IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 6 January 2024, accepted 29 January 2024, date of publication 21 February 2024, date of current version 6 March 2024.
Digital Object Identifier 10.1109/ACCESS.2024.3368526

RESEARCH ARTICLE

Identification of Eco-Friendly Transportation Mode by Employing Complex Intuitionistic Fuzzy Multi-Criteria Decision-Making Approach Based on Probability Aggregation Operators

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This work was supported by the Qinglan Project of Jiangsu Province, Jiangsu Famous Teacher's Studio of Dual Qualification for Vocational Education and Intelligent Equipment Manufacturing.

ABSTRACT A number of factors, including environmental effects, cost efficiency, delivery time, and capacity, must be considered when choosing an eco-friendly method of transportation. This is a multi-criteria decision-making dilemma because it necessitates the simultaneous evaluation and prioritization of multiple considerations. Selecting the most eco-friendly mode of transportation requires weighing trade-offs between these factors to make an informed choice that satisfies particular transportation needs and is in line with sustainability goals. This calls for a thorough and methodical decision-making process. Thus, in this manuscript, first, we anticipate fundamental algebraic properties for the cartesian form of complex intuitionistic fuzzy set and then devise various probability aggregation operators such as probability complex intuitionistic fuzzy weighted averaging, probability complex intuitionistic fuzzy ordered weighted averaging, immediate probability complex intuitionistic fuzzy ordered weighted averaging, probability complex intuitionistic fuzzy weighted geometric, probability complex intuitionistic fuzzy ordered weighted geometric, immediate probability complex intuitionistic fuzzy ordered weighted geometric and linked axioms. Afterward, we deduce a technique of multi-criteria decision-making in the setting of the cartesian framework of the complex intuitionistic fuzzy set by utilizing the anticipated aggregation operators and then analyze a case study ''identification of eco-friendly transportation mode'' with the assistance of the developed technique of multi-criteria decision-making. In the last of the manuscript, we compare the anticipated theory with certain prevailing concepts to reveal the supremacy and preeminence of the originated theory.

INDEX TERMS Eco-friendly transportation mode, probability aggregation operators, multi-criteria decision-making, complex intuitionistic fuzzy set.

I. INTRODUCTION

Any kind of transportation that, when compared to traditional methods, drastically reduces emissions, energy consumption, and pollution is considered eco-friendly. By utilizing cleaner technology, renewable energy sources, or encouraging higher resource efficiency, these approaches promote sustainability. Common examples include walking, bicycling, using renewable energy to power public transportation such as buses and trains, and electric vehicles (EVs). It is impossible to overestimate the significance of eco-friendly transportation options in the fight against climate change and the destruction

The associate editor coordinating the review of this manuscript and approving it for publication was Giovanni Pau.

of the environment. Because typical cars run on fossil fuels, transportation has a major impact on air pollution and greenhouse gas emissions. Societies may lessen the negative consequences of pollution caused by transportation and drastically reduce their carbon footprint by switching to eco-friendly alternatives. Reducing hazardous emissions, such as particulate matter, nitrogen oxides (NOx), and carbon dioxide (CO2), is an important factor. For example, electric cars have no exhaust emissions, which helps to clean the air and lowers the risk of diseases linked to pollution for human health. In a similar vein, public transit networks powered by cleaner fuels or renewable energy sources lessen traffic jams and lower emissions from individual cars.

Adopting eco-friendly forms of transportation also encourages energy saving and efficiency. When renewable energy sources like solar or wind power are used to charge electric automobiles, it creates a more sustainable energy ecology. Furthermore, modes like walking and cycling not only lower emissions but also encourage better lives, addressing problems like obesity and sedentary behavior. Furthermore, by lowering the demand for expansive infrastructure like parking lots and roads, these mobility options contribute to sustainable urban growth. Putting money into bike lanes and public transit promotes compact, walkable neighborhoods, which lowers land use, protects green spaces, and makes cities more livable overall. Businesses, people, and governments all have crucial responsibilities to play in accelerating the transition to eco-friendly transportation. Policies that support public transportation, encourage the use of renewable energy cars, and make investments in infrastructure for walkers and bicycles are essential. Similarly, lowering the environmental effect of transportation is greatly aided by individual and consumer decisions and activities, such as selecting zero-emission automobiles or using public transportation. Ku et al. [1] evaluated the influence of eco-friendly transportation. Ajay et al. [2] devised a management system of eco-friendly transportation relying on IoT in urban areas. Kuzey et al. [3] discussed the CSR method and eco-friendly creativities in transportation. The finest design of the eco-friendly transportation network was deduced by Alshamrani et al. [4]. Ku et al. [5] devised various advantages of eco-friendly transportation. Bencekri et al. [6] discussed eco-friendly guidance for the infrastructure of transportation. Gao et al. [7] deduced various modes of eco-friendly transportation.

Objects in crisp set theory are classified as completely belonging to a set or not. Take a group of ''tall people,'' for instance, where the cutoff is six feet. Anyone above six feet tall belongs in this crisp set; anyone under that doesn't. However, this rigid categorization ignores those who are just under six feet tall but may still be regarded as ''fairly tall.'' The inability of crisp sets to explain this ''in-between'' situation gave rise to the fuzzy set (FS) theory which was devised by Zadeh [8]. Membership grades are recognized and accommodated by FS. In FS, using the example of ''tall people,'' a person 6'2'' may belong strongly, whereas a person 5'10'' would belong less strongly, indicating different levels of ''tallness.'' Unlike crisp sets, which are binary, FS supports items with partial membership and provides gradual transitions. This flexibility in determining membership levels is especially helpful in real-world scenarios where qualities or features may be too complicated to be well represented by sharp divisions. To tackle fuzzy transportation issues, Basirzadeh [9] devised a method. Hsueh et al. [10] originated a procedure of decision-making (DM) relying on a discriminant analysis fuzzy approach for eco-friendly residence design. Aminuddin et al. [11] employed the fuzzy AHP approach for the selection of eco-friendly cars. Although revolutionary in managing uncertainty, FS theory has several drawbacks. Since it only considers membership grades between 0 and 1 without expressly taking non-membership into account, it finds it difficult to accurately depict the lack of knowledge or uncertainty in membership assignments. To overcome these drawbacks, intuitionistic fuzzy sets (IFS) augment FS. IFS, which was first proposed by Atanassov [12], adds two more functions to the mix: membership, non-membership, and hesitancy grades. This gives a more thorough framework for managing uncertainty. This captures the uncertainties not taken into consideration in standard FS and enables a more nuanced representation where an element may have some hesitancy or indeterminacy in its membership or non-membership status. The theory of IFS is a useful tool in many domains, including artificial intelligence, pattern recognition, and decision-making, since it allows for more accurate modeling of ambiguous or uncertain data. Various researchers anticipated various aggregation operators (AOs) for aggregating intuitionistic fuzzy (IF) information such as Xu [13] anticipated IF AOS, Xia et al. [14] deduced IF AOs relying on Archimedean t-norm and conorm, Wei and Merigo [15] interpreted probability AOs for IFS, Liu and Chen [16] devised Heronian AOs for IFS, Senapati et al. [17] devised Aczel-Alsina AOs for IFS, and Huang [18] anticipated Hamacher AOs for IFS. Ebrahimnejad and Verdegay [19] discussed a novel method for tackling IF transportation issues.

The concepts of FS are expanded upon in a complex FS (CFS), firstly by Ramot et al. [20] which allows for membership grades to contain magnitude and phase term, where magnitude term belongs to [0, 1] and phase term can be a crisp value. Afterward, another framework of CFS was anticipated by Tamir et al. [21], that is cartesian form, where the membership grades contain real and unreal parts and both are placed in [0, 1]. This expansion is beneficial for managing multidimensional uncertainty, particularly in domains where complex data representations are essential, such as signal processing, image analysis, and decision-making. By adding both real and imaginary components, CFS provides a more comprehensive framework for representing hazy or unclear information, improving the expressiveness and adaptability of FS theory. A discussion about complex fuzzy logic and CFS was anticipated by Yazdanbakhsh and Dick [22]. Tamir et al. [23] devised the applications of

CFS. Rehman [24] anticipated probability AOs for CFS and Luqman et al. [25] deduced hypergraph for CFS. Mahmood et al. [26] anticipated a complex fuzzy N-soft set. The theory of complex intuitionistic fuzzy (CIF) set (CIFS) expands the notion of CFS and IFS and was devised by Alkouri et al. [27]. In this theory, both membership and non-membership grades are in a polar structure and placed in a unit square of a complex plane. After that, Ali et al. [28] deduced another form of CIFS, where both grades are in the Cartesian form and are placed in a complex plane's unit square. This theory of CIFS is more beneficial. Numerous researchers anticipated various AOs in the setting of the polar framework of CIFS such as Garg and Rani [29] deduced averaging/geometric AOs, Rani and Garg [30] devised power AOs, Akram et al. [31] devised Hamacher AOs, Ali et al. [32] devised Maclaurin symmetric mean (MSM) operators, Garg and Rani [33] deduced Bonferroni mean (BM) operators, etc. Akram et al. [34] devised a graph for the notion of CIFS and Rehman and Mahmood [35] deduced a CIF N-soft set.

It is evident from the preceding discussion that various researchers developed various AOs in the setting of the polar form of CIFS to aggregate CIF information. But, as of yet, no AO has been developed in the Cartesian framework of CIFS that can aggregate CIF information. More, there is no technique in the Cartesian form of CIFS, which can solve the information that is modeled in the Cartesian form of CIFS. As we know the Cartesian structure of CIFS is critical for solving and modeling various real-life dilemmas. Therefore, in this manuscript, we investigate the following notions.

- We devise some fundamental algebraic operations, score and accuracy functions, and related outcomes in the setting of the Cartesian framework of CIFS.
- Employing algebraic operations, we investigate probability AOs in the Cartesian framework of CIFS that is P-CIFWA, P-CIFOWA, IP-CIFOWA, P-CIFWG, P-CIFOWG, and IP-CIFOWG, operators along with basic axioms.
- After that, we devise a procedure of MADM within the Cartesian structure of CIFS by employing deduced operators for solving real-life dilemmas.
- We also analyze a case study "Identification of eco-friendly transportation mode" with the assistance of the proposed work.
- We compare the anticipated work with certain prevailing concepts to reveal the supremacy and advantages of the anticipated work.

The rest of the manuscript is anticipated as: In section II, we analyze the primary notions such as IFS, and CFS along with basic operations and the Cartesian form of CIFS. In section III, we anticipate the score and accuracy function in the setting of the Cartesian framework of CIFN, and after that we fundamental algebraic operations along with the related results. In section IV, we interpret probability AOs in the setting of the Cartesian framework of CIFS. These AOs are probability CIF weighted averaging (P-CIFWA), probability CIF ordered weighted averaging (P-CIFOWA), immediate P-CIFOWA (IP-CIFOWA), probability CIF weighted geometric (P-CIFWG), probability CIF ordered weighted geometric (P-CIFOWG), immediate P-CIFOWG (IP-CIFOWG) operators. Further, we examine their linked properties. In section V, we develop a procedure of MADM in the setting of the Cartesian form of CIFS and then analyze the case study "identification of eco-friendly transportation mode". Section VI of this manuscript contains the comparative analysis and section VII contains the conclusion.

II. PRELIMINARIES

Here, we analyze the primary notions such as IFS, and CFS along with basic operations and the Cartesian form of CIFS.

Definition 1 [12]: The mathematical framework of IFS is anticipated as

$$\mathcal{P}_{IFS}=\left\{\left(\xi,\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS}}(\xi),\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS}}(\xi)\right)\right)\mid \xi\in\check{\mathrm{E}}\right\},$$

where, $\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS}}(\xi)$ is a membership grade and $\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS}}(\xi)$ is a non-membership grade which placed in a $[0,\ 1]$ with the condition that $0\le\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS}}(\xi)+\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS}}(\xi)\le 1$. Further, $\mathcal{F}^{\mathcal{D}}_{\mathcal{P}_{IFS}}(\xi)=1-\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS}}(\xi)+\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS}}(\xi)\right)$ is an indeterminacy. The pair $\mathcal{P}_{IFS}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS}}\right)$ would be anticipated as an IF number (IFN).

Definition 2 [13]: Let $\mathcal{P}_{IFS-1}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS-1}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS-1}}\right)$ and $\mathcal{P}_{IFS-2}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS-2}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS-2}}\right)$ be two IFN with $\eth\ge 0$. then

1. $\mathcal{P}_{IFS-1}\oplus\mathcal{P}_{IFS-2}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS-1}}+\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS-2}}-\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS-1}}\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS-2}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS-1}}\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS-2}}\right)$
2. $\mathcal{P}_{IFS-1}\otimes\mathcal{P}_{IFS-2}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS-1}}\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS-2}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS-1}}+\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS-2}}-\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS-1}}\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS-2}}\right)$
3. $\eth\mathcal{P}_{IFS-1}=\left(1-\left(1-\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS-1}}\right)^{\eth},\ \left(\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS-1}}\right)^{\eth}\right)$
4. $\mathcal{P}^{\eth}_{IFS-1}=\left(\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS-1}}\right)^{\eth},\ 1-\left(1-\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{IFS-1}}\right)^{\eth}\right)$

Definition 3 [21]: The mathematical framework of CFS is anticipated as

$$\mathcal{P}_{CFS}=\left\{\left(\xi,\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CFS}}(\xi)\right)\mid \xi\in\check{\mathrm{E}}\right\}=\left\{\left(\xi,\mathcal{F}^{\mathcal{R}}_{\mathcal{P}_{CFS}}(\xi)+\iota\,\mathcal{F}^{\mathcal{I}}_{\mathcal{P}_{CFS}}(\xi)\right)\mid \xi\in\check{\mathrm{E}}\right\},$$

where, $\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{IFS}}(\xi)$ is a membership grade with real-term $\mathcal{F}^{\mathcal{R}}_{\mathcal{P}_{CFS}}(\xi)$ and unreal term $\mathcal{F}^{\mathcal{I}}_{\mathcal{P}_{CFS}}(\xi)$ and placed in a complex plane's unit square. The complex fuzzy number (CFN) would be anticipated as $\mathcal{P}_{CFS}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CFS}}\right)=\left(\mathcal{F}^{\mathcal{R}}_{\mathcal{P}_{CFS}}+\iota\,\mathcal{F}^{\mathcal{I}}_{\mathcal{P}_{CFS}}\right)$.

Definition 4 [24]: Let $\mathcal{P}_{CFS-1}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CFS-1}}\right)=\left(\mathcal{F}^{\mathcal{R}}_{\mathcal{P}_{CFS-1}}+\iota\,\mathcal{F}^{\mathcal{I}}_{\mathcal{P}_{CFS-1}}\right)$ and $\mathcal{P}_{CFS-2}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CFS-2}}\right)=\left(\mathcal{F}^{\mathcal{R}}_{\mathcal{P}_{CFS-2}}+\iota\,\mathcal{F}^{\mathcal{I}}_{\mathcal{P}_{CFS-2}}\right)$ be two CFNs, and $\eth\ge 0$. Then

1. $\mathcal{P}_{CFS-1}\oplus\mathcal{P}_{CFS-2}$
$$=\left(\begin{array}{c}\mathcal{F}^{\mathcal{R}}_{\mathcal{P}_{CFS-1}}+\mathcal{F}^{\mathcal{R}}_{\mathcal{P}_{CFS-2}}-\mathcal{F}^{\mathcal{R}}_{\mathcal{P}_{CFS-1}}\mathcal{F}^{\mathcal{R}}_{\mathcal{P}_{CFS-2}}\\+\iota\left(\mathcal{F}^{\mathcal{I}}_{\mathcal{P}_{CFS-1}}+\mathcal{F}^{\mathcal{I}}_{\mathcal{P}_{CFS-2}}-\mathcal{F}^{\mathcal{I}}_{\mathcal{P}_{CFS-1}}\mathcal{F}^{\mathcal{I}}_{\mathcal{P}_{CFS-2}}\right)\end{array}\right)$$

2. $\mathcal{P}_{CFS-1}\otimes\mathcal{P}_{CFS-2}=\begin{pmatrix}\mathcal{F}^{\mathcal{R}}_{\mathcal{P}_{CFS-1}}\mathcal{F}^{\mathcal{R}}_{\mathcal{P}_{CFS-2}}\\ +\iota\,\mathcal{F}^{\mathcal{I}}_{\mathcal{P}_{CFS-1}}\mathcal{F}^{\mathcal{I}}_{\mathcal{P}_{CFS-2}}\end{pmatrix}$
3. $\eth\mathcal{P}_{CFS-1}=\begin{pmatrix}1-\left(1-\mathcal{F}^{\mathcal{R}}_{\mathcal{P}_{CFS-1}}\right)^{\eth}\\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{I}}_{\mathcal{P}_{CFS-1}}\right)^{\eth}\right)\end{pmatrix}$
4. $\mathcal{P}^{\eth}_{CFS-1}=\left(\left(\mathcal{F}^{\mathcal{R}}_{\mathcal{P}_{CFS-1}}\right)^{\eth}+\iota\left(\mathcal{F}^{\mathcal{I}}_{\mathcal{P}_{CFS-1}}\right)^{\eth}\right)$

Definition 5 [31]: The mathematical framework of CIFS in cartesian form is anticipated as

$$\mathcal{P}_{CIFS}=\left\{\left(\xi,\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS}}(\xi),\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS}}(\xi)\right)\right)\mid\xi\in\grave{\text{E}}\right\}=\left\{\left(\xi,\begin{pmatrix}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}(\xi)+\iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}}(\xi),\\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}(\xi)+\iota\,\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}(\xi)\end{pmatrix}\right)\mid\xi\in\grave{\text{E}}\right\},$$

where, $\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS}}(\xi)$ is a membership grade and $\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS}}(\xi)$ is a non-membership grade placed in a complex plane's unit square. Further, $\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}(\xi)$, $\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}}(\xi)$, $\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}(\xi)$ and $\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}(\xi)\in[0,\ 1]$ with the condition that $0\leq\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}(\xi)+\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}(\xi)\leq1$ and $0\leq\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}}(\xi)+\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}(\xi)\leq1$. The set $\mathcal{P}_{CIFS}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS}}\right)=\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}+\iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}},\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}+\iota\,\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}\right)$ would be identified as CIF number (CIFN).

III. FUNDAMENTAL OPERATIONS WITHIN CARTESIAN FRAMEWORK OF CIFN

In this section, we anticipate the score and accuracy function in the setting of the cartesian framework of CIFN, and after that we fundamental algebraic operations along with the related results.

Definition 6: For any CIFN $\mathcal{P}_{CIFS}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS}}\right)=\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}+\iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}},\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}+\iota\,\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}\right)$, the score value would be analyzed as

$$\text{Ş}\left(\mathcal{P}_{CIFS}\right)=\frac{1}{2}\begin{pmatrix}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}\\ +\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}}-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}\end{pmatrix},\tag{1}$$

$\text{Ş}\left(\mathcal{P}_{CIFS}\right)\in[-1,\ 1]$

Definition 7: For any CIFN $\mathcal{P}_{CIFS}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS}}\right)=\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}+\iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}},\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}+\iota\,\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}\right)$, the accuracy value would be analyzed as

$$\hat{\text{H}}\left(\mathcal{P}_{CIFS}\right)=\frac{1}{2}\begin{pmatrix}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}+\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}}\\ +\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}+\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}\end{pmatrix},\tag{2}$$

$\hat{\text{H}}\left(\mathcal{P}_{CIFS}\right)\in[0,\ 1]$

Employing Eq. (1) and (2), we achieve,

1. If $\text{Ş}(\mathcal{P}_{CIFS-1})<\text{Ş}(\mathcal{P}_{CIFS-2})$ then $\mathcal{P}_{CIFS-1}<\mathcal{P}_{CIFS-2}$
2. If $\text{Ş}(\mathcal{P}_{CIFS-1})>\text{Ş}(\mathcal{P}_{CIFS-2})$ then $\mathcal{P}_{CIFS-1}>\mathcal{P}_{CIFS-2}$
3. If $\text{Ş}(\mathcal{P}_{CIFS-1})=\text{Ş}(\mathcal{P}_{CIFS-2})$ then we have
   i. If $\hat{\text{H}}(\mathcal{P}_{CIFS-1})<\hat{\text{H}}(\mathcal{P}_{CIFS-2})$ then $\mathcal{P}_{CIFS-1}<\mathcal{P}_{CIFS-2}$
   ii. If $\hat{\text{H}}(\mathcal{P}_{CIFS-1})>\hat{\text{H}}(\mathcal{P}_{CIFS-2})$ then $\mathcal{P}_{CIFS-1}>\mathcal{P}_{CIFS-2}$
   iii. If $\hat{\text{H}}(\mathcal{P}_{CIFS-1})=\hat{\text{H}}(\mathcal{P}_{CIFS-2})$ then $\mathcal{P}_{CIFS-1}=\mathcal{P}_{CIFS-2}$

Definition 8: Let $\mathcal{P}_{CIFS-1}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-1}},\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-1}}\right)=\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}+\iota\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}},\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}+\iota\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)$, and $\mathcal{P}_{CIFS-2}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-2}},\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-2}}\right)=\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}+\iota\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}},\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}+\iota\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)$ as two CIFNs and $\eth\geq0$. Then

1. $\mathcal{P}_{CIFS-1}\oplus\mathcal{P}_{CIFS-2}$
$$=\begin{pmatrix}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}+\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\\ +\iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}+\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right),\\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}+\iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)\end{pmatrix}$$
2. $\mathcal{P}_{CIFS-1}\otimes\mathcal{P}_{CIFS-2}$
$$=\begin{pmatrix}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}+\iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}},\\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}+\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\\ \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}+\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)\end{pmatrix}$$
3. $\eth\mathcal{P}_{CIFS-1}$
$$=\begin{pmatrix}1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}+\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\right),\\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}+\iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\end{pmatrix}$$
4. $\mathcal{P}^{\eth}_{CIFS-1}$
$$=\begin{pmatrix}\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}+\iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth},\\ 1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}+\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\right)\end{pmatrix}$$

Theorem 1: Let $\mathcal{P}_{CIFS-1}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-1}},\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-1}}\right)=\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}+\iota\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}},\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}+\iota\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)$, and $\mathcal{P}_{CIFS-2}=\left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-2}},\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-2}}\right)=\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}+\iota\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}},\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}+\iota\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)$ as two CIFNs and $\eth,\eth_1,\eth_2\geq0$. Then

1. $\mathcal{P}_{CIFS-1}\oplus\mathcal{P}_{CIFS-2}=\mathcal{P}_{CIFS-2}\oplus\mathcal{P}_{CIFS-1}$
2. $\mathcal{P}_{CIFS-1}\otimes\mathcal{P}_{CIFS-2}=\mathcal{P}_{CIFS-2}\otimes\mathcal{P}_{CIFS-1}$
3. $\eth\left(\mathcal{P}_{CIFS-1}\oplus\mathcal{P}_{CIFS-2}\right)=\eth\mathcal{P}_{CIFS-1}\oplus\eth\mathcal{P}_{CIFS-2}$
4. $\left(\mathcal{P}_{CIFS-1}\otimes\mathcal{P}_{CIFS-2}\right)^{\eth}=\mathcal{P}_{CIFS-1}{}^{\eth}\otimes\mathcal{P}_{CIFS-2}{}^{\eth}]$
5. $\eth_1\mathcal{P}_{CIFS-1}\oplus\eth_2\mathcal{P}_{CIFS-1}=\left(\eth_1+\eth_2\right)\mathcal{P}_{CIFS-1}$
6. $\mathcal{P}_{CIFS-1}{}^{\eth_1}\otimes\mathcal{P}_{CIFS-1}{}^{\eth_2}=\mathcal{P}_{CIFS-1}{}^{\eth_1+\eth_2}$
7. $\left(\mathcal{P}_{CIFS-1}{}^{\eth_1}\right)^{\eth_2}=\mathcal{P}_{CIFS-1}{}^{\eth_1\eth_2}$.

Proof: 1. As we have,

$$\mathcal{P}_{CIFS-1}\oplus\mathcal{P}_{CIFS-2}$$
$$=\begin{pmatrix}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}+\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\\ +\iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}+\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)\\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}+\iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)\end{pmatrix}$$
$$=\begin{pmatrix}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}+\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\\ +\iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}+\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)\\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}+\iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)\end{pmatrix}$$
$$=\mathcal{P}_{CIFS-2}\oplus\mathcal{P}_{CIFS-1}$$

2. By Def (9),

$$\mathcal{P}_{CIFS-1}\otimes\mathcal{P}_{CIFS-2} = \begin{pmatrix} \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}, \\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}} + \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}} - \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}} \\ +\iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}} + \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}} - \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}} \\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}} + \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}} - \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}} \\ +\iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}} + \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}} - \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right) \end{pmatrix}$$

$$= \mathcal{P}_{CIFS-2}\otimes\mathcal{P}_{CIFS-1}$$

1. Since we have

$$\eth\mathcal{P}_{CIFS-1} = \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\right), \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} \end{pmatrix}$$

$$\eth\mathcal{P}_{CIFS-2} = \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth}\right), \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \end{pmatrix}$$

Now take the right side,

$$\eth\mathcal{P}_{CIFS-1}\oplus\eth\mathcal{P}_{CIFS-2} = \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\right), \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} \end{pmatrix} \oplus \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth}\right) \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} + 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ -\left(1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\right)\left(1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth}\right) \end{pmatrix} \\ +\iota \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} + 1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ -\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\right)\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth}\right) \end{pmatrix} \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth}\right) \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\left(1-\begin{pmatrix} \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}} + \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}} \\ -\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}} \end{pmatrix}\right)^{\eth} \\ +\iota\left(1-\left(1-\begin{pmatrix} \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}} + \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}} \\ -\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}} \end{pmatrix}\right)^{\eth}\right) \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \end{pmatrix}$$

$$= \eth\left(\mathcal{P}_{CIFS-1}\oplus\mathcal{P}_{CIFS-2}\right)$$

2. Since we have,

$$\left(\mathcal{P}_{CIFS-1}\right)^{\eth} = \begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} + \iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}, \\ 1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\right) \end{pmatrix}$$

and

$$\left(\mathcal{P}_{CIFS-2}\right)^{\eth} = \begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} + \iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ 1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth}\right) \end{pmatrix}$$

Now take the right-hand side,

$$\left(\mathcal{P}_{CIFS-1}\right)^{\eth}\otimes\left(\mathcal{P}_{CIFS-2}\right)^{\eth} = \begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} + \iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}, \\ 1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\right) \end{pmatrix} \otimes \begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} + \iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ 1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ +\iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} \\ +1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ -\left(1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\right) \\ \left(1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth}\right) \end{pmatrix} \\ +\iota \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth} \\ +1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ -\left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\right) \\ \left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth}\right) \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}} \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} + \iota \left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}} \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth}\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}} \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} + \iota \left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}} \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\eth} \\ 1-\left(1-\begin{pmatrix}\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}+\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}} \\ -\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\end{pmatrix}\right)^{\eth} \\ +\iota\left(1-\left(1-\begin{pmatrix}\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}+\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}} \\ -\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\end{pmatrix}\right)^{\eth}\right) \end{pmatrix}$$

$$= \left(\mathcal{P}_{CIFS-1} \otimes \mathcal{P}_{CIFS-2}\right)^{\eth}$$

3. As we have

$$\eth_1 \mathcal{P}_{CIFS-1} = \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\right), \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1} \end{pmatrix}$$

$$\eth_2 \mathcal{P}_{CIFS-1} = \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_2} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_2}\right), \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_2} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_2} \end{pmatrix}$$

Now consider the right-hand side,

$$\eth_1 \mathcal{P}_{CIFS-1} \oplus \eth_2 \mathcal{P}_{CIFS-1}$$

$$= \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\right), \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1} \end{pmatrix} \oplus \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_2} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_2}\right), \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_2} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_2} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}+1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1} \\ -\left(1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\right)\left(1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\right) \end{pmatrix} \\ +\iota\begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}+1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1} \\ -\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\right)\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\right) \end{pmatrix} \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_2}+\iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_2} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_2} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{2}\right) \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1+\eth_2}+\iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1+\eth_2} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1+\eth_2} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1+\eth_2}\right) \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1+\eth_2}+\iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1+\eth_2} \end{pmatrix}$$

$$= \left(\eth_1+\eth_2\right)\mathcal{P}_{CIFS-1}$$

4. Likewise, 5.
5. Since,

$$\mathcal{P}_{CIFS-1}{}^{\eth_1} = \begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}+\iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1} \\ 1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\right) \end{pmatrix}$$

$$\left(\mathcal{P}_{CIFS-1}{}^{\eth_1}\right)^{\eth_2}$$

$$= \begin{pmatrix} \left(\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\right)^{\eth_2}+\iota\left(\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\right)^{\eth_2} \\ 1-\left(\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\right)^{\eth_2} \\ +\iota\left(1-\left(\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1}\right)^{\eth_2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1\eth_2}+\iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1\eth_2} \\ 1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1\eth_2} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\eth_1\eth_2}\right) \end{pmatrix}$$

$$= \mathcal{P}_{CIFS-1}{}^{\eth_1\eth_2}$$

IV. PROBABILISTIC AOS IN THE SETTING OF CIF INFORMATION

Here, we interpret probability AOs in the setting of the Cartesian framework of CIFS. These AOs are P-CIFWA, P-CIFOWA, IP-CIFOWA, P-CIFWG, P-CIFOWG, and IP-CIFOWG operators. Further, we examine their linked properties.

Definition 9: The P-CIFWA operator over a class of CIFNs $\mathcal{P}_{CIFS-\text{ứ}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\text{ứ}}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\text{ứ}}}+\iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\text{ứ}}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\text{ứ}}}+\iota\,\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)$, $\text{ứ} = 1, 2, \ldots, \vartheta$ would be estimated as

$$P-CIFWA\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots, \mathcal{P}_{CIFS-\vartheta}\right) = \bigoplus_{\text{ứ}=1}^{\vartheta} \mathcal{X}_{\mathbb{b}-\text{ứ}} \mathcal{P}_{CIFS-\text{ứ}} \quad (3)$$

In Eq. (3), $\Omega_{\mathbb{wv}} = \left(\Omega_{\mathbb{wv}-1}, \Omega_{\mathbb{wv}-2}, \ldots, \Omega_{\mathbb{wv}-\vartheta}\right)$ is a weight with $0 \le \Omega_{\mathbb{wv}-\text{ứ}} \le 1$, $\sum_{\text{ứ}=1}^{\vartheta} \Omega_{\mathbb{wv}-\text{ứ}} = 1$, $E_{\mathbb{bv}-\text{ứ}} > 0$ with $\sum_{\text{ứ}=1}^{\vartheta} E_{\mathbb{bv}-\text{ứ}} = 1$ is a probability weight and $\mathcal{X}_{\mathbb{b}-\text{ứ}} = \rho E_{\mathbb{bv}-\text{ứ}} + (1-\rho)\,\Omega_{\mathbb{wv}-\text{ứ}}$ fuse the weight vector and probability weight in the same formula with $\rho \in [0, 1]$.

Theorem 2: Let a class of CIFNs

$$\mathcal{P}_{CIFS-ứ} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-ứ}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-ứ}}\right) = \begin{pmatrix} \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ứ}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ứ}}, \\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ứ}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ứ}} \end{pmatrix},$$

$ứ = 1, 2, \ldots, \vartheta$. Then the aggregated outcome after employing the P-CIFWA operator over $\mathcal{P}_{CIFS-ứ}$ would anticipate a CIFN that is

$$P-CIFWA\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots \mathcal{P}_{CIFS-\vartheta}\right) = \begin{pmatrix} 1-\prod_{ứ=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}} \\ +\iota\left(1-\prod_{ứ=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}}\right), \\ \prod_{ứ=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}} \\ +\iota \prod_{ứ=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}} \end{pmatrix} \quad (4)$$

Proof: Consider $\vartheta = 2$, then,

$$\mathcal{X}_{\mathbb{b}-1}\mathcal{P}_{CIFS-1} = \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}}\right), \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}} \\ +\iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}} \end{pmatrix}$$

$$\mathcal{X}_{\mathbb{b}-2}\mathcal{P}_{CIFS-2} = \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}}\right), \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}} \\ +\iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}} \end{pmatrix}$$

Now,

$$\begin{aligned} &P-CIFWA\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}\right) \\ &= \mathcal{X}_{\mathbb{b}-1}\mathcal{P}_{CIFS-1} \oplus \mathcal{X}_{\mathbb{b}-2}\mathcal{P}_{CIFS-2} \\ &= \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}}\right), \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}} \end{pmatrix} \\ &\oplus \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}}\right), \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 1-\prod_{ứ=1}^{2}\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}} \\ +\iota\left(1-\prod_{ứ=1}^{2}\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}}\right), \\ \prod_{ứ=1}^{2}\left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}} \\ +\iota \prod_{ứ=1}^{2}\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}} \end{pmatrix}$$

$\Rightarrow$ Eq. (4) held for $\vartheta = 2$. Consider Eq. (4) is true for $\vartheta = \mathbb{L}$, then

$$P-CIFWA\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots, \mathcal{P}_{CIFS-\mathbb{L}}\right) = \begin{pmatrix} 1-\prod_{ứ=1}^{\mathbb{L}}\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}} \\ +\iota\left(1-\prod_{ứ=1}^{\mathbb{L}}\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}}\right), \\ \prod_{ứ=1}^{\mathbb{L}}\left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}} \\ +\iota \prod_{ứ=1}^{\mathbb{L}}\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}} \end{pmatrix}$$

Next to prove that Eq. (4) is true for $\vartheta = \mathbb{L}+1$, as

$$\begin{aligned} &P-CIFWA\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots \mathcal{P}_{CIFS-\mathbb{L}+1}\right) \\ &= P-CIFWA\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots \mathcal{P}_{CIFS-\mathbb{L}}\right) \\ &\quad \oplus \mathcal{P}_{CIFS-\mathbb{L}+1} \\ &= \begin{pmatrix} 1-\prod_{ứ=1}^{\mathbb{L}}\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}} \\ +\iota\left(1-\prod_{ứ=1}^{\mathbb{L}}\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}}\right), \\ \prod_{ứ=1}^{\mathbb{L}}\left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}} \\ +\iota \prod_{ứ=1}^{\mathbb{L}}\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ứ}}\right)^{\mathcal{X}_{\mathbb{b}-ứ}} \end{pmatrix} \\ &\oplus \begin{pmatrix} 1-\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\mathbb{L}+1}}\right)^{\mathcal{X}_{\mathbb{b}-\mathbb{L}+1}} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\mathbb{L}+1}}\right)^{\mathcal{X}_{\mathbb{b}-\mathbb{L}+1}}\right), \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\mathbb{L}+1}}\right)^{\mathcal{X}_{\mathbb{b}-\mathbb{L}+1}} + \iota\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\mathbb{L}+1}}\right)^{\mathcal{X}_{\mathbb{b}-\mathbb{L}+1}} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 1-\prod_{\acute{u}=1}^{\mathbb{L}+1}\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \\ +\iota\left(1-\prod_{\acute{u}=1}^{\mathbb{L}+1}\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}\right), \\ \prod_{\acute{u}=1}^{\mathbb{L}+1}\left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \\ +\iota\prod_{\acute{u}=1}^{\mathbb{L}+1}\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \end{pmatrix}$$

$$= P-CIFWA\begin{pmatrix}\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots \\ \mathcal{P}_{CIFS-\mathbb{L}},\ \mathcal{P}_{CIFS-\mathbb{L}+1}\end{pmatrix}$$

Thus, Eq. (4) is true for $\mathbb{L}+1$ and consequently true $\forall\ \vartheta$.

Properties 1: Underneath are the properties that the P-CIFWA operator satisfies.

1. *Idempotency:* Let $\mathcal{P}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\acute{u}}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\acute{u}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}}+\iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}},\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}}+\iota\,\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)$, $\acute{u} = 1, 2, \ldots, \vartheta$, as a class of CIFNs. Then if $\forall\ \acute{u}\ \mathcal{P}_{CIFS-\acute{u}} = \mathcal{P}_{CIFS}$ i.e., $\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} = \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}$, $\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}} = \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}}$, $\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} = \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}$ and $\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}} = \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}$, then

$$P-CIFWA\left(\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots\mathcal{P}_{CIFS-\vartheta}\right) = \mathcal{P}_{CIFS-\acute{u}}$$

Proof: Since $\mathcal{P}_{CIFS-\acute{u}} = \mathcal{P}_{CIFS}\ \forall\ \acute{u}$, then we have

$$P-CIFWA\left(\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots\mathcal{P}_{CIFS-\vartheta}\right)$$
$$= \begin{pmatrix} 1-\prod_{\acute{u}=1}^{\mathbb{L}+1}\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \\ +\iota\left(1-\prod_{\acute{u}=1}^{\mathbb{L}+1}\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}\right), \\ \prod_{\acute{u}=1}^{\mathbb{L}+1}\left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \\ +\iota\prod_{\acute{u}=1}^{\mathbb{L}+1}\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \end{pmatrix}$$
$$= \begin{pmatrix}\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}\right)^{\sum_{\acute{u}=1}^{\vartheta}\mathcal{X}_{\mathbb{b}-\acute{u}}} + \iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}}\right)^{\sum_{\acute{u}=1}^{\vartheta}\mathcal{X}_{\mathbb{b}-\acute{u}}}, \\ \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}\right)^{\sum_{\acute{u}=1}^{\vartheta}\mathcal{X}_{\mathbb{b}-\acute{u}}} + \iota\left(\mathcal{FI}^{\mathcal{N}}_{\mathcal{P}_{CIFS}}\right)^{\sum_{\acute{u}=1}^{\vartheta}\mathcal{X}_{\mathbb{b}-\acute{u}}}\end{pmatrix}$$
$$= \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}+\iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}},\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}+\iota\,\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}\right)$$
$$= \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS}}\right) = \mathcal{P}_{CIFS}$$

Note that $\sum_{\acute{u}=1}^{\vartheta}\mathcal{X}_{\mathbb{b}-\acute{u}} = \sum_{\acute{u}=1}^{\vartheta}\begin{pmatrix}\rho E_{\mathbb{b}\mathbb{v}-\acute{u}} \\ +(1-\rho)\,\Omega_{\mathbb{w}\mathbb{v}-\acute{u}}\end{pmatrix} = \rho\sum_{\acute{u}=1}^{\vartheta}E_{\mathbb{b}\mathbb{v}-\acute{u}} + (1-\rho)\sum_{\acute{u}=1}^{\vartheta}\Omega_{\mathbb{w}\mathbb{v}-\acute{u}} = \rho+1-\rho = 1.$

2. *Monotonicity:* Let $\mathcal{P}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\acute{u}}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\acute{u}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}}+\iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}},\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}}+\iota\,\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)$, and

$$\mathcal{P}^{\#}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}+\iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}},\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\ \iota\,\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right),$$

$\acute{u} = 1, 2, \ldots, \vartheta$, as two classes of CIFNs. If $\forall\ \acute{u}\ \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} \leq \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}$, $\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}} \leq \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}$, $\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} \geq \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}$, and $\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}} \geq \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}$ then

$$P-CIFWA\left(\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots\mathcal{P}_{CIFS-\vartheta}\right) \leq P-CIFWA\left(\mathcal{P}^{\#}_{CIFS-1},\ \mathcal{P}^{\#}_{CIFS-2},\ \ldots\mathcal{P}^{\#}_{CIFS-\vartheta}\right)$$

Proof: Since

$$\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} \leq \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}} \Rightarrow 1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} \geq 1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}$$
$$\Rightarrow \left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \geq \left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}$$
$$\Rightarrow \prod_{\acute{u}=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \geq \prod_{\acute{u}=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}$$
$$\Rightarrow 1-\prod_{\acute{u}=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \leq 1-\prod_{\acute{u}=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}$$

Likewise, we can achieve that

$$1-\prod_{\acute{u}=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \leq 1-\prod_{\acute{u}=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}$$

Next, since

$$\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} \geq \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}} \Rightarrow \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right) \geq \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right)$$
$$\Rightarrow \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \geq \left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}$$
$$\Rightarrow \prod_{\acute{u}=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \geq \prod_{\acute{u}=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}$$

Likewise, we can achieve that

$$\prod_{\acute{u}=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \geq \prod_{\acute{u}=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}$$

Thus,

$$P-CIFWA\left(\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots\mathcal{P}_{CIFS-\vartheta}\right) \leq P-CIFWA\left(\mathcal{P}^{\#}_{CIFS-1},\ \mathcal{P}^{\#}_{CIFS-2},\ \ldots\mathcal{P}^{\#}_{CIFS-\vartheta}\right)$$

3. *Boundedness:* Let $\mathcal{P}_{CIFS-ẃ} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-ẃ}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-ẃ}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ẃ}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ẃ}}\right)$, $ẃ = 1, 2, \ldots, \vartheta$ as a class of CIFNs. If $\mathcal{P}^{-}_{CIFS} = \begin{pmatrix} \min_{ẃ}\left\{\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ẃ}}\right\} + \iota \min_{ẃ}\left\{\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ẃ}}\right\}, \\ \max_{ẃ}\left\{\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ẃ}}\right\} + \iota \max_{ẃ}\left\{\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ẃ}}\right\} \end{pmatrix}$ and $\mathcal{P}^{+}_{CIFS} = \begin{pmatrix} \max_{ẃ}\left\{\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ẃ}}\right\} + \iota \max_{ẃ}\left\{\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ẃ}}\right\}, \\ \min_{ẃ}\left\{\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ẃ}}\right\} + \iota \min_{ẃ}\left\{\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ẃ}}\right\} \end{pmatrix}$, then

$$\mathcal{P}^{-}_{CIFS} \leq P-CIFWA\begin{pmatrix} \mathcal{P}_{CIFS-1}, \\ \mathcal{P}_{CIFS-2}, \\ \ldots \mathcal{P}_{CIFS-\vartheta} \end{pmatrix} \leq \mathcal{P}^{+}_{CIFS}$$

Proof: Through properties 1 and 2 we can get this.

Definition 10: The P-CIFOWA operator over a class of CIFNs $\mathcal{P}_{CIFS-ẃ} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-ẃ}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-ẃ}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ẃ}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ẃ}}\right)$, $ẃ = 1, 2, \ldots, \vartheta$ would be estimated as

$$P-CIFOWA\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots, \mathcal{P}_{CIFS-\vartheta}\right) = \bigoplus_{ẃ=1}^{\vartheta} \mathcal{X}_{\mathbb{b}-ẃ}\mathcal{P}_{CIFS-\mathfrak{r}(ẃ)} \tag{5}$$

In Eq. (5), $(\mathfrak{r}(1), \mathfrak{r}(2), \ldots, \mathfrak{r}(\vartheta))$ is a permutation of $(1, 2, \ldots, \vartheta)$ such that $\mathfrak{r}(ẃ-1) \geq \mathfrak{r}(ẃ)$, for $ẃ = 2, 3, \ldots, \vartheta$ and $\mathcal{X}_{\mathbb{b}-ẃ} = \rho E_{\mathbb{bv}-ẃ} + (1-\rho)\,\Omega_{\mathbb{wv}-ẃ}$ fuse the weight vector and probability weight in the same formula with $\rho \in [0, 1]$.

Theorem 3: Let a class of CIFNs $\mathcal{P}_{CIFS-ẃ} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-ẃ}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-ẃ}}\right) = \begin{pmatrix} \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ẃ}}, \\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ẃ}} \end{pmatrix}$, $ẃ = 1, 2, \ldots, \vartheta$. Then the aggregated outcome after employing the P-CIFOWA operator over $\mathcal{P}_{CIFS-ẃ}$ would anticipate a CIFN that is

$$P-CIFOWA\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots \mathcal{P}_{CIFS-\vartheta}\right) = \begin{pmatrix} 1 - \prod_{ẃ=1}^{\vartheta}\left(1 - \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\mathfrak{r}(ẃ)}}\right)^{\mathcal{X}_{\mathbb{b}-ẃ}} \\ +\iota\left(1 - \prod_{ẃ=1}^{\vartheta}\left(1 - \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\mathfrak{r}(ẃ)}}\right)^{\mathcal{X}_{\mathbb{b}-ẃ}}\right), \\ \prod_{ẃ=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\mathfrak{r}(ẃ)}}\right)^{\mathcal{X}_{\mathbb{b}-ẃ}} \\ +\iota \prod_{ẃ=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\mathfrak{r}(ẃ)}}\right)^{\mathcal{X}_{\mathbb{b}-ẃ}} \end{pmatrix}$$

The P-CIFOWA operator satisfies properties such as idempotency, monotonicity, and boundedness.

Definition 11: The IP-CIFOWA operator over a class of CIFNs $\mathcal{P}_{CIFS-ẃ} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-ẃ}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-ẃ}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ẃ}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ẃ}}\right)$, $ẃ = 1, 2, \ldots, \vartheta$ would be estimated as

$$IP-CIFOWA\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots, \mathcal{P}_{CIFS-\vartheta}\right) = \bigoplus_{ẃ=1}^{\vartheta} \mathfrak{J}_{\mathbb{b}-ẃ}\mathcal{P}_{CIFS-\mathfrak{r}(ẃ)} \tag{6}$$

In Eq. (6), $(\mathfrak{r}(1), \mathfrak{r}(2), \ldots, \mathfrak{r}(\vartheta))$ is a permutation of $(1, 2, \ldots, \vartheta)$ such that $\mathfrak{r}(ẃ-1) \geq \mathfrak{r}(ẃ)$, for $ẃ = 2, 3, \ldots, \vartheta$ and $\mathfrak{J}_{\mathbb{b}-ẃ} = \frac{\Omega_{\mathbb{wv}-ẃ}E_{\mathbb{bv}-ẃ}}{\sum_{ẃ=1}^{\vartheta}\Omega_{\mathbb{wv}-ẃ}E_{\mathbb{bv}-ẃ}}$ is an immediate probability (IP) interpreted to CIFN

Theorem 4: Let a class of CIFNs

$$\mathcal{P}_{CIFS-ẃ} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-ẃ}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-ẃ}}\right) = \begin{pmatrix} \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ẃ}}, \\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ẃ}} \end{pmatrix}, \quad ẃ = 1, 2, \ldots, \vartheta.$$

Then the aggregated outcome after employing the IP-CIFOWA operator over $\mathcal{P}_{CIFS-ẃ}$ would anticipate a CIFN that is

$$IP-CIFOWA\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots \mathcal{P}_{CIFS-\vartheta}\right) = \begin{pmatrix} 1 - \prod_{ẃ=1}^{\vartheta}\left(1 - \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\mathfrak{r}(ẃ)}}\right)^{\mathfrak{J}_{\mathbb{b}-ẃ}} \\ +\iota\left(1 - \prod_{ẃ=1}^{\vartheta}\left(1 - \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\mathfrak{r}(ẃ)}}\right)^{\mathfrak{J}_{\mathbb{b}-ẃ}}\right), \\ \prod_{ẃ=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\mathfrak{r}(ẃ)}}\right)^{\mathfrak{J}_{\mathbb{b}-ẃ}} \\ +\iota \prod_{ẃ=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\mathfrak{r}(ẃ)}}\right)^{\mathfrak{J}_{\mathbb{b}-ẃ}} \end{pmatrix}$$

Properties 2: Underneath are the properties that the IP-CIFWA operator satisfies.

1. **Idempotency:** Let $\mathcal{P}_{CIFS-ẃ} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-ẃ}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-ẃ}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ẃ}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ẃ}}\right)$, $ẃ = 1, 2, \ldots, \vartheta$, as a class of CIFNs. Then if $\forall\, ẃ\ \mathcal{P}_{CIFS-ẃ} = \mathcal{P}_{CIFS}$ i.e., $\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ẃ}} = \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}$, $\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ẃ}} = \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}}$, $\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ẃ}} = \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}$ and $\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ẃ}} = \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}$, then

$$IP-CIFOWA\begin{pmatrix} \mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \\ \ldots \mathcal{P}_{CIFS-\vartheta} \end{pmatrix} = \mathcal{P}_{CIFS-ẃ}$$

2. **Monotonicity:** Let $\mathcal{P}_{CIFS-ẃ} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-ẃ}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-ẃ}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-ẃ}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-ẃ}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-ẃ}}\right)$,

and

$$\mathcal{P}^{\#}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}},\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\ \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right),$$

$\acute{u} = 1, 2, \ldots, \vartheta$, as two classes of CIFNs. If $\forall_{\acute{u}}\, \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} \leq \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}$, $\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}} \leq \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}$ $\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} \geq \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}$, and $\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}} \geq \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}$ then

$$IP-CIFOWA\left(\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots \mathcal{P}_{CIFS-\vartheta}\right) \leq IP-CIFOWA\left(\mathcal{P}^{\#}_{CIFS-1},\ \mathcal{P}^{\#}_{CIFS-2},\ \ldots \mathcal{P}^{\#}_{CIFS-\vartheta}\right)$$

3. **Boundedness:** Let $\mathcal{P}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\acute{u}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)$, $\acute{u} = 1, 2, \ldots, \vartheta$ as a class of CIFNs. If $\mathcal{P}^{-}_{CIFS} = \begin{pmatrix} \min_{\acute{u}}\left\{\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\} + \iota \min_{\acute{u}}\left\{\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\}, \\ \max_{\acute{u}}\left\{\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\} + \iota \max_{\acute{u}}\left\{\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\} \end{pmatrix}$ and $\mathcal{P}^{+}_{CIFS} = \begin{pmatrix} \max_{\acute{u}}\left\{\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\} + \iota \max_{\acute{u}}\left\{\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\}, \\ \min_{\acute{u}}\left\{\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\} + \iota \min_{\acute{u}}\left\{\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\} \end{pmatrix}$, then

$$\mathcal{P}^{-}_{CIFS} \leq IP-CIFOWA\begin{pmatrix} \mathcal{P}_{CIFS-1}, \\ \mathcal{P}_{CIFS-2}, \ldots \\ \mathcal{P}_{CIFS-\vartheta} \end{pmatrix} \leq \mathcal{P}^{+}_{CIFS}$$

Definition 12: The P-CIFWG operator over a class of CIFNs $\mathcal{P}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\acute{u}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)$, $\acute{u} = 1, 2, \ldots, \vartheta$ would be estimated as

$$P-CIFWG\left(\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots,\ \mathcal{P}_{CIFS-\vartheta}\right) = \bigotimes_{\acute{u}=1}^{\vartheta} \left(\mathcal{P}_{CIFS-\acute{u}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \tag{7}$$

In Eq. (7), $\Omega_{\mathbb{wv}} = \left(\Omega_{\mathbb{wv}-1}, \Omega_{\mathbb{wv}-2}, \ldots, \Omega_{\mathbb{wv}-\vartheta}\right)$ is a weight with $0 \leq \Omega_{\mathbb{wv}-\acute{u}} \leq 1$, $\sum_{\acute{u}=1}^{\vartheta} \Omega_{\mathbb{wv}-\acute{u}} = 1$, $E_{\mathbb{bv}-\acute{u}} > 0$ with $\sum_{\acute{u}=1}^{\vartheta} E_{\mathbb{bv}-\acute{u}} = 1$ is a probability weight and $\mathcal{X}_{\mathbb{b}-\acute{u}} = \rho E_{\mathbb{bv}-\acute{u}} + (1-\rho)\,\Omega_{\mathbb{wv}-\acute{u}}$ fuse the weight vector and probability weight in the same formula with $\rho \in [0, 1]$.

Theorem 5: Let a class of CIFNs $\mathcal{P}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\acute{u}}}\right) = \begin{pmatrix} \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}, \\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}} \end{pmatrix}$, $\acute{u} = 1, 2, \ldots, \vartheta$. Then the aggregated outcome after employing the P-CIFWG operator over $\mathcal{P}_{CIFS-\acute{u}}$ would anticipate a CIFN that is

$$P-CIFWG\left(\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots \mathcal{P}_{CIFS-\vartheta}\right) = \begin{pmatrix} \prod_{\acute{u}=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \\ +\iota \prod_{\acute{u}=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}, \\ 1-\prod_{\acute{u}=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \\ +\iota\left(1-\prod_{\acute{u}=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}\right) \end{pmatrix} \tag{8}$$

Proof: Consider $\vartheta = 2$, then,

$$\left(\mathcal{P}_{CIFS-1}\right)^{\mathcal{X}_{\mathbb{b}-1}} = \begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}} \\ +\iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}}, \\ 1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}}\right) \end{pmatrix}$$

$$\left(\mathcal{P}_{CIFS-2}\right)^{\mathcal{X}_{\mathbb{b}-2}} = \begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}} \\ +\iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}}, \\ 1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}}\right) \end{pmatrix}$$

now,

$$\begin{aligned} &P-CIFWG\left(\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2}\right) \\ &= \left(\mathcal{P}_{CIFS-1}\right)^{\mathcal{X}_{\mathbb{b}-1}} \otimes \left(\mathcal{P}_{CIFS-2}\right)^{\mathcal{X}_{\mathbb{b}-2}} \\ &= \begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}} + \iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}}, \\ 1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-1}}\right)^{\mathcal{X}_{\mathbb{b}-1}}\right) \end{pmatrix} \\ &\otimes \begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}} + \iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}}, \\ 1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-2}}\right)^{\mathcal{X}_{\mathbb{b}-2}}\right) \end{pmatrix} \\ &= \begin{pmatrix} \prod_{\acute{u}=1}^{2}\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \\ +\iota \prod_{\acute{u}=1}^{2}\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}, \\ 1-\prod_{\acute{u}=1}^{2}\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} \\ +\iota\left(1-\prod_{\acute{u}=1}^{2}\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}\right) \end{pmatrix} \end{aligned}$$

$\Rightarrow$ Eq. (8) held for $\vartheta = 2$. Consider Eq. (8) is true for $\vartheta = \mathbb{L}$, then

$$P-CIFWG\left(\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots\mathcal{P}_{CIFS-\mathbb{L}}\right) = \begin{pmatrix} \prod_{\text{ứ}=1}^{\mathbb{L}}\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)^{\mathcal{X}_{\mathbb{b}-\text{ứ}}} \\ +\iota\prod_{\text{ứ}=1}^{\mathbb{L}}\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)^{\mathcal{X}_{\mathbb{b}-\text{ứ}}}, \\ 1-\prod_{\text{ứ}=1}^{\mathbb{L}}\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)^{\mathcal{X}_{\mathbb{b}-\text{ứ}}} \\ +\iota\left(1-\prod_{\text{ứ}=1}^{\mathbb{L}}\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)^{\mathcal{X}_{\mathbb{b}-\text{ứ}}}\right) \end{pmatrix}$$

Next to prove that Eq. (8) is true for $\vartheta = \mathbb{L}+1$, as

$$\begin{aligned} &P-CIFWG\left(\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots\mathcal{P}_{CIFS-\mathbb{L}+1}\right) \\ &= P-CIFWG\left(\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots\mathcal{P}_{CIFS-\mathbb{L}}\right) \\ &\quad\otimes\mathcal{P}_{CIFS-\mathbb{L}+1} \\ &= \begin{pmatrix} \prod_{\text{ứ}=1}^{\mathbb{L}}\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)^{\mathcal{X}_{\mathbb{b}-\text{ứ}}} \\ +\iota\prod_{\text{ứ}=1}^{\mathbb{L}}\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)^{\mathcal{X}_{\mathbb{b}-\text{ứ}}}, \\ 1-\prod_{\text{ứ}=1}^{\mathbb{L}}\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)^{\mathcal{X}_{\mathbb{b}-\text{ứ}}} \\ +\iota\left(1-\prod_{\text{ứ}=1}^{\mathbb{L}}\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)^{\mathcal{X}_{\mathbb{b}-\text{ứ}}}\right) \end{pmatrix} \\ &\otimes\begin{pmatrix} \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\mathbb{L}+1}}\right)^{\mathcal{X}_{\mathbb{b}-\mathbb{L}+1}} \\ +\iota\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\mathbb{L}+1}}\right)^{\mathcal{X}_{\mathbb{b}-\mathbb{L}+1}}, \\ 1-\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\mathbb{L}+1}}\right)^{\mathcal{X}_{\mathbb{b}-\mathbb{L}+1}} \\ +\iota\left(1-\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\mathbb{L}+1}}\right)^{\mathcal{X}_{\mathbb{b}-\mathbb{L}+1}}\right) \end{pmatrix} \\ &= \begin{pmatrix} \prod_{\text{ứ}=1}^{\mathbb{L}+1}\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)^{\mathcal{X}_{\mathbb{b}-\text{ứ}}} \\ +\iota\prod_{\text{ứ}=1}^{\mathbb{L}+1}\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)^{\mathcal{X}_{\mathbb{b}-\text{ứ}}}, \\ 1-\prod_{\text{ứ}=1}^{\mathbb{L}+1}\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)^{\mathcal{X}_{\mathbb{b}-\text{ứ}}} \\ +\iota\left(1-\prod_{\text{ứ}=1}^{\mathbb{L}+1}\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)^{\mathcal{X}_{\mathbb{b}-\text{ứ}}}\right) \end{pmatrix} \\ &= P-CIFWG\begin{pmatrix}\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots \\ \mathcal{P}_{CIFS-\mathbb{L}},\ \mathcal{P}_{CIFS-\mathbb{L}+1}\end{pmatrix} \end{aligned}$$

Thus, Eq. (8) is true for $\mathbb{L}+1$ and consequently true $\forall\ \vartheta$.

Properties 3: Underneath are the properties that the IP-CIFWG operator satisfies.

1. **Idempotency:** Let $\mathcal{P}_{CIFS-\text{ứ}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\text{ứ}}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\text{ứ}}} + \iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\text{ứ}}},\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\text{ứ}}} + \iota\,\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)$, $\text{ứ} = 1,\ 2,\ \ldots,\ \vartheta$, as a class of CIFNs. Then if $\forall\ _{\text{ứ}}\ \mathcal{P}_{CIFS-\text{ứ}} = \mathcal{P}_{CIFS}$ i.e., $\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\text{ứ}}} = \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}$, $\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\text{ứ}}} = \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}}$, $\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\text{ứ}}} = \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}$ and $\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\text{ứ}}} = \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}$, then
$$P-CIFWG\begin{pmatrix}\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2}, \\ \ldots\mathcal{P}_{CIFS-\vartheta}\end{pmatrix} = \mathcal{P}_{CIFS-\text{ứ}}$$
2. **Monotonicity:** Let $\mathcal{P}_{CIFS-\text{ứ}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\text{ứ}}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\text{ứ}}} + \iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\text{ứ}}},\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\text{ứ}}} + \iota\,\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)$, and $\mathcal{P}^{\#}_{CIFS-\text{ứ}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}^{\#}_{CIFS-\text{ứ}}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}^{\#}_{CIFS-\text{ứ}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}^{\#}_{CIFS-\text{ứ}}} + \iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}^{\#}_{CIFS-\text{ứ}}},\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}^{\#}_{CIFS-\text{ứ}}}\ \iota\,\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}^{\#}_{CIFS-\text{ứ}}}\right)$, $\text{ứ} = 1,\ 2,\ \ldots,\ \vartheta$, as two classes of CIFNs. If $\forall\ _{\text{ứ}}\ \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\text{ứ}}} \leq \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}^{\#}_{CIFS-\text{ứ}}}$, $\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\text{ứ}}} \leq \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}^{\#}_{CIFS-\text{ứ}}}$ $\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\text{ứ}}} \geq \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}^{\#}_{CIFS-\text{ứ}}}$, and $\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\text{ứ}}} \geq \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}^{\#}_{CIFS-\text{ứ}}}$ then
$$\begin{aligned}&P-CIFWG\left(\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots\mathcal{P}_{CIFS-\vartheta}\right) \\ &\leq P-CIFWG\left(\mathcal{P}^{\#}_{CIFS-1},\ \mathcal{P}^{\#}_{CIFS-2},\ \ldots\mathcal{P}^{\#}_{CIFS-\vartheta}\right)\end{aligned}$$
3. **Boundedness:** Let $\mathcal{P}_{CIFS-\text{ứ}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\text{ứ}}},\mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\text{ứ}}} + \iota\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\text{ứ}}},\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\text{ứ}}} + \iota\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)$, $\text{ứ} = 1, 2, \ldots, \vartheta$ as a class of CIFNs. If $\mathcal{P}^{-}_{CIFS} = \begin{pmatrix}\min_{\text{ứ}}\left\{\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right\} + \iota\min_{\text{ứ}}\left\{\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right\}, \\ \max_{\text{ứ}}\left\{\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right\} + \iota\max_{\text{ứ}}\left\{\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right\}\end{pmatrix}$ and $\mathcal{P}^{+}_{CIFS} = \begin{pmatrix}\max_{\text{ứ}}\left\{\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right\} + \iota\max_{\text{ứ}}\left\{\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right\}, \\ \min_{\text{ứ}}\left\{\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right\} + \iota\min_{\text{ứ}}\left\{\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right\}\end{pmatrix}$, then
$$\mathcal{P}^{-}_{CIFS} \leq P-CIFWG\begin{pmatrix}\mathcal{P}_{CIFS-1}, \\ \mathcal{P}_{CIFS-2},\ \ldots \\ \mathcal{P}_{CIFS-\vartheta}\end{pmatrix} \leq \mathcal{P}^{+}_{CIFS}$$

Definition 13: The P-CIFOWG operator over a class of CIFNs $\mathcal{P}_{CIFS-\text{ứ}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\text{ứ}}},\ \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\text{ứ}}} + \iota\,\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\text{ứ}}},\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\text{ứ}}} + \iota\,\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\text{ứ}}}\right)$, $\text{ứ} = 1,\ 2,\ \ldots,\ \vartheta$ would be estimated as

$$\begin{aligned}&P-CIFOWG\left(\mathcal{P}_{CIFS-1},\ \mathcal{P}_{CIFS-2},\ \ldots,\ \mathcal{P}_{CIFS-\vartheta}\right) \\ &= \bigotimes_{\text{ứ}=1}^{\vartheta}\left(\mathcal{P}_{CIFS-\mathfrak{f}(\text{ứ})}\right)^{\mathcal{X}_{\mathbb{b}-\text{ứ}}} \qquad (9)\end{aligned}$$

In Eq. (9), $(\mathfrak{r}(1), \mathfrak{r}(2), \ldots, \mathfrak{r}(\vartheta))$ is a permutation of $(1, 2, \ldots, \vartheta)$ such that $\mathfrak{r}(\acute{u}-1) \geq \mathfrak{r}(\acute{u})$, for $\acute{u} = 2, 3, \ldots, \vartheta$ and $\mathcal{X}_{\mathbb{b}-\acute{u}} = \rho E_{\mathbb{b}\mathbb{v}-\acute{u}} + (1-\rho)\,\Omega_{\mathbb{w}\mathbb{v}-\acute{u}}$ fuse the weight vector and probability weight in the same formula with $\rho \in [0, 1]$.

Theorem 6: Let a class of CIFNs

$$\mathcal{P}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\acute{u}}}\right) = \begin{pmatrix} \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}, \\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}} \end{pmatrix}, \quad \acute{u} = 1, 2, \ldots, \vartheta.$$

Then the aggregated outcome after employing the P-CIFOWG operator over $\mathcal{P}_{CIFS-\acute{u}}$ would anticipate a CIFN that is

$$P-CIFOWG\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots \mathcal{P}_{CIFS-\vartheta}\right)$$
$$= \begin{pmatrix} \prod_{\acute{u}=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\mathfrak{r}(\acute{u})}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} + \iota \prod_{\acute{u}=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\mathfrak{r}(\acute{u})}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}, \\ 1-\prod_{\acute{u}=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\mathfrak{r}(\acute{u})}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}} + \iota\left(1-\prod_{\acute{u}=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\mathfrak{r}(\acute{u})}}\right)^{\mathcal{X}_{\mathbb{b}-\acute{u}}}\right) \end{pmatrix}$$

The P-CIFOWA operator satisfies properties such as idempotency, monotonicity, and boundedness.

Definition 14: The IP-CIFOWG operator over a class of CIFNs $\mathcal{P}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\acute{u}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)$, $\acute{u} = 1, 2, \ldots, \vartheta$ would be estimated as

$$IP-CIFOWG\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots, \mathcal{P}_{CIFS-\vartheta}\right) = \bigotimes_{\acute{u}=1}^{\vartheta}\left(\mathcal{P}_{CIFS-\mathfrak{r}(\acute{u})}\right)^{\mathfrak{J}_{\mathbb{b}-\acute{u}}} \tag{10}$$

In Eq. (10), $(\mathfrak{r}(1), \mathfrak{r}(2), \ldots, \mathfrak{r}(\vartheta))$ is a permutation of $(1, 2, \ldots, \vartheta)$ such that $\mathfrak{r}(\acute{u}-1) \geq \mathfrak{r}(\acute{u})$, for $\acute{u} = 2, 3, \ldots, \vartheta$ and $\mathfrak{J}_{\mathbb{b}-\acute{u}} = \frac{\Omega_{\mathbb{w}\mathbb{v}-\acute{u}} E_{\mathbb{b}\mathbb{v}-\acute{u}}}{\sum_{\acute{u}=1}^{\vartheta} \Omega_{\mathbb{w}\mathbb{v}-\acute{u}} E_{\mathbb{b}\mathbb{v}-\acute{u}}}$ is an immediate probability (IP) interpreted to CIFN

Theorem 7: Let a class of CIFNs $\mathcal{P}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\acute{u}}}\right) = \begin{pmatrix} \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}, \\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}} \end{pmatrix}$, $\acute{u} = 1, 2, \ldots, \vartheta$. Then the aggregated outcome after employing the IP-CIFOWG operator over $\mathcal{P}_{CIFS-\acute{u}}$ would anticipate a CIFN that is

$$IP-CIFOWG\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots \mathcal{P}_{CIFS-\vartheta}\right)$$
$$= \begin{pmatrix} \prod_{\acute{u}=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\mathfrak{r}(\acute{u})}}\right)^{\mathfrak{J}_{\mathbb{b}-\acute{u}}} + \iota \prod_{\acute{u}=1}^{\vartheta}\left(\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\mathfrak{r}(\acute{u})}}\right)^{\mathfrak{J}_{\mathbb{b}-\acute{u}}}, \\ 1-\prod_{\acute{u}=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\mathfrak{r}(\acute{u})}}\right)^{\mathfrak{J}_{\mathbb{b}-\acute{u}}} + \iota\left(1-\prod_{\acute{u}=1}^{\vartheta}\left(1-\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\mathfrak{r}(\acute{u})}}\right)^{\mathfrak{J}_{\mathbb{b}-\acute{u}}}\right) \end{pmatrix}$$

Properties 4: Underneath are the properties that the IP-CIFWG operator satisfies.

1. **Idempotency:** Let $\mathcal{P}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\acute{u}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)$, $\acute{u} = 1, 2, \ldots, \vartheta$, as a class of CIFNs. Then if $\forall\, \acute{u}\ \mathcal{P}_{CIFS-\acute{u}} = \mathcal{P}_{CIFS}$ i.e., $\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} = \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS}}$, $\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}} = \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS}}$, $\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} = \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS}}$ and $\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}} = \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS}}$, then
$$IP-CIFOWG\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots \mathcal{P}_{CIFS-\vartheta}\right) = \mathcal{P}_{CIFS-\acute{u}}$$

2. **Monotonicity:** Let $\mathcal{P}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\acute{u}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)$, and $\mathcal{P}^{\#}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\ \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}\right)$, $\acute{u} = 1, 2, \ldots, \vartheta$, as two classes of CIFNs. If $\forall\, \acute{u}\ \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} \leq \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}$, $\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}} \leq \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}$ $\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} \geq \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}$, and $\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}} \geq \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}^{\#}_{CIFS-\acute{u}}}$ then
$$IP-CIFOWG\left(\mathcal{P}_{CIFS-1}, \mathcal{P}_{CIFS-2}, \ldots \mathcal{P}_{CIFS-\vartheta}\right) \leq IP-CIFOWG\left(\mathcal{P}^{\#}_{CIFS-1}, \mathcal{P}^{\#}_{CIFS-2}, \ldots \mathcal{P}^{\#}_{CIFS-\vartheta}\right)$$

3. **Boundedness:** Let $\mathcal{P}_{CIFS-\acute{u}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\acute{u}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right)$, $\acute{u} = 1, 2, \ldots, \vartheta$ as a class of CIFNs. If $\mathcal{P}^{-}_{CIFS} = \begin{pmatrix} \min_{\acute{u}}\left\{\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\} + \iota \min_{\acute{u}}\left\{\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\}, \\ \max_{\acute{u}}\left\{\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\} + \iota \max_{\acute{u}}\left\{\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\} \end{pmatrix}$ and $\mathcal{P}^{+}_{CIFS} = \begin{pmatrix} \max_{\acute{u}}\left\{\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\} + \iota \max_{\acute{u}}\left\{\mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\}, \\ \min_{\acute{u}}\left\{\mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\} + \iota \min_{\acute{u}}\left\{\mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{u}}}\right\} \end{pmatrix}$, then
$$\mathcal{P}^{-}_{CIFS} \leq IP-CIFOWG\begin{pmatrix} \mathcal{P}_{CIFS-1}, \\ \mathcal{P}_{CIFS-2}, \ldots \\ \mathcal{P}_{CIFS-\vartheta} \end{pmatrix} \leq \mathcal{P}^{+}_{CIFS}$$

V. MULTI-CRITERIA DECISION-MAKING TECHNIQUE IN THE FRAMEWORK OF CIFS

Let us assume a multi-criteria decision-making (MCDM) dilemma, where ϑ alternatives that are $\{\mathfrak{B}_{\mathcal{A}\mathfrak{T}-1}, \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2}, \ldots, \mathfrak{B}_{\mathcal{A}\mathfrak{T}-\vartheta}\}$ and θ criteria that is $\{\mathfrak{Z}_{\mathfrak{C}-1}, \mathfrak{Z}_{\mathfrak{C}-2}, \ldots, \mathfrak{Z}_{\mathfrak{C}-\theta}\}$ are involved. Based on these criteria, the decision analyst has to determine the optimal alternatives. According to the decision analyst, each criterion can have its own significance, thus the decision analyst would interpret the weight of the criteria that is $\Omega_{\mathbb{wv}} = (\Omega_{\mathbb{wv}-1}, \Omega_{\mathbb{wv}-2}, \ldots, \Omega_{\mathbb{wv}-\theta})$ with $0 \leq \Omega_{\mathbb{wv}-\grave{\upsilon}} \leq 1$, $\sum_{\grave{\upsilon}=1}^{\theta} \Omega_{\mathbb{wv}-\grave{\upsilon}} = 1$. Also, the decision analyst would consider the probability of each criterion by revealing the probability weight that is $E_{\mathbb{bv}-\grave{\upsilon}} > 0$ with $\sum_{\grave{\upsilon}=1}^{\theta} E_{\mathbb{bv}-\grave{\upsilon}} = 1$. The decision analyst will anticipate his/her assessment outcomes in the setting of CIFNs i.e., $\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}} = \left(\mathcal{F}^{\mathcal{M}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}}, \mathcal{F}^{\mathcal{N}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}}\right) = \left(\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}}\right)$, where $\mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}}, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}}, \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}}, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} \in [0, 1]$ and $0 \leq \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} + \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} \leq 1$, and $0 \leq \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} + \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} \leq 1$ and will construct a CIF decision matrix $\mathcal{D}_{CIFS}$. Through the underlying phases, this MCDM dilemma will be tackled.

Phase 1: The criterion can be two kinds: benefit and cost type. Thus, there is always the requirement for of normalization the CIF decision matrix and for that, we have the below formula.

$$(\mathcal{D}_{CIFS})^{N} = \begin{cases} \begin{pmatrix} \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}}, \\ \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} \end{pmatrix} & \textit{for benefit kind} \\ \begin{pmatrix} \mathcal{F}^{\mathcal{RN}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} + \iota\, \mathcal{F}^{\mathcal{IN}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} \\ \mathcal{F}^{\mathcal{RM}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} + \iota\, \mathcal{F}^{\mathcal{IM}}_{\mathcal{P}_{CIFS-\acute{r}\grave{\upsilon}}} \end{pmatrix} & \textit{for cost kind} \end{cases}$$

Phase 2: By taking into account the probabilistic information and weight vector, determine $\mathcal{X}_{\mathbb{b}-\grave{\upsilon}} = \rho E_{\mathbb{bv}-\grave{\upsilon}} + (1-\rho)\, \Omega_{\mathbb{wv}-\grave{\upsilon}}$ and $\mathfrak{I}_{\mathbb{b}-\grave{\upsilon}} = \frac{\Omega_{\mathbb{wv}-\grave{\upsilon}} E_{\mathbb{bv}-\grave{\upsilon}}}{\sum_{\acute{r}=1}^{\vartheta} \Omega_{\mathbb{wv}-\grave{\upsilon}} E_{\mathbb{bv}-\grave{\upsilon}}}$

Phase 3: The CIF decision matrix would be aggregated by employing the invented AOs that are P-CIFWA, P-CIFOWA, IP-CIFOWA, P-CIFWA, P-CIFOWA, and IP-CIFOWA operators.

Phase 4: Afterward the aggregation of the CIF decision matrix, the score value or accuracy values of each alternative would be anticipated.

Phase 5: Based on the score or accuracy values, the ranking of alternatives would be devised.

A. CASE STUDY

A manufacturing company is giving eco-friendliness priority when deciding which forms of transportation to use for product distribution. To make an educated choice, the company considers four different transportation options depending on a variety of factors. These options are briefly discussed below

$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-1}$:**Electric vehicles:**Because electric trucks run on electricity, they use fewer fossil fuels overall. They operate with no tailpipe emissions because they use electric motors rather than internal combustion engines. When compared to conventional diesel trucks, these vehicles significantly reduce greenhouse gas emissions since they are powered by electric grids. Their eco-friendliness may be further improved by charging them using renewable energy sources.

$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-2}$: **Rail transportation:**Rail transportation is known for its reduced environmental effect and energy efficiency. An environmentally beneficial option for moving cargo over land is to use diesel-powered locomotives or electrically powered trains operating on electrified lines. In comparison to road-based transportation, trains produce less emissions per ton-mile and are capable of carrying large loads. This makes them particularly efficient for long-distance transportation.

$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-3}$: **Biofuel-powered airplanes:**Biofuel-powered aircraft run on renewable biofuels that come from waste materials, plants, or algae. The goal of these biofuels is to lessen reliance on conventional aviation fuels made from fossil fuels and carbon emissions. Although biofuels have the potential to cut greenhouse gas emissions, the aviation industry is currently experimenting with their utilization due to uneven availability and scalability.

$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4}$: **Hybrid cargo ships:**To lower fuel usage and pollution, hybrid cargo ships combine conventional fuel with alternative energy sources like electricity or wind power. In addition to conventional engines, these ships frequently use technology like sails, solar panels, or batteries. Even if they use cleaner energy sources to lessen their influence on the environment, companies still partially rely on traditional fuels, which results in emissions that are decreased but not eliminated.

The company will assess these modes on the following criteria.

$\mathfrak{Z}_{\mathfrak{C}-1}$**: Environmental Effect:**Taking into account variables like pollution, carbon emissions, and ecological footprint, this feature assesses the total environmental impact of each mode of transportation.

$\mathfrak{Z}_{\mathfrak{C}-2}$**: Cost efficiency:**Calculates the overall cost of using a transportation mode, taking into account all upfront, ongoing, and maintenance expenditures.

$\mathfrak{Z}_{\mathfrak{C}-3}$**: Delivery time:**Evaluates how quickly and dependably each method of transportation can deliver items to their intended location.

$\mathfrak{Z}_{\mathfrak{C}-4}$**: Capacity:**Determines how much weight or volume any form of transportation can effectively manage.

The company would provide weights (0.21, 0.24, 0.26, 0.29) and probability (0.15, 0.25, 0.35, 0.25) to each criterion according to its relative relevance in order to employ the MCDM methodology. The company would next gather assessment values on the performance of each mode of transportation in relation to these criteria and the assessment values would be in the environment of CIFNs to construct a CIF decision matrix as revealed in Table 1.

TABLE 1. The assessment values of various ECO-Friendly transportation modes in the cartesian framework of CIFS.

	$\mathfrak{Z}_{\mathfrak{C}-1}$	$\mathfrak{Z}_{\mathfrak{C}-2}$	$\mathfrak{Z}_{\mathfrak{C}-3}$	$\mathfrak{Z}_{\mathfrak{C}-4}$
$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-1}$	$\begin{pmatrix}0.326+\iota 0.378,\\ 0.278+\iota 0.543\end{pmatrix}$	$\begin{pmatrix}0.516+\iota 0.441,\\ 0.163+\iota 0.471\end{pmatrix}$	$\begin{pmatrix}0.463+\iota 0.366,\\ 0.513+\iota 0.633\end{pmatrix}$	$\begin{pmatrix}0.735+\iota 0.834,\\ 0.233+\iota 0.133\end{pmatrix}$
$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-2}$	$\begin{pmatrix}0.478+\iota 0.435,\\ 0.138+\iota 0.541\end{pmatrix}$	$\begin{pmatrix}0.735+\iota 0.354,\\ 0.165+\iota 0.511\end{pmatrix}$	$\begin{pmatrix}0.364+\iota 0.392,\\ 0.413+\iota 0.572\end{pmatrix}$	$\begin{pmatrix}0.643+\iota 0.523,\\ 0.211+\iota 0.321\end{pmatrix}$
$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-3}$	$\begin{pmatrix}0.206+\iota 0.339,\\ 0.503+\iota 0.586\end{pmatrix}$	$\begin{pmatrix}0.565+\iota 0.111,\\ 0.316+\iota 0.523\end{pmatrix}$	$\begin{pmatrix}0.712+\iota 0.614,\\ 0.146+\iota 0.217\end{pmatrix}$	$\begin{pmatrix}0.613+\iota 0.640,\\ 0.123+\iota 0.150\end{pmatrix}$
$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4}$	$\begin{pmatrix}0.779+\iota 0.325,\\ 0.154+\iota 0.455\end{pmatrix}$	$\begin{pmatrix}0.761+\iota 0.811,\\ 0.164+\iota 0.112\end{pmatrix}$	$\begin{pmatrix}0.484+\iota 0.623,\\ 0.339+\iota 0.132\end{pmatrix}$	$\begin{pmatrix}0.238+\iota 0.432,\\ 0.461+\iota 0.332\end{pmatrix}$

TABLE 2. The aggregated values of ECO-Friendly transportation modes after using investigated operators.

Operators	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-1}$	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-2}$	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-3}$	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4}$
P-CIFWA	$\begin{pmatrix}0.549+\iota 0.571,\\ 0.378+\iota 0.282\end{pmatrix}$	$\begin{pmatrix}0.577+\iota 0.431,\\ 0.224+\iota 0.469\end{pmatrix}$	$\begin{pmatrix}0.583+\iota 0.489,\\ 0.211+\iota 0.291\end{pmatrix}$	$\begin{pmatrix}0.592+\iota 0.601,\\ 0.268+\iota 0.206\end{pmatrix}$
P-CIFOWA	$\begin{pmatrix}0.503+\iota 0.516,\\ 0.278+\iota 0.428\end{pmatrix}$	$\begin{pmatrix}0.563+\iota 0.423,\\ 0.211+\iota 0.492\end{pmatrix}$	$\begin{pmatrix}0.544+\iota 0.434,\\ 0.251+\iota 0.346\end{pmatrix}$	$\begin{pmatrix}0.612+\iota 0.559,\\ 0.255+\iota 0.239\end{pmatrix}$
IP-CIFOWA	$\begin{pmatrix}0.479+\iota 0.482,\\ 0.286+\iota 0.461\end{pmatrix}$	$\begin{pmatrix}0.549+\iota 0.418,\\ 0.214+\iota 0.505\end{pmatrix}$	$\begin{pmatrix}0.531+\iota 0.408,\\ 0.268+\iota 0.273\end{pmatrix}$	$\begin{pmatrix}0.604+\iota 0.531,\\ 0.26+\iota 0.257\end{pmatrix}$
P-CIFWG	$\begin{pmatrix}0.506+\iota 0.48,\\ 0.328+\iota 0.476\end{pmatrix}$	$\begin{pmatrix}0.531+\iota 0423,\\ 0.255+\iota 0.491\end{pmatrix}$	$\begin{pmatrix}0.513+\iota 0.369,\\ 0.262+\iota 0.368\end{pmatrix}$	$\begin{pmatrix}0.484+\iota 0.531,\\ 0.308+\iota 0.256\end{pmatrix}$
P-CIFOWG	$\begin{pmatrix}0.462+\iota 0.447,\\ 0.319+\iota 0.5\end{pmatrix}$	$\begin{pmatrix}0.59+\iota 0.416,\\ 0.244+\iota 0.508\end{pmatrix}$	$\begin{pmatrix}0.458+\iota 0.316,\\ 0.309+\iota 0.425\end{pmatrix}$	$\begin{pmatrix}0.498+\iota 0.487,\\ 0.298+\iota 0.294\end{pmatrix}$
IP-CIFOWG	$\begin{pmatrix}0.444+\iota 0.428,\\ 0.327+\iota 0.519\end{pmatrix}$	$\begin{pmatrix}0.505+\iota 0.413,\\ 0.249+\iota 0.519\end{pmatrix}$	$\begin{pmatrix}0.442+\iota 0.294,\\ 0.325+\iota 0.446\end{pmatrix}$	$\begin{pmatrix}0.484+\iota 0.467,\\ 0.305+\iota 0.31\end{pmatrix}$

TABLE 3. The score values of ECO-Friendly transportation modes.

Operators	Ş$(\mathfrak{B}_{\mathcal{A}\mathfrak{T}-1})$	Ş$(\mathfrak{B}_{\mathcal{A}\mathfrak{T}-2})$	Ş$(\mathfrak{B}_{\mathcal{A}\mathfrak{T}-3})$	Ş$(\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4})$
P-CIFWA	0.23	0.157	0.285	0.359
P-CIFOWA	0.157	0.142	0.19	0.338
IP-CIFOWA	0.107	0.124	0.149	0.309
P-CIFWG	0.091	0.104	0.126	0.226
P-CIFOWG	0.045	0.092	0.02	0.197
IP-CIFOWG	0.013	0.075	−0.02	0.169

To handle this MCDM, we would employ the invented CIF MCDM approach.

*Phase 1:*We are skipping this phase as all the criteria are beneficial.

Phase 2: Here, we achieve

$$\mathcal{X}_{\mathbb{b}-1}= 0.174,\ \mathcal{X}_{\mathbb{b}-2}= 0.246,\ \mathcal{X}_{\mathbb{b}-3}= 0.314,\ \mathcal{X}_{\mathbb{b}-4}= 0.266$$

and

$$\mathfrak{J}_{\mathbb{b}-1}= 0.124,\ \mathfrak{J}_{\mathbb{b}-2}= 0.235,\ \mathfrak{J}_{\mathbb{b}-3}= 0.357,\ \mathfrak{J}_{\mathbb{b}-4}= 0.284$$

where $\rho = 0.6$.

Phase 3: After the utilization of invented operators P-CIFWA, P-CIFOWA, IP-CIFOWA, P-CIFWA, P-CIFOWA, and IP-CIFOWA, the aggregated outcomes are portrayed in Table 2.

Phase 4: Table 3 devised the score values of all transportation modes.

Phase 5: Employing these modes of transportation's score values, we deduce the ranking in Table 4.

The invented MCDM revealed that $\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4}$ that is "hybrid cargo ships" is the optimal eco-friendly transportation mode.

VI. COMPARATIVE ANALYSIS

To reveal the importance and significance of the proposed AOs and a procedure of MADM in the setting of the Cartesian framework of CIFS, we compare this work with various prevailing notions which are briefly discussed below

→ Xu [13] devised AOs for IFS.

→ Wei and Merigo [15] anticipated probability AOs for IFS and an approach for tackling DM issues in IFS.

→ Rehman [24] anticipated probability AOs and MADM procedures for CFS.

→ Garg and Rani [29] deduced AOs and MCDM approaches for the polar framework of CIFS.

TABLE 4. The ranking of ECO-Friendly transportation modes.

Operators	Ranking
P-CIFWA	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-3} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-1} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2}$
P-CIFOWA	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-3} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-1} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2}$
IP-CIFOWA	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-3} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-1}$
P-CIFWG	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-3} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2}$
P-CIFOWG	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-1} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-3}$
IP-CIFOWG	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-1} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-3}$

TABLE 5. The result after employing the deduced and considered theories.

Reference	Ş($\boldsymbol{\mathfrak{B}}_{\mathcal{A}\mathfrak{T}-1}$)	Ş($\boldsymbol{\mathfrak{B}}_{\mathcal{A}\mathfrak{T}-2}$)	Ş($\boldsymbol{\mathfrak{B}}_{\mathcal{A}\mathfrak{T}-3}$)	Ş($\boldsymbol{\mathfrak{B}}_{\mathcal{A}\mathfrak{T}-4}$)
Xu [13[	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×
Wei and Merigo [15]	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×
Rehman [24]	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×
Garg and Rani [29]	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×
Rani and Garg [30]	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×
Akram et al. [31]	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×
Ali et al. [32]	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×
Garg and Rani [33]	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×	×⇒×⇒×
P-CIFWA	0.23	0.157	0.285	0.359
P-CIFOWA	0.157	0.142	0.19	0.338
IP-CIFOWA	0.107	0.124	0.149	0.309
P-CIFWG	0.091	0.104	0.126	0.226
P-CIFOWG	0.045	0.092	0.02	0.197
IP-CIFOWG	0.013	0.075	−0.02	0.169

TABLE 6. The ranking is based on the result of table 5.

Reference	Ranking
Xu [13[	×⇒×⇒×
Wei and Merigo [15]	×⇒×⇒×
Rehman [24]	×⇒×⇒×
Garg and Rani [29]	×⇒×⇒×
Rani and Garg [30]	×⇒×⇒×
Akram et al. [31]	×⇒×⇒×
Ali et al. [32]	×⇒×⇒×
Garg and Rani [33]	×⇒×⇒×
P-CIFWA	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-3} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-1} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2}$
P-CIFOWA	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-3} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-1} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2}$
IP-CIFOWA	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-3} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-1}$
P-CIFWG	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-3} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2}$
P-CIFOWG	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-1} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-3}$
IP-CIFOWG	$\mathfrak{B}_{\mathcal{A}\mathfrak{T}-4} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-2} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-1} > \mathfrak{B}_{\mathcal{A}\mathfrak{T}-3}$

→ Rani and Garg [30] anticipated power AOs and MCDM for the polar framework of CIFS.

→ Akram et al. [31] devised Hamacher AOs for the polar framework of CIFS.

→ Ali et al. [32] devised an MSM operator for the polar framework of CIFS.

→ Garg and Rani [33] deduced BM operators and the DM approach for the polar framework of CIFS.

Now to test the accuracy and significance of these considered theories and the anticipated theory, we reconsider the information in Table 1, which is in the Cartesian framework of CIFS. After employing these theories to aggregate and tackle the information in Table 1, we the result which is interpreted in Tables 5 and 6.

Both Tables make it clear that no theory now in use directly addresses or resolves the data in Table 1. Rather, we use the work done in this manuscript to address and resolve the information given in Table 1 and show the results. Every theory offers unique explanations for why it doesn't work. The theory devised by Xu [13] and Wei and Merigo [15] is in the environment of IFS, and the framework of IFS can't cope with 2^{nd} dimension (additional fuzzy information) and also the AOs of Xu [13] can't address the probability of the information. Thus, these theories failed to address the CIF information. The notion devised by Rehman [24] within CFS, but the model of CFS can't model with non-membership and this became the reason for its failure. The theories devised by Garg and Rani [29], Rani and Garg [20], Akram et al. [31], Ali et al. [32], and Garg and Rani [33] are in the polar framework of CIFS which is devised by Alkouri et al. [27] but the polar framework of CIFS different framework than the Cartesian framework of CIFS and each theory devised in the polar framework of CIFS can't model the CIF information in Cartesian form. This implies that none of the prevailing theories in the literature can model the CIF information in the Cartesian framework. Further, the anticipated AOs and MADM can be degenerated in the various other mathematical frameworks such as the Cartesian framework of CFS, IFS, and FS.

VII. CONCLUSION

Many factors need to be considered when choosing an environmentally friendly means of transportation, including cost-effectiveness, capacity, delivery time, and environmental impact. Due to the need to simultaneously weigh and rank several aspects, this is an MCDM conundrum. To find the most ecologically friendly mode of transportation that yet satisfies certain needs and sustainability goals, trade-offs between these factors must be considered. This calls for a methodical and thorough DM process. We initially anticipated fundamental algebraic properties for the CIFS before constructing the various probability AOs in this manuscript. These AOs include P-CIFWA, P-CIFOWA, IP-CIFOWA, P-CIFWG, P-CIFOWG, IP-CIFOWG operators, and linked axioms. Then, within a Cartesian framework of CIFS, we deduced an MCDM technique based on the anticipated AOs. We then applied our method to the analysis of a case study named "Identification of eco-friendly transportation mode." In the last section of the manuscript, we demonstrated the superiority and preeminence of the original idea by contrasting it with a few popular conceptions.

The developed theory is in the framework of CIFS and can't cope with information in various generalized structures of CIFS. Thus, in the future, we hope to investigate and work on the following structures: bipolar complex fuzzy set [36], bipolar complex fuzzy soft set [37], picture FS [38], [39], and Fermatean fuzzy set [40], [41].

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