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RESEARCH ARTICLE

Research on Enterprise Investment Decision Based on Linear Quadratic Jump Uncertainty Stochastic Differential Game

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ABSTRACT Aiming at the objective uncertainty, subjective uncertainty, and extreme events may be in a dynamic system simultaneously. This paper focuses on the differential game problem of a linear quadratic jump uncertain stochastic system. The system is described by both a jump uncertain differential equation and a stochastic differential equation. The principle of optimality and equation of optimality are established. Then, a differential game model based on linear quadratic jump uncertain stochastic system is constructed. Furthermore, Nash equilibrium are discussed by using the obtained equation. Finally, its application in the dynamic investment decision of enterprises is given, and numerical simulations are performed. This approach offers a new method for quantitative analysis in future studies.

INDEX TERMS Linear quadratic, jump uncertainty, stochastic, Nash equilibrium, enterprise investment.

I. INTRODUCTION

In the real world, non-deterministic information is everywhere. The emergence of information or events is subject to various accidental and unpredictable factors. Non-deterministic information can be categorized as objective uncertainty or subjective uncertainty. The objective indeterminacy is viewed as a random variable or a stochastic process whose probability distribution can be estimated by amounts of historical data. In practice, there is non-deterministic information that stems from human subjective consciousness or is expressed in human language. The sample data of these indeterminate factors maybe not enough to estimate. Estimating the distributions of these uncertain factors may be challenging due to insufficient sample data. To tackle this kind of subjective non-deterministic information, Liu proposed the uncertainty theory [1], [2].

In many cases, objective indeterminacy and subjective indeterminacy together. Therefore, only considering randomness is not enough, as uncertainty also needs to be considered. To describe this phenomenon, Liu first proposed the concept of uncertain random variables [3]. Managing such a complex system cannot be easily achieved through probability theory or uncertainty theory alone. Consequently, Liu [3], [4] put forward chance theory as a modeling tool for uncertain stochastic systems. Chance theory is a useful tool to deal with the analysis of indeterminacy including both uncertainty and randomness. However, objective uncertainty, subjective uncertainty, and extreme events may occur simultaneously. There are limited studies focusing on uncertain stochastic optimal control problems and jump uncertain

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stochastic optimal control problems. For instance, Chen and Jin [5] researched the optimal control of a multistage uncertain random system. Chen et al. [6] also studied the optimal control problem of jump uncertain stochastic dynamic systems. Chen and Zhu [7] developed an optimal control model for multistage uncertain random dynamic systems with multiple time delays. Subsequently, Chen et al. [8] investigated uncertain random discrete-time noncausal systems and optimal control problems.

The linear quadratic problem is a commonly used model in scientific research and industrial production. Linear quadratic are widely adopted as a general framework for easy numerical computation in analysis [10]. In 1960, Kalman [11] first studied the linear quadratic optimal control problem for deterministic systems. Then, in 1968, Wonham [12] extended the linear quadratic optimal control problem to stochastic systems. Deng and Zhu [9] conducted detailed research on linear quadratic uncertain optimal control with jumps. Zhang [13] carried out in-depth analysis on linear quadratic uncertain differential games with jumps. Chen et al. [14] proposed an linear quadratic model based on multistage uncertain random systems. Later, Chen and Zhu investigated linear quadratic optimal control problems for two types of uncertain random systems that consider the coefficient of the perturbed term as either a constant vector or a vector-valued function of state vector and control vector [15]. Chen and Zhu [16] also studied two types of linear quadratic optimal control models for multistage uncertain random systems.

Financial activities are the results of the participation of two or more parties in transactions, and they are a game behavior that involves mutual influence, interdependence, and mutual constraint. Differential game specifically refers to the continuous evolution process of multiple participants engaging in ongoing games, which is reflected as a change trajectory described by a differential equation. Under such constraints on state evolution, each participant strives to optimize their own independent and conflicting goals and ultimately obtain strategies that evolve over time for all participants to achieve Nash equilibrium. Therefore, the differential game model is an appropriate tool for studying decision-making problems in enterprises. By using it to construct enterprise differential game models, dynamic competition and strategic interactions among enterprises can be effectively characterized.

Differential games originated with Rufus Isaacs in the early 1940s as a pursuit and evasion problem for missile guidance [17]. Its adequate explanation of dynamics in conflict [18] leads to a proliferation of studies. The nonzero-sum situations explored by Starr and Ho [19] extend the utility to non-antagonistic conflicts where multi players minimize their own quadratic performance criteria differently. These more suitable features cover diverse application scenarios [20], [21], [22].

To sum up, Chen et al. [6] investigated the optimal control problem of jump uncertain stochastic dynamic systems and its application in advertising. However, there is still a lack of research results and applications regarding the theory of linear quadratic jump uncertain stochastic differential games. A jump uncertain stochastic differential game problem is established by further studying a jump uncertain stochastic optimal control problem. This differs from other problems such as the stochastic differential game problem [23], linear quadratic uncertain differential game [24], jump linear quadratic uncertain differential game problem [8], discrete-time uncertain differential game problem [25], uncertain stochastic optimal control problem [6], two-person games for uncertain random singular dynamic systems [26], and discrete-time uncertain stochastic optimal control problem [17].

The innovations of this paper are as follows. Firstly, it presents a continuous-time jump uncertain stochastic differential game problem, which is distinct from the works in Chen et al. [6]. Secondly, it extends the equilibrium equations obtained in [6] and [25]. Lastly, it discusses an investment decision problem for enterprises using the equation acquired based on [13].

Consequently, there are two main obstacles. The first challenge is to derive the Nash equilibrium in jump uncertainty stochastic system, which are common in the real world. Secondly, due to mathematical complexities, we utilized Simulink to solve the coupled Ricardi differential equation and provide a numerical simulation of the equilibrium investment strategy.

The rest of this paper is organized as follows. In Section II, the uncertain stochastic dynamic system with jump is introduced, and then principle of optimality and equation of optimality are get. The equilibrium strategy and equilibrium value function are discussed of jump uncertain stochastic differential game in Section III. Finally, we apply the aforementioned conclusions to address an investment decision problem, solve the analytical expression of the equilibrium investment strategy and value function of the enterprise, and give a numerical simulation in Sections IV.

Notation: $\mathbb{R}$ reprsents the 1-dimensional real Euclidean space, E_{Ch} denotes the expected value of the uncertain random variable in the sense of chance measure.

II. JUMP UNCERTAINTY STOCHASTIC DIFFERENTIAL GAME MODEL

In real situations, companies invest in both traditional and emerging products. For the traditional products whose probability distributions of price can be estimated by a large amount of historical data, but for the emerging products, we need invited experts to make an assessment and gived belief degree. Such system behaves both randomly and uncertainly, and cannot explain clearly by stochastic system or uncertain system. They can modeled by random variables while some others by uncertain variables.

The dynamic phenomena with objective randomness is described by as follows

$$\begin{cases} dx(t) = f(t,x,u,v)\,dt + g_1(t,x,u,v)\,dW(t), \\ x(0) = x_0, \end{cases} \tag{1}$$

where $x \in \mathbb{R}$ is the state vector of the system at time t with the initial condition x_0, $u \in U$ and $v \in V$ are the control input of the system at time t, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g_1 : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are valued function. In addition, $W(t)$ is a standard Wiener processes.

Uncertain differential equations are used to describe dynamic phenomena with subjective uncertainty. In the case of sudden changes, a jump uncertain differential equation can be written as follows

$$\begin{aligned} dy(t) &= \bar{f}(t,y,u,v)\,dt + g_2(t,y,u,v)\,dC(t) \\ &\quad + g_3(t,y,u,v)\,dV(t), \; y(0) = y_0, \end{aligned} \tag{2}$$

where $y \in \mathbb{R}$ is the state vector of the system at time t with the initial value y_0, $\bar{f} : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $g_2 : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $g_3 : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are valued function. In addition, $C(t)$ is a canonical Liu processes, $V(t)$ is a V jump uncertainty process with a parameter of r_1 and r_2, $W(t)$, $C(t)$ and $V(t)$ are independent. For further literature on jump uncertainty, we can refer to [9] and [21]. The chance theory is the more suitable approach for uncertain stochastic systems. Next, we assume that the objective function is

$$\begin{cases} J_1(0,x_0,y_0) \equiv \\ \min\limits_u E_{Ch}\left\{\int_0^T g\left(t,x,y,u,v^*\right)\, dt + N_{1T}\left(x_T^1\right)^2 + \bar{N}_{1T}\left(y_T^1\right)^2\right\}, \\ J_2(0,x_0,y_0) \equiv \\ \min\limits_v E_{Ch}\left\{\int_0^T \bar{g}\left(t,x,y,u^*,v\right)\, dt + N_{2T}\left(x_T^2\right)^2 + \bar{N}_{2T}\left(y_T^2\right)^2\right\}, \end{cases} \tag{3}$$

where $g : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\bar{g} : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are the objective function, $N_{iT}\left(x_T^i\right)^2$ and $\bar{N}_{iT}\left(y_T^i\right)^2$, $i = 1,2$ are the terminal-reward function.

Next, we present the principle of equilibrium equations for the jump uncertain stochastic differential game model.

Theorem 2.1: (Equilibrium Equations) Let $J(t,x,y)$ be a twice continuously differentiable function on $[0,T] \times \mathbb{R} \times \mathbb{R}$. Then, the Nash equilibrium solution of the uncertain stochastic differential game problem with jump for two players satisfy the following equations

$$\begin{aligned} -J_{1t}(t,x,y) &= \max_{u\in U}\Big[g\left(t,x,y,u,v^*\right) + f\left(t,x,y,u,v^*\right) J_{1x} \\ &\quad + \bar{f}\left(t,x,y,u,v^*\right)J_{1y} + \frac{1}{2}g_1^2\left(t,x,y,u,v^*\right)J_{1xx} \\ &\quad + \frac{3-r_1-r_2}{4} g_3\left(t,x,u,v^*\right) J_{1x}\Big], \\ -J_{2t}(t,x,y) &= \max_{v\in V}\Big[\bar{g}\left(t,x,y,u^*,v\right) + f\left(t,x,y,u^*,v\right) J_{2x} \\ &\quad + \bar{f}\left(t,x,y,u,v^*\right)J_{2y} + \frac{1}{2}g_1^2\left(t,x,y,u^*,v\right)J_{2xx} \\ &\quad + \frac{3-r_1-r_2}{4} g_3\left(t,x,y,u^*,v\right) J_{2x}\Big]. \end{aligned} \tag{4}$$

where $J_{it}(t,x,y)$ is the partial derivative of function $J_i(t,x,y)$ in t, $J_{ix}(t,x,y)$ and $J_{iy}(t,x,y)$ are the partial derivative of function $J_i(t,x,y)$ in x and y, respectively, $J_{ixx}(t,x,y)$, $i = 1,2$ are the quadratic differentiable of function $J_i(t,x,y)$ in x.

Proof: Similar to Theorem 3.2 in [6], the proof is omitted here.

III. LINEAR QUADRATIC JUMP UNCERTAINTY STOCHASTIC NON-ZERO-SUM DIFFERENTIAL GAME

The linear quadratic differential game is one of the most significant and fundamental classes of differential game theory. In this part, the linear quadratic differential game for jump uncertain stochastic dynamic systems are discussed. Then, its application in corporate investment decisions is given. In practice, it is an interesting and important model. Considering the following linear quadratic jump uncertain stochastic system

$$\begin{aligned} dx(t) &= \left[\beta_1(t)x(t) + \beta_2(t)u(t) + \beta_3(t)v(t)\right]dt \\ &\quad + \gamma_1(t)x(t)\,dW(t), \; x(0) = x_0, \end{aligned} \tag{5}$$

$$\begin{aligned} dy(t) &= \left[\bar{\beta}_1(t)y(t) + \bar{\beta}_2(t)u(t) + \bar{\beta}_3(t)v(t)\right]dt \\ &\quad + \gamma_2(t)y(t)\,dC(t) + \delta(t)y(t)\,dV(t), \; y(0) = y_0, \end{aligned} \tag{6}$$

where $\beta_1(t)$, $\beta_2(t)$, $\beta_3(t)$, $\bar{\beta}_1(t)$, $\bar{\beta}_2(t)$, $\bar{\beta}_3(t)$, $\delta(t)$ and $\gamma_i(t)$, $i = 1,2$ are continuous functions at time t.

We define the performance indicators of two players as

$$\begin{cases} J_1(0,x_0,y_0) \equiv \inf\limits_u E_{Ch}\left\{\int_0^T \Big[\alpha_1(t)x^2(t) + \alpha_2(t)u^2(t)\right. \\ +\alpha_3(t)x(t)u(t) + \alpha_4(t)x(t) + \alpha_5(t)u(t) + \alpha_6(t) \\ +\xi_1(t)y^2(t) + \xi_2(t)y(t)u(t) + \xi_3(t)y(t)\Big]dt \\ \left. +F_{1T}\left(x_T^1\right)^2 + \bar{F}_{1T}\left(y_T^1\right)^2\right\}, \\ J_2(0,x_0,y_0) \equiv \inf\limits_v E_{Ch}\left\{\int_0^T \Big[\bar{\alpha}_1(t)x^2(t) + \bar{\alpha}_2(t)v^2(t)\right. \\ +\bar{\alpha}_3(t)x(t)v(t) + \bar{\alpha}_4(t)x(t) + \bar{\alpha}_5(t)v(t) + \bar{\alpha}_6(t) \\ +\bar{\xi}_1(t)y^2(t) + \bar{\xi}_2(t)y(t)v(t) + \bar{\xi}_3(t)y\Big]dt \\ \left. +F_{2T}\left(x_T^2\right)^2 + \bar{F}_{2T}\left(y_T^2\right)^2\right\}, \end{cases} \tag{7}$$

where the initial value $x_0 > 0$, $y_0 > 0$, $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$, $\bar{\xi}_1(t)$, $\bar{\xi}_2(t)$, $\bar{\xi}_3(t)$, $\alpha_i(t)$ and $\bar{\alpha}_i(t)$ $(i = 1,2,\cdots,6)$ are all continuous function. F_{1T}, $\bar{F}_{1T}$, F_{2T} and $\bar{F}_{2T}$ are positive semidefinite matrices. For simplicity, we omit t.

Define Nash equilibrium of linear quadratic jump uncertain stochastic differential game problems [5], [6], [7] as follows

Definition 1: ([16], [27]) If the admissible control pair (u^*, v^*) satisfies

$$\begin{cases} J_1\left(u^*,v^*\right) \le J_1\left(u,v^*\right), & \forall u \in U, \\ J_2\left(u^*,v^*\right) \le J_2\left(u^*,v\right), & \forall v \in V, \end{cases} \tag{8}$$

and the state $x^*(t)$ and $y^*(t)$ are defined as

$$\begin{cases} dx^*(t) = \left(\beta_1 x^* + \beta_2 u^* + \beta_3 v^*\right) dt + \gamma_1 x^* dW(t), \\ x(0) = x_0, \end{cases} \tag{9}$$

$$\begin{cases} dy^*(t) = \left(\bar\beta_1 y^* + \bar\beta_2 u^* + \bar\beta_3 v^*\right) dt + \gamma_2 y^* dC(t) \\ +\delta y^* dV(t), \\ y(0) = y_0. \end{cases} \tag{10}$$

The strategy (u^*, v^*) satisfying [8], [9], [10] are called Nash equilibrium for linear quadratic jump uncertainty stochastic differential game problem, the state $x^*(t)$ and $y^*(t)$ are the optimal trajectory, $J_1(u^*, v^*)$ and $J_2(u^*, v^*)$ are respectively the optimal performance index of the two players.

Next, this paper presents the Nash equilibrium solution for the non-zero-sum differential game problem of uncertain stochastic systems with jumps for two players.

Theorem 3.1: Suppose that the performance indices $J_1(u, v)$ and $J_2(u, v)$ is a twice continuously differentiable function, $\alpha_i(t)$, $\bar\alpha_i(t)(i = 1, 2, \cdots, 6)$, $\beta_j(t)$,$\bar\beta_j(t)$, $\xi_j(t)$, $\bar\xi_j(t)$(j=1,2,3), $\delta(t)$, $a_2^{-1}(t)$ and $\bar\alpha_2^{-1}(t)$ are all continuous and bounded functions. Then, the Nash equilibrium strategy (u^*, v^*) of linear quadratic uncertain stochastic differential games problem with jumps for two players are

$$\begin{cases} u^* = -\dfrac{A_1 x + B_1 y + C_1}{2\alpha_2}, \\ v^* = -\dfrac{A_2 x + B_2 y + C_2}{2\bar\alpha_2}, \end{cases} \tag{11}$$

where

$$\begin{aligned} &A_1 = \alpha_3 + \beta_2 P_1 + \bar\beta_2 N_1, A_2 = \bar\alpha_3 + \beta_3 P_2 + \bar\beta_3 N_2, \\ &B_1 = \bar\beta_2 R_1 + \beta_2 N_1 + \xi_3, B_2 = \bar\xi_3 + \bar\beta_3 R_2 + \beta_3 N_2, \\ &C_1 = \alpha_5 + \beta_2 Q_1 + \bar\beta_2 S_1, C_2 = \bar\alpha_5 + \beta_3 Q_2 + \bar\beta_3 S_2, \end{aligned}$$

and $P_i(t)$, $Q_i(t)$, $R_i(t)$, $S_i(t)$ and $N_i(t), i = 1, 2$ follow the coupled Riccati differential equations

$$\begin{aligned} &\frac{1}{2}\frac{dP_1}{dt} + \left(\beta_1 + \frac{1}{2}\gamma_2^2 - \frac{\beta_3 A_2}{2\bar\alpha_2}\right) P_1 - \frac{A_1^2}{4\alpha_2} - \frac{\bar\beta_3}{2\bar\alpha_2} N_1 A_2 \\ &\quad + \alpha_1 = 0, P_1(T) = F_{1T}, \\ &\frac{1}{2}\frac{dR_1}{dt} - \frac{1}{4\alpha_2} B_1^2 - \left(\bar\beta_3 R_1 + \beta_3 N_1\right) \frac{B_2}{2\bar\alpha_2} + \bar\beta_1 R_1 + \xi_1 \\ &\quad + r\delta R_1 = 0, R_1(T) = \bar F_{1T}, \\ &\frac{dN_1}{dt} - \frac{A_1 B_1}{2\alpha_2} - \left(\bar\beta_3 N_1 + \beta_3 P_1\right) \frac{B_2}{2\bar\alpha_2} - \left(\bar\beta_3 R_1 + \beta_3 N_1\right) \frac{A_2}{2\bar\alpha_2} \\ &\quad + \left(\bar\beta_1 + \beta_1 + r\delta\right) N_1 = 0, N_1(T) = 0, \\ &\frac{dQ_1}{dt} - \frac{A_1 C_1}{2\alpha_2} - \left(\beta_3 Q_1 + \bar\beta_3 S_1\right) \frac{A_2}{2\bar\alpha_2} - \left(\bar\beta_3 N_1 + \beta_3 P_1\right) \frac{C_2}{2\bar\alpha_2} \\ &\quad + \beta_1 Q_1 + \alpha_4 = 0, Q_1(T) = 0, \\ &\frac{dS_1}{dt} - \frac{B_1 C_1}{2\alpha_2} - \left(\bar\beta_3 S_1 + \beta_3 Q_1\right) \frac{B_2}{2\bar\alpha_2} - \left(\bar\beta_3 R_1 + \beta_3 N_1\right) \frac{C_2}{2\bar\alpha_2} \\ &\quad + \bar\beta_1 S_1 + r\delta S_1 + \xi_4 = 0, S_1(T) = 0, \\ &\frac{dM_1}{dt} - \frac{C_1^2}{4\alpha_2} - \left(\bar\beta_3 S_1 + \beta_3 Q_1\right) \frac{C_2}{2\bar\alpha_2} + \alpha_6 = 0, M_1(T) = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &\frac{1}{2}\frac{dP_2}{dt} + \left(\beta_1 - \frac{A_1}{2\alpha_2}\beta_2 + \frac{1}{2}\gamma_2^2\right) P_2 \\ &\quad - \frac{A_2^2}{4\bar\alpha_2} - \frac{A_1}{2\alpha_2} N_2 \bar\beta_2 + \bar\alpha_1 = 0, P_2(T) = F_{2T}, \\ &\dot Q_2 - \frac{1}{2\bar\alpha_2} A_2 C_2 - \frac{\beta_2}{2\alpha_2}\left(A_1 Q_2 + C_1 P_2\right) - \frac{\bar\beta_2}{2\alpha_2}\left(A_1 S_2 + C_1 N_2\right) \\ &\quad + \beta_1 Q_2 + \bar\alpha_4 = 0, Q_2(T) = 0, \\ &\frac{1}{2}\frac{dR_2}{dt} - \frac{B_2^2}{4\bar\alpha_2} - \frac{B_1}{2\alpha_2}\left(\beta_2 N_2 + \bar\beta_2 R_2\right) + \bar\xi_1 + \bar\beta_1 R_2 + r\delta R_2 \\ &\quad = 0, R_2(T) = \bar F_{2T}, \\ &\dot N_2 - \frac{A_1 B_1}{2\bar\alpha_2} + \beta_1 N_2 + \bar\beta_1 N_2 - \frac{\beta_2}{2\alpha_2}\left(A_1 N_2 + B_1 P_2\right) \\ &\quad - \frac{\bar\beta_2}{2\alpha_2}\left(A_1 R_2 + B_1 N_2\right) + r\delta N_2 \\ &\quad = 0, N_2(T) = 0, \frac{dS_2}{dt} - \frac{B_2 C_2}{2\bar\alpha_2} - \frac{B_1}{2\alpha_2}\left(\beta_2 Q_2 + \bar\beta_2 S_2\right) \\ &\quad - \frac{C_1}{2\alpha_2}\left(\beta_2 N_2 + \bar\beta_2 R_2\right) \\ &\quad + \bar\xi_4 + \bar\beta_1 S_2 + r\delta S_2 = 0, S_2(T) = 0, \\ &\frac{dM_2}{dt} - \frac{C_2^2}{4\bar\alpha_2} + \bar\alpha_6 - \frac{C_1}{2\alpha_2}\left(Q_2\beta_2 + S_2\bar\beta_2\right) = 0, M_2(T) = 0, \end{aligned} \tag{13}$$

here $r = \frac{(3 - r_1 - r_2)}{4}$.

The corresponding optimal value for the system are

$$\begin{aligned} J_i(u^*, v^*) &= \frac{1}{2} P_i(0) x_0^2 + Q_i(0) x_0 + \frac{1}{2} R_i(0) y_0^2 + N_1(0) x_0 y_0 \\ &\quad + S_i(0) y_0 + M_i(0), i = 1, 2. \end{aligned} \tag{14}$$

Proof: According to the optimality equation, we have

$$\begin{aligned} -J_{1_t} &= \inf_u \{\alpha_1 x^2 + \alpha_2 u^2 + \alpha_3 xu + \alpha_4 x + \alpha_5 u + \alpha_6 \\ &\quad + \xi_1 y^2 + \xi_2 yu + \xi_3 y + \left(\bar\beta_1 y + \bar\beta_2 u + \bar\beta_3 v^*\right) J_{1y} \\ &\quad + \left[\beta_1 x + \beta_2 u + \beta_3 v^*\right] J_{1x} + \frac{1}{2}\gamma_1^2 x^2 J_{1xx} + r\delta y J_{1y}\} \\ &= \inf_u L_1(u, v^*), \\ -J_{2_t} &= \inf_v \{\bar\alpha_1 x^2 + \bar\alpha_2 v^2 + \bar\alpha_3 xv + \bar\alpha_4 x + \bar\alpha_5 v + \bar\alpha_6 \\ &\quad + \bar\xi_1 y^2 + \bar\xi_2 yv + \bar\xi_3 y + \left(\bar\beta_1 y + \bar\beta_2 u^* + \bar\beta_3 v\right) J_{2y} \\ &\quad + \left[\beta_1 x + \beta_2 u^* + \beta_3 v\right] J_{2x} + \frac{1}{2}\gamma_1^2 x^2 J_{2xx} + r\delta y J_{2y}\} \\ &= \inf_v L_2(u^*, v). \end{aligned} \tag{15}$$

Then the optimal control satisfies

$$\begin{aligned} \frac{\partial L_1}{\partial u} &= 2\alpha_2 u + \alpha_3 x + \xi_2 y + \alpha_5 + \beta_2 J_{1x} + \bar\beta_2 J_{1y} = 0, \\ \frac{\partial L_2}{\partial v} &= 2\bar\alpha_2 v + \bar\alpha_3 x + \bar\xi_2 y + \bar\alpha_5 + \beta_3 J_{2x} + \bar\beta_3 J_{2y} = 0. \end{aligned}$$

Moreover, because of $\frac{\partial^2 L_1}{\partial u^2} = 2\alpha_2 > 0$ and $\frac{\partial^2 L_2}{\partial v^2} = 2\bar{\alpha}_2 > 0$, then

$$u^* = -\frac{\alpha_3 x + \xi_2 y + \alpha_5 + \beta_2 J_{1x} + \bar{\beta}_2 J_{1y}}{2\alpha_2},$$
$$v^* = -\frac{\bar{\alpha}_3 x + \bar{\xi}_2 y + \bar{\alpha}_5 + \beta_3 J_{2x} + \bar{\beta}_3 J_{2y}}{2\bar{\alpha}_2}, \quad (16)$$

are the minimum point of $L_1(u, v)$ and $L_2(u, v)$, respectively.

Substituting (16) into (15), we obtain

$$\begin{aligned} & J_{1t} + \alpha_1 x^2 - \alpha_2 (u^*)^2 + \alpha_4 x + \alpha_6 + \xi_1 y^2 + \xi_3 y \\ & \quad + (\bar{\beta}_1 y + \bar{\beta}_3 v^*) J_{1y} + (\beta_1 x + \beta_3 v^*) J_{1x} \\ & \quad + \frac{1}{2}\gamma_1^2 x^2 J_{1xx} + r\delta y J_{1y} = 0, \\ & J_{2t} + \bar{\alpha}_1 x^2 - \bar{\alpha}_2 (v^*)^2 + \bar{\alpha}_4 x + \bar{\xi}_3 y + \bar{\alpha}_6 + \bar{\xi}_1 y^2 \\ & \quad + (\bar{\beta}_1 y + \bar{\beta}_2 u^*) J_{2y} + (\beta_1 x + \beta_2 u^*) J_{2x} \\ & \quad + \frac{1}{2}\gamma_1^2 x^2 J_{2xx} + r\delta y J_{2y} = 0. \end{aligned} \quad (17)$$

Since $J_i(T, x_T, y_T) = F_{iT}(x_T^i)^2 + \bar{F}_{iT}(y_T^i)^2$, we conjecture that

$$J_i = \frac{1}{2}P_i x^2 + Q_i x + \frac{1}{2}R_i y^2 + N_i xy + S_i y + M_i, \; i = 1, 2,$$

then

$$J_{ix} = P_i x + Q_i + N_i y, \; J_{iy} = R_i y + S_i + N_i x, \; i = 1, 2. \quad (18)$$

Substituting equation (18) into (16), we obtain

$$u^* = -\frac{A_1 x + B_1 y + C_1}{2\alpha_2}, \; v^* = -\frac{A_2 x + B_2 y + C_2}{2\bar{\alpha}_2}, \quad (19)$$

where

$$\begin{aligned} & A_1 = \alpha_3 + \beta_2 P_1 + \bar{\beta}_2 N_1, A_2 = \bar{\alpha}_3 + \beta_3 P_2 + \bar{\beta}_3 N_2, \\ & B_1 = \bar{\beta}_2 R_1 + \beta_2 N_1 + \xi_3, B_2 = \bar{\xi}_3 + \bar{\beta}_3 R_2 + \beta_3 N_2, \\ & C_1 = \alpha_5 + \beta_2 Q_1 + \bar{\beta}_2 S_1, C_2 = \bar{\alpha}_5 + \beta_3 Q_2 + \bar{\beta}_3 S_2. \end{aligned}$$

Substituting equations (19) and (18) into equation (17) yields as (20) and (21), shown at the bottom of the page and next page.

To separate the variables of the above formula, equations (12) and (13) are proven. Thus, we know that u^* and v^* are the solution of equation (6). Since the objective function is a convex function, equation (7) will generate a minimum value. Therefore, u^* and v^* represent the Nash equilibrium of the linear quadratic uncertainty stochastic differential game with jumps. We obtain the optimal value as equation (14)

$$\begin{aligned} & J_{1t} + \alpha_1 x^2 + \alpha_4 x + \alpha_6 + \xi_1 y^2 + \xi_3 y - \alpha_2 \left(\frac{A_1 x + B_1 y + C_1}{2\alpha_2}\right)^2 \\ & \quad - (R_1 y + N_1 x + S_1)\bar{\beta}_3 \frac{A_2 x + B_2 y + C_2}{2\bar{\alpha}_2} \\ & \quad - \beta_3 (P_1 x + N_1 y + Q_1)\frac{A_2 x + B_2 y + C_2}{2\bar{\alpha}_2} \\ & + \bar{\beta}_1 y (R_1 y + N_1 x + S_1) + \beta_1 x (P_1 x + N_1 y + Q_1) \\ & \quad + \frac{1}{2}\gamma_1^2 x^2 P_1 + r\delta y (R_1 y + N_1 x + S_1) \\ & = x^2 \left[\frac{1}{2}\frac{dP_1}{dt} + \left(\beta_1 + \frac{1}{2}\gamma_1^2 - \frac{\beta_3 A_2}{2\bar{\alpha}_2}\right) P_1 - \frac{\bar{\beta}_3 N_1 A_2}{2\bar{\alpha}_2} - \frac{A_1^2}{4\alpha_2} + \alpha_1\right] \\ & \quad + y^2 \left[\frac{1}{2}\frac{dR_1}{dt} - \frac{B_1^2}{4\alpha_2} - (\bar{\beta}_3 R_1 + \beta_3 N_1)\frac{B_2}{2\bar{\alpha}_2} + (\bar{\beta}_1 + r\delta) R_1 + \xi_1\right] \\ & \quad + xy \left[\frac{dN_1}{dt} - \frac{A_1 B_1}{2\alpha_2} - (\bar{\beta}_3 N_1 + \beta_3 P_1)\frac{B_2}{2\bar{\alpha}_2} - (\bar{\beta}_3 R_1 + \beta_3 N_1)\frac{A_2}{2\bar{\alpha}_2} \right. \\ & \quad \left. + (\bar{\beta}_1 + \beta_1 + r\delta) N_1\right] \\ & + x \left[\frac{dQ_1}{dt} - \frac{A_1 C_1}{2\alpha_2} - \frac{(\beta_3 Q_1 + \bar{\beta}_3 S_1) A_2}{2\bar{\alpha}_2} - \frac{(\bar{\beta}_3 N_1 + \beta_3 P_1) C_2}{2\bar{\alpha}_2} + \beta_1 Q_1 + \alpha_4\right] \\ & \quad + y \left[\frac{dS_1}{dt} - \frac{B_1 C_1}{2\alpha_2} - \frac{(\bar{\beta}_3 S_1 + \beta_3 Q_1) B_2}{2\bar{\alpha}_2} - \frac{(\bar{\beta}_3 R_1 + \beta_3 N_1) C_2}{2\bar{\alpha}_2} \right. \\ & \quad \left. + (\bar{\beta}_1 + r\delta) S_1 + \xi_3\right] \\ & \quad + \frac{dM_1}{dt} - \frac{C_1^2}{4\alpha_2} - (\bar{\beta}_3 S_1 + \beta_3 Q_1)\frac{C_2}{2\bar{\alpha}_2} + \alpha_6 = 0, \end{aligned} \quad (20)$$

simultaneously in the jump uncertainty stochastic differential game model.

This completes the proof of Theorem 3.1.

IV. CORPORATE INVESTMENT DECISION-MAKING

Dynamic portfolio management is a trending research topic. In the realm of corporate investment decision-making, external extreme events or sudden disturbances greatly influence uncertain stochastic dynamic systems. Hence, it becomes imperative to account for jump scenarios.

The basic idea behind using probability theory or uncertainty theory is to determine if the available data is sufficient for estimating its distribution. In the investment decision-making of enterprises, both traditional products and emerging products are invested, and the investments between the two will have mutual influence. Traditional products have sufficient historical data, and their capital accumulation process can be described by stochastic differential equations. Emerging products lack historical data and require experts from relevant fields to estimate their credibility. The capital accumulation process of emerging products follows uncertain differential equations.

A. MODEL SETUP

For investment of traditional products, the capital of enterprise is with the initial capital k_0

$$\begin{aligned} dK(t) = {} & [\beta_1(t) K(t) + \beta_2(t) I_1(t) + \beta_3(t) I_2(t)]\, dt \\ & + \gamma_1(t) K(t)\, dW(t), K(0) = k_0, \end{aligned} \tag{22}$$

for investment investment of emerging products, the capital follows with the initial capital $\bar{k}_0$

$$\begin{aligned} d\bar{K}(t) = {} & \left[\bar{\beta}_1(t) \bar{K}(t) + \bar{\beta}_2(t) I_1(t) + \bar{\beta}_3(t) I_2(t)\right] dt \\ & + \gamma_2(t) \bar{K}(t)\, dC(t) + \delta(t) \bar{K}(t)\, dV(t), \bar{K}(0) = \bar{k}_0, \end{aligned} \tag{23}$$

where $I_1(t)$ and $I_2(t)$ are the investment of traditional products and emerging products, respectively. β_1 and $\bar{\beta}_1$ are the constant rate of change of capital, β_2 and $\bar{\beta}_3$ represent the rate of investment change of traditional products and

$$\begin{aligned} & J_{2t} + \bar{\alpha}_1 x^2 - \bar{\alpha}_2 \left(\frac{A_2 x + B_2 y + C_2}{2\bar{\alpha}_2} \right)^2 + \bar{\alpha}_4 x + \bar{\alpha}_6 + \bar{\xi}_1 y^2 + \bar{\xi}_4 y \\ & \quad + \beta_1 x (P_2 x + Q_2 + N_2 y) - \frac{A_1 x + B_1 y + C_1}{2\alpha_2} \beta_2 (P_2 x + Q_2 + N_2 y) \\ & \quad + \bar{\beta}_1 y (R_2 y + S_2 + N_2 x) - \frac{A_1 x + B_1 y + C_1}{2\alpha_2} \bar{\beta}_2 (R_2 y + S_2 + N_2 x) \\ & \quad + \frac{1}{2} \gamma_1^2 x^2 P_2 + r\delta y (R_2 y + S_2 + N_2 x) \\ & = x^2 \left[\frac{1}{2} \frac{dP_2}{dt} + \left(\beta_1 - \frac{A_1}{2\alpha_2} \beta_2 + \frac{1}{2} \gamma_1^2 \right) P_2 - \frac{A_2^2}{4\bar{\alpha}_2} - \frac{A_1}{2\alpha_2} N_2 \bar{\beta}_2 + \bar{\alpha}_1 \right] \\ & \quad + x \left[\frac{dQ_2}{dt} - \frac{A_2 C_2}{2\bar{\alpha}_2} + \bar{\alpha}_4 + \beta_1 Q_2 - \left(Q_2 \beta_2 + S_2 \bar{\beta}_2 \right) \frac{A_1}{2\alpha_2} \right. \\ & \quad \left. - \frac{C_1}{2\alpha_2} \left(P_2 \beta_2 + N_2 \bar{\beta}_2 \right) \right] \\ & \quad + y^2 \left[\frac{1}{2} \frac{dR_2}{dt} - \frac{B_2^2}{4\bar{\alpha}_2} - \frac{B_1 \left(\beta_2 N_2 + \bar{\beta}_2 R_2 \right)}{2\alpha_2} + \bar{\xi}_1 + \bar{\beta}_1 R_2 + r\delta R_2 \right] \\ & \quad + xy \left[\frac{dN_2}{dt} - \frac{A_2 B_2}{2\bar{\alpha}_2} + \left(r\delta + \beta_1 + \bar{\beta}_1 \right) N_2 - \frac{A_1 \left(N_2 \beta_2 + R_2 \bar{\beta}_2 \right)}{2\alpha_2} \right. \\ & \quad \left. - \frac{B_1 \left(P_2 \beta_2 + N_2 \bar{\beta}_2 \right)}{2\alpha_2} \right] \\ & \quad + y \left[\frac{dS_2}{dt} - \frac{B_2 C_2}{2\bar{\alpha}_2} - \frac{B_1 \left(\beta_2 Q_2 + \bar{\beta}_2 S_2 \right)}{2\alpha_2} - \frac{C_1 \left(\beta_2 N_2 + \bar{\beta}_2 R_2 \right)}{2\alpha_2} \right. \\ & \quad \left. + \bar{\xi}_3 + \left(\bar{\beta}_1 + r\delta \right) S_2 \right] \\ & \quad + \frac{dM_2}{dt} - \frac{C_2^2}{4\bar{\alpha}_2} + \bar{\alpha}_6 - \frac{C_1}{2\alpha_2} \left(Q_2 \beta_2 + S_2 \bar{\beta}_2 \right) = 0. \end{aligned} \tag{21}$$

emerging products, β_3 and $\bar{\beta}_2$ are the competition coefficient of emerging products and traditional products. γ_1, γ_2 and δ denote respectively the diffusion coefficient and jump coefficient of unchanging capital fluctuation.

Assuming that the decision maker's objective is to maximize the expected value of investment profits for the enterprise, then the dynamic profit function are

$$L_i(t) = p_i Q_i(K(t)) + \bar{p}_i \bar{Q}_i\left(\bar{K}(t)\right) - w - I_i(t) - C_i(I_i(t)), \quad i = 1, 2, \tag{24}$$

where $L_i(t)$ are the dynamic profit function of two enterprises respectively, $Q_i(K(t))$ and $\bar{Q}_i\left(\bar{K}(t)\right)$ represent the output of two companies respectively, $C_i(I_i(t))$ are the adjustment costs of the two enterprises respectively, $p_i(t)$ represent the traditional product price which can be get a large amount of data, $\bar{p}_i(t), i = 1, 2$ stand for the emerging product price which can't be obtained much data, w denotes constant labor wages, a_1 and a_2 stand for the production technology level of two enterprises, t represents the time span of investment, b is the coefficient of the adjusted cost function.

Assuming the production function are $Q_i(K) = a_i K\ (a_i > 0)$ and $\bar{Q}_i\left(\bar{K}\right) = \bar{a}_i \bar{K}\ (\bar{a}_i > 0)$, the adjusted cost function are $C_i(I_i(t)) = \frac{b}{2} I_i^2(t), i = 1, 2$, then the profit function are expressed as

$$L_i(t) = p_i a_i K + \bar{p}_i \bar{a}_i \bar{K} - w - I_i(t) - \frac{b}{2} I_i^2(t), i = 1, 2. \tag{25}$$

Therefore, the investment decision-making model for the enterprise 1 and 2 are expressed as follows

$$\begin{cases} J_1\left(0, k_0, \bar{k}_0\right) \equiv \max\limits_{I_1} E\left\{\int_0^T \left[p_1 a_1 K + \bar{p}_1 \bar{a}_1 \bar{K} - w - I_1(t) \right.\right. \\ \qquad \left.\left. - \frac{b}{2} I_1^2(t)\ \right] dt + S_{1T}\left(k_T^1\right)^2 + \bar{S}_{1T}\left(\bar{k}_T^1\right)^2\right\}, \\ J_2\left(0, k_0, \bar{k}_0\right) \equiv \max\limits_{I_2} E\left\{\int_0^T \left[p_2 a_2 K + \bar{p}_2 \bar{a}_2 \bar{K} - w - I_2(t) \right.\right. \\ \qquad \left.\left. - \frac{b}{2} I_2^2(t)\right] dt + S_{2T}\left(k_T^2\right)\left(k_T^1\right)^2 + \bar{S}_{2T}\left(\bar{k}_T^2\right)^2\right\}, \\ s.t. \\ dK(t) = \left[\beta_1(t) K(t) + \beta_2(t) I_1(t) + \beta_3(t) I_2(t)\right] dt \\ \qquad + \gamma_1(t) K(t)\, dW(t), \\ d\bar{K}(t) = \left[\bar{\beta}_1(t) \bar{K}(t) + \bar{\beta}_2(t) I_1(t) + \bar{\beta}_3(t) I_2(t)\right] dt \\ \qquad + \gamma_2(t) \bar{K}(t)\, dC(t) + \delta(t) \bar{K}(t)\, dV(t), \\ K(0) = k_0, \bar{K}(0) = \bar{k}_0. \end{cases} \tag{26}$$

B. MODEL SOLVING

Comparing equation (26) with equations (5)-(7), we have $u = I_1, v = I_2, x = K, y = \bar{K}, \alpha_1 = \alpha_3 = \bar{\alpha}_1 = \bar{\alpha}_3 = \xi_1 = \xi_2 = \bar{\xi}_1 = \bar{\xi}_2 = 0, \alpha_2 = -\frac{b}{2}, \bar{\alpha}_2 = -\frac{b}{2}, \alpha_4 = p_1 a_1, \xi_3 = \bar{p}_1 \bar{a}_1, \bar{\alpha}_4 = p_2 a_2, \bar{\xi}_3 = \bar{p}_2 \bar{a}_2, \alpha_5 = \bar{\alpha}_5 = -1, \alpha_6 = \bar{\alpha}_6 = -w$.

According to Theorem 3.1, we obtain the optimal investment strategies for traditional products and emerging products as

$$I_1^*(t) = \frac{A_1 K(t) + B_1 \bar{K}(t) + C_1}{b}, \quad I_2^*(t) = \frac{A_2 K(t) + B_2 \bar{K}(t) + C_2}{b}, \tag{27}$$

where

$$A_1 = \beta_2 P_1 + \bar{\beta}_2 N_1, A_2 = \beta_3 P_2 + \bar{\beta}_3 N_2,$$
$$B_1 = \bar{\beta}_2 R_1 + \beta_2 N_1, B_2 = \bar{\beta}_3 R_2 + \beta_3 N_2,$$
$$C_1 = \beta_2 Q_1 + \bar{\beta}_2 S_1 - 1, C_2 = \beta_3 Q_2 + \bar{\beta}_3 S_2 - 1,$$

$P_i(t)$, $Q_i(t)$,$R_i(t)$,$N_i(t)$,$S_i(t)$ and $M_i(t)$ $(i = 1, 2)$ satisfy the following Riccati differential equations and boundary conditions

$$\begin{aligned} & \frac{1}{2}\frac{dP_1}{dt} + \left(\beta_1 + \frac{1}{2}\gamma_1^2 + \frac{\beta_3 A_2}{b}\right) P_1 + \frac{A_1^2}{2b} + \frac{\bar{\beta}_3}{b} N_1 A_2 = 0, \\ & \quad P_1(T) = F_{1T}, \\ & \frac{1}{2}\frac{dR_1}{dt} + \frac{1}{2b} B_1^2 + \left(\bar{\beta}_3 R_1 + \beta_3 N_1\right)\frac{B_2}{b} + (\bar{\beta}_1 + r\delta) R_1 = 0, \\ & R_1(T) = \bar{F}_{1T}, \\ & \frac{dN_1}{dt} + \frac{A_1 B_1}{b} + \left(\bar{\beta}_3 N_1 + \beta_3 P_1\right)\frac{B_2}{b} + \left(\bar{\beta}_3 R_1 + \beta_3 N_1\right)\frac{A_2}{b} \\ & \quad + \left(\bar{\beta}_1 + \beta_1 + r\delta\right) N_1 = 0, N_1(T) = 0, \\ & \frac{dQ_1}{dt} + \frac{A_1 C_1}{b} + \left(\beta_3 Q_1 + \bar{\beta}_3 S_1\right)\frac{A_2}{b} + \left(\bar{\beta}_3 N_1 + \beta_3 P_1\right)\frac{C_2}{b} \\ & \quad + \beta_1 Q_1 + p_1 a_1 = 0, Q_1(T) = 0, \\ & \frac{dS_1}{dt} + \frac{B_1 C_1}{b} + \left(\bar{\beta}_3 S_1 + \beta_3 Q_1\right)\frac{B_2}{b} + \left(\bar{\beta}_3 R_1 + \beta_3 N_1\right)\frac{C_2}{b} \\ & \quad + (\bar{\beta}_1 + r\delta) S_1 + \bar{p}_1 \bar{a}_1 = 0, S_1(T) = 0, \\ & \frac{dM_1}{dt} + \frac{C_1^2}{2b} + \left(\bar{\beta}_3 S_1 + \beta_3 Q_1\right)\frac{C_2}{b} - w = 0, M_1(T) = 0, \end{aligned} \tag{28}$$

$$\begin{aligned} & \frac{1}{2}\frac{dP_2}{dt} + \left(\beta_1 + \frac{A_1}{b}\beta_2 + \frac{1}{2}\gamma_1^2\right) P_2 + \frac{A_2^2}{2b} + \frac{A_1}{b} N_2 \bar{\beta}_2 = 0, \\ & P_2(T) = 2F_{2T}, \\ & \dot{Q}_2 + \frac{1}{b} A_2 C_2 + \frac{\beta_2}{b}(A_1 Q_2 + C_1 P_2) + \frac{\bar{\beta}_2}{b}(A_1 S_2 + C_1 N_2) \\ & \quad + \beta_1 Q_2 + p_2 a_2 = 0, Q_2(T) = 0, \\ & \frac{1}{2}\frac{dR_2}{dt} + \frac{B_2^2}{2b} + \frac{B_1}{b}\left(\beta_2 N_2 + \bar{\beta}_2 R_2\right) + \bar{\beta}_1 R_2 + r\delta R_2 = 0, \\ & R_2(T) = \bar{F}_{2T}, \\ & \dot{N}_2 + \frac{A_1 B_1}{b} + \beta_1 N_2 + \bar{\beta}_1 N_2 + \frac{\beta_2}{b}(A_1 N_2 + B_1 P_2) \\ & \quad + \frac{\bar{\beta}_2}{b}(A_1 R_2 + B_1 N_2) + r\delta N_2 = 0, N_2(T) = 0, \\ & \frac{dS_2}{dt} + \frac{B_2 C_2}{b} + \frac{B_1}{b}\left(\beta_2 Q_2 + \bar{\beta}_2 S_2\right) + \frac{C_1}{b}\left(\beta_2 N_2 + \bar{\beta}_2 R_2\right) \\ & \quad + \bar{p}_2 \bar{a}_2 + \bar{\beta}_1 S_2 + r\delta S_2 = 0, S_2(T) = 0, \end{aligned}$$

$$\frac{dM_2}{dt} + \frac{C_2^2}{2b} + \frac{C_1}{b}\left(Q_2\beta_2 + S_2\bar{\beta}_2\right) - w = 0, M_2(T) = 0. \tag{29}$$

The optimal value function are

$$J_1\left(I_1(t)^*, I_2(t)^*\right) = \frac{1}{2}P_1(0)k_0^2 + Q_1(0)k_0 + \frac{1}{2}R_1(0)\bar{k}_0^2 + S_1(0)\bar{k}_0 + M_1(0),$$
$$J_2\left(I_1(t)^*, I_2(t)^*\right) = \frac{1}{2}P_2(0)k_0^2 + Q_2(0)k_0 + \frac{1}{2}R_2(0)\bar{k}_0^2 + N_2(0)k_0\bar{k}_0 + S_2(0)\bar{k}_0 + M_2(0). \tag{30}$$

C. NUMERICAL ANALYSIS

In this section, due to the difficulty of solving analytically, we present numerical examples to illustrate our results. Moreover, the basic parameters are given by $S_{1T} = S_{2T} = \frac{1}{2}, \beta_1 = -0.3, \beta_2 = \delta = 0.4, \beta_3 = 0.6, r = 0.5, b = T = p_1 = p_2 = w = k_0 = \bar{k}_0 = 1, \alpha = 0.8, \gamma_1 = 0.1, \gamma_2 = 0.2, r_1 = 0.3, r_2 = 0.4, \alpha_1 = \alpha_3 = \xi_1 = \xi_2 = \bar{\xi}_1 = \bar{\xi}_2 = 0, \alpha_2 = \bar{\alpha}_2 = -\frac{1}{2}, \alpha_4 = a_1, \xi_3 = \bar{a}_1, \bar{\alpha}_4 = a_2, \bar{\xi}_3 = \bar{a}_2, \alpha_5 = \bar{\alpha}_5 = -1, \alpha_6 = -1, \bar{\alpha}_6 = -1, \bar{\alpha}_1 = \bar{\alpha}_3 = 0, a_1 = a_2 = \bar{a}_1 = \bar{a}_2 = 1$.

Then, the optimal investment strategies and value function are respectively

$$\begin{cases} I_1^*(t) = 0.4(P_1+N_1)K(t) + 0.4(R_1+N_1)\bar{K}(t) \\ +0.4(Q_1+S_1) - 1, \\ I_2^*(t) = 0.6(P_2+N_2)K(t) + 0.6(R_2+N_2)\bar{K}(t) \\ +0.6(Q_2+S_2) - 1, \end{cases}$$

where

$$0.5\frac{dP_1}{dt} + [0.36(P_2-N_2)+0.35]P_1 + 0.08(P_1+N_1)^2 + 0.36N_1(-P_2+N_2) = 0, P_1(T) = 1,$$
$$\frac{dN_1}{dt} + 0.16(P_1+N_1)(R_1+N_1) + 0.36(R_1-N_1)\times (N_2-P_2) + 0.36(N_1+P_1)(R_2+N_2) - 0.8N_1 = 0, N_1(T)=0,$$
$$0.5\frac{dR_1}{dt} + 0.08(R_1+N_1)^2 + 0.36(R_1+N_1)(R_2+N_2) - 0.1R_1 = 0, R_1(T) = 1,$$
$$\frac{dQ_1}{dt} + 0.4(P_1+N_1)[0.4(Q_1+S_1)-1] + 0.36(Q_1+S_1) \times (P_2+N_2) + 0.6(N_1-P_1)[0.6(S_2-Q_2)-1] + 0.3Q_1 + 1 = 0, Q_1(T) = 0,$$
$$\frac{dS_1}{dt} + 0.4(R_1+N_1)[0.4(Q_1+S_1)-1] + 0.36(Q_1-S_1) \times (N_2-R_2) + 0.6(R_1-N_1)[0.6(-Q_2+S_2)-1] - 0.5S_1 + 1 = 0, S_1(T) = 0,$$
$$\frac{dM_1}{dt} + 0.5[0.4(Q_1+S_1)-1]^2 + 0.6(S_1-Q_1) \times [0.6(S_2-Q_2)-1] - 1 = 0, M_1(T) = 0.$$
$$0.5\frac{dP_2}{dt} + [0.16(P_1+N_1)+0.305]P_2 + 0.18(P_1+N_1)^2 + 0.16(P_1+N_1)N_2 = 0, P_2(T) = 1,$$
$$\dot{Q}_2 + 0.6(N_2-P_2)[0.6(S_2-Q_2)-1] + 0.16(P_1+N_1)(Q_2+S_2) + 0.4[0.4(Q_1+S_1)-1] \times (P_2+N_2) - 0.3Q_2 + 1 = 0, Q_2(T) = 0,$$
$$0.5\frac{dR_2}{dt} + 0.18(R_2+N_2)^2 + 0.16(R_1+N_1)(N_2+R_2) + 0.5R_2 = 0, R_2(T) = 1,$$
$$\frac{dN_2}{dt} + 0.16(P_1+N_1)(R_1+N_1) + 0.16(P_1+N_1)(N_2+R_2) + 0.6N_2 + 0.16(R_1+N_1)(P_2+N_2) = 0, N_2(T) = 0,$$
$$\frac{dS_2}{dt} + 0.6(R_2+N_2)[0.6(Q_2+S_2)-1] + 0.16(R_1+N_1)(Q_2+S_2) + 0.4(N_2+R_2) \times [0.4(Q_1+S_1)-1] + 1 + 0.5S_2 = 0, S_2(T) = 0,$$
$$\frac{dM_2}{dt} + 0.5[0.6(S_2-Q_2)-1]^2 + 0.4[0.4(Q_1+S_1)-1] \times (Q_2+S_2) - 1 = 0, M_2(T) = 0.$$

The capital of traditional products and emerging products follows

$$dK(t) = [0.3K(t) + 0.4I_1(t) - 0.6I_2(t)]dt + 0.2K(t)dW(t), K(0) = 100,$$
$$d\bar{K}(t) = \left[0.3\bar{K}(t) - 0.4(t)I_1(t) + 0.6I_2(t)\right]dt + 0.1\bar{K}(t)dC(t) + 0.4\bar{K}(t)dV(t), \bar{K}(0) = 100,$$

the optimal value function are

$$J_1\left(I_1(t)^*, I_2(t)^*\right) = \frac{1}{2}P_1(0) + Q_1(0) + \frac{1}{2}R_1(0) + N_1(0) + S_1(0) + M_1(0),$$
$$J_2\left(I_1(t)^*, I_2(t)^*\right) = \frac{1}{2}P_2(0) + Q_2(0) + \frac{1}{2}R_2(0) + N_2(0) + S_2(0) + M_2(0).$$

The equilibrium strategies and capital accumulation of traditional products and emerging products over time are shown in Figures 1-2.

Figure 1 shows the process of changes in the optimal investment strategy for enterprises of traditional products and emerging products. Enterprises are gradually decreasing their investment in traditional products, and they are more cautious about emerging products at the beginning, showing a trend of first decreasing and then increasing.

Figure 2 illustrates the process of capital accumulation. Through the interaction of both parties' optimal strategies and the calculation of the above-mentioned performance indicators for both players, it can be concluded that both parties can achieve their respective optimal performance indicators. Under the joint action of optimal control strategies from both players in the game, there is an overall upward trend. The capital accumulation of traditional products increased relatively slowly, while the capital accumulation of emerging products increased more. Therefore, it can

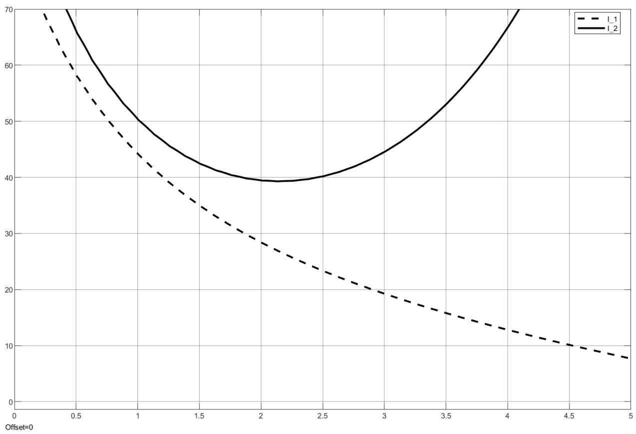

FIGURE 1. Equilibrium investment strategy I_1 and I_2.

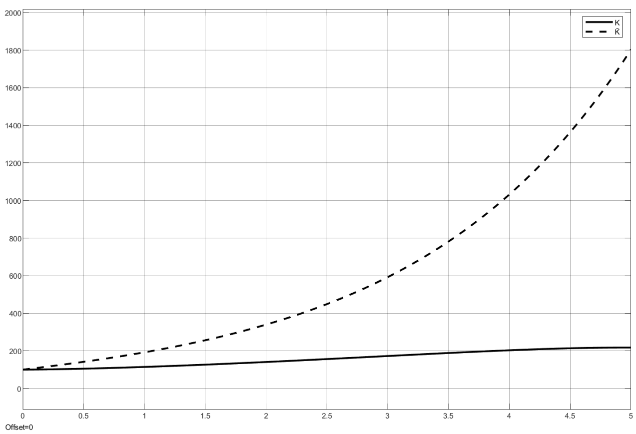

FIGURE 2. The capital accumulation K and $\bar{K}$.

be inferred that the uncertain stochastic differential system model with jumps is effective under the joint control of both players in the game.

V. CONCLUSION

In order to better describe the dynamic characteristics of enterprise investment decision-making in this paper, assuming the capital accumulation process of the enterprises are subject to the uncertain stochastic process with jumps. We investigate the dynamic portfolio of two enterprises based on a jump uncertain stochastic differential game model. We derive the equilibrium investment strategy and equilibrium value function through dynamic programming principle.

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