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RESEARCH ARTICLE

Distributed Finite-Time and Fixed-Time Containment Control for Nonlinear Multi-Agent Systems Under Heterogeneous Networks

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ABSTRACT In this work, for nonlinear multi-agent systems, we mainly study distributed double-integrator finite-time and fixed-time containment control with undirected heterogeneous networks. At the first, two controllers to realize finite-time containment and two controllers to achieve fixed-time containment were designed. Second, we design appropriate finite-time and fixed-time containment protocols to ensure followers can move to geometric region formed by leaders, and settling time can be estimated. We can obtain that the calculation of convergence time of finite-time containment control is associated to the initial state. And the settling time of the fixed-time containment control is related to the parameters. Finally, correctness of the theory is verified by mathematical simulation experiments.

INDEX TERMS Finite-time containment, fixed-time containment, heterogeneous networks, sliding mode control, second-order systems.

I. INTRODUCTION

Over the decade years, due to the wide application [1], [2] of multi-agent systems [3], [4], research on distributed cooperative control has attracted a lot of attention from researchers. Depend on the number of leader, we can study consensus from following aspects: leaderless consensus, leader-following consensus and containment control. Containment control, as a special type of multi leader consensus, has great application value in fields such as ground monitoring, investigation, and environmental exploration. Containment control has become one of the key contents of multi leader collaborative control research. The aim of containment control is all follower agents will reach convex hull spanned by leaders.

It can be observed that, studies on multi-agent systems were represented in the past by first-order models [5], [6], [7], and agents dynamics are linear model [8], [9], [10]. In [5], distributed containment control was studied with many dynamic or stationary leaders and switching and fixed topology. Proportional-derivative(PD) control containment was considered with directed graph and time-delay in work [6]. In [7], authors studied first-order bipartite containment control with static leaders and switching communication graph. In [8], containment control for linear dynamics systems was mentioned with graph which is directed. In [9], authors talked about formation-containment control with output problem based on hybrid active controller and linear systems is heterogeneous. Containment control of linear continuous-time single-integrator and double-integrator system were investigated in work [10].

Nonetheless, in engineering examples, multi-agent systems are mostly use nonlinear models [11], [12], [13]. This is because the nonlinear models are more complex and more realistic. Therefore, the rise of first-order systems to second-order systems [14], [15], [16] will be beneficial to obtain a

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wide range of applications. In [11], nonlinear containment was investigated for second-order system with and without centralized event-triggered strategy. Containment control for second-order nonlinear system with intermittent position measurements at irregular intervals, which introduced a filter to deal with problem of relative velocity measurement and used relative positive measurement was investigated in work [12]. In [13], it proved that containment for nonlinear system with leaders and followers will exponentially converge into convex hull spanned by dynamics leaders. In [14], authors studied tracking consensus for second-order systems affected by disturbance. In [15], authors given controller of second-order dynamics systems under nonlinear function for reaching asymptotic leader-following consensus. Second-order systems containment was investigated in [16] under directed graph with and without communication delay.

When talking about containment control, convergence time is the focus of discussion. However, the above controllers only guarantee to achieve asymptotic convergence. So the top priority of the research is to find a proper method to obtain a faster convergence speed for better practical system performance. Thus, finite-time containment control turns out to be a popular research problem. Researchers have given distributed controllers for linear and nonlinear system for accomplishing finite-time containment [17], [18], [19], [20]. In [17], problem of fuzzy control of nonlinear dynamics systems with time-delay and adaptive was discussed. Adaptive containment has been studied for uncertain manipulator systems with novel distributed adaptive backstepping strategy in [18]. In [19], authors studied finite-time containment control for systems with disturbances and uncertainties. In the study of multi-agent systems, we generally focus on continuous-time control, which may waste communication resources. Based on this, event-triggered intermittent control strategies can be studied. Therefore, in [20], authors investigated the finite-time synchronization control problem of complex networks with time-delay using event-triggered control methods. It is thought-provoking that convergence time of finite-time containment control is related to initial value. Based on this, many scholars nowadays prefer to study fixed-time controllers. The reason is that a fixed-time controller controller will not depend on initial state compared to finite-time controller laws. In [21], state observer and input observer proposed by authors are used to observe state and input. Distributed fixed-time controllers are proposed to enable follower states to track the leader. In [22], a suitable fixed-time consensus controller is proposed to solve heterogeneous tracking consensus problem for nonlinear systems. Several agents are studied in above papers, some of which are first-order systems and some of which are second-order systems. In [23], the distributed optimization problem was posed, and this optimization problem includes global and local optimization.

Moreover, it should be noted that containment control of second-order system is mostly with same position and velocity topology graph. But in some practical situations, different methods may be used to measure position and velocity, for example, using different sensors. Therefore, the position and velocity measurements will communicate between the agents under the different network topologies through the different communication devices. Even if the same way strategy, and the same way of transmission, information loss leads to a different topology of position and velocity. To solve this problem, second-order systems which communicating with topologies of different position and velocity are studied in [24], [25], and [26], which are called heterogeneous networks.

In this work, finite-time and fixed-time containment are studied for second-order system with nonlinear function under heterogeneous networks. Regarding the design of finite-time and fixed-time control protocols, it depends mainly on states of neighboring agents and relative state errors. By means of graph theory, Lyapunov theory and other related control knowledge, the corresponding controllers are designed to realize finite-time and fixed-time containment control of second-order dynamics systems. Our control objective is to enable followers to enter the convex hull formed by leaders in finite-time or fixed time. Finite-time containment control [27] refers to the follower entering the convex hull formed by the leader within a finite time. Fixed-time containment control [28] refers to the follower entering the convex hull formed by the leader within a fixed time. The convergence time of finite-time containment control depends on the initial state of the system, while fixed-time containment control does not depend on the initial state of the system. Design controllers in this work by using nonsingular terminal sliding mode control way, and role of sliding mode is to suppress disturbance.

Main contributions of this work:

(1) Compared to position and velocity using the same topologies, we study heterogeneous topologies about position and velocity. In real life, it is necessary to study heterogeneous networks because of the different measurement methods, or the same measurement methods but different sensors. To the best of our knowledge, there are few studies on finite-time and fixed-time of heterogeneous networks, so it is necessary to study containment control in finite-time and fixed-time under heterogeneous networks.

(2) In this paper, we establish sufficient conditions for control protocols designed under fixed topology with finite-time and fixed-time by using nonsingular terminal sliding mode control methods. The results show that the settling time of convex hull formed by leaders reached by follower can be calculated explicitly under the proposed control protocol.

The remaining articles are organized as follows. Introducing graph theory and useful lemmas in Section II. In section III, problem to be studied is described. Finite-time containment control with and terminal sliding mode nonsingular terminal sliding mode was proposed in Section IV. Section V proposed fixed-time containment control with and terminal sliding mode nonsingular terminal sliding mode. Section VI verifies the correctness of theory by giving some examples. At the end, conclusion is drawn in Section VII.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. NOTATIONS

R^p represents set of p dimensional vectors, $R^{p\times q}$ represents $p \times q$ dimensional matrix. Define I_{pm} to be pm dimensional identity matrix. $\|\bullet\|$ denotes norm of vector. Given a vector $Q = \left[Q_1, Q_2, \cdots, Q_p\right]^T$, $diag\left(Q\right)$ denotes a $p \times p$ diagonal matrix where $Q_1, Q_2, \cdots, Q_p$ are diagonal elements. Define $sign\left(Q\right) = \left[sign\left(Q_1\right), sgin\left(Q_2\right), \cdots, sign\left(Q_n\right)\right]^T$, where $sign\left(\bullet\right)$ is signum function. $\lambda_{\min}\left(M\right)$ and $\lambda_{\max}\left(M\right)$ represent minimum and maximum eigenvalues of a real symmetric matrix M. The max and min represent maximum and minimum value of a real symmetric matrix M.

B. GRAPH THEORY

An undirected graph $G_b = \{V_b, E_b, A_b\}$ is to describe information interaction of agents. Set of nodes is defined as $V_b = \{v_1, v_2, \cdots, v_{M+N}\}$, set of edges is defined as $E_b \subseteq V_b \times V_b$. Denotes adjacent matrix of G_b is $A_b = \left[a_{ij}\right] \in R^{(M+N)\times(M+N)}$, where $a_{ij} > 0$ if $\left(v_i, v_j\right) \in E_b$ and $a_{ij} = 0$ otherwise. $L = \left[l_{ij}\right] \in R^{(M+N)\times(M+N)}$ is Laplacian matrix which related to adjacency matrix, where $l_{ii} = \sum_{j=1,j\neq i}^{M+N} a_{ij}, l_{ij} = -a_{ij}, j \neq i, i, j = \{1, 2, \cdots, M+N\}$. The topologies of position and velocity can be described by $G^w = \left\{V, E^w, A^w = \left[a_{ij}^w\right]\right\}$ and $G^s = \left\{V, E^s, A^s = \left[a_{ij}^s\right]\right\}$, respectively. And Laplacian matrices of position and velocity can be expressed by

$$L_w = \begin{pmatrix} L_{w1} & L_{w2} \\ 0_{M\times N} & 0_{M\times M} \end{pmatrix}$$

and

$$L_s = \begin{pmatrix} L_{s1} & L_{s2} \\ 0_{M\times N} & 0_{M\times M} \end{pmatrix}$$

where $L_w = \left[l_{ij}^w\right] \in R^{(M+N)\times(M+N)}, A_w = \left[a_{ij}^w\right] \in R^{(M+N)\times(M+N)}$, $L_s = \left[l_{ij}^s\right] \in R^{(M+N)\times(M+N)}, A_s = \left[a_{ij}^s\right] \in R^{(M+N)\times(M+N)}$, and $-L_{w1}^{-1}L_{w2}1_{M\times 1} = 1_{N\times 1}$, $-L_{s1}^{-1}L_{s2}1_{M\times 1} = 1_{N\times 1}$.

Suppose it is a system with N followers and M leaders. An agent is called leader if agent has no neighbor, and a follower if agent has at least one neighbor. We assume that followers indexed by $1, \cdots, N$ and leaders indexed by $N+1, \cdots, N+M$. Follower set is $F = \{1, \cdots, N\}$ and leader set is $L = \{N+1, \cdots, N+M\}$.

C. SOME USEFUL LEMMAS AND DEFINITIONS

Definition 1 [29]: Let C_h be a set in a real vector space $R \subseteq R^P$. The set is convex if, for any x_h and y_h in C_h, point $\left(1-\alpha_h\right)x_h + \alpha_h y_h \in C_h$ for any $\alpha_h \in [0, 1]$. Convex hull for a set of points $\chi_p := \left\{x_{1p}, \cdots, x_{mp}\right\}$, denoted by $Co\left(\chi_p\right)$, that is, $Co\left(\chi_p\right) := \left\{\sum_{i=1}^m \beta_{ip}x_{ip} \left| x_{ip} \in \chi_p, \beta_{ip} \geq 0, \sum_{i=1}^m \beta_{ip} = 1\right.\right\}$.

Definition 2 [30]: For (2) and (3), containment will be get if and only if there is a control laws u_i and $T_n > 0$, such that followers' positions and velocities enter convex hull $Co\left(X_bL\right)$ and $Co\left(V_bL\right)$, respectively. And

$$Co\left(X_bL\right) = \left\{\sum\nolimits_{N+1}^{N+M} \theta_i x_i \left|\theta_i \geq 0, \sum\nolimits_{N+1}^{N+M} \theta_i = 1\right.\right\}$$

$$Co\left(V_bL\right) = \left\{\sum\nolimits_{N+1}^{N+M} \theta_i v_i \left|\theta_i \geq 0, \sum\nolimits_{N+1}^{N+M} \theta_i = 1\right.\right\}.$$

Definition 3 [31]: Nonlinear dynamics system is

$$\dot{z} = h\left(z\right), h\left(0\right) = 0, z\left(0\right) = z_0, \tag{1}$$

where $z \in R^q$ is state, $h : R^q \to R^q$ is continuous. Equilibrium point of (1) is finite-time stable, if for all $z_0 \in R^q$, there is a settling time $T_h\left(z_0\right)$, such that $z\left(t_h\right) = 0$ is get for all $t_h > T_h\left(z_0\right)$.

Lemma 1 [31]: If there is a positive define continuous function such that

$$\dot{V}_p\left(p\right) \leq -\lambda_p V_p^{\alpha_p}\left(p\right),$$

where $\lambda_p > 0, 0 < \alpha_p < 1$. Then, system (1) is finite-time stable, and convergence time $T_f\left(p_0\right)$ is set as

$$T_f\left(p_0\right) \leq \frac{V_p^{1-\alpha_p}\left(p_0\right)}{\lambda_p\left(1-\alpha_p\right)}.$$

Lemma 2 [32]: If there is a continuous function which is positive define such that

$$\dot{V}_h\left(x\right) \leq -\alpha_h V^{p_h}\left(x\right) - \beta_h V^{q_h}\left(x\right),$$

where $\alpha_h, \beta_h > 0, 0 < p_h < 1, q_h > 1$. Then origin is fixed-time stable equilibrium of system (1) and convergence time $T_h\left(x_0\right)$ is defined as

$$T_h \leq \frac{1}{\alpha_h\left(1-p_h\right)} + \frac{1}{\beta_h\left(1-q_h\right)}.$$

III. PROBLEM FORMULATION

Followers are represented by the model

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = u_i + f_i\left(x_i, v_i, t\right), \end{cases}$$
$$i \in F = \{1, \cdots, N\}, \tag{2}$$

where $x_i \in R^n$ is position,$v_i \in R^n$ is velocity, $u_i \in R^n$ is control input of ith follower. $f_i\left(x_i, v_i, t\right)$ is intrinsic nonlinear dynamics function.

Leaders are represented by the model

$$\begin{cases} \dot{x}_j = v_j \\ \dot{v}_j = f_j\left(x_j, v_j, t\right), \end{cases}$$
$$j \in L = \{N+1, \cdots, N+M\}, \tag{3}$$

where $x_j \in R^n$ is position, $v_j \in R^n$ is velocity, $u_j \in R^n$ is control input of jth leader. $f_j\left(x_j, v_j, t\right)$ is intrinsic nonlinear dynamics.

Assumption 1 [33]: Given $\varpi_1^s, \cdots, \varpi_M^s$, satisfying $\sum_{j=1}^{M} \varpi_j^s = 1$ and $\varpi_j^s \geq 0$, there are two constants $\rho_1^s, \rho_2^s > 0$, such that for $m^s, n^s, m_j^s, n_j^s \in R^n, j = 1, 2, \cdots, M$,

$$\left\| f\left(m^s, n^s, t\right) - \sum_{j=1}^{M} \varpi_j^s f_j\left(m_j^s, n_j^s, t\right) \right\| \leq \rho_1^s \left\| m^s - \sum_{j=1}^{M} \varpi_j^s m_j^s \right\| + \rho_2^s \left\| n^s - \sum_{j=1}^{M} \varpi_j^s n_j^s \right\|.$$

Assumption 2: For every follower, at least a leader is connected to follower.

For markup purposes, define

$$\begin{aligned} X_f(t) &= \{x_1, \cdots, x_N\}^T, \\ V_f(t) &= \{v_1, \cdots, v_N\}^T, \\ U_f(t) &= \{u_1, \cdots, u_N\}^T, \\ X_l(t) &= \{x_{N+1}, \cdots, x_{N+M}\}^T, \\ F_f(t) &= \{f_1, \cdots, f_N\}^T, \\ V_l(t) &= \{v_{N+1}, \cdots, v_{N+M}\}^T, \\ F_l(t) &= \{f_{N+1}, \cdots, f_{N+M}\}^T. \end{aligned}$$

Define error functions

$$\begin{cases} e_{xi} = \sum_{j=1}^{N+M} a_{ij}^w \left(x_i(t) - x_j(t)\right) \\ e_{vi} = \sum_{j=1}^{N+M} a_{ij}^s \left(v_i(t) - v_j(t)\right), \end{cases} \quad i \in \{1, 2, \cdots, N\}, \tag{4}$$

and $e_x = \{e_{x1}, \cdots, e_{xN}\}^T$, $e_v = \{e_{v1}, \cdots, e_{vN}\}^T$.

We can describe (4) in abbreviated form

$$\begin{cases} e_x = (L_{w1} \otimes I_m) X_f(t) + (L_{w2} \otimes I_m) X_l(t) \\ e_v = (L_{s1} \otimes I_m) V_f(t) + (L_{s2} \otimes I_m) V_l(t) \end{cases}$$

and let

$$\begin{aligned} \eta_x &= X_f(t) + \left(L_{w1}^{-1} L_{w2} \otimes I_m\right) X_l(t) \\ \eta_v &= V_f(t) + \left(L_{s1}^{-1} L_{s2} \otimes I_m\right) V_l(t) \end{aligned} \tag{5}$$

where $\eta_x = \{\eta_{x1}, \eta_{x2}, \cdots \eta_{xN}\}^T$, $\eta_v = \{\eta_{v1}, \eta_{v2}, \cdots \eta_{vN}\}^T$.

Take derivative of (5)

$$\begin{aligned} \dot{\eta}_x &= \eta_v \\ \dot{\eta}_v &= \dot{V}_f(t) + \left(L_{s1}^{-1} L_{s2} \otimes I_m\right) \dot{V}_l(t) \\ &= U_f(t) + F_f(t) + \left(L_{s1}^{-1} L_{s2} \otimes I_m\right) F_l(t) \end{aligned} \tag{6}$$

Definition $-L_{s1}^{-1} L_{s2} = \left(\xi_1^T, \xi_2^T, \cdots, \xi_N^T\right)^T$, where $\xi_i = (\xi_{i1}, \xi_{i2}, \cdots, \xi_{iM})$ and $\xi_{ij} \geq 0$. According to Assumption 1, we have

$$\begin{aligned} &\left\| F_f + \left(L_{s1}^{-1} L_{s2} \otimes I_n\right) F_l \right\| \\ &= \left\| \begin{pmatrix} \left[f_1(x_1, v_1, t) - \sum_{j=1}^{M} \xi_{1j} f_j\left(x_j, v_j, t\right) \right]^T, \\ \cdots, \\ \left[f_N(x_N, v_N, t) - \sum_{j=1}^{M} \xi_{Nj} f_j\left(x_j, v_j, t\right) \right]^T \end{pmatrix} \right\| \\ &\leq \left\| \begin{pmatrix} \rho_1 \left\| x_1 - \sum_{j=1}^{M} \xi_{1j} x_j \right\| + \rho_2 \left\| v_1 - \sum_{j=1}^{M} \xi_{1j} v_j \right\|, \\ \cdots, \\ \rho_1 \left\| x_N - \sum_{j=1}^{M} \xi_{Nj} x_j \right\| + \rho_2 \left\| v_N - \sum_{j=1}^{M} \xi_{Nj} v_j \right\| \end{pmatrix} \right\| \\ &\leq \left\| \rho_1 \left(X_f + \left(L_{s1}^{-1} L_{s2} \otimes I_n\right) X_l \right) \right\| \\ &\quad + \left\| \rho_2 \left(V_f + \left(L_{s1}^{-1} L_{s2} \otimes I_n\right) V_l \right) \right\| \\ &= \rho_1 \left\| \eta_{xv} \right\| + \rho_2 \left\| \eta_v \right\| \end{aligned} \tag{7}$$

where $\eta_{xv} = X_f + \left(L_{s1}^{-1} L_{s2} \otimes I_n\right) X_l$ and $\eta_{xv} = \{\eta_{xv1}, \eta_{xv2}, \cdots \eta_{xvN}\}^T$.

IV. FINITE-TIME CONTAINMENT CONTROL

Generally speaking, leaders are assumed to be dynamic in system. In fact, static leaders can be regards as special form of dynamic leaders which make velocity is equal to zero.

A. FINITE-TIME CONTAINMENT WITH TERMINAL SLIDING MODE

Sliding mode control can eliminate disturbances, therefore, we introduce terminal sliding mode. Compared with the linear sliding mode, the traditional terminal sliding mode control improves the speed of convergence to the equilibrium state due to the introduction of the nonlinear part, and the farther away from the equilibrium state the faster the convergence speed.

At first, we introduce terminal sliding mode. So sliding mode can be expressed

$$s_{i1} = \eta_{vi} + \beta_1 |\eta_{xi}|^{\alpha_1}, i \in \{1, 2, \cdots, N\}, \tag{8}$$

with $0 < \alpha_1 < 1$, and $\beta_1 > 0$.

Sliding mode (8) can be expressed in a simple form

$$S_1 = \eta_v + \beta_1 |\eta_x|^{\alpha_1},$$

with $S_1 = \{s_{11}, s_{21}, \cdots, s_{N1}\}^T$.

Take derivative (8) get

$$\dot{s}_{i1} = \dot{\eta}_{vi} + \beta_1 \alpha_1 |\eta_{xi}|^{\alpha_1 - 1} \eta_{vi}.$$

According to terminal sliding mode error, following containment controller is proposed as

$$u_{i1} = \left(\sum_{j=1}^{N+M} a_{ij}^{s}\right)^{-1} \left(-k_{21}\left|\eta_{vi}\right| sign\left(s_{i1}\right) - k_{11}\left|\eta_{xvi}\right| sign\left(s_{i1}\right) - k_{41}\left(\eta_{xi}\right)^{\alpha_1 - 1}\eta_{vi} - sign\left(s_{i1}\right) + \sum_{j=1}^{N} a_{ij}^{s} u_{j1}\right) \tag{9}$$

with $k_{11} > 0, k_{21} > 0, k_{41} > 0$.

Controller (9) can be rewritten in a simple form

$$U_{f1} = \left(L_{s1}^{-1} \otimes I_m\right)\left(-k_{41}\eta_v(\eta_x)^{\alpha_1-1} - diag\left(k_{11}\left|\eta_{xv}\right| + k_{21}\left|\eta_v\right| + 1\right) sign\left(S_1\right)\right)$$

with $U_{f1} = \{u_{11}, u_{21}, \cdots, u_{N1}\}^T$

Theorem 1: Consider the Assumption 1 to 2 hold and the communication graph is undirected. There is a controller (9) and terminal sliding mode (8), which make system (2) and (3) can achieve second-order multi-agent containment in finite time, if following inequalities are satisfied

$$k_{11} \geq \rho_1 \lambda_{\max}^{\frac{1}{2}}(P) \tag{10}$$

$$k_{21} \geq \rho_2 \lambda_{\max}^{\frac{1}{2}}(P),$$

$$k_{41} > \beta_1 \alpha_1 \lambda_{\max}^{\frac{1}{2}}(P). \tag{11}$$

where $P = L_{s1}^T L_{s1}$.

Proof: Design following Lyapunov function

$$V_1 = S_1^T\left(L_{s1} \otimes I\right) S_1. \tag{12}$$

Differentiating (12), one can gets

$$\begin{aligned}\dot{V}_1 &= S_1^T\left(L_{s1} \otimes I\right)\dot{S}_1\\ &= S_1^T\left(L_{s1} \otimes I\right)\left(\dot{\eta}_v + \beta_1\alpha_1\left|\eta_x\right|^{\alpha_1-1}\eta_v\right)\\ &= S_1^T\left(L_{s1} \otimes I_m\right)\left(U_f(t) + F_f(t)\right)\\ &\quad + S_1^T\left(L_{s2} \otimes I_m\right)F_l(t)\\ &\quad + S_1^T\beta_1\alpha_1\left(e_x\right)^{\alpha_1-1}\left(L_{s1} \otimes I\right)\eta_v.\end{aligned} \tag{13}$$

Substituting (9) into (13), one has

$$\begin{aligned}\dot{V}_1 &= S_1^T\left(L_{s1} \otimes I\right)\dot{S}_1\\ &= S_1^T\left(L_{s1} \otimes I_m\right)F_f(t) + S_1^T\left(L_{s2} \otimes I_m\right)F_l(t)\\ &\quad + S_1^T\beta_1\alpha_1\left(\eta_x\right)^{\alpha_1-1}\left(L_{s1} \otimes I\right)\eta_v\\ &\quad - S_1^T k_{41}\left(\eta_x\right)^{\alpha_1-1}\eta_v\\ &\quad - S_1^T\left(diag\left(k_{11}\left|\eta_{xv}\right| + k_{21}\left|\eta_v\right| + 1\right) sign\left(S_1\right)\right).\end{aligned} \tag{14}$$

Combine (4), (7), with (10), one can proved that

$$\begin{aligned}&S_1^T\left(L_{s1} \otimes I_m\right)F_f(t) + S_1^T\left(L_{s2} \otimes I_m\right)F_l(t)\\ &= S_1^T\left(L_{s1} \otimes I_m\right)\left(F_f(t) + \left(L_{s1}^{-1}L_{s2} \otimes I_m\right)F_l(t)\right)\\ &\leq \left\|S_1^T\right\|\left\|\left(L_{s1} \otimes I_m\right)\right\|\\ &\quad \times \left\|\left(F_f(t) + \left(L_{s1}^{-1}L_{s2} \otimes I_m\right)F_l(t)\right)\right\|\\ &\leq \left\|S_1\right\|\left\|\left(L_{s1} \otimes I_m\right)\right\|\left(\rho_1\left\|\eta_{xv}\right\| + \rho_2\left\|\eta_v\right\|\right)\\ &\leq \lambda_{\max}^{1/2}\left(L_{s1}^T L_{s1}\right)\left(\rho_1\left|\eta_{xv}\right|\left\|S_1\right\| + \rho_2\left|\eta_v\right|\left\|S_1\right\|\right).\end{aligned} \tag{15}$$

Otherwise, by (11),

$$\begin{aligned}&S_1^T\beta_1\alpha_1\left(\eta_x\right)^{\alpha_1-1}\left(L_{s1} \otimes I\right)\eta_v\\ &\quad - S_1^T k_{41}\left(\eta_x\right)^{\alpha_1-1}\eta_v\\ &\quad \leq \lambda_{\max}^{\frac{1}{2}}\left(L_{s1}^T L_{s1}\right)\beta_1\alpha_1 S_1^T\left(\eta_x\right)^{\alpha_1-1}\eta_v\\ &\quad\quad - k_{41}S_1^T\left(\eta_x\right)^{\alpha_1-1}\eta_v \leq 0.\end{aligned} \tag{16}$$

Combine (15) and (16), (14) can lead to

$$\dot{V}_1 \leq -S_1^T sign\left(S_1\right) = -\sqrt{2}V_1^{\frac{1}{2}}.$$

By Lemma 1, it can get conclusion that system will reach sliding mode $s_{i1} = 0$ surface

$$t_1 = \sqrt{2}V_1^{\frac{1}{2}}(0).$$

When $t > t_1$, terminal sliding mode be expressed as

$$s_{i1} = \eta_{vi} + \beta_1 sig\left(\eta_{xi}\right)^{\alpha_1} = 0, i \in F.$$

That is,

$$\eta_{vi} = \dot{\eta}_{xi} = -\beta_1 sig\left(e_{xi}\right)^{\alpha_1}.$$

We choose proper Lyapunov function as

$$V_2 = \frac{1}{2}\eta_x^T\eta_x.$$

By Definition 2, one gets

$$\begin{aligned}\dot{V}_2 &= \eta_x^T\dot{\eta}_x = -\eta_x^T\beta_1 sig\left(\eta_x\right)^{\alpha_1}\\ &= -\beta_1\left(\eta_x\right)^{\alpha_1+1} \leq -\beta_1 2^{\frac{\alpha_1+1}{2}}V_2^{\frac{\alpha_1+1}{2}}\end{aligned}$$

One can obtains

$$t_2 = \frac{V_2^{\frac{1-\alpha_1}{2}}(t_1)}{\beta_1 2^{\frac{\alpha_1+1}{2}}\left(\frac{1-\alpha_1}{2}\right)}.$$

In same way, one can gets

$$\lim_{t \to t_2}\eta_v = 0.$$

So, it can get converge time is

$$T_1 = t_1 + 2t_2.$$

B. FINITE-TIME CONTAINMENT CONTROL

We can see when $\eta_x = 0, \eta_v \neq 0, \eta_x \to 0$, control protocol (9) will be zero.

In this subsection, a nonsingular terminal sliding mode control is applied to deal with this issue. Furthermore, we investigate finite-time containment control under a undirected interaction topology.

We introduce nonsingular terminal sliding mode. So sliding mode can be expressed

$$s_{i2} = \eta_{xi} + \beta_2 sig(\eta_{vi})^{\alpha_2}, i \in \{1, 2, \cdots, N\}, \tag{17}$$

with $\beta_2 > 0, 1 < \alpha_2 < 2$.

(17) can be described in a simple form

$$S_2 = \eta_x + \beta_2 sig(\eta_v)^{\alpha_2},$$

with $S_2 = \{s_{12}, s_{22}, \cdots, s_{N2}\}^T$.

Deriving the derivative for the sliding mode (17) yields

$$\dot{s}_{i2} = \dot{\eta}_{xi} + \alpha_2\beta_2 diag\left((\eta_{vi})^{\alpha_2-1}\right)\dot{e}_{vi}.$$

Based on nonsingular terminal sliding mode, controller is able to be expressed as

$$\begin{aligned} u_{i2} \;\; &= \left(\sum_{j=1}^{N+M} a_{ij}^s\right)^{-1} (-sign(s_{i2}) \\ &- \frac{e_{vi}}{\beta_2\alpha_2(\eta_{vi})^{\alpha_2-1}} + \sum_{j=1}^{N} a_{ij}^s u_{j2} \\ &-k_{12}|\eta_{xvi}|\, sign(s_{i2}) - k_{22}|\eta_{vi}|\, sign(s_{i2})), \end{aligned} \tag{18}$$

with $k_{12}, k_{22} > 0$.

Controller (18) can be simply described as

$$\begin{aligned} U_{f2} = &\left(L_{s1}^{-1} \otimes I_m\right)\left(-diag\left(\frac{\eta_v (L_{s1} \otimes I_m)}{\beta_2\alpha_2(\eta_v)^{\alpha-1}}\right)\right. \\ &\left.-diag(1 + k_{12}|\eta_{xv}| + k_{22}|\eta_v|) sign(S_2)\right) \end{aligned}$$

with $U_{f2} = \{u_{12}, u_{22}, \cdots, u_{N2}\}^T$.

Theorem 2: Based on Assumptions 1 and 2, as well as topology graph is undirected, the existence of controllers (18) and nonsingular terminal sliding modes (17) enables the second-order system (2) and (3) to achieve finite-time containment control if following inequalities are satisfied

$$k_{12} \geq \rho_1 \lambda_{\max}^{1/2}\left(L_{s1}^T L_{s1}\right),$$

$$k_{22} \geq \rho_2 \lambda_{\max}^{1/2}\left(L_{s1}^T L_{s1}\right)$$

Proof: Design following Lyapunov function

$$V_3 = S_2^T (L_{s1} \otimes I) S_2.$$

Differentiating V_1, we can obtain

$$\begin{aligned} \dot{V}_3 &= S_2^T (L_{s1} \otimes I) \dot{S}_2 \\ &= S_2^T \alpha_2\beta_2 diag\left((\eta_v)^{\alpha_2-1}\right)(L_{s1} \otimes I_m) F_f(t) \\ &\quad + S_2^T \alpha_2\beta_2 diag\left((\eta_v)^{\alpha_2-1}\right)(L_{s2} \otimes I_m) F_l(t) \\ &\quad + S_2^T (L_{s1} \otimes I) \eta_v \\ &\quad + S_2^T \alpha_2\beta_2 diag\left((\eta_v)^{\alpha_2-1}\right)\left(-diag\left(\frac{\eta_v (L_{s1} \otimes I_m)}{\beta_2\alpha_2(\eta_v)^{\alpha_2-1}}\right)\right. \\ &\quad \left.-diag(1 + k_{12}|\eta_{xv}| + k_{22}|\eta_v|) sign(S_2)\right), \end{aligned}$$

Similar to Theorem 1, we have

$$\dot{V}_3 \leq -\alpha_2\beta_2 diag\left((\eta_v)^{\alpha_2-1}\right) S_2^T sign(S_2).$$

When $\eta_{vr} \neq 0, r = 1, \cdots, N$, define

$$\begin{aligned} R = \min &\left\{\alpha_2\beta_2 diag\left((\eta_{v1})^{\alpha_2-1}\right),\right. \\ &\left.\cdots, \alpha_2\beta_2 diag\left((\eta_{vN})^{\alpha_2-1}\right)\right\}, \end{aligned}$$

and $R > 0$, one can gets $\dot{V}_3 \leq -RS_2^T sign(S_2) = -\sqrt{2}RV_3^{\frac{1}{2}}$.

When $\eta_{vr} = 0, r = 1, 2, \cdots, N$, from the equation (6) and (18) a new equation can be obtained

$$\begin{aligned} \dot{\eta}_v &= U_{f2}(t) + F_f(t) + \left(L_{s1}^{-1} L_{s2} \otimes I_m\right) F_l(t) \\ &= \left(L_{s1}^{-1} \otimes I_m\right)(-diag(1 + k_{12}|\eta_{xv}|) sign(S_2)) \\ &\quad + F_f(t) + \left(L_{s1}^{-1} L_{s2} \otimes I_m\right) F_l(t) \end{aligned}$$

It can gets when $s_{i2} > 0, \dot{\eta}_{vr} < 1$ and $s_{i2} < 0, \dot{\eta}_{vr} > 1$ according to Assumption 1-2.

Consider situation first $s_{i2} > 0$. There will be $diag\left(\dot{S}_2(t)\right) = diag(\eta_v(t)) \leq diag(-t1_n)$, and $S_2(t_3) = 0$, t_3 is setting time of finite-time containment control, in other words, when $t > t_3$, sliding mode s_{i2} will converge to 0 and remain on the sliding mode surface. For the other case, the same analysis is used. Thus, by Definition 2, it can get conclusion that sliding mode surface $s_{i2} = 0$ can reached in finite time, and convergence time is $t_3 = \frac{\sqrt{2}V_3^{\frac{1}{2}}(0)}{R}$.

When $t > t_3$, sliding mode be described as

$$s_{i2} = \eta_{xi} + \beta_2 sig(\eta_{vi})^{\alpha_2} = 0. \tag{19}$$

Derivative of (19) is

$$\dot{s}_{i2} = \dot{\eta}_{xi} + \beta_2\alpha_2(\eta_{vi})^{\alpha_2-1}\dot{\eta}_{vi} = 0.$$

That is

$$\dot{\eta}_{vi} = -\frac{\dot{\eta}_{xi}}{\beta_2\alpha_2(\eta_{vi})^{\alpha_2-1}}.$$

One selects the proper Lyapunov function as

$$V_4 = \frac{1}{2}\eta_v^T \eta_v.$$

One can acquires

$$\dot{V}_4 = \eta_v^T \dot{\eta}_v = -\eta_v^T \frac{\dot{\eta}_x}{\beta_2\alpha_2(\eta_v)^{\alpha_2-1}}$$
$$= \frac{1}{\beta_2\alpha_2}(\eta_v)^{3-\alpha_2} \leq -\frac{1}{\beta_2\alpha_2}2^{\frac{3-\alpha_2}{2}}V_4^{\frac{3-\alpha_2}{2}}$$

By Definition 2, one gets

$$t_4 = \frac{\beta_2\alpha_2 V_4^{\frac{\alpha_2-1}{2}}(t_3)}{2^{\frac{3-\alpha_2}{2}}\left(\frac{\alpha_2-1}{2}\right)}.$$

In same way, one will obtains

$$\lim_{t\to t_4}\eta_x = 0.$$

So, it can get converge time is

$$T_2 = t_3 + 2t_4.$$

V. FIXED-TIME CONTAINMENT CONTROL

A. FIXED-TIME CONTAINMENT WITH TERMINAL SLIDING MODE

Terminal sliding mode is expressed as

$$s_{i3} = \eta_{vi} + \beta_3|\eta_{xi}|^{\alpha_3} + \beta_4|\eta_{xi}|^{\alpha_4}, \tag{20}$$

with $i \in \{1, 2, \cdots, N\}$, $0 < \alpha_3 < 1$, $\alpha_4 > 1$, and $\beta_3, \beta_4 > 0$.

Sliding mode (20) can be expressed in a simple form

$$S_3 = \eta_v + \beta_3|\eta_x|^{\alpha_3} + \beta_4|\eta_x|^{\alpha_4},$$

with $S_3 = \{s_{13}, s_{23}, \cdots, s_{N3}\}^T$.

Take derivative of (20) has following form

$$\dot{s}_{i3} = \dot{\eta}_{vi} + \beta_3\alpha_3|\eta_{xi}|^{\alpha_3-1}\dot{\eta}_{xi} + \beta_4\alpha_4|\eta_{xi}|^{\alpha_4-1}\dot{\eta}_{xi}.$$

Containment controller is proposed as

$$\begin{aligned}u_{i3} = &\left(\sum_{j=1}^{N+M} a_{ij}^s\right)^{-1}\Bigg(-k_{13}\left|\eta_{xvi}\right| sign\left(s_{i3}\right)\\ &- k_{23}\left|\eta_{vi}\right| sign\left(s_{i3}\right) - \left(s_{i3}^{\frac{m}{n}} + s_{i3}^{\frac{p}{q}}\right) sign\left(s_{i3}\right)\\ &+ \sum_{j=1}^{N} a_{ij}^s u_{j3} - k_{43}(\eta_{xi})^{\alpha_3-1}\dot{\eta}_{xi}\\ &-k_{33}(\eta_{xi})^{\alpha_4-1}\dot{\eta}_{xi}\Bigg)\end{aligned} \tag{21}$$

with $k_{13} > 0, k_{23} > 0, k_{33} > 0, k_{43} > 0, p > q, m < n$.

Controller (21) can be rewritten in a simple form

$$\begin{aligned}U_{f3} = &\left(L_{s1}^{-1}\otimes I_m\right)\Bigg(-\left(S_3^{\frac{m}{n}} + S_3^{\frac{p}{q}}\right) sign\left(S_3\right)\\ &- k_{33} diag\left(\dot{\eta}_x\right)\left(\eta_x\right)^{\alpha_4-1}\\ &- k_{43} diag\left(\dot{\eta}_x\right)\left(\eta_x\right)^{\alpha_3-1}\\ &-diag\left(k_{13}\left|\eta_{xv}\right| + k_{23}\left|\eta_v\right|\right) sign\left(S_3\right)\Bigg)\end{aligned}$$

with $U_{f3} = \{u_{13}, u_{23}, \cdots, u_{N3}\}^T$.

Theorem 3: Consider the Assumption 1 to 2 hold and the communication graph is undirected. There is a controller (21) and terminal sliding mode (20), which make system (2) and (3) can achieve second-order multi-agent containment in fixed time, if following inequalities are satisfied

$$\begin{aligned}k_{13} &\geq \rho_1\lambda_{\max}^{\frac{1}{2}}(P)\\ k_{23} &\geq \rho_2\lambda_{\max}^{\frac{1}{2}}(P),\\ k_{33} &> \alpha_4\beta_4\lambda_{\max}^{\frac{1}{2}}(P)\\ k_{43} &> \alpha_3\beta_3\lambda_{\max}^{\frac{1}{2}}(P)\end{aligned}$$

where $P = L_{s1}^T L_{s1}$.

Proof: Design following Lyapunov function

$$V_5 = S_3^T\left(L_{s1}\otimes I\right)S_3. \tag{22}$$

Differentiating (22), it can get

$$\begin{aligned}\dot{V}_5 &= S_3^T\left(L_{s1}\otimes I\right)\dot{S}_3\\ &= S_3^T\left(L_{s1}\otimes I\right)\left(\dot{\eta}_v + \beta_3\alpha_3|\eta_x|^{\alpha_3-1}\dot{\eta}_x\right)\\ &\quad + S_3^T\left(L_{s1}\otimes I\right)\beta_4\alpha_4|\eta_x|^{\alpha_4-1}\dot{\eta}_x\\ &= S_3^T\left(L_{s1}\otimes I\right)\left(L_{s1}^{-1}L_{s2}\otimes I_m\right)F_l(t)\\ &\quad + S_3^T\left(L_{s1}\otimes I\right)\left(U_f(t) + F_f(t)\right)\\ &\quad + S_3^T\left(L_{s1}\otimes I\right)\beta_3\alpha_3|\eta_x|^{\alpha_3-1}\dot{\eta}_x\\ &\quad + S_3^T\left(L_{s1}\otimes I\right)\beta_4\alpha_4|\eta_x|^{\alpha_4-1}\dot{\eta}_x\\ &= S_3^T\left(L_{s1}\otimes I_m\right)F_f(t) + S_3^T\left(L_{s2}\otimes I_m\right)F_l(t)\\ &\quad + S_3^T\left(L_{s1}\otimes I\right)\beta_3\alpha_3(\eta_x)^{\alpha_3-1} diag\left(\dot{\eta}_x\right)\\ &\quad + S_3^T\left(L_{s1}\otimes I\right)\beta_4\alpha_4|\eta_x|^{\alpha_4-1} diag\left(\dot{\eta}_x\right)\\ &\quad + S_3^T\Bigg(-k_{43} diag\left(\dot{\eta}_x\right)\left(\eta_x\right)^{\alpha_3-1}\\ &\quad - k_{33} diag\left(\dot{\eta}_x\right)\left(\eta_x\right)^{\alpha_4-1} - \left(S_3^{\frac{m}{n}} + S_3^{\frac{p}{q}}\right) sign\left(S_3\right)\\ &\quad -diag\left(k_{13}\left|\eta_{xv}\right| + k_{23}\left|\eta_v\right|\right) sign\left(S_3\right)\Bigg)\end{aligned}$$

Similar to Theorem 1, on can leads to

$$\begin{aligned}\dot{V}_5 &\leq -S_3^T S_3^{\frac{m}{n}} sign\left(S_3\right) - S_3^T S_3^{\frac{p}{q}} sign\left(S_3\right)\\ &= -\sqrt{2}^{\frac{m}{n}+1}V_5^{\frac{m+n}{2n}} - \sqrt{2}^{\frac{p}{q}+1}V_5^{\frac{p+q}{2q}}.\end{aligned}$$

By Lemma 2, it can get conclusion that system will reach sliding mode $s_{i3} = 0$ surface

$$t_5 = \frac{1}{\sqrt{2}^{\frac{m}{n}+1}\left(1 - \frac{m+n}{2n}\right)} + \frac{1}{\sqrt{2}^{\frac{p}{q}+1}\left(1 - \frac{p+q}{2q}\right)}.$$

When $t > t_5$, terminal sliding mode be expressed as

$$s_{i5} = \eta_{vi} + \beta_3|\eta_{xi}|^{\alpha_3} + \beta_4|\eta_{xi}|^{\alpha_4} = 0, i \in F.$$

That is,

$$\eta_{vi} = \dot{\eta}_{xi} = -\beta_3|\eta_{xi}|^{\alpha_3} - \beta_4|\eta_{xi}|^{\alpha_4}.$$

One chooses proper Lyapunov function as

$$V_6 = \frac{1}{2}\eta_x^T \eta_x.$$

One can obtains

$$\begin{aligned}\dot{V}_6 &= \eta_x^T \dot{\eta}_x \\ &= \eta_x^T \left(-\beta_3|\eta_{xi}|^{\alpha_3} - \beta_4|\eta_{xi}|^{\alpha_4}\right) \\ &= -\beta_3(\eta_x)^{\alpha_3+1} - \beta_4(\eta_x)^{\alpha_4+1} \\ &\le -\beta_3 2^{\frac{\alpha_3+1}{2}} V_6^{\frac{\alpha_3+1}{2}} - \beta_4 2^{\frac{\alpha_4+1}{2}} V_6^{\frac{\alpha_4+1}{2}}.\end{aligned}$$

By Definition 2, one gets

$$\begin{aligned}t_6 = &\frac{1}{\beta_3 2^{\frac{\alpha_3+1}{2}}\left(1-\frac{\alpha_3+1}{2}\right)} \\ &+ \frac{1}{\beta_4 2^{\frac{\alpha_4+1}{2}}\left(1-\frac{\alpha_4+1}{2}\right)}.\end{aligned}$$

In same way, one can owns

$$\lim_{t\to t_6} \eta_v = 0.$$

So, it can get converge time is

$$T_3 = t_5 + 2t_6.$$

B. FIXED-TIME CONTAINMENT CONTROL

Define nonsingular terminal sliding mode as

$$s_{i4} = \eta_{xi} + \beta_5|\eta_{vi}|^{\alpha_5} + \beta_6|\eta_{vi}|^{\alpha_6}, \tag{23}$$

with $0 < \alpha_5 < 1, \alpha_6 > 1, \beta_5, \beta_6 > 0$ and $i \in \{1, 2, \cdots, N\}$.

Sliding mode (23) can be expressed in a simple form

$$S_4 = \eta_x + \beta_5|\eta_v|^{\alpha_5} + \beta_6|\eta_v|^{\alpha_6},$$

with $S_4 = \{s_{14}, s_{24}, \cdots, s_{N4}\}^T$.

Find the derivative of the (23) and obtain

$$\dot{s}_{i4} = \dot{\eta}_{xi} + \beta_5\alpha_5|\eta_{vi}|^{\alpha_5-1}\dot{\eta}_{vi} + \beta_6\alpha_6|\eta_{vi}|^{\alpha_6-1}\dot{\eta}_{vi}.$$

Thus, controller is designed to

$$\begin{aligned}u_{i4} = &\left(\sum_{j=1}^{N+M} a_{ij}^s\right)^{-1} \left(-k_{14}\left|\eta_{xvi}\right| sign\left(s_{i4}\right)\right. \\ &- k_{24}\left|\eta_{vi}\right| sign\left(s_{i4}\right) \\ &- \left(s_{i4}^{\frac{m}{n}} + s_{i4}^{\frac{p}{q}}\right) sign\left(s_{i4}\right) + \sum_{j=1}^{N} a_{ij}^s u_{j4} \\ &\left.- \frac{e_{vi}}{\beta_5\alpha_5|\eta_{vi}|^{\alpha_5-1} + \beta_6\alpha_6|\eta_{vi}|^{\alpha_6-1}}\right)\end{aligned} \tag{24}$$

with $k_{14} > 0, k_{24} > 0, k_{34} > 0, k_{44} > 0, p > q, m < n$.

The controller (24) can be abbreviated as

$$\begin{aligned}U_{f4} = &\left(L_{s1}^{-1} \otimes I_n\right)\left(-\left(S_4^{\frac{m}{n}} + S_4^{\frac{p}{q}}\right) sign\left(S_4\right)\right. \\ &- diag\left(k_{14}\left|\eta_{xv}\right| + k_{24}\left|\eta_v\right|\right) sign\left(S_4\right) \\ &\left.- diag\left(\frac{e_v}{\beta_5\alpha_5 diag\left(|\eta_v|^{\alpha_5-1}\right) + \beta_6\alpha_6 diag\left(|\eta_v|^{\alpha_6-1}\right)}\right)\right)\end{aligned}$$

with $U_{f4} = \{u_{14}, u_{24}, \cdots, u_{N4}\}^T$.

Theorem 4: Under Assumption 1 to 2 hold, consider second-order system (2) and (3) with proposed controller (24) and nonsingular terminal sliding mode (23). Then, connectivity among agents can be preserved and the containment control will reach in finite time, if following inequalities are satisfied

$$\begin{aligned}k_{14} &\ge \rho_1\lambda_{\max}^{1/2}\left(L_{s1}^T L_{s1}\right), \\ k_{24} &\ge \rho_2\lambda_{\max}^{1/2}\left(L_{s1}^T L_{s1}\right).\end{aligned}$$

Proof: Design following Lyapunov function

$$V_7 = S_4^T\left(L_{s1} \otimes I\right) S_4. \tag{25}$$

Differentiating V_7, we can get (26), as shown at the bottom of the next page.

Similar to Theorem 2, we have

$$\begin{aligned}\dot{V}_7 \le &-S_4^T\left(S_4^{\frac{m}{n}} + S_4^{\frac{p}{q}}\right) sign\left(S_4\right) \\ &\left(\beta_5\alpha_5 diag\left(|\eta_v|^{\alpha_5-1}\right) + \beta_6\alpha_6 diag\left(|\eta_v|^{\alpha_6-1}\right)\right).\end{aligned}$$

When $\eta_{vm} \ne 0, r = 1, 2, \cdots, N$, define $M =$

$$\begin{aligned}\min&\left\{\beta_5\alpha_5 diag\left(|\eta_{v1}|^{\alpha_5-1}\right) + \beta_6\alpha_6 diag\left(|\eta_{v1}|^{\alpha_6-1}\right),\right. \\ &\left.\cdots, \beta_5\alpha_5 diag\left(|\eta_{vN}|^{\alpha_5-1}\right) + \beta_6\alpha_6 diag\left(|\eta_{vN}|^{\alpha_6-1}\right)\right\},\end{aligned}$$

and $M > 0$, we can get

$$\begin{aligned}\dot{V}_7 &\le -MS_4^T\left(S_4^{\frac{m}{n}} + S_4^{\frac{p}{q}}\right) sign\left(S_4\right) \\ &\le -M\sqrt{2}^{\frac{m}{n}+1} V_7^{\frac{m+n}{2n}} - M\sqrt{2}^{\frac{p}{q}+1} V_7^{\frac{p+q}{2q}}.\end{aligned}$$

When $\eta_{vm} = 0, m = 1, 2, \cdots, N$, similar to Theorem 1, when $t > t_3$, all sliding mode $s_{i4} = 0$ could be obtained. And settling time is $t_7 = \frac{1}{M\sqrt{2}^{\frac{m}{n}+1}\left(1-\frac{m+n}{2n}\right)} + \frac{1}{M\sqrt{2}^{\frac{p}{q}+1}\left(1-\frac{p+q}{2q}\right)}$.

When $t > t_7$, nonsingular terminal sliding mode be expressed as

$$s_{i4} = \eta_{xi} + \beta_5|\eta_{vi}|^{\alpha_5} + \beta_6|\eta_{vi}|^{\alpha_6} = 0, i \in F. \tag{27}$$

Derivative of (27) is

$$\dot{\eta}_{xi} = -\alpha_5\beta_5|\eta_{vi}|^{\alpha_5-1}\dot{\eta}_{vi} - \alpha_6\beta_6|\eta_{vi}|^{\alpha_6-1}\dot{\eta}_{vi}.$$

That is,

$$\begin{aligned}\dot{\eta}_{vi} &= \frac{\dot{\eta}_{xi}}{-\alpha_5\beta_5|\eta_{vi}|^{\alpha_5-1} - \alpha_6\beta_6|\eta_{vi}|^{\alpha_6-1}} \\ &= \frac{\eta_{vi}}{-\alpha_5\beta_5|\eta_{vi}|^{\alpha_5-1} - \alpha_6\beta_6|\eta_{vi}|^{\alpha_6-1}}.\end{aligned}$$

One chooses Lyapunov function as

$$V_8 = \frac{1}{2}\eta_v^T \eta_v.$$

It can obtain

$$\begin{aligned}
\dot{V}_8 &= \eta_v^T \dot{\eta}_v \\
&= \eta_v^T \left(\frac{\eta_v}{-\alpha_5\beta_5 |\eta_v|^{\alpha_5 - 1} - \alpha_6\beta_6 |\eta_v|^{\alpha_6 - 1}} \right) \\
&= -\frac{\eta_v^{3-\alpha_5}}{\alpha_5\beta_5} - \frac{\eta_v^{3-\alpha_6}}{\alpha_6\beta_6} \\
&\leq -\frac{2^{\frac{3-\alpha_5}{2}} V_8^{\frac{3-\alpha_5}{2}}}{\alpha_5\beta_5} - \frac{2^{\frac{3-\alpha_6}{2}} V_8^{\frac{3-\alpha_6}{2}}}{\alpha_6\beta_6}.
\end{aligned}$$

By Definition 2, one gets

$$t_8 = \frac{\alpha_5\beta_5}{2^{\frac{3-\alpha_5}{2}}\left(1 - \frac{3-\alpha_5}{2}\right)} + \frac{\alpha_6\beta_6}{2^{\frac{3-\alpha_6}{2}}\left(1 - \frac{3-\alpha_6}{2}\right)}.$$

In same way, one gets

$$\lim_{t \to t_4} \eta_x = 0.$$

So, it can obtain converge time is

$$T_4 = t_7 + 2t_8.$$

Remark 1: For controller (9), "sign(.)" is used, which can cause chattering behavior. This is a common problem in dynamical systems with non-Lipschitz right-hand side. The sat function is usually used as a Lipschitz approximation of the sign function and is used to eliminate chattering. sat function can be designed as

$$sat(s_i) = \begin{cases} -1 & s_i < -a \\ \frac{s_i}{a} & |s_i| \leq a \\ 1 & s_i > a \end{cases}$$

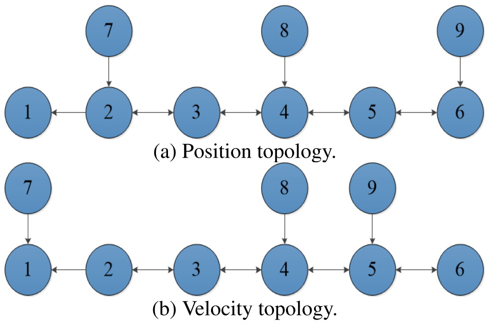


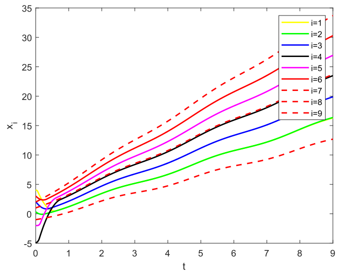
FIGURE 1. **The position and velocity topology graph.**

Remark 2: Theorem 1 is to solve finite-time containment control using terminal sliding mode method, and Theorem 2 is to solve finite-time containment control using nonsingular terminal sliding mode method. Terminal sliding mode may cause singularity in the system, therefore Theorem 2 uses nonsingular terminal sliding mode to solve finite-time containment control problems. Theorem 3 is to solve fixed-time containment control using terminal sliding mode method, and Theorem 4 is to solve fixed-time containment control using nonsingular terminal sliding mode method. The comparison between Theorem 3 and Theorem 1, as well as the comparison between Theorem 4 and Theorem 2, are aimed at highlighting the advantages of fixed-time containment control. Terminal sliding mode may cause singularity in the system, therefore Theorem 4 uses nonsingular terminal sliding mode to solve fixed-time containment control problems. ■

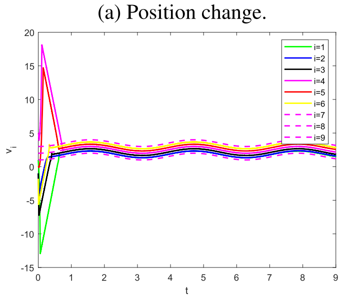
VI. SIMULATIONS

In this section, assume nine agents with 6 followers and 3 leaders. Communication topology graph of position and velocity be designed as Figure 1. We implement finite-time and fixed-time containment controllers in four cases.

$$\begin{aligned}
\dot{V}_7 &= S_4^T (L_{s1} \otimes I) \dot{S}_4 \\
&= S_4^T (L_{s1} \otimes I) \left(\dot{\eta}_x + \beta_5\alpha_5 diag\left(|\eta_v|^{\alpha_5 - 1}\right) \dot{\eta}_v + \beta_6\alpha_6 diag\left(|\eta_v|^{\alpha_6 - 1}\right) \dot{\eta}_v \right) \\
&= S_4^T (L_{s1} \otimes I) \eta_v \\
&\quad + S_4^T (L_{s1} \otimes I) \left(\beta_5\alpha_5 diag\left(|\eta_v|^{\alpha_5 - 1}\right) + \beta_6\alpha_6 diag\left(|\eta_v|^{\alpha_6 - 1}\right)\right) \left(U_{f4}(t) + F_f(t) + \left(L_{s1}^{-1} L_{s2} \otimes I_m\right) F_l(t)\right) \\
&= S_4^T \left(\beta_5\alpha_5 diag\left(|\eta_v|^{\alpha_5 - 1}\right) + \beta_6\alpha_6 diag\left(|\eta_v|^{\alpha_6 - 1}\right)\right) \left((L_{s1} \otimes I_m)\left(U_{f4}(t) + F_f(t)\right) + (L_{s2} \otimes I_m) F_l(t)\right) \\
&\quad + S_4^T (L_{s1} \otimes I) \eta_v \\
&= S_4^T \left(\beta_5\alpha_5 diag\left(|\eta_v|^{\alpha_5 - 1}\right) + \beta_6\alpha_6 diag\left(|\eta_v|^{\alpha_6 - 1}\right)\right) \left((L_{s1} \otimes I_m) F_f(t) + (L_{s2} \otimes I_m) F_l(t)\right) \\
&\quad + S_4^T \left(\beta_5\alpha_5 diag\left(|\eta_v|^{\alpha_5 - 1}\right) + \beta_6\alpha_6 diag\left(|\eta_v|^{\alpha_6 - 1}\right)\right) \\
&\quad \left(-diag\left(\frac{e_v}{\beta_5\alpha_5 diag\left(|\eta_v|^{\alpha_5 - 1}\right) + \beta_6\alpha_6 diag\left(|\eta_v|^{\alpha_6 - 1}\right)} \right) - diag\left(k_{14} |\eta_{xv}| + k_{24} |\eta_v|\right) sign(S_4) - \left(S_4^{\frac{m}{n}} + S_4^{\frac{p}{q}}\right) sign(S_4) \right) \\
&\quad + S_4^T (L_{s1} \otimes I_m) \eta_v
\end{aligned} \tag{26}$$

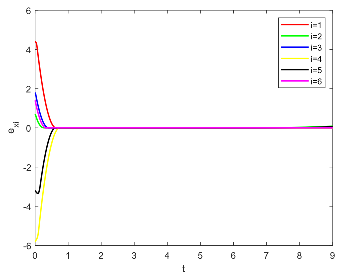


(a) Position change.

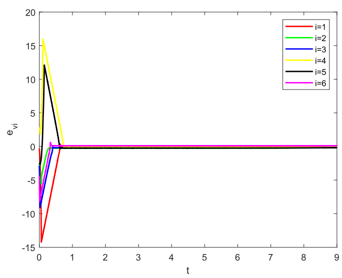


(b) Velocity change.

FIGURE 2. The position and velocity change of eight agents.

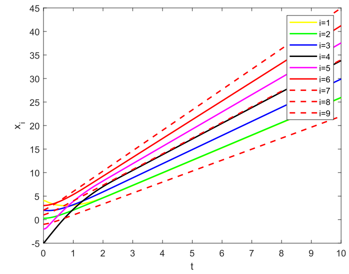


(a) Position error.

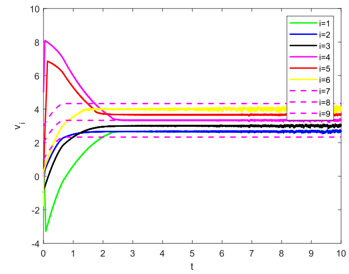


(b) Velocity error.

FIGURE 3. The position and velocity error of eight agents.

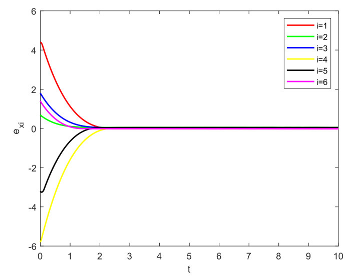


(a) Position change.

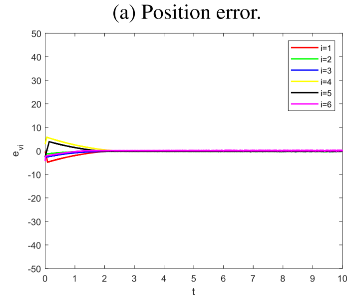


(b) Velocity change.

FIGURE 4. The position and velocity change of eight agents.



(a) Position error.



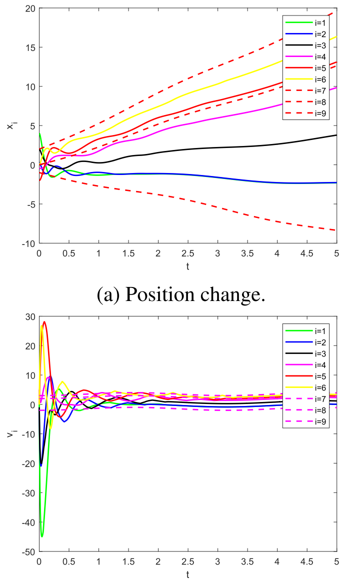
(b) Velocity error.

FIGURE 5. The position and velocity error of eight agents.

Example 1: We consider second-order system with six followers and three leaders. Label 7,8,9 are leaders and 1,2,3,4,5,6 are followers. In Theorem 1, the initial value of followers' position is $x(0) = \{4, 0.3, 2, -5, -2, 3\}, i = 1, \cdots 6$. The initial value of velocity of followers is $v(0) = \{-1, -1, 5, -0.05, -1.8, 1\}, i = 1, \cdots 6$. And leaders' position initial value was set $x(0) = \{-1, 1, 1\}, i = 7, 8, 9$, and leaders' velocity initial value was set $v(0) = \{2, 2, 3\}, i = 7, 8, 9$. Controller parameters can be designed as $k_{11} = 30, k_{21} = 3, k_{41} = 3$, and sliding mode parameters can be designed $\beta_1 = 7, \alpha_1 = 0.5$.

We can get Figure 2 to Figure 3. Figure 2 depicts position and velocity change of six followers and two leaders. Figure 3 presents position error and velocity error for leaders.From the Figure 2 to Figure 3, we can see that the position and velocity of the followers will enter the convex hull formed by the position and velocity of the leaders, and the position and velocity error between the followers and the leaders will converge to 0. Therefore, our proposed controller is effective.

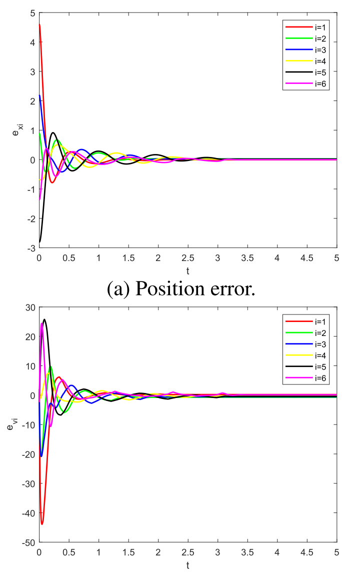
Example 2: We consider same system in Example 1 by (2) and (3). We use the same initial values as Example 1. We design controller parameters as $k_{12} = 20, k_{22} = 1$, and sliding mode parameters $\beta_2 = 0.4, \alpha_2 = 1.5$ and get Figure 4 to Figure 5. Figure 4 shows position and velocity change of all agents. Figure 5 presents position error and velocity error.



(a) Position change.

(b) Velocity change.

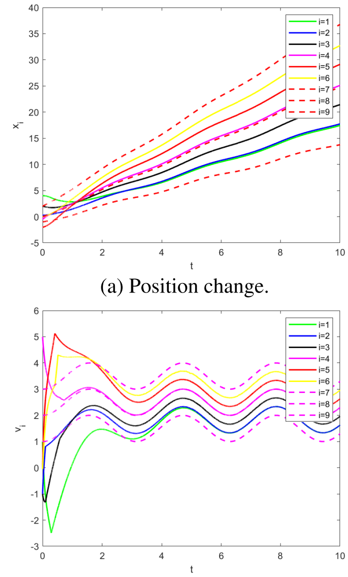
FIGURE 6. The position and velocity change of eight agents.



(a) Position error.

(b) Velocity error.

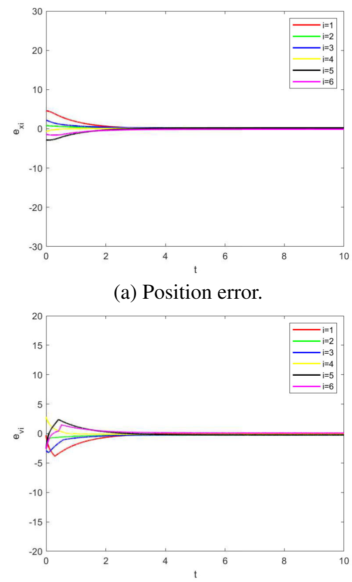
FIGURE 7. The position and velocity error of eight agents.



(a) Position change.

(b) Velocity change.

FIGURE 8. The position and velocity change of eight agents.



(a) Position error.

(b) Velocity error.

FIGURE 9. The position and velocity error of eight agents.

Figure 4 expresses the followers can enter into the convex hull spanned by leaders, and from Figure 5 can see that the error can converge to 0 in finite-time. Therefore, the controller proposed by Theorem 2 is correct and suitable.

Example 3: We also consider same system in Example 1 by (2) and (3). We use the same initial values as Example 1. We design controller parameters as $k_{13} = 3, k_{23} = 5, k_{33} = 3, k_{43} = 5$, and sliding mode parameters $\beta_3 = 20, \alpha_3 = 0.9, \beta_4 = 0.9, \alpha_4 = 1.5$. One can gets Figure 6 to Figure 7. Figure 6 depicts position and velocity change of six followers and two leaders. Figure 6 presents position error and velocity error for leaders and followers. From the Figure 6, we can see that the red dashed line represents the leader's position trajectory, while the magenta dashed line represents the leader's velocity trajectory. The position of the followers can enter the convex hull formed by the red dashed line, and the velocity of the followers can enter the convex hull formed by the magenta dashed line. From the Figure 7, the position and velocity error between the followers and the leaders will converge to 0. Therefore, the controller proposed in Theorem 3 is effective.

Example 4: We consider same system in Example 1 by (2) and (3). We use the same initial values as Example 1. We design controller parameters as $k_{14} = 2, k_{24} = 3, k_{34} = 3, k_{44} = 4$, and sliding mode parameters $\beta_5 = 0.1, \alpha_5 = 0.9, \beta_6 = 0.9, \alpha_6 = 1$. We can get Figure 8 to Figure 9.

Figure 8 shows position and velocity change of all agents. Figure 9 presents position error and velocity error. From the Figure 8, we can see that followers can enter the convex hull formed by the position and velocity of leaders. From the Figure 9, the position and velocity error between the followers and the leaders will converge to 0. Therefore, the controller proposed in Theorem 4 is effective.

VII. CONCLUSION

In the paper, two different controllers were proposed to solve the finite-time and fixed-time containment control problems for multi-agent systems with nonlinear dynamic function under heterogeneous networks. Controllers were designed according to sliding mode control to deal with containment problem. By using relevant control theories such as Lyapunov theory and graph theory, four containment controls are verified and the effects of containment are achieved. We also calculate the convergence time of the converged final containment state. Finally, using simulations, the validity of both theories is verified.

The prospect of this work is to consider that sliding mode control can eliminate the effects of disturbances. In a heterogeneous topology, finite-time and fixed-time containment control were achieved by designing corresponding controllers. Due to various reasons, the communication topology of position and velocity between multi-agent systems can adopt different topologies, making heterogeneous topologies more meaningful for research. The limitation is that the controller may experience nesting, increasing computational complexity. Therefore, we hope that the proposed controller can reduce computational complexity to achieve the goal.

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