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RESEARCH ARTICLE

Time-Domain Analysis of Electromagnetic Fields in Conical Cavities With Modified EAE

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ABSTRACT In this study, we apply a novel format of Maxwell's equations in SI Units for analyzing electromagnetic fields in conical cavities, with a special focus on time-domain analysis. This unique approach aligns the dimensions of electric (E) and magnetic (H) fields as inverse meters, thereby facilitating theoretical investigations into complex geometrical field behaviors. Our primary focus is on deriving evolutionary equations for electromagnetic fields within conical structures. Additionally, this study serves as a stepping stone for future in-depth research into the mechanical properties of electromagnetic fields, particularly due to the unified dimensional approach of E and H fields. This method is expected to provide more insightful perspectives in understanding the dynamics of electromagnetic fields in conical cavities. The implications of this research extend to practical applications, notably in the design and analysis of microwave resonant cavities and conical antennas, enhancing our comprehension of electromagnetic phenomena in specialized structures within the broader scope of electrodynamics.

INDEX TERMS Maxwell's equations, conical cavities, evolutionary electrodynamics, time domain.

I. INTRODUCTION

The exploration of electromagnetic fields within conical geometries has garnered significant attention due to its implications in advanced technologies, notably biconical transmission lines [1], conical antennas [2], and the concepts like the EMDrive (RF resonant cavity thrusters) [3], [4]. Stemming from the seminal works of Smith and Tai [5], [6], research in this area has continually evolved to demystify the complex behaviors exhibited by fields in these unique structures [7]. Our study builds upon these foundational theories and subsequent advancements in electromagnetic field analysis.

In the realm of conical cavity electrodynamics, the Evolutionary Approach to Electrodynamics (EAE) has played an essential role in enhancing our understanding of these fields in the time domain [8], [9], [10], [11], [12], [13], [14], [15], [16]. However, previous investigations utilizing the EAE focused on electromagnetic fields represented in regular SI units, which somewhat constrained the analysis of mechanical properties like mass and inertia. Inspired by Kaiser's theoretical exploration [17], which reexamines the mechanical properties of electromagnetic fields in non-SI units—specifically employing a unique perspective where electromagnetic fields are considered with common physical dimensions to facilitate the analysis of their mechanical properties—our research adopts an innovative approach. Unlike Kaiser, who uses the CGS system's flexibility to align the dimensions of electric and magnetic fields by setting ϵ_0 and μ_0 for vacuum to 1, our methodology utilizes an inverse meter, $\lfloor 1/\mathrm{m} \rfloor$, unit representation for electric, $\vec{\mathbb{E}}(\mathbf{r},t)$, and magnetic, $\vec{\mathbb{H}}(\mathbf{r},t)$, fields [18], [19]. This novel alignment within the SI unit system diverges from Kaiser's approach by ensuring the physical dimensions of $\vec{\mathbb{E}}$ and $\vec{\mathbb{H}}$ are unified for a broader, more universally accessible analysis in conical structures. While this paper lays the groundwork for future study on the mechanical characteristics of electromagnetic fields, aligning with Kaiser's goal of

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dimensional commonality, it establishes the infrastructure within the SI framework, ready for detailed exploration of these mechanical properties in subsequent research.

A key focus of our work is the derivation of evolutionary equations for electromagnetic fields in conical geometries, an essential step toward solving modal amplitudes in such structures. While the development of analytical and numerical solution techniques has been instrumental in advancing the field [20], [21], [22], [23], [24], [25], [26], our study primarily concentrates on the theoretical formulation of evolutionary equations in conical configurations using the novel format of Maxwell's equations [19]. Rather than presenting numerical solutions, our focus is on the conceptual and analytical framework that lays the groundwork for future numerical analyses and applications in complex electromagnetic scenarios.

This work contributes to the ongoing discourse in the field, providing a foundational framework for future research and explorations. It opens doors for potential practical applications in areas such as advanced antenna design and microwave technology, thus contributing to both the theoretical and practical realms of electromagnetic research.

II. DEFINITION OF THE PROBLEM AND METHOD

In this section, we define the problem of analyzing electromagnetic fields in conical cavities in time domain.

The focus is on conical geometry converging towards a single vertex, made of perfectly electric conducting (PEC) material, see Figure 1. We introduce a novel approach by scaling the dimensions of electric and magnetic fields into a common unit, inverse meters, and electric and magnetic current densities into a common unit, inverse meters squared. This scaling is fundamental to our method also as in [17]. We then apply these scaled novel fields and densities to reformulate Maxwell's equations in SI units.

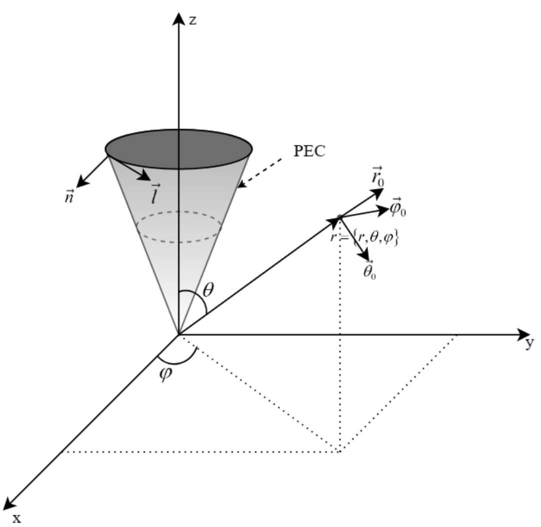

FIGURE 1. **Geometry of the problem.**

This representation is not only mathematically elegant but also offers a clearer insight into the dynamics of electromagnetic fields in conical structures. The culmination of this section is the establishment of a theoretical framework, setting the stage for the derivation of evolutionary equations that are crucial to obtaining the amplitudes of the fields.

The redefined scaling of standard electric and magnetic field vectors, $\vec{\mathcal{E}}(\vec{r},t)$ and $\vec{\mathcal{H}}(\vec{r},t)$, along with standard electric and magnetic current densities, $\vec{\mathcal{J}}(\vec{r},t)$ and $\vec{\mathcal{M}}(\vec{r},t)$, has been introduced as a transformative approach in electrodynamics research. This scaling adjusts the dimensions of these field vectors to a uniform inverse meter unit for electric and magnetic fields, marking a significant shift in how electromagnetic problems are approached and analyzed, as highlighted in various seminal studies [18], [19] as:

$$\left.\begin{aligned}
\underbrace{\vec{\mathcal{E}}(\mathbf{r},t)}_{\lfloor \mathrm{V/m} \rfloor} &= \underbrace{\epsilon_0^{\mathrm{V}}}_{\lfloor \mathrm{V} \rfloor}\underbrace{\vec{\mathbb{E}}(\mathbf{r},t)}_{\lfloor 1/\mathrm{m} \rfloor} = \underbrace{3.361\times 10^5}_{\lfloor \mathrm{V} \rfloor} \times \underbrace{\vec{\mathbb{E}}(\mathbf{r},t)}_{\lfloor 1/\mathrm{m} \rfloor}\\
\underbrace{\vec{\mathcal{H}}(\mathbf{r},t)}_{\lfloor \mathrm{A/m} \rfloor} &= \underbrace{\mu_0^{\mathrm{A}}}_{\lfloor \mathrm{A} \rfloor}\underbrace{\vec{\mathbb{H}}(\mathbf{r},t)}_{\lfloor 1/\mathrm{m} \rfloor} = \underbrace{8.921\times 10^2}_{\lfloor \mathrm{A} \rfloor} \times \underbrace{\vec{\mathbb{H}}(\mathbf{r},t)}_{\lfloor 1/\mathrm{m} \rfloor}\\
\underbrace{\vec{\mathcal{J}}(\mathbf{r},t)}_{\lfloor \mathrm{A/m^2} \rfloor} &= \underbrace{\mu_0^{\mathrm{A}}}_{\lfloor \mathrm{A} \rfloor}\underbrace{\vec{\mathbb{J}}(\mathbf{r},t)}_{\lfloor 1/\mathrm{m^2} \rfloor} = \underbrace{8.921\times 10^2}_{\lfloor \mathrm{A} \rfloor} \times \underbrace{\vec{\mathbb{J}}(\mathbf{r},t)}_{\lfloor 1/\mathrm{m^2} \rfloor}\\
\underbrace{\vec{\mathcal{M}}(\mathbf{r},t)}_{\lfloor \mathrm{V/m^2} \rfloor} &= \underbrace{\epsilon_0^{\mathrm{V}}}_{\lfloor \mathrm{V} \rfloor}\underbrace{\vec{\mathbb{M}}(\mathbf{r},t)}_{\lfloor 1/\mathrm{m^2} \rfloor} = \underbrace{3.361\times 10^5}_{\lfloor \mathrm{V} \rfloor} \times \underbrace{\vec{\mathbb{M}}(\mathbf{r},t)}_{\lfloor 1/\mathrm{m^2} \rfloor}
\end{aligned}\right\} \quad (1)$$

where ϵ_0^{V} and μ_0^{A} represent the new permittivity and permeability values, respectively, with the former having a *volt,* $\lfloor \mathrm{V} \rfloor$, dimension and the latter an *ampere,* $\lfloor \mathrm{A} \rfloor$ dimension. The field vectors $\vec{\mathbb{E}}$ and $\vec{\mathbb{H}}$ are now characterized by a unified dimension of inverse meters, while the current vectors $\vec{\mathbb{J}}$ and $\vec{\mathbb{M}}$ both share a dimensionality of inverse meters squared.

Incorporating the scaled dimensions as outlined in Equation (1) into the traditional framework of Maxwell's equations leads to an innovative and reformulated version of these equations within the framework of SI units as

$$\nabla\times\vec{\mathbb{H}}(\mathbf{r},t) = \frac{1}{c}\frac{\partial}{\partial t}\vec{\mathbb{E}}(\mathbf{r},t) + \vec{\mathbb{J}}(\mathbf{r},t) \quad (2a)$$

$$\nabla\times\vec{\mathbb{E}}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{\mathbb{H}}(\mathbf{r},t) - \vec{\mathbb{M}}(\mathbf{r},t) \quad (2b)$$

$$\nabla\cdot\vec{\mathbb{E}}(\mathbf{r},t) = \varrho;\ \nabla\cdot\vec{\mathbb{H}}(\mathbf{r},t) = \varrho_m. \quad (2c)$$

In the revised Maxwell's equations (2), the quantities ϱ and ϱ_m are introduced, both having a dimensional specification of inverse square meters. These quantities are related to the conventional densities ρ and ρ_m through the relationships $\rho = \sqrt{\mathrm{N}\epsilon_0}\varrho$ and $\rho_m = \sqrt{\mathrm{N}\mu_0}\varrho_m$, respectively, where $\lfloor \mathrm{N} \equiv \mathrm{kgm/s^2} \rfloor$, representing *newtons*, is the unit of force.

Consider the case where a wave source is positioned near the central point of our spherical coordinate framework. This source generates electric and magnetic fields, symbolized by $\vec{\mathbb{E}}(\mathbf{r},t)$ and $\vec{\mathbb{H}}(\mathbf{r},t)$, in addition to electric and magnetic current densities, represented by $\vec{\mathbb{J}}(\mathbf{r},t)$ and $\vec{\mathbb{M}}(\mathbf{r},t)$, along the surfaces of our conical structure. These generated quantities serve as auxiliary sources, instigating transient electromagnetic fields around the primary wave source.

In this specific case, to accurately address the problem, it's necessary to supplement Maxwell's equations, as referenced in (2), with well-defined boundary conditions suitable for PEC cone surfaces:

$$\vec{l}\cdot\vec{\mathbb{E}}|_L = 0; \quad \vec{n}\cdot\vec{\mathbb{H}}|_L = 0; \vec{r}_0\cdot\vec{\mathbb{E}}|_L = 0 \tag{3}$$

where the normal vector relative to the conical geometry's side surface is denoted as $\vec{n}$, the unit tangential vector along the contour is represented by $\vec{l}$, and the radial unit vector is indicated as $\vec{r}_0$. The problem also necessitates the addition of initial conditions, which specify the starting state of the electromagnetic fields in the conical geometry as

$$\vec{\mathbb{E}}(\mathbf{r},t)|_{t=0} = \mathbf{0}, \vec{\mathbb{H}}(\mathbf{r},t)|_{t=0} = \mathbf{0}. \tag{4}$$

III. MODAL BASIS AND FIELD DECOMPOSITIONS

A. MAXWELL'S EQUATIONS IN ANGULAR-RADIAL FORM

The new field vectors, $\vec{\mathbb{E}}(\mathbf{r},t)$ and $\vec{\mathbb{H}}(\mathbf{r},t)$, and the new density vectors, $\vec{\mathbb{J}}(\mathbf{r},t)$, and $\vec{\mathbb{M}}(\mathbf{r},t)$, are decomposed into two parts: a two-dimensional angular (transverse) vector with subscript $_\perp$ and a separate radial component with subscript $_r$ as:

$$\vec{\mathbb{E}} = \vec{\mathbb{E}}_\perp + \vec{r}_0\mathbb{E}_r; \vec{\mathbb{H}} = \vec{\mathbb{H}}_\perp + \vec{r}_0\mathbb{H}_r;$$
$$\vec{\mathbb{J}} = \vec{\mathbb{J}}_\perp + \vec{r}_0\mathbb{J}_r; \vec{\mathbb{M}} = \vec{\mathbb{M}}_\perp + \vec{r}_0\mathbb{M}_r \tag{5}$$

for the problem under study defined in the spherical coordinate system. The operator ∇ is also divided into a combination of the angular derivative operator $\nabla_\perp$ and the radial derivative $\frac{\partial}{\partial r}$ as:

$$\nabla = \vec{\theta}_0\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{\varphi}_0\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} + \vec{r}_0\frac{\partial}{\partial r} = \frac{1}{r}\nabla_\perp + \vec{r}_0\frac{\partial}{\partial r} \tag{6}$$

where

$$\nabla_\perp = \vec{\theta}_0\partial_\theta + \vec{\varphi}_0\frac{1}{\sin\theta}\partial_\varphi. \tag{7}$$

When reformulated in angular-radial terms, the divergent Maxwell's equations (2c) yields the following:

$$r^{-2}\partial_r\left(r^2\mathbb{E}_r\right) + r^{-1}\nabla_\perp\cdot\vec{\mathbb{E}}_\perp = \varrho$$
$$r^{-2}\partial_r\left(r^2\mathbb{H}_r\right) + r^{-1}\nabla_\perp\cdot\vec{\mathbb{H}}_\perp = \varrho_m. \tag{8}$$

Projecting curl Maxwell's equations (2a)-(2b) in the radial direction yields the following:

$$r^{-1}\nabla_\perp\cdot\left[\vec{\mathbb{H}}_\perp\times\vec{r}_0\right] = \frac{1}{c}\partial_t\mathbb{E}_r + \mathbb{J}_r$$
$$-r^{-1}\nabla_\perp\cdot\left[\vec{\mathbb{E}}_\perp\times\vec{r}_0\right] = \frac{1}{c}\partial_t\mathbb{H}_r + \mathbb{M}_r, \tag{9}$$

and projecting of them (2a)-(2b) in the angular direction yields:

$$-r^{-1}\left(\partial_r\left(r\left[\vec{\mathbb{H}}_\perp\times\vec{r}_0\right]\right) + \left[\vec{r}_0\times\nabla_\perp\mathbb{H}_r\right]\right)$$
$$= \frac{1}{c}\partial_t\vec{\mathbb{E}}_\perp + \vec{\mathbb{J}}_\perp$$
$$r^{-1}\left(\partial_r\left(r\left[\vec{\mathbb{E}}_\perp\times\vec{r}_0\right]\right) + \left[\vec{r}_0\times\nabla_\perp\mathbb{E}_r\right]\right)$$
$$= \frac{1}{c}\partial_t\vec{\mathbb{H}}_\perp + \vec{\mathbb{M}}_\perp. \tag{10}$$

By applying Equations (8) and (9), we can eliminate the radial field components in Equation (10). This process leads to a set of second-order equations primarily involving the angular components:

$$\left[\vec{r}_0\times\nabla_\perp\right]\nabla_\perp\cdot\vec{\mathbb{H}}_\perp$$
$$= r^{-1}\partial_r r^3\left\{\frac{1}{c}\partial_t\vec{\mathbb{E}}_\perp + r^{-1}\partial_r\left[r\vec{\mathbb{H}}_\perp\times\vec{r}_0\right]\right\}$$
$$+\left\{r^{-1}\partial_r r^3\vec{\mathbb{J}}_\perp + r\left[\vec{r}_0\times\nabla_\perp\varrho_m\right]\right\} \tag{11}$$

$$\left[\nabla_\perp\times\vec{r}_0\right]\nabla_\perp\cdot\vec{\mathbb{E}}_\perp$$
$$= r^{-1}\partial_r r^3\left\{\frac{1}{c}\partial_t\vec{\mathbb{H}}_\perp + r^{-1}\partial_r\left[\vec{r}_0\times r\vec{\mathbb{E}}_\perp\right]\right\}$$
$$+\left\{r^{-1}\partial_r\left(r^3\vec{\mathbb{M}}_\perp\right) + r\left[\nabla_\perp\varrho\times\vec{r}_0\right]\right\} \tag{12}$$

$$\nabla_\perp\left[\vec{r}_0\times\nabla_\perp\right]\cdot\vec{\mathbb{E}}_\perp$$
$$= -r^2\partial_t\left\{\frac{1}{c}r^{-1}\partial_r\left(r\vec{\mathbb{H}}_\perp\right) + \frac{1}{c^2}\partial_t\left(\left[\vec{r}_0\times\vec{\mathbb{E}}_\perp\right]\right)\right\}$$
$$-r\left\{\frac{1}{c}r\partial_t\left[\vec{r}_0\times\vec{\mathbb{J}}_\perp\right] + \nabla_\perp\mathbb{M}_r\right\} \tag{13}$$

$$\nabla_\perp\left[\nabla_\perp\times\vec{r}_0\right]\cdot\vec{\mathbb{H}}_\perp$$
$$= -r^2\partial_t\left\{\frac{1}{c}r^{-1}\partial_r\left(r\vec{\mathbb{E}}_\perp\right) + \frac{1}{c^2}\partial_t\left[\vec{\mathbb{H}}_\perp\times\vec{r}_0\right]\right\}$$
$$-r\left\{\nabla_\perp\mathbb{J}_r + r\frac{1}{c}\partial_t\left[\vec{\mathbb{M}}_\perp\times\vec{r}_0\right]\right\}. \tag{14}$$

Correspondingly, the boundary conditions detailed in (3) are represented as follows

$$\vec{l}\cdot\vec{\mathbb{E}}_\perp|_L = 0, \quad \vec{n}\cdot\vec{\mathbb{H}}_\perp|_L = 0,$$
$$\nabla_\perp\cdot\vec{\mathbb{E}}_\perp|_L = 0, \quad \nabla_\perp\cdot\left[\vec{\mathbb{H}}_\perp\times\vec{r}_0\right]|_L = 0. \tag{15}$$

By combining the two-dimensional angular vector components into a single four-dimensional vector, labeled as $\vec{\mathbb{X}}_\perp$ which is formed by the collation of $\vec{\mathbb{X}}_\perp = col(\vec{\mathbb{E}}_\perp,\vec{\mathbb{H}}_\perp)$, the Hilbert space $\mathfrak{L}_2^4(\mathrm{S})$, where S is the sphere surface with the center in the origin, is introduced. This space serves as the domain for resolutions to the initial boundary value problem outlined in this study, as referenced in (18) of [11].

The reformulation of the novel format of Maxwell's equations presented in (2) into an operator framework in the spherical coordinates of the problem, while integrating the boundary conditions (15), leads to the development of two distinct linear operators, designated as W_H and W_E:

$$W_H\vec{\mathbb{X}}_\perp = \begin{pmatrix} 0 & \left[\vec{r}_0\times\nabla_\perp\right]\nabla_\perp\cdot \\ \nabla_\perp\left[\vec{r}_0\times\nabla_\perp\right]\cdot & 0 \end{pmatrix}\begin{pmatrix}\vec{\mathbb{E}}_\perp \\ \vec{\mathbb{H}}_\perp\end{pmatrix} \tag{16}$$

$$W_E\vec{\mathbb{X}}_\perp = \begin{pmatrix} 0 & \nabla_\perp\left[\nabla_\perp\times\vec{r}_0\right]\cdot \\ \left[\nabla_\perp\times\vec{r}_0\right]\nabla_\perp\cdot & 0 \end{pmatrix}\begin{pmatrix}\vec{\mathbb{E}}_\perp \\ \vec{\mathbb{H}}_\perp\end{pmatrix}. \tag{17}$$

The equations from (11)-(15) are arranged into a compact operator format, placing transverse (*angular*) derivatives (16)-(17) to the left side, and radial and time derivatives along with sources (18)-(19), on the right side:

$$W_H\vec{\mathbb{X}}_\perp=\begin{pmatrix} r^{-1}\partial_r r^3\left\{\frac{1}{c}\partial_t\vec{\mathbb{E}}_\perp+r^{-1}\partial_r\left[r\vec{\mathbb{H}}_\perp\times\vec{r}_0\right]\right\} \\ +\left\{r^{-1}\partial_r\left(r^3\vec{\mathbb{J}}_\perp\right)+r\left[\vec{r}_0\times\nabla_\perp\varrho_m\right]\right\} \\ -r^2\partial_t\left\{\frac{1}{c}r^{-1}\partial_r\left(r\vec{\mathbb{H}}_\perp\right)+\frac{1}{c}\partial_t\left[\vec{r}_0\times\vec{\mathbb{E}}_\perp\right]\right\} \\ -r\left\{\frac{1}{c}r\partial_t\left[\vec{r}_0\times\vec{\mathbb{J}}_\perp\right]+\nabla_\perp\mathbb{M}_r\right\}\end{pmatrix} \tag{18}$$

$$W_E\vec{\mathbb{X}}_\perp=\begin{pmatrix} -r^2\partial_t\left\{\frac{1}{c}r^{-1}\partial_r\left(r\vec{\mathbb{E}}_\perp\right)+\frac{1}{c}\partial_t\left[\vec{\mathbb{H}}_\perp\times\vec{r}_0\right]\right\} \\ -r\left\{\nabla_\perp\mathbb{J}_r+r\frac{1}{c}\partial_t\left[\vec{\mathbb{M}}_\perp\times\vec{r}_0\right]\right\} \\ r^{-1}\partial_r r^3\left\{\frac{1}{c}\partial_t\vec{\mathbb{H}}_\perp+r^{-1}\partial_r\left[\vec{r}_0\times r\vec{\mathbb{E}}_\perp\right]\right\} \\ +\left\{r^{-1}\partial_r\left(r^3\vec{\mathbb{M}}_\perp\right)+r\left[\nabla_\perp\varrho\times\vec{r}_0\right]\right\}\end{pmatrix} \tag{19}$$

This structure simplifies the equations' representation and facilitates a clearer understanding of their interplay and implications. Note that equation (18) corresponds directly to equations (11), (13), and (15), while equation (19) aligns with equations (12), (14), and (15).

B. MODAL BASIS FOR THE CONICAL GEOMETRY

It is demonstrated that operators W_H and W_E exhibit self-adjoint properties [11]. The eigenfunctions corresponding to these operators constitute the modal basis within the Hilbert functional space $\mathfrak{L}_2^4(\mathrm{S})$.

1) MODAL BASIS FOR THE TE-WAVES

The eigenvalue equation holds for operator W_H as

$$W_H\vec{\mathbb{X}}_{\perp m}^H=p_m^2\vec{\mathbb{X}}_{\perp m}^H \tag{20}$$

where p_m signifies an eigenvalue associated with the eigenfunction $\vec{\mathbb{X}}_{\perp m}^H$. The eigenvalue problem (20) can be simplified into a scalar one by introducing two scalar functions Φ_m^H and Ψ_m^H, which are connected to the corresponding eigenfunctions $\vec{\mathbb{E}}_{\perp m}^H$ and $\vec{\mathbb{H}}_{\perp m}^H$ as follows:

$$\vec{\mathbb{E}}_{\perp m}^H=p_m^{-1}\left[\nabla_\perp\Phi_m^H\times\vec{r}_0\right];\vec{\mathbb{H}}_{\perp m}^H=p_m^{-1}\nabla_\perp\Psi_m^H. \tag{21}$$

The basis for *TE*-waves is constituted by the eigenfunctions of the operator W_H. Substituting the scalar functions from (21) into (20), a scalar boundary eigenvalue problem emerges, focusing on *TE*-waves particularly when $\mathbb{E}_r$ equals zero:

$$\begin{cases}\nabla_\perp\cdot\nabla_\perp\Psi_m^H+p_m^2\Phi_m^H=0\\ \nabla_\perp\cdot\nabla_\perp\Phi_m^H+p_m^2\Psi_m^H=0\\ \dfrac{\partial\Phi_m^H}{\partial\vec{n}}|_L=0,\quad \dfrac{\partial\Psi_m^H}{\partial\vec{n}}|_L=0.\end{cases} \tag{22}$$

2) MODAL BASIS FOR THE TM-WAVES

Similarly, the eigenvalue equation holds for operator W_E as

$$W_E\vec{\mathbb{X}}_{\perp n}^E=q_n^2\vec{\mathbb{X}}_{\perp n}^E \tag{23}$$

where q_n signifies an eigenvalue associated with the eigenfunction $\vec{\mathbb{X}}_{\perp n}^E$. The eigenvalue problem (23) can be simplified into a scalar one by introducing scalar functions Ψ_n^E and Φ_n^E, which are connected to the corresponding eigenfunctions $\vec{\mathbb{E}}_{\perp n}^E$ and $\vec{\mathbb{H}}_{\perp n}^E$ as follows:

$$\vec{\mathbb{E}}_{\perp n}^E=q_n^{-1}\nabla_\perp\Psi_n^E;\vec{\mathbb{H}}_{\perp n}^E=q_n^{-1}\left[\vec{r}_0\times\nabla_\perp\Phi_n^E\right]. \tag{24}$$

The basis for *TM*-waves is constituted by the eigenfunctions of the operator W_E. Substituting the scalar functions from (24) into (23), a scalar boundary eigenvalue problem emerges, focusing on *TM*-waves particularly when $\mathbb{H}_r$ equals zero:

$$\begin{cases}\nabla_\perp\cdot\nabla_\perp\Phi_n^E+q_n^2\Psi_n^E=0\\ \nabla_\perp\cdot\nabla_\perp\Psi_n^E+q_n^2\Phi_n^E=0\\ \Psi_n^E|_L=0,\quad \Phi_n^E|_L=0.\end{cases} \tag{25}$$

Consequently, the angular components of the electric, $\mathbb{E}_\perp$, and magnetic, $\mathbb{H}_\perp$, fields can be expressed in the following manner:

$$\vec{\mathbb{E}}_\perp(r,\theta,\varphi,t)=r^{-1}\left(\sum_m e_m^H(r,t)\,\vec{\mathbb{E}}_{\perp m}^H(\theta,\varphi)+\sum_n e_n^E(r,t)\,\vec{\mathbb{E}}_{\perp n}^E(\theta,\varphi)\right) \tag{26}$$

$$\vec{\mathbb{H}}_\perp(r,\theta,\varphi,t)=r^{-1}\left(\sum_m h_m^H(r,t)\,\vec{\mathbb{H}}_{\perp m}^H(\theta,\varphi)+\sum_n h_n^E(r,t)\,\vec{\mathbb{H}}_{\perp n}^E(\theta,\varphi)\right) \tag{27}$$

The component of the electric field in the radial direction, denoted as $\mathbb{E}_r$, is expressed via an expansion using the basis functions Φ_n^E, where Ψ_n^E serving as projectors:

$$\mathbb{E}_r(r,\theta,\varphi,t)=r^{-2}\sum_n e_n^r(r,t)\,q_n\Phi_n^E(\theta,\varphi). \tag{28}$$

Similarly, the component of the magnetic field in the radial direction, denoted as $\mathbb{H}_r$, is expressed via an expansion using the basis functions Φ_m^H, where Ψ_n^H serving as projectors:

$$\mathbb{H}_r(r,\theta,\varphi,t)=r^{-2}\sum_m h_m^r(r,t)\,p_m\Phi_m^H(\theta,\varphi). \tag{29}$$

The coefficients dependent on time and radial coordinates, designated as e_m^H, e_n^E, h_m^H, h_n^E, e_n^r, and h_m^r in equations (26)-(29), can be determined later following the derivation and solution of the evolutionary equations.

IV. EVOLUTIONARY EQUATIONS

The coefficients e_m^H, e_n^E, h_m^H, h_n^E, e_n^r, and h_m^r represent the modal amplitudes in the electromagnetic field expansion on modal bases, as detailed in equations (26)-(29). These coefficients are key to the expansion and characterization of electromagnetic fields within the specified modal framework.

By projecting Maxwell's equations (2) onto the bases defined in (21) and (24), a set of evolutionary equations, specifically (30) and (31), emerges. Solving these evolutionary equations enables the determination of the magnitudes of the modes for the angular and radial elements of the transient electromagnetic fields, e_m^H, e_n^E, h_m^H, h_n^E, e_n^r, and h_m^r, offering insights into their dynamic properties:

$$\begin{aligned} \frac{\partial}{\partial r}\left(e_n^r\right) &= e_n^E + r^2 \frac{1}{4\pi}\int_S \varrho q_n^{-1}\Psi_n^E dS \\ \frac{\partial}{\partial r}\left(h_m^r\right) &= h_m^H + r^2 \frac{1}{4\pi}\int_S \varrho_m p_m^{-1}\Psi_m^H dS \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{1}{c}\frac{\partial}{\partial t}\left(e_n^r\right) &= -\sum_{n'} L_{nn'}^{EE} h_{n'}^E - r^2\frac{1}{4\pi}\int_S \mathbb{J}_r q_n^{-1}\Psi_n^E dS \\ \frac{1}{c}\frac{\partial}{\partial t}\left(h_m^r\right) &= -\sum_{m'} L_{m'm}^{HH} e_{m'}^H - r^2\frac{1}{4\pi}\int_S \mathbb{M}_r p_m^{-1}\Psi_m^H dS. \end{aligned} \tag{31}$$

In the presented equations, (30) and (31), the matrices denoted as L represent the coupling of modes influenced by the inhomogeneity of the medium. In situations where the medium's structure is homogeneous, these matrices simplify to identity matrices. Consequently, the evolutionary equations become a set of uncoupled equations, reducing the complexity of the problem.

The analysis of *TE*- and *TM*-waves, in light of evolutionary equations (30) and (31), leads to specific sets of equations: (32)-(36) for *TE*-waves, and (37)-(41) for *TM*-waves.

A. FOR THE TE-WAVES

$$\begin{aligned} &\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + \frac{p_{mk}}{r^2}\right\}\left(r^2 e_m^H\right) \\ &\quad = -\frac{1}{2\pi}\left\{\frac{\partial}{\partial r}\left(r^2\int_S \varrho_m \Psi_m^H dS\right)\right. \\ &\qquad - r\int_S \vec{\mathbb{J}}_\perp\cdot\left[\nabla_\perp\times\vec{r}_0\right]\Psi_m^H dS \\ &\qquad \left. + \frac{r^2}{c^2}\frac{\partial}{\partial t}\left(\int_S \mathbb{M}_r\Psi_m^H dS\right)\right\} \end{aligned} \tag{32}$$

$$\begin{aligned} &\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right\}\left(r^2 e_0^H\right) \\ &\quad = -\frac{1}{4\pi}\frac{\partial}{\partial r}\left(r^2\int_S \varrho_m dS\right) \\ &\qquad - \frac{r^2}{4\pi}\frac{\partial}{\partial t}\left(\int_S \mathbb{M}_r dS\right) \end{aligned} \tag{33}$$

$$\begin{aligned} \vec{\mathbb{E}}_\perp = \sum_{m=1}^{\infty}\sum_{k=-m}^{m}\left[\nabla_\perp\Psi_{mk}\times\vec{r}_0\right]&\left\{-r\frac{\partial}{\partial t}\left(e_m^H\right)\right. \\ &\left. - \frac{r}{2\pi}\int_S \mathbb{M}_r\Psi_{mk}^* dS\right\} \end{aligned} \tag{34}$$

$$\begin{aligned} \vec{\mathbb{H}}_\perp = \sum_{m=1}^{\infty}\sum_{k=-m}^{m}\nabla_\perp\Psi_{mk}&\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 e_m^H\right)\right. \\ &\left. - \frac{r}{2\pi}\int_S \varrho_m\Psi_{mk}^* dS\right\} \end{aligned} \tag{35}$$

$$\mathbb{H}_r = \sum_{m=1}^{\infty}\sum_{k=-m}^{m} h_m^r p_m \Psi_{mk} + e_0^H \tag{36}$$

B. FOR THE TM-WAVES

$$\begin{aligned} &\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + \frac{q_{nl}}{r^2}\right\}\left(r^2 h_n^E\right) \\ &\quad = -\frac{1}{2\pi}\left\{\frac{\partial}{\partial r}\left(r^2\int_S \varrho\Psi_{nl}^* dS\right)\right. \\ &\qquad + r\int_S \vec{\mathbb{M}}_\perp\cdot\left[\nabla_\perp\times\vec{r}_0\right]\Psi_{nl}^* dS \\ &\qquad \left. + \frac{r^2}{c^2}\frac{\partial}{\partial t}\left(\int_S \mathbb{J}_r\Psi_{nl}^* dS\right)\right\} \end{aligned} \tag{37}$$

$$\begin{aligned} &\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right\}\left(r^2 h_0^E\right) \\ &\quad = -\frac{1}{4\pi}\frac{\partial}{\partial r}\left(r^2\int_S \varrho dS\right) \\ &\qquad - \frac{r^2}{4\pi}\frac{\partial}{\partial t}\left(\int_S \mathbb{J}_r dS\right) \end{aligned} \tag{38}$$

$$\begin{aligned} \vec{\mathbb{E}}_\perp = \sum_{n=1}^{\infty}\sum_{l=-n}^{n}\nabla_\perp\Psi_{nl}&\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 h_n^E\right)\right. \\ &\left. - \frac{r}{2\pi}\int_S \varrho\Psi_{nl}^* dS\right\} \end{aligned} \tag{39}$$

$$\begin{aligned} \vec{\mathbb{H}}_\perp = \sum_{n=1}^{\infty}\sum_{l=-n}^{n}\left[\vec{r}_0\times\nabla_\perp\Psi_{nl}\right]&\left\{-r\frac{\partial}{\partial t}\left(h_n^E\right)\right. \\ &\left. - \frac{r}{2\pi}\int_S \mathbb{J}_r\Psi_{nl}^* dS\right\} \end{aligned} \tag{40}$$

$$\mathbb{E}_r = \sum_{n=1}^{\infty}\sum_{l=-n}^{n} e_n^r q_n \Psi_{nl} + h_0^E \tag{41}$$

where $p_{\pm m}$ and $q_{\pm n}$ are defined as $\pm m(m+1)$ and $\pm n(n+1)$ respectively, n and m take values from the set $1, 2, \ldots$. The variables k and l range over $-m$ to m and $-n$ to n, denoted as $k = \overline{-m, m}$ and $l = \overline{-n, n}$. The function Ψ_{mk} is given by $\sqrt{\frac{2m+1}{2}\frac{(m-|k|)!}{(m+|k|)!}}Pm^{|k|}(\cos\theta e^{ik\varphi})$, where $P_m^{|k|}(x)$ represents the associated Legendre functions.

V. CONCLUSION

The Evolutionary Approach to Electrodynamics (EAE), developed initially by Tretyakov in the early 1990s [27], [28], has been acknowledged as an alternative to the time-harmonic field method [29]. The EAE has seen successful applications across various cavity [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], and waveguide problems [40], [41], [42], [43], [44], [45], [46], [47]. This includes both hollow structures and those with diverse media, with surfaces ranging from electrically perfect conducting to lossy [48], [49]. Recently, with the introduction of a novel format of Maxwell's equations in SI units [19], the EAE has been enhanced to address complexities within cavity problems more effectively [50].

In this study, we present an innovative method that employs revised Maxwell's equations for deriving evolutionary equations of transient electromagnetic field amplitudes in conical cavities. This approach offers new perspectives in analyzing complex electromagnetic behaviors, enhancing our understanding of their dynamics in these structures. This paper focuses mainly on developing the foundational evolutionary equations and the theoretical framework of our method.

The resolution of these equations will pave the way for revealing the modal amplitudes of the fields, enabling a more nuanced illustration of the temporal dynamics of both electric and magnetic fields in conical cavities. These modal amplitudes are crucial for understanding the mechanical properties of electromagnetic fields [51], particularly in exploring the inertial characteristics within these geometries.

This analysis not only enhances our comprehension of the time-dependent behavior of electromagnetic fields in specialized structures but also sets a foundation for future advancements in electromagnetic research. The insights gained from this study have potential applications in advanced electromagnetic systems. This application of factorization of physical dimensions can also be used to further develop modern topics such as the quantization of electromagnetic fields [52].

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