]}, \check{\check{\mathfrak{z}}}_2^{+[\tilde{\mathbb{R}}]}\right] + \iota max \left[\check{\check{\mathfrak{z}}}_1^{+[\tilde{\mathbb{I}}]}, \check{\check{\mathfrak{z}}}_2^{+[\tilde{\mathbb{I}}]}\right] \right), \\ \left(min\left[\check{\check{\mathfrak{z}}}_1^{-[\tilde{\mathbb{R}}]}, \check{\check{\mathfrak{z}}}_2^{-[\tilde{\mathbb{R}}]}\right] + \iota min \left[\check{\check{\mathfrak{z}}}_1^{-[\tilde{\mathbb{I}}]}, \check{\check{\mathfrak{z}}}_2^{-[\tilde{\mathbb{I}}]}\right] \right) \end{array} \right\}$$

3) $\overline{\mathfrak{U}}_1 \cap \overline{\mathfrak{U}}_2$
$$= \left\{ \begin{array}{c} \left(min\left[\check{\check{\mathfrak{z}}}_1^{+[\tilde{\mathbb{R}}]}, \check{\check{\mathfrak{z}}}_2^{+[\tilde{\mathbb{R}}]}\right] + \iota min \left[\check{\check{\mathfrak{z}}}_1^{+[\tilde{\mathbb{I}}]}, \check{\check{\mathfrak{z}}}_2^{+[\tilde{\mathbb{I}}]}\right] \right), \\ \left(max\left[\check{\check{\mathfrak{z}}}_1^{-[\tilde{\mathbb{R}}]}, \check{\check{\mathfrak{z}}}_2^{-[\tilde{\mathbb{R}}]}\right] + \iota max \left[\check{\check{\mathfrak{z}}}_1^{-[\tilde{\mathbb{I}}]}, \check{\check{\mathfrak{z}}}_2^{-[\tilde{\mathbb{I}}]}\right] \right) \end{array} \right\}$$

Definition 6 ([47]): Let $\overline{\mathfrak{U}}$, $\overline{\mathfrak{U}}_1$ and $\overline{\mathfrak{U}}_2$ be three BCFNs and $\lambda > 0$ Then.

1) $\overline{\mathfrak{U}}_1 \oplus \overline{\mathfrak{U}}_2$
$$= \left(\begin{array}{c} \left(\begin{array}{c} \left[\check{\check{\mathfrak{z}}}_1^{+[\tilde{\mathbb{R}}]} + \check{\check{\mathfrak{z}}}_2^{+[\tilde{\mathbb{R}}]} - \check{\check{\mathfrak{z}}}_1^{+[\tilde{\mathbb{R}}]} \check{\check{\mathfrak{z}}}_2^{+[\tilde{\mathbb{R}}]}\right] + \\ \iota \left[\check{\check{\mathfrak{z}}}_1^{+[\tilde{\mathbb{I}}]} + \check{\check{\mathfrak{z}}}_2^{+[\tilde{\mathbb{I}}]} - \check{\check{\mathfrak{z}}}_1^{+[\tilde{\mathbb{I}}]} \check{\check{\mathfrak{z}}}_2^{+[\tilde{\mathbb{I}}]}\right] \end{array} \right) \\ \left(\left[- \left(\check{\check{\mathfrak{z}}}_1^{-[\tilde{\mathbb{R}}]} \check{\check{\mathfrak{z}}}_2^{-[\tilde{\mathbb{R}}]} \right) \right] + \iota \left[- \left(\check{\check{\mathfrak{z}}}_1^{-[\tilde{\mathbb{I}}]} \check{\check{\mathfrak{z}}}_2^{-[\tilde{\mathbb{I}}]} \right) \right] \right) \end{array} \right)$$

2) $$\overline{\mathfrak{U}}_1 \otimes \overline{\mathfrak{U}}_2 = \left(\begin{array}{c} \left(\left[\check{\check{\mathfrak{z}}}_1^{+[\tilde{\mathbb{R}}]} \check{\check{\mathfrak{z}}}_2^{+[\tilde{\mathbb{R}}]}\right] + \iota \left[\check{\check{\mathfrak{z}}}_1^{+[\tilde{\mathbb{I}}]} \check{\check{\mathfrak{z}}}_2^{+[\tilde{\mathbb{I}}]}\right] \right), \\ \left(\begin{array}{c} \left[\check{\check{\mathfrak{z}}}_1^{-[\tilde{\mathbb{R}}]} + \check{\check{\mathfrak{z}}}_2^{-[\tilde{\mathbb{R}}]} + \check{\check{\mathfrak{z}}}_1^{-[\tilde{\mathbb{R}}]} \check{\check{\mathfrak{z}}}_2^{-[\tilde{\mathbb{R}}]}\right] + \\ \iota \left[\check{\check{\mathfrak{z}}}_1^{-[\tilde{\mathbb{I}}]} + \check{\check{\mathfrak{z}}}_2^{-[\tilde{\mathbb{I}}]} + \check{\check{\mathfrak{z}}}_1^{-[\tilde{\mathbb{I}}]} \check{\check{\mathfrak{z}}}_2^{-[\tilde{\mathbb{I}}]}\right] \end{array} \right) \end{array} \right)$$

3) $\overline{\mathfrak{U}}^{\lambda}$
$$= \left(\begin{array}{c} \left(\left[\left(\check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{R}}]} \right)^{\lambda} \right] + \iota \left[\left(\check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{I}}]} \right)^{\lambda} \right] \right), \\ \left(\left[-1 + \left(1 + \check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{R}}]} \right)^{\lambda} \right] + \iota \left[-1 + \left(1 + \check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{I}}]} \right)^{\lambda} \right] \right) \end{array} \right)$$

4) $\lambda \overline{\mathfrak{U}}$
$$= \left(\begin{array}{c} \left(\left[1 - \left(1 - \check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{R}}]} \right)^{\lambda} \right] + \iota \left[1 - \left(1 - \check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{I}}]} \right)^{\lambda} \right] \right), \\ \left(\left[- \left| \check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{R}}]} \right|^{\lambda} \right] + \iota \left[- \left| \check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{I}}]} \right|^{\lambda} \right] \right) \end{array} \right)$$

Definition 7 ([47]): Let $\overline{\mathfrak{U}} = \left(\S^{+},\ \S^{-}\right) = \left(\check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{R}}]} + \iota \check{\check{\mathfrak{z}}}^{+[\tilde{\mathbb{I}}]}, \check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{R}}]} + \iota \check{\check{\mathfrak{z}}}^{-[\tilde{\mathbb{I}}]}\right)$ be a BCFN then the score and accuracy functions are:

$\check{s}(\overline{\mathfrak{U}}) = \frac{1}{4}\left(2 + \check{\check{\mathfrak{z}}}^{+[\tilde{R}]} + \check{\check{\mathfrak{z}}}^{+[\tilde{I}]} + \check{\check{\mathfrak{z}}}^{-[\tilde{R}]} + \check{\check{\mathfrak{z}}}^{-[\tilde{I}]}\right)$ where $\check{s}(\overline{\mathfrak{U}}) \in [0,\ 1]$.

$\check{\alpha}(\overline{\mathfrak{U}}) = \frac{1}{4}\left(\check{\check{\mathfrak{z}}}^{+[\tilde{R}]} + \check{\check{\mathfrak{z}}}^{+[\tilde{I}]} - \check{\check{\mathfrak{z}}}^{-[\tilde{R}]} - \check{\check{\mathfrak{z}}}^{-[\tilde{I}]}\right)$ where $\check{\alpha}(\overline{\mathfrak{U}}) \in [0,\ 1]$.

The above discussion of different notions or definitions shows the basics of the proposed work and also shows why we need the theory of HBCFSs. We can make a simple comparison like HFSs can only deal with the membership values in the form of a set and cannot deal with all of our theory components like imaginary parts etc. Moreover, the notions of BCFSs are closer to our work but cannot exactly solve our information and data due to some restrictions. BCFSs can deal positive and negative aspects of any object but cannot solve all BCF information in the form of a set. So, we can say that is why we need the theory of HBCFSs.

III. HESITANT BIPOLAR COMPLEX FUZZY SET AND THEIR OPERATIONS

In this section, we studied the basic definition, operations, score, and accuracy functions related to HBCFSs.

Definition 8: A HBCFS hat ($\hat{\mathfrak{B}}$ over fixed hat ($\hat{\mathfrak{G}}$ is noted as

$$\begin{aligned}\hat{\mathfrak{B}} &= \left\{ <\mathfrak{a}, \overline{\mathfrak{U}}_{\hat{\mathfrak{B}}}(\mathfrak{a})> \mid \mathfrak{a}\in\hat{\mathfrak{G}}\right\}\\ &= \left\{<\mathfrak{a}, \left(\S^{+}_{\hat{\mathfrak{B}}}(\mathfrak{a}),\ \S^{-}_{\hat{\mathfrak{B}}}(\mathfrak{a})\right)> \mid \mathfrak{a}\in\hat{\mathfrak{G}}\right\} \qquad (3)\end{aligned}$$

where $\S^{+}_{\hat{\mathfrak{B}}}(\mathfrak{a}) = \left\{\check{\check{\mathfrak{z}}}^{+[\tilde{R}]}_{\hat{\mathfrak{B}}_j}(\mathfrak{a}) + \iota\check{\check{\mathfrak{z}}}^{+[\tilde{I}]}_{\hat{\mathfrak{B}}_j}(\mathfrak{a}),\ j = 1,\ 2,\ \ldots m\right\}$ is the set of finite values that lies in the unit square of a complex plane denotes the positive part of the membership grade (PPMG) and $\S^{-}_{\hat{\mathfrak{B}}}(\mathfrak{a}) = \{\check{\check{\mathfrak{z}}}^{-[\tilde{R}]}_{\hat{\mathfrak{B}}_k}(\mathfrak{a}) + \iota\check{\check{\mathfrak{z}}}^{-[\tilde{I}]}_{\hat{\mathfrak{B}}_k}(\mathfrak{a}),\ k = 1,\ 2,\ \ldots \hat{n}\}$ is the set of finite values that lies in the unit square of a complex plane denotes the negative part of the membership grade (NPMG) of each $\mathfrak{a}\in\hat{\mathfrak{G}}$. For easiness, the hesitant bipolar complex fuzzy number (HBCFN) is symbolized by $\overline{\mathfrak{U}} = (\S^{+},\ \S^{-}) = \left(\check{\check{\mathfrak{z}}}^{+[\tilde{R}]} + \iota\check{\check{\mathfrak{z}}}^{+[\tilde{I}]}, \check{\check{\mathfrak{z}}}^{-[\tilde{R}]} + \iota\check{\check{\mathfrak{z}}}^{-[\tilde{I}]}\right)$.

Remark 1: When the imaginary component (IC) approaches zero, we receive HBFS. If we disregard the negative degree, we obtain CHFS. If we disregard the negative degree and assume that the IC is equal to zero, we obtain HFS.

Example 1: Let hat $\hat{\mathfrak{G}} = \{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3\}$ be a fixed set. Let $\S^{+}_{\hat{\mathfrak{B}}}(\mathfrak{a}_1) = \{0.5 + \mathrm{i}0.3,\ 0.4 + \mathrm{i}0.2,\ 0.3 + \mathrm{i}0.5\}$, $\S^{+}_{\hat{\mathfrak{B}}}(\mathfrak{a}_2) = \{0.4 + \mathrm{i}0.5,\ 0.2 + \mathrm{i}0.6,\ 0.8 + \mathrm{i}0.9\}$, $\S^{+}_{\hat{\mathfrak{B}}}(\mathfrak{a}_3) = \{0.7 + \mathrm{i}0.3,\ 0.6 + \mathrm{i}0.7,\ 0.5 + \mathrm{i}0.5\}$ and also $\S^{-}_{\hat{\mathfrak{B}}}(\mathfrak{a}_1) = \{-0.1 - \mathrm{i}0.4,\ -0.7 - \mathrm{i}0.2\}$, $\S^{-}_{\hat{\mathfrak{B}}}(\mathfrak{a}_2) = \{-0.12 - \mathrm{i}0.7,\ -0.4 - \mathrm{i}0.3,\ -0.2 - \mathrm{i}0.7\}$, $\S^{-}_{\hat{\mathfrak{B}}}(\mathfrak{a}_3) = \{-0.11 - \mathrm{i}0.4,\ -0.6 - \mathrm{i}0.3,\ -0.21 - \mathrm{i}0.16\}$ be the hesitant complex fuzzy positive elements (HCFPEs) and hesitant complex fuzzy negative elements (HCFNEs) of $\mathfrak{a}_i\ (i = 1,\ 2,\ 3)$ respectively. Then the HBCFS is,

$$\hat{\mathfrak{B}} = \left\{\begin{array}{l} <\mathfrak{a}_1, \left(\begin{array}{c}\left\{\begin{array}{c}0.5 + \mathrm{i}0.3,\\ 0.4 + \mathrm{i}0.2,\\ 0.3 + \mathrm{i}0.5\end{array}\right\},\\ \left\{\begin{array}{c}-0.1 - \mathrm{i}0.4,\\ -0.7 - \mathrm{i}0.2\end{array}\right\}\end{array}\right)>,\\ <\mathfrak{a}_2, \left(\begin{array}{c}\left\{\begin{array}{c}0.4 + \mathrm{i}0.5,\\ 0.2 + \mathrm{i}0.6,\\ 0.8 + \mathrm{i}0.9\end{array}\right\},\\ \left\{\begin{array}{c}-0.12 - \mathrm{i}0.7,\\ -0.4 - \mathrm{i}0.3,\\ -0.2 - \mathrm{i}0.7\end{array}\right\}\end{array}\right)>,\\ <\mathfrak{a}_3, \left(\begin{array}{c}\left\{\begin{array}{c}0.7 + \mathrm{i}0.3,\\ 0.6 + \mathrm{i}0.7,\\ 0.5 + \mathrm{i}0.5\end{array}\right\},\\ \left\{\begin{array}{c}-0.11 - \mathrm{i}0.4,\\ -0.6 - \mathrm{i}0.3,\\ -0.21 - \mathrm{i}0.16\end{array}\right\}\end{array}\right)> \end{array}\right\}$$

Definition 9: Let $\overline{\mathfrak{U}} = (\S^{+},\ \S^{-}) = (\check{\check{\mathfrak{z}}}^{+[\tilde{R}]} + \iota\check{\check{\mathfrak{z}}}^{+[\tilde{I}]}, \check{\check{\mathfrak{z}}}^{-[\tilde{R}]} + \iota\check{\check{\mathfrak{z}}}^{-[\tilde{I}]})$, $\overline{\mathfrak{U}}_1 = (\S^{+}_1,\ \S^{-}_1) = \left(\check{\check{\mathfrak{z}}}_1^{+[\tilde{R}]} + \iota\check{\check{\mathfrak{z}}}_1^{+[\tilde{I}]}, \check{\check{\mathfrak{z}}}_1^{-[\tilde{R}]} + \iota\check{\check{\mathfrak{z}}}_1^{-[\tilde{I}]}\right)$ and $\overline{\mathfrak{U}}_2 = (\S^{+}_2,\ \S^{-}_2) = \left(\check{\check{\mathfrak{z}}}_2^{+[\tilde{R}]} + \iota\check{\check{\mathfrak{z}}}_2^{+[\tilde{I}]}, \check{\check{\mathfrak{z}}}_2^{-[\tilde{R}]} + \iota\check{\check{\mathfrak{z}}}_2^{-[\tilde{I}]}\right)$ be three HBCFNs, then

1. $\overline{\mathfrak{U}}^{c} = \left(\begin{array}{l}\cup_{\check{\check{\mathfrak{z}}}^{+}\in\S^{+}}\left\{\left[1 - \check{\check{\mathfrak{z}}}^{+[\tilde{R}]}\right] + \iota\left[1 - \check{\check{\mathfrak{z}}}^{+[\tilde{I}]}\right]\right\},\\ \cup_{\check{\check{\mathfrak{z}}}^{-}\in\S^{-}}\left\{\left[-1 - \check{\check{\mathfrak{z}}}^{-[\tilde{R}]}\right] + \iota\left[-1 - \check{\check{\mathfrak{z}}}^{-[\tilde{I}]}\right]\right\}\end{array}\right)$

2. $\overline{\mathfrak{U}}_1 \cup \overline{\mathfrak{U}}_2 = \left(\begin{array}{l}\cup_{\check{\check{\mathfrak{z}}}_1^{+}\in\S_1^{+}, \check{\check{\mathfrak{z}}}_2^{+}\in\S_2^{+}}\left\{\begin{array}{l} max\left[\check{\check{\mathfrak{z}}}_1^{+[\tilde{R}]}, \check{\check{\mathfrak{z}}}_2^{+[\tilde{R}]}\right] +\\ \iota max\left[\check{\check{\mathfrak{z}}}_1^{+[\tilde{I}]}, \check{\check{\mathfrak{z}}}_2^{+[\tilde{I}]}\right]\end{array}\right\},\\ \cup_{\check{\check{\mathfrak{z}}}_1^{-}\in\S_1^{-}, \check{\check{\mathfrak{z}}}_2^{-}\in\S_2^{-}}\left\{\begin{array}{l} min\left[\check{\check{\mathfrak{z}}}_1^{-[\tilde{R}]}, \check{\check{\mathfrak{z}}}_2^{-[\tilde{R}]}\right] +\\ \iota min\left[\check{\check{\mathfrak{z}}}_1^{-[\tilde{I}]}, \check{\check{\mathfrak{z}}}_2^{-[\tilde{I}]}\right]\end{array}\right\}\end{array}\right)$

3. $\overline{\mathfrak{U}}_1 \cap \overline{\mathfrak{U}}_2 = \left(\begin{array}{l}\cup_{\check{\check{\mathfrak{z}}}_1^{+}\in\S_1^{+}, \check{\check{\mathfrak{z}}}_2^{+}\in\S_2^{+}}\left\{\begin{array}{l} min\left[\check{\check{\mathfrak{z}}}_1^{+[\tilde{R}]}, \check{\check{\mathfrak{z}}}_2^{+[\tilde{R}]}\right] +\\ \iota min\left[\check{\check{\mathfrak{z}}}_1^{+[\tilde{I}]}, \check{\check{\mathfrak{z}}}_2^{+[\tilde{I}]}\right]\end{array}\right\},\\ \cup_{\check{\check{\mathfrak{z}}}_1^{-}\in\S_1^{-}, \check{\check{\mathfrak{z}}}_2^{-}\in\S_2^{-}}\left\{\begin{array}{l} max\left[\check{\check{\mathfrak{z}}}_1^{-[\tilde{R}]}, \check{\check{\mathfrak{z}}}_2^{-[\tilde{R}]}\right] +\\ \iota max\left[\check{\check{\mathfrak{z}}}_1^{-[\tilde{I}]}, \check{\check{\mathfrak{z}}}_2^{-[\tilde{I}]}\right]\end{array}\right\}\end{array}\right)$

Definition 10: Let $\overline{\mathfrak{U}}_1 = (\S^{+}_1,\ \S^{-}_1) = \left(\begin{array}{l}\check{\check{\mathfrak{z}}}_1^{+[\tilde{R}]} + \iota\check{\check{\mathfrak{z}}}_1^{+[\tilde{I}]},\\ \check{\check{\mathfrak{z}}}_1^{-[\tilde{R}]} + \iota\check{\check{\mathfrak{z}}}_1^{-[\tilde{I}]}\end{array}\right)$ and $\overline{\mathfrak{U}}_2 = (\S^{+}_2,\ \S^{-}_2) = \left(\begin{array}{l}\check{\check{\mathfrak{z}}}_2^{+[\tilde{R}]} + \iota\check{\check{\mathfrak{z}}}_2^{+[\tilde{I}]},\\ \check{\check{\mathfrak{z}}}_2^{-[\tilde{R}]} + \iota\check{\check{\mathfrak{z}}}_2^{-[\tilde{I}]}\end{array}\right)$ be two HBCFNs, then

1. $\overline{\text{Ꝗ}}_1 \oplus \overline{\text{Ꝗ}}_2$

$$= \left(\begin{array}{l} \cup_{\text{ǯ}_1^{+} \in §_1^{+}, \text{ǯ}_2^{+} \in §_2^{+}} \left\{ \begin{array}{l} \left[\text{ǯ}_1^{+[\tilde{R}]} + \text{ǯ}_2^{+[\tilde{R}]} - \text{ǯ}_1^{+[\tilde{R}]} \text{ǯ}_2^{+[\tilde{R}]} \right] + \\ \iota \left[\text{ǯ}_1^{+[\tilde{I}]} + \text{ǯ}_2^{+[\tilde{I}]} - \text{ǯ}_1^{+[\tilde{I}]} \text{ǯ}_2^{+[\tilde{I}]} \right] \end{array} \right\}, \\ \cup_{\text{ǯ}_1^{-} \in §_1^{-}, \text{ǯ}_2^{-} \in §_2^{-}} \left\{ \left[-\text{ǯ}_1^{-[\tilde{R}]} \text{ǯ}_2^{-[\tilde{R}]} \right] + \iota \left[-\text{ǯ}_1^{-[\tilde{I}]} \text{ǯ}_2^{-[\tilde{I}]} \right] \right\} \end{array} \right)$$

2. $\overline{\text{Ꝗ}}_1 \otimes \overline{\text{Ꝗ}}_2$

$$= \left(\begin{array}{l} \cup_{\text{ǯ}_1^{+} \in §_1^{+}, \text{ǯ}_2^{+} \in §_2^{+}} \left\{ \left[\text{ǯ}_1^{+[\tilde{R}]} \text{ǯ}_2^{+[\tilde{R}]} \right] + \iota \left[\text{ǯ}_1^{+[\tilde{I}]} \text{ǯ}_2^{+[\tilde{I}]} \right] \right\}, \\ \cup_{\text{ǯ}_1^{-} \in §_1^{-}, \text{ǯ}_2^{-} \in §_2^{-}} \left\{ \begin{array}{l} \left[\text{ǯ}_1^{-[\tilde{R}]} + \text{ǯ}_2^{-[\tilde{R}]} + \text{ǯ}_1^{-[\tilde{R}]} \text{ǯ}_2^{-[\tilde{R}]} \right] + \\ \iota \left[\text{ǯ}_1^{-[\tilde{I}]} + \text{ǯ}_2^{-[\tilde{I}]} + \text{ǯ}_1^{-[\tilde{I}]} \text{ǯ}_2^{-[\tilde{I}]} \right] \end{array} \right\} \end{array} \right)$$

Definition 11: Let $\overline{\text{Ꝗ}} = (§^{+}, §^{-}) = (\text{ǯ}^{+[\tilde{R}]} + \iota \text{ǯ}^{+[\tilde{I}]}, \text{ǯ}^{-[\tilde{R}]} + \iota \text{ǯ}^{-[\tilde{I}]})$ be an HBCFN and $\lambda > 0$, then.

1.

$$\overline{\text{Ꝗ}}^{\lambda} = \left(\begin{array}{l} \cup_{\text{ǯ}^{+} \in §^{+}} \left\{ \left[\left(\text{ǯ}^{+[\tilde{R}]} \right)^{\lambda} \right] + \iota \left[\left(\text{ǯ}^{+[\tilde{I}]} \right)^{\lambda} \right] \right\}, \\ \cup_{\text{ǯ}^{-} \in §^{-}} \left\{ \left[-1 + \left(1 + \text{ǯ}^{-[\tilde{R}]} \right)^{\lambda} \right] \right. \\ \left. + \iota \left[-1 + \left(1 + \text{ǯ}^{-[\tilde{I}]} \right)^{\lambda} \right] \right\} \end{array} \right)$$

2.

$$\overline{\lambda \text{Ꝗ}} = \left(\begin{array}{l} \cup_{\text{ǯ}^{+} \in §^{+}} \left\{ \left[1 - \left(1 - \text{ǯ}^{+[\tilde{R}]} \right)^{\lambda} \right] \right. \\ \left. + \iota \left[1 - \left(1 - \text{ǯ}^{+[\tilde{I}]} \right)^{\lambda} \right] \right\}, \\ \cup_{\text{ǯ}^{-} \in §^{-}} \left\{ \left[- \left| \text{ǯ}^{-[\tilde{R}]} \right|^{\lambda} \right] \right. \\ \left. + \iota \left[- \left| \text{ǯ}^{-[\tilde{I}]} \right|^{\lambda} \right] \right\} \end{array} \right)$$

Definition 12: Let $\overline{\text{Ꝗ}} = (§^{+}, §^{-}) = (\text{ǯ}^{+[\tilde{R}]} + \iota \text{ǯ}^{+[\tilde{I}]}, \text{ǯ}^{-[\tilde{R}]} + \iota \text{ǯ}^{-[\tilde{I}]})$ be an HBCFN then the score function is originated by:

$$\mathbb{S}(\overline{\text{Ꝗ}}) = \frac{1}{4} \left(\begin{array}{l} 2 + \frac{1}{\text{ɭ}_{\text{ǯ}^{+[\tilde{R}]}}} \sum_{\text{ǯ}^{+} \in §^{+}} \text{ǯ}^{+[\tilde{R}]} + \frac{1}{\text{ɭ}_{\text{ǯ}^{+[\tilde{I}]}}} \sum_{\text{ǯ}^{+} \in §^{+}} \text{ǯ}^{+[\tilde{I}]} \\ + \frac{1}{\text{ɭ}_{\text{ǯ}^{-[\tilde{R}]}}} \sum_{\text{ǯ}^{-} \in §^{-}} \text{ǯ}^{-[\tilde{R}]} + \frac{1}{\text{ɭ}_{\text{ǯ}^{-[\tilde{I}]}}} \sum_{\text{ǯ}^{-} \in §^{-}} \text{ǯ}^{-[\tilde{I}]} \end{array} \right),$$
$$\mathbb{S}(\overline{\text{Ꝗ}}) \in [0, 1] \tag{4}$$

Definition 13: Let $\overline{\text{Ꝗ}} = (§^{+}, §^{-}) = (\text{ǯ}^{+[\tilde{R}]} + \iota \text{ǯ}^{+[\tilde{I}]}, \text{ǯ}^{-[\tilde{R}]} + \iota \text{ǯ}^{-[\tilde{I}]})$ be an HBCFN then the accuracy function is originated by:

$$\mathbb{A}(\overline{\text{Ꝗ}}) = \frac{1}{4} \left(\begin{array}{l} \frac{1}{\text{ɭ}_{\text{ǯ}^{+[\tilde{R}]}}} \sum_{\text{ǯ}^{+} \in §^{+}} \text{ǯ}^{+[\tilde{R}]} + \frac{1}{\text{ɭ}_{\text{ǯ}^{+[\tilde{I}]}}} \sum_{\text{ǯ}^{+} \in §^{+}} \text{ǯ}^{+[\tilde{I}]} \\ - \frac{1}{\text{ɭ}_{\text{ǯ}^{-[\tilde{R}]}}} \sum_{\text{ǯ}^{-} \in §^{-}} \text{ǯ}^{-[\tilde{R}]} - \frac{1}{\text{ɭ}_{\text{ǯ}^{-[\tilde{I}]}}} \sum_{\text{ǯ}^{-} \in §^{-}} \text{ǯ}^{-[\tilde{I}]} \end{array} \right),$$
$$\mathbb{A}(\overline{\text{Ꝗ}}) \in [0, 1] \tag{5}$$

where ɭ is the length of positive real, positive imaginary, and negative real, negative imaginary numbers, using the above equations (4) and (5) for two HBCFNs $\overline{\text{Ꝗ}}_1$ and $\overline{\text{Ꝗ}}_2$ we have the following criteria for comparison:

If $\mathbb{S}(\overline{\text{Ꝗ}}_1) < \mathbb{S}(\overline{\text{Ꝗ}}_2)$, then $\overline{\text{Ꝗ}}_1 < \overline{\text{Ꝗ}}_2$, If $\mathbb{S}(\overline{\text{Ꝗ}}_1) > \mathbb{S}(\overline{\text{Ꝗ}}_2)$, then $\overline{\text{Ꝗ}}_1 > \overline{\text{Ꝗ}}_2$, If $\mathbb{S}(\overline{\text{Ꝗ}}_1) = \mathbb{S}(\overline{\text{Ꝗ}}_2)$, then If $\mathbb{A}(\overline{\text{Ꝗ}}_1) < \mathbb{A}(\overline{\text{Ꝗ}}_2)$, then $\overline{\text{Ꝗ}}_1 < \overline{\text{Ꝗ}}_2$, If $\mathbb{A}(\overline{\text{Ꝗ}}_1) > \mathbb{A}(\overline{\text{Ꝗ}}_2)$, then $\overline{\text{Ꝗ}}_1 > \overline{\text{Ꝗ}}_2$, If $\mathbb{A}(\overline{\text{Ꝗ}}_1) = \mathbb{A}(\overline{\text{Ꝗ}}_2)$ then $\overline{\text{Ꝗ}}_1 = \overline{\text{Ꝗ}}_2$.

Theorem 1: Let us assume that $\overline{\text{Ꝗ}} = (§^{+}, §^{-}) = \left(\begin{array}{l} \text{ǯ}^{+[\tilde{R}]} + \iota \text{ǯ}^{+[\tilde{I}]}, \\ \text{ǯ}^{-[\tilde{R}]} + \iota \text{ǯ}^{-[\tilde{I}]} \end{array} \right)$, $\overline{\text{Ꝗ}}_1 = (§_1^{+}, §_1^{-}) = \left(\begin{array}{l} \text{ǯ}_1^{+[\tilde{R}]} + \iota \text{ǯ}_1^{+[\tilde{I}]}, \\ \text{ǯ}_1^{-[\tilde{R}]} + \iota \text{ǯ}_1^{-[\tilde{I}]} \end{array} \right)$ and $\overline{\text{Ꝗ}}_2 = (§_2^{+}, §_2^{-}) = \left(\begin{array}{l} \text{ǯ}_2^{+[\tilde{R}]} + \iota \text{ǯ}_2^{+[\tilde{I}]}, \\ \text{ǯ}_2^{-[\tilde{R}]} + \iota \text{ǯ}_2^{-[\tilde{I}]} \end{array} \right)$ be three HBCFNs and $\lambda > 0$, then the following holds

a) $\left(\overline{\text{Ꝗ}}^{c} \right)^{\lambda} = \left(\overline{\lambda \text{Ꝗ}} \right)^{c}$
b) $\lambda \overline{\text{Ꝗ}}^{c} = \left(\overline{\text{Ꝗ}}^{\lambda} \right)^{c}$
c) $\overline{\text{Ꝗ}}_1^{c} \cup \overline{\text{Ꝗ}}_2^{c} = \left(\overline{\text{Ꝗ}}_1 \cap \overline{\text{Ꝗ}}_2 \right)^{c}$
d) $\overline{\text{Ꝗ}}_1^{c} \cap \overline{\text{Ꝗ}}_2^{c} = \left(\overline{\text{Ꝗ}}_1 \cup \overline{\text{Ꝗ}}_2 \right)^{c}$
e) $\overline{\text{Ꝗ}}_1^{c} \oplus \overline{\text{Ꝗ}}_2^{c} = \left(\overline{\text{Ꝗ}}_1 \otimes \overline{\text{Ꝗ}}_2 \right)^{c}$
f) $\overline{\text{Ꝗ}}_1^{c} \otimes \overline{\text{Ꝗ}}_2^{c} = \left(\overline{\text{Ꝗ}}_1 \oplus \overline{\text{Ꝗ}}_2 \right)^{c}$
g) $\lambda \left(\overline{\text{Ꝗ}}_1 \oplus \overline{\text{Ꝗ}}_2 \right) = \overline{\lambda \text{Ꝗ}_1} \oplus \overline{\lambda \text{Ꝗ}_2}$
h) $\left(\overline{\text{Ꝗ}}_1 \otimes \overline{\text{Ꝗ}}_2 \right)^{\lambda} = \overline{\text{Ꝗ}}_1^{\lambda} \otimes \overline{\text{Ꝗ}}_2^{\lambda}$.

A. DOMBI OPERATIONS

Dombi product and Dombi sum which are particular cases of t-norm and t-conforms are discussed below.

Definition 14 ([48]): Let Υ_1, Υ_2 be two real numbers, then the Dombi t-norm and Dombi t-conorm are defined as;

$$Dom(\Upsilon_1, \Upsilon_2) = \frac{1}{1 + \left\{ \left(\frac{1-\Upsilon_1}{\Upsilon_1} \right)^{\text{ɷ}} + \left(\frac{1-\Upsilon_2}{\Upsilon_2} \right)^{\text{ɷ}} \right\}^{\frac{1}{\text{ɷ}}}} \tag{6}$$

$$Dom*(\Upsilon_1, \Upsilon_2) = 1 - \frac{1}{1 + \left\{ \left(\frac{\Upsilon_1}{1-\Upsilon_1} \right)^{\text{ɷ}} + \left(\frac{\Upsilon_2}{1-\Upsilon_2} \right)^{\text{ɷ}} \right\}^{\frac{1}{\text{ɷ}}}} \tag{7}$$

where $\text{ɷ} \geq 1$ and $(\Upsilon_1, \Upsilon_2) \in [0, 1] \times [0, 1]$.

B. DOMBI OPERATIONSON HBCFNS

In this subsection, we developed the Dombi operations on HBCFNs.

Definition 15: Let $\bar{Џ} = (\S^{+}, \S^{-}) = (\breve{\breve{\mathfrak{Z}}}^{+[\tilde{R}]} + \iota\breve{\breve{\mathfrak{Z}}}^{+[\tilde{I}]}, \breve{\breve{\mathfrak{Z}}}^{-[\tilde{R}]} + \iota\breve{\breve{\mathfrak{Z}}}^{-[\tilde{I}]})$, $\bar{Џ}_1 = (\S_1^{+}, \S_1^{-}) = (\breve{\breve{\mathfrak{Z}}}_1^{+[\tilde{R}]} + \iota\breve{\breve{\mathfrak{Z}}}_1^{+[\tilde{I}]}, \breve{\breve{\mathfrak{Z}}}_1^{-[\tilde{R}]} + \iota\breve{\breve{\mathfrak{Z}}}_1^{-[\tilde{I}]})$ and $\bar{Џ}_2 = (\S_2^{+}, \S_2^{-}) = (\breve{\breve{\mathfrak{Z}}}_2^{+[\tilde{R}]} + \iota\breve{\breve{\mathfrak{Z}}}_2^{+[\tilde{I}]}, \breve{\breve{\mathfrak{Z}}}_2^{-[\tilde{R}]} + \iota\breve{\breve{\mathfrak{Z}}}_2^{-[\tilde{I}]})$ be three HBCFNs and $\lambda > 0$ then

1. As shown at the bottom of the page.
2. As shown at the bottom of the next page.
3. As shown at the bottom of page 9.
4. As shown at the bottom of page 10.

IV. HESITANT BIPOLAR COMPLEX FUZZY DOMBI AGGREGATION OPERATORS

In this section of the article, we developed the Dombi arithmetic AOs and Dombi geometric AOs in the environment of HBCFNs

A. HBCF DOMBI ARITHMETIC AGGREGATION OPERATIONS

This section consists of some very necessary AOs in the environment of HBCFNs, such as hesitant bipolar complex fuzzy Dombi (HBCFD) weighted averaging (HBCFDWA), HBCFD ordered weighted averaging (HBCFDOWA), HBCFD hybrid averaging (HBCFDHA), HBCFH weighted geometric (HBCFDWG), HBCFD ordered weighted geometric (HBCFDOWG) and HBCFD hybrid geometric (HBCFDHG) operators.

Definition 16: Let $\bar{Џ}_{\mathfrak{s}} = (\S_{\mathfrak{s}}^{+}, \S_{\mathfrak{s}}^{-}) = (\breve{\breve{\mathfrak{Z}}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\breve{\breve{\mathfrak{Z}}}_{\mathfrak{s}}^{+[\tilde{I}]}, \breve{\breve{\mathfrak{Z}}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\breve{\breve{\mathfrak{Z}}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, \acute{n})$ be a collection of HBCFNs, then the HBCFDWA operator is a function of HBCFDWA $: \bar{Џ}^{\acute{n}} \longrightarrow \bar{Џ}$ such that.

$$\text{HBCFDWA}_{\omega}\left(\bar{Џ}_1, \bar{Џ}_2, \ldots, \bar{Џ}_{\acute{n}}\right) = \bigoplus_{\mathfrak{s}=1}^{\acute{n}} \left(\omega_{\mathfrak{s}}\bar{Џ}_{\mathfrak{s}}\right) \quad (8)$$

$$\bar{Џ}_1 \oplus \bar{Џ}_2 = \left(\begin{array}{l} \bigcup_{\breve{\breve{\mathfrak{Z}}}_1^{+} \in \S_1^{+}, \breve{\breve{\mathfrak{Z}}}_2^{+} \in \S_2^{+}} \left\{ \begin{array}{l} \left(1 - \dfrac{1}{1 + \left\{ \left(\dfrac{\breve{\breve{\mathfrak{Z}}}_1^{+[\tilde{R}]}}{1 - \breve{\breve{\mathfrak{Z}}}_1^{+[\tilde{R}]}} \right)^{\text{б}} + \left(\dfrac{\breve{\breve{\mathfrak{Z}}}_2^{+[\tilde{R}]}}{1 - \breve{\breve{\mathfrak{Z}}}_2^{+[\tilde{R}]}} \right)^{\text{б}} \right\}^{\frac{1}{\text{б}}}} \right) + \\ \iota \left(1 - \dfrac{1}{1 + \left\{ \left(\dfrac{\breve{\breve{\mathfrak{Z}}}_1^{+[\tilde{I}]}}{1 - \breve{\breve{\mathfrak{Z}}}_1^{+[\tilde{I}]}} \right)^{\text{б}} + \left(\dfrac{\breve{\breve{\mathfrak{Z}}}_2^{+[\tilde{I}]}}{1 - \breve{\breve{\mathfrak{Z}}}_2^{+[\tilde{I}]}} \right)^{\text{б}} \right\}^{\frac{1}{\text{б}}}} \right) \end{array} \right\}, \\ \bigcup_{\breve{\breve{\mathfrak{Z}}}_1^{-} \in \S_1^{-}, \breve{\breve{\mathfrak{Z}}}_2^{-} \in \S_2^{-}} \left\{ \begin{array}{l} \left(\dfrac{-1}{1 + \left\{ \left(\dfrac{1 + \breve{\breve{\mathfrak{Z}}}_1^{-[\tilde{R}]}}{\left| \breve{\breve{\mathfrak{Z}}}_1^{-[\tilde{R}]} \right|} \right)^{\text{б}} + \left(\dfrac{1 + \breve{\breve{\mathfrak{Z}}}_2^{-[\tilde{R}]}}{\left| \breve{\breve{\mathfrak{Z}}}_2^{-[\tilde{R}]} \right|} \right)^{\text{б}} \right\}^{\frac{1}{\text{б}}}} \right) + \iota \\ \left(\dfrac{-1}{1 + \left\{ \left(\dfrac{1 + \breve{\breve{\mathfrak{Z}}}_1^{-[\tilde{I}]}}{\left| \breve{\breve{\mathfrak{Z}}}_1^{-[\tilde{I}]} \right|} \right)^{\text{б}} + \left(\dfrac{1 + \breve{\breve{\mathfrak{Z}}}_2^{-[\tilde{I}]}}{\left| \breve{\breve{\mathfrak{Z}}}_2^{-[\tilde{I}]} \right|} \right)^{\text{б}} \right\}^{\frac{1}{\text{б}}}} \right) \end{array} \right\} \end{array} \right)$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_{ń})^T$ be the weight vector (WV) of $\overline{Џ}_{\mathfrak{s}} = \left(\S^+_{\mathfrak{s}}, \S^-_{\mathfrak{s}}\right)\left(\mathfrak{s} = 1, 2, \ldots, ń\right)$ with $\omega_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{ń} \omega_{\mathfrak{s}} = 1$.

Theorem 2: Let $\overline{Џ}_{\mathfrak{s}} = (\S^+_{\mathfrak{s}}, \S^-_{\mathfrak{s}}) = (\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, ń)$ be a collection of HBCFNs, then by employing the above eq. (14), we get the HBCFN and as (9), shown at the bottom of page 11.

where $\omega = (\omega_1, \omega_2, \ldots, \omega_{ń})^T$ be the weight vector (WV) of $\overline{Џ}_{\mathfrak{s}} = \left(\S^+_{\mathfrak{s}}, \S^-_{\mathfrak{s}}\right)\left(\mathfrak{s} = 1, 2, \ldots, ń\right)$ with $\omega_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{ń} \omega_{\mathfrak{s}} = 1$.

Proof: By mathematical induction to prove the above statement.

Let $ń = 2$, the left side of (9) becomes $HBCFDWA_{\omega}(\overline{Џ}_1, \overline{Џ}_2) = \overline{Џ}_1 \oplus \overline{Џ}_2$ and the Right side of (9) becomes, as shown at the bottom of page 12.

This shows that (9) holds for $ń = 2$.

Now assume that (9) holds for $ń = ķ$

Next, we show that (9) holds for $ń = ķ + 1$

This shows that (9) holds for $ń = ķ + 1$. This implies that (9) holds for every $ń$.

Given below properties can be proved in the simplest way

Theorem 3 (Idempotency): Let $\overline{Џ}_{\mathfrak{s}} = \left(\S^+_{\mathfrak{s}}, \S^-_{\mathfrak{s}}\right) = \left(\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]}\right)\left(\mathfrak{s} = 1, 2, 3, \ldots, ń\right)$ be a collection of HBCFNs, $\omega = (\omega_1, \omega_2, \ldots, \omega_{ń})^T$ be the weight

$$\overline{Џ}_1 \otimes \overline{Џ}_2 = \left(\bigcup_{\check{\mathfrak{Z}}_1^{+} \in \S_1^+, \check{\mathfrak{Z}}_2^{+} \in \S_2^+} \left\{ \left(\frac{1}{1+\left\{ \left(\frac{1-\check{\mathfrak{Z}}_1^{+[\tilde{R}]}}{\check{\mathfrak{Z}}_1^{+[\tilde{R}]}} \right)^{6} + \left(\frac{1-\check{\mathfrak{Z}}_2^{+[\tilde{R}]}}{\check{\mathfrak{Z}}_2^{+[\tilde{R}]}} \right)^{6} \right\}^{\frac{1}{6}}} \right) + \iota \left(\frac{1}{1+\left\{ \left(\frac{1-\check{\mathfrak{Z}}_1^{+[\tilde{I}]}}{\check{\mathfrak{Z}}_1^{+[\tilde{I}]}} \right)^{6} + \left(\frac{1-\check{\mathfrak{Z}}_2^{+[\tilde{I}]}}{\check{\mathfrak{Z}}_2^{+[\tilde{I}]}} \right)^{6} \right\}^{\frac{1}{6}}} \right) \right\}, \bigcup_{\check{\mathfrak{Z}}_1^{-} \in \S_1^-, \check{\mathfrak{Z}}_2^{-} \in \S_2^-} \left\{ \left(-1 + \frac{1}{1+\left\{ \left(\frac{\left|\check{\mathfrak{Z}}_1^{-[\tilde{R}]}\right|}{1+\check{\mathfrak{Z}}_1^{-[\tilde{R}]}} \right)^{6} + \left(\frac{\left|\check{\mathfrak{Z}}_2^{-[\tilde{R}]}\right|}{1+\check{\mathfrak{Z}}_2^{-[\tilde{R}]}} \right)^{6} \right\}^{\frac{1}{6}}} \right) + \iota \left(-1 + \frac{1}{1+\left\{ \left(\frac{\left|\check{\mathfrak{Z}}_1^{-[\tilde{I}]}\right|}{1+\check{\mathfrak{Z}}_1^{-[\tilde{I}]}} \right)^{6} + \left(\frac{\left|\check{\mathfrak{Z}}_2^{-[\tilde{I}]}\right|}{1+\check{\mathfrak{Z}}_2^{-[\tilde{I}]}} \right)^{6} \right\}^{\frac{1}{6}}} \right) \right\} \right)$$

vector (WV) of $\overline{\text{Џ}}_{\mathfrak{s}} = \left(\S_{\mathfrak{s}}^{+},\ \S_{\mathfrak{s}}^{-}\right)\left(\mathfrak{s} = 1,\ 2, \ldots, \acute{n}\right)$ with $\omega_{\mathfrak{s}} \in [0,\ 1]$ and $\sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} = 1$. If $\overline{\text{Џ}}_{\mathfrak{s}} = \overline{\text{Џ}}\ \forall \mathfrak{s}$ then

$$\text{HBCFDWA}_{\omega}\left(\overline{\text{Џ}}_1, \overline{\text{Џ}}_2, \ldots, \overline{\text{Џ}}_{\acute{n}}\right) = \overline{\text{Џ}}$$

Theorem 4 (Boundedness): Let $\overline{\text{Џ}}_{\mathfrak{s}} = (\S_{\mathfrak{s}}^{+},\ \S_{\mathfrak{s}}^{-}) = (\check{\breve{\text{Ʒ}}}_{\mathfrak{s}}^{+[\check{R}]} + \iota\check{\breve{\text{Ʒ}}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\breve{\text{Ʒ}}}_{\mathfrak{s}}^{-[\check{R}]} + \iota\check{\breve{\text{Ʒ}}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, \acute{n})$ be a collection of HBCFNs, and $\overline{\text{Џ}}^{-} = \min(\overline{\text{Џ}}_1,\ \overline{\text{Џ}}_2,\ \overline{\text{Џ}}_3, \ldots, \overline{\text{Џ}}_{\acute{n}}), \overline{\text{Џ}}^{+} = \max(\overline{\text{Џ}}_1,\ \overline{\text{Џ}}_2,\ \overline{\text{Џ}}_3, \ldots, \overline{\text{Џ}}_{\acute{n}})$ then

$$\overline{\text{Џ}}^{-} \leq \text{HBCFDWA}_{\omega}\left(\overline{\text{Џ}}_1, \overline{\text{Џ}}_2, \ldots, \overline{\text{Џ}}_{\acute{n}}\right) \leq \overline{\text{Џ}}^{+}$$

Theorem 5 (Monotonicity): Let $\overline{\text{Џ}}_{\mathfrak{s}} = (\S_{\mathfrak{s}}^{+},\ \S_{\mathfrak{s}}^{-}) = (\check{\breve{\text{Ʒ}}}_{\mathfrak{s}}^{+[\check{R}]} + \iota\check{\breve{\text{Ʒ}}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\breve{\text{Ʒ}}}_{\mathfrak{s}}^{-[\check{R}]} + \iota\check{\breve{\text{Ʒ}}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, \acute{n})$ and $\overline{\text{Џ}}'_{\mathfrak{s}} = (\S'^{+}_{\mathfrak{s}},\ \S'^{-}_{\mathfrak{s}}) = (\check{\breve{\text{Ʒ}}}'^{+[\check{R}]}_{\mathfrak{s}} + \iota\check{\breve{\text{Ʒ}}}'^{+[\tilde{I}]}_{\mathfrak{s}}, \check{\breve{\text{Ʒ}}}'^{-[\check{R}]}_{\mathfrak{s}} + \iota\check{\breve{\text{Ʒ}}}'^{-[\tilde{I}]}_{\mathfrak{s}})(\mathfrak{s} = 1, 2, 3, \ldots, \acute{n})$ be a collection of two HBCFNs, then

$$\text{HBCFDWA}_{\omega}\left(\overline{\text{Џ}}_1, \overline{\text{Џ}}_2, \ldots, \overline{\text{Џ}}_{\acute{n}}\right) \leq \text{HBCFDWA}_{\omega}\left(\overline{\text{Џ}}'_1, \overline{\text{Џ}}'_2, \ldots, \overline{\text{Џ}}'_{\acute{n}}\right)$$

If $\overline{\text{Џ}}_{\mathfrak{s}} \leq \overline{\text{Џ}}'_{\mathfrak{s}}\ \forall \mathfrak{s}$.

Definition 17: Let $\overline{\text{Џ}}_{\mathfrak{s}} = (\S_{\mathfrak{s}}^{+},\ \S_{\mathfrak{s}}^{-}) = (\check{\breve{\text{Ʒ}}}_{\mathfrak{s}}^{+[\check{R}]} + \iota\check{\breve{\text{Ʒ}}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\breve{\text{Ʒ}}}_{\mathfrak{s}}^{-[\check{R}]} + \iota\check{\breve{\text{Ʒ}}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, \acute{n})$ be a collection of HBCFNs, then the HBCFDOWA operator is a function of HBCFDOWA $: \overline{\text{Џ}}^{\acute{n}} \longrightarrow \overline{\text{Џ}}$ such that.

$$\text{HBCFDOWA}_{\psi}\left(\overline{\text{Џ}}_1, \overline{\text{Џ}}_2, \ldots, \overline{\text{Џ}}_{\acute{n}}\right) = \bigoplus_{\mathfrak{s}=1}^{\acute{n}} \left(\psi_{\mathfrak{s}} \overline{\text{Џ}}_{\mho(\mathfrak{s})}\right) \quad (10)$$

$$\overline{\lambda \text{Џ}} = \left(\begin{array}{c} \cup_{\check{\breve{\text{Ʒ}}}^{+} \in \S^{+}} \left\{ \begin{array}{c} \left(1 - \dfrac{1}{1+\left\{\lambda\left(\dfrac{\check{\breve{\text{Ʒ}}}^{+[\check{R}]}}{1-\check{\breve{\text{Ʒ}}}^{+[\check{R}]}}\right)^{\text{ϭ}}\right\}^{\frac{1}{\text{ϭ}}}}\right) + \\ \iota\left(1 - \dfrac{1}{1+\left\{\lambda\left(\dfrac{\check{\breve{\text{Ʒ}}}^{+[\tilde{I}]}}{1-\check{\breve{\text{Ʒ}}}^{+[\tilde{I}]}}\right)^{\text{ϭ}}\right\}^{\frac{1}{\text{ϭ}}}}\right) \end{array} \right\}, \\ \cup_{\check{\breve{\text{Ʒ}}}^{-} \in \S^{-}} \left\{ \begin{array}{c} \left(\dfrac{1}{1+\left\{\lambda\left(\dfrac{1+\check{\breve{\text{Ʒ}}}^{-[\check{R}]}}{\left|\check{\breve{\text{Ʒ}}}^{-[\check{R}]}\right|}\right)^{\text{ϭ}}\right\}^{\frac{1}{\text{ϭ}}}}\right) + \\ \iota\left(\dfrac{1}{1+\left\{\lambda\left(\dfrac{1+\check{\breve{\text{Ʒ}}}^{-[\tilde{I}]}}{\left|\check{\breve{\text{Ʒ}}}^{-[\tilde{I}]}\right|}\right)^{\text{ϭ}}\right\}^{\frac{1}{\text{ϭ}}}}\right) \end{array} \right\} \end{array} \right)$$

where $\psi = (\psi_1, \psi_2, \ldots, \psi_{\acute{n}})^T$ be WV of $\overline{\text{Ʉ}}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}})$ $(\mathfrak{s} = 1, 2, \ldots, \acute{n})$ with $\psi_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mathfrak{s}} = 1$, and $\mho(1), \mho(2), \mho(3), \ldots, \mho(\acute{n})$ are the permutation of $\mho(\mathfrak{s})$ $(\mathfrak{s} = 1, 2, 3, \ldots, \acute{n})$ such that $\overline{\text{Ʉ}}_{\mho(\mathfrak{s}-1)} \geq \overline{\text{Ʉ}}_{\mho(\mathfrak{s})}$ $\forall \mathfrak{s}$.

Theorem 6: Let $\overline{\text{Ʉ}}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}}) = (\check{\text{Ƶ}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\check{\text{Ƶ}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\text{Ƶ}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\check{\text{Ƶ}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, \acute{n})$ be a collection of HBCFNs, then by employing the above eq. (10), we get the HBCFN and as (11), shown at the bottom of page 18.

Where $\psi = (\psi_1, \psi_2, \ldots, \psi_{\acute{n}})^T$ be WV of $\overline{\text{Ʉ}}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}})$ $(\mathfrak{s} = 1, 2, \ldots, \acute{n})$ with $\psi_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mathfrak{s}} = 1$, and $\mho(1), \mho(2), \mho(3), \ldots, \mho(\acute{n})$ are the permutation of $\mho(\mathfrak{s})$ $(\mathfrak{s} = 1, 2, 3, \ldots, \acute{n})$ such that $\overline{\text{Ʉ}}_{\mho(\mathfrak{s}-1)} \geq \overline{\text{Ʉ}}_{\mho(\mathfrak{s})}$ $\forall \mathfrak{s}$.

Given below properties can be proved most simply.

Theorem 7 (Idempotency): Let $\overline{\text{Ʉ}}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}}) = (\check{\text{Ƶ}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\check{\text{Ƶ}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\text{Ƶ}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\check{\text{Ƶ}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, \acute{n})$ be a collection of HBCFNs, $\psi = (\psi_1, \psi_2, \ldots, \psi_{\acute{n}})^T$ be the WV of $\overline{\text{Ʉ}}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}})(\mathfrak{s} = 1, 2, \ldots, \acute{n})$ with $\psi_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mathfrak{s}} = 1$. If $\overline{\text{Ʉ}}_{\mathfrak{s}} = \overline{\text{Ʉ}}$ $\forall \mathfrak{s}$ then

$$\text{HBCFDOWA}_{\psi}\left(\overline{\text{Ʉ}}_1, \overline{\text{Ʉ}}_2, \ldots, \overline{\text{Ʉ}}_{\acute{n}}\right) = \overline{\text{Ʉ}}$$

Theorem 8 (Boundedness): Let $\overline{\text{Ʉ}}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}}) = (\check{\text{Ƶ}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\check{\text{Ƶ}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\text{Ƶ}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\check{\text{Ƶ}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, \acute{n})$ be a collection of HBCFNs, and $\overline{\text{Ʉ}}^{-} = \min(\overline{\text{Ʉ}}_1, \overline{\text{Ʉ}}_2, \overline{\text{Ʉ}}_3, \ldots, \overline{\text{Ʉ}}_{\acute{n}}), \overline{\text{Ʉ}}^{+} =$

$$\overline{\text{Ʉ}}^{\lambda} = \left(\bigcup_{\check{\text{Ƶ}}^{+} \in \S^{+}} \left\{ \left[\left(\frac{1}{1+\left\{\lambda\left(\frac{1-\check{\text{Ƶ}}^{+[\tilde{R}]}}{\check{\text{Ƶ}}^{+[\tilde{R}]}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}} \right) + \iota\left(\frac{1}{1+\left\{\lambda\left(\frac{1-\check{\text{Ƶ}}^{+[\tilde{I}]}}{\check{\text{Ƶ}}^{+[\tilde{I}]}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}} \right) \right] \right\}, \bigcup_{\check{\text{Ƶ}}^{-} \in \S^{-}} \left\{ \left[\left(-1 + \frac{1}{1+\left\{\lambda\left(\frac{\left|\check{\text{Ƶ}}^{-[\tilde{R}]}\right|}{1+\check{\text{Ƶ}}^{-[\tilde{R}]}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}} \right) + \iota\left(-1 + \frac{1}{1+\left\{\lambda\left(\frac{\left|\check{\text{Ƶ}}^{-[\tilde{I}]}\right|}{1+\check{\text{Ƶ}}^{-[\tilde{I}]}}\right)^{\varsigma}\right\}^{\frac{1}{\varsigma}}} \right) \right] \right\} \right)$$

$\max(\bar{Џ}_1, \bar{Џ}_2, \bar{Џ}_3, \ldots, \bar{Џ}_{ń})$ then

$$\bar{Џ}^{-} \leq \text{HBCFDOWA}_{\psi}\left(\bar{Џ}_1, \bar{Џ}_2, \ldots, \bar{Џ}_{ń}\right) \leq \bar{Џ}^{+}$$

Theorem 9 (Monotonicity): Let $\bar{Џ}_{\mathfrak{s}} = (\S_{\mathfrak{s}}^{+}, \S_{\mathfrak{s}}^{-}) = (\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}, \breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, ń)$ and $\bar{Џ}'_{\mathfrak{s}} = (\S'^{+}_{\mathfrak{s}}, \S'^{-}_{\mathfrak{s}}) = (\breve{\mathfrak{Z}}'^{+[\tilde{R}]}_{\mathfrak{s}} + \iota\breve{\mathfrak{Z}}'^{+[\tilde{I}]}_{\mathfrak{s}}, \breve{\mathfrak{Z}}'^{-[\tilde{R}]}_{\mathfrak{s}} + \iota\breve{\mathfrak{Z}}'^{-[\tilde{I}]}_{\mathfrak{s}})(\mathfrak{s} = 1, 2, 3, \ldots, ń)$ be a collection of two HBCFNs, then

$$\text{HBCFDOWA}_{\psi}\left(\bar{Џ}_1, \bar{Џ}_2, \ldots, \bar{Џ}_{ń}\right) \leq \text{HBCFDOWA}_{\psi}\left(\bar{Џ}'_1, \bar{Џ}'_2, \ldots, \bar{Џ}'_{ń}\right)$$

If $\bar{Џ}_{\mathfrak{s}} \leq \bar{Џ}'_{\mathfrak{s}}\ \forall\ \mathfrak{s}$.

The HBCFDWA operator only deals with HBCF values, while the HBCFDOWA operator only focuses on the ordered position of the HBCF values, as can be seen from definitions (15) and (16). The HBCFDWA and HBCFDOWA operators were then combined to create the HBCFDHA operator.

Definition 18: Let $\bar{Џ}_{\mathfrak{s}} = (\S_{\mathfrak{s}}^{+}, \S_{\mathfrak{s}}^{-}) = (\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}, \breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, ń)$ be a collection of HBCFNs, then the HBCFDHWA operator is a function of HBCFDHWA $:\bar{Џ}^{ń} \longrightarrow \bar{Џ}$ such that as (12), shown at the bottom of page 19.

Where $\psi = (\psi_1, \psi_2, \ldots, \psi_{ń})^{T}$ be WV of $\bar{Џ}_{\mathfrak{s}} = (\S_{\mathfrak{s}}^{+}, \S_{\mathfrak{s}}^{-})(\mathfrak{s} = 1, 2, \ldots, ń)$ with $\psi_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{ń} \psi_{\mathfrak{s}} = 1$, $\breve{\mathfrak{Z}}'_{\mho(\mathfrak{s})}$ is the $\mathfrak{s}$ th largest weighted HBCF value of $\breve{\mathfrak{Z}}'_{\mathfrak{s}}(\breve{\mathfrak{Z}}'_{\mathfrak{s}} = (ń\omega)\breve{\mathfrak{Z}}_{\mathfrak{s}}, \mathfrak{s} = 1, 2, \ldots, ń)$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_{ń})$ be the WV of $\breve{\mathfrak{Z}}_{\mathfrak{s}}$ $(\mathfrak{s} = 1, 2, \ldots, ń)$ where $\omega_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{ń} \omega_{\mathfrak{s}} = 1$, and $ń$ is the balancing coefficient.

Remark:

(2) When $\psi = \left(\frac{1}{ń}, \frac{1}{ń}, \frac{1}{ń}, \ldots, \frac{1}{ń}\right)$ the HBCFDHWA operators convert into the HBCFDWA operators.

(3) When $\omega = \left(\frac{1}{ń}, \frac{1}{ń}, \frac{1}{ń}, \ldots, \frac{1}{ń}\right)$ then the HBCFDHA operators convert into the HBCFDOWA.

B. HBCF DOMBI GEOMETRIC AGGREGATION OPERATIONS

This section consists of the following AOs in the environment of HBCFNs. Operators are HBCFDWG operator, HBCFDOWG operator and HBCFDHG operator.

Definition 19: Let $\bar{Џ}_{\mathfrak{s}} = (\S_{\mathfrak{s}}^{+}, \S_{\mathfrak{s}}^{-}) = (\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}, \breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, ń)$ be a collection of HBCFNs, then the HBCFDWG operator is a function of

$$\text{HBCFDWA}_{\omega}\left(\bar{Џ}_1, \bar{Џ}_2, \ldots, \bar{Џ}_{ń}\right) = \bigoplus_{\mathfrak{s}=1}^{ń}\left(\omega_{\mathfrak{s}}\bar{Џ}_{\mathfrak{s}}\right)$$

$$= \left(\begin{array}{c} \bigcup\limits_{\breve{\mathfrak{Z}}_1^{+} \in \S_1^{+}, \ldots, \breve{\mathfrak{Z}}_{ń}^{+} \in \S_{ń}^{+}} \left\{\left(1 - \frac{1}{1 + \left\{\sum_{\mathfrak{s}=1}^{ń} \omega_{\mathfrak{s}} \left(\frac{\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]}}{1 - \breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]}}\right)^{\mathfrak{G}}\right\}^{\frac{1}{\mathfrak{G}}}}\right) + \iota\left(1 - \frac{1}{1 + \left\{\sum_{\mathfrak{s}=1}^{ń} \omega_{\mathfrak{s}} \left(\frac{\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}}{1 - \breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}}\right)^{\mathfrak{G}}\right\}^{\frac{1}{\mathfrak{G}}}}\right)\right\}, \\ \bigcup\limits_{\breve{\mathfrak{Z}}_1^{-} \in \S_1^{-}, \ldots, \breve{\mathfrak{Z}}_{ń}^{-} \in \S_{ń}^{-}} \left\{\left(\frac{-1}{1 + \left\{\sum_{\mathfrak{s}=1}^{ń} \omega_{\mathfrak{s}} \left(\frac{1 + \breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]}}{\left|\breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]}\right|}\right)^{\mathfrak{G}}\right\}^{\frac{1}{\mathfrak{G}}}}\right) + \iota\left(\frac{-1}{1 + \left\{\sum_{\mathfrak{s}=1}^{ń} \omega_{\mathfrak{s}} \left(\frac{1 + \breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]}}{\left|\breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]}\right|}\right)^{\mathfrak{G}}\right\}^{\frac{1}{\mathfrak{G}}}}\right)\right\} \end{array}\right) \quad (9)$$

HBCFDWG $:\overline{Ц}^{ή} \longrightarrow \overline{Ц}$ such that.

$$\mathrm{HBCFDWG}_{\omega}\left(\overline{Ц}_1, \overline{Ц}_2, \ldots, \overline{Ц}_{ή}\right) = \bigotimes_{\mathfrak{s}=1}^{ή} \left(\overline{Ц}_{\mathfrak{s}}\right)^{\omega_{\mathfrak{s}}} \tag{13}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_{ή})^T$ be the WV of $\overline{Ц}_{\mathfrak{s}} = \left(\S_{\mathfrak{s}}^{+}, \S_{\mathfrak{s}}^{-}\right)$ $\left(\mathfrak{s} = 1, 2, \ldots, ή\right)$ with $\omega_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{ή} \omega_{\mathfrak{s}} = 1$.

Theorem 10: Let $\overline{Ц}_{\mathfrak{s}} = (\S_{\mathfrak{s}}^{+}, \S_{\mathfrak{s}}^{-}) = (\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\check{R}]} + \iota\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\check{R}]} + \iota\check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, ή)$ be a collection of HBCFNs, then by employing the above eq. (11), we get the HBCFN and as (14), shown at the bottom of page 19.

Where $\omega = (\omega_1, \omega_2, \ldots, \omega_{ή})^T$ be the WV of $\overline{Ц}_{\mathfrak{s}} = \left(\S_{\mathfrak{s}}^{+}, \S_{\mathfrak{s}}^{-}\right)$ $\left(\mathfrak{s} = 1, 2, \ldots, ή\right)$ with $\omega_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{ή} \omega_{\mathfrak{s}} = 1$.

Proof: We will use mathematical induction to prove the above statement.

Suppose $ή = 2$, the left side of (15) becomes as shown at the bottom of page 20.

$$\mathrm{HBCFDWG}_{\omega}\left(\overline{Ц}_1, \overline{Ц}_2\right) = \overline{Ц}_1 \otimes \overline{Ц}_2$$

and the Right side of (15) becomes

This shows that (15) holds for $ή = 2$.

Now assume that (15) holds for $ή = ķ$

As shown at the bottom of page 21.

Next, we show that (15) holds for $ή = ķ+$

$$= \left(\bigcup_{\check{\mathfrak{Z}}_1^{+} \in \S_1^{+}, \check{\mathfrak{Z}}_2^{+} \in \S_2^{+}} \left\{ \left(1 - \frac{1}{1 + \left\{ \begin{array}{l} \omega_1 \left(\frac{\check{\mathfrak{Z}}_1^{+[\check{R}]}}{1 - \check{\mathfrak{Z}}_1^{+[\check{R}]}} \right)^{б} + \\ \omega_2 \left(\frac{\check{\mathfrak{Z}}_2^{+[\check{R}]}}{1 - \check{\mathfrak{Z}}_2^{+[\check{R}]}} \right)^{б} \end{array} \right\}^{\frac{1}{б}}} \right) + \iota \left(1 - \frac{1}{1 + \left\{ \begin{array}{l} \omega_1 \left(\frac{\check{\mathfrak{Z}}_1^{+[\tilde{I}]}}{1 - \check{\mathfrak{Z}}_1^{+[\tilde{I}]}} \right)^{б} + \\ \omega_1 \left(\frac{\check{\mathfrak{Z}}_1^{+[\tilde{I}]}}{1 - \check{\mathfrak{Z}}_1^{+[\tilde{I}]}} \right)^{б} \end{array} \right\}^{\frac{1}{б}}} \right) \right\}, \right.$$

$$\left. \bigcup_{\check{\mathfrak{Z}}_1^{-} \in \S_1^{-}, \check{\mathfrak{Z}}_2^{-} \in \S_2^{-}} \left\{ \left(\frac{-1}{1 + \left\{ \begin{array}{l} \omega_1 \left(\frac{1 + \check{\mathfrak{Z}}_1^{-[\check{R}]}}{\left|\check{\mathfrak{Z}}_1^{-[\check{R}]}\right|} \right)^{б} + \\ \omega_2 \left(\frac{1 + \check{\mathfrak{Z}}_2^{-[\check{R}]}}{\left|\check{\mathfrak{Z}}_2^{-[\check{R}]}\right|} \right)^{б} \end{array} \right\}^{\frac{1}{б}}} \right) + \iota \left(\frac{-1}{1 + \left\{ \begin{array}{l} \omega_2 \left(\frac{1 + \check{\mathfrak{Z}}_2^{-[\tilde{I}]}}{\left|\check{\mathfrak{Z}}_2^{-[\tilde{I}]}\right|} \right)^{б} + \\ \omega_2 \left(\frac{1 + \check{\mathfrak{Z}}_2^{-[\tilde{I}]}}{\left|\check{\mathfrak{Z}}_2^{-[\tilde{I}]}\right|} \right)^{б} \end{array} \right\}^{\frac{1}{б}}} \right) \right\} \right)$$

As shown at the bottom of page 22.

This shows that (15) holds for $\acute{n} = \mathfrak{k} + 1$. This implies that (15) holds for every $\acute{n}$.

Given below properties can be proved in the simplest way

Theorem 11 (Idempotency): Let $\overline{\text{Ц}}_{\mathfrak{s}} = \left(\S_{\mathfrak{s}}^{+}, \S_{\mathfrak{s}}^{-}\right) = \left(\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]}\right) \left(\mathfrak{s} = 1, 2, 3, \ldots, \acute{n}\right)$ be a collection of HBCFNs, $\omega = \left(\omega_1, \omega_2, \ldots, \omega_{\acute{n}}\right)^T$ be the WV of $\overline{\text{Ц}}_{\mathfrak{s}} = \left(\S_{\mathfrak{s}}^{+}, \S_{\mathfrak{s}}^{-}\right) \left(\mathfrak{s} = 1, 2, \ldots, \acute{n}\right)$ with $\omega_{\mathfrak{s}} \in [0, 1]$ and

$$= \left(\bigcup_{\check{\mathfrak{Z}}_1^{+} \in \S_1^{+}, \check{\mathfrak{Z}}_2^{+} \in \S_2^{+}} \left\{ \begin{array}{l} \left(1 - \dfrac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{2} \omega_{\mathfrak{s}} \left(\dfrac{\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]}}{1 - \check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]}} \right)^{\varsigma} \right\}^{\frac{1}{\varsigma}}} \right) + \\ \iota \left(1 - \dfrac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{2} \omega_{\mathfrak{s}} \left(\dfrac{\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}}{1 - \check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}} \right)^{\varsigma} \right\}^{\frac{1}{\varsigma}}} \right) \end{array} \right\}, \bigcup_{\check{\mathfrak{Z}}_1^{-} \in \S_1^{-}, \check{\mathfrak{Z}}_2^{-} \in \S_2^{-}} \left\{ \begin{array}{l} \left(\dfrac{-1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{2} \omega_{\mathfrak{s}} \left(\dfrac{1 + \check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]}}{\left| \check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]} \right|} \right)^{\varsigma} \right\}^{\frac{1}{\varsigma}}} \right) + \\ \iota \left(\dfrac{-1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{2} \omega_{\mathfrak{s}} \left(\dfrac{1 + \check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]}}{\left| \check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]} \right|} \right)^{\varsigma} \right\}^{\frac{1}{\varsigma}}} \right) \end{array} \right\} \right)$$

$\sum_{\text{ꞩ}=1}^{\acute{n}} \omega_{\text{ꞩ}} = 1$. If $\overline{\text{Ꝕ}}_{\text{ꞩ}} = \overline{\text{Ꝕ}}\ \forall \text{ꞩ}$ then

$$\text{HBCFDWG}_{\omega}\left(\overline{\text{Ꝕ}}_1, \overline{\text{Ꝕ}}_2, \ldots, \overline{\text{Ꝕ}}_{\acute{n}}\right) = \overline{\text{Ꝕ}}$$

Theorem 12 (Boundedness): Let $\overline{\text{Ꝕ}}_{\text{ꞩ}} = (\S_{\text{ꞩ}}^{+}, \S_{\text{ꞩ}}^{-}) = (\breve{\text{Ʒ}}_{\text{ꞩ}}^{+[\tilde{R}]} + \iota \breve{\text{Ʒ}}_{\text{ꞩ}}^{+[\tilde{I}]}, \breve{\text{Ʒ}}_{\text{ꞩ}}^{-[\tilde{R}]} + \iota \breve{\text{Ʒ}}_{\text{ꞩ}}^{-[\tilde{I}]})(\text{ꞩ} = 1, 2, 3, \ldots, \acute{n})$ be a collection

$$\text{HBCFDWA}_{\omega}\left(\overline{\text{Ꝕ}}_1, \overline{\text{Ꝕ}}_2, \ldots, \overline{\text{Ꝕ}}_{\text{ƙ}}\right) = \underset{\text{ꞩ}=1}{\overset{\text{ƙ}}{\oplus}} \left(\omega_{\text{ꞩ}} \overline{\text{Ꝕ}}_{\text{ꞩ}}\right)$$

$$= \left(\begin{array}{l} \bigcup_{\breve{\text{Ʒ}}_1^{+} \in \S_1^{+}, \ldots, \breve{\text{Ʒ}}_{\text{ƙ}}^{+} \in \S_{\text{ƙ}}^{+}} \left\{ \begin{array}{l} \left(1 - \dfrac{1}{1 + \left\{ \sum_{\text{ꞩ}=1}^{\text{ƙ}} \omega_{\text{ꞩ}} \left(\dfrac{\breve{\text{Ʒ}}_{\text{ꞩ}}^{+[\tilde{R}]}}{1 - \breve{\text{Ʒ}}_{\text{ꞩ}}^{+[\tilde{R}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) + \\ \iota \left(1 - \dfrac{1}{1 + \left\{ \sum_{\text{ꞩ}=1}^{\text{ƙ}} \omega_{\text{ꞩ}} \left(\dfrac{\breve{\text{Ʒ}}_{\text{ꞩ}}^{+[\tilde{I}]}}{1 - \breve{\text{Ʒ}}_{\text{ꞩ}}^{+[\tilde{I}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) \end{array} \right\}, \\ \bigcup_{\breve{\text{Ʒ}}_1^{-} \in \S_1^{-}, \ldots, \breve{\text{Ʒ}}_{\acute{n}}^{-} \in \S_{\acute{n}}^{-}} \left\{ \begin{array}{l} \left(\dfrac{-1}{1 + \left\{ \sum_{\text{ꞩ}=1}^{\text{ƙ}} \omega_{\text{ꞩ}} \left(\dfrac{1 + \breve{\text{Ʒ}}_{\text{ꞩ}}^{-[\tilde{R}]}}{\left| \breve{\text{Ʒ}}_{\text{ꞩ}}^{-[\tilde{R}]} \right|} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) + \\ \iota \left(\dfrac{-1}{1 + \left\{ \sum_{\text{ꞩ}=1}^{\text{ƙ}} \omega_{\text{ꞩ}} \left(\dfrac{1 + \breve{\text{Ʒ}}_{\text{ꞩ}}^{-[\tilde{I}]}}{\left| \breve{\text{Ʒ}}_{\text{ꞩ}}^{-[\tilde{I}]} \right|} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) \end{array} \right\} \end{array} \right)$$

of HBCFNs, and $\overline{Џ}^{-} = \min(\overline{Џ}_1, \overline{Џ}_2, \overline{Џ}_3, \ldots, \overline{Џ}_{ń}), \overline{Џ}^{+} = \max(\overline{Џ}_1, \overline{Џ}_2, \overline{Џ}_3, \ldots, \overline{Џ}_{ń})$

$$\mathrm{HBCFDWA}_{\omega}\left(\overline{Џ}_1, \overline{Џ}_2, \ldots, \overline{Џ}_{ķ}, \overline{Џ}_{ķ+1}\right) = \underset{\mathfrak{s}=1}{\overset{ķ}{\oplus}} \left(\omega_{\mathfrak{s}}\overline{Џ}_{\mathfrak{s}}\right) \oplus \left(\omega_{ķ+1}\overline{Џ}_{ķ+1}\right)$$

$$= \left(\begin{array}{c} \bigcup_{\breve{\mathfrak{Z}}_1^{+} \in \S_1^{+}, \ldots, \breve{\mathfrak{Z}}_{ķ} \in \S_{ķ}^{+}} \left\{ \begin{array}{c} \left(1 - \dfrac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{ķ} \omega_{\mathfrak{s}} \left(\dfrac{\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]}}{1 - \breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) + \\ \iota \left(1 - \dfrac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{ķ} \omega_{\mathfrak{s}} \left(\dfrac{\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}}{1 - \breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) \end{array} \right\}, \\ \bigcup_{\breve{\mathfrak{Z}}_1^{-} \in \S_1^{-}, \ldots, \breve{\mathfrak{Z}}_{ń}^{-} \in \S_{ń}^{-}} \left\{ \begin{array}{c} \left(\dfrac{-1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{ķ} \omega_{\mathfrak{s}} \left(\dfrac{1 + \breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]}}{\left| \breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]} \right|} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) + \\ \iota \left(\dfrac{-1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{ķ} \omega_{\mathfrak{s}} \left(\dfrac{1 + \breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]}}{\left| \breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]} \right|} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) \end{array} \right\} \end{array} \right)$$

$$\oplus \left(\begin{array}{c} \bigcup_{\check{\mathfrak{Z}}_1^{+} \in \S_1^{+}, \ldots, \check{\mathfrak{Z}}_{\text{ķ}}^{+} \in \S_{\text{ķ}}^{+}, \check{\mathfrak{Z}}_{\text{ķ}+1}^{+} \in \S_{\text{ķ}+1}^{+}} \left\{ \begin{array}{c} \left(1 - \dfrac{1}{1+ \left\{ \omega_{\text{ķ}+1} \left(\dfrac{\check{\mathfrak{Z}}_{\text{ķ}+1}^{+[\tilde{R}]}}{1 - \check{\mathfrak{Z}}_{\text{ķ}+1}^{+[\tilde{R}]}} \right)^{\text{ϭ}} \right\}^{\frac{1}{\text{ϭ}}}} \right) + \\ \iota \left(1 - \dfrac{1}{1+ \left\{ \omega_{\text{ķ}+1} \left(\dfrac{\check{\mathfrak{Z}}_{\text{ķ}+1}^{+[\tilde{I}]}}{1 - \check{\mathfrak{Z}}_{\text{ķ}+1}^{+[\tilde{I}]}} \right)^{\text{ϭ}} \right\}^{\frac{1}{\text{ϭ}}}} \right) \end{array} \right\}, \\ \bigcup_{\check{\mathfrak{Z}}_1^{-} \in \S_1^{-}, \ldots, \check{\mathfrak{Z}}_{\text{ň}}^{-} \in \S_{\text{ň}}^{-}, \check{\mathfrak{Z}}_{\text{ķ}+1}^{-} \in \S_{\text{ķ}+1}^{-}} \left\{ \begin{array}{c} \left(\dfrac{-1}{1+ \left\{ \omega_{\text{ķ}+1} \left(\dfrac{1 + \check{\mathfrak{Z}}_{\text{ķ}+1}^{-[\tilde{R}]}}{\left| \check{\mathfrak{Z}}_{\text{ķ}+1}^{-[\tilde{R}]} \right|} \right)^{\text{ϭ}} \right\}^{\frac{1}{\text{ϭ}}}} \right) + \\ \iota \left(\dfrac{-1}{1+ \left\{ \omega_{\text{ķ}+1} \left(\dfrac{1 + \check{\mathfrak{Z}}_{\text{ķ}+1}^{-[\tilde{I}]}}{\left| \check{\mathfrak{Z}}_{\text{ķ}+1}^{-[\tilde{I}]} \right|} \right)^{\text{ϭ}} \right\}^{\frac{1}{\text{ϭ}}}} \right) \end{array} \right\} \end{array} \right)$$

$$= \left(\begin{array}{c} \bigcup_{\breve{\mathfrak{Z}}_1^{+} \in \S_1^{+}, \ldots, \breve{\mathfrak{Z}}_{\mathrm{k}}^{+} \in \S_{\mathrm{k}}^{+}, \breve{\mathfrak{Z}}_{\mathrm{k}+1}^{+} \in \S_{\mathrm{k}+1}^{+}} \left\{ \begin{array}{c} \left(1 - \frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\mathrm{k}+1} \omega_{\mathfrak{s}} \left(\frac{\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]}}{1 - \breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]}} \right)^{\mathfrak{G}} \right\}^{\frac{1}{\mathfrak{G}}}} \right) + \\ \iota \left(1 - \frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\mathrm{k}+1} \omega_{\mathfrak{s}} \left(\frac{\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}}{1 - \breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}} \right)^{\mathfrak{G}} \right\}^{\frac{1}{\mathfrak{G}}}} \right) \end{array} \right\}, \\ \bigcup_{\breve{\mathfrak{Z}}_1^{-} \in \S_1^{-}, \ldots, \breve{\mathfrak{Z}}_{\mathrm{n}}^{-} \in \S_{\mathrm{n}}^{-}, \breve{\mathfrak{Z}}_{\mathrm{k}+1}^{-} \in \S_{\mathrm{k}+1}^{-}} \left\{ \begin{array}{c} \left(\frac{-1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\mathrm{k}+1} \omega_{\mathfrak{s}} \left(\frac{1 + \breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]}}{\left| \breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]} \right|} \right)^{\mathfrak{G}} \right\}^{\frac{1}{\mathfrak{G}}}} \right) + \\ \iota \left(\frac{-1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\mathrm{k}+1} \omega_{\mathfrak{s}} \left(\frac{1 + \breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]}}{\left| \breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]} \right|} \right)^{\mathfrak{G}} \right\}^{\frac{1}{\mathfrak{G}}}} \right) \end{array} \right\} \end{array} \right)$$

$$\mathrm{HBCFDOWA}_{\psi}\left(\overline{\text{Џ}}_1,\overline{\text{Џ}}_2,\ldots,\overline{\text{Џ}}_{ń}\right)=\bigoplus_{\mathfrak{s}=1}^{ń}\left(\psi_{\mathfrak{s}}\overline{\text{Џ}}_{\mho(\mathfrak{s})}\right)$$

$$=\left(\begin{array}{c}\bigcup_{\check{\mathfrak{Z}}_1^{+}\in\S_1^{+},\ldots,\check{\mathfrak{Z}}_{ń}^{+}\in\S_{ń}^{+}}\left\{\begin{array}{c}\left(1-\dfrac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{ń}\psi_{\mho(\mathfrak{s})}\left(\dfrac{\check{\mathfrak{Z}}_{\mho(\mathfrak{s})}^{+[\check{R}]}}{1-\check{\mathfrak{Z}}_{\mho(\mathfrak{s})}^{+[\check{R}]}}\right)^{\zeta}\right\}^{\frac{1}{\zeta}}}\right)+\\ \iota\left(1-\dfrac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{ń}\psi_{\mho(\mathfrak{s})}\left(\dfrac{\check{\mathfrak{Z}}_{\mho(\mathfrak{s})}^{+[\tilde{I}]}}{1-\check{\mathfrak{Z}}_{\mho(\mathfrak{s})}^{+[\tilde{I}]}}\right)^{\zeta}\right\}^{\frac{1}{\zeta}}}\right)\end{array}\right\},\\ \bigcup_{\check{\mathfrak{Z}}_1^{-}\in\S_1^{-},\ldots,\check{\mathfrak{Z}}_{ń}^{-}\in\S_{ń}^{-}}\left\{\begin{array}{c}\left(\dfrac{-1}{1+\left\{\sum_{\mathfrak{s}=1}^{ń}\psi_{\mho(\mathfrak{s})}\left(\dfrac{1+\check{\mathfrak{Z}}_{\mho(\mathfrak{s})}^{-[\check{R}]}}{\left|\check{\mathfrak{Z}}_{\mho(\mathfrak{s})}^{-[\check{R}]}\right|}\right)^{\zeta}\right\}^{\frac{1}{\zeta}}}\right)+\\ \iota\left(\dfrac{-1}{1+\left\{\sum_{\mathfrak{s}=1}^{ń}\psi_{\mho(\mathfrak{s})}\left(\dfrac{1+\check{\mathfrak{Z}}_{\mho(\mathfrak{s})}^{-[\tilde{I}]}}{\left|\check{\mathfrak{Z}}_{\mho(\mathfrak{s})}^{-[\tilde{I}]}\right|}\right)^{\zeta}\right\}^{\frac{1}{\zeta}}}\right)\end{array}\right\}\end{array}\right) \tag{11}$$

then

$$\overline{Ц}^{-} \leq \text{HBCFDWG}_{\omega}\left(\overline{Ц}_1, \overline{Ц}_2, \ldots, \overline{Ц}_{\acute{n}}\right) \leq \overline{Ц}^{+}$$

Theorem 13 (Monotonicity): Let $\overline{Ц}_{\mathfrak{s}} = (\S_{\mathfrak{s}}^{+}, \S_{\mathfrak{s}}^{-}) = (\breve{\check{\mathfrak{Z}}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\breve{\check{\mathfrak{Z}}}_{\mathfrak{s}}^{+[\tilde{I}]}, \breve{\check{\mathfrak{Z}}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\breve{\check{\mathfrak{Z}}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, \acute{n})$ and $\overline{Ц}'_{\mathfrak{s}} = (\S'^{+}_{\mathfrak{s}}, \S'^{-}_{\mathfrak{s}}) = (\breve{\check{\mathfrak{Z}}}'^{+[\tilde{R}]}_{\mathfrak{s}} + \iota\breve{\check{\mathfrak{Z}}}'^{+[\tilde{I}]}_{\mathfrak{s}}, \breve{\check{\mathfrak{Z}}}'^{-[\tilde{R}]}_{\mathfrak{s}} + \iota\breve{\check{\mathfrak{Z}}}'^{-[\tilde{I}]}_{\mathfrak{s}})(\mathfrak{s} =$

$$\text{HBCFDHWA}_{\psi,\ \omega}\left(\overline{Ц}_1, \overline{Ц}_2, \ldots, \overline{Ц}_{\acute{n}}\right) = \bigoplus_{\mathfrak{s}=1}^{\acute{n}} \left(\psi_{\mathfrak{s}}\overline{Ц}_{\mho(\mathfrak{s})}\right)$$

$$= \left(\begin{array}{c} \bigcup\limits_{\breve{\check{\mathfrak{Z}}}_1^{+} \in \S_1^{+}, \ldots, \breve{\check{\mathfrak{Z}}}_{\acute{n}}^{+} \in \S_{\acute{n}}^{+}} \left\{ \left[\left(1 - \frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mho(\mathfrak{s})} \left(\frac{\breve{\check{\mathfrak{Z}}}_{\mho(\mathfrak{s})}^{+[\tilde{R}]}}{1 - \breve{\check{\mathfrak{Z}}}_{\mho(\mathfrak{s})}^{+[\tilde{R}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) + \iota \left(1 - \frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mho(\mathfrak{s})} \left(\frac{\breve{\check{\mathfrak{Z}}}_{\mho(\mathfrak{s})}^{+[\tilde{I}]}}{1 - \breve{\check{\mathfrak{Z}}}_{\mho(\mathfrak{s})}^{+[\tilde{I}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) \right] \right\}, \\ \bigcup\limits_{\breve{\check{\mathfrak{Z}}}_1^{-} \in \S_1^{-}, \ldots, \breve{\check{\mathfrak{Z}}}_{\acute{n}}^{-} \in \S_{\acute{n}}^{-}} \left\{ \left[\left(\frac{-1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mho(\mathfrak{s})} \left(\frac{1 + \breve{\check{\mathfrak{Z}}}_{\mho(\mathfrak{s})}^{-[\tilde{R}]}}{\left| \breve{\check{\mathfrak{Z}}}_{\mho(\mathfrak{s})}^{-[\tilde{R}]} \right|} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) + \iota \left(\frac{-1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mho(\mathfrak{s})} \left(\frac{1 + \breve{\check{\mathfrak{Z}}}_{\mho(\mathfrak{s})}^{-[\tilde{I}]}}{\left| \breve{\check{\mathfrak{Z}}}_{\mho(\mathfrak{s})}^{-[\tilde{I}]} \right|} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) \right] \right\} \end{array} \right) \tag{12}$$

$$\text{HBCFDWG}_{\omega}\left(\overline{Ц}_1, \overline{Ц}_2, \ldots, \overline{Ц}_{\acute{n}}\right) = \bigotimes_{\mathfrak{s}=1}^{\acute{n}} \left(\overline{Ц}_{\mathfrak{s}}\right)^{\omega_{\mathfrak{s}}}$$

$$= \left(\begin{array}{c} \bigcup\limits_{\breve{\check{\mathfrak{Z}}}_1^{+} \in \S_1^{+}, \ldots, \breve{\check{\mathfrak{Z}}}_{\acute{n}}^{+} \in \S_{\acute{n}}^{+}} \left\{ \left[\left(\frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} \left(\frac{1 - \breve{\check{\mathfrak{Z}}}_{\mathfrak{s}}^{+[\tilde{R}]}}{\breve{\check{\mathfrak{Z}}}_{\mathfrak{s}}^{+[\tilde{R}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) + \iota \left(\frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} \left(\frac{1 - \breve{\check{\mathfrak{Z}}}_{\mathfrak{s}}^{+[\tilde{I}]}}{\breve{\check{\mathfrak{Z}}}_{\mathfrak{s}}^{+[\tilde{I}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) \right] \right\}, \\ \bigcup\limits_{\breve{\check{\mathfrak{Z}}}_1^{-} \in \S_1^{-}, \ldots, \breve{\check{\mathfrak{Z}}}_{\acute{n}}^{-} \in \S_{\acute{n}}^{-}} \left\{ \left[\left(-1 + \frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} \left(\frac{\left| \breve{\check{\mathfrak{Z}}}_{\mathfrak{s}}^{-[\tilde{R}]} \right|}{1 + \breve{\check{\mathfrak{Z}}}_{\mathfrak{s}}^{-[\tilde{R}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) + \iota \left(-1 + \frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} \left(\frac{\left| \breve{\check{\mathfrak{Z}}}_{\mathfrak{s}}^{-[\tilde{I}]} \right|}{1 + \breve{\check{\mathfrak{Z}}}_{\mathfrak{s}}^{-[\tilde{I}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) \right] \right\} \end{array} \right) \tag{14}$$

$$
= \left(\begin{array}{l}
\bigcup_{\check{\breve{\mathfrak{Z}}}_1^{+} \in \S_1^{+}, \check{\breve{\mathfrak{Z}}}_2^{+} \in \S_2^{+}} \left\{ \begin{array}{l}
\left(\frac{1}{1+\left\{\omega_1 \left(\frac{1-\check{\breve{\mathfrak{Z}}}_1^{+[\check{\mathrm{R}}]}}{\check{\breve{\mathfrak{Z}}}_1^{+[\check{\mathrm{R}}]}}\right)^{\sigma} + \omega_2 \left(\frac{1-\check{\breve{\mathfrak{Z}}}_2^{+[\check{\mathrm{R}}]}}{\check{\breve{\mathfrak{Z}}}_2^{+[\check{\mathrm{R}}]}}\right)^{\sigma}\right\}^{\frac{1}{\sigma}}} \right) + \\
\iota \left(\frac{1}{1+\left\{\omega_1 \left(\frac{1-\check{\breve{\mathfrak{Z}}}_1^{+[\tilde{\mathrm{I}}]}}{\check{\breve{\mathfrak{Z}}}_1^{+[\tilde{\mathrm{I}}]}}\right)^{\sigma} + \omega_1 \left(\frac{1-\check{\breve{\mathfrak{Z}}}_1^{+[\tilde{\mathrm{I}}]}}{\check{\breve{\mathfrak{Z}}}_1^{+[\tilde{\mathrm{I}}]}}\right)^{\sigma}\right\}^{\frac{1}{\sigma}}} \right)
\end{array} \right\}, \\
\bigcup_{\check{\breve{\mathfrak{Z}}}_1^{-} \in \S_1^{-}, \check{\breve{\mathfrak{Z}}}_2^{-} \in \S_2^{-}} \left\{ \begin{array}{l}
\left(-1 + \frac{1}{1+\left\{\omega_1 \left(\frac{\left|\check{\breve{\mathfrak{Z}}}_1^{-[\check{\mathrm{R}}]}\right|}{1+\check{\breve{\mathfrak{Z}}}_1^{-[\check{\mathrm{R}}]}}\right)^{\sigma} + \omega_2 \left(\frac{\left|\check{\breve{\mathfrak{Z}}}_2^{-[\check{\mathrm{R}}]}\right|}{1+\check{\breve{\mathfrak{Z}}}_2^{-[\check{\mathrm{R}}]}}\right)^{\sigma}\right\}^{\frac{1}{\sigma}}} \right) + \\
\iota \left(-1 + \frac{1}{1+\left\{\omega_2 \left(\frac{\left|\check{\breve{\mathfrak{Z}}}_2^{-[\tilde{\mathrm{I}}]}\right|}{1+\check{\breve{\mathfrak{Z}}}_2^{-[\tilde{\mathrm{I}}]}}\right)^{\sigma} + \omega_2 \left(\frac{\left|\check{\breve{\mathfrak{Z}}}_2^{-[\tilde{\mathrm{I}}]}\right|}{1+\check{\breve{\mathfrak{Z}}}_2^{-[\tilde{\mathrm{I}}]}}\right)^{\sigma}\right\}^{\frac{1}{\sigma}}} \right)
\end{array} \right\}
\end{array} \right)
$$

$$
= \left(\begin{array}{l}
\bigcup_{\check{\breve{\mathfrak{Z}}}_1^{+} \in \S_1^{+}, \check{\breve{\mathfrak{Z}}}_2^{+} \in \S_2^{+}} \left\{ \left(\frac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{2} \omega_{\mathfrak{s}} \left(\frac{1-\check{\breve{\mathfrak{Z}}}_{\mathfrak{s}}^{+[\check{\mathrm{R}}]}}{\check{\breve{\mathfrak{Z}}}_{\mathfrak{s}}^{+[\check{\mathrm{R}}]}}\right)^{\sigma}\right\}^{\frac{1}{\sigma}}} \right) + \iota \left(\frac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{2} \omega_{\mathfrak{s}} \left(\frac{1-\check{\breve{\mathfrak{Z}}}_{\mathfrak{s}}^{+[\tilde{\mathrm{I}}]}}{\check{\breve{\mathfrak{Z}}}_{\mathfrak{s}}^{+[\tilde{\mathrm{I}}]}}\right)^{\sigma}\right\}^{\frac{1}{\sigma}}} \right) \right\}, \\
\bigcup_{\check{\breve{\mathfrak{Z}}}_1^{-} \in \S_1^{-}, \check{\breve{\mathfrak{Z}}}_2^{-} \in \S_2^{-}} \left\{ \left(-1 + \frac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{2} \omega_{\mathfrak{s}} \left(\frac{\left|\check{\breve{\mathfrak{Z}}}_{\mathfrak{s}}^{-[\check{\mathrm{R}}]}\right|}{1+\check{\breve{\mathfrak{Z}}}_{\mathfrak{s}}^{-[\check{\mathrm{R}}]}}\right)^{\sigma}\right\}^{\frac{1}{\sigma}}} \right) + \iota \left(-1 + \frac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{2} \omega_{\mathfrak{s}} \left(\frac{\left|\check{\breve{\mathfrak{Z}}}_{\mathfrak{s}}^{-[\tilde{\mathrm{I}}]}\right|}{1+\check{\breve{\mathfrak{Z}}}_{\mathfrak{s}}^{-[\tilde{\mathrm{I}}]}}\right)^{\sigma}\right\}^{\frac{1}{\sigma}}} \right) \right\}
\end{array} \right)
$$

$1, 2, 3, \ldots, ń)$ be a collection of two HBCFNs, then

$$\begin{aligned}&HBCFDWG_{\omega}\left(\overline{Џ}_{1}, \overline{Џ}_{2}, \ldots, \overline{Џ}_{ń}\right)\\ &\quad \leq HBCFDWG_{\omega}\left(\overline{Џ}'_{1}, \overline{Џ}'_{2}, \ldots, \overline{Џ}'_{ń}\right)\end{aligned}$$

If $\overline{Џ}_{\mathfrak{s}} \leq \overline{Џ}'_{\mathfrak{s}}\ \forall \mathfrak{s}$.

Definition 20: Let $\overline{Џ}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}}) = (\check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, ń)$ be a collection of HBCFNs, then the HBCFDOWG operator is a function of HBCFDOWG $: \overline{Џ}^{ń} \longrightarrow \overline{Џ}$ such that.

$$HBCFDOWG_{\psi}\left(\overline{Џ}_{1}, \overline{Џ}_{2}, \ldots, \overline{Џ}_{ń}\right) = \underset{\mathfrak{s}=1}{\overset{ń}{\otimes}} \left(\overline{Џ}_{\mho(\mathfrak{s})}\right)^{\psi_{\mathfrak{s}}} \quad (15)$$

where $\psi = (\psi_{1}, \psi_{2}, \ldots, \psi_{ń})^{T}$ be WV of $\overline{Џ}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}})$ $(\mathfrak{s} = 1, 2, \ldots, ń)$ with $\psi_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{ń} \psi_{\mathfrak{s}} = 1$, and $\mho(1), \mho(2), \mho(3), \ldots, \mho(ń)$ are the permutation of $\mho(\mathfrak{s})$ $(\mathfrak{s} = 1, 2, 3, \ldots, ń)$ such that $\overline{Џ}_{\mho(\mathfrak{s}-1)} \geq \overline{Џ}_{\mho(\mathfrak{s})}\ \forall \mathfrak{s}$.

Theorem 17: Let $\overline{Џ}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}}) = (\check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, ń)$ be a collection of HBCFNs, then by employing the above eq. (26), we get the HBCFN and as (16), shown at the bottom of page 24.

where $\psi = (\psi_{1}, \psi_{2}, \ldots, \psi_{ń})^{T}$ be WV of $\overline{Џ}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}})$ $(\mathfrak{s} = 1, 2, \ldots, ń)$ with $\psi_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{ń} \psi_{\mathfrak{s}} = 1$, and $\mho(1), \mho(2), \mho(3), \ldots, \mho(ń)$ are the permutation of $\mho(\mathfrak{s})$ $(\mathfrak{s} = 1, 2, 3, \ldots, ń)$ such that $\overline{Џ}_{\mho(\mathfrak{s}-1)} \geq \overline{Џ}_{\mho(\mathfrak{s})}\ \forall \mathfrak{s}$.

Given below properties can be proved most simply.

Theorem 14 (Idempotency): Let $\overline{Џ}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}}) = \left(\check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{I}]}\right)$ $(\mathfrak{s} = 1, 2, 3, \ldots, ń)$ be a collection of HBCFNs, $\psi = (\psi_{1}, \psi_{2}, \ldots, \psi_{ń})^{T}$ be WV of $\overline{Џ}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}})$ $(\mathfrak{s} = 1, 2, \ldots, ń)$ with $\psi_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{ń} \psi_{\mathfrak{s}} = 1$. If $\overline{Џ}_{\mathfrak{s}} = \overline{Џ}\ \forall \mathfrak{s}$ then

$$HBCFDOWG_{\psi}\left(\overline{Џ}_{1}, \overline{Џ}_{2}, \ldots, \overline{Џ}_{ń}\right) = \overline{Џ}$$

Theorem 15 (Boundedness): Let $\overline{Џ}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}}) = (\check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, ń)$ be a collection of HBCFNs, and $\overline{Џ}^{-} = \min(\overline{Џ}_{1}, \overline{Џ}_{2}, \overline{Џ}_{3}, \ldots, \overline{Џ}_{ń})$, $\overline{Џ}^{+} = \max(\overline{Џ}_{1}, \overline{Џ}_{2}, \overline{Џ}_{3}, \ldots, \overline{Џ}_{ń})$ then

$$\overline{Џ}^{-} \leq HBCFDOWG_{\psi}\left(\overline{Џ}_{1}, \overline{Џ}_{2}, \ldots, \overline{Џ}_{ń}\right) \leq \overline{Џ}^{+}$$

Theorem 16 (Monotonicity): Let $\overline{Џ}_{\mathfrak{s}} = (\S^{+}_{\mathfrak{s}}, \S^{-}_{\mathfrak{s}}) = (\check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, ń)$ and $\overline{Џ}'_{\mathfrak{s}} = (\S'^{+}_{\mathfrak{s}}, \S'^{-}_{\mathfrak{s}}) = (\check{\mathfrak{z}}'^{+[\tilde{R}]}_{\mathfrak{s}} + \iota\check{\mathfrak{z}}'^{+[\tilde{I}]}_{\mathfrak{s}}, \check{\mathfrak{z}}'^{-[\tilde{R}]}_{\mathfrak{s}} + \iota\check{\mathfrak{z}}'^{-[\tilde{I}]}_{\mathfrak{s}})(\mathfrak{s} = 1, 2, 3, \ldots, ń)$ be a collection of two HBCFNs, then

$$\begin{aligned}&HBCFDOWG_{\psi}\left(\overline{Џ}_{1}, \overline{Џ}_{2}, \ldots, \overline{Џ}_{ń}\right)\\ &\quad \leq HBCFDOWG_{\psi}\left(\overline{Џ}'_{1}, \overline{Џ}'_{2}, \ldots, \overline{Џ}'_{ń}\right)\end{aligned}$$

If $\overline{Џ}_{\mathfrak{s}} \leq \overline{Џ}'_{\mathfrak{s}}\ \forall \mathfrak{s}$.

$$HBCFDWG_{\omega}\left(\overline{Џ}_{1}, \overline{Џ}_{2}, \ldots, \overline{Џ}_{ķ}\right) = \underset{\mathfrak{s}=1}{\overset{ķ}{\otimes}} \left(\overline{Џ}_{\mathfrak{s}}\right)^{\omega_{\mathfrak{s}}}$$

$$= \left(\begin{array}{c} \bigcup_{\check{\mathfrak{z}}_{1}^{+} \in \S_{1}^{+}, \ldots, \check{\mathfrak{z}}_{ķ}^{+} \in \S_{ķ}^{+}} \left\{\left(\dfrac{1}{\left[1 + \left\{\sum_{\mathfrak{s}=1}^{ķ} \omega_{\mathfrak{s}} \left(\dfrac{1 - \check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{R}]}}{\check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{R}]}}\right)^{\delta}\right\}\right]^{\frac{1}{\delta}}}\right) + \iota\left(\dfrac{1}{\left[1 + \left\{\sum_{\mathfrak{s}=1}^{ķ} \omega_{\mathfrak{s}} \left(\dfrac{1 - \check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{I}]}}{\check{\mathfrak{z}}_{\mathfrak{s}}^{+[\tilde{I}]}}\right)^{\delta}\right\}\right]^{\frac{1}{\delta}}}\right)\right\}, \\ \bigcup_{\check{\mathfrak{z}}_{1}^{-} \in \S_{1}^{-}, \ldots, \check{\mathfrak{z}}_{ń}^{-} \in \S_{ń}^{-}} \left\{\left(-1 + \dfrac{1}{\left[1 + \left\{\sum_{\mathfrak{s}=1}^{ķ} \omega_{\mathfrak{s}} \left(\dfrac{\left|\check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{R}]}\right|}{1 + \check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{R}]}}\right)^{\delta}\right\}\right]^{\frac{1}{\delta}}}\right) + \iota\left(-1 + \dfrac{1}{\left[1 + \left\{\sum_{\mathfrak{s}=1}^{ķ} \omega_{\mathfrak{s}} \left(\dfrac{\left|\check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{I}]}\right|}{1 + \check{\mathfrak{z}}_{\mathfrak{s}}^{-[\tilde{I}]}}\right)^{\delta}\right\}\right]^{\frac{1}{\delta}}}\right)\right\} \end{array}\right)$$

The HBCFDOWG operator only addresses the ordered position of the HBCF values, and the HBCFDWG operator only deals with HBCF values, as can be seen from definitions (18) and (19). After that, we combined the previously described HBCFDWG and HBCFDOWG operators to create the HBCFDHWG operator.

Definition 21: Let $\overline{\text{Ц}}_{\mathfrak{s}} = (\S_{\mathfrak{s}}^{+}, \S_{\mathfrak{s}}^{-}) = (\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]} + \iota\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}, \check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]} + \iota\check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]})(\mathfrak{s} = 1, 2, 3, \ldots, \acute{n})$ be a collection of HBCFNs, then the HBCFDHW operator is a function of HBCFDHWG $: \overline{\text{Ц}}^{\acute{n}} \longrightarrow \overline{\text{Ц}}$ such that as (17), shown at the bottom of page 24.

Where $\psi = (\psi_1, \psi_2, \ldots, \psi_{\acute{n}})^T$ be WV of $\overline{\text{Ц}}_{\mathfrak{s}} = (\S_{\mathfrak{s}}^{+}, \S_{\mathfrak{s}}^{-})(\mathfrak{s} = 1, 2, \ldots, \acute{n})$ with $\psi_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mathfrak{s}} =$

$$\mathrm{HBCFDWG}_{\omega}\left(\overline{\text{Ц}}_1, \overline{\text{Ц}}_2, \ldots, \overline{\text{Ц}}_{\text{ķ}}, \overline{\text{Ц}}_{\text{ķ}+1}\right) = \underset{\mathfrak{s}=1}{\overset{\text{ķ}}{\otimes}} \left(\overline{\text{Ц}}_{\mathfrak{s}}\right)^{\omega_{\mathfrak{s}}} \otimes \left(\overline{\text{Ц}}_{\text{ķ}+1}\right)^{\omega_{\text{ķ}+1}}$$

$$= \left(\bigcup_{\check{\mathfrak{Z}}_1^{+} \in \S_1^{+}, \ldots, \check{\mathfrak{Z}}_{\text{ķ}} \in \S_{\text{ķ}}^{+}} \left\{ \begin{array}{l} \left(\dfrac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\text{ķ}} \omega_{\mathfrak{s}} \left(\dfrac{1 - \check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]}}{\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]}} \right)^{6} \right\}^{\frac{1}{6}}} \right) \\ + \iota \left(\dfrac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\text{ķ}} \omega_{\mathfrak{s}} \left(\dfrac{1 - \check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}}{\check{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}} \right)^{6} \right\}^{\frac{1}{6}}} \right) \end{array} \right\}, \bigcup_{\check{\mathfrak{Z}}_1^{-} \in \S_1^{-}, \ldots, \check{\mathfrak{Z}}_{\acute{n}}^{-} \in \S_{\acute{n}}^{-}} \left\{ \begin{array}{l} \left(-1 + \dfrac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\text{ķ}} \omega_{\mathfrak{s}} \left(\dfrac{\left| \check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]} \right|}{1 + \check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]}} \right)^{6} \right\}^{\frac{1}{6}}} \right) + \\ \iota \left(-1 + \dfrac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\text{ķ}} \omega_{\mathfrak{s}} \left(\dfrac{\left| \check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]} \right|}{1 + \check{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]}} \right)^{6} \right\}^{\frac{1}{6}}} \right) \end{array} \right\} \right)$$

1, $\breve{\mathfrak{Z}}'_{\mho(\mathfrak{s})}$ is the $\mathfrak{s}$ th largest weighted HBCF value of $\breve{\mathfrak{Z}}'_{\mathfrak{s}}(\breve{\mathfrak{Z}}'_{\mathfrak{s}} = (\acute{n}\omega)\breve{\mathfrak{Z}}_{\mathfrak{s}}$, $\mathfrak{s} = 1, 2, \ldots, \acute{n})$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_{\acute{n}})$ be the WV of $\breve{\mathfrak{Z}}_{\mathfrak{s}}$ $(\mathfrak{s} = 1, 2, \ldots, \acute{n})$ where $\omega_{\mathfrak{s}} \in [0, 1]$ and $\sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} = 1$, and $\acute{n}$ is the balancing coefficient.

Remark:

(1) When $\psi = \left(\frac{1}{\acute{n}}, \frac{1}{\acute{n}}, \frac{1}{\acute{n}}, \ldots, \frac{1}{\acute{n}}\right)$ the HBCFDHWG operators convert into the HBCFDWG operators.

(2) When $\omega = \left(\frac{1}{\acute{n}}, \frac{1}{\acute{n}}, \frac{1}{\acute{n}}, \ldots, \frac{1}{\acute{n}}\right)$ then the HBCFDHA operators convert into the HBCFDOWG.

V. MADM AOOROACH UNDER THE ENVIRONMENT OF HBCF INFORMATION

In this section, we propose a MADM approach based on the proposed operators and in the environment of HBCFNs.

$$
\otimes\left(\begin{array}{l}
\bigcup_{\breve{\mathfrak{Z}}_1^{+}\in\S_1^{+},\ldots,\breve{\mathfrak{Z}}_{k}^{+}\in\S_{k}^{+},\breve{\mathfrak{Z}}_{k+1}^{+}\in\S_{k+1}^{+}}\left\{\left(\frac{1}{1+\left\{\omega_{k+1}\left(\frac{1-\breve{\mathfrak{Z}}_{k+1}^{+[\tilde{R}]}}{\breve{\mathfrak{Z}}_{k+1}^{+[\tilde{R}]}}\right)^{ƍ}\right\}^{\frac{1}{ƍ}}}\right)+\iota\left(\frac{1}{1+\left\{\omega_{k+1}\left(\frac{1-\breve{\mathfrak{Z}}_{k+1}^{+[\tilde{I}]}}{\breve{\mathfrak{Z}}_{k+1}^{+[\tilde{I}]}}\right)^{ƍ}\right\}^{\frac{1}{ƍ}}}\right)\right\},\\
\bigcup_{\breve{\mathfrak{Z}}_1^{-}\in\S_1^{-},\ldots,\breve{\mathfrak{Z}}_{\acute{n}}^{-}\in\S_{\acute{n}}^{-},\breve{\mathfrak{Z}}_{k+1}^{-}\in\S_{k+1}^{-}}\left\{\left(-1+\frac{1}{1+\left\{\omega_{k+1}\left(\frac{\left|\breve{\mathfrak{Z}}_{k+1}^{-[\tilde{R}]}\right|}{1+\breve{\mathfrak{Z}}_{k+1}^{-[\tilde{R}]}}\right)^{ƍ}\right\}^{\frac{1}{ƍ}}}\right)+\iota\left(-1+\frac{1}{1+\left\{\omega_{k+1}\left(\frac{\left|\breve{\mathfrak{Z}}_{k+1}^{-[\tilde{I}]}\right|}{1+\breve{\mathfrak{Z}}_{k+1}^{-[\tilde{I}]}}\right)^{ƍ}\right\}^{\frac{1}{ƍ}}}\right)\right\}
\end{array}\right)
$$

$$
=\left(\begin{array}{l}
\bigcup_{\breve{\mathfrak{Z}}_1^{+}\in\S_1^{+},\ldots,\breve{\mathfrak{Z}}_{k}^{+}\in\S_{k}^{+},\breve{\mathfrak{Z}}_{k+1}^{+}\in\S_{k+1}^{+}}\left\{\left(\frac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{k+1}\omega_{\mathfrak{s}}\left(\frac{1-\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]}}{\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{R}]}}\right)^{ƍ}\right\}^{\frac{1}{ƍ}}}\right)+\iota\left(\frac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{k+1}\omega_{\mathfrak{s}}\left(\frac{1-\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}}{\breve{\mathfrak{Z}}_{\mathfrak{s}}^{+[\tilde{I}]}}\right)^{ƍ}\right\}^{\frac{1}{ƍ}}}\right)\right\},\\
\bigcup_{\breve{\mathfrak{Z}}_1^{-}\in\S_1^{-},\ldots,\breve{\mathfrak{Z}}_{\acute{n}}^{-}\in\S_{\acute{n}}^{-},\breve{\mathfrak{Z}}_{k+1}^{-}\in\S_{k+1}^{-}}\left\{\left(-1+\frac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{k+1}\omega_{\mathfrak{s}}\left(\frac{\left|\breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]}\right|}{1+\breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{R}]}}\right)^{ƍ}\right\}^{\frac{1}{ƍ}}}\right)+\iota\left(-1+\frac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{k+1}\omega_{\mathfrak{s}}\left(\frac{\left|\breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]}\right|}{1+\breve{\mathfrak{Z}}_{\mathfrak{s}}^{-[\tilde{I}]}}\right)^{ƍ}\right\}^{\frac{1}{ƍ}}}\right)\right\}
\end{array}\right)
$$

Let us assume that there are $\bar{\Upsilon}$ alternatives $\breve{A}_{\varpi}(\varpi = 1, 2, \ldots, \bar{\Upsilon})$ and $\flat$ criteria $\mathcal{B}_{\eta}(\eta = 1, 2, \ldots, \flat)$ along with criteria WV $\omega = (\omega_1, \omega_2, \ldots, \omega_{\flat})^T$ $\omega_{\flat} \in [0, 1]$ $\forall$ $\flat$ and $\sum_{\eta=1}^{\flat} \omega_{\eta} = 1$ Now we assume that the HBCF decision matrix is $\mathcal{M} = (\Omega_{\varpi\eta})_{\bar{\Upsilon}\times\flat} = (\S^{+}_{\varpi\eta}, \S^{-}_{\varpi\eta})_{\mathfrak{T}\times\varsigma} =$

$$\mathrm{HBCFDOWG}_{\psi}\left(\bar{\mathfrak{U}}_1, \bar{\mathfrak{U}}_2, \ldots, \bar{\mathfrak{U}}_{\acute{n}}\right) = \bigotimes_{\mathfrak{s}=1}^{\acute{n}} \left(\bar{\mathfrak{U}}_{\mho(\mathfrak{s})}\right)^{\psi_{\mathfrak{s}}}$$

$$= \left(\begin{array}{c} \bigcup_{\check{\mathfrak{z}}_1^{+} \in \S_1^{+}, \ldots, \check{\mathfrak{z}}_{\acute{n}}^{+} \in \S_{\acute{n}}^{+}} \left\{ \left[\left(\frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mho(\mathfrak{s})} \left(\frac{1 - \check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{+[\tilde{R}]}}{\check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{+[\tilde{R}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) + \iota \left(\frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mho(\mathfrak{s})} \left(\frac{1 - \check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{+[\tilde{I}]}}{\check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{+[\tilde{I}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) \right] \right\}, \\ \bigcup_{\check{\mathfrak{z}}_1^{-} \in \S_1^{-}, \ldots, \check{\mathfrak{z}}_{\acute{n}}^{-} \in \S_{\acute{n}}^{-}} \left\{ \left[\left(-1 + \frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mho(\mathfrak{s})} \left(\frac{\left| \check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{-[\tilde{R}]} \right|}{1 + \check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{-[\tilde{R}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) + \iota \left(-1 + \frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mho(\mathfrak{s})} \left(\frac{\left| \check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{-[\tilde{I}]} \right|}{1 + \check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{-[\tilde{I}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) \right] \right\} \end{array} \right) \tag{16}$$

$$\mathrm{HBCFDHWG}_{\psi,\ \omega}\left(\bar{\mathfrak{U}}_1, \bar{\mathfrak{U}}_2, \ldots, \bar{\mathfrak{U}}_{\acute{n}}\right) = \bigotimes_{\mathfrak{s}=1}^{\acute{n}} \left(\bar{\mathfrak{U}}_{\mho(\mathfrak{s})}\right)^{\psi_{\mathfrak{s}}}$$

$$= \left(\begin{array}{c} \bigcup_{\check{\mathfrak{z}}_1^{+} \in \S_1^{+}, \ldots, \check{\mathfrak{z}}_{\acute{n}}^{+} \in \S_{\acute{n}}^{+}} \left\{ \left[\left(\frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mho(\mathfrak{s})} \left(\frac{1 - \check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{+[\tilde{R}]}}{\check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{+[\tilde{R}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) + \iota \left(1 - \frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mho(\mathfrak{s})} \left(\frac{1 - \check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{+[\tilde{I}]}}{\check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{+[\tilde{I}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) \right] \right\}, \\ \bigcup_{\check{\mathfrak{z}}_1^{-} \in \S_1^{-}, \ldots, \check{\mathfrak{z}}_{\acute{n}}^{-} \in \S_{\acute{n}}^{-}} \left\{ \left[\left(-1 + \frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mho(\mathfrak{s})} \left(\frac{\left| \check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{-[\tilde{R}]} \right|}{1 + \check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{-[\tilde{R}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) + \iota \left(-1 + \frac{1}{1 + \left\{ \sum_{\mathfrak{s}=1}^{\acute{n}} \psi_{\mho(\mathfrak{s})} \left(\frac{\left| \check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{-[\tilde{I}]} \right|}{1 + \check{\mathfrak{z}}_{\mho(\mathfrak{s})}^{-[\tilde{I}]}} \right)^{\sigma} \right\}^{\frac{1}{\sigma}}} \right) \right] \right\} \end{array} \right) \tag{17}$$

$(\breve{\mathfrak{Z}}^{+[\tilde{R}]}_{\text{@}\mathfrak{m}} + \iota\breve{\mathfrak{Z}}^{+[\tilde{I}]}_{\text{@}\mathfrak{m}}, \breve{\mathfrak{Z}}^{-[\tilde{R}]}_{\text{@}\mathfrak{m}} + \iota\breve{\mathfrak{Z}}^{-[\tilde{I}]}_{\text{@}\mathfrak{m}})_{\bar{Y}\times\mathfrak{p}}$ Where $\S^{+}_{\text{@}\mathfrak{m}}$ is the PPMD for which the alternative $\breve{A}_{\mathfrak{m}}$ satisfies the attribute $\mathfrak{B}_{\mathfrak{m}}$ given by the DM, and $\S^{-}_{\text{@}\mathfrak{m}}$ is the NPMD for which the alternative $\breve{A}_{\mathfrak{m}}$ do not satisfy the attribute $\mathfrak{B}_{\mathfrak{m}}$ given by the DM, where $\S^{+}_{\text{@}\mathfrak{m}} \in [0,\ 1]$ and $\S^{-}_{\text{@}\mathfrak{m}} \in [-1, 0]$.

Algorithm:

With the help of the suggested HBCFDWA and HBCFDWG operators, we assume the algorithm for solving the MADM issue within the context of HBCFNs.

Step-1: In each MADM process, the attribute may be in two types, one is benefit type and the second is cost type. In any MADM procedure if the attributes are given in the form of cost type, then we use the following formula to make it benefit type:

$$\Omega_{\text{@}\mathfrak{m}} = \begin{cases} \left(\breve{\mathfrak{Z}}^{+[\tilde{R}]}_{\text{@}\mathfrak{m}} + \iota\breve{\mathfrak{Z}}^{+[\tilde{I}]}_{\text{@}\mathfrak{m}}, \breve{\mathfrak{Z}}^{-[\tilde{R}]}_{\text{@}\mathfrak{m}} + \iota\breve{\mathfrak{Z}}^{-[\tilde{I}]}_{\text{@}\mathfrak{m}}\right) \\ \textit{for benefit type of attribute} \\ \textit{and} \\ \left(\begin{matrix} 1 - \breve{\mathfrak{Z}}^{+[\tilde{R}]}_{\text{@}\mathfrak{m}} + \iota\left(1 - \breve{\mathfrak{Z}}^{+[\tilde{I}]}_{\text{@}\mathfrak{m}}\right), \\ 1 - \breve{\mathfrak{Z}}^{-[\tilde{R}]}_{\text{@}\mathfrak{m}} + \iota\left(1 - \breve{\mathfrak{Z}}^{-[\tilde{I}]}_{\text{@}\mathfrak{m}}\right) \end{matrix}\right) \\ \textit{for cost type of attribute} \end{cases}$$

Step-2: By employing the HBCFDWA to the supposed decision information provided in the matrix $\mathcal{M}$, derive all the values $\Omega_{\text{@}}$, $(\text{@} = 1, 2, \ldots, \bar{Y})$ of the alternative $\mathfrak{B}_{\text{@}}$. As (18), shown at the bottom of the page.

If we select the HBCFDWG operator then we have as (19), shown at the bottom of the next page.

Step 3: Compute the score values $\mathbb{S}(\Omega_{\text{@}})(\text{@} = 1, 2, \ldots, \bar{Y})$**.**

Step 4: Rank all the alternatives $\breve{A}_{\text{@}}$ $(\text{@} = 1, 2, \ldots, \bar{Y})$ in terms of $\mathbb{S}\left(\Omega_{\text{@}}\right)$ $(\text{@} = 1, 2, \ldots, \bar{Y})$**.** If two score functions $\mathbb{S}\left(\Omega_{\text{@}}\right)$ and $\mathbb{S}\left(\Omega_{\mathfrak{m}}\right)$ have the same values, then we use the accuracy function.

Step 5: Choose the best alternative**:**

Step 6: End.

A. CASE STUDY

Let's take a hypothetical situation where a small software development business, X, is trying to decide which cloud services provider to use to host their apps and manage their infrastructure. Four attributes have been defined for evaluating the four possible cloud service providers that are $\breve{A}_1$, $\breve{A}_2$, $\breve{A}_3$, and $\breve{A}_4$. The following are the attributes:

1. Performance: This attribute assesses the performance of the cloud service provider in terms of speed, dependability, and uptime. X wants to make sure that their applications function properly and are available to users with little to no downtime.
2. Cost: The price of services given by each supplier must be taken into account because X is a small business with a constrained budget. They want a company that offers affordable prices without sacrificing the caliber of services.
3. Security: For X, data security is essential since it deals with confidential client data and intellectual property. To shield their data from potential breaches and hacks, they seek a cloud service provider that delivers strong security safeguards.
4. Scalability: As a developing business, X seeks a cloud services provider that can readily meet its expanding demands. They seek a scalable solution that can adjust to their shifting needs without creating interruptions to their applications.

   Let's now examine four prospective cloud service providers and assess them in light of the aforementioned attributes. The examined values are demonstrated in Table 1.

Step 1: The given data in Table 1 is benefit type so there is no need to normalize it.

Step 2: For $\mathfrak{G} = 2$ use the HBCFDWA operators to determine all the preference values $\Omega_{\text{@}}$ of the cloud service providers $\breve{A}_{\text{@}}$ $(\text{@} = 1,\ 2,\ 3,\ 4)$.

$$\Omega_1 = \left(\begin{matrix} \left\{\begin{matrix} (0.010151 + \iota0.045173), \\ (0.093679 + \iota170704), \\ (0.162709 + \iota0.302074) \end{matrix}\right\}, \\ \left\{\begin{matrix} (-0.05856 - \iota0.12539), \\ (-0.317 - \iota0.50035), \\ (-0.32865 - \iota51527) \end{matrix}\right\} \end{matrix}\right)$$

$$\Omega_2 = \left(\begin{matrix} \left\{\begin{matrix} (0.010491 + \iota0.043546), \\ (0.149829 + \iota205853), \\ (0.244894 + \iota385972) \end{matrix}\right\}, \\ \left\{\begin{matrix} (-0.06396 - \iota0.21493), \\ (-0.09634 - \iota0.3461), \\ (-0.40441 - \iota60524) \end{matrix}\right\} \end{matrix}\right)$$

$$\Omega_3 = \left(\begin{matrix} \left\{\begin{matrix} (0.010324 + \iota0.41019), \\ (0.099947 + \iota0.237896), \\ (0.290512 + \iota521025) \end{matrix}\right\}, \\ \left\{\begin{matrix} (-0.04029 - \iota14852), \\ (-0.42814 - \iota0.52571), \\ (-0.17811 - \iota0.45874) \end{matrix}\right\} \end{matrix}\right)$$

$$\Omega_4 = \left(\begin{matrix} \left\{\begin{matrix} (0.005563 + \iota0.026615), \\ (0.062261 + \iota0.111564), \\ (0.199135 + \iota0.322602) \end{matrix}\right\}, \\ \left\{\begin{matrix} (-0.04135 - \iota0.17921), \\ (-0.36829 - \iota0.49384), \\ (-0.65492 - \iota0.74863) \end{matrix}\right\} \end{matrix}\right)$$

Step 3: The obtained score values of $\mathbb{S}\left(\Omega_{\text{@}}\right)$ $(\text{@} = 1,\ 2,\ 3,\ 4)$ of the overall HBCFNs $\left(\Omega_{\text{@}}\right)$ $(\text{@} = 1,\ 2,\ 3,\ 4)$ are

$$\mathbb{S}\left(\Omega_1\right) = 0.411606,\ \mathbb{S}\left(\Omega_2\right) = 0.442468,$$
$$\mathbb{S}\left(\Omega_3\right) = 0.451767,\ \mathbb{S}\left(\Omega_4\right) = 0.353458$$

Step 4: Rank all the cloud service providers $\breve{A}_{\text{@}}$ $(\text{@} = 1,\ 2,\ 3,\ 4)$ with the following score values

$\mathbb{S}\left(\Omega_{\text{ɷ}}\right)$ ($\text{ɷ} = 1, 2, 3, 4$) of the overall HBCFNs:

$$\breve{A}_3 > \breve{A}_2 > \breve{A}_1 > \breve{A}_4.$$

Step 5: $\breve{A}_3$ is selected as the best cloud service provider.

Step 6: End.

If we use the HBCFDWG operator instead of the HBCFDWA operator then all the above steps are similar for the HBCFDWG framework.

Step 1: The given data in Table 1 is benefit type so there is no need to normalize it.

Step 2: For $\text{ɓ} = 2$ use the HBCFDWG operator to determine all the preferences values $\Omega_{\text{ɷ}}$ of the cloud service providers $\breve{A}_{\text{ɷ}}$ ($\text{ɷ} = 1, 2, 3, 4$).

$$\Omega_1 = \left(\begin{array}{c}\left\{\begin{array}{c}(0.03537 + \iota 0.154307), \\ (0.291803 + \iota 431053), \\ (0.331454 + \iota 0.427463)\end{array}\right\}, \\ \left\{\begin{array}{c}(-0.01746 - \iota 0.03513), \\ (-0.11425 - \iota 0.20132), \\ (-0.15966 - \iota 25671)\end{array}\right\}\end{array}\right)$$

$$\Omega_2 = \left(\begin{array}{c}\left\{\begin{array}{c}(0.040433 + \iota 0.152446), \\ (0.411503 + \iota 504791), \\ (0.381996 + \iota 580109)\end{array}\right\}, \\ \left\{\begin{array}{c}(-0.11188 - \iota 0.24327), \\ (-0.10857 - \iota 0.22521), \\ (-0.20668 - \iota 40546)\end{array}\right\}\end{array}\right)$$

$$\Omega_3 = \left(\begin{array}{c}\left\{\begin{array}{c}(0.036813 + \iota 0.139119), \\ (0.296548 + \iota 0.525907), \\ (0.352216 + \iota 655157)\end{array}\right\}, \\ \left\{\begin{array}{c}(-0.01194 - \iota 0.04344), \\ (-0.15871 - \iota 0.2302), \\ (-0.33881 - \iota 0.39481)\end{array}\right\}\end{array}\right)$$

$$\Omega_4 = \left(\begin{array}{c}\left\{\begin{array}{c}(0.043432 + \iota 0.192445), \\ (0.451968 + \iota 0.59084), \\ (0.469065 + \iota 0.607338)\end{array}\right\}, \\ \left\{\begin{array}{c}(-0.01334 - \iota 0.6238), \\ (-0.14181 - \iota 0.19626), \\ (-0.39174 - \iota 0.57336)\end{array}\right\}\end{array}\right)$$

Step 3: The obtained score values of $\mathbb{S}\left(\Omega_{\text{ɷ}}\right)$ ($\text{ɷ}=1, 2, 3, 4$) of the overall HBCFNs $\left(\Omega_{\text{ɷ}}\right)$ ($\text{ɷ} = 1, 2, 3, 4$) are

$$\mathbb{S}(\Omega_1) = 0.57391,\ \mathbb{S}(\Omega_2) = 0.564184,$$
$$\mathbb{S}(\Omega_3) = 0.568988,\ \mathbb{S}(\Omega_4) = 0.581351$$

Step 4: Rank all the cloud service providers $\breve{A}_{\text{ɷ}}$ ($\text{ɷ} = 1, 2, 3, 4$) with the following score values $\mathbb{S}\left(\Omega_{\text{ɷ}}\right)$ ($\text{ɷ} = 1, 2, 3, 4$) of the overall HBCFNs:

$$\breve{A}_4 > \breve{A}_3 > \breve{A}_1 > \breve{A}_2.$$

Step 5: $\breve{A}_4$ is selected as the cloud service provider.

Step 6: End.

$$\begin{aligned}\Omega_{\text{ɷ}} &= \left(\S^{+}_{\text{ɷ}}, \S^{-}_{\text{ɷ}}\right) = \left(\breve{\mathfrak{z}}^{+[\tilde{R}]}_{\text{ɷ}} + \iota \breve{\mathfrak{z}}^{+[\tilde{I}]}_{\text{ɷ}}, \breve{\mathfrak{z}}^{-[\tilde{R}]}_{\text{ɷ}} + \iota \breve{\mathfrak{z}}^{-[\tilde{I}]}_{\text{ɷ}}\right) \\ &= \text{HBCFDWA}_{\omega}\left(\Omega_{\text{ɷ}1}, \Omega_{\text{ɷ}2}, \ldots, \Omega_{\text{ɷ}\acute{n}}\right) \\ &= \bigoplus_{\mathfrak{s}=1}^{\acute{n}} \left(\omega_{\mathfrak{s}} \Omega_{\text{ɷ}\mathfrak{m}}\right) \\ &= \left(\begin{array}{c} \bigcup_{\breve{\mathfrak{z}}^{+}_{1} \in \S^{+}_{1}, \ldots, \breve{\mathfrak{z}}^{+}_{\acute{n}} \in \S^{+}_{\acute{n}}} \left\{\left(1 - \frac{1}{1 + \left\{\sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} \left(\frac{\breve{\mathfrak{z}}^{+[\tilde{R}]}_{\text{ɷ}\mathfrak{m}}}{1 - \breve{\mathfrak{z}}^{+[\tilde{R}]}_{\text{ɷ}\mathfrak{m}}}\right)^{\text{ɓ}}\right\}^{\frac{1}{\text{ɓ}}}}\right) + \iota \left(1 - \frac{1}{1 + \left\{\sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} \left(\frac{\breve{\mathfrak{z}}^{+[\tilde{I}]}_{\text{ɷ}\mathfrak{m}}}{1 - \breve{\mathfrak{z}}^{+[\tilde{I}]}_{\text{ɷ}\mathfrak{m}}}\right)^{\text{ɓ}}\right\}^{\frac{1}{\text{ɓ}}}}\right)\right\}, \\ \bigcup_{\breve{\mathfrak{z}}^{-}_{1} \in \S^{-}_{1}, \ldots, \breve{\mathfrak{z}}^{-}_{\acute{n}} \in \S^{-}_{\acute{n}}} \left\{\left(\frac{-1}{1 + \left\{\sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} \left(\frac{1 + \breve{\mathfrak{z}}^{-[\tilde{R}]}_{\text{ɷ}\mathfrak{m}}}{\left|\breve{\mathfrak{z}}^{-[\tilde{R}]}_{\text{ɷ}\mathfrak{m}}\right|}\right)^{\text{ɓ}}\right\}^{\frac{1}{\text{ɓ}}}}\right) + \iota \left(\frac{-1}{1 + \left\{\sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} \left(\frac{1 + \breve{\mathfrak{z}}^{-[\tilde{I}]}_{\text{ɷ}\mathfrak{m}}}{\left|\breve{\mathfrak{z}}^{-[\tilde{I}]}_{\text{ɷ}\mathfrak{m}}\right|}\right)^{\text{ɓ}}\right\}^{\frac{1}{\text{ɓ}}}}\right)\right\} \end{array}\right)\end{aligned} \tag{18}$$

VI. COMPARATIVE ANALYSIS

To highlight the benefits and guiding principles of the diagnosed theory, we make a comparison of diagnosed work with previously published studies. This is because comparison is essential for comprehending the importance and effectiveness of any newly produced work. We are unable to discriminate between the superior and inferior without comparison. For this purpose, we take some previously published concepts and after that, we apply all these selected concepts to the proposed theory and observe the results that are discussed in Table 2. First of all, we make a list of selected theories:

- MADM using HF Dombi–Archimedean weighted (HFDAW) AOs by Liu et al. [21].
- IF geometric AOs based on Einstein (IFGE) by Wang and Liu [23].
- MADM methods based on AA power (AAP) AOs for managing complex IFSs by Mahmood and Ali [28].
- BF Hamacher (BFH) AOs in MADM by Wei et al. [38].
- HBVFSs and BVHFSs and their applications in MAGDM by Mandal and Ranadive [40].
- Identification and classification of AOs using BCF settings and their application in decision support systems by Mahmood et al. [47].

The above-selected theories are mostly aggregation-related theories like Dombi–Archimedean weighted AOs, geometric AOs based on Einstein, AA power AOs, Hamacher AOs, etc. Using these AOs, we make a comparison with our proposed theory of Dombi AOs. Using our Dombi AOs, we find out the score values of HBCFNs and then we also find out these score values using the above-discussed operators and after that, we try to compare all of them.

The above-given Table 2 is the resultant table, and based on the given observation of Table 2 we conclude that all supposed theories cannot support us and cannot follow our data and results. The reasons why these ideas do not support us and do not support our findings and data are covered in detail below. First of all, we start with the concept of HF Dombi–Archimedean weighted (HFDAW) AOs by Liu et al. [21]. We find that this theory is unable to accommodate our terms and conditions since HFS and their AOs can only manage the set of possible GM and cannot accommodate two-dimensional information such as bipolarity with complex form. While the theory of IF geometric AOs based on Einstein (IFGE) proposed by Wang and Liu. [23] can easily manage both the GM and GNM but cannot handle the information related to hesitant and also cannot afford two-dimensional complex information. The theory of MADM methods based

$$
\begin{aligned}
\Omega_{\wp} &= \left(\S_{\wp}^{+},\ \S_{\wp}^{-}\right) \\
&= \left(\check{\mathfrak{Z}}_{\wp}^{+[\tilde{R}]} + \iota\check{\mathfrak{Z}}_{\wp}^{+[\tilde{I}]}, \check{\mathfrak{Z}}_{\wp}^{-[\tilde{R}]} + \iota\check{\mathfrak{Z}}_{\wp}^{-[\tilde{I}]}\right) \\
&= \mathrm{HBCFDWG}_{\omega}\left(\Omega_{\wp_1}, \Omega_{\wp_2}, \ldots, \Omega_{\wp_{\acute{n}}}\right) \\
&= \bigotimes_{\mathfrak{s}=1}^{\acute{n}} \left(\Omega_{\wp\mathfrak{m}}\right)^{\omega_{\mathfrak{s}}} \\
&= \left(\begin{array}{c}
\bigcup_{\check{\mathfrak{Z}}_1^{+} \in \S_1^{+}, \ldots, \check{\mathfrak{Z}}_{\acute{n}}^{+} \in \S_{\acute{n}}^{+}} \left\{\left[\left(\frac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} \left(\frac{1-\check{\mathfrak{Z}}_{\wp\mathfrak{m}}^{+[\tilde{R}]}}{\check{\mathfrak{Z}}_{\wp\mathfrak{m}}^{+[\tilde{R}]}}\right)^{\vartheta}\right\}^{\frac{1}{\vartheta}}}\right) + \iota\left(\frac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} \left(\frac{1-\check{\mathfrak{Z}}_{\wp\mathfrak{m}}^{+[\tilde{I}]}}{\check{\mathfrak{Z}}_{\wp\mathfrak{m}}^{+[\tilde{I}]}}\right)^{\vartheta}\right\}^{\frac{1}{\vartheta}}}\right)\right]\right\}, \\
\bigcup_{\check{\mathfrak{Z}}_1^{-} \in \S_1^{-}, \ldots, \check{\mathfrak{Z}}_{\acute{n}}^{-} \in \S_{\acute{n}}^{-}} \left\{\left[\left(-1+\frac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} \left(\frac{\left|\check{\mathfrak{Z}}_{\wp\mathfrak{m}}^{-[\tilde{R}]}\right|}{1+\check{\mathfrak{Z}}_{\wp\mathfrak{m}}^{-[\tilde{R}]}}\right)^{\vartheta}\right\}^{\frac{1}{\vartheta}}}\right) + \iota\left(-1+\frac{1}{1+\left\{\sum_{\mathfrak{s}=1}^{\acute{n}} \omega_{\mathfrak{s}} \left(\frac{\left|\check{\mathfrak{Z}}_{\wp\mathfrak{m}}^{-[\tilde{I}]}\right|}{1+\check{\mathfrak{Z}}_{\wp\mathfrak{m}}^{-[\tilde{I}]}}\right)^{\vartheta}\right\}^{\frac{1}{\vartheta}}}\right)\right]\right\}
\end{array}\right)
\end{aligned}
\tag{19}
$$

TABLE 1. Expert decision matrix in the environment of HBCFNs.

	Ƀ$_1$	Ƀ$_2$	Ƀ$_3$	Ƀ$_4$
Ă$_1$	$\left(\left\{(0.124+\iota 0.210),(0.309+\iota 0.337)\right\},\left\{(-0.172-\iota 0.217),(-0.308-\iota 0.411)\right\}\right)$	$\left(\left\{(0.109+\iota 0.233),(0.309+\iota 0.424),(0.513+\iota 0.630)\right\},\left\{(-0.126-\iota 0.209),(-0.395-\iota 0.411)\right\}\right)$	$\left(\left\{(0.114+\iota 0.249),(0.312+\iota 0.399)\right\},\left\{(-0.170-\iota 0.203),(-0.307-\iota 0.412)\right\}\right)$	$\left(\left\{(0.181+\iota 0.249),(0.331+\iota 0.413)\right\},\left\{(-0.114-\iota 0.239),(-0.371-\iota 0.445),(-0.556-\iota 0.613)\right\}\right)$
Ă$_2$	$\left(\left\{(0.131+\iota 0.245),(0.363+\iota 0.411),(0.544+\iota 0.632)\right\},\left\{(-0.193-\iota 0.231),(-0.383-\iota 0.401),(-0.508-\iota 0.604)\right\}\right)$	$\left(\left\{(0.119+\iota 0.232),(0.387+\iota 0.448)\right\},\left\{(-0.514-\iota 0.624),(-0.101-\iota 0.220),(-0.381-\iota 0.542)\right\}\right)$	$\left(\left\{(0.127+\iota 0.225),(0.367+\iota 0.406)\right\},\left\{(-0.135-\iota 0.258),(-0.315-\iota 0.49)\right\}\right)$	$\left(\left\{(0.131+\iota 0.215),(0.390+\iota 0.423)\right\},\left\{(-0.104-\iota 0.290),(-0.390-\iota 0.440)\right\}\right)$
Ă$_3$	$\left(\left\{(0.115+\iota 0.220),(0.321+\iota 0.499)\right\},\left\{(-0.132-\iota 0.213),(-0.390-\iota 0.424)\right\}\right)$	$\left(\left\{(0.159+\iota 0.264),(0.309+\iota 0.402)\right\},\left\{(-0.104-\iota 0.261),(-0.376-\iota 0.498)\right\}\right)$	$\left(\left\{(0.115+\iota 0.206),(0.301+\iota 0.408),(0.570+\iota 0.676)\right\},\left\{(-0.154-\iota 0.233),(-0.370-\iota 0.408),(-0.590-\iota 0.605)\right\}\right)$	$\left(\left\{(0.121+\iota 0.231),(0.390+\iota 0.405)\right\},\left\{(-0.102-\iota 0.209),(-0.399-\iota 0.411)\right\}\right)$
Ă$_4$	$\left(\left\{(0.191+\iota 0.211),(0.389+\iota 0.442)\right\},\left\{(-0.181-\iota 0.209),(-0.309-\iota 0.412),(-0.506-\iota 0.639)\right\}\right)$	$\left(\left\{(0.119+\iota 0.282),(0.398+\iota 0.442)\right\},\left\{(-0.121-\iota 0.289),(-0.391-\iota 0.407)\right\}\right)$	$\left(\left\{(0.111+\iota 0.292),(0.389+\iota 0.482)\right\},\left\{(-0.119-\iota 0.298),(-0.393-\iota 0.415),(-0.566-\iota 0.646)\right\}\right)$	$\left(\left\{(0.181+\iota 0.299),(0.392+\iota 0.471),(0.529+\iota 0.631)\right\},\left\{(-0.102-\iota 0.209),(-0.311-\iota 0.403),(-0.592-\iota 0.642)\right\}\right)$

on AA power (AAP) AOs for managing complex IFSs by Mahmood and Ali [28] can solve all complex information of both GM and GNM. However, it cannot handle the hesitant and bipolarity information related to both grades. Moreover, the theory of BF Hamacher (BFH) AOs in MADM by Wei et al. [38] is capable of dealing with the positive and negative aspects of any object, which means that framework with bipolarity is not a problem with this theory but there are also many problems with this theory when information comes in bipolar complex form then this theory cannot work properly. We assume another theory of HBVFSs and BVHFSs and their applications in MAGDM by Mandal and Ranadive [40] and try to use it for solving proposed HBCF information. It is very easy to handle hesitancy and bipolarity with this theory but it is also one-dimensional and cannot support the information of hesitant and complex with two dimensions at the same time. In the end, we compare our work with the theory of Identification and classification of AOs using BCF

TABLE 2. Comparison between the existing and proposed work.

Theories	Methods	$\mathbb{S}(\breve{A}_1)$	$\mathbb{S}(\breve{A}_2)$	$\mathbb{S}(\breve{A}_3)$	$\mathbb{S}(\breve{A}_4)$	Ranking
Liu et al. [21]	HFDAW AOs	×××××	×××××	×××××	×××××	×××××
Wang and Liu. [23]	IFGE AOs	×××××	×××××	×××××	×××××	×××××
Mahmood and Ali [28]	IFAAP AOs	×××××	×××××	×××××	×××××	×××××
Wei et al. [38]	BFH AOs	×××××	×××××	×××××	×××××	×××××
Mandal and Ranadive [40]	HBVF AOs	×××××	×××××	×××××	×××××	×××××
Mahmood and Rehman [47]	BCF AOs	×××××	×××××	×××××	×××××	×××××
Proposed work	HBCFDWA	0.411606	0.442468	0.451767	0.353458	$\breve{A}_3 > \breve{A}_2 > \breve{A}_1 > \breve{A}_4$
Proposed work	HBCFDWG	0.57391	0.564184	0.568988	0.581351	$\breve{A}_4 > \breve{A}_3 > \breve{A}_1 > \breve{A}_2$

settings and their application in decision support systems by Mahmood et al. [47]. This theory is close to our framework it can handle easily two-dimensional information with positive and negative aspects of any object. But BCFS theory cannot work with hesitant form so all the above comparison shows that all the above-discussed theories in Table 2 cannot solve our HBCF information because our information contains all the following characteristics hesitancy, bipolarity, complex numbers data, and also two-dimensional information. So, the above comparison shows the effectiveness and superiority of our work.

VII. CONCLUSION

The term CS describes a range of computer tools and programs that are transmitted via the internet by the servers of unaffiliated third parties. Without having to buy or maintain actual gear and infrastructure, these services let customers access and use computer resources including storage, processing power, networking, and software applications. Because of their adaptability, scalability, and affordability, CS has become extremely popular. In this manuscript, we developed the theory of HBCFSs and their related Dombi AOs to discuss the best CS provider. Using the framework of HBCFSs theory we try to invent a tool that can easily solve two-dimensional information with the properties of bipolarity. HBCFSs offer DMs a helpful framework for resolving day-to-day issues. We invent the HBCFSs theory by combining the two basic concepts of HFSs and BCFSs. Operational laws related to HBCFSs are also discussed for solving several Dombi aggregation operators with Dombi t-norm and Dombi t-conorm. The main developed concepts in this manuscript are HBCFDWA, HBCFDOWA, HBCFDHA, HBCFDWG, HBCFDOWG, and HBCFDHG operators. We also invent comparison rules to compare two HBCFNs. The MADM technique is one of the most popular techniques to make decisions freely. We resolve the MADM technique with their algorithm under the environment of HBCF information and after that using the MADM algorithm we resolve the numerical application. At the end of the manuscript, we proposed a comparison with different theories and we conclude, that our theory is more effective and prominent for solving two-dimensional issues in the form of a set.

A. FUTURE WORK AND LIMITATIONS

Our theory of HBCFSs can be extended into many other frameworks but there are some limitations and restrictions available in our theory. The theory of HBCFSs cannot handle simple fuzzy, intuitionistic, and complex information until hesitation and 2^{nd} dimensions are not involved. In the future, we will convert the HBCF idea into soft sets, rough sets, intuitionistic complex hesitant FSs, picture FSs, and spherical or T-spherical FSs. We will also extend the above-presented idea for Schweizer-Sklar AOs [49] and Hamacher and Muirhead mean operators [50], [51], [52]. The above-defined idea of HBCFSs can be extended into many other fields of mathematics and can be converted into different mathematical structures to solve hesitation and uncertainty.

Some more directions are discussed as follows. We can extend the HBCF framework into confidence levels BCF AOs [53]. We can use different techniques of DM like TOPSIS techniques under BCF intuitionistic fuzzy N-SSs [54]. Moreover, our idea can be extended to the following concepts of bipolar neutrosophic SSs and applications in DM [55]. Linguistic AOs, Cubic BF Dombi averaging AOs bipolar picture fuzzy operators, and new distance measures are very good options for the extensions of HBCFSs theory [37], [56], [57].

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