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RESEARCH ARTICLE

Analysis and Applications of Bipolar Complex Fuzzy Soft Power Dombi Aggregation Operators for Robot Selection in Artificial Intelligence

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ABSTRACT AI-powered robots contain many sensors, such as vision devices like 2D/3D cameras, vibration sensors, proximity sensors, accelerometers, and other environmental sensors. These sensors enable real-time sensing data to be obtained and analyzed. Here, the discussion of robot selection AI-field. In this work, we aim to examine AI robotic systems based on a bipolar complex fuzzy soft set (BCFSS) with Power dombi aggregation operators (PDAO). Moreover, we aim to examine BCFS power dombi average AOs and BCFS power dombi geometric AOs. Three BCFS power dombi average (PDA) AOs are identified in this work, namely the BCFS power dombi weighted average (BCFSPDWA) AO, BCFS power dombi hybrid average (BCFSPDHA) AO and BCFS power dombi ordered weighted average (BCFSPDOWA) AO. Likewise, three types of BCFS power dombi geometric AOs have been identified, namely: BCFS power dombi weighted geometric (BCFSPDWG) AO, BCFS power dombi order-weighted geometric (BCFSPDOWG) AO and BCFS power dombi hybrid geometric (BCFSPDHG) AO. Subsequently, we will propose a numerical model for the proposed operators, as well as a multi-attribute decision-making (MADM) model that can be used to select the best robot in AI. Finally, we will compare the results of the numerical model with the prevailing outcomes in terms of supremacy and dominance.

INDEX TERMS Bipolar complex fuzzy soft set, fuzzy set, MADM method, power dombi aggregation operators, soft set.

INTRODUCTION

Within artificial intelligence, robotics is a distinct field that studies the development of intelligent machines or robots. Robotics combines computer science, engineering, electrical and mechanical engineering, and programming language. Many people believe that robotics is a subset of artificial intelligence (AI), even though the two have different purposes and objectives. Robots that mimic humans in both appearance and logic can be created using AI. Robotic applications were extremely limited in the past, but with the integration of artificial intelligence, they have grown in sophistication and efficiency. AI has become indispensable in the industrial sector, surpassing human productivity and quality. The numerous uses, benefits, distinctions, and other aspects of robotics and AI will be covered in these problems.

The uncertainty and intricacy typically contain a great deal of conundrums. The professionals also attempt to solve these conundrums. In response to this, Zadeh [1] started FS. Truth grade (TG) in FS is classified as belonging to [1, 0]. FS deals with every problem that has complexity and uncertainty involved. The majority of things are handled by more than just FS. Consequently, Mardani et al. [2] started using the DM approach to apply some AOs over FS. Nowadays, FS is failing to handle an increasing number of issues. We only

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have one TG in the FS. Many conundrums are composed of two-sided operations. Intuitionistic FS (IFS), which addresses issues of ambiguity and uncertainty, was first introduced by Atanassov [3]. IFS currently consists of false grades (FG) and TG, where FG and TG belong to [1, 0]. Moreover, Xu [4] initiated the deployment of the IFS AO. To make the AOs' difficulties easier to understand, use mathematics to solve them. Prioritized IFAOs (PIFAOs) were demonstrated by Yu and Xu [5]. Improved language IFAOs and their use with MADM were indicated by Liu and Wang [6], based on IFSs, Xu and Yager [7] created a few geometric AOs (GAOs). The IFS Dombi AOs (DAOs) and their application to MADM were invented by Seikh and Mandal [8]. Kutlu Gundogdu and Kahraman [9] introduced the use of spherical FSs (SFSs) in the extension of WASPAS. In addition, many problems have both positive and negative aspects. Zhang [10] used a computational framework for multi-agent decision analysis and cognitive modeling along with bipolar FSs (BFSs) and relations. Positive TG (PTG) and negative TG (NTG) in BFS are defined as follows: PTG belongs to [1, 0], NTG to [−1, 0], and their sum to [−1, 1]. BF DAOs (BFDAOs) and its application in MADM approach were introduced by Jana et al. [11]. BF Hamacher AOs (BFHAOs) in MADM by Wei et al. [12]. In MADM, BFD prioritized AOs (BFDPAOs) were shown by Jana et al. [13]. In MADM, Wei et al. [12] denoted (BFHAOs). BFDPAOs were introduced in MADM by Jana et al. [13]. Bipolar neutrosophic DAOs (BNDAOs) are new and can be used in MADM problems, according to Mahmood et al. [14]. Furthermore, BF graphs (BFGs) were started by Akram [15].

We also noticed that the majority of the problems have two dimensions. Ramot et al. [16] addressed two-dimensional conundrums, demonstrating complex FSs (CFSs). Both real and unreal components are present in CFS. Real and unreal parts are located in the complex membership grade [1, 0], where $\iota = \sqrt{-1}$. There is a polar shape in this CFS. Following this, a new interpretation of the complex membership grade was proposed by Tamir et al. [17]. This CFS is composed of real and unreal components that pertain to [1, 0] and is in Cartesian form. Additionally, their sum contains [1, 0]. An outline of the theory and uses of CFSs and CF logic were indicated by Rishe et al. [18]. CF AAOs and GAOs were started by Bi et al. [19], [20]. Complex hesitant FSs (CHFSs) and their applications to DM with various and creative distance measures (DM) were denoted by Garg et al. [21]. An innovative method for complex dual hesitant FSs (CDHFSs) and their applications in pattern recognition (PR) and medical diagnosis (MD) were started by Rehman et al. [22]. The DM algorithm and new CF N-soft sets were introduced by Mahmood et al. [23]. Additionally, under the influence of BCFSs, Mahmood and Rehman [24], [25] developed DAOs and generalized similarity measures (GSMs). The categorization of renewable energy and its sources using a DM technique based on BCF frank power AOs was indicated by Naeem et al. [26]. Mahmood et al. [27] also looked into BCFHAOs and how they could be used in MADM.

The soft set (SS), which plays a crucial role in resolving ambiguity and vagueness concerns, is one of the greatest achievements after FS. Due to requirements of dilemmas, Molodtsov [28] initiated the first results of the SS theory. Due to SS all dilemmas have been handled easily and fruitfully. Maji et al. [29] demonstrated some operations in SS. Ali et al. [30] signified some novel operations in SS theory. By defining the operations of SS is so easy for extension in SS. Therefore most of the experts are started work on the SS. Babitha and Sunil [31] initiated SS relations and mappings. Herawan and Deris [32] demonstrated SS technique for related conditions mining. Moreover, SS can sort out all dilemmas which are associated with SS, although many of them have drawbacks. Therefore, the significance of bipolar SS (BSS) is fruitful. For this Mahmood [33] presented a novel technique of BSS and their applications. There is a new notion developed by combining FS and SS. This work has been done by Roy and Maji [34] and created fuzzy soft (FSS). For more extension Abdullah et al. [35] demonstrated bipolar FSS (BFSS) and its applications in DM dilemmas. There were facing one-dimensional FSS but Thirunavukarasu et al. [36] remove these dilemmas and propound by adding complex to FSS and created complex FSS (CFSS) with applications. Alcantud [37] signified FSS in the basis of DM. Selvachandran and Singh [38] designed interval-valued CFSS with application. FSS was consist of one dimension but by adding complex become 2-dimensional. Now, the work is not running by only 2-dimensional and required negative aspect. For this, Mahmood et al. [39] initiated bipolar CFSSs (BCFSSs) and DM applications. In this article, experts defined all operational laws for extension of this idea. Keeping in mind the above idea Mahmood et al. [40] designed PR and MD based on trigonometric SMs for BCFSS. On this idea Jaleel [41] used WASPAS approach on the DAOs under the effect of BCFSS for Agricultural robot technology. This publication aims to identify the most influential robot selection in the AI field by first deducing AOs under the BCFSS environment, which are BCFSPDGA and BCFSPDAA operators. Additionally, this pertains to the power aggregation operator (PAO). The main advantage of the inferred operators is that the assistance parameterization aspect allows them to mimic real-world problems. The supplied data, as described by the experts or decision-makers, is aggregated by AOs to produce a single value used to rank the alternatives. The score values are then obtained by using the scoring function. As far as we can tell, the literature on BCF soft power Dombi aggregation information is lacking; research on BCFSS has up to this point only addressed its basic theory and applications. As such, it is a new conundrum with potential for future establishment in decision science. The DM conundrums that have been mentioned above in the context of BCFSS under AOs sufficiently motivate us to write this manuscript. This manuscript's main objective is to build specific AOs in the context of BCFSS, build a DM approach based on the inferred operators, and then identify the robot that has the greatest influence in the AI-field. The discussion above also led us to the conclusion that different authors have discussed various kinds of robots and that they serve a very important purpose in AI. Furthermore, we deduced from the preceding discussion that BCFSSs are a more advanced and effective structure than the theories mentioned above since they manage the challenging and intricate data associated with the second dimension of robots in the AI-field. It therefore clearly fulfils and generates remarkable outcomes from other notions.

The article has been constructed as follows: Basic definitions were introduced in section II. Meanwhile, a few BCFSS operations were defined in Section III. Subsequently, section IV dealt with Dombi operators for BCFSNs. Bipolar complex fuzzy soft power dombi aggregation operators are covered in section V. The application of MADM with a numerical example is covered in section VI. We performed a comparative analysis of prevalent concepts of ascendancy and hegemony in section VII. We reached a conclusion regarding the advantages in section 8.

I. PRELIMINARIES

This section consists of some prevailing basic definitions and their properties which we want to discuss.

Definition 1 ([1]): A FS $\mathfrak{B}$ in the form of $\mathfrak{B} = \left\{\left(\bar{\mathfrak{r}}, \mathfrak{y}_{\bar{\mathfrak{r}}}\right), \forall \bar{\mathfrak{r}} \in \tilde{\mho}\right\}$ on a fixed set $\tilde{\mho}$, where, $\mathfrak{y}_{\mathfrak{B}} : \tilde{\mho} \to [\mathbf{0}, \mathbf{1}]$ signifies the TG of each element $\bar{\mathfrak{r}} \in \tilde{\mho}$.

Definition 2 ([10]): A BFS $\mathfrak{B}$ over $\tilde{\mho}$ is of the form $\mathfrak{B} = \left\{\left(\bar{\mathfrak{r}}, \mathfrak{y}^+_{\mathfrak{B}}, \mathfrak{y}^-_{\mathfrak{B}}\right), \bar{\mathfrak{r}} \in \tilde{\mho}\right\}$, where $\mathfrak{y}^+_{\mathfrak{B}} : \tilde{\mho} \to [\mathbf{0}, \mathbf{1}]$, $\mathfrak{y}^-_{\mathfrak{B}} : \tilde{\mho} \to [-\mathbf{1}, \mathbf{0}]$ are the PTG and NTG.

Definition 3 ([16]): A CFS $\mathfrak{B}$ in the form of $\mathfrak{B} = \left\{\left(\bar{\mathfrak{r}}, \mathfrak{y}_{\mathfrak{B}}\right), \bar{\mathfrak{r}} \in \tilde{\mho}\right\} = \left\{\bar{\mathfrak{r}}, \mathrm{P}_{\mathfrak{B}} + \iota \mathrm{P}_{\mathfrak{B}}, \bar{\mathfrak{r}} \in \tilde{\mho}\right\}$ where $\mathfrak{y}_{\mathfrak{B}}$ is Complex TG and $\mathrm{P}_{\mathfrak{B}}, \mathrm{P}_{\mathfrak{B}} \in [\mathbf{0}, \mathbf{1}]$, and $\iota = \sqrt{-\mathbf{1}}$.

Definition 4 ([24]): A BCFS $\mathfrak{B}$ is of the form $\mathfrak{B} = \left\{\left(\bar{\mathfrak{r}}, \mathfrak{y}^+_{\mathfrak{B}}, \mathfrak{z}^-_{\mathfrak{B}}\right), \bar{\mathfrak{r}} \in \tilde{\mho}\right\}$ where $\mathfrak{y}^+_{\mathfrak{B}} = \mathrm{P}^+_{\mathfrak{B}} + \iota \mathrm{P}^+_{\mathfrak{B}}$ indicate the PTG and $\mathfrak{z}^-_{\mathfrak{B}} = \mathrm{P}^-_{\mathfrak{B}} + \iota \mathrm{P}^-_{\mathfrak{B}}$ indicate the NTG. The values of $\mathfrak{y}^+_{\mathfrak{B}}$ and $\mathfrak{z}^-_{\mathfrak{B}}$ can be take which is lies the unit square of complex plane and $\mathrm{P}^+_{\mathfrak{B}}, \mathrm{P}^+_{\mathfrak{B}} \in [\mathbf{0}, \mathbf{1}]$ and $\mathrm{P}^-_{\mathfrak{B}}, \mathrm{P}^-_{\mathfrak{B}} \in [-\mathbf{1}, \mathbf{0}]$.

Definition 5 ([28]): Suppose $\tilde{\mho}$ is a fixed set, $\boldsymbol{\Theta}$ is a set of attribute, $\mathfrak{e} \subset \boldsymbol{\Theta}$, then the pair $(\mathfrak{B}, \mathfrak{e})$ is known as SS, where $\mathfrak{B} : \mathfrak{e} \to \mathfrak{B}\left(\tilde{\mho}\right)$, $\mathfrak{B}\left(\tilde{\mho}\right)$ is the power set of $\tilde{\mho}$.

Definition 6 ([39]): Suppose $\tilde{\mho}$ is a fixed set, $\boldsymbol{\Theta}$ is a set of attribute, $\mathfrak{e} \subset \boldsymbol{\Theta}$, then the pair $(\mathfrak{B}, \mathfrak{e})$ is known as bipolar complex fuzzy soft set (BCFSS) over $\tilde{\mho}$, where $\mathfrak{B} : \mathfrak{e} \to$ **BCFS** $\left(\tilde{\mho}\right)$, ***BCFS*** $\left(\tilde{\mho}\right)$ is the collection of all BCFSs of $\tilde{\mho}$. It is demonstrated as

$$
\begin{aligned}
(\mathfrak{B}, \mathfrak{e}) = \mathfrak{B}\left(\mathfrak{e}_{\mathfrak{l}}\right) &= \left\{\left(\bar{\mathfrak{r}}_{\mathfrak{k}}, \mathfrak{y}^+_{\mathfrak{B}}, \mathfrak{z}^-_{\mathfrak{B}}\right) | \forall \bar{\mathfrak{r}}_{\mathfrak{k}} \in \tilde{\mho}, \forall \mathfrak{e}_{\mathfrak{l}} \in \Theta\right\} \\
&= \left\{\left(\bar{\mathfrak{r}}_{\mathfrak{k}}, \mathrm{P}^+_{\mathfrak{B}} + \iota \mathrm{P}^+_{\mathfrak{B}}, \mathrm{P}^-_{\mathfrak{B}} + \iota \mathrm{P}^-_{\mathfrak{B}}\right) | \forall \bar{\mathfrak{r}}_{\mathfrak{k}} \in \tilde{\mho}, \forall \mathfrak{e}_{\mathfrak{l}} \in \Theta\right\} \quad (1)
\end{aligned}
$$

For easiness in this article, we employ $(\mathfrak{B}, \mathfrak{e}) = \mathfrak{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = \left(\mathfrak{y}^+_{\mathfrak{k}\mathfrak{l}}, \mathfrak{z}^-_{\mathfrak{k}\mathfrak{l}}\right) = \left(\mathrm{P}^+_{\mathfrak{k}\mathfrak{l}} + \iota \mathrm{P}^+_{\mathfrak{k}\mathfrak{l}}, \mathrm{P}^-_{\mathfrak{k}\mathfrak{l}} + \iota \mathrm{P}^-_{\mathfrak{k}\mathfrak{l}}\right)$ $(\mathfrak{k} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \mathfrak{P}; \mathfrak{l} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \aleph)$ as BCFSS. Here we have $\mathfrak{H} = \mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_{\mathfrak{P}}$ be the weight vector (WV) of experts $\bar{\mathfrak{r}}_{\mathfrak{k}}$ and $\omega = \omega_1, \omega_2, \ldots, \omega_{\aleph}$ be the WV of attributes $\mathfrak{e}_{\mathfrak{l}}$ holding $\mathfrak{H}_{\mathfrak{k}} \geq \mathbf{0}, \omega_{\mathfrak{l}} \geq \mathbf{0}$ such that $\sum_{\mathfrak{k}=\mathbf{1}}^{\mathfrak{P}} \mathfrak{H}_{\mathfrak{k}} = \mathbf{1}$ and $\sum_{\mathfrak{l}=\mathbf{1}}^{\aleph} \omega_{\mathfrak{l}} = \mathbf{1}$.

II. SOME OPERATIONS ON BCFSSs

In this section, we describe score and accuracy mappings in the setting of BCFSNs. Later on, we extend some fundamental operations based on BCFSNs.

Definition 7: The score mapping of a BCFSN is

$$\tilde{S_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) : \mathfrak{e} \to BCFS\left(\tilde{\mho}\right)$$

where $(\mathfrak{B}, \mathfrak{e}) = \mathfrak{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = \left(\mathfrak{y}^+_{\mathfrak{k}\mathfrak{l}}, \mathfrak{z}^-_{\mathfrak{k}\mathfrak{l}}\right) = \left(\mathrm{P}^+_{\mathfrak{k}\mathfrak{l}} + \iota \mathrm{P}^+_{\mathfrak{k}\mathfrak{l}}, \mathrm{P}^-_{\mathfrak{k}\mathfrak{l}} + \iota \mathrm{P}^-_{\mathfrak{k}\mathfrak{l}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{P} : \mathfrak{l} = 1, 2, \ldots, \aleph)$ and expresses as

$$\tilde{S_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) = \frac{1}{4}\left(2 + \mathrm{P}^+_{\mathfrak{k}\mathfrak{l}} + \mathrm{P}^+_{\mathfrak{k}\mathfrak{l}} + \mathrm{P}^-_{\mathfrak{k}\mathfrak{l}} + \mathrm{P}^-_{\mathfrak{k}\mathfrak{l}}\right) \quad (2)$$

Definition 8: The accuracy mapping of a BCFSN is

$$\tilde{H_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) : \mathfrak{e} \to BCFS\left(\tilde{\mho}\right)$$

where $(\mathfrak{B}, \mathfrak{e}) = \mathfrak{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = \left(\mathfrak{y}^+_{\mathfrak{k}\mathfrak{l}}, \mathfrak{z}^-_{\mathfrak{k}\mathfrak{l}}\right) = \left(\mathrm{P}^+_{\mathfrak{k}\mathfrak{l}} + \iota \mathrm{P}^+_{\mathfrak{k}\mathfrak{l}}, \mathrm{P}^-_{\mathfrak{k}\mathfrak{l}} + \iota \mathrm{P}^-_{\mathfrak{k}\mathfrak{l}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{P} : \mathfrak{l} = 1, 2, \ldots, \aleph)$ and expresses as

$$\tilde{H_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) = \frac{\mathrm{P}^+_{\mathfrak{k}\mathfrak{l}} + \iota \mathrm{P}^+_{\mathfrak{k}\mathfrak{l}} + \mathrm{P}^-_{\mathfrak{k}\mathfrak{l}} + \iota \mathrm{P}^-_{\mathfrak{k}\mathfrak{l}}}{4} \quad (3)$$

It is clear that $\tilde{S_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) \in [0, 1]$ and $\tilde{H_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) \in [0, 1]$.

Definition 9: Suppose $\mathfrak{B}_{\mathfrak{e}_{11}} = \left(\mathrm{P}^+_{11} + \iota \mathrm{P}^+_{11}, \mathrm{P}^-_{11} + \iota \mathrm{P}^-_{11}\right)$ and $\mathfrak{B}_{\mathfrak{e}_{12}} = \left(\mathrm{P}^+_{12} + \iota \mathrm{P}^+_{12}, \mathrm{P}^-_{12} + \iota \mathrm{P}^-_{12}\right)$ are two BCFSNs, then we signify the demand relation as follows:

1. If $\tilde{S_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{11}}\right) < \tilde{S_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{12}}\right)$, then $\mathfrak{B}_{\mathfrak{e}_{11}} < \mathfrak{B}_{\mathfrak{e}_{12}}$
2. If $\tilde{S_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{11}}\right) > \tilde{S_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{12}}\right)$, then $\mathfrak{B}_{\mathfrak{e}_{11}} > \mathfrak{B}_{\mathfrak{e}_{12}}$
3. If $\tilde{S_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{11}}\right) = \tilde{S_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{12}}\right)$, then
   i. If $\tilde{H_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{11}}\right) < \tilde{H_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{12}}\right)$, then $\mathfrak{B}_{\mathfrak{e}_{11}} < \mathfrak{B}_{\mathfrak{e}_{12}}$
   ii. If $\tilde{H_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{11}}\right) > \tilde{H_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{12}}\right)$, then $\mathfrak{B}_{\mathfrak{e}_{11}} > \mathfrak{B}_{\mathfrak{e}_{12}}$
   iii. If $\tilde{H_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{11}}\right) = \tilde{H_{\mathfrak{B}}}\left(\mathfrak{B}_{\mathfrak{e}_{12}}\right)$, then $\mathfrak{B}_{\mathfrak{e}_{11}} = \mathfrak{B}_{\mathfrak{e}_{12}}$

Definition 10: Suppose $\mathfrak{B}_{\mathfrak{e}_{11}} = \left(\mathrm{P}^+_{11} + \iota \mathrm{P}^+_{11}, \mathrm{P}^-_{11} + \iota \mathrm{P}^-_{11}\right)$ and $\mathfrak{B}_{\mathfrak{e}_{12}} = \left(\mathrm{P}^+_{12} + \iota \mathrm{P}^+_{12}, \mathrm{P}^-_{12} + \iota \mathrm{P}^-_{12}\right)$ are two BCFSNs and $\tau > 0$, is any real number then.

1.

$$\mathfrak{B}_{\mathfrak{e}_{11}} \oplus \mathfrak{B}_{\mathfrak{e}_{12}} = \begin{pmatrix} \mathrm{P}^+_{11} + \mathrm{P}^+_{12} - \mathrm{P}^+_{11}\mathrm{P}^+_{12} + \iota\left(\mathrm{P}^+_{11} + \mathrm{P}^+_{12} - \mathrm{P}^+_{11}\mathrm{P}^+_{12}\right), \\ -\left(\mathrm{P}^-_{11}\mathrm{P}^-_{12}\right) + \iota\left(-\left(\mathrm{P}^-_{11}\mathrm{P}^-_{12}\right)\right) \end{pmatrix}$$

2.

$$\mathfrak{B}_{\mathfrak{e}_{11}} \otimes \mathfrak{B}_{\mathfrak{e}_{12}} = \begin{pmatrix} \left(\mathrm{P}^+_{11}\mathrm{P}^+_{12}\right) + \iota\left(\mathrm{P}^+_{11}\mathrm{P}^+_{12}\right), \\ \mathrm{P}^-_{11} + \mathrm{P}^-_{12} - \mathrm{P}^-_{11}\mathrm{P}^-_{12} + \iota\left(\mathrm{P}^-_{11} + \mathrm{P}^-_{12} - \mathrm{P}^-_{11}\mathrm{P}^-_{12}\right) \end{pmatrix}$$

3.

$$\tau \mathfrak{B}_{\mathfrak{e}} = \left(1 - \left(1 - \mathrm{P}^+\right)^{\tau} + \iota\left(1 - \left(1 - \mathrm{P}^+\right)^{\tau}\right), -\left|\mathrm{P}^-\right|^{\tau} + \iota\left(-\left|\mathrm{P}^-\right|^{\tau}\right)\right)$$

4.

$$\mathfrak{B}_{\mathfrak{e}}^{\tau} = \left(\left(\mathrm{P}^+\right)^{\tau} + \iota\left(\mathrm{P}^+\right)^{\tau}, -1 + \left(1 + \mathrm{P}^-\right)^{\tau}\right.$$

$$+\iota\left(-1+\left(1+\mathrm{P}^{-}\right)^{\tau}\right)\Big)$$

Theorem 1: Suppose $\mathcal{B}_{e_{11}}=\left(\mathrm{P}_{11}^{+}+\iota\mathrm{P}_{11}^{+},\mathrm{P}_{11}^{-}+\iota\mathrm{P}_{11}^{-}\right)$ and $\mathcal{B}_{e_{12}}=\left(\mathrm{P}_{12}^{+}+\iota\mathrm{P}_{12}^{+},\mathrm{P}_{12}^{-}+\iota\mathrm{P}_{12}^{-}\right)$ are two BCFSNs, and τ, τ_1, $\tau_2>0$, are real numbers, then the following holds

1. $\mathcal{B}_{e_{11}}\oplus\mathcal{B}_{e_{12}}=\mathcal{B}_{e_{12}}\oplus\mathcal{B}_{e_{11}}$
2. $\mathcal{B}_{e_{11}}\otimes\mathcal{B}_{e_{12}}=\mathcal{B}_{e_{12}}\otimes\mathcal{B}_{e_{11}}$
3. $\tau\left(\mathcal{B}_{e_{11}}\oplus\mathcal{B}_{e_{12}}\right)=\tau\mathcal{B}_{e_{11}}\oplus\tau\mathcal{B}_{e_{12}}$
4. $\left(\mathcal{B}_{e_{11}}\otimes\mathcal{B}_{e_{12}}\right)^{\tau}=\mathcal{B}_{e_{11}}^{\tau}\otimes\mathcal{B}_{e_{12}}^{\tau}$
5. $\tau_1\mathcal{B}_{e}\oplus\tau_2\mathcal{B}_{e}=\left(\tau_1+\tau_2\right)\mathcal{B}_{e}$
6. $\mathcal{B}_{e}^{\tau_1}\otimes\mathcal{B}_{e}^{\tau_2}=\mathcal{B}_{e}^{\tau_1+\tau_2}$
7. $\left(\mathcal{B}_{e}^{\tau_1}\right)^{\tau_2}=\mathcal{B}_{e}^{\tau_1\tau_2}$

Proof: Trivial.

Definition 11: Suppose $\mathcal{B}_{e_{11}}=\left(\mathrm{P}_{11}^{+}+\iota\mathrm{P}_{11}^{+},\mathrm{P}_{11}^{-}+\iota\mathrm{P}_{11}^{-}\right)$ and $\mathcal{B}_{e_{12}}=\left(\mathrm{P}_{12}^{+}+\iota\mathrm{P}_{12}^{+},\mathrm{P}_{12}^{-}+\iota\mathrm{P}_{12}^{-}\right)$ are two BCFSNs, then the distance between two BCFSNs is defined by

$$d\left(\mathcal{B}_{e_{11}},\mathcal{B}_{e_{12}}\right)=\frac{1}{4}\left(\left|\mathrm{P}_{11}^{+}-\mathrm{P}_{12}^{+}\right|+\left|\mathrm{P}_{11}^{+}-\mathrm{P}_{12}^{+}\right|+\left|\mathrm{P}_{11}^{-}-\mathrm{P}_{12}^{-}\right|+\left|\mathrm{P}_{11}^{-}-\mathrm{P}_{12}^{-}\right|\right)\tag{4}$$

Definition 12: For the collection of $\left(\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_{Ъ}\right)$ BCFSNs, then PAO is determined as

$$PA\left(\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_{Ъ}\right)=\oplus_{\mathrm{I}=1}^{Ъ}\frac{\left(1+\mathbb{T}\left(\mathcal{B}_{\mathrm{I}}\right)\right)}{\sum_{\mathrm{I}=1}^{Ъ}\left(1+\mathbb{T}\left(\mathcal{B}_{\mathrm{I}}\right)\right)}\tag{5}$$

where $\mathbb{T}\left(\mathcal{B}_{\mathrm{I}}\right)=\sum_{\substack{\wp=1,\\ \wp\neq\mathrm{I}}}^{Ъ}Sup\left(\mathcal{B}_{Ъ},\mathcal{B}_{\aleph}\right)$ and $Sup\left(\mathcal{B}_{Ъ},\mathcal{B}_{\aleph}\right)=1-d\left(\mathcal{B}_{Ъ},\mathcal{B}_{\aleph}\right)$ demonstrate the support among $\mathcal{B}_{Ъ}$ and $\mathcal{B}_{\aleph}$ with following properties

1. $Sup\left(\mathcal{B}_{Ъ},\mathcal{B}_{\aleph}\right)\in[0,1]$
2. $Sup\left(\mathcal{B}_{Ъ},\mathcal{B}_{\aleph}\right)=Sup\left(\mathcal{B}_{\aleph},\mathcal{B}_{Ъ}\right)$
3. $Sup\left(\mathcal{B}_{Ъ},\mathcal{B}_{\aleph}\right)\geq Sup\left(\mathcal{B}_{\mathbb{p}},\mathcal{B}_{\mathbb{q}}\right)$ if $d\left(\mathcal{B}_{Ъ},\mathcal{B}_{\aleph}\right)<d\left(\mathcal{B}_{\mathbb{p}},\mathcal{B}_{\mathbb{q}}\right)$, d is any distance measure among them.

III. DOMBI OPERATORS ON BCFSNs

In this Section, we will signify the DOs in Sect IV-A with operation on BCFSNs.

A. DOMBI OPERATION

Dombi understood operations Dombi product and sum which are particular cases of t-norms and t-co-norms provided underneath.

Definition 13: Suppose $\mathrm{\bar{r}}_1$, $\mathrm{\bar{r}}_2$ are two real numbers, formerly the Dombi t-norms and t-conorms are specified as follows.

$$Dom\left(\mathrm{\bar{r}}_1,\mathrm{\bar{r}}_2\right)=\frac{1}{1+\left\{\left(\frac{1-\mathrm{\bar{r}}_1}{\mathrm{\bar{r}}_1}\right)^{o}+\left(\frac{1-\mathrm{\bar{r}}_2}{\mathrm{\bar{r}}_2}\right)^{o}\right\}^{\frac{1}{o}}}\tag{6}$$

$$Dom^{*}\left(\mathrm{\bar{r}}_1,\mathrm{\bar{r}}_2\right)=1-\frac{1}{1+\left\{\left(\frac{\mathrm{\bar{r}}_1}{1-\mathrm{\bar{r}}_1}\right)^{o}+\left(\frac{\mathrm{\bar{r}}_2}{1-\mathrm{\bar{r}}_2}\right)^{o}\right\}^{\frac{1}{o}}}\tag{7}$$

where $o\geq1$ and $\left(\mathrm{\bar{r}}_1,\mathrm{\bar{r}}_2\right)\in\lfloor0,1\rfloor\times\lfloor0,1\rfloor$.

Definition 14: Suppose $\mathcal{B}_{e_{11}}=\left(\mathrm{P}_{11}^{+}+\iota\mathrm{P}_{11}^{+},\mathrm{P}_{11}^{-}+\iota\mathrm{P}_{11}^{-}\right)$ and $\mathcal{B}_{e_{12}}=\left(\mathrm{P}_{12}^{+}+\iota\mathrm{P}_{12}^{+},\mathrm{P}_{12}^{-}+\iota\mathrm{P}_{12}^{-}\right)$ are two BCFSNs, then

IV. BIPOLAR COMPLEX FUZZY SOFT POWER DOMBI AGGREGATION OPERATORS

In this section, we will design Power dombi arithmetic AOs and Power dombi geometric AOs with BCFSNs.

A. BIPOLAR COMPLEX FUZZY SOFT POWER DOMBI ARITHMETIC AGGREGATION OPERATORS

This segment of the article consists of Bipolar complex fuzzy soft (BCFS) Power dombi averaging (BCFSPDA) operator, Bipolar complex fuzzy soft (BCFS) Power dombi weighted averaging (BCFSPDWA) operator, BCFS Power dombi ordered weighted averaging (BCFSPDOWA) operator, and BCFS Power dombi hybrid averaging (BCFSPDHA) operator.

Definition 15: Suppose $\mathcal{B}_{e_{\mathrm{fI}}}=\left(\mathfrak{h}_{\mathrm{fI}}^{+},\mathfrak{Z}_{\mathrm{fI}}^{-}\right)=\left(\mathrm{P}_{\mathrm{fI}}^{+}+\iota\mathrm{P}_{\mathrm{fI}}^{+},\mathrm{P}_{\mathrm{fI}}^{-}+\iota\mathrm{P}_{\mathrm{fI}}^{-}\right)$ $(\mathrm{f}=1,2,\ldots,Ъ,\mathrm{I}=1,2,\ldots,\aleph)$ is the collection of BCFSNs, then the BCFSPDA operator is a mapping $\mathcal{B}_{e}^{n}\rightarrow\mathcal{B}_{e}$ such that

$$BCFSPDA\left(\mathcal{B}_{e_{11}},\mathcal{B}_{e_{12}},\mathcal{B}_{e_{13}},\ldots,\mathcal{B}_{e_{Ъ\aleph}}\right)=\oplus_{\mathrm{I}=1}^{\aleph}\omega_{\mathrm{I}}\left(\oplus_{\mathrm{f}=1}^{Ъ}\mathfrak{h}_{\mathrm{f}}\mathcal{B}_{e_{\mathrm{fI}}}\right)\tag{8}$$

where $\omega_{\mathrm{I}}=\left(\frac{\left(1+\mathbb{T}_{\mathrm{I}}\right)}{\sum_{\mathrm{I}=1}^{\aleph}\left(1+\mathbb{T}_{\mathrm{I}}\right)}\right)$, $\mathfrak{h}_{\mathrm{f}}=\left(\frac{\left(1+\mathbb{R}_{\mathrm{f}}\right)}{\sum_{\mathrm{f}=1}^{Ъ}\left(1+\mathbb{R}_{\mathrm{f}}\right)}\right)$, $\mathbb{R}_{\mathrm{f}}=\sum_{\substack{\mathrm{r}=1\\ \mathrm{r}\neq\mathrm{f}}}^{Ъ}Sup\left(\mathcal{B}_{e_{\mathrm{fI}}},\mathcal{B}_{e_{\mathrm{rI}}}\right)$, and $\mathbb{T}_{\mathrm{I}}=\sum_{\substack{\mathfrak{s}=1\\ \mathfrak{s}\neq\mathrm{I}}}^{\aleph}Sup\left(\mathcal{B}_{e_{\mathrm{fI}}},\mathcal{B}_{e_{\mathrm{f}\mathfrak{s}}}\right)$, $Sup\left(\mathcal{B}_{e_{\mathrm{fI}}},\mathcal{B}_{e_{\mathrm{rI}}}\right)$ denotes the support for $\mathcal{B}_{e_{\mathrm{fI}}}$ from $\mathcal{B}_{e_{\mathrm{rI}}}$.

Theorem 2: Suppose $\mathcal{B}_{e_{\mathrm{fI}}}=\left(\mathfrak{h}_{\mathrm{fI}}^{+},\mathfrak{Z}_{\mathrm{fI}}^{-}\right)=\left(\mathrm{P}_{\mathrm{fI}}^{+}+\iota\mathrm{P}_{\mathrm{fI}}^{+},\mathrm{P}_{\mathrm{fI}}^{-}+\iota\mathrm{P}_{\mathrm{fI}}^{-}\right)$ $(\mathrm{f}=1,2,\ldots,Ъ,\mathrm{I}=1,2,\ldots,\aleph)$ is the collection of BCFSNs, then by using BCFSPDA operator their calculated is again a BCFSN in (9), as shown at the bottom of page 6.

Proof: Here, the mathematical induction method is used, as follows:

$$\mathcal{B}_{e_{11}}\oplus\mathcal{B}_{e_{12}}=\left(\begin{array}{c}1-\dfrac{1}{1+\left\{\left(\frac{\mathrm{P}_{11}^{+}}{1-\mathrm{P}_{11}^{+}}\right)^{o}+\left(\frac{\mathrm{P}_{12}^{+}}{1-\mathrm{P}_{12}^{+}}\right)^{o}\right\}^{\frac{1}{o}}}\\ +\iota\left(1-\dfrac{1}{1+\left\{\left(\frac{\mathrm{P}_{11}^{+}}{1-\mathrm{P}_{11}^{+}}\right)^{o}+\left(\frac{\mathrm{P}_{12}^{+}}{1-\mathrm{P}_{12}^{+}}\right)^{o}\right\}^{\frac{1}{o}}}\right),\\ \dfrac{-1}{1+\left\{\left(\frac{1+\mathrm{P}_{11}^{-}}{\left|\mathrm{P}_{11}^{-}\right|}\right)^{o}+\left(\frac{1+\mathrm{P}_{12}^{-}}{\left|\mathrm{P}_{12}^{-}\right|}\right)^{o}\right\}^{\frac{1}{o}}}\\ +\iota\left(\dfrac{-1}{1+\left\{\left(\frac{1+\mathrm{P}_{11}^{-}}{\left|\mathrm{P}_{11}^{-}\right|}\right)^{o}+\left(\frac{1+\mathrm{P}_{12}^{-}}{\left|\mathrm{P}_{12}^{-}\right|}\right)^{o}\right\}^{\frac{1}{o}}}\right)\end{array}\right)$$

First, we prove that the result accessible for $\text{Ꝉ} = 2$ and $\text{ꝝ} = 2$, as follows:

$BCFSPDA\left(\mathcal{B}_{\underset{\sim}{e}_{11}}, \mathcal{B}_{\underset{\sim}{e}_{12}}\right)$

$$= \oplus_{\text{I}=1}^{2} \text{Ꝏ}_{\text{I}} \left(\oplus_{\text{ꝉ}=1}^{2} \text{Ƕ}_{\text{ꝉ}} \mathcal{B}_{\underset{\sim}{e}_{\text{ꝉI}}}\right)$$
$$= \text{Ꝏ}_1 \left(\text{Ƕ}_1 \mathcal{B}_{\underset{\sim}{e}_{11}} \oplus \text{Ƕ}_2 \mathcal{B}_{\underset{\sim}{e}_{21}}\right) \oplus \text{Ꝏ}_2 \left(\text{Ƕ}_1 \mathcal{B}_{\underset{\sim}{e}_{12}} \oplus \text{Ƕ}_2 \mathcal{B}_{\underset{\sim}{e}_{22}}\right)$$

1.

$$\mathcal{B}_{\underset{\sim}{e}_{11}} \oplus \mathcal{B}_{\underset{\sim}{e}_{12}} = \left(\begin{array}{c} 1 - \frac{1}{1+\left\{\left(\frac{\text{₱}^{+}_{11}}{1-\text{₱}^{+}_{11}}\right)^{o}+\left(\frac{\text{₱}^{+}_{12}}{1-\text{₱}^{+}_{12}}\right)^{o}\right\}^{\frac{1}{o}}} + \\ \iota\left(1 - \frac{1}{1+\left\{\left(\frac{\text{Ꝑ}^{+}_{11}}{1-\text{Ꝑ}^{+}_{11}}\right)^{o}+\left(\frac{\text{Ꝑ}^{+}_{12}}{1-\text{Ꝑ}^{+}_{12}}\right)^{o}\right\}^{\frac{1}{o}}}\right), \\ \frac{-1}{1+\left\{\left(\frac{1+\text{₱}^{-}_{11}}{\left|\text{₱}^{-}_{11}\right|}\right)^{o}+\left(\frac{1+\text{₱}^{-}_{12}}{\left|\text{₱}^{-}_{12}\right|}\right)^{o}\right\}^{\frac{1}{o}}} + \iota\left(\frac{-1}{1+\left\{\left(\frac{1+\text{Ꝑ}^{-}_{11}}{\left|\text{Ꝑ}^{-}_{11}\right|}\right)^{o}+\left(\frac{1+\text{Ꝑ}^{-}_{12}}{\left|\text{Ꝑ}^{-}_{12}\right|}\right)^{o}\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

2.

$$\mathcal{B}_{\underset{\sim}{e}_{11}} \otimes \mathcal{B}_{\underset{\sim}{e}_{12}} = \left(\begin{array}{c} \frac{1}{1+\left\{\left(\frac{1-\text{₱}^{+}_{11}}{\text{₱}^{+}_{11}}\right)^{o}+\left(\frac{1-\text{₱}^{+}_{12}}{\text{₱}^{+}_{12}}\right)^{o}\right\}^{\frac{1}{o}}} + \iota\left(\frac{1}{1+\left\{\left(\frac{1-\text{Ꝑ}^{+}_{11}}{\text{Ꝑ}^{+}_{11}}\right)^{o}+\left(\frac{1-\text{Ꝑ}^{+}_{12}}{\text{Ꝑ}^{+}_{12}}\right)^{o}\right\}^{\frac{1}{o}}}\right), \\ -1 + \frac{1}{1+\left\{\left(\frac{\left|\text{₱}^{-}_{11}\right|}{1+\text{₱}^{-}_{11}}\right)^{o}+\left(\frac{\left|\text{₱}^{-}_{11}\right|}{1+\text{₱}^{-}_{11}}\right)^{o}\right\}^{\frac{1}{o}}} + \\ \iota\left(-1 + \frac{1}{1+\left\{\left(\frac{\left|\text{Ꝑ}^{-}_{11}\right|}{1+\text{Ꝑ}^{-}_{11}}\right)^{o}+\left(\frac{\left|\text{Ꝑ}^{-}_{12}\right|}{1+\text{Ꝑ}^{-}_{12}}\right)^{o}\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

3.

$$\tau\mathcal{B}_{\underset{\sim}{e}} = \left(\begin{array}{c} 1 - \frac{1}{1+\left\{\tau\left(\frac{\text{₱}^{+}}{1-\text{₱}^{+}}\right)^{o}\right\}^{\frac{1}{o}}} + \iota\left(1 - \frac{1}{1+\left\{\tau\left(\frac{\text{Ꝑ}^{+}}{1-\text{Ꝑ}^{+}}\right)^{o}\right\}^{\frac{1}{o}}}\right), \\ \frac{-1}{1+\left\{\tau\left(\frac{1+\text{₱}^{-}}{\left|\text{₱}^{-}\right|}\right)^{o}\right\}^{\frac{1}{o}}} + \iota\left(\frac{-1}{1+\left\{\tau\left(\frac{1+\text{Ꝑ}^{-}}{\left|\text{Ꝑ}^{-}\right|}\right)^{o}\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

4.

$$\mathcal{B}_{\underset{\sim}{e}}^{\tau} = \left(\begin{array}{c} \frac{1}{1+\left\{\tau\left(\frac{1-\text{₱}^{+}}{\text{₱}^{+}}\right)^{o}\right\}^{\frac{1}{o}}} + \iota\left(\frac{1}{1+\left\{\tau\left(\frac{1-\text{Ꝑ}^{+}}{\text{Ꝑ}^{+}}\right)^{o}\right\}^{\frac{1}{o}}}\right), \\ -1 + \frac{1}{1+\left\{\tau\left(\frac{\left|\text{₱}^{-}\right|}{1+\text{₱}^{-}}\right)^{o}\right\}^{\frac{1}{o}}} + \iota\left(-1 + \frac{1}{1+\left\{\tau\left(\frac{\left|\text{Ꝑ}^{-}\right|}{1+\text{Ꝑ}^{-}}\right)^{o}\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

$$
= \varpi_1 \left(\begin{array}{c}
1-\dfrac{1}{1+\left\{\hbar_1\left(\frac{\mathcal{P}^+_{11}}{1-\mathcal{P}^+_{11}}\right)^o+\hbar_2\left(\frac{\mathcal{P}^+_{21}}{1-\mathcal{P}^+_{21}}\right)^o\right\}^{\frac{1}{o}}} \\
+\iota\left(1-\dfrac{1}{1+\left\{\hbar_1\left(\frac{\mathbb{P}^+_{11}}{1-\mathbb{P}^+_{11}}\right)^o+\hbar_2\left(\frac{\mathbb{P}^+_{21}}{1-\mathbb{P}^+_{21}}\right)^o\right\}^{\frac{1}{o}}}\right), \\
\dfrac{-1}{1+\left\{\hbar_1\left(\frac{1+\mathcal{P}^-_{11}}{\left|\mathcal{P}^-_{11}\right|}\right)^o+\hbar_2\left(\frac{1+\mathcal{P}^-_{21}}{\left|\mathcal{P}^-_{21}\right|}\right)^o\right\}^{\frac{1}{o}}} \\
+\iota\left(\dfrac{-1}{1+\left\{\hbar_1\left(\frac{1+\mathbb{P}^-_{11}}{\left|\mathbb{P}^-_{11}\right|}\right)^o+\hbar_2\left(\frac{1+\mathbb{P}^-_{21}}{\left|\mathbb{P}^-_{21}\right|}\right)^o\right\}^{\frac{1}{o}}}\right)
\end{array} \right)
\oplus \varpi_2 \left(\begin{array}{c}
1-\dfrac{1}{1+\left\{\hbar_1\left(\frac{\mathcal{P}^+_{12}}{1-\mathcal{P}^+_{12}}\right)^o+\hbar_2\left(\frac{\mathcal{P}^+_{22}}{1-\mathcal{P}^+_{22}}\right)^o\right\}^{\frac{1}{o}}} \\
+\iota\left(1-\dfrac{1}{1+\left\{\hbar_1\left(\frac{\mathbb{P}^+_{12}}{1-\mathbb{P}^+_{12}}\right)^o+\hbar_2\left(\frac{\mathbb{P}^+_{22}}{1-\mathbb{P}^+_{22}}\right)^o\right\}^{\frac{1}{o}}}\right), \\
\dfrac{-1}{1+\left\{\hbar_1\left(\frac{1+\mathcal{P}^-_{12}}{\left|\mathcal{P}^-_{12}\right|}\right)^o+\hbar_2\left(\frac{1+\mathcal{P}^-_{22}}{\left|\mathcal{P}^-_{22}\right|}\right)^o\right\}^{\frac{1}{o}}} \\
+\iota\left(\dfrac{-1}{1+\left\{\hbar_1\left(\frac{1+\mathbb{P}^-_{12}}{\left|\mathbb{P}^-_{12}\right|}\right)^o+\hbar_2\left(\frac{1+\mathbb{P}^-_{22}}{\left|\mathbb{P}^-_{22}\right|}\right)^o\right\}^{\frac{1}{o}}}\right)
\end{array} \right)
$$

$$
= \varpi_1 \left(\begin{array}{c}
1-\dfrac{1}{1+\left\{\sum_{\mathfrak{t}=1}^{2}\hbar_{\mathfrak{t}}\left(\frac{\mathcal{P}^+_{\mathfrak{t}1}}{1-\mathcal{P}^+_{\mathfrak{t}1}}\right)^o\right\}^{\frac{1}{o}}} \\
+\iota\left(1-\dfrac{1}{1+\left\{\sum_{\mathfrak{t}=1}^{2}\hbar_{\mathfrak{t}}\left(\frac{\mathbb{P}^+_{\mathfrak{t}1}}{1-\mathbb{P}^+_{\mathfrak{t}1}}\right)^o\right\}^{\frac{1}{o}}}\right), \\
\dfrac{-1}{1+\left\{\sum_{\mathfrak{t}=1}^{2}\hbar_{\mathfrak{t}}\left(\frac{1+\mathcal{P}^-_{\mathfrak{t}1}}{\left|\mathcal{P}^-_{\mathfrak{t}1}\right|}\right)^o\right\}^{\frac{1}{o}}} \\
+\iota\left(\dfrac{-1}{1+\left\{\sum_{\mathfrak{t}=1}^{2}\hbar_{\mathfrak{t}}\left(\frac{1+\mathbb{P}^-_{\mathfrak{t}1}}{\left|\mathbb{P}^-_{\mathfrak{t}1}\right|}\right)^o\right\}^{\frac{1}{o}}}\right)
\end{array} \right)
\oplus \varpi_2 \left(\begin{array}{c}
1-\dfrac{1}{1+\left\{\sum_{\mathfrak{t}=1}^{2}\hbar_{\mathfrak{t}}\left(\frac{\mathcal{P}^+_{\mathfrak{t}2}}{1-\mathcal{P}^+_{\mathfrak{t}2}}\right)^o\right\}^{\frac{1}{o}}} \\
+\iota\left(1-\dfrac{1}{1+\left\{\sum_{\mathfrak{t}=1}^{2}\hbar_{\mathfrak{t}}\left(\frac{\mathbb{P}^+_{\mathfrak{t}2}}{1-\mathbb{P}^+_{\mathfrak{t}2}}\right)^o\right\}^{\frac{1}{o}}}\right), \\
\dfrac{-1}{1+\left\{\sum_{\mathfrak{t}=1}^{2}\hbar_{\mathfrak{t}}\left(\frac{1+\mathcal{P}^-_{\mathfrak{t}2}}{\left|\mathcal{P}^-_{\mathfrak{t}2}\right|}\right)^o\right\}^{\frac{1}{o}}} \\
+\iota\left(\dfrac{-1}{1+\left\{\sum_{\mathfrak{t}=1}^{2}\hbar_{\mathfrak{t}}\left(\frac{1+\mathbb{P}^-_{\mathfrak{t}2}}{\left|\mathbb{P}^-_{\mathfrak{t}2}\right|}\right)^o\right\}^{\frac{1}{o}}}\right)
\end{array} \right)
$$

$$
\begin{aligned}
&BCFSPDA_{\varpi}\left(\mathcal{B}e_{11},\mathcal{B}e_{12},\mathcal{B}_{13},\ldots,\mathcal{B}e_{\mathfrak{P}\vartheta}\right) \\
&= \oplus_{\mathfrak{l}=1}^{\vartheta}\varpi_{\mathfrak{l}}\left(\oplus_{\mathfrak{t}=1}^{\mathfrak{P}}\hbar_{\mathfrak{t}}\mathcal{B}e_{\mathfrak{t}\mathfrak{l}}\right) \\
&= \left(\begin{array}{c}
1-\dfrac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\vartheta}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{t}=1}^{\mathfrak{P}}\hbar_{\mathfrak{t}}\left(\frac{\mathcal{P}^+_{\mathfrak{t}\mathfrak{l}}}{1-\mathcal{P}^+_{\mathfrak{t}\mathfrak{l}}}\right)^o\right)\right\}^{\frac{1}{o}}} \\
+\iota\left(1-\dfrac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\vartheta}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{t}=1}^{\mathfrak{P}}\hbar_{\mathfrak{t}}\left(\frac{\mathbb{P}^+_{\mathfrak{t}\mathfrak{l}}}{1-\mathbb{P}^+_{\mathfrak{t}\mathfrak{l}}}\right)^o\right)\right\}^{\frac{1}{o}}}\right), \\
\dfrac{-1}{1+\left\{\sum_{\mathfrak{l}=1}^{\vartheta}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{t}=1}^{\mathfrak{P}}\hbar_{\mathfrak{t}}\left(\frac{1+\mathcal{P}^-_{\mathfrak{t}\mathfrak{l}}}{\left|\mathcal{P}^-_{\mathfrak{t}\mathfrak{l}}\right|}\right)^o\right)\right\}^{\frac{1}{o}}}+\iota\left(\dfrac{-1}{1+\left\{\sum_{\mathfrak{l}=1}^{\vartheta}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{t}=1}^{\mathfrak{P}}\hbar_{\mathfrak{t}}\left(\frac{1+\mathbb{P}^-_{\mathfrak{t}\mathfrak{l}}}{\left|\mathbb{P}^-_{\mathfrak{t}\mathfrak{l}}\right|}\right)^o\right)\right\}^{\frac{1}{o}}}\right)
\end{array} \right)
\end{aligned} \tag{9}
$$

$$
= \begin{pmatrix}
1-\dfrac{1}{1+\left\{\begin{matrix}\omega_1\left(\sum_{\mathfrak{t}=1}^{2}\mathfrak{h}_{\mathfrak{t}}\left(\frac{\mathbb{P}^{+}_{\mathfrak{t}1}}{1-\mathbb{P}^{+}_{\mathfrak{t}1}}\right)^{o}\right)\\ +\omega_2\left(\sum_{\mathfrak{t}=1}^{2}\mathfrak{h}_{\mathfrak{t}}\left(\frac{\mathbb{P}^{+}_{\mathfrak{t}2}}{1-\mathbb{P}^{+}_{\mathfrak{t}2}}\right)^{o}\right)\end{matrix}\right\}^{\frac{1}{o}}} \\
+\iota\left(1-\dfrac{1}{1+\left\{\begin{matrix}\omega_1\left(\sum_{\mathfrak{t}=1}^{2}\mathfrak{h}_{\mathfrak{t}}\left(\frac{\mathcal{P}^{+}_{\mathfrak{t}1}}{1-\mathcal{P}^{+}_{\mathfrak{t}1}}\right)^{o}\right)\\ +\omega_2\left(\sum_{\mathfrak{t}=1}^{2}\mathfrak{h}_{\mathfrak{t}}\left(\frac{\mathcal{P}^{+}_{\mathfrak{t}2}}{1-\mathcal{P}^{+}_{\mathfrak{t}2}}\right)^{o}\right)\end{matrix}\right\}^{\frac{1}{o}}}\right), \\
\dfrac{-1}{1+\left\{\begin{matrix}\omega_1\left(\sum_{\mathfrak{t}=1}^{2}\mathfrak{h}_{\mathfrak{t}}\left(\frac{1+\mathbb{P}^{-}_{\mathfrak{t}1}}{\left|\mathbb{P}^{-}_{\mathfrak{t}1}\right|}\right)^{o}\right)\\ +\omega_2\left(\sum_{\mathfrak{t}=1}^{2}\mathfrak{h}_{\mathfrak{t}}\left(\frac{1+\mathbb{P}^{-}_{\mathfrak{t}2}}{\left|\mathbb{P}^{-}_{\mathfrak{t}2}\right|}\right)^{o}\right)\end{matrix}\right\}^{\frac{1}{o}}} \\
+\iota\left(\dfrac{-1}{1+\left\{\begin{matrix}\omega_1\left(\sum_{\mathfrak{t}=1}^{2}\mathfrak{h}_{\mathfrak{t}}\left(\frac{1+\mathcal{P}^{-}_{\mathfrak{t}1}}{\left|\mathcal{P}^{-}_{\mathfrak{t}1}\right|}\right)^{o}\right)\\ +\omega_2\left(\sum_{\mathfrak{t}=1}^{2}\mathfrak{h}_{\mathfrak{t}}\left(\frac{1+\mathcal{P}^{-}_{\mathfrak{t}2}}{\left|\mathcal{P}^{-}_{\mathfrak{t}2}\right|}\right)^{o}\right)\end{matrix}\right\}^{\frac{1}{o}}}\right)
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1-\dfrac{1}{1+\left\{\sum_{\mathfrak{I}=1}^{2}\omega_{\mathfrak{I}}\left(\sum_{\mathfrak{t}=1}^{2}\mathfrak{h}_{\mathfrak{t}}\left(\frac{\mathbb{P}^{+}_{\mathfrak{tI}}}{1-\mathbb{P}^{+}_{\mathfrak{tI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\
+\iota\left(1-\dfrac{1}{1+\left\{\sum_{\mathfrak{I}=1}^{2}\omega_{\mathfrak{I}}\left(\sum_{\mathfrak{t}=1}^{2}\mathfrak{h}_{\mathfrak{t}}\left(\frac{\mathcal{P}^{+}_{\mathfrak{tI}}}{1-\mathcal{P}^{+}_{\mathfrak{tI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\
\dfrac{-1}{1+\left\{\sum_{\mathfrak{I}=1}^{2}\omega_{\mathfrak{I}}\left(\sum_{\mathfrak{t}=1}^{2}\mathfrak{h}_{\mathfrak{t}}\left(\frac{1+\mathbb{P}^{-}_{\mathfrak{tI}}}{\left|\mathbb{P}^{-}_{\mathfrak{tI}}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\
+\iota\left(\dfrac{-1}{1+\left\{\sum_{\mathfrak{I}=1}^{2}\omega_{\mathfrak{I}}\left(\sum_{\mathfrak{t}=1}^{2}\mathfrak{h}_{\mathfrak{t}}\left(\frac{1+\mathcal{P}^{-}_{\mathfrak{tI}}}{\left|\mathcal{P}^{-}_{\mathfrak{tI}}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right)
\end{pmatrix}
$$

Therefore, this result is true for $\mathfrak{L} = 2$ and $\aleph = 2$.

Now, this result is valid for $\mathfrak{L} = \wp_1$ and $\aleph = \wp_2$, as follows:

$$
BCFSPDA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{e_{\wp_1\wp_2}}\right)
= \oplus_{\mathfrak{I}=1}^{\wp_2}\omega_{\mathfrak{I}}\left(\oplus_{\mathfrak{t}=1}^{\wp_1}\mathfrak{h}_{\mathfrak{t}}\mathcal{B}_{e_{\mathfrak{tI}}}\right)
$$

$$
= \begin{pmatrix}
1-\dfrac{1}{1+\left\{\sum_{\mathfrak{I}=1}^{\wp_2}\omega_{\mathfrak{I}}\left(\sum_{\mathfrak{t}=1}^{\wp_1}\mathfrak{h}_{\mathfrak{t}}\left(\frac{\mathbb{P}^{+}_{\mathfrak{tI}}}{1-\mathbb{P}^{+}_{\mathfrak{tI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\
+\iota\left(1-\dfrac{1}{1+\left\{\sum_{\mathfrak{I}=1}^{\wp_2}\omega_{\mathfrak{I}}\left(\sum_{\mathfrak{t}=1}^{\wp_1}\mathfrak{h}_{\mathfrak{t}}\left(\frac{\mathcal{P}^{+}_{\mathfrak{tI}}}{1-\mathcal{P}^{+}_{\mathfrak{tI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\
\dfrac{-1}{1+\left\{\sum_{\mathfrak{I}=1}^{\wp_2}\omega_{\mathfrak{I}}\left(\sum_{\mathfrak{t}=1}^{\wp_1}\mathfrak{h}_{\mathfrak{t}}\left(\frac{1+\mathbb{P}^{-}_{\mathfrak{tI}}}{\left|\mathbb{P}^{-}_{\mathfrak{tI}}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\
+\iota\left(\dfrac{-1}{1+\left\{\sum_{\mathfrak{I}=1}^{\wp_2}\omega_{\mathfrak{I}}\left(\sum_{\mathfrak{t}=1}^{\wp_1}\mathfrak{h}_{\mathfrak{t}}\left(\frac{1+\mathcal{P}^{-}_{\mathfrak{tI}}}{\left|\mathcal{P}^{-}_{\mathfrak{tI}}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right)
\end{pmatrix}
$$

Next, we consider this result is valid for $\mathfrak{L} = \wp_1 + 1$ and $\aleph = \wp_2 + 1$, as follows:

$$
BCFSPDA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{e_{\wp_1\wp_2}}, \mathcal{B}_{e_{(\wp_1+1)(\wp_2+1)}}\right)
= \oplus_{\mathfrak{I}=1}^{\wp_2}\omega_{\mathfrak{I}}\left(\oplus_{\mathfrak{t}=1}^{\wp_1}\mathfrak{h}_{\mathfrak{t}}\mathcal{B}_{e_{\mathfrak{tI}}}\right)\oplus\left(\omega_{\wp_2+1}\left(\mathfrak{h}_{\wp_1+1}\mathcal{B}_{e_{(\wp_1+1)(\wp_2+1)}}\right)\right)
$$

$$
= \begin{pmatrix}
1-\dfrac{1}{1+\left\{\sum_{\mathfrak{I}=1}^{\wp_2}\omega_{\mathfrak{I}}\left(\sum_{\mathfrak{t}=1}^{\wp_1}\mathfrak{h}_{\mathfrak{t}}\left(\frac{\mathbb{P}^{+}_{\mathfrak{tI}}}{1-\mathbb{P}^{+}_{\mathfrak{tI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\
+\iota\left(1-\dfrac{1}{1+\left\{\sum_{\mathfrak{I}=1}^{\wp_2}\omega_{\mathfrak{I}}\left(\sum_{\mathfrak{t}=1}^{\wp_1}\mathfrak{h}_{\mathfrak{t}}\left(\frac{\mathcal{P}^{+}_{\mathfrak{tI}}}{1-\mathcal{P}^{+}_{\mathfrak{tI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\
\dfrac{-1}{1+\left\{\sum_{\mathfrak{I}=1}^{\wp_2}\omega_{\mathfrak{I}}\left(\sum_{\mathfrak{t}=1}^{\wp_1}\mathfrak{h}_{\mathfrak{t}}\left(\frac{1+\mathbb{P}^{-}_{\mathfrak{tI}}}{\left|\mathbb{P}^{-}_{\mathfrak{tI}}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\
+\iota\left(\dfrac{-1}{1+\left\{\sum_{\mathfrak{I}=1}^{\wp_2}\omega_{\mathfrak{I}}\left(\sum_{\mathfrak{t}=1}^{\wp_1}\mathfrak{h}_{\mathfrak{t}}\left(\frac{1+\mathcal{P}^{-}_{\mathfrak{tI}}}{\left|\mathcal{P}^{-}_{\mathfrak{tI}}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right)
\end{pmatrix}
\oplus
\begin{pmatrix}
1-\dfrac{1}{1+\left\{\omega_{\wp_2+1}\left(\mathfrak{h}_{\wp_1+1}\left(\frac{\mathbb{P}^{+}_{\mathfrak{tI}}}{1-\mathbb{P}^{+}_{\mathfrak{tI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\
+\iota\left(1-\dfrac{1}{1+\left\{\omega_{\wp_2+1}\left(\mathfrak{h}_{\wp_1+1}\left(\frac{\mathcal{P}^{+}_{\mathfrak{tI}}}{1-\mathcal{P}^{+}_{\mathfrak{tI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\
\dfrac{-1}{1+\left\{\omega_{\wp_2+1}\left(\mathfrak{h}_{\wp_1+1}\left(\frac{1+\mathbb{P}^{-}_{\mathfrak{tI}}}{\left|\mathbb{P}^{-}_{\mathfrak{tI}}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\
+\iota\left(\dfrac{-1}{1+\left\{\omega_{\wp_2+1}\left(\mathfrak{h}_{\wp_1+1}\left(\frac{1+\mathcal{P}^{-}_{\mathfrak{tI}}}{\left|\mathcal{P}^{-}_{\mathfrak{tI}}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right)
\end{pmatrix}
$$

$$= \left(\begin{array}{c} 1 - \dfrac{1}{1+\left\{ \sum_{\mathfrak{l}=1}^{\wp_2} \begin{array}{c} \varpi_{\mathfrak{l}} \left(\sum_{\mathfrak{k}=1}^{\wp_1} \hbar_{\mathfrak{k}} \left(\frac{\mathfrak{p}^+_{\mathfrak{kl}}}{1-\mathfrak{p}^+_{\mathfrak{kl}}}\right)^o\right) \\ +\varpi_{\wp_2+1}\left(\hbar_{\wp_1+1}\left(\frac{\mathfrak{p}^+_{\mathfrak{kl}}}{1-\mathfrak{p}^+_{\mathfrak{kl}}}\right)^o\right)\end{array}\right\}^{\frac{1}{o}}} \\ + \iota \left(1 - \dfrac{1}{1+\left\{ \sum_{\mathfrak{l}=1}^{\wp_2} \begin{array}{c} \varpi_{\mathfrak{l}} \left(\sum_{\mathfrak{k}=1}^{\wp_1} \hbar_{\mathfrak{k}} \left(\frac{\mathfrak{P}^+_{\mathfrak{kl}}}{1-\mathfrak{P}^+_{\mathfrak{kl}}}\right)^o\right) \\ +\varpi_{\wp_2+1}\left(\hbar_{\wp_1+1}\left(\frac{\mathfrak{P}^+_{\mathfrak{kl}}}{1-\mathfrak{P}^+_{\mathfrak{kl}}}\right)^o\right)\end{array}\right\}^{\frac{1}{o}}}\right), \\ \dfrac{-1}{1+\left\{ \sum_{\mathfrak{l}=1}^{\wp_2} \begin{array}{c} \varpi_{\mathfrak{l}} \left(\sum_{\mathfrak{k}=1}^{\wp_1} \hbar_{\mathfrak{k}} \left(\frac{1+\mathfrak{p}^-_{\mathfrak{kl}}}{\left|\mathfrak{p}^-_{\mathfrak{kl}}\right|}\right)^o\right) \\ +\varpi_{\wp_2+1}\left(\hbar_{\wp_1+1}\left(\frac{1+\mathfrak{p}^-_{\mathfrak{kl}}}{\left|\mathfrak{p}^-_{\mathfrak{kl}}\right|}\right)^o\right)\end{array}\right\}^{\frac{1}{o}}} \\ +\iota\left(\dfrac{-1}{1+\left\{ \sum_{\mathfrak{l}=1}^{\wp_2} \begin{array}{c} \varpi_{\mathfrak{l}} \left(\sum_{\mathfrak{k}=1}^{\wp_1} \hbar_{\mathfrak{k}} \left(\frac{1+\mathfrak{P}^-_{\mathfrak{kl}}}{\left|\mathfrak{P}^-_{\mathfrak{kl}}\right|}\right)^o\right) \\ +\varpi_{\wp_2+1}\left(\hbar_{\wp_1+1}\left(\frac{1+\mathfrak{P}^-_{\mathfrak{kl}}}{\left|\mathfrak{P}^-_{\mathfrak{kl}}\right|}\right)^o\right)\end{array}\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

$$= \left(\begin{array}{c} 1 - \dfrac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\wp_2+1}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{\wp_1+1}\hbar_{\mathfrak{k}}\left(\frac{\mathfrak{p}^+_{\mathfrak{kl}}}{1-\mathfrak{p}^+_{\mathfrak{kl}}}\right)^o\right)\right\}^{\frac{1}{o}}} \\ +\iota\left(1 - \dfrac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\wp_2+1}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{\wp_1+1}\hbar_{\mathfrak{k}}\left(\frac{\mathfrak{P}^+_{\mathfrak{kl}}}{1-\mathfrak{P}^+_{\mathfrak{kl}}}\right)^o\right)\right\}^{\frac{1}{o}}}\right), \\ \dfrac{-1}{1+\left\{\sum_{\mathfrak{l}=1}^{\wp_2+1}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{\wp_1+1}\hbar_{\mathfrak{k}}\left(\frac{1+\mathfrak{p}^-_{\mathfrak{kl}}}{\left|\mathfrak{p}^-_{\mathfrak{kl}}\right|}\right)^o\right)\right\}^{\frac{1}{o}}} \\ +\iota\left(\dfrac{-1}{1+\left\{\sum_{\mathfrak{l}=1}^{\wp_2+1}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{\wp_1+1}\hbar_{\mathfrak{k}}\left(\frac{1+\mathfrak{P}^-_{\mathfrak{kl}}}{\left|\mathfrak{P}^-_{\mathfrak{kl}}\right|}\right)^o\right)\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

Therefore, the result is valid for $\mathfrak{b} = \wp_1 + 1$ and $\mho = \wp_2 + 1$. Hence, the result is true $\mathfrak{b}, \mho \geq 1$. It is obvious from above expression the aggregated results established BCFSPDA operator is again BCFSPDA operator.

Theorem 3: Suppose $\mathcal{B}_{e_{\mathfrak{kl}}} = \left(\mathfrak{y}^+_{\mathfrak{kl}}, \mathfrak{z}^-_{\mathfrak{kl}}\right) = \left(\mathfrak{p}^+_{\mathfrak{kl}} + \iota\mathfrak{P}^+_{\mathfrak{kl}}, \mathfrak{p}^-_{\mathfrak{kl}} + \iota\mathfrak{P}^-_{\mathfrak{kl}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{b} : \mathfrak{l} = 1, 2, \ldots, \mho)$ is the collection of BCFSNs, where $\varpi_{\mathfrak{l}} = \left(\frac{(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\mho}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\hbar_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{\mathfrak{b}}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1 \\ \mathfrak{r}\neq\mathfrak{k}}}^{\mathfrak{b}} Sup\left(\mathcal{B}_{e_{\mathfrak{kl}}}, \mathcal{B}_{e_{\mathfrak{rl}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1 \\ \mathfrak{s}\neq\mathfrak{l}}}^{\mho} Sup\left(\mathcal{B}_{e_{\mathfrak{kl}}}, \mathcal{B}_{e_{\mathfrak{ks}}}\right)$, $Sup\left(\mathcal{B}_{e_{\mathfrak{kl}}}, \mathcal{B}_{e_{\mathfrak{rl}}}\right)$ denotes the support for $\mathcal{B}_{e_{\mathfrak{kl}}}$ from $\mathcal{B}_{e_{\mathfrak{rl}}}$.

Then, *BCFSPDA* AO satisfies the following properties:

1) (Idempotency): Suppose $\mathcal{B}_{e_{\mathfrak{kl}}} = \left(\mathfrak{y}^+_{\mathfrak{kl}}, \mathfrak{z}^-_{\mathfrak{kl}}\right) = \left(\mathfrak{p}^+_{\mathfrak{kl}} + \iota\mathfrak{P}^+_{\mathfrak{kl}}, \mathfrak{p}^-_{\mathfrak{kl}} + \iota\mathfrak{P}^-_{\mathfrak{kl}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{b} : \mathfrak{l} = 1, 2, \ldots, \mho)$ is the collection of BCFSNs, and all are the same i.e. $\mathcal{B}_{e_{\mathfrak{kl}}} = \mathcal{B}_e, \forall\mathfrak{k}, \mathfrak{l}$, then the following is obtained:

$$BCFSPDA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \ldots, \mathcal{B}_{e_{\mathfrak{kl}}}\right) = \mathcal{B}_e$$

2) (Boundedness): Suppose $\mathcal{B}^-_{e_{\mathfrak{kl}}} = \left(\min_{\mathfrak{l}}\min_{\mathfrak{k}} \mathfrak{y}^+_{\mathfrak{kl}}, \max_{\mathfrak{l}}\max_{\mathfrak{k}} \mathfrak{z}^-_{\mathfrak{kl}}\right)$ and $\mathcal{B}^+_{e_{\mathfrak{kl}}} = \left(\max_{\mathfrak{l}}\max_{\mathfrak{k}} \mathfrak{y}^+_{\mathfrak{kl}}, \min_{\mathfrak{l}}\min_{\mathfrak{k}} \mathfrak{z}^-_{\mathfrak{kl}}\right)$ then,

$$\mathcal{B}^-_{e_{\mathfrak{kl}}} \leq BCFSPDA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \ldots, \mathcal{B}_{e_{\mathfrak{kl}}}\right) \leq \mathcal{B}^+_{e_{\mathfrak{kl}}}$$

3) (Monotonicity): Suppose $\mathcal{B}_{e_{\mathfrak{kl}}} = \left(\mathfrak{y}^+_{\mathfrak{kl}}, \mathfrak{z}^-_{\mathfrak{kl}}\right) = \left(\mathfrak{p}^+_{\mathfrak{kl}} + \iota\mathfrak{P}^+_{\mathfrak{kl}}, \mathfrak{p}^-_{\mathfrak{kl}} + \iota\mathfrak{P}^-_{\mathfrak{kl}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{b} : \mathfrak{l} = 1, 2, \ldots, \mho)$ and $\phi_{e_{\mathfrak{kl}}} = \left(\mathcal{R}^+_{\mathfrak{kl}}, \mathcal{S}^-_{\mathfrak{kl}}\right) = \left(\vartheta^+_{\mathfrak{kl}} + \iota\mu^+_{\mathfrak{kl}}, \vartheta^-_{\mathfrak{kl}} + \iota\mu^-_{\mathfrak{kl}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{b} : \mathfrak{l} = 1, 2, \ldots, \mho)$ are two collections of BCFSNs, then the following is obtained:

$$BCFSPDA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \ldots, \mathcal{B}_{e_{\mathfrak{kl}}}\right) \leq BCFSPDA\left(\varphi_{e_{11}}, \varphi_{e_{12}}, \ldots, \varphi_{e_{\mathfrak{kl}}}\right)$$

If $\mathcal{B}_{e_{\mathfrak{kl}}} \leq \phi_{e_{\mathfrak{kl}}}, \forall\mathfrak{k}, \mathfrak{l}$.

Definition 16: Suppose $\mathcal{B}_{e_{\mathfrak{kl}}} = \left(\mathfrak{y}^+_{\mathfrak{kl}}, \mathfrak{z}^-_{\mathfrak{kl}}\right) = \left(\mathfrak{p}^+_{\mathfrak{kl}} + \iota\mathfrak{P}^+_{\mathfrak{kl}}, \mathfrak{p}^-_{\mathfrak{kl}} + \iota\mathfrak{P}^-_{\mathfrak{kl}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{b}, \mathfrak{l} = 1, 2, \ldots, \mho)$ is the collection of BCFSNs, then the BCFSPDWA operator is a mapping $\mathcal{B}^n_e \to \mathcal{B}_e$ such that

$$BCFSPDWA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{e_{\mathfrak{b}\mho}}\right) = \oplus_{\mathfrak{l}=1}^{\mho}\varpi_{\mathfrak{l}}\left(\oplus_{\mathfrak{k}=1}^{\mathfrak{b}}\hbar_{\mathfrak{k}}\mathcal{B}_{e_{\mathfrak{kl}}}\right) \tag{10}$$

where $\varpi_{\mathfrak{l}} = \left(\frac{\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\mho}\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\hbar_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{\mathfrak{b}}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1 \\ \mathfrak{r}\neq\mathfrak{k}}}^{\mathfrak{b}} Sup\left(\mathcal{B}_{e_{\mathfrak{kl}}}, \mathcal{B}_{e_{\mathfrak{rl}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1 \\ \mathfrak{s}\neq\mathfrak{l}}}^{\mho} Sup\left(\mathcal{B}_{e_{\mathfrak{kl}}}, \mathcal{B}_{e_{\mathfrak{ks}}}\right)$, $Sup\left(\mathcal{B}_{e_{\mathfrak{kl}}}, \mathcal{B}_{e_{\mathfrak{rl}}}\right)$ denotes the support for $\mathcal{B}_{e_{\mathfrak{kl}}}$ from $\mathcal{B}_{e_{\mathfrak{rl}}}$, and $\sum_{\mathfrak{l}=1}^{\mho}\omega_{\mathfrak{l}} = 1$.

Theorem 4: Suppose $\mathcal{B}_{e_{\mathfrak{kl}}} = \left(\mathfrak{y}^+_{\mathfrak{kl}}, \mathfrak{z}^-_{\mathfrak{kl}}\right) = \left(\mathfrak{p}^+_{\mathfrak{kl}} + \iota\mathfrak{P}^+_{\mathfrak{kl}}, \mathfrak{p}^-_{\mathfrak{kl}} + \iota\mathfrak{P}^-_{\mathfrak{kl}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{b} : \mathfrak{l} = 1, 2, \ldots, \mho)$ is the collection of BCFSNs, then by using BCFSPDWA operator their calculated is again a BCFSN and

$$BCFSPDWA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{e_{\mathfrak{b}\mho}}\right)$$

$$= \oplus_{\mathfrak{l}=1}^{\aleph}\varpi_{\mathfrak{l}}\left(\oplus_{\mathfrak{k}=1}^{Ᵽ}\mathfrak{h}_{\mathfrak{k}}\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}\right)$$

$$= \left(\begin{array}{c} 1-\frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{Ᵽ}\mathfrak{h}_{\mathfrak{k}}\left(\frac{\mathfrak{p}_{\mathfrak{kl}}^{+}}{1-\mathfrak{p}_{\mathfrak{kl}}^{+}}\right)^{o}\right)\right\}^{\frac{1}{o}}}+ \\ \iota\left(1-\frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{Ᵽ}\mathfrak{h}_{\mathfrak{k}}\left(\frac{\mathbb{P}_{\mathfrak{kl}}^{+}}{1-\mathbb{P}_{\mathfrak{kl}}^{+}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\ \frac{-1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{Ᵽ}\mathfrak{h}_{\mathfrak{k}}\left(\frac{1+\mathfrak{p}_{\mathfrak{kl}}^{-}}{\left|\mathfrak{p}_{\mathfrak{kl}}^{-}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}}+ \\ \iota\left(\frac{-1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{Ᵽ}\mathfrak{h}_{\mathfrak{k}}\left(\frac{1+\mathbb{P}_{\mathfrak{kl}}^{-}}{\left|\mathbb{P}_{\mathfrak{kl}}^{-}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right) \end{array}\right) \tag{11}$$

Proof: Proof is similar to Theorem 2.

Theorem 5: Suppose $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}} = \left(\eta_{\mathfrak{kl}}^{+}, \mathfrak{z}_{\mathfrak{kl}}^{-}\right) = \left(\mathfrak{p}_{\mathfrak{kl}}^{+} + \iota\mathbb{P}_{\mathfrak{kl}}^{+}, \mathfrak{p}_{\mathfrak{kl}}^{-} + \iota\mathbb{P}_{\mathfrak{kl}}^{-}\right)$ $(\mathfrak{k} = 1, 2, \ldots, Ᵽ : \mathfrak{l} = 1, 2, \ldots, \aleph)$ is the collection of BCFSNs, where $\varpi_{\mathfrak{l}} = \left(\frac{\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\aleph}\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\mathfrak{h}_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{Ᵽ}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1\\ \mathfrak{r}\neq\mathfrak{k}}}^{Ᵽ} Sup\left(\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}, \mathcal{B}_{\underset{\sim}{e}_{\mathfrak{rl}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1\\ \mathfrak{s}\neq\mathfrak{l}}}^{\aleph} Sup\left(\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}, \mathcal{B}_{\underset{\sim}{e}_{\mathfrak{ks}}}\right)$, $Sup\left(\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}, \mathcal{B}_{\underset{\sim}{e}_{\mathfrak{rl}}}\right)$ denotes the support for $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}$ from $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{rl}}}$, and $\sum_{\mathfrak{l}=1}^{\aleph}\omega_{\mathfrak{l}} = 1$. Then, *BCFSPDWA* AO satisfies the following properties:

1) (Idempotency): Suppose $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}} = \left(\eta_{\mathfrak{kl}}^{+}, \mathfrak{z}_{\mathfrak{kl}}^{-}\right) = \left(\mathfrak{p}_{\mathfrak{kl}}^{+} + \iota\mathbb{P}_{\mathfrak{kl}}^{+}, \mathfrak{p}_{\mathfrak{kl}}^{-} + \iota\mathbb{P}_{\mathfrak{kl}}^{-}\right)$ $(\mathfrak{k} = 1, 2, \ldots, Ᵽ : \mathfrak{l} = 1, 2, \ldots, \aleph)$ is the collection of BCFSNs, and all are the same i.e. $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}} = \mathcal{B}_{\underset{\sim}{e}}, \forall\mathfrak{k}, \mathfrak{l}$, then the following is obtained:

$$BCFSPDWA\left(\mathcal{B}_{\underset{\sim}{e}_{11}}, \mathcal{B}_{\underset{\sim}{e}_{12}}, \ldots, \mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}\right) = \mathcal{B}_{\underset{\sim}{e}}$$

2) (Boundedness): Suppose $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}^{-} = \left(\min_{\mathfrak{l}}\min_{\mathfrak{k}}\eta_{\mathfrak{kl}}^{+}, \max_{\mathfrak{l}}\max_{\mathfrak{k}}\mathfrak{z}_{\mathfrak{kl}}^{-}\right)$ and $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}^{+} = \left(\max_{\mathfrak{l}}\max_{\mathfrak{k}}\eta_{\mathfrak{kl}}^{+}, \min_{\mathfrak{l}}\min_{\mathfrak{k}}\mathfrak{z}_{\mathfrak{kl}}^{-}\right)$ then,

$$\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}^{-} \leq BCFSPDWA\left(\mathcal{B}_{\underset{\sim}{e}_{11}}, \mathcal{B}_{\underset{\sim}{e}_{12}}, \ldots, \mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}\right) \leq \mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}^{+}$$

3) (Monotonicity): Suppose $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}} = \left(\eta_{\mathfrak{kl}}^{+}, \mathfrak{z}_{\mathfrak{kl}}^{-}\right) = \left(\mathfrak{p}_{\mathfrak{kl}}^{+} + \iota\mathbb{P}_{\mathfrak{kl}}^{+}, \mathfrak{p}_{\mathfrak{kl}}^{-} + \iota\mathbb{P}_{\mathfrak{kl}}^{-}\right)$ $(\mathfrak{k} = 1, 2, \ldots, Ᵽ : \mathfrak{l} = 1, 2, \ldots, \aleph)$ and $\varphi_{\underset{\sim}{e}_{\mathfrak{kl}}} = \left(\mathcal{R}_{\mathfrak{kl}}^{+}, \mathcal{S}_{\mathfrak{kl}}^{-}\right) = \left(\vartheta_{\mathfrak{kl}}^{+} + \iota\mu_{\mathfrak{kl}}^{+}, \vartheta_{\mathfrak{kl}}^{-} + \iota\mu_{\mathfrak{kl}}^{-}\right)$ $(\mathfrak{k} = 1, 2, \ldots, Ᵽ : \mathfrak{l} = 1, 2, \ldots, \aleph)$ are two collections of BCFSNs, then the following is obtained:

$$BCFSPDWA\left(\mathcal{B}_{\underset{\sim}{e}_{11}}, \mathcal{B}_{\underset{\sim}{e}_{12}}, \ldots, \mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}\right) \leq BCFSPDWA\left(\varphi_{\underset{\sim}{e}_{11}}, \varphi_{\underset{\sim}{e}_{12}}, \ldots, \varphi_{\underset{\sim}{e}_{\mathfrak{kl}}}\right)$$

If $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}} \leq \varphi_{\underset{\sim}{e}_{\mathfrak{kl}}}, \forall\mathfrak{k}, \mathfrak{l}$.

Definition 17: Suppose $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}} = \left(\eta_{\mathfrak{kl}}^{+}, \mathfrak{z}_{\mathfrak{kl}}^{-}\right) = \left(\mathfrak{p}_{\mathfrak{kl}}^{+} + \iota\mathbb{P}_{\mathfrak{kl}}^{+}, \mathfrak{p}_{\mathfrak{kl}}^{-} + \iota\mathbb{P}_{\mathfrak{kl}}^{-}\right)$ $(\mathfrak{k} = 1, 2, \ldots, Ᵽ, \mathfrak{l} = 1, 2, \ldots, \aleph)$ is the collection of BCFSNs, then the BCFSPDOWA operator is a mapping $\mathcal{B}_{\underset{\sim}{e}}^{n} \to \mathcal{B}_{\underset{\sim}{e}}$ such that

$$BCFSPDOWA\left(\mathcal{B}_{\underset{\sim}{e}_{11}}, \mathcal{B}_{\underset{\sim}{e}_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{\underset{\sim}{e}_{Ᵽ\aleph}}\right) = \oplus_{\mathfrak{l}=1}^{\aleph}\varpi_{\mathfrak{l}}\left(\oplus_{\mathfrak{k}=1}^{Ᵽ}\mathfrak{h}_{\mathfrak{k}}\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}\right) \tag{12}$$

where $\varpi_{\mathfrak{l}} = \left(\frac{\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\aleph}\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\mathfrak{h}_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{Ᵽ}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1\\ \mathfrak{r}\neq\mathfrak{k}}}^{Ᵽ} Sup\left(\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}, \mathcal{B}_{\underset{\sim}{e}_{\mathfrak{rl}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1\\ \mathfrak{s}\neq\mathfrak{l}}}^{\aleph} Sup\left(\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}, \mathcal{B}_{\underset{\sim}{e}_{\mathfrak{ks}}}\right)$, $Sup\left(\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}, \mathcal{B}_{\underset{\sim}{e}_{\mathfrak{rl}}}\right)$ denotes the support for $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}$ from $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{rl}}}$, and $\sum_{\mathfrak{l}=1}^{\aleph}\omega_{\mathfrak{l}} = 1$, and $\mathcal{B}_{\underset{\sim}{e}_{\Xi(\mathfrak{kl})}} = \left(\eta_{\Xi(\mathfrak{kl})}^{+}, \mathfrak{z}_{\Xi(\mathfrak{kl})}^{-}\right)$ is the permutation of the $\mathfrak{k}th$ row and $\mathfrak{l}th$ largest elements of the collection for $\mathfrak{k} \times \mathfrak{l}$ BCFSNs $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}} = \left(\eta_{\mathfrak{kl}}^{+}, \mathfrak{z}_{\mathfrak{kl}}^{-}\right)$ for $\mathfrak{k} = 1, 2, \ldots, Ᵽ$ *and* $\mathfrak{l} = 1, 2, \ldots, \aleph$. Now, using equation (8) we can define *BCFSPDOWA* AOs, as follows:

Theorem 6: Suppose $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}} = \left(\eta_{\mathfrak{kl}}^{+}, \mathfrak{z}_{\mathfrak{kl}}^{-}\right) = \left(\mathfrak{p}_{\mathfrak{kl}}^{+} + \iota\mathbb{P}_{\mathfrak{kl}}^{+}, \mathfrak{p}_{\mathfrak{kl}}^{-} + \iota\mathbb{P}_{\mathfrak{kl}}^{-}\right)$ $(\mathfrak{k} = 1, 2, \ldots, Ᵽ : \mathfrak{l} = 1, 2, \ldots, \aleph)$ is the collection of BCFSNs, then by using BCFSPDOWA operator their calculated is again a BCFSN and

$$BCFSPDOWA\left(\mathcal{B}_{\underset{\sim}{e}_{11}}, \mathcal{B}_{\underset{\sim}{e}_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{\underset{\sim}{e}_{Ᵽ\aleph}}\right) = \oplus_{\mathfrak{l}=1}^{\aleph}\varpi_{\mathfrak{l}}\left(\oplus_{\mathfrak{k}=1}^{Ᵽ}\mathfrak{h}_{\mathfrak{k}}\mathcal{B}_{\underset{\sim}{e}_{\Xi(\mathfrak{kl})}}\right)$$

$$= \left(\begin{array}{c} 1-\frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{Ᵽ}\mathfrak{h}_{\mathfrak{k}}\left(\frac{\mathfrak{p}_{\Xi(\mathfrak{kl})}^{+}}{1-\mathfrak{p}_{\Xi(\mathfrak{kl})}^{+}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\ +\iota\left(1-\frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{Ᵽ}\mathfrak{h}_{\mathfrak{k}}\left(\frac{\mathbb{P}_{\Xi(\mathfrak{kl})}^{+}}{1-\mathbb{P}_{\Xi(\mathfrak{kl})}^{+}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\ \frac{-1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{Ᵽ}\mathfrak{h}_{\mathfrak{k}}\left(\frac{1+\mathfrak{p}_{\Xi(\mathfrak{kl})}^{-}}{\left|\mathfrak{p}_{\Xi(\mathfrak{kl})}^{-}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\ +\iota\left(\frac{-1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph}\varpi_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{Ᵽ}\mathfrak{h}_{\mathfrak{k}}\left(\frac{1+\mathbb{P}_{\Xi(\mathfrak{kl})}^{-}}{\left|\mathbb{P}_{\Xi(\mathfrak{kl})}^{-}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right) \end{array}\right) \tag{13}$$

Proof: Proof is similar to Theorem 2.

Theorem 7: Suppose $\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}} = \left(\eta_{\mathfrak{kl}}^{+}, \mathfrak{z}_{\mathfrak{kl}}^{-}\right) = \left(\mathfrak{p}_{\mathfrak{kl}}^{+} + \iota\mathbb{P}_{\mathfrak{kl}}^{+}, \mathfrak{p}_{\mathfrak{kl}}^{-} + \iota\mathbb{P}_{\mathfrak{kl}}^{-}\right)$ $(\mathfrak{k} = 1, 2, \ldots, Ᵽ : \mathfrak{l} = 1, 2, \ldots, \aleph)$ is the collection of BCFSNs, where $\varpi_{\mathfrak{l}} = \left(\frac{\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\aleph}\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\mathfrak{h}_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{Ᵽ}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1\\ \mathfrak{r}\neq\mathfrak{k}}}^{Ᵽ} Sup\left(\mathcal{B}_{\underset{\sim}{e}_{\mathfrak{kl}}}, \mathcal{B}_{\underset{\sim}{e}_{\mathfrak{rl}}}\right)$, and

$\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1 \\ \mathfrak{s} \neq \mathfrak{l}}}^{\aleph} Sup\left(\mathcal{B}_{e_{\mathfrak{tl}}}, \mathcal{B}_{e_{\mathfrak{ts}}}\right)$, $Sup\left(\mathcal{B}_{e_{\mathfrak{tl}}}, \mathcal{B}_{e_{\mathfrak{rl}}}\right)$ denotes the support for $\mathcal{B}_{e_{\mathfrak{tl}}}$ from $\mathcal{B}_{e_{\mathfrak{rl}}}$, and $\sum_{\mathfrak{l}=1}^{\aleph} \omega_{\mathfrak{l}} = 1$. Then, *BCFSPDOWA* AO satisfies the following properties:

1) (Idempotency): Suppose $\mathcal{B}_{e_{\mathfrak{tl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{tl}}, \mathfrak{z}^{-}_{\mathfrak{tl}}\right) = \left(\mathfrak{p}^{+}_{\mathfrak{tl}} + \iota \mathfrak{P}^{+}_{\mathfrak{tl}}, \mathfrak{p}^{-}_{\mathfrak{tl}} + \iota \mathfrak{P}^{-}_{\mathfrak{tl}}\right)$ $(\mathfrak{t} = 1, 2, \ldots, \mathfrak{L} : \mathfrak{l} = 1, 2, \ldots, \aleph)$ is the collection of BCFSNs, and all are the same i.e. $\mathcal{B}_{e_{\mathfrak{tl}}} = \mathcal{B}_{e}, \forall \mathfrak{t}, \mathfrak{l}$, then the following is obtained:
$$BCFSPDOWA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \ldots, \mathcal{B}_{e_{\mathfrak{tl}}}\right) = \mathcal{B}_{e}$$
2) (Boundedness): Suppose $\mathcal{B}^{-}_{e_{\mathfrak{tl}}} = \left(\min_{\mathfrak{l}} \min_{\mathfrak{t}} \mathfrak{y}^{+}_{\mathfrak{tl}}, \max_{\mathfrak{l}} \max_{\mathfrak{t}} \mathfrak{z}^{-}_{\mathfrak{tl}}\right)$ and $\mathcal{B}^{+}_{e_{\mathfrak{tl}}} = \left(\max_{\mathfrak{l}} \max_{\mathfrak{t}} \mathfrak{y}^{+}_{\mathfrak{tl}}, \min_{\mathfrak{l}} \min_{\mathfrak{t}} \mathfrak{z}^{-}_{\mathfrak{tl}}\right)$ then,
$$\mathcal{B}^{-}_{e_{\mathfrak{tl}}} \leq BCFSPDOWA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \ldots, \mathcal{B}_{e_{\mathfrak{tl}}}\right) \leq \mathcal{B}^{+}_{e_{\mathfrak{tl}}}$$
3) (Monotonicity): Suppose $\mathcal{B}_{e_{\mathfrak{tl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{tl}}, \mathfrak{z}^{-}_{\mathfrak{tl}}\right) = \left(\mathfrak{p}^{+}_{\mathfrak{tl}} + \iota \mathfrak{P}^{+}_{\mathfrak{tl}}, \mathfrak{p}^{-}_{\mathfrak{tl}} + \iota \mathfrak{P}^{-}_{\mathfrak{tl}}\right)$ $(\mathfrak{t} = 1, 2, \ldots, \mathfrak{L} : \mathfrak{l} = 1, 2, \ldots, \aleph)$ and $\varphi_{e_{\mathfrak{tl}}} = \left(\mathcal{R}^{+}_{\mathfrak{tl}}, \mathcal{S}^{-}_{\mathfrak{tl}}\right) = \left(\vartheta^{+}_{\mathfrak{tl}} + \iota \mu^{+}_{\mathfrak{tl}}, \vartheta^{-}_{\mathfrak{tl}} + \iota \mu^{-}_{\mathfrak{tl}}\right)$ $(\mathfrak{t} = 1, 2, \ldots, \mathfrak{L} : \mathfrak{l} = 1, 2, \ldots, \aleph)$ are two collections of BCFSNs, then the following is obtained:
$$BCFSPDOWA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \ldots, \mathcal{B}_{e_{\mathfrak{tl}}}\right) \leq BCFSPDOWA\left(\varphi_{e_{11}}, \varphi_{e_{12}}, \ldots, \varphi_{e_{\mathfrak{tl}}}\right)$$
If $\mathcal{B}_{e_{\mathfrak{tl}}} \leq \varphi_{e_{\mathfrak{tl}}}, \forall \mathfrak{t}, \mathfrak{l}$.

We can see from definitions (16) and (17) that the BCFSPDWA operator solely targets BCFS values, whereas BCFSPDOWA only targets ordered positions of BCFS values rather than the weights of the BCFS values themselves. By combining the qualities of the BCFSPDWA and BCFSPDOWA, the BCFSPDHA operator is defined below.

Definition 18: Suppose $\mathcal{B}_{e_{\mathfrak{tl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{tl}}, \mathfrak{z}^{-}_{\mathfrak{tl}}\right) = \left(\mathfrak{p}^{+}_{\mathfrak{tl}} + \iota \mathfrak{P}^{+}_{\mathfrak{tl}}, \mathfrak{p}^{-}_{\mathfrak{tl}} + \iota \mathfrak{P}^{-}_{\mathfrak{tl}}\right)$ $(\mathfrak{t} = 1, 2, \ldots, \mathfrak{L}, \mathfrak{l} = 1, 2, \ldots, \aleph)$ is the collection of BCFSNs, then the BCFSPDHWA operator is a mapping $\mathcal{B}^{n}_{e} \rightarrow \mathcal{B}_{e}$ such that
$$BCFSPDHWA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{e_{\mathfrak{L}\aleph}}\right) = \oplus_{\mathfrak{l}=1}^{\aleph} \mathfrak{O}_{\mathfrak{l}} \left(\oplus_{\mathfrak{t}=1}^{\mathfrak{L}} \mathfrak{h}_{\mathfrak{t}} \mathcal{B}_{e_{\mathfrak{tl}}}\right) \tag{14}$$
where $\mathfrak{O}_{\mathfrak{l}} = \left(\frac{\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\aleph} \omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\mathfrak{h}_{\mathfrak{t}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{t}})}{\sum_{\mathfrak{t}=1}^{\mathfrak{L}} (1+\mathbb{R}_{\mathfrak{t}})}\right)$, $\mathbb{R}_{\mathfrak{t}} = \sum_{\substack{\mathfrak{r}=1 \\ \mathfrak{r} \neq \mathfrak{t}}}^{\mathfrak{L}} Sup\left(\mathcal{B}_{e_{\mathfrak{tl}}}, \mathcal{B}_{e_{\mathfrak{rl}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1 \\ \mathfrak{s} \neq \mathfrak{l}}}^{\aleph} Sup\left(\mathcal{B}_{e_{\mathfrak{tl}}}, \mathcal{B}_{e_{\mathfrak{ts}}}\right)$, $Sup\left(\mathcal{B}_{e_{\mathfrak{tl}}}, \mathcal{B}_{e_{\mathfrak{rl}}}\right)$ denotes the support for $\mathcal{B}_{e_{\mathfrak{tl}}}$ from $\mathcal{B}_{e_{\mathfrak{rl}}}$, and $\sum_{\mathfrak{l}=1}^{\aleph} \omega_{\mathfrak{l}} = 1$. Here, $\mathcal{B}_{e_{\Xi(\mathfrak{tl})}} = \left(\mathfrak{y}^{+}_{\Xi(\mathfrak{tl})}, \mathfrak{z}^{-}_{\Xi(\mathfrak{tl})}\right)$ is the permutation of the $\mathfrak{t}th$ row and $\mathfrak{l}th$ largest elements of the collection for $\mathfrak{t} \times \mathfrak{l}$ BCFSNs $\mathcal{B}_{e_{\mathfrak{tl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{tl}}, \mathfrak{z}^{-}_{\mathfrak{tl}}\right)$ for $\mathfrak{t} = 1, 2, \ldots, \mathfrak{L} and \mathfrak{l} = 1, 2, \ldots, \aleph$. Now, using equation (8) we can define *BCFSPDHWA* AOs, as follows:

Theorem 8: Suppose $\mathcal{B}_{e_{\mathfrak{tl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{tl}}, \mathfrak{z}^{-}_{\mathfrak{tl}}\right) = \left(\mathfrak{p}^{+}_{\mathfrak{tl}} + \iota \mathfrak{P}^{+}_{\mathfrak{tl}}, \mathfrak{p}^{-}_{\mathfrak{tl}} + \iota \mathfrak{P}^{-}_{\mathfrak{tl}}\right)$ $(\mathfrak{t} = 1, 2, \ldots, \mathfrak{L} : \mathfrak{l} = 1, 2, \ldots, \aleph)$ is the collection of BCFSNs, then by using BCFSPDHWA operator their calculated is again a BCFSN and
$$\begin{aligned} &BCFSPDHWA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{e_{\mathfrak{L}\aleph}}\right) \\ &= \oplus_{\mathfrak{l}=1}^{\aleph} \mathfrak{O}_{\mathfrak{l}} \left(\oplus_{\mathfrak{t}=1}^{\mathfrak{L}} \mathfrak{h}_{\mathfrak{t}} \mathcal{B}_{e_{\Xi(\mathfrak{tl})}}\right) \\ &= \begin{pmatrix} 1 - \dfrac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph} \mathfrak{O}_{\mathfrak{l}} \left(\sum_{\mathfrak{t}=1}^{\mathfrak{L}} \mathfrak{h}_{\mathfrak{t}} \left(\frac{\mathfrak{p}^{+}_{\Xi(\mathfrak{tl})}}{1-\mathfrak{p}^{+}_{\Xi(\mathfrak{tl})}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\ +\iota \left(1 - \dfrac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph} \mathfrak{O}_{\mathfrak{l}} \left(\sum_{\mathfrak{t}=1}^{\mathfrak{L}} \mathfrak{h}_{\mathfrak{t}} \left(\frac{\mathfrak{P}^{+}_{\Xi(\mathfrak{tl})}}{1-\mathfrak{P}^{+}_{\Xi(\mathfrak{tl})}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\ \dfrac{-1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph} \mathfrak{O}_{\mathfrak{l}} \left(\sum_{\mathfrak{t}=1}^{\mathfrak{L}} \mathfrak{h}_{\mathfrak{t}} \left(\frac{1+\mathfrak{p}^{-}_{\Xi(\mathfrak{tl})}}{\left|\mathfrak{p}^{-}_{\Xi(\mathfrak{tl})}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\ +\iota \left(\dfrac{-1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph} \mathfrak{O}_{\mathfrak{l}} \left(\sum_{\mathfrak{t}=1}^{\mathfrak{L}} \mathfrak{h}_{\mathfrak{t}} \left(\frac{1+\mathfrak{P}^{-}_{\Xi(\mathfrak{tl})}}{\left|\mathfrak{P}^{-}_{\Xi(\mathfrak{tl})}\right|}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right) \end{pmatrix} \end{aligned} \tag{15}$$

Proof: Proof is similar to Theorem 2.

Theorem 9: Suppose $\mathcal{B}_{e_{\mathfrak{tl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{tl}}, \mathfrak{z}^{-}_{\mathfrak{tl}}\right) = \left(\mathfrak{p}^{+}_{\mathfrak{tl}} + \iota \mathfrak{P}^{+}_{\mathfrak{tl}}, \mathfrak{p}^{-}_{\mathfrak{tl}} + \iota \mathfrak{P}^{-}_{\mathfrak{tl}}\right)$ $(\mathfrak{t} = 1, 2, \ldots, \mathfrak{L} : \mathfrak{l} = 1, 2, \ldots, \aleph)$ is the collection of BCFSNs, where $\mathfrak{O}_{\mathfrak{l}} = \left(\frac{\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\aleph} \omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\mathfrak{h}_{\mathfrak{t}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{t}})}{\sum_{\mathfrak{t}=1}^{\mathfrak{L}} (1+\mathbb{R}_{\mathfrak{t}})}\right)$, $\mathbb{R}_{\mathfrak{t}} = \sum_{\substack{\mathfrak{r}=1 \\ \mathfrak{r} \neq \mathfrak{t}}}^{\mathfrak{L}} Sup\left(\mathcal{B}_{e_{\mathfrak{tl}}}, \mathcal{B}_{e_{\mathfrak{rl}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1 \\ \mathfrak{s} \neq \mathfrak{l}}}^{\aleph} Sup\left(\mathcal{B}_{e_{\mathfrak{tl}}}, \mathcal{B}_{e_{\mathfrak{ts}}}\right)$, $Sup\left(\mathcal{B}_{e_{\mathfrak{tl}}}, \mathcal{B}_{e_{\mathfrak{rl}}}\right)$ denotes the support for $\mathcal{B}_{e_{\mathfrak{tl}}}$ from $\mathcal{B}_{e_{\mathfrak{rl}}}$, and $\sum_{\mathfrak{l}=1}^{\aleph} \omega_{\mathfrak{l}} = 1$. Then, *BCFSPDHWA* AO satisfies the following properties:

1) (Idempotency): Suppose $\mathcal{B}_{e_{\mathfrak{tl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{tl}}, \mathfrak{z}^{-}_{\mathfrak{tl}}\right) = \left(\mathfrak{p}^{+}_{\mathfrak{tl}} + \iota \mathfrak{P}^{+}_{\mathfrak{tl}}, \mathfrak{p}^{-}_{\mathfrak{tl}} + \iota \mathfrak{P}^{-}_{\mathfrak{tl}}\right)$ $(\mathfrak{t} = 1, 2, \ldots, \mathfrak{L} : \mathfrak{l} = 1, 2, \ldots, \aleph)$ is the collection of BCFSNs, and all are the same i.e. $\mathcal{B}_{e_{\mathfrak{tl}}} = \mathcal{B}_{e}, \forall \mathfrak{t}, \mathfrak{l}$, then the following is obtained:
$$BCFSPDHWA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \ldots, \mathcal{B}_{e_{\mathfrak{tl}}}\right) = \mathcal{B}_{e}$$
2) (Boundedness): Suppose $\mathcal{B}^{-}_{e_{\mathfrak{tl}}} = \left(\min_{\mathfrak{l}} \min_{\mathfrak{t}} \mathfrak{y}^{+}_{\mathfrak{tl}}, \max_{\mathfrak{l}} \max_{\mathfrak{t}} \mathfrak{z}^{-}_{\mathfrak{tl}}\right)$ and $\mathcal{B}^{+}_{e_{\mathfrak{tl}}} = \left(\max_{\mathfrak{l}} \max_{\mathfrak{t}} \mathfrak{y}^{+}_{\mathfrak{tl}}, \min_{\mathfrak{l}} \min_{\mathfrak{t}} \mathfrak{z}^{-}_{\mathfrak{tl}}\right)$ then,
$$\mathcal{B}^{-}_{e_{\mathfrak{tl}}} \leq BCFSPDHWA\left(\mathcal{B}_{e_{11}}, \mathcal{B}_{e_{12}}, \ldots, \mathcal{B}_{e_{\mathfrak{tl}}}\right) \leq \mathcal{B}^{+}_{e_{\mathfrak{tl}}}$$
3) (Monotonicity): Suppose $\mathcal{B}_{e_{\mathfrak{tl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{tl}}, \mathfrak{z}^{-}_{\mathfrak{tl}}\right) = \left(\mathfrak{p}^{+}_{\mathfrak{tl}} + \iota \mathfrak{P}^{+}_{\mathfrak{tl}}, \mathfrak{p}^{-}_{\mathfrak{tl}} + \iota \mathfrak{P}^{-}_{\mathfrak{tl}}\right)$ $(\mathfrak{t} = 1, 2, \ldots, \mathfrak{L} : \mathfrak{l} = 1, 2, \ldots, \aleph)$ and $\varphi_{e_{\mathfrak{tl}}} = \left(\mathcal{R}^{+}_{\mathfrak{tl}}, \mathcal{S}^{-}_{\mathfrak{tl}}\right) = \left(\vartheta^{+}_{\mathfrak{tl}} + \iota \mu^{+}_{\mathfrak{tl}}, \vartheta^{-}_{\mathfrak{tl}} + \iota \mu^{-}_{\mathfrak{tl}}\right)$

$(\mathfrak{k} = 1, 2, \ldots, \mathfrak{P} : \mathfrak{l} = 1, 2, \ldots, \mathfrak{N})$ are two collections of BCFSNs, then the following is obtained:

$$BCFSPDHWA\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) \le BCFSPDHWA\left(\varphi_{\mathfrak{e}_{11}}, \varphi_{\mathfrak{e}_{12}}, \ldots, \varphi_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right)$$

If $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} \le \varphi_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \forall \mathfrak{k}, \mathfrak{l}$.

B. BIPOLAR COMPLEX FUZZY SOFT POWER DOMBI GEOMETRIC AGGREGATION OPERATORS

This segment of the article consist of Bipolar complex fuzzy soft (BCFS) Power dombi geometric (BCFSPDG) operator, Bipolar complex fuzzy soft (BCFS) Power dombi weighted geometric (BCFSPDWG) operator, BCFS Power dombi ordered weighted geometric (BCFSPDOWG) operator, and BCFS Power dombi hybrid geometric (BCFSPDHG) operator.

Definition 19: Suppose $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = \left(\mathfrak{y}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{z}^{-}_{\mathfrak{k}\mathfrak{l}}\right) = \left(\mathfrak{p}^{+}_{\mathfrak{k}\mathfrak{l}} + \iota \mathrm{P}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{p}^{-}_{\mathfrak{k}\mathfrak{l}} + \iota \mathrm{P}^{-}_{\mathfrak{k}\mathfrak{l}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{P}, \mathfrak{l} = 1, 2, \ldots, \mathfrak{N})$ is the collection of BCFSNs, then the BCFSPDG operator is a mapping $\mathcal{B}^{n}_{\mathfrak{e}} \to \mathcal{B}_{\mathfrak{e}}$ such that

$$BCFSPDG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{P}\mathfrak{N}}}\right) = \oplus^{\mathfrak{N}}_{\mathfrak{l}=1} \varpi_{\mathfrak{l}} \left(\oplus^{\mathfrak{P}}_{\mathfrak{k}=1} \hbar_{\mathfrak{k}} \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) \tag{16}$$

where $\varpi_{\mathfrak{l}} = \left(\frac{(1+\mathbb{T}_{\mathfrak{l}})}{\sum^{\mathfrak{N}}_{\mathfrak{l}=1}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\hbar_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum^{\mathfrak{P}}_{\mathfrak{k}=1}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum^{\mathfrak{P}}_{\substack{\mathfrak{r}=1\\ \mathfrak{r}\neq\mathfrak{k}}} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum^{\mathfrak{N}}_{\substack{\mathfrak{s}=1\\ \mathfrak{s}\neq\mathfrak{l}}} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{s}}}\right)$, $Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}\right)$ denotes the support for $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}$ from $\mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}$.

Theorem 10: Suppose $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = \left(\mathfrak{y}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{z}^{-}_{\mathfrak{k}\mathfrak{l}}\right) = \left(\mathfrak{p}^{+}_{\mathfrak{k}\mathfrak{l}} + \iota \mathrm{P}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{p}^{-}_{\mathfrak{k}\mathfrak{l}} + \iota \mathrm{P}^{-}_{\mathfrak{k}\mathfrak{l}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{P} : \mathfrak{l} = 1, 2, \ldots, \mathfrak{N})$ is the collection of BCFSNs, then by using BCFSPDG operator their calculated is again a BCFSN and

$$BCFSPDG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{P}\mathfrak{N}}}\right) = \otimes^{\mathfrak{N}}_{\mathfrak{l}=1} \varpi_{\mathfrak{l}} \left(\otimes^{\mathfrak{P}}_{\mathfrak{k}=1} \hbar_{\mathfrak{k}} \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right)$$
$$= \left(\begin{array}{c} \dfrac{1}{1+\left\{\sum^{\mathfrak{N}}_{\mathfrak{l}=1}\varpi_{\mathfrak{l}}\left(\sum^{\mathfrak{P}}_{\mathfrak{k}=1}\hbar_{\mathfrak{k}}\left(\frac{1-\mathfrak{p}^{+}_{\mathfrak{k}\mathfrak{l}}}{\mathfrak{p}^{+}_{\mathfrak{k}\mathfrak{l}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} + \iota\left(\dfrac{1}{1+\left\{\sum^{\mathfrak{N}}_{\mathfrak{l}=1}\varpi_{\mathfrak{l}}\left(\sum^{\mathfrak{P}}_{\mathfrak{k}=1}\hbar_{\mathfrak{k}}\left(\frac{1-\mathrm{P}^{+}_{\mathfrak{k}\mathfrak{l}}}{\mathrm{P}^{+}_{\mathfrak{k}\mathfrak{l}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\ -1+\dfrac{1}{1+\left\{\sum^{\mathfrak{N}}_{\mathfrak{l}=1}\varpi_{\mathfrak{l}}\left(\sum^{\mathfrak{P}}_{\mathfrak{k}=1}\hbar_{\mathfrak{k}}\left(\frac{|\mathfrak{p}^{-}_{\mathfrak{k}\mathfrak{l}}|}{1+\mathfrak{p}^{-}_{\mathfrak{k}\mathfrak{l}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} + \iota\left(-1+\dfrac{1}{1+\left\{\sum^{\mathfrak{N}}_{\mathfrak{l}=1}\varpi_{\mathfrak{l}}\left(\sum^{\mathfrak{P}}_{\mathfrak{k}=1}\hbar_{\mathfrak{k}}\left(\frac{|\mathrm{P}^{-}_{\mathfrak{k}\mathfrak{l}}|}{1+\mathrm{P}^{-}_{\mathfrak{k}\mathfrak{l}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right) \end{array}\right) \tag{17}$$

where $\varpi_{\mathfrak{l}} = \left(\frac{(1+\mathbb{T}_{\mathfrak{l}})}{\sum^{\mathfrak{N}}_{\mathfrak{l}=1}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\hbar_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum^{\mathfrak{P}}_{\mathfrak{k}=1}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum^{\mathfrak{P}}_{\substack{\mathfrak{r}=1\\ \mathfrak{r}\neq\mathfrak{k}}} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum^{\mathfrak{N}}_{\substack{\mathfrak{s}=1\\ \mathfrak{s}\neq\mathfrak{l}}} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{s}}}\right)$, $Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}\right)$ denotes the support for $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}$ from $\mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}$.

Proof: Here, the mathematical induction method is used, as follows:

$$\mathcal{B}_{\mathfrak{e}_{11}} \otimes \mathcal{B}_{\mathfrak{e}_{12}}$$
$$= \left(\begin{array}{c} \dfrac{1}{1+\left\{\left(\frac{1-\mathfrak{p}^{+}_{11}}{\mathfrak{p}^{+}_{11}}\right)^{o}+\left(\frac{1-\mathfrak{p}^{+}_{12}}{\mathfrak{p}^{+}_{12}}\right)^{o}\right\}^{\frac{1}{o}}} + \iota\left(\dfrac{1}{1+\left\{\left(\frac{1-\mathrm{P}^{+}_{11}}{\mathrm{P}^{+}_{11}}\right)^{o}+\left(\frac{1-\mathrm{P}^{+}_{12}}{\mathrm{P}^{+}_{12}}\right)^{o}\right\}^{\frac{1}{o}}}\right), \\ -1+\dfrac{1}{1+\left\{\left(\frac{|\mathfrak{p}^{-}_{11}|}{1+\mathfrak{p}^{-}_{11}}\right)^{o}+\left(\frac{|\mathfrak{p}^{-}_{12}|}{1+\mathfrak{p}^{-}_{12}}\right)^{o}\right\}^{\frac{1}{o}}} + \iota\left(-1+\dfrac{1}{1+\left\{\left(\frac{|\mathrm{P}^{-}_{11}|}{1+\mathrm{P}^{-}_{11}}\right)^{o}+\left(\frac{|\mathrm{P}^{-}_{12}|}{1+\mathrm{P}^{-}_{12}}\right)^{o}\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

First, we prove that the result holds for $\mathfrak{P} = 2$ and $\mathfrak{N} = 2$, as follows:

$$BCFSPDG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}\right) = \otimes^{2}_{\mathfrak{l}=1} \varpi_{\mathfrak{l}} \left(\otimes^{2}_{\mathfrak{k}=1} \hbar_{\mathfrak{k}} \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right)$$
$$= \varpi_{1}\left(\hbar_{1}\mathcal{B}_{\mathfrak{e}_{11}} \otimes \hbar_{2}\mathcal{B}_{\mathfrak{e}_{21}}\right) \otimes \varpi_{2}\left(\hbar_{1}\mathcal{B}_{\mathfrak{e}_{12}} \otimes \hbar_{2}\mathcal{B}_{\mathfrak{e}_{22}}\right)$$
$$= \varpi_{1}\left(\begin{array}{c} \dfrac{1}{1+\left\{\hbar_{1}\left(\frac{1-\mathfrak{p}^{+}_{11}}{\mathfrak{p}^{+}_{11}}\right)^{o}+\hbar_{2}\left(\frac{1-\mathfrak{p}^{+}_{21}}{\mathfrak{p}^{+}_{21}}\right)^{o}\right\}^{\frac{1}{o}}} + \iota\left(\dfrac{1}{1+\left\{\hbar_{1}\left(\frac{1-\mathrm{P}^{+}_{11}}{\mathrm{P}^{+}_{11}}\right)^{o}+\hbar_{2}\left(\frac{1-\mathrm{P}^{+}_{21}}{\mathrm{P}^{+}_{21}}\right)^{o}\right\}^{\frac{1}{o}}}\right), \\ -1+\dfrac{1}{1+\left\{\hbar_{1}\left(\frac{|\mathfrak{p}^{-}_{11}|}{1+\mathfrak{p}^{-}_{11}}\right)^{o}+\hbar_{2}\left(\frac{|\mathfrak{p}^{-}_{21}|}{1+\mathfrak{p}^{-}_{21}}\right)^{o}\right\}^{\frac{1}{o}}} + \iota\left(-1+\dfrac{1}{1+\left\{\hbar_{1}\left(\frac{|\mathrm{P}^{-}_{11}|}{1+\mathrm{P}^{-}_{11}}\right)^{o}+\hbar_{2}\left(\frac{|\mathrm{P}^{-}_{21}|}{1+\mathrm{P}^{-}_{21}}\right)^{o}\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

$$\otimes \text{Ꙍ}_2 \left(\begin{array}{c} \frac{1}{1+\left\{\text{Ђ}_1\left(\frac{1-\text{Ᵽ}^+_{12}}{\text{Ᵽ}^+_{12}}\right)^o + \text{Ђ}_2\left(\frac{1-\text{Ᵽ}^+_{22}}{\text{Ᵽ}^+_{22}}\right)^o\right\}^{\frac{1}{o}}} \\ +\iota\left(\frac{1}{1+\left\{\text{Ђ}_1\left(\frac{1-\text{Ꝑ}^+_{12}}{\text{Ꝑ}^+_{12}}\right)^o + \text{Ђ}_2\left(\frac{1-\text{Ꝑ}^+_{22}}{\text{Ꝑ}^+_{22}}\right)^o\right\}^{\frac{1}{o}}}\right), \\ -1+\frac{1}{1+\left\{\text{Ђ}_1\left(\frac{\left|\text{Ᵽ}^-_{12}\right|}{1+\text{Ᵽ}^-_{12}}\right)^o + \text{Ђ}_2\left(\frac{\left|\text{Ᵽ}^-_{22}\right|}{1+\text{Ᵽ}^-_{22}}\right)^o\right\}^{\frac{1}{o}}} \\ +\iota\left(-1+\frac{1}{1+\left\{\text{Ђ}_1\left(\frac{\left|\text{Ꝑ}^-_{12}\right|}{1+\text{Ꝑ}^-_{12}}\right)^o + \text{Ђ}_2\left(\frac{\left|\text{Ꝑ}^-_{22}\right|}{1+\text{Ꝑ}^-_{22}}\right)^o\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

$$= \text{Ꙍ}_1 \left(\begin{array}{c} \frac{1}{1+\left\{\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{1-\text{Ᵽ}^+_{\text{ƚ}1}}{\text{Ᵽ}^+_{\text{ƚ}1}}\right)^o\right\}^{\frac{1}{o}}} \\ +\iota\left(\frac{1}{1+\left\{\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{1-\text{Ꝑ}^+_{\text{ƚ}1}}{\text{Ꝑ}^+_{\text{ƚ}1}}\right)^o\right\}^{\frac{1}{o}}}\right), \\ -1+\frac{1}{1+\left\{\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{\left|\text{Ᵽ}^-_{\text{ƚ}1}\right|}{1+\text{Ᵽ}^-_{\text{ƚ}1}}\right)^o\right\}^{\frac{1}{o}}} \\ +\iota\left(-1+\frac{1}{1+\left\{\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{\left|\text{Ꝑ}^-_{\text{ƚ}1}\right|}{1+\text{Ꝑ}^-_{\text{ƚ}1}}\right)^o\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

$$\otimes \text{Ꙍ}_2 \left(\begin{array}{c} \frac{1}{1+\left\{\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{1-\text{Ᵽ}^+_{\text{ƚ}2}}{\text{Ᵽ}^+_{\text{ƚ}2}}\right)^o\right\}^{\frac{1}{o}}} \\ +\iota\left(\frac{1}{1+\left\{\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{1-\text{Ꝑ}^+_{\text{ƚ}2}}{\text{Ꝑ}^+_{\text{ƚ}2}}\right)^o\right\}^{\frac{1}{o}}}\right), \\ -1+\frac{1}{1+\left\{\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{\left|\text{Ᵽ}^-_{\text{ƚ}2}\right|}{1+\text{Ᵽ}^-_{\text{ƚ}2}}\right)^o\right\}^{\frac{1}{o}}} \\ +\iota\left(-1+\frac{1}{1+\left\{\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{\left|\text{Ꝑ}^-_{\text{ƚ}2}\right|}{1+\text{Ꝑ}^-_{\text{ƚ}2}}\right)^o\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

$$= \left(\begin{array}{c} \frac{1}{1+\left\{\begin{array}{c}\text{Ꙍ}_1\left(\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{1-\text{Ᵽ}^+_{\text{ƚ}1}}{\text{Ᵽ}^+_{\text{ƚ}1}}\right)^o\right) \\ +\text{Ꙍ}_2\left(\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{1-\text{Ᵽ}^+_{\text{ƚ}2}}{\text{Ᵽ}^+_{\text{ƚ}2}}\right)^o\right)\end{array}\right\}^{\frac{1}{o}}} \\ +\iota\left(\frac{1}{1+\left\{\begin{array}{c}\text{Ꙍ}_1\left(\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{1-\text{Ꝑ}^+_{\text{ƚ}1}}{\text{Ꝑ}^+_{\text{ƚ}1}}\right)^o\right) \\ +\text{Ꙍ}_2\left(\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{1-\text{Ꝑ}^+_{\text{ƚ}2}}{\text{Ꝑ}^+_{\text{ƚ}2}}\right)^o\right)\end{array}\right\}^{\frac{1}{o}}}\right), \\ -1+\frac{1}{1+\left\{\begin{array}{c}\text{Ꙍ}_1\left(\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{\left|\text{Ᵽ}^-_{\text{ƚ}1}\right|}{1+\text{Ᵽ}^-_{\text{ƚ}1}}\right)^o\right) \\ +\text{Ꙍ}_2\left(\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{\left|\text{Ᵽ}^-_{\text{ƚ}2}\right|}{1+\text{Ᵽ}^-_{\text{ƚ}2}}\right)^o\right)\end{array}\right\}^{\frac{1}{o}}} \\ +\iota\left(-1+\frac{1}{1+\left\{\begin{array}{c}\text{Ꙍ}_1\left(\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{\left|\text{Ꝑ}^-_{\text{ƚ}1}\right|}{1+\text{Ꝑ}^-_{\text{ƚ}1}}\right)^o\right) \\ +\text{Ꙍ}_2\left(\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{\left|\text{Ꝑ}^-_{\text{ƚ}2}\right|}{1+\text{Ꝑ}^-_{\text{ƚ}2}}\right)^o\right)\end{array}\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

$$= \left(\begin{array}{c} \frac{1}{1+\left\{\sum_{\mathrm{I}=1}^2 \text{Ꙍ}_\mathrm{I}\left(\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{1-\text{Ᵽ}^+_{\text{ƚ}\mathrm{I}}}{\text{Ᵽ}^+_{\text{ƚ}\mathrm{I}}}\right)^o\right)\right\}^{\frac{1}{o}}} \\ +\iota\left(\frac{1}{1+\left\{\sum_{\mathrm{I}=1}^2 \text{Ꙍ}_\mathrm{I}\left(\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{1-\text{Ꝑ}^+_{\text{ƚ}\mathrm{I}}}{\text{Ꝑ}^+_{\text{ƚ}\mathrm{I}}}\right)^o\right)\right\}^{\frac{1}{o}}}\right), \\ -1+\frac{1}{1+\left\{\sum_{\mathrm{I}=1}^2 \text{Ꙍ}_\mathrm{I}\left(\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{\left|\text{Ᵽ}^-_{\text{ƚ}\mathrm{I}}\right|}{1+\text{Ᵽ}^-_{\text{ƚ}\mathrm{I}}}\right)^o\right)\right\}^{\frac{1}{o}}} \\ +\iota\left(-1+\frac{1}{1+\left\{\sum_{\mathrm{I}=1}^2 \text{Ꙍ}_\mathrm{I}\left(\sum_{\text{ƚ}=1}^2 \text{Ђ}_\text{ƚ}\left(\frac{\left|\text{Ꝑ}^-_{\text{ƚ}\mathrm{I}}\right|}{1+\text{Ꝑ}^-_{\text{ƚ}\mathrm{I}}}\right)^o\right)\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

Therefore, this result is true for $\text{Ƀ} = 2$ and $\text{Ꝩ} = 2$. Now, this result is valid for $\text{Ƀ} = \wp_1$ and $\text{Ꝩ} = \wp_2$, as follows:

$$BCFSDG\left(\mathcal{B}_{\underset{\sim}{e}_{11}}, \mathcal{B}_{\underset{\sim}{e}_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{\underset{\sim}{e}_{\wp_1\wp_2}}\right)$$

$$= \otimes_{\text{I}=1}^{\wp_2} \text{Ѡ}_{\text{I}} \left(\otimes_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}} \mathcal{B}e_{\text{ŧI}}\right)$$

$$= \begin{pmatrix} \frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}}\left(\frac{1-\text{Ᵽ}^{+}_{\text{ŧI}}}{\text{Ᵽ}^{+}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} + \iota\left(\frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}}\left(\frac{1-\text{Ꝑ}^{+}_{\text{ŧI}}}{\text{Ꝑ}^{+}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\ -1+\frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}}\left(\frac{|\text{Ᵽ}^{-}_{\text{ŧI}}|}{1+\text{Ᵽ}^{-}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} + \iota\left(-1+\frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}}\left(\frac{|\text{Ꝑ}^{-}_{\text{ŧI}}|}{1+\text{Ꝑ}^{-}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right) \end{pmatrix}$$

Next, we consider this result is valid for $\text{Ꞵ} = \wp_1 + 1$ and $\text{Ȣ} = \wp_2 + 1$, as follows:

$$BCFSDG\left(\mathcal{B}e_{11}, \mathcal{B}e_{12}, \mathcal{B}_{13}, \ldots, \mathcal{B}e_{\wp_1\wp_2}, \mathcal{B}e_{(\wp_1+1)(\wp_2+1)}\right)$$
$$= \otimes_{\text{I}=1}^{\wp_2} \text{Ѡ}_{\text{I}} \left(\otimes_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}} \mathcal{B}e_{\text{ŧI}}\right) \otimes \left(\text{Ѡ}_{\wp_2+1}\left(\text{Ƃ}_{\wp_1+1} \mathcal{B}e_{(\wp_1+1)(\wp_2+1)}\right)\right)$$

$$= \begin{pmatrix} \frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}}\left(\frac{1-\text{Ᵽ}^{+}_{\text{ŧI}}}{\text{Ᵽ}^{+}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} + \iota\left(\frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}}\left(\frac{1-\text{Ꝑ}^{+}_{\text{ŧI}}}{\text{Ꝑ}^{+}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\ -1+\frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}}\left(\frac{|\text{Ᵽ}^{-}_{\text{ŧI}}|}{1+\text{Ᵽ}^{-}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} + \iota\left(-1+\frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}}\left(\frac{|\text{Ꝑ}^{-}_{\text{ŧI}}|}{1+\text{Ꝑ}^{-}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right) \end{pmatrix}$$
$$\otimes \begin{pmatrix} \frac{1}{1+\left\{\text{Ѡ}_{\wp_2+1}\left(\text{Ƃ}_{\wp_1+1}\left(\frac{1-\text{Ᵽ}^{+}_{\text{ŧI}}}{\text{Ᵽ}^{+}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} + \iota\left(\frac{1}{1+\left\{\text{Ѡ}_{\wp_2+1}\left(\text{Ƃ}_{\wp_1+1}\left(\frac{1-\text{Ꝑ}^{+}_{\text{ŧI}}}{\text{Ꝑ}^{+}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\ -1+\frac{1}{1+\left\{\text{Ѡ}_{\wp_2+1}\left(\text{Ƃ}_{\wp_1+1}\left(\frac{|\text{Ᵽ}^{-}_{\text{ŧI}}|}{1+\text{Ᵽ}^{-}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} + \iota\left(-1+\frac{1}{1+\left\{\text{Ѡ}_{\wp_2+1}\left(\text{Ƃ}_{\wp_1+1}\left(\frac{|\text{Ꝑ}^{-}_{\text{ŧI}}|}{1+\text{Ꝑ}^{-}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2} \begin{matrix} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}}\left(\frac{1-\text{Ᵽ}^{+}_{\text{ŧI}}}{\text{Ᵽ}^{+}_{\text{ŧI}}}\right)^{o}\right) \\ +\text{Ѡ}_{\wp_2+1}\left(\text{Ƃ}_{\wp_1+1}\left(\frac{1-\text{Ᵽ}^{+}_{\text{ŧI}}}{\text{Ᵽ}^{+}_{\text{ŧI}}}\right)^{o}\right)\end{matrix}\right\}^{\frac{1}{o}}} \\ +\iota\left(\frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2} \begin{matrix} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}}\left(\frac{1-\text{Ꝑ}^{+}_{\text{ŧI}}}{\text{Ꝑ}^{+}_{\text{ŧI}}}\right)^{o}\right) \\ +\text{Ѡ}_{\wp_2+1}\left(\text{Ƃ}_{\wp_1+1}\left(\frac{1-\text{Ꝑ}^{+}_{\text{ŧI}}}{\text{Ꝑ}^{+}_{\text{ŧI}}}\right)^{o}\right)\end{matrix}\right\}^{\frac{1}{o}}}\right) \\ -1+\frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2} \begin{matrix} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}}\left(\frac{|\text{Ᵽ}^{-}_{\text{ŧI}}|}{1+\text{Ᵽ}^{-}_{\text{ŧI}}}\right)^{o}\right) \\ +\text{Ѡ}_{\wp_2+1}\left(\text{Ƃ}_{\wp_1+1}\left(\frac{|\text{Ᵽ}^{-}_{\text{ŧI}}|}{1+\text{Ᵽ}^{-}_{\text{ŧI}}}\right)^{o}\right)\end{matrix}\right\}^{\frac{1}{o}}} \\ +\iota\left(-1+\frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2} \begin{matrix} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1} \text{Ƃ}_{\text{ŧ}}\left(\frac{|\text{Ꝑ}^{-}_{\text{ŧI}}|}{1+\text{Ꝑ}^{-}_{\text{ŧI}}}\right)^{o}\right) \\ +\text{Ѡ}_{\wp_2+1}\left(\text{Ƃ}_{\wp_1+1}\left(\frac{|\text{Ꝑ}^{-}_{\text{ŧI}}|}{1+\text{Ꝑ}^{-}_{\text{ŧI}}}\right)^{o}\right)\end{matrix}\right\}^{\frac{1}{o}}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2+1} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1+1} \text{Ƃ}_{\text{ŧ}}\left(\frac{1-\text{Ᵽ}^{+}_{\text{ŧI}}}{\text{Ᵽ}^{+}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} + \iota\left(\frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2+1} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1+1} \text{Ƃ}_{\text{ŧ}}\left(\frac{1-\text{Ꝑ}^{+}_{\text{ŧI}}}{\text{Ꝑ}^{+}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\ -1+\frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2+1} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1+1} \text{Ƃ}_{\text{ŧ}}\left(\frac{|\text{Ᵽ}^{-}_{\text{ŧI}}|}{1+\text{Ᵽ}^{-}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} + \iota\left(-1+\frac{1}{1+\left\{\sum_{\text{I}=1}^{\wp_2+1} \text{Ѡ}_{\text{I}}\left(\sum_{\text{ŧ}=1}^{\wp_1+1} \text{Ƃ}_{\text{ŧ}}\left(\frac{|\text{Ꝑ}^{-}_{\text{ŧI}}|}{1+\text{Ꝑ}^{-}_{\text{ŧI}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right) \end{pmatrix}$$

Therefore, the result is valid for $\text{Ꞵ} = \wp_1 + 1$ and $\text{Ȣ} = \wp_2 + 1$. Hence, the result is true $\text{Ꞵ}, \text{Ȣ} \geq 1$. It is obvious from above expression the aggregated results established BCFSPDG operator is again BCFSPDG operator.

Theorem 11: Suppose $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = (\mathfrak{y}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{z}^{-}_{\mathfrak{k}\mathfrak{l}}) = (\mathfrak{p}^{+}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{p}^{-}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{-}_{\mathfrak{k}\mathfrak{l}})$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{D}, \mathfrak{l} = 1, 2, \ldots, \mho)$ is the collection of BCFSNs, where $\mathfrak{W}_{\mathfrak{l}} = \left(\frac{(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\mho}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\mathfrak{h}_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{\mathfrak{D}}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1 \\ \mathfrak{r}\neq\mathfrak{k}}}^{\mathfrak{D}} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1 \\ \mathfrak{s}\neq\mathfrak{l}}}^{\mho} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{s}}}\right)$, $Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}\right)$ denotes the support for $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}$ from $\mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}$. Then, *BCFSPDG* AO satisfies the following properties:

1) (Idempotency): Suppose $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = (\mathfrak{y}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{z}^{-}_{\mathfrak{k}\mathfrak{l}}) = (\mathfrak{p}^{+}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{p}^{-}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{-}_{\mathfrak{k}\mathfrak{l}})$ $(\mathfrak{k}=1, 2, \ldots, \mathfrak{D}, \mathfrak{l}=1, 2, \ldots, \mho)$ is the collection of BCFSNs, and all are the same i.e. $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = \mathcal{B}_{\mathfrak{e}}$, $\forall \mathfrak{k}, \mathfrak{l}$, then the following is obtained:
$$BCFSPDG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) = \mathcal{B}_{\mathfrak{e}}$$

2) (Boundedness): Suppose $\mathcal{B}^{-}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = \left(\min_{\mathfrak{l}}\min_{\mathfrak{k}} \mathfrak{y}^{+}_{\mathfrak{k}\mathfrak{l}}, \max_{\mathfrak{l}}\max_{\mathfrak{k}} \mathfrak{z}^{-}_{\mathfrak{k}\mathfrak{l}}\right)$ and $\mathcal{B}^{+}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = \left(\max_{\mathfrak{l}}\max_{\mathfrak{k}} \mathfrak{y}^{+}_{\mathfrak{k}\mathfrak{l}}, \min_{\mathfrak{l}}\min_{\mathfrak{k}} \mathfrak{z}^{-}_{\mathfrak{k}\mathfrak{l}}\right)$ then,
$$\mathcal{B}^{-}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} \leq BCFSPDG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) \leq \mathcal{B}^{+}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}$$

3) (Monotonicity): Suppose $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = (\mathfrak{y}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{z}^{-}_{\mathfrak{k}\mathfrak{l}}) = (\mathfrak{p}^{+}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{p}^{-}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{-}_{\mathfrak{k}\mathfrak{l}})$ $(\mathfrak{k}=1, 2, \ldots, \mathfrak{D}, \mathfrak{l}=1, 2, \ldots, \mho)$ and $\varphi_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = (\mathcal{R}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathcal{S}^{-}_{\mathfrak{k}\mathfrak{l}}) = (\vartheta^{+}_{\mathfrak{k}\mathfrak{l}} + \iota\mu^{+}_{\mathfrak{k}\mathfrak{l}}, \vartheta^{-}_{\mathfrak{k}\mathfrak{l}} + \iota\mu^{-}_{\mathfrak{k}\mathfrak{l}})$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{D}, \mathfrak{l} = 1, 2, \ldots, \mho)$ are two collections of BCFSNs, then the following is obtained:
$$BCFSPDG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) \leq BCFSDG\left(\varphi_{\mathfrak{e}_{11}}, \varphi_{\mathfrak{e}_{12}}, \ldots, \varphi_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right)$$

If $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} \leq \phi_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}$, $\forall \mathfrak{k}, \mathfrak{l}$.

Definition 20: Suppose $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = (\mathfrak{y}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{z}^{-}_{\mathfrak{k}\mathfrak{l}}) = (\mathfrak{p}^{+}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{p}^{-}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{-}_{\mathfrak{k}\mathfrak{l}})$ $(\mathfrak{k}=1, 2, \ldots, \mathfrak{D}, \mathfrak{l} = 1, 2, \ldots, \mho)$ is the collection of BCFSNs, then the BCFSPDWG operator is a mapping $\mathcal{B}^{n}_{\mathfrak{e}} \rightarrow \mathcal{B}_{\mathfrak{e}}$ such that
$$BCFSPDWG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{D}\mho}}\right) = \oplus_{\mathfrak{l}=1}^{\mho}\mathfrak{W}_{\mathfrak{l}}\left(\oplus_{\mathfrak{k}=1}^{\mathfrak{D}}\mathfrak{h}_{\mathfrak{k}}\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) \tag{18}$$
where $\mathfrak{W}_{\mathfrak{l}} = \left(\frac{\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\mho}\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\mathfrak{h}_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{\mathfrak{D}}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1 \\ \mathfrak{r}\neq\mathfrak{k}}}^{\mathfrak{D}} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1 \\ \mathfrak{s}\neq\mathfrak{l}}}^{\mho} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{s}}}\right)$, $Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}\right)$ denotes the support for $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}$ from $\mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}$, and $\sum_{\mathfrak{l}=1}^{\mho}\omega_{\mathfrak{l}} = 1$.

Theorem 12: Suppose $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = (\mathfrak{y}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{z}^{-}_{\mathfrak{k}\mathfrak{l}}) = (\mathfrak{p}^{+}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{p}^{-}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{-}_{\mathfrak{k}\mathfrak{l}})$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{D} : \mathfrak{l} = 1, 2, \ldots, \mho)$ is the collection of BCFSNs, then by using BCFSPDWG operator their calculated is again a BCFSS and
$$BCFSPDWG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{D}\mho}}\right) = \otimes_{\mathfrak{l}=1}^{\mho}\mathfrak{W}_{\mathfrak{l}}\left(\otimes_{\mathfrak{k}=1}^{\mathfrak{D}}\mathfrak{h}_{\mathfrak{k}}\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right)$$
$$= \left(\begin{array}{c} \frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\mho}\mathfrak{W}_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{\mathfrak{D}}\mathfrak{h}_{\mathfrak{k}}\left(\frac{1-\mathfrak{p}^{+}_{\mathfrak{k}\mathfrak{l}}}{\mathfrak{p}^{+}_{\mathfrak{k}\mathfrak{l}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\ +\iota\left(\frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\mho}\mathfrak{W}_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{\mathfrak{D}}\mathfrak{h}_{\mathfrak{k}}\left(\frac{1-\mathfrak{P}^{+}_{\mathfrak{k}\mathfrak{l}}}{\mathfrak{P}^{+}_{\mathfrak{k}\mathfrak{l}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\ -1+\frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\mho}\mathfrak{W}_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{\mathfrak{D}}\mathfrak{h}_{\mathfrak{k}}\left(\frac{|\mathfrak{p}^{-}_{\mathfrak{k}\mathfrak{l}}|}{1+\mathfrak{p}^{-}_{\mathfrak{k}\mathfrak{l}}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\ +\iota\left(-1+\frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\mho}\mathfrak{W}_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{\mathfrak{D}}\mathfrak{h}_{\mathfrak{k}}\left(\frac{|\mathfrak{P}^{-}_{\mathfrak{k}\mathfrak{l}}|}{1+\mathfrak{P}^{-}_{\mathfrak{k}\mathfrak{l}}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right) \end{array}\right) \tag{19}$$
where $\mathfrak{W}_{\mathfrak{l}} = \left(\frac{\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\mho}\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\mathfrak{h}_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{\mathfrak{D}}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1 \\ \mathfrak{r}\neq\mathfrak{k}}}^{\mathfrak{D}} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1 \\ \mathfrak{s}\neq\mathfrak{l}}}^{\mho} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{s}}}\right)$, $Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}\right)$ denotes the support for $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}$ from $\mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}$, and $\sum_{\mathfrak{l}=1}^{\mho}\omega_{\mathfrak{l}} = 1$.

Proof: The proof is similar to Theorem 9.

Theorem 13: Suppose $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = (\mathfrak{y}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{z}^{-}_{\mathfrak{k}\mathfrak{l}}) = (\mathfrak{p}^{+}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{p}^{-}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{-}_{\mathfrak{k}\mathfrak{l}})$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{D} : \mathfrak{l} = 1, 2, \ldots, \mho)$ is the collection of BCFSNs, where $\mathfrak{W}_{\mathfrak{l}} = \left(\frac{\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\mho}\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\mathfrak{h}_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{\mathfrak{D}}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1 \\ \mathfrak{r}\neq\mathfrak{k}}}^{\mathfrak{D}} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1 \\ \mathfrak{s}\neq\mathfrak{l}}}^{\mho} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{s}}}\right)$, $Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}\right)$ denotes the support for $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}$ from $\mathcal{B}_{\mathfrak{e}_{\mathfrak{r}\mathfrak{l}}}$. Then, *BCFSPWDG* AO satisfies the following properties:

1) (Idempotency): Suppose $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = (\mathfrak{y}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{z}^{-}_{\mathfrak{k}\mathfrak{l}}) = (\mathfrak{p}^{+}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{+}_{\mathfrak{k}\mathfrak{l}}, \mathfrak{p}^{-}_{\mathfrak{k}\mathfrak{l}} + \iota \mathfrak{P}^{-}_{\mathfrak{k}\mathfrak{l}})$ $(\mathfrak{k}=1, 2, \ldots, \mathfrak{D} : \mathfrak{l}=1, 2, \ldots, \mho)$ is the collection of BCFSNs, and all are the same i.e. $\mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = \mathcal{B}_{\mathfrak{e}}$, $\forall \mathfrak{k}, \mathfrak{l}$, then the following is obtained:
$$BCFSPDWG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) = \mathcal{B}_{\mathfrak{e}}$$

2) (Boundedness): Suppose $\mathcal{B}^{-}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = \left(\min_{\mathfrak{l}}\min_{\mathfrak{k}} \mathfrak{y}^{+}_{\mathfrak{k}\mathfrak{l}}, \max_{\mathfrak{l}}\max_{\mathfrak{k}} \mathfrak{z}^{-}_{\mathfrak{k}\mathfrak{l}}\right)$ and $\mathcal{B}^{+}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} = \left(\max_{\mathfrak{l}}\max_{\mathfrak{k}} \mathfrak{y}^{+}_{\mathfrak{k}\mathfrak{l}}, \min_{\mathfrak{l}}\min_{\mathfrak{k}} \mathfrak{z}^{-}_{\mathfrak{k}\mathfrak{l}}\right)$ then,
$$\mathcal{B}^{-}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}} \leq BCFSPDWG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}\right) \leq \mathcal{B}^{+}_{\mathfrak{e}_{\mathfrak{k}\mathfrak{l}}}$$

3) (Monotonicity): Suppose $\mathcal{B}e_{\mathfrak{kl}} = (ŋ^{+}_{\mathfrak{kl}}, Ʒ^{-}_{\mathfrak{kl}}) = (Ᵽ^{+}_{\mathfrak{kl}} + \iota Ᵽ^{+}_{\mathfrak{kl}}, Ᵽ^{-}_{\mathfrak{kl}} + \iota Ᵽ^{-}_{\mathfrak{kl}})$ $(\mathfrak{k} = 1, 2, \ldots, Ƀ : \mathfrak{l} = 1, 2, \ldots, ℨ)$ and $\phi e_{\mathfrak{kl}} = (\mathcal{R}^{+}_{\mathfrak{kl}}, \mathcal{S}^{-}_{\mathfrak{kl}}) = (\vartheta^{+}_{\mathfrak{kl}} + \iota\mu^{+}_{\mathfrak{kl}}, \vartheta^{-}_{\mathfrak{kl}} + \iota\mu^{-}_{\mathfrak{kl}})$ $(\mathfrak{k} = 1, 2, \ldots, Ƀ : \mathfrak{l} = 1, 2, \ldots, ℨ)$ are two collections of BCFSNs, then the following is obtained:

$$BCFSPDWG\left(\mathcal{B}e_{11}, \mathcal{B}e_{12}, \ldots, \mathcal{B}e_{\mathfrak{kl}}\right) \leq BCFSDWG\left(\phi e_{11}, \phi e_{12}, \ldots, \phi e_{\mathfrak{kl}}\right)$$

If $\mathcal{B}e_{\mathfrak{kl}} \leq \phi e_{\mathfrak{kl}}, \forall \mathfrak{k}, \mathfrak{l}$.

Definition 21: Suppose $\mathcal{B}e_{\mathfrak{kl}} = (ŋ^{+}_{\mathfrak{kl}}, Ʒ^{-}_{\mathfrak{kl}}) = (Ᵽ^{+}_{\mathfrak{kl}} + \iota Ᵽ^{+}_{\mathfrak{kl}}, Ᵽ^{-}_{\mathfrak{kl}} + \iota Ᵽ^{-}_{\mathfrak{kl}})$ $(\mathfrak{k} = 1, 2, \ldots, Ƀ, \mathfrak{l} = 1, 2, \ldots, ℨ)$ is the collection of BCFSNs, then the BCFSPDOWG operator is a mapping $\mathcal{B}e^{n} \to \mathcal{B}e$ such that

$$BCFSPDOWG\left(\mathcal{B}e_{11}, \mathcal{B}e_{12}, \mathcal{B}_{13}, \ldots, \mathcal{B}e_{Ƀℨ}\right) = \oplus_{\mathfrak{l}=1}^{ℨ} \mathbb{Q}_{\mathfrak{l}} \left(\oplus_{\mathfrak{k}=1}^{Ƀ} \mathfrak{H}_{\mathfrak{k}} \mathcal{B}e_{\mathfrak{kl}}\right) \tag{20}$$

where $\mathbb{Q}_{\mathfrak{l}} = \left(\frac{\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{ℨ}\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\mathfrak{H}_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{Ƀ}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1\\ \mathfrak{r}\neq\mathfrak{k}}}^{Ƀ} Sup\left(\mathcal{B}e_{\mathfrak{kl}}, \mathcal{B}e_{\mathfrak{rl}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1\\ \mathfrak{s}\neq\mathfrak{l}}}^{ℨ} Sup\left(\mathcal{B}e_{\mathfrak{kl}}, \mathcal{B}e_{\mathfrak{ks}}\right)$, $Sup\left(\mathcal{B}e_{\mathfrak{kl}}, \mathcal{B}e_{\mathfrak{rl}}\right)$ denotes the support for $\mathcal{B}e_{\mathfrak{kl}}$ from $\mathcal{B}e_{\mathfrak{rl}}$, and $\sum_{\mathfrak{l}=1}^{ℨ}\omega_{\mathfrak{l}} = 1$. Here, $\mathcal{B}e_{\Xi(\mathfrak{kl})} = \left(ŋ^{+}_{\Xi(\mathfrak{kl})}, Ʒ^{-}_{\Xi(\mathfrak{kl})}\right)$ is the permutation of the $\mathfrak{k}th$ row and $\mathfrak{l}th$ largest elements of the collection for $\mathfrak{k} \times \mathfrak{l}$ BCFSNs $\mathcal{B}e_{\mathfrak{kl}} = (ŋ^{+}_{\mathfrak{kl}}, Ʒ^{-}_{\mathfrak{kl}})$ for $\mathfrak{k} = 1, 2, \ldots, Ƀ$ *and* $\mathfrak{l} = 1, 2, \ldots, ℨ$. Now, using equation (16) we can define *BCFSPDOWG* AOs, as follows:

Theorem 14: Suppose $\mathcal{B}e_{\mathfrak{kl}} = (ŋ^{+}_{\mathfrak{kl}}, Ʒ^{-}_{\mathfrak{kl}}) = (Ᵽ^{+}_{\mathfrak{kl}} + \iota Ᵽ^{+}_{\mathfrak{kl}}, Ᵽ^{-}_{\mathfrak{kl}} + \iota Ᵽ^{-}_{\mathfrak{kl}})$ $(\mathfrak{k} = 1, 2, \ldots, Ƀ : \mathfrak{l} = 1, 2, \ldots, ℨ)$ is the collection of BCFSNs, then by using BCFSPDOWG operator their calculated is again a BCFSS and

$$BCFSPDOWG\left(\mathcal{B}e_{11}, \mathcal{B}e_{12}, \mathcal{B}_{13}, \ldots, \mathcal{B}e_{Ƀℨ}\right) = \oplus_{\mathfrak{l}=1}^{ℨ} \mathbb{Q}_{\mathfrak{l}} \left(\oplus_{\mathfrak{k}=1}^{Ƀ} \mathfrak{H}_{\mathfrak{k}} \mathcal{B}e_{\Xi(\mathfrak{kl})}\right)$$

$$= \left(\begin{array}{l} \frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{ℨ}\mathbb{Q}_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{Ƀ}\mathfrak{H}_{\mathfrak{k}}\left(\frac{1-Ᵽ^{+}_{\Xi(\mathfrak{kl})}}{Ᵽ^{+}_{\Xi(\mathfrak{kl})}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\ +\iota\left(\frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{ℨ}\mathbb{Q}_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{Ƀ}\mathfrak{H}_{\mathfrak{k}}\left(\frac{1-Ᵽ^{+}_{\Xi(\mathfrak{kl})}}{Ᵽ^{+}_{\Xi(\mathfrak{kl})}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right), \\ -1+\frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{ℨ}\mathbb{Q}_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{Ƀ}\mathfrak{H}_{\mathfrak{k}}\left(\frac{\left|Ᵽ^{-}_{\Xi(\mathfrak{kl})}\right|}{1+Ᵽ^{-}_{\Xi(\mathfrak{kl})}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\ +\iota\left(-1+\frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{ℨ}\mathbb{Q}_{\mathfrak{l}}\left(\sum_{\mathfrak{k}=1}^{Ƀ}\mathfrak{H}_{\mathfrak{k}}\left(\frac{\left|Ᵽ^{-}_{\Xi(\mathfrak{kl})}\right|}{1+Ᵽ^{-}_{\Xi(\mathfrak{kl})}}\right)^{o}\right)\right\}^{\frac{1}{o}}}\right) \end{array}\right)$$

where $\mathcal{B}e_{\Xi(\mathfrak{kl})} = n\partial_{\mathfrak{k}}\rho_{\mathfrak{l}}\mathcal{B}e_{\mathfrak{kl}}$ is the permutation of the $\mathfrak{k}th$ row and $\mathfrak{l}th$ largest elements of the collection for $\mathfrak{k} \times \mathfrak{l}$ BCFSNs $\mathcal{B}e_{\mathfrak{kl}} = (ŋ^{+}_{\mathfrak{kl}}, Ʒ^{-}_{\mathfrak{kl}})$ and $'n'$ is the balancing coefficient.

Proof: The proof theorem is like to Theorem 9.

Theorem 15: Suppose $\mathcal{B}e_{\mathfrak{kl}} = (ŋ^{+}_{\mathfrak{kl}}, Ʒ^{-}_{\mathfrak{kl}}) = (Ᵽ^{+}_{\mathfrak{kl}} + \iota Ᵽ^{+}_{\mathfrak{kl}}, Ᵽ^{-}_{\mathfrak{kl}} + \iota Ᵽ^{-}_{\mathfrak{kl}})$ $(\mathfrak{k} = 1, 2, \ldots, Ƀ : \mathfrak{l} = 1, 2, \ldots, ℨ)$ is the collection of BCFSNs, where $\mathbb{Q}_{\mathfrak{l}} = \left(\frac{\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{ℨ}\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\mathfrak{H}_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{Ƀ}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1\\ \mathfrak{r}\neq\mathfrak{k}}}^{Ƀ} Sup\left(\mathcal{B}e_{\mathfrak{kl}}, \mathcal{B}e_{\mathfrak{rl}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1\\ \mathfrak{s}\neq\mathfrak{l}}}^{ℨ} Sup\left(\mathcal{B}e_{\mathfrak{kl}}, \mathcal{B}e_{\mathfrak{ks}}\right)$, $Sup\left(\mathcal{B}e_{\mathfrak{kl}}, \mathcal{B}e_{\mathfrak{rl}}\right)$ denotes the support for $\mathcal{B}e_{\mathfrak{kl}}$ from $\mathcal{B}e_{\mathfrak{rl}}$, and $\sum_{\mathfrak{l}=1}^{ℨ}\omega_{\mathfrak{l}} = 1$. Then, *BCFSPDOWG* AO satisfies the following properties:

1) (Idempotency): Suppose $\mathcal{B}e_{\mathfrak{kl}} = (ŋ^{+}_{\mathfrak{kl}}, Ʒ^{-}_{\mathfrak{kl}}) = (Ᵽ^{+}_{\mathfrak{kl}} + \iota Ᵽ^{+}_{\mathfrak{kl}}, Ᵽ^{-}_{\mathfrak{kl}} + \iota Ᵽ^{-}_{\mathfrak{kl}})$ $(\mathfrak{k} = 1, 2, \ldots, Ƀ : \mathfrak{l} = 1, 2, \ldots, ℨ)$ is the collection of BCFSNs, and all are the same i.e. $\mathcal{B}e_{\mathfrak{kl}} = \mathcal{B}e, \forall \mathfrak{k}, \mathfrak{l}$, then the following is obtained:

$$BCFSPDOWG\left(\mathcal{B}e_{11}, \mathcal{B}e_{12}, \ldots, \mathcal{B}e_{\mathfrak{kl}}\right) = \mathcal{B}e$$

2) (Boundedness): Suppose $\mathcal{B}^{-}_{e_{\mathfrak{kl}}} = \left(\min_{\mathfrak{l}}\min_{\mathfrak{k}} ŋ^{+}_{\mathfrak{kl}}, \max_{\mathfrak{l}}\max_{\mathfrak{k}} Ʒ^{-}_{\mathfrak{kl}}\right)$ and $\mathcal{B}^{+}_{e_{\mathfrak{kl}}} = \left(\max_{\mathfrak{l}}\max_{\mathfrak{k}} ŋ^{+}_{\mathfrak{kl}}, \min_{\mathfrak{l}}\min_{\mathfrak{k}} Ʒ^{-}_{\mathfrak{kl}}\right)$ then,

$$\mathcal{B}^{-}_{e_{\mathfrak{kl}}} \leq BCFSPDOWG\left(\mathcal{B}e_{11}, \mathcal{B}e_{12}, \ldots, \mathcal{B}e_{\mathfrak{kl}}\right) \leq \mathcal{B}^{+}_{e_{\mathfrak{kl}}}$$

3) (Monotonicity): Suppose $\mathcal{B}e_{\mathfrak{kl}} = (ŋ^{+}_{\mathfrak{kl}}, Ʒ^{-}_{\mathfrak{kl}}) = (Ᵽ^{+}_{\mathfrak{kl}} + \iota Ᵽ^{+}_{\mathfrak{kl}}, Ᵽ^{-}_{\mathfrak{kl}} + \iota Ᵽ^{-}_{\mathfrak{kl}})$ $(\mathfrak{k} = 1, 2, \ldots, Ƀ : \mathfrak{l} = 1, 2, \ldots, ℨ)$ and $\varphi e_{\mathfrak{kl}} = (\mathcal{R}^{+}_{\mathfrak{kl}}, \mathcal{S}^{-}_{\mathfrak{kl}}) = (\vartheta^{+}_{\mathfrak{kl}} + \iota\mu^{+}_{\mathfrak{kl}}, \vartheta^{-}_{\mathfrak{kl}} + \iota\mu^{-}_{\mathfrak{kl}})$ $(\mathfrak{k} = 1, 2, \ldots, Ƀ : \mathfrak{l} = 1, 2, \ldots, ℨ)$ are two collections of BCFSNs, then the following is obtained:

$$BCFSPDOWG\left(\mathcal{B}e_{11}, \mathcal{B}e_{12}, \ldots, \mathcal{B}e_{\mathfrak{kl}}\right) \leq BCFSPDOWG\left(\varphi e_{11}, \varphi e_{12}, \ldots, \varphi e_{\mathfrak{kl}}\right)$$

If $\mathcal{B}e_{\mathfrak{kl}} \leq \phi e_{\mathfrak{kl}}, \forall \mathfrak{k}, \mathfrak{l}$.

We can see from definitions (20) and (21) that the BCFSPDWG operator solely targets BCFS values, whereas BCFSPDOWG only targets ordered positions of BCFS values rather than the weights of the BCFS values themselves. By combining the qualities of the BCFSPDWG and BCFSPDOWG, the BCFSPDHG operator is defined below.

Definition 22: Suppose $\mathcal{B}e_{\mathfrak{kl}} = (ŋ^{+}_{\mathfrak{kl}}, Ʒ^{-}_{\mathfrak{kl}}) = (Ᵽ^{+}_{\mathfrak{kl}} + \iota Ᵽ^{+}_{\mathfrak{kl}}, Ᵽ^{-}_{\mathfrak{kl}} + \iota Ᵽ^{-}_{\mathfrak{kl}})$ $(\mathfrak{k} = 1, 2, \ldots, Ƀ, \mathfrak{l} = 1, 2, \ldots, ℨ)$ is the collection of BCFSNs, then the BCFSPDHWG operator is a mapping $\mathcal{B}e^{n} \to \mathcal{B}e$ such that

$$BCFSPDHWG\left(\mathcal{B}e_{11}, \mathcal{B}e_{12}, \mathcal{B}_{13}, \ldots, \mathcal{B}e_{Ƀℨ}\right)$$

$$= \oplus_{\mathfrak{l}=1}^{\aleph} \varpi_{\mathfrak{l}} \left(\oplus_{\mathfrak{k}=1}^{\mathfrak{P}} \mathfrak{h}_{\mathfrak{k}} \mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}} \right) \tag{21}$$

where $\varpi_{\mathfrak{l}} = \left(\frac{\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\aleph} \omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})} \right)$, $\mathfrak{h}_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{\mathfrak{P}} (1+\mathbb{R}_{\mathfrak{k}})} \right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1 \\ \mathfrak{r} \neq \mathfrak{k}}}^{\mathfrak{P}} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{rl}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1 \\ \mathfrak{s} \neq \mathfrak{l}}}^{\aleph} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{ks}}}\right)$, $Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{rl}}}\right)$ denotes the support for $\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}}$ from $\mathcal{B}_{\mathfrak{e}_{\mathfrak{rl}}}$, and $\sum_{\mathfrak{l}=1}^{\aleph} \omega_{\mathfrak{l}} = 1$. Here, $\mathcal{B}_{\mathfrak{e}_{\Xi(\mathfrak{kl})}} = \left(\mathfrak{y}^{+}_{\Xi(\mathfrak{kl})}, \mathfrak{z}^{-}_{\Xi(\mathfrak{kl})}\right)$ is the permutation of the $\mathfrak{k}th$ row and $\imath th$ largest elements of the collection for $\mathfrak{k} \times \mathfrak{l}$ BCFSNs $\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{kl}}, \mathfrak{z}^{-}_{\mathfrak{kl}}\right)$ for $\mathfrak{k} = 1, 2, \ldots, \mathfrak{P}$ *and* $\imath = 1, 2, \ldots, \aleph$. Now, using equation (16) we can define *BCFSPDOWG* AOs, as follows:

Theorem 16: Suppose $\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{kl}}, \mathfrak{z}^{-}_{\mathfrak{kl}}\right) = \left(\mathfrak{p}^{+}_{\mathfrak{kl}} + \iota \mathfrak{P}^{+}_{\mathfrak{kl}}, \mathfrak{p}^{-}_{\mathfrak{kl}} + \iota \mathfrak{P}^{-}_{\mathfrak{kl}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{P} : \imath = 1, 2, \ldots, \aleph)$ is the collection of BCFSNs, then by using BCFSPDHWG operator their calculated is again a BCFSS and

$$BCFSPDHWG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \mathcal{B}_{13}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{P}\aleph}}\right)$$
$$= \oplus_{\mathfrak{l}=1}^{\aleph} \varpi_{\mathfrak{l}} \left(\oplus_{\mathfrak{k}=1}^{\mathfrak{P}} \mathfrak{h}_{\mathfrak{k}} \mathcal{B}_{\mathfrak{e}_{\Xi(\mathfrak{kl})}} \right)$$
$$= \left(\begin{array}{c} \frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph} \varpi_{\mathfrak{l}} \left(\sum_{\mathfrak{k}=1}^{\mathfrak{P}} \mathfrak{h}_{\mathfrak{k}} \left(\frac{1-\mathfrak{p}^{+}_{\Xi(\mathfrak{kl})}}{\mathfrak{p}^{+}_{\Xi(\mathfrak{kl})}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\ +\iota \left(\frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph} \varpi_{\mathfrak{l}} \left(\sum_{\mathfrak{k}=1}^{\mathfrak{P}} \mathfrak{h}_{\mathfrak{k}} \left(\frac{1-\mathfrak{P}^{+}_{\Xi(\mathfrak{kl})}}{\mathfrak{P}^{+}_{\Xi(\mathfrak{kl})}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \right), \\ -1+\frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph} \varpi_{\mathfrak{l}} \left(\sum_{\mathfrak{k}=1}^{\mathfrak{P}} \mathfrak{h}_{\mathfrak{k}} \left(\frac{\left|\mathfrak{p}^{-}_{\Xi(\mathfrak{kl})}\right|}{1+\mathfrak{p}^{-}_{\Xi(\mathfrak{kl})}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \\ +\iota \left(-1+\frac{1}{1+\left\{\sum_{\mathfrak{l}=1}^{\aleph} \varpi_{\mathfrak{l}} \left(\sum_{\mathfrak{k}=1}^{\mathfrak{P}} \mathfrak{h}_{\mathfrak{k}} \left(\frac{\left|\mathfrak{P}^{-}_{\Xi(\mathfrak{kl})}\right|}{1+\mathfrak{P}^{-}_{\Xi(\mathfrak{kl})}}\right)^{o}\right)\right\}^{\frac{1}{o}}} \right) \end{array} \right)$$

where $\mathcal{B}_{\mathfrak{e}_{\Xi(\mathfrak{kl})}} = n\partial_{\mathfrak{k}}\rho_{\mathfrak{l}}\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}}$ is the permutation of the $\mathfrak{k}th$ row and $\imath th$ largest elements of the collection for $\mathfrak{k} \times \mathfrak{l}$ BCFSNs $\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{kl}}, \mathfrak{z}^{-}_{\mathfrak{kl}}\right)$ and $'n'$ is the balancing coefficient.

Proof: The proof theorem is like to Theorem 9.

Theorem 17: Suppose $\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{kl}}, \mathfrak{z}^{-}_{\mathfrak{kl}}\right) = \left(\mathfrak{p}^{+}_{\mathfrak{kl}} + \iota \mathfrak{P}^{+}_{\mathfrak{kl}}, \mathfrak{p}^{-}_{\mathfrak{kl}} + \iota \mathfrak{P}^{-}_{\mathfrak{kl}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{P} : \imath = 1, 2, \ldots, \aleph)$ is the collection of BCFSNs, where $\varpi_{\mathfrak{l}} = \left(\frac{\omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\aleph} \omega_{\mathfrak{l}}(1+\mathbb{T}_{\mathfrak{l}})} \right)$, $\mathfrak{h}_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{\mathfrak{P}} (1+\mathbb{R}_{\mathfrak{k}})} \right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1 \\ \mathfrak{r} \neq \mathfrak{k}}}^{\mathfrak{P}} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{rl}}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1 \\ \mathfrak{s} \neq \mathfrak{l}}}^{\aleph} Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{ks}}}\right)$, $Sup\left(\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}}, \mathcal{B}_{\mathfrak{e}_{\mathfrak{rl}}}\right)$ denotes the support for $\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}}$ from $\mathcal{B}_{\mathfrak{e}_{\mathfrak{rl}}}$, and $\sum_{\mathfrak{l}=1}^{\aleph} \omega_{\mathfrak{l}} = 1$. Then, *BCFSPDHWG* AO satisfies the following properties:

1) (Idempotency): Suppose $\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{kl}}, \mathfrak{z}^{-}_{\mathfrak{kl}}\right) = \left(\mathfrak{p}^{+}_{\mathfrak{kl}} + \iota \mathfrak{P}^{+}_{\mathfrak{kl}}, \mathfrak{p}^{-}_{\mathfrak{kl}} + \iota \mathfrak{P}^{-}_{\mathfrak{kl}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{P} : \imath = 1, 2, \ldots, \aleph)$ is the collection of BCFSNs, and all are the same i.e. $\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}} = \mathcal{B}_{\mathfrak{e}}, \forall \mathfrak{k}, \mathfrak{l}$, then the following is obtained:
$$BCFSPDHWG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}}\right) = \mathcal{B}_{\mathfrak{e}}$$

2) (Boundedness): Suppose $\mathcal{B}^{-}_{\mathfrak{e}_{\mathfrak{kl}}} = \left(\min_{\mathfrak{l}} \min_{\mathfrak{k}} \mathfrak{y}^{+}_{\mathfrak{kl}}, \max_{\mathfrak{l}} \max_{\mathfrak{k}} \mathfrak{z}^{-}_{\mathfrak{kl}}\right)$ and $\mathcal{B}^{+}_{\mathfrak{e}_{\mathfrak{kl}}} = \left(\max_{\mathfrak{l}} \max_{\mathfrak{k}} \mathfrak{y}^{+}_{\mathfrak{kl}}, \min_{\mathfrak{l}} \min_{\mathfrak{k}} \mathfrak{z}^{-}_{\mathfrak{kl}}\right)$ then,
$$\mathcal{B}^{-}_{\mathfrak{e}_{\mathfrak{kl}}} \leq BCFSPDHWG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}}\right) \leq \mathcal{B}^{+}_{\mathfrak{e}_{\mathfrak{kl}}}$$

3) (Monotonicity): Suppose $\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}} = \left(\mathfrak{y}^{+}_{\mathfrak{kl}}, \mathfrak{z}^{-}_{\mathfrak{kl}}\right) = \left(\mathfrak{p}^{+}_{\mathfrak{kl}} + \iota \mathfrak{P}^{+}_{\mathfrak{kl}}, \mathfrak{p}^{-}_{\mathfrak{kl}} + \iota \mathfrak{P}^{-}_{\mathfrak{kl}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{P}, \imath = 1, 2, \ldots, \aleph)$ and $\phi_{\mathfrak{e}_{\mathfrak{kl}}} = \left(\mathcal{R}^{+}_{\mathfrak{kl}}, \mathcal{S}^{-}_{\mathfrak{kl}}\right) = \left(\vartheta^{+}_{\mathfrak{kl}} + \iota \mu^{+}_{\mathfrak{kl}}, \vartheta^{-}_{\mathfrak{kl}} + \iota \mu^{-}_{\mathfrak{kl}}\right)$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{P} : \imath = 1, 2, \ldots, \aleph)$ are two collections of BCFSNs, then the following is obtained:
$$BCFSPDHWG\left(\mathcal{B}_{\mathfrak{e}_{11}}, \mathcal{B}_{\mathfrak{e}_{12}}, \ldots, \mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}}\right) \leq BCFSPDHWG\left(\phi_{\mathfrak{e}_{11}}, \phi_{\mathfrak{e}_{12}}, \ldots, \phi_{\mathfrak{e}_{\mathfrak{kl}}}\right)$$

If $\mathcal{B}_{\mathfrak{e}_{\mathfrak{kl}}} \leq \phi_{\mathfrak{e}_{\mathfrak{kl}}}, \forall \mathfrak{k}, \mathfrak{l}$.

V. APPLICATION

In this segment, we demonstrate the application of robotics-AI in the business. We have ten types but we discuss only four types of robotics-AI. We also use the MADM method for the solution of robotics-AI.

- **Starship Delivery Robots**

The use of delivery robots is becoming increasingly common, and Starship Technologies Company offers some of the most well-liked models in this field. Starship robots can transport packages to customers and businesses, navigate streets on their own, and carry objects up to a distance of four miles (six km). In order for the robots to comprehend their surroundings and their own location, they are outfitted with mapping systems, AI, and sensors. They weigh no more than one hundred pounds and move at a leisurely pace. The robots increase the efficiency and lower the cost of local delivery thanks to collaborations with numerous retailers and eateries. Food and packages are delivered straight from stores to customers via a smartphone app upon request. Once programmed, a smartphone can be used to track the path and location of the robots.

- **Pepper Humanoid Robot**

A humanoid robot called Pepper was created to engage with humans, offering assistance, information sharing, and customer service in retail settings. About four feet tall, with a table displaying information in the middle of its breast, Pepper has multilingual speech and gesture capabilities. AI for emotion recognition is used by the robot to comprehend human behavior and react accordingly. It can identify human emotions, like as happiness, and react appropriately by grinning, for instance. When a customer is in a store, Pepper can assist them in finding the things they want by providing tailored recommendations. Additionally, it can interact with the human staff and sell and cross-sell. Pepper works to enhance customer experience and assist businesses in cutting expenses in settings such as banks, hotels, pizzerias, and hospitals.

• **Penny Restaurant Robot**

Penny is an artificially intelligent food-service robot that resembles a bowling pin. It has the ability to independently transport food and beverages from the kitchen of a restaurant to the tables and return the dishes for cleaning. Penny in able to run in a variety of food service settings, including dining rooms, pizzerias, sizable event halls, gaming floors in casinos, restaurants, and cafes. Multiple drinks may be smoothly delivered at once by this self-driving robot. This device can mapping during peak hours or night shifts because of its long-lasting battery, which has an 8 to 12 hour lifespan. Penny's mission is to shorten wait times by bringing the plates to the busing station. As a result, the waiters may concentrate more on enhancing the dining experience for patrons, attending to their needs, and enquiring about the quality of the food.

• **Nimbo Security Robot**

Nimbo is a robotic security guard with a variety of security uses and asset protection capabilities. It is built on cutting-edge AI technology. The robot observes what people are doing and its surroundings as it investigates and patrols designated areas, paths, or self-optimized routes. When Nimbo notices a security breach, it can alert the surrounding area by light, sound, and video. It records video evidence and notifies the human guard in real time. Additionally, the human security personnel can choose where to patrol. The robot will then go around the area, continuously scanning, from one location to another. Nimbo operates in places like retail malls, warehouses, offices, educational institutions, etc. and integrates with VMSs (Video Monitoring Systems) with ease.

A. ALGORITHM MADM

Suppose $\mathfrak{B} = \{\mathfrak{B}_1, \mathfrak{B}_1, \mathfrak{B}_1, \ldots, \mathfrak{B}_l\}$ being the discrete set of options determined by the set of $\mathfrak{P}$ experts $\{\bar{\mathfrak{r}}_1, \bar{\mathfrak{r}}_2, \bar{\mathfrak{r}}_3, \ldots, \bar{\mathfrak{r}}_{\mathfrak{P}}\}$ by the conditions of $\mathfrak{V}$ attributes $\{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \ldots, \mathfrak{e}_{\mathfrak{V}}\}$. Suppose $\mathfrak{H} = (\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3, \ldots, \mathfrak{H}_{\mathfrak{P}})^T$ and $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_{\mathfrak{V}})^T$ signify correspondingly the weight vectors of the $\mathfrak{P}$ experts $\bar{\mathfrak{r}}'_{\mathfrak{k}}$ s and $\mathfrak{V}$ attributes $\mathfrak{e}'_{\mathfrak{l}}$. We use the power formula which is defined by $\omega_{\mathfrak{l}} = \left(\frac{(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\mathfrak{V}}(1+\mathbb{T}_{\mathfrak{l}})}\right)$, $\mathfrak{H}_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{\mathfrak{P}}(1+\mathbb{R}_{\mathfrak{k}})}\right)$, $\mathbb{R}_{\mathfrak{k}} = \sum_{\substack{\mathfrak{r}=1\\ \mathfrak{r}\neq\mathfrak{k}}}^{\mathfrak{P}} Sup\left(\mathcal{B}e_{\mathfrak{kl}}, \mathcal{B}e_{\mathfrak{rl}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{r}=1\\ \mathfrak{r}\neq\mathfrak{k}}}^{\mathfrak{P}} Sup\left(\mathcal{B}e_{\mathfrak{kl}}, \mathcal{B}e_{\mathfrak{rl}}\right)$, and $\mathbb{T}_{\mathfrak{l}} = \sum_{\substack{\mathfrak{s}=1\\ \mathfrak{s}\neq\mathfrak{l}}}^{\mathfrak{V}} Sup\left(\mathcal{B}e_{\mathfrak{kl}}, \mathcal{B}e_{\mathfrak{ks}}\right)$, $Sup\left(\mathcal{B}e_{\mathfrak{kl}}, \mathcal{B}e_{\mathfrak{rl}}\right)$ denotes the support for $\mathcal{B}e_{\mathfrak{kl}}$ from $\mathcal{B}e_{\mathfrak{rl}}$. Moreover the $\sum_{\mathfrak{l}=1}^{\mathfrak{V}} \omega_{\mathfrak{l}} = 1$, and $\sum_{\mathfrak{k}=1}^{\mathfrak{P}} \mathfrak{H}_{\mathfrak{k}} = 1$ must be required. To select for the determination of most superb alternative, the expert expresses their assessment in the setting of BCF-SNs $\mathcal{B}e_{\mathfrak{kl}} = (\mathfrak{y}^+_{\mathfrak{kl}}, \mathfrak{Z}^-_{\mathfrak{kl}}) = (\mathfrak{p}^+_{\mathfrak{kl}} + \iota\mathfrak{P}^+_{\mathfrak{kl}}, \mathfrak{p}^-_{\mathfrak{kl}} + \iota\mathfrak{P}^-_{\mathfrak{kl}})$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{P}, \mathfrak{l} = 1, 2, \ldots, \mathfrak{V})$ where $-1 \leq \mathfrak{y}^+_{\mathfrak{kl}} + \mathfrak{Z}^-_{\mathfrak{kl}} \leq 1$ and combined complete decision matrix is stated as M$=\left(\mathcal{B}e_{\mathfrak{kl}}\right)_{\mathfrak{P}\times\mathfrak{V}}$ and formed a BCFS decision-matrix M $= \left(\mathcal{B}e_{\mathfrak{kl}}\right)_{\mathfrak{P}\times\mathfrak{V}}$. For tackling the BCFS decision matrix, we have the underneath steps

Step 1: Collect all data in the shape of BCF soft matrix $\mathcal{B} = (\mathfrak{y}^+_{\mathfrak{kl}}, \mathfrak{Z}^-_{\mathfrak{kl}}) = (\mathfrak{p}^+_{\mathfrak{kl}} + \iota\mathfrak{P}^+_{\mathfrak{kl}}, \mathfrak{p}^-_{\mathfrak{kl}} + \iota\mathfrak{P}^-_{\mathfrak{kl}})$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{P}, \mathfrak{l} = 1, 2, \ldots, \mathfrak{V})$ associated to each option under the attributes $\omega_{\mathfrak{H}}$ $(\mathfrak{H} = 1, 2, 3, \ldots, l)$ as

$$\mathcal{B}_{\mathfrak{P}\times\mathfrak{V}} = M = \begin{bmatrix} (\mathfrak{y}^+_{11}, \mathfrak{Z}^-_{11})\ (\mathfrak{y}^+_{12}, \mathfrak{Z}^-_{12}) \ldots \left(\mathfrak{y}^+_{1\mathfrak{V}}, \mathfrak{Z}^-_{1\mathfrak{V}}\right) \\ (\mathfrak{y}^+_{21}, \mathfrak{Z}^-_{21})\ (\mathfrak{y}^+_{22}, \mathfrak{Z}^-_{22}) \ldots \left(\mathfrak{y}^+_{2\mathfrak{V}}, \mathfrak{Z}^-_{2\mathfrak{V}}\right) \\ \cdot \\ \cdot \\ \cdot \\ \left(\mathfrak{y}^+_{1\mathfrak{P}}, \mathfrak{Z}^-_{1\mathfrak{P}}\right)\ \left(\mathfrak{y}^+_{2\mathfrak{P}}, \mathfrak{Z}^-_{2\mathfrak{P}}\right) \ldots \left(\mathfrak{y}^+_{\mathfrak{P}\mathfrak{V}}, \mathfrak{Z}^-_{\mathfrak{P}\mathfrak{V}}\right) \end{bmatrix}$$

Step 2: For the computing the WVs for experts and attributes we signify the power formula defined by

$$\omega_{\mathfrak{l}} = \left(\frac{(1+\mathbb{T}_{\mathfrak{l}})}{\sum_{\mathfrak{l}=1}^{\mathfrak{V}}(1+\mathbb{T}_{\mathfrak{l}})}\right) \text{and} \mathfrak{H}_{\mathfrak{k}} = \left(\frac{(1+\mathbb{R}_{\mathfrak{k}})}{\sum_{\mathfrak{k}=1}^{\mathfrak{P}}(1+\mathbb{R}_{\mathfrak{k}})}\right)$$

Step 3: For normalization of BCFS decision-matrix, employ the following formula.

$$\mathfrak{S}_{|} = \begin{cases} (\mathfrak{y}^+_{\mathfrak{kl}}, \mathfrak{Z}^-_{\mathfrak{kl}}) = (\mathfrak{p}^+_{\mathfrak{kl}} + \iota\mathfrak{P}^+_{\mathfrak{kl}}, \mathfrak{p}^-_{\mathfrak{kl}} + \iota\mathfrak{P}^-_{\mathfrak{kl}})\ forbenefittype \\ (\mathfrak{y}^+_{\mathfrak{kl}}, \mathfrak{Z}^-_{\mathfrak{kl}})^c = \begin{pmatrix} 1 + \mathfrak{p}^+_{\mathfrak{kl}} + i\left(1 - \mathfrak{P}^+_{\mathfrak{kl}}\right), \\ -1 - \mathfrak{p}^-_{\mathfrak{kl}} + i\left(-1 - \mathfrak{P}^-_{\mathfrak{kl}}\right) \end{pmatrix} \\ \quad forcosttype \end{cases}$$

Step 4: Aggregate the BCFSNs $\mathcal{B}e_{\mathfrak{kl}}$ $(\mathfrak{k} = 1, 2, \ldots, \mathfrak{P}, \mathfrak{l} = 1, 2, \ldots, \mathfrak{V})$ for every alternative $\mathsf{A}_{\mathfrak{H}}$ $(\mathfrak{H} = 1, 2, 3, \ldots, l)$ into combined decision-matrix by employing BCFSPDAA or BCFSPDGA operators.

Step 5: By the assistance of Eq. (2), we determine the score value of each alternative.

Step 6: We would rank the alternatives by employing the determined score values.

Step 7: End.

B. NUMERICAL EXAMPLE

Some experts decide to select best robot in the AI field. For this AI experts that is $\bar{\mathfrak{r}}_1$, $\bar{\mathfrak{r}}_2$, $\bar{\mathfrak{r}}_3$, $\bar{\mathfrak{r}}_4$, and $\bar{\mathfrak{r}}_5$ with their WVs $\mathfrak{H} = (\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3, \ldots, \mathfrak{H}_{\mathfrak{k}})$, who will examine and

TABLE 1. BCFSS data table.

$\mathcal{B}_{e_{\mathfrak{t}I}}$	$\underset{\sim}{e}_1$	$\underset{\sim}{e}_2$	$\underset{\sim}{e}_3$	$\underset{\sim}{e}_4$
$\bar{r}_1$	$(0.1+\iota 0.3, -0.5-\iota 0.7)$	$(0.9+\iota 0.7, -0.5-\iota 0.3)$	$(0.1+\iota 0.5, -0.9-\iota 0.13)$	$(0.2+\iota 0.3, -0.4-\iota 0.5)$
$\bar{r}_2$	$(0.2+\iota 0.4, -0.6-\iota 0.8)$	$(0.11+\iota 0.8, -0.5-\iota 0.2)$	$(0.17+\iota 0.21, -0.25-\iota 0.29)$	$(0.12+\iota 0.13, -0.15-\iota 0.14)$
$\bar{r}_3$	$(0.1+\iota 0.4, -0.6-\iota 0.8)$	$(0.8+\iota 0.6, -0.4-\iota 0.1)$	$(0.33+\iota 0.37, -0.41-\iota 0.45)$	$(0.22+\iota 0.23, -0.25-\iota 0.24)$
$\bar{r}_4$	$(0.2+\iota 0.5, -0.8-\iota 0.11)$	$(0.8+\iota 0.6, -0.4-\iota 0.2)$	$(0.49+\iota 0.53, -0.57-\iota 0.61)$	$(0.32+\iota 0.33, -0.35-\iota 0.34)$
$\bar{r}_5$	$(0.3+\iota 0.5, -0.7-\iota 0.9)$	$(0.7+\iota 0.5, -0.3-\iota 0.1)$	$(0.65+\iota 0.69, -0.73-\iota 0.77)$	$(0.21+\iota 0.32, -0.53-\iota 0.44)$

TABLE 2. BCFSS data table.

$\mathcal{B}_{e_{\mathfrak{t}I}}$	$\underset{\sim}{e}_1$	$\underset{\sim}{e}_2$	$\underset{\sim}{e}_3$	$\underset{\sim}{e}_4$
$\bar{r}_1$	$(0.21+\iota 0.32, -0.53-\iota 0.44)$	$(0.65+\iota 0.69, -0.73-\iota 0.77)$	$(0.7+\iota 0.5, -0.3-\iota 0.1)$	$(0.3+\iota 0.5, -0.7-\iota 0.9)$
$\bar{r}_2$	$(0.32+\iota 0.33, -0.35-\iota 0.34)$	$(0.49+\iota 0.53, -0.57-\iota 0.61)$	$(0.8+\iota 0.6, -0.4-\iota 0.2)$	$(0.2+\iota 0.5, -0.8-\iota 0.11)$
$\bar{r}_3$	$(0.12+\iota 0.13, -0.15-\iota 0.14)$	$(0.17+\iota 0.21, -0.25-\iota 0.29)$	$(0.11+\iota 0.8, -0.5-\iota 0.2)$	$(0.2+\iota 0.4, -0.6-\iota 0.8)$
$\bar{r}_4$	$(0.22+\iota 0.23, -0.25-\iota 0.24)$	$(0.33+\iota 0.37, -0.41-\iota 0.45)$	$(0.8+\iota 0.6, -0.4-\iota 0.1)$	$(0.1+\iota 0.4, -0.6-\iota 0.8)$
$\bar{r}_5$	$(0.2+\iota 0.3, -0.4-\iota 0.5)$	$(0.1+\iota 0.5, -0.9-\iota 0.13)$	$(0.9+\iota 0.7, -0.5-\iota 0.3)$	$(0.1+\iota 0.3, -0.5-\iota 0.7)$

provide their opinion about the 4 considered robots that are ß_1, ß_2, ß_3, and ß_4 under the parameters $E = (\underset{\sim}{e}_1 = Starshipdeliveryrobot, \underset{\sim}{e}_2 = Pepperhumanoidrobot, \underset{\sim}{e}_3 = Pennyrestraurantrobot, \underset{\sim}{e}_4 = Nimbosecurityrobot)$ in the exemplary of BCFSNs. The associated with parameter's WVs are computed with the rest power formula where the $\sum_{I=1}^{\aleph} \omega_I = 1$. To determine the most effective robot, we employ the underneath techniques.

C. BCFSPDAA OPERATOR

Step 1: W e collect two tables for BCFSN, then we represented as blow:

Step 2: Computed WVs for experts and attributes by using power formula we obtained the following WVs:

$$\mathfrak{H}_{\mathfrak{k}} = (0.206112, 0.195926, 0.190518, 0.201333, 0.206112)$$
$$\omega_I = (0.255694, 0.243114, 0.244968, 0.256224)$$

Step 3: For normalization of BCFS decision-matrix, we consider Table 1 for the benefits type

Step 4: Aggregate the BCFSNs $\mathcal{B}_{e_{\mathfrak{t}I}}$ ($\mathfrak{t} = 1, 2, \ldots, \mathfrak{E}$, $I = 1, 2, \ldots, \aleph$) for every alternative $\text{ß}_{\mathfrak{K}}$ ($\mathfrak{K} = 1, 2, 3, \ldots, l$) into combined decision-matrix by employing BCFSPDAA operators by equation (8), we attained:

$$\mathcal{B}_{\underset{\sim}{e}_{11}} = (2.9877 \times 10^{-06} + 0.001675296\iota, -0.99958 - 0.99978\iota),$$
$$\mathcal{B}_{\underset{\sim}{e}_{12}} = (0.272838984 + 0.156126228\iota, -0.95563 - 0.3527\iota),$$
$$\mathcal{B}_{\underset{\sim}{e}_{13}} = (0.000180673 + 0.003440338\iota, -0.99882 - 0.96657\iota),$$
$$\mathcal{B}_{\underset{\sim}{e}_{14}} = (1.08675 \times 10^{-05} + 3.90595 \times 10^{-05}\iota, -0.72918 - 0.69703\iota)$$

Step 5: By the assistance of Eq. (2), we determine the score value of each alternative,

$$\text{ß}_1 = 0.00079244, \text{ß}_2 = 0.359515325, \text{ß}_3 = 0.009559022, \text{ß}_4 = 0.14359854$$

Step 6: We would rank the alternatives by employing the determined score values:

$$\text{ß}_2 > \text{ß}_4 > \text{ß}_3 > \text{ß}_1$$

Hence, we have the best and powerful robot ß_2 in AI.

Step 7: End.

D. BCFSPDGA OPERATOR

Step 1 to Step 3: Here, we have the same process as above.

Step 4: Aggregate the BCFSNs $\mathcal{B}_{e_{\mathfrak{t}I}}$ ($\mathfrak{t} = 1, 2, \ldots, \mathfrak{E}$, $I = 1, 2, \ldots, \aleph$) for every alternative $\text{ß}_{\mathfrak{K}}$ ($\mathfrak{K} = 1, 2, 3, \ldots, l$) into combined decision-matrix by employing BCFSPDGA operators by equation (16), we obtained:

$$\mathcal{B}_{\underset{\sim}{e}_{11}} = (0.035308 + 0.955792\iota, -0.084180706 - 0.159750721\iota),$$
$$\mathcal{B}_{\underset{\sim}{e}_{12}} = (0.999782 + 0.999581\iota, -0.008033033 - 1.51395 \times 10^{-06}\iota),$$
$$\mathcal{B}_{\underset{\sim}{e}_{13}} = (0.688825 + 0.978011\iota, -0.031668404 - 0.00119568\iota),$$
$$\mathcal{B}_{\underset{\sim}{e}_{14}} = (0.117486 + 0.334776\iota, -0.000104215 - 09.52529 \times 10^{-05}\iota)$$

Step 5: By the assistance of Eq. (2), we determine the score value of each alternative,

$$\text{ß}_1 = 1.373584, \text{ß}_2 = 1.999264, \text{ß}_3 = 1.816986,$$

TABLE 3. Comparative data of BCFSPDAO.

Techniques	Score values	Ranking
Mahmood and Rehman [25]	Unsuccessful	Unsuccessful
Naeem et al [26]	Unsuccessful	Unsuccessful
Jaleel [41]	Unsuccessful	Unsuccessful
BCFSDAAO	$ß_1 =$ 0.00079244, $ß_2 =$ 0.359515325, $ß_3 =$ 0.009559022, $ß_4 =$ 0.14359854	$ß_2 > ß_4 > ß_3 > ß_1$
BCFSDGAO	$ß_1 = 1.373584$, $ß_2 = 1.999264$, $ß_3 = 1.816986$, $ß_4 = 1.226031$	$ß_2 > ß_3 > ß_1 > ß_4$

$ß_4 = 1.226031$

Step 6: We would rank the alternatives by employing the determined score values:

$$ß_2 > ß_3 > ß_1 > ß_4$$

Hence, we have the best and powerful robot $ß_2$ in AI.

Step 7: End.

VI. COMPARATIVE ANALYSIS

We will now start the process of comparing the accepted ideas with our proposed work. Three concepts have been identified in relevant studies. Dombi AOs are used in Mahmood and Rehman's [25] MADM approach while working with BCF data. This study addresses most of the unclear and complex concerns. Then, Naeem et al. [26] described how a DM approach based on BCF frank power AOs was used to categorize renewable energy and its sources. Most of these problems have been overcome in our proposed model. Thirdly, Jaleel [41] developed the WASPAS technique for use in agricultural robotics systems based on Dombi AOs under BCFS Information. This work also discusses these issues. After this, we may conclude that we solved every issue that the first three articles failed to cover. Mahmood and Rehman were the first researchers to examine this topic [25]. They examined the impacts of BCFS on dombi AO. Our study incorporates BCFSPDAO, which outperforms Mahmood and Rehman [25] due to the latter's inefficient power and SS. Subsequently, we discussed our research and compared it with the study performed by Naeem et al. [26]. We concluded that our work has a more significant influence than that of Naeem et al. [26] since no SS was employed in this work. Thirdly, we discussed the work of Jaleel [41], who compared BCFSDAO and BCFSPDAO, the latter of which is influenced to a lesser extent by our work since no power is employed. Based on the above explanation, we can conclude that our research efforts have been more fruitful than previous endeavors. This discussion is summarized in Table 3.

VII. CONCLUSION

In this work, we explored how BCFSPDAO was established by incorporating authority over BCFSDAO. We started by examining BCFSPDG and BCFSPDA AOs, after which we identified three varieties of BCFSPDA AOs and BCFSPDG AOs. Subsequently, we were able to narrow down three categories of BCFSPDA AOs: BCFSPDWA AO, BCFSPDOWA AO, and BCFSPDHA AO. Similarly, there are three types of BCFSPDG AO: BCFSPDWG AO, BCFSPDOWG AO, and BCFSPDHG AO. Subsequently, the topic of artificial intelligence and robots was discussed. In the study, the previously specified procedures were performed. We employed a numerical model that was resolved by the MADM method. Robots make up the numerical model in AI. We thus selected the robot with the best performance. The BCFSPDGAO and BCFSPDAAO ranking series are found here. The $ß_2$ that is generated in BCFSPDGAO and BCFSPDAAO is the same, despite differences in the ranking series. We wrapped up this study by drawing comparisons with common notions of supremacy and dominance. Our results demonstrate that every problem caused by BCFSS can be resolved and managed with ease.

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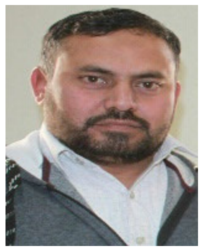

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