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THEORY

Feedback Stabilization of Cyber-Physical Systems for Sampled-Data Control: Synthesizing the Cyber and the Physical With Closed-Loop Interactions

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ABSTRACT Inspired by the cyber-physical systems (CPS) of numerical methods for stochastic differential equations, we present a CPS model of sampled-data control systems (typically a synonym for computer control systems), which regards the intersection of the physical and the cyber (the key feature of CPS). As a theoretic foundation, we develop by the Lyapunov method a stability theory for a general class of stochastic impulsive differential equations (SiDE) which is formulated as a canonical form for CPS that may work in feedback loops and thus include those of sampled-data control systems. Applying the fundamental theory, we study stability of the CPS, which implies that of the sampled-data control system. By our CPS approach, we not only obtain stability criteria for the CPS of sampled-data control systems but also reveal the equivalence and intrinsic relationship between the two main approaches (viz. controller emulation and discrete-time approximation) in the literature. As the applications of our CPS theory, we propose a control design method for feedback stabilization of the CPS of sampled-data stochastic systems. Illustrative examples are conducted to verify that our method significantly improves the existing results. In this paper, we initiate the study of a systems science of design for CPS. This provokes many open and interesting problems.

INDEX TERMS Cyber-physical systems, exponential stability, feedback stabilization, Lyapunov method, sampled-data control, stochastic impulsive differential equations.

I. INTRODUCTION

Feedback mechanisms were discovered and exploited at all levels in nature, which are crucial to homeostasis and life [3], [10], [72]. As a technology, feedback control can be found in many examples from ancient times. In the modern era, it was fundamental to the industrial evolution that James Watt successfully adapted the centrifugal governor for the steam engine and, in the later designs, the governor became an integral part of all steam engines. Theorectic investigation on the mechanical systems of governors started with the classical paper of Maxwell that placed stability at the core of his analysis of feedback mechanisms [47]. Stability analysis and feedback stabilization of dynamical systems are at the core of systems and control theory [2], [3], [5], [8], [32], [33], [34], [37], [39], [40], [44], [65], [74]. As is well known, the Lyapunov method is an efficient and powerful tool for stability analysis and synthesis of control systems. The Lyapunov-type theorems have been developed for stability analysis and application to feedback stabilization of various systems [5], [14], [15], [25], [28], [37], [40], [66]. In the enormous literature, there is a number of Lyapunov-type theorems on stability and feedback stabilization of impulsive systems [13], [21], [31], [58], [74].

Practically all control systems that are implemented today are based on computer control, which contain both continuous-time signals and sampled, or discrete-time, signals. Such systems have traditionally been termed

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sampled-data systems and have motivated the study of sampled-data control systems [2], [49]. There is a wealth of impressive results along two main approaches [2], [13], [43], [48], [49], [50], [51], [52], [54], [62]. The first starts with a designed continuous controller and focuses on discretizing the controller on a sampler and zero-order-hold (ZOH) device, which employs the strategy of controller emulation and is called the process-oriented view. The second disccretizes a continuous plant given implementation-dependent sampling times and designs a controller for the discretized plant, which utilizes some approximate discrete-time model for controller design and is called the computer-oriented view. There is another approach based on the hybrid modelling of sampled-data systems [18], [48], [63] which describes the sampled state as a pure jump process and is a special case of our canonical form (see Remark 2 below). Over the recent years, sampled-data control of stochastic systems has also been studied [12], [45], [46], [76] since stochastic modelling has come to play an important role in engineering and science. In fact, sampled-data control has an enormous range of applications (with a lot outside the field of computer control) as many practical systems are inherently sampled due to the measurement procedure and/or pulsed operation [2], [28], [29], [60]. For instance, biological systems are fundamently sampled since the signal transmission in a nervous system is in the form of pulses [2], [72].

Sampled-data control systems are generally used as a synonym for computer control systems and have an exemplary structure of cyber-physical systems (CPS) [38, Figure 1], where computers are embeded as components in control systems to monitor and control physical processes with feedback loops [1], [2], [29], [52], [60]. The science of design for CPS has been identified as a key research priority due to the utmost importance and urgency of CPS in the age of networking and information technology [1], [10], [11], [35]. Cyberphysicality spans the gamut of engineering domains. Among the hardest problems, the CPS is not the union of the physical and the cyber but their seamless, fully synergistic integration [35], [38]. As such the CPS demands a model/theory that comprehends both the cyber and the physical sides [11], [38]. The author [30] has constructed the CPS theory of numerical methods for stochastic differential equations (SDE). The CPS of a numerical method, say, the widely-used Euler method is a seamless integration of the SDE and the Euler scheme, unlike in the literature where they are two separate systems united by inequalities [22]. As the physical subsystem of the CPS is driven by itself only, the canonical form (1) in [30] does not involve any impact of the cyber on the physical and thus the established theory may not apply to synthesized CPS that are considered in the study of stabilization problems (see [30] for more detials). The CPS theory [30] of numerical methods for SDE has transformed the way we understand the relationship between the physical and the cyber, which is fundamental to a much bigger holistic worldview emerging in the age of networking and information technology.

As has been recognized [11], [35], [38], it is the intersection of the physcial and the cyber that is the key feature and renders it a challenging problem to create a theoretic foundation for CPS. To cope with the challenging problem, we formulate a general class of stochastic impulsive differential equations (SiDE) to serve as a canonical form for CPS that involve interactions between the physical and the cyber subsystems, which is a substantial generalization of [30, SiDE (1)]. The general class of SiDE can represent CPS that may work in feedback loops such as those of sampled-data control systems while the particular class [30, SiDE (1)] may not be used for such a purpose. As a theoretic foundation, we develop a Lyapunov stability theory for the general class of SiDE. As has been noted, sampled-data control systems have an exemplary structure of synthesized CPS [38, Figure 1] and thus they can naturally be expressed in our canonical form of SiDE. As a matter of fact, a special case of our canonical form has been employed to study sampled-data systems for a few decades (see [13], [18], [48], [63] and the references therein). Since it just depicts its cyber subsystem as a pure jump process, such special form of CPS may disregard some intersample dynamics. It may actually be the union rather than the intersection of its physical and cyber subsystems (see Remark 2 below). In light of the CPS theory [30] of numerical methods for SDE, we present our CPS of the sampled-data control system in which its physical subsystem represents the state of the control system and its cyber subsystem describes the error between the state and the sampled state. Thus the stability of the CPS implies that of the sampled-data control system. As the CPS [30] of numerical methods for SDE, our CPS of the sampled-data control system regards the intersection of the physical and the cyber, which is the key feature of CPS.

Our CPS of sampled-data control systems are definitely expressed by means of the canonical form of SiDE which we formulate for sythesized CPS. Applying the Lyapunov stability theory, we study stability of the CPS of sampled-data control systems and address the key questions in the two main approaches, respectively. By our CPS approach, we not only develop stability criteria for the CPS of sampled-data control systems but also reveal the equivalence and intrinsic relationship between the two main design methods in the literature. As the applications of our CPS theory, we study state-feedback stabilization of the CPS of linear sampled-data stochastic systems and propose a control design method for feedback stabilization of the CPS. Particularly, we present an algorithm for the control design in the form of generic linear matrix inequalities (LMI) so that it can conveniently be implemented with some toolboxes [8], [16], [17]. Numerical examples are conducted to illustate the applications of our established results and verify that our proposed method substantially improves the exsting results. In this paper, we construct a foundational theory for CPS of sampled-data control and initiate a systems science for CPS. This provokes many interesting and challenging problems.

II. STABILITY OF A CLASS OF STOCHASTIC IMPULSIVE DIFFERENTIAL EQUATIONS FOR SYNTHESIS OF CPS

Throughout this paper, unless otherwise specified, we employ the following notation. Denote by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions [44] and by $\mathbb{E}[\cdot]$ the expectation operator with respect to the probability measure. Let $B(t) = \big[B_1(t) \cdots B_m(t)\big]^T$ be an m-dimensional Brownian motion defined on the probability space. If x, y are real numbers, then $x \vee y$ (resp. $x \wedge y$) denotes the maximum (resp. minimum) of x and y. Denote by A^T the transpose of a vector or a matrix A. If P is a square matrix, $P > 0$ (resp. $P < 0$) means that P is a symmetric positive (resp. negative) definite matrix of appropriate dimensions while $P \geq 0$ (resp. $P \leq 0$) is a symmetric positive (resp. negative) semidefinite matrix. Let $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ be a matrix's eigenvalues with the maximum and the minimum real parts, respectively, and $|\cdot|$ the Euclidean norm of a vector and the trace (or Frobenius) norm of a matrix. Denote by I_n the $n \times n$ identity matrix and by $0_{n\times m}$ the $n \times m$ the zero matrix, or, simply, by 0 the zero matrix of appropriate dimensions. Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ be the family of all nonnegative functions $V(x, t)$ on $\mathbb{R}^n \times \mathbb{R}_+$ that are continuously twice differentiable in x and once in t, and $C^2(\mathbb{R}^n; \mathbb{R}_+)$ the special class of $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ that is independent of t. Let $\mathcal{K}$ be the class of continuous strictly increasing functions φ from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\varphi(0) = 0$ and $\mathcal{K}_\infty$ be a family of functions $\varphi \in \mathcal{K}$ with $\mu(r) \to \infty$ as $r \to \infty$. Let $\mathbb{N}$ be the set of all natural numers and $\Xi^m_{\mathbb{N}}$ be the set of all independent and identically distributed sequences $\{\xi(k)\}_{k\in\mathbb{N}}$ with $\xi(k) = \big[\xi_1(k) \cdots \xi_m(k)\big]^T$ and $\xi_j(k)$ obeying the standard Gaussian distribution for $j = 1, 2, \cdots, m$. Denote by $\{t_k\}_{k\in\mathbb{N}}$ with $t_1 > t_0 := 0$ a strictly increasing sequence on $(0, \infty)$ that satisfies

$$0 < \underline{\Delta t} := \inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\} \leq \overline{\Delta t} := \sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} < \infty$$

and hence $t_k \to \infty$ as $k \to \infty$. Upon the sequence $\{t_k\}_{k\in\mathbb{N}}$ and $t_0 = 0$, for each $t \in [0, \infty)$, define

$$t_* := \sup\{t_k : t \geq t_k, k \geq 0\}$$

and therefore $t_* = t_{k-1}$ if $t \in [t_{k-1}, t_k)$ for some $k \in \mathbb{N}$.

Let us consider a stochastic impulsive system

$$\mathrm{d}x(t) = f(x(t), y(t), t)\mathrm{d}t + g(x(t), y(t), t)\mathrm{d}B(t) \quad t \in [0, \infty) \tag{1a}$$

$$\mathrm{d}y(t) = \tilde{f}(x(t), y(t), t)\mathrm{d}t + \tilde{g}(x(t), y(t), t)\mathrm{d}B(t) \quad t \in [0, \infty) \setminus \{t_k\}_{k\in\mathbb{N}} \tag{1b}$$

$$\begin{aligned}\tilde{\Delta}(x(t_k^-), x(t_{k-1}), y(t_k^-), y(t_{k-1}), k) &:= y(t_k) - y(t_k^-)\\ &= \tilde{h}_f(x(t_k^-), x(t_{k-1}), y(t_k^-), y(t_{k-1}), k)\\ &\quad + \tilde{h}_g(x(t_k^-), x(t_{k-1}), y(t_k^-), y(t_{k-1}), k)\bar{\xi}(k) \quad k \in \mathbb{N}\end{aligned} \tag{1c}$$

with initial values $x(0) \in \mathbb{R}^n$ and $y(0) \in \mathbb{R}^{\tilde{n}}$, where $\bar{\xi} \in \Xi^{\tilde{n}}_{\mathbb{N}}$ is the measurement noise with $\bar{\xi}(k)$ being independent of $\{x(t), y(t), B(t) : 0 \leq t < t_k\}$ for every $k \in \mathbb{N}$; $f : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \times \mathbb{R}_+ \mapsto \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \times \mathbb{R}_+ \mapsto \mathbb{R}^{n\times m}$, $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \times \mathbb{R}_+ \mapsto \mathbb{R}^{\tilde{n}}$, $\tilde{g} : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \times \mathbb{R}_+ \mapsto \mathbb{R}^{\tilde{n}\times m}$, $\tilde{h}_f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \times \mathbb{R}^{\tilde{n}} \times \mathbb{N} \mapsto \mathbb{R}^{\tilde{n}}$ and $\tilde{h}_g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \times \mathbb{R}^{\tilde{n}} \times \mathbb{N} \mapsto \mathbb{R}^{\tilde{n}\times\tilde{n}}$ are measurable functions. To study the stability, assume that the functions obey

$$\begin{aligned}&f(0, 0, t) = 0, g(0, 0, t) = 0, \tilde{f}(0, 0, t) = 0,\\ &\tilde{g}(0, 0, t) = 0, \tilde{h}_f(0, 0, 0, 0, k) = 0, \tilde{h}_g(0, 0, 0, 0, k) = 0\end{aligned}$$

for all $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$ and thus the trivial solution is an equilibrium of SiDE (1). Moreover, the functions $f, g, \tilde{f}, \tilde{g}, \tilde{h}_f, \tilde{h}_g$ all satisfy the local Lipschitz condition.

Assumption 1: For every integer $\hat{n} \geq 1$, there is a constant $L_{\hat{n}} > 0$ such that

$$\begin{aligned}&|f(x, y, t) - f(\bar{x}, \bar{y}, t)| \vee |g(x, y, t) - g(\bar{x}, \bar{y}, t)|\\ &\quad \vee |\tilde{f}(x, y, t) - \tilde{f}(\bar{x}, \bar{y}, t)| \vee |\tilde{g}(x, y, t) - \tilde{g}(\bar{x}, \bar{y}, t)|\\ &\quad \vee |\tilde{h}_f(x, \tilde{x}, y, \tilde{y}, k) - \tilde{h}_f(\bar{x}, \bar{\tilde{x}}, \bar{y}, \bar{\tilde{y}}, k)|\\ &\quad \vee |\tilde{h}_g(x, \tilde{x}, y, \tilde{y}, k) - \tilde{h}_g(\bar{x}, \bar{\tilde{x}}, \bar{y}, \bar{\tilde{y}}, k)|\\ &\leq L_{\hat{n}}(|x - \bar{x}| \vee |\tilde{x} - \bar{\tilde{x}}| \vee |y - \bar{y}| \vee |\tilde{y} - \bar{\tilde{y}}|)\end{aligned}$$

for all $(x, \tilde{x}, y, \tilde{y}, \bar{x}, \bar{\tilde{x}}, \bar{y}, \bar{\tilde{y}}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \times \mathbb{R}^{\tilde{n}} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \times \mathbb{R}^{\tilde{n}}$ with $|x| \vee |\tilde{x}| \vee |y| \vee |\tilde{y}| \vee |\bar{x}| \vee |\bar{\tilde{x}}| \vee |\bar{y}| \vee |\bar{\tilde{y}}| \leq \hat{n}$, $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$.

The SiDE (1) is formulated as a canonical form for synthesized CPS that may work in feedback loops, in which the cyber $y(t)$ interacts with the physical $x(t)$ and the impulses on the cyber $y(t)$ involve delayed as well as current states. It is a substantial generalization of [30, SiDE (1)] as the impluses on the physical $x(t)$ and the simulation sequence are omitted for the sake of simplicity. The interactions between the subsystems clearly demonstrate that our knowledge/resources on both the physical and the cyber sides may be utilized to control the physical processes. The canonical form (1) for synthesized CPS has a wide range of applications, which can represent the CPS dynamics for sampled-data control systems and also for observer-based control of systems with impulse effects such as a robot model in [19].

Let $z(t) = [x^T(t)\ y^T(t)]^T \in \mathbb{R}^{n+\tilde{n}}$, $C = [I_n\ 0_{n\times\tilde{n}}]$ and $D = [0_{\tilde{n}\times n}\ I_{\tilde{n}}]$, then $x(t) = Cz(t)$ and $y(t) = Dz(t)$ for all $t \geq 0$. SiDE (1) can be written in a compact form

$$\mathrm{d}z(t) = F(z(t), t)\mathrm{d}t + G(z(t), t)\mathrm{d}B(t) \quad t \neq t_k \tag{2a}$$

$$\begin{aligned}\bar{\Delta}(z(t_k^-), z(t_{k-1}), k) &:= z(t_k) - z(t_k^-)\\ &= H_F(z(t_k^-), z(t_{k-1}), k)\\ &\quad + H_G(z(t_k^-), z(t_{k-1}), k)\bar{\xi}(k) \quad k \in \mathbb{N}\end{aligned} \tag{2b}$$

with initial data $z(0) = [x^T(0)\ \ y^T(0)]^T \in \mathbb{R}^{n+\tilde{n}}$, where functions $F : \mathbb{R}^{n+\tilde{n}} \times \mathbb{R}_+ \mapsto \mathbb{R}^{n+\tilde{n}}$, $G : \mathbb{R}^{n+\tilde{n}} \times \mathbb{R}_+ \mapsto \mathbb{R}^{(n+\tilde{n})\times m}$, $H_F : \mathbb{R}^{n+\tilde{n}} \times \mathbb{R}^{n+\tilde{n}} \times \mathbb{N} \mapsto \mathbb{R}^{n+\tilde{n}}$ and

$H_G : \mathbb{R}^{n+\tilde{n}} \times \mathbb{R}^{n+\tilde{n}} \times \mathbb{N} \mapsto \mathbb{R}^{(n+\tilde{n})\times\tilde{n}}$ are given as

$$F(z,t) = \begin{bmatrix} f(Cz,Dz,t) \\ \tilde{f}(Cz,Dz,t) \end{bmatrix}, \quad G(z,t) = \begin{bmatrix} g(Cz,Dz,t) \\ \tilde{g}(Cz,Dz,t) \end{bmatrix},$$
$$H_F(z,\tilde{z},k) = \begin{bmatrix} 0_{n\times 1} \\ \tilde{h}_f(Cz,C\tilde{z},Dz,D\tilde{z},k) \end{bmatrix},$$
$$H_G(z,\tilde{z},k) = \begin{bmatrix} 0_{n\times \tilde{n}} \\ \tilde{h}_g(Cz,C\tilde{z},Dz,D\tilde{z},k) \end{bmatrix};$$

and they obey $F(0,t)=0$, $G(0,t)=0$, $H_F(0,0,k)=0$ and $H_G(0,0,k)=0$ for all $t\in\mathbb{R}_+$ and $k\in\mathbb{N}$. Clearly, the trivial solution is an equilibrium of SiDE (2). For simplicity only, fix any $z(0) = [x^T(0) \;\; y^T(0)]^T \in \mathbb{R}^{n+\tilde{n}}$. These functions satisfy the local Lipschitz condition, that is, there is a constant $L_{z,\hat{n}} > 0$ for every integer $\hat{n}\geq 1$ such that

$$\begin{aligned} &|F(z,t)-F(\bar{z},t)| \vee |G(z,t)-G(\bar{z},t)| \\ &\vee |H_F(z,\tilde{z},k)-H_F(\bar{z},\bar{\tilde{z}},k)| \vee |H_G(z,\tilde{z},k)-H_G(\bar{z},\bar{\tilde{z}},k)| \\ &\leq L_{z,\hat{n}}(|z-\bar{z}| \vee |\tilde{z}-\bar{\tilde{z}}|) \end{aligned} \tag{3}$$

for all $(z,\tilde{z},\bar{z},\bar{\tilde{z}}) \in \mathbb{R}^{n+\tilde{n}}\times\mathbb{R}^{n+\tilde{n}}\times\mathbb{R}^{n+\tilde{n}}\times\mathbb{R}^{n+\tilde{n}}$ with $|z| \vee |\tilde{z}| \vee |\bar{z}| \vee |\bar{\tilde{z}}| \leq \hat{n}$, $t\in\mathbb{R}_+$ and $k\in\mathbb{N}$, which is exactly the compact form of Assumption 1.

For a function $\bar{V}\in C^{2,1}(\mathbb{R}^{n+\tilde{n}}\times\mathbb{R}_+;\mathbb{R}_+)$, the infinitesimal generator $\overline{\mathscr{L}}\bar{V} : \mathbb{R}^{n+\tilde{n}}\times\mathbb{R}_+ \mapsto \mathbb{R}$ associated with system (2a) is defined as

$$\begin{aligned} \overline{\mathscr{L}}\bar{V}(z,t) = \bar{V}_t(z,t) + \bar{V}_z(z,t)F(z,t) \\ + \frac{1}{2}tr\left[G^T(z,t)\bar{V}_{zz}(z,t)G(z,t)\right] \end{aligned} \tag{4}$$

where $\bar{V}_t(z,t) = \frac{\partial \bar{V}(z,t)}{\partial t}$, $\bar{V}_{zz}(z,t) = \left[\frac{\partial^2 V(z,t)}{\partial z_i \partial z_j}\right]_{(n+\tilde{n})\times(n+\tilde{n})}$, $\bar{V}_z(z,t) = \left[\frac{\partial \bar{V}(z,t)}{\partial z_1} \cdots \frac{\partial \bar{V}(z,t)}{\partial z_{(n+\tilde{n})}}\right]$ and trA stands for the trace of matrix A. Under condition (3), we obtain a result on the existence and uniqueness of solutions to SiDE (2).

Proposition 1: Under the local Lipschitz condition (3), SiDE (2) has a unique (right-continuous) solution on $[0,\infty)$ if there is a function $\bar{V}\in C^{2,1}(\mathbb{R}^{n+\tilde{n}}\times\mathbb{R}_+;\mathbb{R}_+)$ and positive constants $\bar{c}_1, p, \bar{K}$ such that, for all $(z,\tilde{z})\in\mathbb{R}^{n+\tilde{n}}\times\mathbb{R}^{n+\tilde{n}}$,

$$\bar{V}(z,t) \geq \bar{c}_1|z|^p, \qquad t\geq 0 \tag{5a}$$
$$\overline{\mathscr{L}}\bar{V}(z,t) \leq 2\bar{K}\left(1+\bar{V}(z,t)\right), \qquad t\neq t_k \tag{5b}$$
$$\begin{aligned} &\mathbb{E}[\bar{V}(z+\bar{\Delta}(z,\tilde{z},k),t_k)|z,\tilde{z}] - \bar{V}(z,t_k) \\ &\leq \bar{K}\underline{\Delta t}\left[2+\left(\bar{V}(z,t_k)+\bar{V}(\tilde{z},t_{k-1})\right)/2\right], \quad k\in\mathbb{N}. \end{aligned} \tag{5c}$$

The proof of is relegated to Appendix A. Now that we have the existence and uniqueness of solutions to SiDE (2), namely, SiDE (1), we shall exploit the structure and study the stability of the unique solution of the SiDE. Similarly, for a function $V\in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)$, the differential generator $\overline{\mathscr{L}}V : \mathbb{R}^{n+\tilde{n}}\times\mathbb{R}_+ \to \mathbb{R}$ associated with subsystem (1a) is defined as

$$\begin{aligned} \overline{\mathscr{L}}V(z,t) = V_t(x,t) + V_x(x,t)f(x,Dz,t) \\ + \frac{1}{2}tr\left[g^T(x,Dz,t)V_{xx}(x,t)g(x,Dz,t)\right] \end{aligned} \tag{6}$$

with $x = Cz$ for all $t\geq 0$ and, for a $\tilde{V}\in C^{2,1}(\mathbb{R}^{\tilde{n}}\times\mathbb{R}_+;\mathbb{R}_+)$, the differential generator $\overline{\mathscr{L}}\tilde{V} : \mathbb{R}^{n+\tilde{n}}\times\mathbb{R}_+ \to \mathbb{R}$ associated with subsystem (1b) is defined as

$$\begin{aligned} \overline{\mathscr{L}}\tilde{V}(z,t) = \tilde{V}_t(y,t) + \tilde{V}_y(y,t)\tilde{f}(Cz,y,t) \\ + \frac{1}{2}tr\left[\tilde{g}^T(Cz,y,t)\tilde{V}_{yy}(y,t)\tilde{g}(Cz,y,t)\right] \end{aligned} \tag{7}$$

with $y = Dz$ for all $t\in[0,\infty)\setminus\{t_k\}_{k\in\mathbb{N}}$. The definition of exponential stability is cited from the literature [44].

Definition 1: The system (2) is said to be pth $(p>0)$ moment exponentially stable if there is a pair of positive constants K and c with $K\geq|z(0)|^p$ such that $\mathbb{E}|z(t)|^p \leq Ke^{-ct}$ for all $t\geq 0$, which implies $\limsup_{t\to\infty}\frac{1}{t}\ln(\mathbb{E}|z(t)|^p) \leq -c < 0$ for all $z(0)\in\mathbb{R}^{n+\tilde{n}}$.

Let us establish by the Lyapunov method a stability theory for the general class (1) of SiDE.

Theorem 1: Suppose that Assumption 1 holds and there is a pair of candidate Lyapunov functions $V\in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)$ and $\bar{V}\in C^{2,1}(\mathbb{R}^{n+\tilde{n}}\times\mathbb{R}_+;\mathbb{R}_+)$ for the subsystem (1a) and the whole system (2), respectively, such that

(i) for all $(x,z,t)\in\mathbb{R}^n\times\mathbb{R}^{n+\tilde{n}}\times\mathbb{R}_+$, some positive constants $p, c_1, \bar{c}_1, \bar{c}$ and some functions $c_2, \bar{c}_2$ of $\mathcal{K}_\infty$ class,

$$c_1|x|^p \leq V(x,t) \leq c_2(|x|^p), \tag{8a}$$
$$\bar{c}_1|z|^p \leq \bar{V}(z,t) \leq \bar{c}_2(|z|^p), \tag{8b}$$
$$V(Cz,t) \leq \bar{c}\,\bar{V}(z,t); \tag{8c}$$

(ii) for all $(z,t)\in\mathbb{R}^{n+\tilde{n}}\times\mathbb{R}_+$ and some constants $\alpha_1>0$, $\alpha_2\geq 0$, $\bar{\alpha}_1\geq 0$, $\bar{\alpha}_2>0$,

$$\overline{\mathscr{L}}V(z,t) \leq -\alpha_1 V(Cz,t) + \alpha_2\bar{V}(z,t), \quad t\geq 0 \tag{9a}$$
$$\overline{\mathscr{L}}\bar{V}(z,t) \leq \bar{\alpha}_1 V(Cz,t) + \bar{\alpha}_2\bar{V}(z,t), \quad t\neq t_k; \tag{9b}$$

(iii) at $t=t_k$ for each $k\in\mathbb{N}$,

$$\begin{aligned} &\mathbb{E}\left[\bar{V}(z+\bar{\Delta}(z,\tilde{z},k),t_k)\middle|z,\tilde{z}\right] \leq \beta_0 V(Cz,t_k) \\ &+ \beta_1 V(C\tilde{z},t_{k-1}) + \bar{\beta}_0\bar{V}(z,t_k) + \bar{\beta}_1\bar{V}(\tilde{z},t_{k-1}) \end{aligned} \tag{10}$$

for all $(z,\tilde{z})\in\mathbb{R}^{n+\tilde{n}}\times\mathbb{R}^{n+\tilde{n}}$, where $\beta_0, \beta_1, \bar{\beta}_0, \bar{\beta}_1$ are nonnegative constants such that

$$0 < \frac{\alpha_2}{\alpha_1}(\beta_0+\beta_1) + \bar{\beta}_0 + \bar{\beta}_1 < 1. \tag{11}$$

SiDE (2) has a unique (right-continuous) global solution. Moreover, it is pth moment exponentially stable provided that the impulse time sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfies

$$0 < \underline{\Delta t} \leq \overline{\Delta t} < \widehat{\tau}(q) := \frac{-\ln q}{(\alpha_1 q)^{-1}\alpha_2\bar{\alpha}_1 + \bar{\alpha}_2} \tag{12}$$

for some $q\in\left(\alpha_1^{-1}\alpha_2(\beta_0+\beta_1)+\bar{\beta}_1+\bar{\beta}_2, 1\right)$.

Theorem 1 is a substantial advancement of [30, Theorem 1], which copes with not only delayed states but also the interactions between the physical and the cyber subsystems that should/must be taken into account by a model/theory of CPS that may work in feedback loops [11], [35], [38]. This distinguishes its proof from the one of [30, Theorem 1] as the latter shows the stability of the cyber subsystem and hence of the whole CPS based on the stability of the physical subsystem (that is exactly the SDE itself). The proof of Theorem 1 is rather more technical, so we give it in Appendix B. As the general class (1) of SiDE is formulated as a canonical form for CPS that may work in feedback loops, Theorem 1 is the fundamental result in our study on stability of CPS of sampled-data control systems.

Remark 1: In application of Theorem 1, the time sequence $\{t_k\}_{k\in\mathbb{N}}$ is chosen/designed to satisfy inequality (12) (see Sections III-V below). In (12), $\widehat{\tau}(q) > 0$ for every $q \in (0,1)$. If constants $\alpha_1, \alpha_2, \bar{\alpha}_1, \bar{\alpha}_2$, are all positive, $\widehat{\tau}(q)$ is a continuously differentiable function on $(0, 1)$ with derivative

$$\frac{\mathrm{d}\widehat{\tau}(q)}{\mathrm{d}q} = -\Big(\frac{\alpha_2\bar{\alpha}_1}{\alpha_1\sqrt{\bar{\alpha}_2}} + \sqrt{\bar{\alpha}_2 q}\Big)^{-2}\tau'(q), \tag{13}$$

where $\tau'(q) = \frac{\alpha_2\bar{\alpha}_1}{\alpha_1\bar{\alpha}_2}\big(1+\ln q\big) + q$. Notice that $\tau'(q)$ is increasing on $(0,\infty)$ and the maximum of $\widehat{\tau}(q)$ is achieved at $q = q_*$ given by

$$\tau'(q_*) = \frac{\alpha_2\bar{\alpha}_1}{\alpha_1\bar{\alpha}_2}\big(1+\ln q_*\big) + q_* = 0 \tag{14}$$

and $q_* \in (e^{-(\alpha_1\bar{\alpha}_2+\alpha_2\bar{\alpha}_1)/(\alpha_2\bar{\alpha}_1)}, 1)$ since $\tau'(1) = \frac{\alpha_2\bar{\alpha}_1}{\alpha_1\bar{\alpha}_2} + 1 > 0 > \tau'(e^{-(\alpha_1\bar{\alpha}_2+\alpha_2\bar{\alpha}_1)/(\alpha_2\bar{\alpha}_1)})$. One can compute q_* by solving (14) with initial guess

$$q_0 = \Big[\frac{\alpha_2(\beta_0+\beta_1)}{\alpha_1} + \bar{\beta}_0 + \bar{\beta}_1\Big] \vee e^{-(\alpha_1\bar{\alpha}_2+\alpha_2\bar{\alpha}_1)/(\alpha_2\bar{\alpha}_1)}.$$

Obviously, $0 < q_0 < 1$. It is observed from condition (12) that, for expoonential stability of SiDE (2), the choice of q is confined to $q \in (q_0, 1)$. By (13) and (14),

$$\sup_{q\in(q_0,1)} \widehat{\tau}(q) = \begin{cases} \widehat{\tau}(q_*), & 0 < q_0 \le q_* < 1 \\ \widehat{\tau}(q_0), & 0 < q_* < q_0 < 1. \end{cases}$$

So the condition (12) in Theorem 1 can be specified as

$$0 < \underline{\Delta t} \le \overline{\Delta t} < \widehat{\tau}(q_* \vee q_0) = \sup_{q\in(q_0,1)} \widehat{\tau}(q).$$

Recall that $\widehat{\tau}(q)$ is continuously differentiable on $(0, 1)$. Hence there is a number $q \in (q_0, 1)$ sufficiently close to $q_* \vee q_0$ such that (12) holds.

III. STABILITY OF CPS OF SAMPLED-DATA CONTROL SYSTEMS

Let us consider a sampled-data control system

$$\mathrm{d}x(t) = [\bar{f}(x(t)) + \bar{u}(x(t_*))]\mathrm{d}t + \bar{g}(x(t))\mathrm{d}B(t) \quad t \ge 0 \tag{15}$$

with initial value $x(0) = x_0 \in \mathbb{R}^n$ and sampling sequence $\{t_k\}_{k\in\mathbb{N}}$, where $\bar{u} \in C^2(\mathbb{R}^n;\mathbb{R}^n)$ with $\bar{u}(0) = 0$ is the control input; $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $\bar{g} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable functions with $\bar{f}(0) = 0$ and $\bar{g}(0) = 0$, which both satisfy the local Lipschitz condition, that is, there is a constant $\bar{L}_{\hat{n}} > 0$ for every integer $\hat{n} \ge 1$ such that

$$|\bar{f}(x) - \bar{f}(\bar{x})| \vee |\bar{g}(x) - \bar{g}(\bar{x})| \le \bar{L}_{\hat{n}}|x - \bar{x}| \tag{16}$$

for all $(x, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^n$ with $|x| \vee |\bar{x}| \le \hat{n}$. A notable case of (16) with $\bar{L}_{\hat{n}} = \sqrt{\bar{L}} > 0$ for all $\hat{n} \ge 1$ is called the global Lipschitz condition

$$|\bar{f}(x) - \bar{f}(\bar{x})| \vee |\bar{g}(x) - \bar{g}(\bar{x})| \le \sqrt{\bar{L}}\,|x - \bar{x}| \tag{17}$$

for all $(x, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^n$, which also implies the linear growth condition. Let us consider sampled-data system (15) that has a linear feedback control $\bar{u}(x) = \bar{B}x$ with matrix $\bar{B} \in \mathbb{R}^{n\times n}$

$$\mathrm{d}x(t) = [\bar{f}(x(t)) + \bar{B}x(t_*)]\mathrm{d}t + \bar{g}(x(t))\mathrm{d}B(t) \quad t \ge 0. \tag{18}$$

The linear feedback control law can easily be implemented and it preserves some important properties of the uncontrolled system such as the global Lipschitz condition and the linear growth condition [31], [45], [55], [68]. We shall illustrate with a classical example the application of Theorem 1 to sampled-data system (15) that has a nonlinear feedback control law (see Example 3 below).

Let $y(t) = x(t) - x(t_*)$ for all $t \ge 0$. This implies that $\mathrm{d}y(t) = \mathrm{d}x(t)$ on (t_{k-1}, t_k) and $y(t_k) = 0$ for all $k \in \mathbb{N}$. So we obtain a CPS model of sampled-data control system (18)

$$\mathrm{d}x(t) = [\bar{f}(x(t)) + \bar{B}(x(t) - y(t))]\mathrm{d}t + \bar{g}(x(t))\mathrm{d}B(t), \quad t \in [0,\infty) \tag{19a}$$

$$\mathrm{d}y(t) = [\bar{f}(x(t)) + \bar{B}(x(t) - y(t))]\mathrm{d}t + \bar{g}(x(t))\mathrm{d}B(t), \quad t \in [0,\infty) \setminus \{t_k\}_{k\in\mathbb{N}} \tag{19b}$$

$$y(t_k) - y(t_k^-) = x(t_{k-1}) - x(t_k^-), \quad k \in \mathbb{N} \tag{19c}$$

with $x(0) = x_0 \in \mathbb{R}^n$ and $y(0) = 0$. Clearly, it is the closed-loop interactions that synthesize the physical subsystem $x(t)$ and the cyber subsystem $y(t)$ into a seamless, fully synergistic integration, namely the CPS (19) of sampled-data control system (18). The CPS (19) is definitely a particular case of our canonical form (1) for sythesized CPS that satisfies Assumption 1, in which $\tilde{n} = n$, $f(x, y, t) = \tilde{f}(x, y, t) = \bar{f}(x) + \bar{B}(x - y)$, $g(x, y, t) = \tilde{g}(x, y, t) = \bar{g}(x)$, $\tilde{h}_f(x(t_k^-), x(t_{k-1}), y(t_k^-), y(t_{k-1}), k) = x(t_{k-1}) - x(t_k^-)$ and $\tilde{h}_g(x(t_k^-), x(t_{k-1}), y(t_k^-), y(t_{k-1}), k) = 0$ for all $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$. A particular version of Theorem 1 is given as follows.

Theorem 2: Suppose that the conditions (8)-(11) hold for CPS (19). If the sampling sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfies (12), then the CPS (19) is pth moment exponentially stable, which implies that the sampled-data control system (18) is pth moment exponentially stable.

Remark 2: A sampled-data system has been expressed by means of impulsive differential equations in the [18], [48], [63], which is also a special case of our canonical form (1) for CPS. For example, the hybrid system [48, Eq.(13)] is a special case of our proposed SiDE (2) with $y(t) = x(t_*)$ and hence $z(t) = [x^T(t)\ x^T(t_*)]^T$, which just depicts its cyber subsystem as a pure jump process. Clearly, our CPS (19) is distinct from those with a pure jump cyber subsystem in the literature, see also Remark 3. It is worth noting that the cyber subsystem of our CPS (19) regards the intersample behaviour of the error $y(t) = x(t) - x(t_*)$ while that of [48, Eq.(13)] is set to be the sampled state and is thus constant between the samples. The hybrid system [48, Eq.(13)] may essentially be the union of its physical and cyber subsystems, in which the subsystems never change simultaneously but alternately. It is easy to observe that the time interval $(0, \infty)$ is broken into a disjoint union $T_p \cup T_c$, where $T_p = \cup_{k\in\mathbb{N}}(t_{k-1}, t_k)$ is the set of sampling intervals on which the physical subsystem of [48, Eq.(13)] could vary as the cyber subsystem is constant and $T_c = \{t_k : k \in \mathbb{N}\}$ is the set of sampling instants at which the cyber could jump as the physical is continuous.

A. CONTROLLER EMULATION (PROCESS-ORIENTED MODELS)

By approach of controller emulation that is from the viewpoint of process-oriented models, a continuous-time state-feedback controller is designed based on the continuous-time plant model for stability of the closed-loop system

$$\mathrm{d}x(t) = \bar{f}_u(x)\mathrm{d}t + \bar{g}(x(t))\mathrm{d}B(t) \quad t \geq 0 \tag{20}$$

with $\bar{f}_u(x) = \bar{f}(x) + \bar{u}(x) = \bar{f}(x) + \bar{B}x$ (being the drift of the closed-loop system) and then the state-feedback controller is discretized and implemented using a sampler and ZOH device. This leads to the sampled-data control system (18) and its cyber-physical dynamics is described by (19). The main question in the design method is [2], [48], and [49]

for what sampling sequence $\{t_k\}_{k\in\mathbb{N}}$ do the CPS (19) and hence the sampled-data control system (18) preserve the stability property of the continuous-time system (20)?

We apply our established CPS theory and address the main question. Specifically, we apply Theorem 2 and find the conditions on $\{t_k\}_{k\in\mathbb{N}}$ for exponential stability of the CPS (19), which implies that of the sampled-data system (18), when the feedback control $\bar{u}(x) = \bar{B}x$ is designed such that

$$\mathscr{L}V(x) \leq -\alpha V(x) \quad \forall x \in \mathbb{R}^n \tag{21}$$

and thus the closed-loop system (20) is exponentially stable [34], [44], where α is a positive constant, $V \in C^2(\mathbb{R}^n; \mathbb{R}_+)$ is a Lyapunov function with property (8a) and the infinitesimal generator $\mathscr{L}V : \mathbb{R}^n \to \mathbb{R}$ associated with the controlled system (20) is defined as

$$\mathscr{L}V(x) = V_x(x)\bar{f}_u(x) + \frac{1}{2}tr\left[\bar{g}^T(x)V_{xx}(x)\bar{g}(x)\right]. \tag{22}$$

Generally, for a function $\tilde{V} \in C^2(\mathbb{R}^n; \mathbb{R}_+)$, we introduce the differential operator $co\mathscr{L}\tilde{V} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ acting on $\tilde{V}(y)$ jointly associated with the controlled system (20)

$$co\mathscr{L}\tilde{V}(y, x) = \tilde{V}_y(y)\bar{f}_u(x) + \frac{1}{2}tr\left[\bar{g}^T(x)\tilde{V}_{yy}(y)\bar{g}(x)\right] \tag{23}$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. From (22) and (23), it is easy to see that $\mathscr{L}V(x) = co\mathscr{L}V(x, x)$ for all $x \in \mathbb{R}^n$.

Remark 3: The input delay approach [13] basically uses the differential operator (22) (see also [24, Eq. (2)]) to study a sampled-data system. The approach [18], [48] employs such a special case of SiDE (2) that it can just use the operator (6) for stability analysis of the system as well. It is observed that the operator (22) is a major part of (6) and the sampled state is treated as a delayed state in [13]. Hence, these approaches essentially focus on part of the system behaviour that is representd by (6). But the CPS (19) involves the closed-loop interactions which need to be described by both the operators (6) and (7). Thus we introduce the general differential operator (23) for the representation of intersample dynamics of the CPS, which constitutes a major part of (7).

Let us now consider the CPS (19) of sample-data system (18) with global Lipschitz condition (17). It is very helpful for exposing not only the interactions between the subsystems (see also [30]) but also the intrinsic relationship between the two main approaches, see Theorem 4 and Remark 5 below. Moreover, under the global Lipschitz condition, the mean-square exponential stability of the system implies that it is also almost surely exponentially stable [22], [25], [30], [44].

Theorem 3: Let the Lyapunov function $V(x)$ in (21) for the controlled system (20) be a quadratic function

$$V(x) = x^T Px \tag{24}$$

with some positive definite matrix $P \in \mathbb{R}^{n\times n}$. Under the condition (17), the CPS (19) and thus the sampled-data control system (18) are mean-square exponentially stable provided that the sampling sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfies

$$0 < \underline{\Delta t} \leq \overline{\Delta t} < \frac{\ln\alpha - \ln\alpha_2}{\bar{\alpha}_2}, \tag{25}$$

where $\alpha_2 \in (0, \alpha)$ and $\bar{\alpha}_2 > 0$ are constants such that

$$\begin{bmatrix} 0 & -P\bar{B} \\ -\bar{B}^T P & 0 \end{bmatrix} \leq \alpha_2\bar{P}, \quad \bar{P} := \begin{bmatrix} P & 0 \\ 0 & \tilde{P} \end{bmatrix} > 0, \tag{26a}$$

$$\begin{bmatrix} -\alpha P + \tilde{\eta}_1\tilde{P} & -P\bar{B} \\ -\bar{B}^T P & \tilde{\eta}_2\tilde{P} - \bar{B}^T\tilde{P} - \tilde{P}\bar{B} \end{bmatrix} \leq \bar{\alpha}_2\bar{P} \tag{26b}$$

for some positive definite matrix $\tilde{P} \in \mathbb{R}^{n\times n}$ and some constants $\tilde{\eta}_1, \tilde{\eta}_2 \in \mathbb{R}$ being such that, for the quadratic function $\tilde{V}(y) = y^T\tilde{P}y$,

$$co\mathscr{L}\tilde{V}(y, x) \leq \tilde{\eta}_1\tilde{V}(x) + \tilde{\eta}_2\tilde{V}(y) \quad \forall x, y \in \mathbb{R}^n. \tag{27}$$

Proof: Given $P > 0$ and $\alpha > 0$ by (21)-(24), one can choose any positive number α_2 on $(0, \alpha)$ and then find a positive definite matrix $\tilde{P} \in \mathbb{R}^{n\times n}$ with $\lambda_m(\tilde{P}) > 0$ sufficiently large for $|P\bar{B}|^2/\lambda_m(P) \leq \alpha_2^2\lambda_m(\tilde{P})$, which implies (26a). Since functions $\bar{f}, \bar{g}$ both satisfy the global Lipschitz condition (17), so does $\bar{f}_u(x) = \bar{f}(x) + \bar{u}(x) = \bar{f}(x) + \bar{B}x$, that is, $|\bar{f}_u(x) - \bar{f}_u(\bar{x})|^2 \leq 2(\bar{L} + |\bar{B}|^2)|x - \bar{x}|^2$ for all $(x, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^n$. Therefore, $\bar{f}_u(x) = \bar{f}(x) + \bar{B}x$ and $\bar{g}(x)$ satisfy the linear growth conditions $|\bar{f}_u(x)|^2 = |\bar{f}(x) + \bar{B}x|^2 \leq 2(\bar{L} + |\bar{B}|^2)|x|^2$ and $|\bar{g}(x)|^2 \leq \bar{L}|x|^2$ for all $x \in \mathbb{R}^n$, respectively. By [25, Lemma 3.1 and Lemma 5.1],

$$\begin{aligned} co\mathscr{L}\tilde{V}(y, x) &= 2y^T\tilde{P}\bar{f}_u(x) + tr\left[\bar{g}^T(x)\tilde{P}\bar{g}(x)\right] \\ &\leq \tilde{V}(y) + \tilde{V}(\bar{f}_u(x)) + \lambda_M(\tilde{P})|\bar{g}(x)|^2 \\ &\leq \lambda_M(\tilde{P})\big(|\bar{f}_u(x)|^2 + |\bar{g}(x)|^2\big) + \tilde{V}(y) \\ &\leq \frac{(3\bar{L} + 2|\bar{B}|^2)\lambda_M(\tilde{P})}{\lambda_m(\tilde{P})}\tilde{V}(x) + \tilde{V}(y) \end{aligned}$$

for all $x, y \in \mathbb{R}^n$. This implies that there is a pair of real numbers $\tilde{\eta}_1 \leq (3\bar{L} + 2|\bar{B}|^2)\lambda_M(\tilde{P})/\lambda_m(\tilde{P})$ and $\tilde{\eta}_2 \leq 1$ for (27). Given the real constants $\alpha, \tilde{\eta}_1, \tilde{\eta}_2$ and the positive definite matrices $P, \tilde{P}$ as above, it is easy to find a positive number $\bar{\alpha}_2$ sufficiently large for (26b). Thus the control design method (21) with quadratic Lyapunov function (24) ensures that there exists a positive definite matrix $\tilde{P} \in \mathbb{R}^{n\times n}$ and some positive numbers $\alpha_2, \bar{\alpha}_2, \tilde{\eta}_1, \tilde{\eta}_2$ with $\alpha_2 \in (0, \alpha)$ such that the conditions (25), (26) and (27) hold.

It will follow from Theorem 2 that the CPS (19) is mean-square exponentially stable if the conditions (8)-(12) hold with $p = 2$ for (19), which yields that the sampled-data control system (18) is mean-square exponentially stable.

Let $\bar{V}(z) = z^T\bar{P}z = V(x) + \tilde{V}(y)$ for all $z = [x^T\ y^T]^T \in \mathbb{R}^{2n}$. Hence, $\lambda_m(P)|x|^2 \le V(x) \le \lambda_M(P)|x|^2$ for all $x \in \mathbb{R}^n$, $\big(\lambda_m(P) \wedge \lambda_m(\tilde{P})\big)|z|^2 \le \bar{V}(z) \le \big(\lambda_M(P) \vee \lambda_M(\tilde{P})\big)|z|^2$ and $V(Cz) \le \bar{V}(z)$ for all $z \in \mathbb{R}^{2n}$. Thus the conditions (8) hold with positive numbers $p = 2, c_1 = \lambda_m(P), \bar{c}_1 = \lambda_m(P) \wedge \lambda_m(\tilde{P}), \bar{c} = 1$ and functions $c_2(|x|^2) = \lambda_M(P)|x|^2, \bar{c}_2(|z|^2) = \big[\lambda_M(P) \vee \lambda_M(\tilde{P})\big]|z|^2$ of $\mathcal{K}_\infty$ class. Applying (6), (21) and (26a) produces

$$\begin{aligned}\overline{\mathscr{L}}V(z) &= 2\,x^TP[\bar{f}(x) + \bar{B}x - \bar{B}y] + tr\big[\bar{g}^T(x)P\bar{g}(x)\big]\\ &= 2\,x^TP\bar{f}_u(x) + tr\big[\bar{g}^T(x)P\bar{g}(x)\big] - 2x^TP\bar{B}y\\ &\le -\alpha V(Cz) - 2z^TC^TP\bar{B}Dz\\ &\le -\alpha V(Cz) + \alpha_2\bar{V}(z) \quad \forall z = [x^T\ y^T]^T \in \mathbb{R}^{2n}.\end{aligned}$$

Hence, (9a) holds with $\alpha_1 = \alpha$. Similarly,

$$\begin{aligned}\overline{\mathscr{L}}\bar{V}(z) &= \overline{\mathscr{L}}V(z) + \overline{\mathscr{L}}\tilde{V}(z)\\ &\le -\alpha V(Cz) - 2z^TC^TP\bar{B}Dz\\ &\quad + 2\,y^T\tilde{P}[\bar{f}(x) + \bar{B}x - \bar{B}y] + tr\big[\bar{g}^T(x)\tilde{P}\bar{g}(x)\big]\\ &= -\alpha V(Cz) - 2z^TC^TP\bar{B}Dz - 2\,y^T\tilde{P}\bar{B}y\\ &\quad + 2y^T\tilde{P}\bar{f}_u(x) + tr\big[\bar{g}^T(x)\tilde{P}\bar{g}(x)\big] \end{aligned} \tag{28}$$

Using (27) and then (26b), one can deduce from (28) that

$$\begin{aligned}\overline{\mathscr{L}}\bar{V}(z) &\le -\alpha V(Cz) - 2z^TC^TP\bar{B}Dz\\ &\quad + co\mathscr{L}\tilde{V}(Dz, Cz) - 2z^TD^T\tilde{P}\bar{B}Dz\\ &\le -\alpha V(Cz) + \tilde{\eta}_1\tilde{V}(Cz) - 2z^TC^TP\bar{B}Dz\\ &\quad + \tilde{\eta}_2\tilde{V}(Dz) - 2z^TD^T\tilde{P}\bar{B}Dz\\ &\le \bar{\alpha}_2 z^T\bar{P}z = \bar{\alpha}_2\bar{V}(z) \qquad \forall z \in \mathbb{R}^{2n}\end{aligned}$$

which is the condition (9b) with $\bar{\alpha}_1 = 0$.

Observe that (19c) and $y(t) = x(t) - x(t_*)$ for all $t \ge 0$ give $y(t_k) = y(t_k^-) + x(t_{k-1}) - x(t_k^-) = 0$ for all $k \in \mathbb{N}$. This immediately produces $\bar{V}(z(t_k)) = V(Cz(t_k))$ and thus the condition (10) with $\beta_0 = 1$ and $\beta_1 = \tilde{\beta}_0 = \tilde{\beta}_1 = 0$. So the conditions (10)-(11) hold due to $0 < \alpha_2\beta_0/\alpha = \alpha_2/\alpha < 1$. Moreover, the inequality (25) means that the condition (12) holds with $\alpha_1 = \alpha > \alpha_2 > 0, \bar{\alpha}_2 > \bar{\alpha}_1 = 0, \beta_0 = 1$ and $\beta_1 = \tilde{\beta}_0 = \tilde{\beta}_1 = 0$. By Theorem 2, the CPS (19) and hence the sampled-data control system (18) are mean-square exponentially stable. □

B. DISCRETE-TIME APPROXIMATION (COMPUTER-ORIENTED MODELS)

As periodic sampling ($\{t_k\}_{k\in\mathbb{N}}$ with sampling period $\Delta t = \underline{\Delta t} = \overline{\Delta t}$) is normally used [2], [49], [51], a sampling interval $t_k - t_{k-1}$ could vary in the design method based on computer-oriented models which are discrete-time approximation of the underlying continuous-time plants [50], [54]. By approach of discrete-time approximation, one employs some approximate discrete-time model, say, the Euler-Maruyama approximation of the continuous-time plant (due to the usual unavailability of the exact discrete-time model), and designs a discrete-time state-feedback controller $\bar{u}(X) = \bar{B}X$ for stability of the closed-loop system, which is the Euler-Maruyama approximation [22], [44], [51] of the closed-loop system (20),

$$X_k = X_{k-1} + \bar{f}_u(X_{k-1})h + \bar{g}(X_{k-1})\Delta B_k \tag{29}$$

with stepsize $h > 0$ and initial value $X_0 = x_0 \in \mathbb{R}^n$, where both the functions $\bar{f}_u, \bar{g}$ satisfy the global Lipschitz condition (17) and $\Delta B_k = B(kh) - B((k-1)h)$ for all $k \in \mathbb{N}$. Specifically, a state-feedback controller $\bar{u}(X) = \bar{B}X$ is designed such that

$$\mathbb{E}[V(X_k)|X_{k-1}] \le (1-c)V(X_{k-1}) \quad \forall X_{k-1} \in \mathbb{R}^n \tag{30}$$

and, therefore, the closed-loop system (29) is exponentially stable [8], [30], [34], where $c \in (0, 1)$ is a constant and $V : \mathbb{R}^n \to \mathbb{R}_+$ is a Lyapunov function with property (8a), say, the quadratic Lyapunov function (24). The obtained controller $\bar{u}(x) = \bar{B}x$ is then implemented in the continuous-time plant using ZOH sampling, that is, $\bar{u}(t) = \bar{u}(x(t_*)) = \bar{B}x(t_*)$ for all $t \ge 0$. This leads to the sampled-data control system (18) and its CPS model (19) as well. The central question in the design method (30) is, see [2], [49], [50], and [51],

for what sampling sequence $\{t_k\}_{k\in\mathbb{N}}$ do the CPS (19) and hence the sampled-data control system (18) share the stability property of approximate discrete-time system (29)?

We address this question with Theorem 3 and reveal the intrinsic relationship between the control design methods (21) and (30).

Theorem 4: Let the Lyapunov function $V(X)$ in (30) for the discrete-time system (29) be of the quadratic form (24) with some positive definite matrix $P \in \mathbb{R}^{n\times n}$. Under the condition (17), the CPS (19) and thus the sampled-data control system (18) are mean-square exponentially stable provided that the sampling sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfies

$$0 < \underline{\Delta t} \le \overline{\Delta t} < \frac{\ln(ch^{-1} + \alpha_u h) - \ln\alpha_2}{\bar{\alpha}_2}, \tag{31}$$

where $\alpha_2 \in (0, c\,h^{-1} + \alpha_u h)$ and $\bar{\alpha}_2 > 0$ are the constants given by (26) with $\alpha = ch^{-1} + \alpha_u h$ as well as (27), and $\alpha_u > 0$ is a constant such that

$$V(\bar{f}_u(X)) \le \alpha_u V(X) \quad \forall X \in \mathbb{R}^n. \tag{32}$$

Proof: By control design method (30) with quadratic Lyapunov function (24) and condition (32),

$$\begin{aligned}&\mathbb{E}[V(X_k)|X_{k-1}]\\ &= \mathbb{E}\big[X_k^TPX_k|X_{k-1}\big]\\ &= \mathbb{E}\Big[\big(X_{k-1} + \bar{f}_u(X_{k-1})h + \bar{g}(X_{k-1})\Delta B_k\big)^TP\end{aligned}$$

$$
\begin{aligned}
&\cdot \big(X_{k-1} + \bar{f}_u(X_{k-1})h + \bar{g}(X_{k-1})\Delta B_k\big) \big| X_{k-1}\Big] \\
&= V(X_{k-1}) + h\Big[X_{k-1}^T P\bar{f}_u(X_{k-1}) + \bar{f}_u^T(X_{k-1})PX_{k-1} \\
&\quad + tr\big[\bar{g}^T(X_{k-1})P\bar{g}(X_{k-1})\big] + hV(\bar{f}_u(X_{k-1}))\Big] \\
&\le V(X_{k-1}) + h\Big[2X_{k-1}^T P\bar{f}_u(X_{k-1}) \\
&\quad + tr\big[\bar{g}^T(X_{k-1})P\bar{g}(X_{k-1})\big] + \alpha_u hV(X_{k-1})\Big] \\
&\le (1-c)V(X_{k-1}) \quad \forall X_{k-1} \in \mathbb{R}^n \qquad (33)
\end{aligned}
$$

and, therefore,

$$
\begin{aligned}
&2X_{k-1}^T P\bar{f}_u(X_{k-1}) + tr\big[\bar{g}^T(X_{k-1})P\bar{g}(X_{k-1})\big] \\
&\quad + \alpha_u hV(X_{k-1}) \\
&\le -\frac{c}{h}V(X_{k-1}) \qquad (34)
\end{aligned}
$$

for all $X_{k-1} \in \mathbb{R}^n$, where the global Lipschitz condition implies the linear growth condition and hence that there is a positive number $\alpha_u \in (0, 2(\bar{L} + |B|^2)\lambda_M(P)/\lambda_m(P)]$ such that (32) holds due to the fact that, for all $X \in \mathbb{R}^n$,

$$
V(\bar{f}_u(X)) \le \lambda_M(P)|\bar{f}_u(X)|^2 \le 2(\bar{L} + |\bar{B}|^2)\frac{\lambda_M(P)}{\lambda_m(P)}V(X).
$$

Let $V(x) = x^T Px$ also be the candidate Lyapunov function for the continuous-time system (20). From (22) and (34), one can deduce that

$$
\begin{aligned}
\mathscr{L}V(x) &= 2\,x^T P\bar{f}_u(x) + tr\big[\bar{g}^T(x)P\bar{g}(x)\big] \\
&\le -\big(\frac{c}{h} + \alpha_u h\big)V(x) \qquad \forall x \in \mathbb{R}^n.
\end{aligned}
$$

This is exactly the control design method (21) with Lyapunov exponent, or say, decay rate $-\alpha$ as

$$
\alpha = \frac{c}{h} + \alpha_u h. \qquad (35)
$$

On the other hand, if a controller is designed by the method (21) for stability of the controlled system (20), one can choose some constant stepsize $h \in \big(0, (\alpha/\alpha_u) \wedge (1/\alpha)\big)$ for the discrete-time approximation (29). Then, by (33)-(35), one obtains the condition (30) with $c = (\alpha - \alpha_u h)h \in (0, 1)$ for the design method based on discrete-time approximation. Clearly, the intrinsic relationship (35) shows the equivalence between the design methods (21) and (30).

Let $\alpha = ch^{-1} + \alpha_u h > 0$ in the conditions of Theorem 3. It immediately follows from Theorem 3 that the assertion of Theorem 4 holds. □

Remark 4: In the literature, periodic sampling is normally used and it is usually assumed that the sampling period Δt is also the stepsize h of the discrete-time model (i.e., $h = \Delta t$) [2], [49], [50], [51]. When the both are set to be the same $h = \Delta t$, the controlled system may be globally stable if the exact discrete-time model can be utilized, e.g., in linear deterministic systems [2], [62]; otherwise, some discrete-time approximation is employed and the control design may achieve semiglobal practical stabilization [50], [51]. We stress that the stepsize h and the sampling period Δt are essentially two different parameters of the controller. The former is one of the design parameters and the latter a parameter for the implementation of ZOH sampling. For stability of the resulting control system (18), we clearly show by (31) how the design parameters impose the maximum alllowable sampling interval on the implementation.

Remark 5: Under the global Lipschitz condition (17), we have shown the equivalence between the design methods (21) and (30) for sampled-data control system (18). More specifically, we not only provide the link [62] but also reveal the intrinsic relationship (35) between the two main approaches. This is a unique fundamental contribution to sampled-data control systems. It is also observed that, in addtion to P, $\tilde{P}$, α_2, $\bar{\alpha}_2$ shared by both (21) and (30), a few parameters h, c, α_u are involved in the design method (30) as only α in the other.

C. HIGHLY NONLINEAR SYSTEMS WITH LOCAL LIPSCHITZ CONDITION

Under the global Lipschitz condition (17), we construct the quadratic Lyapunov function $\bar{V}(z) = z^T\bar{P}z$ with $\bar{P} = diag\{P, \tilde{P}\} > 0$ for the CPS (19) of sampled-data control system (18) based on the one (24) for the controlled system (20) so that we can exploit the control design (21) with Lyapunov function (24) as well as the linear growth condition in the proof of Theorem 3. But such functions of quadratic form may not be the suitable Lyapunov candidates for a highly nonlinear sampled-data system (18) and its CPS (19), where sampled-data system (18) satisfies the local Lipschitz condition (16) instead of the global one (17). In these cases, we may have to construct an appropriate candidate Lyapunov function for the CPS (19) based on the one used in the control design (21) for the highly nonlinear system (20). Let us proceed to make use of the Lyapunov-based control design (21) and study the CPS (19) of sampled-data control system (18) with local Lipschitz condition (16).

Theorem 5: Suppose that there is a Lyapunov function $V(x)$ designed as (21) for stability of the controlled system (20) with local Lipschitz condition (16) and there is also a candidate Lyapunov function $\tilde{V} \in C^2(\mathbb{R}^n; \mathbb{R}_+)$ with

$$
\tilde{c}_1|y|^p \le \tilde{V}(y) \le \tilde{c}_2(|y|^p) \quad \forall y \in \mathbb{R}^n \qquad (36)
$$

such that

$$
\mathscr{L}V(x) - V_x(x)\bar{B}y \le -\alpha V(x) + \alpha_2\bar{V}(z), \qquad (37a)
$$

$$
\mathscr{L}V(x) - \big[V_x(x) + \tilde{V}_y(y)\big]\bar{B}y + co\mathscr{L}\tilde{V}(y, x) \le \bar{\alpha}_2\bar{V}(z) \qquad (37b)
$$

for all $z = [x^T\ y^T]^T \in \mathbb{R}^{2n}$, where $\bar{V}(z) = V(x) + \tilde{V}(y)$, $\tilde{c}_1, \alpha, \alpha_2, \bar{\alpha}_2$ are positive constants with $\alpha_2 \in (0, \alpha)$ and function $\tilde{c}_2$ is of $\mathcal{K}_\infty$ class. The CPS (19) and thus the sampled-data control system (18) are pth moment exponentially stable provided that the sampling sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfies

$$
0 < \underline{\Delta t} \le \overline{\Delta t} < \frac{\ln\alpha - \ln\alpha_2}{\bar{\alpha}_2}. \qquad (38)
$$

Proof: The assertion follows from Theorem 2 if we verify that the conditions (8)-(12) are satsified. Let $\tilde{V}(y)$ and $\bar{V}(z) = V(x) + \tilde{V}(y)$ be the candidate Lyapunov functions for the cyber subsystem (19b,19c) and the CPS (19), respectively. Obviously, condition (8c) holds with $\bar{c} = 1$. Note that $|z|^2 = |x|^2 + |y|^2$ and, therefore,

$$|x|^p \vee |y|^p \le |z|^p = (|x|^2 + |y|^2)^{p/2} \le k_p(|x|^p + |y|^p)$$

for all $z = [x^T\ y^T]^T \in \mathbb{R}^{2n}$, where $k_p = 1$ when $0 < p < 2$ and $k_p = 2^{(p-2)/2}$ if $p \ge 2$. It is easy to observe from the properties (8a) and (36) of $V(x)$ and $\tilde{V}(y)$ that

$$\begin{aligned}(c_1 \wedge \tilde{c}_1)k_p^{-1}|z|^p &\le (c_1 \wedge \tilde{c}_1)(|x|^p + |y|^p) \le c_1|x|^p + \tilde{c}_1|y|^p \\ &\le \bar{V}(z) = V(x) + \tilde{V}(y) \\ &\le c_2(|x|^p) + \tilde{c}_2(|y|^p) \le c_2(|z|^p) + \tilde{c}_2(|z|^p)\end{aligned}$$

and thus the condition (8b) with $\bar{c}_1 = (c_1 \wedge \tilde{c}_1)/k_p > 0$ and $\bar{c}_2(\cdot) = c_2(\cdot) + \tilde{c}_2(\cdot)$ of $\mathcal{K}_\infty$ class.

As above, by (6), (21) and (37a),

$$\begin{aligned}\overline{\mathscr{L}}V(z) &= V_x(x)[\bar{f}_u(x) - \bar{B}y] + \frac{1}{2}tr\left[\bar{g}^T(x)V_{xx}(x)\bar{g}(x)\right] \\ &= \mathscr{L}V(x) - V_x(x)\bar{B}y \\ &\le -\alpha V(Cz) + \alpha_2\bar{V}(z) \quad \forall z = [x^T\ y^T]^T \in \mathbb{R}^{2n}\end{aligned}$$

which is the condition (9a) with $\alpha_1 = \alpha$.

As (28), by (6), (7) and (37b),

$$\begin{aligned}\overline{\mathscr{L}}\bar{V}(z) &= \overline{\mathscr{L}}V(z) + \overline{\mathscr{L}}\tilde{V}(z) \\ &= \mathscr{L}V(x) - V_x(x)\bar{B}y + co\mathscr{L}\tilde{V}(y,x) - \tilde{V}_y(y)\bar{B}y \\ &\le \bar{\alpha}_2\bar{V}(z)\end{aligned}$$

on $[0,\infty) \setminus \{t_k\}_{k\in\mathbb{N}}$, which gives (9b) with $\bar{\alpha}_1 = 0$.

Due to $y(t_k) = 0$ for all $k \in \mathbb{N}$, $\bar{V}(z(t_k)) = V(x(t_k)) = V(Cz(t_k))$ and thus the condition (10) holds with $\beta_0 = 1$ and $\beta_1 = \tilde{\beta}_0 = \tilde{\beta}_1 = 0$. This with $\alpha > \alpha_2 > 0$ gives the condition (11), namely, $0 < \alpha_2\beta_0/\alpha < 1$. Also observe that (38) implies the condition (12) with $\alpha_1 = \alpha > \alpha_2 > 0$, $\bar{\alpha}_2 > \bar{\alpha}_1 = 0$, $\beta_0 = 1$ and $\beta_1 = \tilde{\beta}_0 = \tilde{\beta}_1 = 0$. This completes the proof. □

For a highly nonlinear system (20) with local Lipschitz condition (16), assume that the candidate Lyapunov functions $V(x)$, $\tilde{V}(y)$ in Theorem 5 are of the forms

$$V(x) = x^TPx + \hat{c}|x|^{\hat{p}}, \quad \forall x \in \mathbb{R}^n \tag{39a}$$
$$\tilde{V}(y) = y^T\tilde{P}y, \quad \forall y \in \mathbb{R}^n \tag{39b}$$

respectively, where $P, \tilde{P} \in \mathbb{R}^{n\times n}$ are both positive definite matrices and $\hat{p} > 2$, $\hat{c} > 0$ are positive constants. A particular version of Theorem 5 is specified as follows.

Corollary 1: Suppose that (39) is a pair of candidate Lyapunov functions such that

$$\begin{aligned}&\mathscr{L}V(x) - \left(2x^TP + \hat{c}\hat{p}|x|^{\hat{p}-2}x^T\right)\bar{B}y \\ &\quad \le -(\alpha - \alpha_2)V(x) + \alpha_2\tilde{V}(y),\end{aligned} \tag{40a}$$
$$\begin{aligned}&\mathscr{L}V(x) - \left(2x^TP + \hat{c}\hat{p}|x|^{\hat{p}-2}x^T + 2\,y^T\tilde{P}\right)\bar{B}y \\ &\quad + co\mathscr{L}\tilde{V}(y,x) \le \bar{\alpha}_2[V(x) + \tilde{V}(y)]\end{aligned} \tag{40b}$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, where $\alpha, \alpha_2, \bar{\alpha}_2$ are positive constants with $\alpha > \alpha_2$. The CPS (19) and thus the sampled-data control system (18) are mean-square exponentially stable provided that the sampling sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfies

$$0 < \underline{\Delta t} \le \overline{\Delta t} < \frac{\ln\alpha - \ln\alpha_2}{\bar{\alpha}_2}. \tag{41}$$

Proof: The candidate Lyapunov functions (39a) and (39b) give $\lambda_m(P)|x|^2 \le V(x) \le c_2(|x|^2) = \lambda_M(P)|x|^2 + \hat{c}(|x|^2)^{\hat{p}/2}$ and $\lambda_m(\tilde{P})|y|^2 \le \tilde{V}(y) \le \tilde{c}_2(|y|^2) = \lambda_M(\tilde{P})|y|^2$, respectively. Hence, conditions (8a) and (36) hold. Let $\bar{V}(z) = V(x) + \tilde{V}(y)$ for all $z = [x^T\ y^T]^T \in \mathbb{R}^{2n}$. Then (40a) and (40b) are the very specifications of conditions (37a) and (37b), respectively. It immediately follows from Theorem 5 that the sampled-data control system (18) and its CPS (19) are mean-square exponentially stable if condition (41) is satisfied. □

IV. STATE-FEEDBACK STABILIZATION OF CPS OF LINEAR SAMPLED-DATA SYSTEMS

As applications of our established theory, we study stability and stabilization of CPS of linear sampled-data stochastic systems in this section. Let us consider the CPS

$$\begin{aligned}\mathrm{d}x(t) &= [Ax(t) + \bar{B}(x(t) - y(t))]\mathrm{d}t \\ &\quad + \sum_{j=1}^m G_jx(t)\mathrm{d}B_j(t), \\ &\quad t \in [0,\infty)\end{aligned} \tag{42a}$$
$$\begin{aligned}\mathrm{d}y(t) &= [Ax(t) + \bar{B}(x(t) - y(t))]\mathrm{d}t \\ &\quad + \sum_{j=1}^m G_jx(t)\mathrm{d}B_j(t), \\ &\quad t \in [0,\infty) \setminus \{t_k\}_{k\in\mathbb{N}}\end{aligned} \tag{42b}$$
$$y(t_k) - y(t_k^-) = x(t_{k-1}) - x(t_k^-), \quad k \in \mathbb{N} \tag{42c}$$

of a linear sampled-data control system

$$\mathrm{d}x(t) = [Ax(t) + \bar{B}x(t_*)]\mathrm{d}t + \sum_{j=1}^m G_jx(t)\mathrm{d}B_j(t) \tag{43}$$

for $t \ge 0$ with initial values $x(0) \in \mathbb{R}^n$ and $y(0) = 0$, where $A \in \mathbb{R}^{n\times n}$ and $G_j \in \mathbb{R}^{n\times n}$, $j = 1,\cdots,m$, are constant matrices. Obviously, the linear CPS (42) is a particular case of (19) with $\bar{f}(x) = Ax$ and $\bar{g}(x) = \left[G_1 \cdots G_m\right]x$, which sastify the global Lipschitz condition (17). As is well known, the continuous-time plant

$$\mathrm{d}x(t) = Fx(t)\mathrm{d}t + \sum_{j=1}^m G_jx(t)\mathrm{d}B_j(t) \quad t \ge 0 \tag{44}$$

with $F = A + \bar{B}$ is mean-square exponentially stable if and only if there is a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$F^TP + PF + \sum_{j=1}^m G_j^TPG_j \le -\alpha P \tag{45}$$

for some constant $\alpha > 0$. This is the Lyapunov-Itô inequality [8], the LMI equivalent to the classical Lyapunov-Itô equation [40]. The Lyapunov-Itô LMI (45) can be reformulated into a generalized eigenvalue minimization problem (GEP) (see also [30])

$$\min \lambda \quad s.t.\ P > 0,$$
$$P < \lambda\big(-F^TP - PF - \sum_{j=1}^{m} G_j^T PG_j\big). \tag{46}$$

The minimum Lyapunov exponent $-1/\lambda$ can be obtained by solving the GEP (46) with some toolboxes such as [16]. By [34, Theorem 5.15, p175] (see also [44, Theorem 4.2, p128]), the mean-square exponential stability of linear SDE (44) implies that it is also almost surely exponentially stable. Unlike the linear deterministic systems, the design methods based on exact discrete-time models [2], [54], [62] are inapplicable to stochastic system (43). Some discrete-time approximation of the continuous-time plant has to be employed instead. As a particular case of (29), the Euler-Maruyama approximation of linear system (44) is

$$X_k = X_{k-1} + FX_{k-1}h + \sum_{j=1}^{m} G_jX_{k-1}\Delta B_{j,k} \tag{47}$$

with constant stepsize $h > 0$ and initial value $X_0 = x(0) \in \mathbb{R}^n$, where $\Delta B_{j,k} = B_j(kh) - B_j((k-1)h)$ for all $k \in \mathbb{N}$. It is also well-known that the discrete-time system (47) is mean-square exponentially stable if and only if there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that, see, e.g., [8],

$$(I_n + hF)^TP(I_n + hF) + h\sum_{j=1}^{m} G_j^T PG_j \le (1-c)P \tag{48}$$

for some $c \in (0, 1)$. Note that (45) and (48) are the linear cases of design methods (21) and (30), respectively. The equivalence between (45) and (48) has been shown by the relationship (35) for any stepsize $h \in \big(0, (\alpha_u^{-1}\alpha) \wedge \alpha^{-1}\big)$, where α_u is a positive constant such that $F^TPF \le \alpha_u P$ for the linear system.

Since we have shown the equivalence between the two main approaches (21) and (30), in the sequel, we focus on sampled-data control systems, say, by approach of controller emulation (process-oriented models). A special version of Theorem 3 is slightly modified for the CPS (42) of linear sampled-data control system (43).

Theorem 6: Suppose that there is a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that LMI (45) holds for some $\alpha > 0$. The CPS (42) and thus the sampled-data control system (43) are mean-square exponentially stable provided that the sampling sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfies

$$0 < \underline{\Delta t} \le \overline{\Delta t} < \frac{\ln\alpha - \ln\alpha_2}{\bar\alpha_2}, \tag{49}$$

where $\alpha_2 \in (0, \alpha)$ and $\bar\alpha_2 > 0$ are constants such that

$$\begin{bmatrix} 0 & -P\bar B \\ -\bar B^TP & 0 \end{bmatrix} \le \alpha_2\bar P, \quad \bar P := \begin{bmatrix} P & 0 \\ 0 & \tilde P \end{bmatrix} > 0, \tag{50a}$$

$$\begin{bmatrix} R_{11} & -P\bar B + F^T\tilde P \\ -\bar B^TP + \tilde PF & -\bar B^T\tilde P - \tilde P\bar B \end{bmatrix} \le \bar\alpha_2\bar P \tag{50b}$$

with $R_{11} = F^TP + PF + \sum_{j=1}^{m} G_j^T(P+\tilde P)G_j$ for some positive definite matrix $\tilde P \in \mathbb{R}^{n\times n}$.

Proof: As in the proof of Theorem 3, it can be shown that the control design method (45) ensures that there is a positive definite matrix $\tilde P \in \mathbb{R}^{n\times n}$ and a pair of real numbers $\alpha_2 \in (0, \alpha)$, $\bar\alpha_2 > 0$ such that the conditions (49) and (50) hold.

We choose, $\forall z = [x^T\ y^T]^T \in \mathbb{R}^{2n}$,

$$\bar V(z) = z^T\bar Pz = V(x) + \tilde V(y) = x^TPx + y^T\tilde Py$$

as the candidate Lyapunov function for the CPS (42). All the conditions (8)-(12) but (9b) in Theorem 2 can be verified in the same way as the proof of Theorem 3. By (6), (21), (23), (45) and (50b),

$$\begin{aligned}
&\overline{\mathscr{L}}\bar V(z)\\
&= \overline{\mathscr{L}}V(z) + \overline{\mathscr{L}}\tilde V(z)\\
&= \mathscr{L}V(x) - V_x(x)\bar By + co\mathscr{L}\tilde V(y,x) - \tilde V_y(y)\bar By\\
&= x^T(F^TP + PF + \sum_{j=1}^{m} G_j^T PG_j)x - 2\,x^TP\bar By\\
&\quad + 2\,y^T\tilde PFx + x^T(\sum_{j=1}^{m} G_j^T\tilde PG_j)x - 2\,y^T\tilde P\bar By\\
&= z^TC^T\big[F^TP + PF + \sum_{j=1}^{m} G_j^T(P+\tilde P)G_j\big]Cz\\
&\quad + 2\,z^TC^T(-P\bar B + F^T\tilde P)Dz - 2\,z^TD^T\tilde P\bar BDz\\
&\le \bar\alpha_2 z^T\bar Pz = \bar\alpha_2\bar V(z), \quad \forall z \in \mathbb{R}^{2n}
\end{aligned}$$

which gives the condition (9b) with $\bar\alpha_1 = 0$. Thus the assertion follows from Theorem 2. □

Given $\alpha \in (0, 1/\lambda)$ by the control design (45) as well as (46), we can choose $\alpha_2 = \breve\kappa\alpha$ with some $\breve\kappa \in (0,1)$ and reformulate the set of LMI (45) and (50) into the GEP

$$\min \bar\alpha_2\ \ s.t.\ P > 0, \tilde P > 0,\ LMI\ (45), (50a), (50b). \tag{51}$$

A solution with $\bar\alpha_2 > 0$ to problem (51) gives the allowable sampling intervals $0 < \underline{\Delta t} \le \overline{\Delta t} < -\ln\breve\kappa/\bar\alpha_2$.

Remark 6: Alternatively, the LMI (45) and (50) can be reformulated into the GEP

$$\min \alpha_2\ \ s.t.\ P > 0, \tilde P > 0,\ LMI\ (45), (50b), (50a), \tag{52}$$

where $\alpha \in (0, 1/\lambda)$ and $\bar\alpha_2 > 0$ are a pair of prescribed parameters. A solution with $0 < \alpha_2 < \alpha$ to the problem yields the allowable sampling intervals (49). Similarly, the set of LMI (54)-(55) for control design can be reformulated into a GEP that minimizes α_2 instead of $\bar\alpha_2$. It may be taken as an exercise to derive such a GEP for control design and apply it to feedback stabilization of the CPS (42) of systems (71) and (72) in Example 2, respectively.

Letting $\bar B = \hat B\hat K$ with some given matrix $\hat B \in \mathbb{R}^{n\times\hat m}$ in the CPS (42) of linear sampled-data control system (43) leads to the state-feedback stabilization problem of the CPS of the sampled-data system, which requires to find a state-feedback gain matrix $\hat{K} \in \mathbb{R}^{\hat{m}\times n}$ and some conditions on the sampling intervals for stability of the CPS (42) of the closed-loop system

$$\mathrm{d}x(t) = [Ax(t) + \hat{B}\hat{K}x(t_*)]\mathrm{d}t + \sum_{j=1}^{m} G_j x(t)\mathrm{d}B_j(t) \quad (53)$$

for all $t \geq 0$. It is reasonable in some sense to set $\tilde{P} = \tilde{c}P$ for some $\tilde{c} > 0$ due to the interrelation of the physical and the cyber subsystems [30]. Applying Theorem 6, we obtain a useful result on state-feedback stabilization of the CPS (42) of linear sampled-data stochastic system (53), which is formulated as a set of LMI with the prescribed number $\tilde{c} > 0$ (see also [13], [25], [48]).

Theorem 7: Suppose that there is a pair of matrices $Q \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{\hat{m}\times n}$ such that $Q > 0$ and

$$\begin{bmatrix} Q_{11} + \alpha Q & * & \cdots & * \\ G_1 Q & -Q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_m Q & 0 & \cdots & -Q \end{bmatrix} \leq 0 \quad (54)$$

for some $\alpha > 0$, where $Q_{11} = QA^T + Y^T\hat{B}^T + AQ + \hat{B}Y$ and entries denoted by $*$ can be readily inferred from symmetry of a matrix. Let sampling sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfy (49) with positive numbers $\alpha_2 \in (0, \alpha)$ and $\bar{\alpha}_2$ being such that

$$\begin{bmatrix} 0 & * \\ -Y^T\hat{B}^T & 0 \end{bmatrix} \leq \alpha_2 \begin{bmatrix} Q & 0 \\ 0 & \tilde{c}\,Q \end{bmatrix}, \quad (55a)$$

$$\begin{bmatrix} Q_{11} - \bar{\alpha}_2 Q & * & * & \cdots & * \\ \tilde{R}_{21} & \tilde{R}_{22} - \bar{\alpha}_2\tilde{c}Q & 0 & \cdots & 0 \\ \tilde{\gamma}\,G_1 Q & 0 & -Q & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{\gamma}\,G_m Q & 0 & 0 & \cdots & -Q \end{bmatrix} \leq 0, \quad (55b)$$

where $\tilde{R}_{21} = -Y^T\hat{B}^T + \tilde{c}\,(AQ + \hat{B}Y)$, $\tilde{R}_{22} = -\tilde{c}\,(Y^T\hat{B}^T + \hat{B}Y)$, $\tilde{\gamma} = \sqrt{1+\tilde{c}}$ and $\tilde{c} > 0$ is a prescribed number. Then the CPS (42) of sampled-data control system (53) with feedback gain matrix $\hat{K} = YQ^{-1}$ is mean-square exponentially stable, which implies that the closed-loop system (53) with $\hat{K} = YQ^{-1}$ is mean-square exponentially stable.

Proof: Let $P = Q^{-1}$ and $\tilde{P} = \tilde{c}P$. Hence, $P > 0$ and $\tilde{P} > 0$. By the Schur complement lemma, LMI (54) produces

$$Q_{11} + \sum_{j=1}^{m} QG_j^T PG_j Q + \alpha Q \leq 0$$

$$\Leftrightarrow Q(A + \hat{B}\hat{K})^T + (A + \hat{B}\hat{K})Q + \sum_{j=1}^{m} QG_j^T PG_j Q \leq -\alpha Q.$$

Premultiplying by P and postmultiplying by P the LMI above gives the LMI (45) with $F = A + \hat{B}\hat{K}$. By the Schur complement lemma, the LMI (55b) implies

$$\begin{bmatrix} \tilde{R}_{11} + (1+\tilde{c}\,)Q\sum_{j=1}^{m} G_j^T PG_j Q & * \\ \tilde{R}_{21} & -\tilde{c}\tilde{R}_{22} \end{bmatrix} \leq 0 \quad (56)$$

with $\tilde{R}_{11} = Q_{11} - \bar{\alpha}_2 Q = QA^T + Y^T\hat{B}^T + AQ + \hat{B}Y - \bar{\alpha}_2 Q$. Premultiplying by $diag\{P, P\}$ and postmultiplying by $diag\{P, P\}$ the LMI (55a) and (56) yield (50a) and (50b) with $\tilde{P} = \tilde{c}\,P$, respectively. It follows from Theorem 6 that the CPS (42) and thus the sampled-data control system (53) with $\hat{K} = YQ^{-1}$ are mean-square exponentially stable, □

Given $\alpha > 0$, $Q > 0$, Y by LMI (54) and $\alpha_2 = \tilde{\kappa}\,\alpha$ with some $\tilde{\kappa} \in (0, 1)$, one can find a number $\tilde{c} > 0$ sufficiently large for (55a) and then a $\bar{\alpha}_2 > 0$ sufficiently large for (55b). Thus the control design method (54), which is equivalent to (45), ensures that there is a set of postive constants $\alpha_2 = \tilde{\kappa}\,\alpha$, $\tilde{\kappa} \in (0, 1)$, $\tilde{c}$ and $\bar{\alpha}_2$ such that the conditions (49) and (55) hold (see also the proofs of Theorems 3 and 6). As an implementation of Theorem 7, we propose an algorithm in the form of generic LMI [8], [16] to find a feasible solution to the set of LMI (54)-(55), which thus yields a feedback gain matrix and its allowable sampling intervals for stabilization of the CPS (42) of system (53). In the interest of simplicity of the formulas, assume here $m = 1$ and $G_1 = G$.

Algorithm 1: Design a state-feedback controller and compute its maximum allowable sampling interval for stabilization of the CPS (42) of sampled-data control system (53).

1) Compute the minimum Lyapunov exponent $-1/\lambda$ by solving the GEP

$$\min \lambda \;\; s.t.\; \bar{Q} > 0, \; \begin{bmatrix} \bar{Q} & 0 \\ 0 & 0 \end{bmatrix} < \lambda \begin{bmatrix} -\bar{Q}_{11} & * \\ -G\bar{Q} & \bar{Q} \end{bmatrix}$$

with $\bar{Q}_{11} = \bar{Q}A^T + \bar{Y}^T\hat{B}^T + A\bar{Q} + \hat{B}\bar{Y}$.

2) Choose a Lyapunov exponent $-\alpha \in (-1/\lambda, 0)$ and obtain matrices $Q > 0$ and Y by solving the LMI (54).
3) Find α_2 and $\alpha_{\tilde{c}}$ by solving the LMI derived from (55a)

$$\alpha_2 < \alpha, \; \begin{bmatrix} 0 & * \\ -Y^T\hat{B}^T & 0 \end{bmatrix} < \begin{bmatrix} \alpha_2 Q & 0 \\ 0 & \alpha_{\tilde{c}}\,Q \end{bmatrix}$$

with $\alpha \in (0, 1/\lambda)$, $Q > 0$ and Y obtained in the previous step and then set the prescribed positive numbers $\tilde{c} = \alpha_{\tilde{c}}/\alpha_2$ and $\tilde{\kappa} = \alpha_2/\alpha < 1$.

4) Compute $\bar{\alpha}_2 > 0$ by solving the linear objective minimization problem

$$\min \bar{\alpha}_2 \;\; s.t.\; \bar{\alpha}_2 > 0, \; LMI\ (55b)$$

with $Q > 0$, Y and $\tilde{c} > 0$ obtained in the previous steps.

5) Compute the feedback gain $\hat{K} = YQ^{-1}$ and its allowable sampling intervals $0 < \underline{\Delta t} \leq \overline{\Delta t} < (-\ln\tilde{\kappa})/\bar{\alpha}_2$ with $Q > 0$, Y, $\tilde{\kappa} = \alpha_2/\alpha$, $\bar{\alpha}_2 > 0$ obtained above.

Each step of Algorithm 1 finds its solution if and only if LMI (54) is feasible, or say, linear SDE (44) with $\bar{B} = \hat{B}\hat{K}$ is mean-square stabilizable. The matrices $Q > 0$, Y and $[\lambda,\; \alpha,\; \alpha_2,\; \bar{\alpha}_2,\; \tilde{c},\; \tilde{\kappa}]$ obtained above not only produce a feasible solution to stabilization problem of the CPS (42) of sampled-data system (53) but also provide a starting point to find some other feasible solutions with larger allowable sampling intervals using some toolboxes such as [16] and [17]. Specifically, the maximum allowable sampling interval may be improved by feedback gain matrix $\hat{K} = YQ^{-1}$ if the matrices $Q > 0$ and Y are obtained by solving the GEP derived from the set of LMI (54)-(55) (see, e.g., [16, p.8-41])

$$\begin{aligned} &\min \bar{\alpha}_2 \quad s.t. \;\; Q > 0, \; \begin{bmatrix} Q_{11} + \alpha Q & * \\ GQ & -Q \end{bmatrix} < 0, \\ &\begin{bmatrix} 0 & * \\ -Y^T \hat{B}^T & 0 \end{bmatrix} < \tilde{\kappa}\,\alpha \begin{bmatrix} Q & 0 \\ 0 & \tilde{c}\,Q \end{bmatrix}, \\ &\begin{bmatrix} Q_{11} - \tilde{Q} & * & * \\ \tilde{R}_{21} - \tilde{G} & \tilde{R}_{22} - \tilde{R} & 0 \\ \tilde{\gamma} GQ & 0 & -Q \end{bmatrix} < 0, \\ &\begin{bmatrix} \tilde{Q} & * \\ \tilde{G} & \tilde{R} \end{bmatrix} > 0, \; \begin{bmatrix} \tilde{Q} & * \\ \tilde{G} & \tilde{R} \end{bmatrix} < \bar{\alpha}_2 \begin{bmatrix} Q & 0 \\ 0 & \tilde{c} Q \end{bmatrix}, \end{aligned} \tag{57}$$

where matrices $\tilde{Q}, \tilde{R}, \tilde{G} \in \mathbb{R}^{n\times n}$ are part of the decision variables and positive numbers $\alpha \in (0, 1/\lambda)$, $\tilde{\kappa} \in (0, 1)$, $\tilde{c} > 0$ are the prescribed paramters. One can search for paramters $\alpha \in (0, 1/\lambda)$, $\tilde{\kappa} \in (0, 1)$, $\tilde{c} > 0$ that produce a larger maximum allowable sampling interval $\overline{\Delta t}$ with the starting point from a feasible solution given by Algorithm 1. The solution with $Q > 0, Y, \bar{\alpha}_2 > 0$ to GEP (57) yields a feedback gain $\hat{K} = YQ^{-1}$ and its allowable sampling intervals $0 < \underline{\Delta t} \le \overline{\Delta t} < (-\ln\tilde{\kappa})/\bar{\alpha}_2$. It is also worth noting that $\overline{\Delta t}$ could be enhanced by solving the GEP (51) for the CPS (42) of sampled-data system (43) with $\bar{B} = \hat{B}\hat{K} = \hat{B}YQ^{-1}$. Our control design method can be applied with Theorem 3 to nonlinear systems such as [68, Example 2.2] (see also [55]).

V. ILLUSTRATIVE EXAMPLES

In this section, we illustrate the applications of our CPS theory with numerical examples. More specifically, we shall apply our established results to stabilization problems of the CPS of sampled-data stochastic systems and verify the advantages as well as effectiveness of our proposed methods with some numerical examples from the literature.

Example 1: Consider the scalar sampled-data control system (18) and its CPS (19) (i.e., $n = 1$ and $x \in \mathbb{R}$) with $m = 1$ as well. Two numerical cases are given as [12, Example 6.1]

$$\bar{f}_u(x) = -2x - 3\,x^3, \quad \bar{B} = -3, \quad \bar{g}(x) = |x|^{3/2}; \tag{58}$$

$$\bar{f}_u(x) = -x - 2\,x^3, \quad \bar{B} = -2, \quad \bar{g}(x) = \frac{1}{2}|x|^{3/2} \tag{59}$$

both which satisfy the local Lipschitz condition (16).

We shall apply Corollary 1 to the CPS (19) of sampled-data systems (58) and (59), respectively. In either case, let

$$V(x) = x^2 + \hat{c}|x|^4 \;\; and \;\; \tilde{V}(y) = \hat{\alpha}y^2 \quad \forall x, y \in \mathbb{R} \tag{60}$$

be the candidate Lyapunov functions for the physical and the cyber subsystems, respectively, where $\hat{c}$ and $\hat{\alpha}$ are both positive numbers to be determined. Obviously, (60) is the sclar case of (39) with $\hat{p} = 4$. For either case, set $\hat{\alpha} = 2\hat{c}$ in this example.

For system (58), using [25, Lemma 3.1], we have

$$\begin{aligned} &\mathscr{L}V(x) - V_x(x)\bar{B}y \le -\Big(4 - \frac{1}{24} - \frac{3}{\sqrt{\hat{\alpha}}}\Big)x^2 \\ &\quad - \Big[8 - \frac{3}{2(2-r)}\Big]\hat{c}x^4 + \Big(\frac{3}{\sqrt{\hat{\alpha}}} + \frac{3}{r}\Big)\hat{\alpha}y^2, \end{aligned} \tag{61a}$$

$$\begin{aligned} &\mathscr{L}V(x) - [V_x(x) + \tilde{V}_y(y)]\bar{B}y + co\mathscr{L}\tilde{V}(y,x) \le \Big[\frac{(1+\hat{\alpha})^2}{24} \\ &\quad + \frac{|2\hat{\alpha}-3|}{\sqrt{\hat{\alpha}}} - 4\Big]x^2 + \Big(\frac{|2\hat{\alpha}-3|}{\sqrt{\hat{\alpha}}} + 6\Big)\hat{\alpha}y^2 \end{aligned} \tag{61b}$$

for all $x, y \in \mathbb{R}$, where r is some postive number on $(0, 2)$. Choose the postive numbers $\hat{\alpha}$ and r such that

$$4 - \frac{1}{24} - \frac{3}{\sqrt{\hat{\alpha}}} = 8 - \frac{3}{2(2-r)} > 0,$$

$$\frac{(1+\hat{\alpha})^2}{24} + \frac{|2\hat{\alpha}-3|}{\sqrt{\hat{\alpha}}} - 4 = \frac{|2\hat{\alpha}-3|}{\sqrt{\hat{\alpha}}} + 6 > 0,$$

which immediately produces

$$\begin{cases} \hat{\alpha} = 4\sqrt{15} - 1 = 14.4919, \\ r = 2 - 36\sqrt{\hat{\alpha}}/\big(97\sqrt{\hat{\alpha}} + 72\big) = 1.6894. \end{cases} \tag{62}$$

Substituting the positive numbers chosen as (62) into the inequalities (61), we obtain the conditions (40) with $\alpha = 4 + 3/r - 1/24 = 5.7341$, $\alpha_2 = 3/\sqrt{\hat{\alpha}} + 3/r = 2.5638$ and $\bar{\alpha}_2 = 6 + (2\hat{\alpha} - 3)/\sqrt{\hat{\alpha}} = 12.8256$. According to Corollary 1, the CPS (19) of sampled-data system (58) is mean-square exponentially stable if

$$0 < \underline{\Delta t} \le \overline{\Delta t} < \frac{\ln(\alpha/\alpha_2)}{\bar{\alpha}_2} = 0.0628. \tag{63}$$

Similarly, for system (59), the inequalities

$$\begin{aligned} &\mathscr{L}V(x) - V_x(x)\bar{B}y \le -\Big(2 - \frac{1}{16^2} - \frac{2}{\sqrt{\hat{\alpha}}}\Big)x^2 \\ &\quad - \Big[4 - \frac{9}{64(2-r)}\Big]\hat{c}x^4 + \Big(\frac{2}{\sqrt{\hat{\alpha}}} + \frac{2}{r}\Big)\hat{\alpha}y^2, \\ &\mathscr{L}V(x) - [V_x(x) + \tilde{V}_y(y)]\bar{B}y + co\mathscr{L}\tilde{V}(y,x) \\ &\quad \le \Big[\frac{(1+\hat{\alpha})^2}{16^2} + \frac{|\hat{\alpha}-2|}{\sqrt{\hat{\alpha}}} - 2\Big]x^2 + \Big(\frac{|\hat{\alpha}-2|}{\sqrt{\hat{\alpha}}} + 4\Big)\hat{\alpha}y^2 \end{aligned}$$

with postive numbers $\hat{\alpha}$ and $r \in (0, 2)$ chosen as

$$\begin{cases} \hat{\alpha} = 16\sqrt{6} - 1 = 38.1918, \\ r = 2 - 36\sqrt{\hat{\alpha}}/\big(513\sqrt{\hat{\alpha}} + 512\big) = 1.9396 \end{cases}$$

immediately imply that the conditions (40) hold with $\alpha = 2 + 2/r - 1/16^2 = 3.0272$, $\alpha_2 = 2/\sqrt{\hat{\alpha}} + 2/r = 1.3548$ and $\bar{\alpha}_2 = 4 + (\hat{\alpha} - 2)/\sqrt{\hat{\alpha}} = 9.8563$. Thus the CPS (19) of sampled-data system (59) is mean-square exponentially stable if

$$0 < \underline{\Delta t} \le \overline{\Delta t} < \frac{\ln(\alpha/\alpha_2)}{\bar{\alpha}_2} = 0.0816. \tag{64}$$

Note that both the allowable sampling intervals (63) and (64) are much better than the result $\tau < 0.0262$ given in [12, Example 6.1] for stability of the Markov jump system [12, Eq. (75)] that switches between (58) and (59).

Example 2: Here we consider two specific cases of the CPS (42) of linear sampled-data stochastic system (43) with $n = 2$ and $m = 1$. In one case,

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -5 \end{bmatrix}, \; G = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \; \hat{B} = \begin{bmatrix} -10 & 0 \\ 0 & 0 \end{bmatrix}, \tag{65}$$

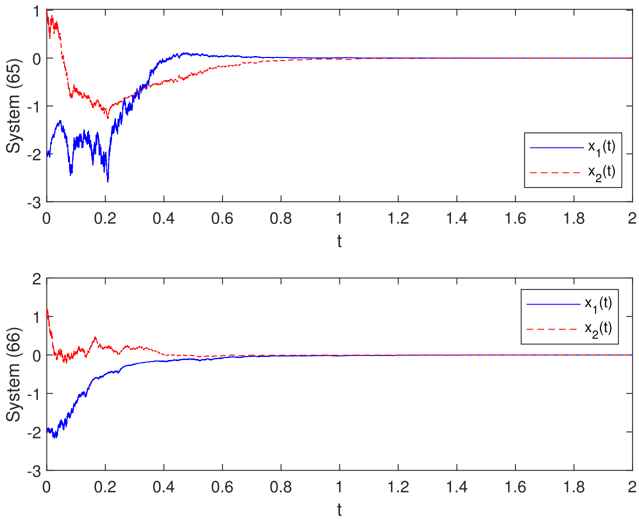

Figure 1. **The trajectory samples of sampled-data control systems (65) (the higher) and (66) (the lower), $\Delta t = 0.0282$ and $x(0) = [-2\ \ 1]^T$.**

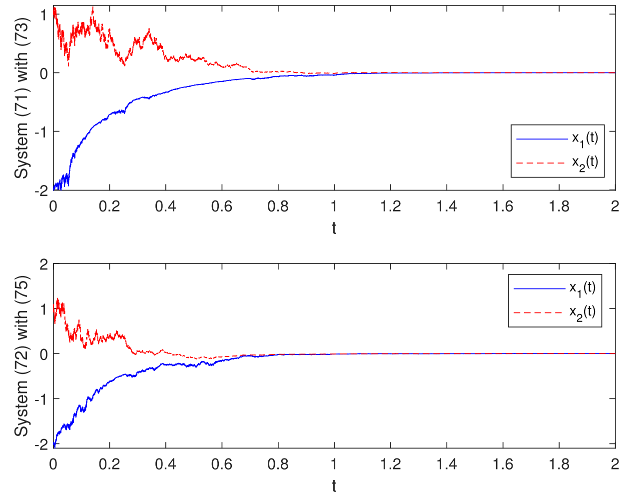

Figure 2. **The trajectory samples of closed-loop systems (71) with (73) (the higher) and (72) with (75) (the lower), $\Delta t = 0.0726$ and $x(0) = [-2\ \ 1]^T$.**

and in the other,

$$A = \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix}, G = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & -10 \end{bmatrix}. \quad (66)$$

Sampled-data stochastic systems (65) and (66) with sampling period $\tau > 0$ have been studied in quite a few works [12], [45], [46]. It is observed from [12, Example 6.1] that, by [12, Corollary 5.4] with $N = 1, Q = I_2, K_1 = 5.236, K_2 = \sqrt{2}, K_3 = 10, c_1 = c_2 = \lambda_1 = 1, \lambda_2 = 4$ and $\lambda_3 = 8$, both the sampled-data systems (65) and (66) are mean-square exponentially stable if the sampling period $\tau < \tau^* = 0.0074$, which gives a better bound than those in [45] and [46].

Let us apply Theorem 6 to the CPS (42) of sampled-data systems (65) and (66), respectively. For system (65), solving the GEP (46) (with some toolboxes such as [16]) produces the minimum Lyapunov exponent $-1/\lambda = -8.8769$. One may use the toolbox [17] to search for a pair of prescribed parameters $\alpha \in (0, 8.8769)$ and $\tilde{\kappa} \in (0, 1)$ of the GEP (51). Here we choose $\alpha = 8.3236, \tilde{\kappa} = 0.4706$ and solve the GEP (51), which produces $\bar{\alpha}_2 = 29.4326$ and hence

$$0 < \underline{\Delta t} \leq \overline{\Delta t} < \frac{-\ln\tilde{\kappa}}{\bar{\alpha}_2} = \frac{-\ln(0.4706)}{29.4326} = 0.0256 \quad (67)$$

for mean-square exponential stability of the CPS (42) of sampled-data system (65). Alternatively, solving the GEP (52) with $\alpha = 8.7257, \bar{\alpha}_2 = 34.1381$ produces $\alpha_2 = 3.3178$ and the maximum allowable sampling interval

$$\overline{\Delta t} < \frac{\ln(\alpha/\alpha_2)}{\bar{\alpha}_2} = \frac{\ln(8.7257/3.3178)}{34.1381} = 0.0283. \quad (68)$$

Similarly, for system (66), the minimum Lyapunov exponent is $-1/\lambda = -8.8769$ as well. Solving the GEP (51) with prescribed parameters $\alpha = 8.2984$ and $\tilde{\kappa} = 0.4632$ yields $\bar{\alpha}_2 = 29.8869$ and hence

$$0 < \underline{\Delta t} \leq \overline{\Delta t} < \frac{-\ln\tilde{\kappa}}{\bar{\alpha}_2} = \frac{-\ln(0.4632)}{29.8869} = 0.0258 \quad (69)$$

for mean-square exponential stability of the CPS (42) of sampled-data system (66). The maximum allowable sampling interval can also be obtained by solving the GEP (52) wtih $\alpha = 8.7851, \bar{\alpha}_2 = 36.4380$, which yields $\alpha_2 = 3.1016$ and the maximum allowable sampling interval

$$\overline{\Delta t} < \frac{\ln(\alpha/\alpha_2)}{\bar{\alpha}_2} = \frac{\ln(8.7851/3.1016)}{36.4380} = 0.0286. \quad (70)$$

Our results (67)-(70) have significantly improved the existing ones [12], [45], [46], which require the sampling period $\tau < \tau^* = 0.0074$. Figure 1 shows the trajectory samples of the sampled systems (65) and (66) in the higher and the lower, respectively, where the sampling period $\Delta t = 0.0282 < 0.0283 < 0.0286$ satifies the conditions (68) and (70), and the initial value $x(0) = [-2\ \ 1]^T$ is the same as the one used in the [12], [45], [46].

Furthermore, as application of Algorithm 1 for our proposed control design method, we study state-feedback stabilization problems of the CPS (42) of sampled-data stochastic system (53) with

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -5 \end{bmatrix}, G = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \hat{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (71)$$

$$\text{and } A = \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix}, G = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, \hat{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (72)$$

respectively, see [12], [45], and [46] and also [25].

Applying Algorithm 1 to linear sampled-data stochastic system (71) and solving the GEP (57) with prescirbed parameters $\alpha = 4.4154, \tilde{\kappa} = 0.4175, \tilde{c} = 4.9913$, we obtain a state-feedback gain matrix

$$\hat{K} = \begin{bmatrix} -4.0851 & 0.3231 \end{bmatrix} \quad (73)$$

and its allowable sampling intervals

$$0 < \underline{\Delta t} \leq \overline{\Delta t} < -\ln\tilde{\kappa}/\bar{\alpha}_2 = 0.0699$$

as well as $\bar{\alpha}_2 = 12.5038$ for mean-square exponential stability of the CPS (42) of sampled-data system (71) with the feedback gain (73). As is noted above, the maximum allowable sampling interval may be enhanced by solving the GEP (51) with prescribed numbers $\alpha = 4.5854, \tilde{\kappa} = 0.4113$ for the CPS (42) of the closed-loop system (71) with (73), which gives $\bar{\alpha}_2 = 12.2193$ and hence

$$0 < \underline{\Delta t} \leq \overline{\Delta t} < \frac{-\ln \tilde{\kappa}}{\bar{\alpha}_2} = \frac{-\ln(0.4113)}{12.2193} = 0.0727. \quad (74)$$

Similarly, for stabilization of the CPS (42) of sampled-data system (43) with (72), solving the GEP (57) with prescribed parameters $\alpha = 4.7233, \tilde{\kappa} = 0.4121, \tilde{c} = 5.1404$, which are obtained by using the toolbox [17] with a starting point from Algorithm 1, yields a feedback gain matrix

$$\hat{K} = \begin{bmatrix} -0.6929 & -4.3621 \end{bmatrix} \quad (75)$$

and its allowable sampling intervals

$$0 < \underline{\Delta t} \leq \overline{\Delta t} < -\ln \tilde{\kappa}/\bar{\alpha}_2 = 0.0680$$

as well as $\bar{\alpha}_2 = 13.0343$. The improved allowable sampling intervals can be obtained by solving the GEP (51) with prescribed numbers $\alpha = 4.8118, \tilde{\kappa} = 0.4425$, which produes $\bar{\alpha}_2 = 10.7483$ and thus

$$0 < \underline{\Delta t} \leq \overline{\Delta t} < \frac{-\ln \tilde{\kappa}}{\bar{\alpha}_2} = \frac{-\ln(0.4425)}{10.7483} = 0.0759 \quad (76)$$

for stability of the CPS (42) of the closed-loop system.

It is clear that our proposed results provide the state-feedback controllers (73) and (75) that not only have smaller gains $|\hat{K}| < 10$ but also offer much larger maximum allowable sampling intervals (74) and (76) than those in the literature [12], [45], [46]. This demonstrates that our CPS theory has substantially improved the control design method for sampled-data stochastic systems. The trajectory samples of the closed-loop systems (71) with (73) and (72) with (75) are shown as the higher and lower ones in Figure 2, respectively, where the sampling period $\Delta t = 0.0726 < 0.0727 < 0.0759$ is required by the conditions (74) and (76), and the initial value $x(0) = [-2 \ \ 1]^T$ is the same as above.

Example 3: The pendulum equation

$$\ddot{\theta} = -a \sin\theta - b\dot{\theta} + c\hat{T}$$

is a classical model in the science, technology, engineering and mathematics literature (see [30], [33], [61], [69], [71] and the references therein), where $a = g/l > 0$, $b = k/m \geq 0$, $c = 1/(ml^2) > 0$, θ is the angle subtended by the rod and the vertical axis, and $\hat{T}$ is the torque applied to the pendulum. For the pendulum to maintain equilibrium at the angle $\theta = \delta$, the torque must have a steady component $\hat{T}_{ss}$ that satisfies $0 = -a\sin\delta + c\hat{T}_{ss}$. Choose the state variables as $x = [x_1 \ x_2]^T = [\theta - \delta \ \ \dot{\theta}]^T$ and the control input as $u = \hat{T} - \hat{T}_{ss}$. The state equation [33, Example 12.2]

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -a[\sin(x_1(t) + \delta) - \sin\delta] - bx_2(t) \end{bmatrix} + \hat{C}u(t) \quad (77)$$

has equilibrium at the origin $[0 \ \ 0]^T$, where $\hat{C} = [0 \quad c]^T$. Several unrelated physical systems such as [33, Exercises 1.8-1.11] are modelled by equations similar to the pendulum equation. Consequently, system (77) is of great practical importance [30], [33], [61], [69], [71]. The nonlinear control system (77) is also a typical example of feedback linearization [33, Chapter 13]. That is, the state feedback control law $u(t) = u(x(t))$ is given by

$$u(t) = \frac{a}{c}[\sin(x_1(t) + \delta) - \sin\delta] + \frac{1}{c}\hat{K}x(t) \quad (78)$$

such that the closed-loop system

$$\dot{x}(t) = (A + \hat{B}\hat{K})x(t) \quad (79)$$

is linear as well as (exponentially) stable, where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \hat{K} = \begin{bmatrix} \hat{k}_1 & \hat{k}_2 \end{bmatrix}.$$

Interestingly, the state feedback control law (78) has transformed the control problem of the nonlinear system into the classical one of linear systems [8], [16], [32], [39], [40], which requires to find a feedback gain matrix $\hat{K}$ such that the closed-loop system (79) is (exponentially) stable. For instance, the feedback gain $\hat{K}$ can conveniently be designed by the classical pole placement approach for stabilization of linear control system (79).

Introducing the sampling and ZOH mechanism into the controlled system (77)-(78) leads to a specific case of sampled-data system (15), in which $n = 2$, $m = 1$,

$$\bar{f}(x(t)) = \begin{bmatrix} x_2(t) \\ -a[\sin(x_1(t) + \delta) - \sin\delta] - bx_2(t) \end{bmatrix},$$
$$\bar{g}(x(t)) = [0 \ \ 0]^T, \quad \bar{u}(t_*) = \hat{C}u(t_*) = \hat{B}cu(t_*) \quad (80)$$

with nonlinear feedback control $u(t_*)$ being given by (78). The closed-loop system (80) resumes a nonlinear control problem due to the sampled-data mechanism.

It is worth noting that one may obtain some CPS of sampled-data system (80) in the canonical form (1) with $\tilde{n} = n$ by letting $y(t) = \bar{u}(t) \in \mathbb{R}^n$ (particularly when the elements of $\bar{u}(t)$ are linearly independent) and develop some result on stabilization of the CPS from Theorem 1 and the techniques with respect to the jointly differential operator $co\mathscr{L}\tilde{V}(y, x)$. But, in this case, we have $\bar{u}(t_*) = \hat{B}cu(t_*)$ for all $t \geq 0$. So we can alternatively set $y(t) = cu(t) - cu(t_*)$ for all $t \geq 0$, which implies that $\dot{y}(t) = c\dot{u}(t)$ on (t_{k-1}, t_k) and $y(t_k) = 0$ for all $k \in \mathbb{N}$. From (78) and (80), we deduce a CPS of sampled-data control system (80) in the standard form (1), where $n = 2$, $m = 1$, $\tilde{n} = 1$,

$$f(x, y, t) = (A + \hat{B}\hat{K})x - \hat{B}y, \quad g(x, y, t) = [0 \ 0]^T,$$
$$\tilde{f}(x, y, t) = [a\cos(x_1 + \delta) + \hat{k}_1 + \hat{k}_2(\hat{k}_2 - b)]x_2 + \hat{k}_1\hat{k}_2x_1 - \hat{k}_2y, \quad \tilde{g}(x, y, t) = 0,$$
$$\tilde{h}_f(x(t_k^-), x(t_{k-1}), y(t_k^-), y(t_{k-1}), k) = cu(x(t_{k-1})) - cu(x(t_k^-)) = a[\sin(x_1(t_{k-1}) + \delta) - \sin(x_1(t_k^-) + \delta)]$$

$$+\hat{K}\left[x(t_{k-1})-x(t_k^-)\right],$$
$$\tilde{h}_g(x(t_k^-), x(t_{k-1}), y(t_k^-), y(t_{k-1}), k) = 0 \quad (81)$$

for all $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$.

As application of Theorem 1, we study the stabilization problem of CPS (81) with feedback gain $\hat{K} = [\hat{k}_1\ \hat{k}_2]$ being typically designed by the pole placement in linear control system (79) and find the allowable sampling intervals of state-feedback controller (78). Let $V(x,t) = V(x) = x^T Px$ and $\bar{V}(z,t) = \bar{V}(z) = V(x) + \tilde{V}(y)$ with $\tilde{V}(y) = y^2$ for all $x \in \mathbb{R}^2$ and $z = [x^T\ y]^T \in \mathbb{R}^3$, where $P \in \mathbb{R}^{2\times 2}$ is a positive definite matrix. This immediately gives the conditions (8) with $p = 2$, $c_1 = \lambda_m(P)$, $c_2(|x|^2) = \lambda_M(P)|x|^2$, $\bar{c}_1 = 1 \wedge \lambda_m(P)$, $\bar{c}_2(|z|^2) = [1 \vee \lambda_M(P)]|z|^2$ and $\bar{c} = 1$. As in the proofs of Theorem 3 and Theorem 6, it can be verified that conditions (9a), (10) and (11) of Theorem 1 hold with $\alpha_1 > \alpha_2 = \tilde{\kappa}\alpha_1 > 0$, $\beta_0 = 1$, $\beta_1 = \tilde{\beta}_0 = \tilde{\beta}_1 = 0$ if

$$(A+\hat{B}\hat{K})^T P + P(A+\hat{B}\hat{K}) \le -\alpha_1 P, \quad (82a)$$
$$\begin{bmatrix} 0 & -P\hat{B} \\ -\hat{B}^T P & 0 \end{bmatrix} \le \tilde{\kappa}\alpha_1 \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \quad (82b)$$

for some positive numbers $\tilde{\kappa} \in (0,1)$ and $\alpha_1 \in (0, 1/\lambda)$ with $\lambda > 0$ being the solution to the GEP

$$\min \lambda \quad s.t.\ \ \bar{P} > 0,$$
$$\bar{P} < -\lambda[(A+\hat{B}\hat{K})^T \bar{P} + \bar{P}(A+\hat{B}\hat{K})]. \quad (83)$$

According to (6) and (7), we calculate

$$\begin{aligned}\overline{\mathscr{L}}\bar{V}(z) &= \overline{\mathscr{L}}V(z) + \overline{\mathscr{L}}\tilde{V}(z) \\ &= z^T \begin{bmatrix} (A+\hat{B}\hat{K})^T P + P(A+\hat{B}\hat{K}) & -P\hat{B} \\ -\hat{B}^T P & 0 \end{bmatrix} z \\ &\quad + 2y\big[\hat{k}_1\hat{k}_2 x_1 + (\hat{k}_1 + \hat{k}_2^2 - b\hat{k}_2)x_2 \\ &\quad + a x_2 \cos(x_1+\delta) - \hat{k}_2 y\big] \\ &\le z^T \begin{bmatrix} (A+\hat{B}\hat{K})^T P + P(A+\hat{B}\hat{K}) & -P\hat{B} \\ -\hat{B}^T P & 0 \end{bmatrix} z \\ &\quad + 2y\big[\hat{k}_1\hat{k}_2 x_1 + (\hat{k}_1 + \hat{k}_2^2 - b\hat{k}_2)x_2 - \hat{k}_2 y\big] \\ &\quad + a(x_2^2 + y^2)\end{aligned}$$

and have condition (9b) with $\bar{\alpha}_1 = 0$ and $\bar{\alpha}_2 > 0$ being sufficiently large for

$$\begin{bmatrix} P_{11} & -P\hat{B} + \hat{K}_{12} \\ -\hat{B}^T P + \hat{K}_{12}^T & a - 2\hat{k}_2 \end{bmatrix} \le \bar{\alpha}_2 \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}, \quad (84)$$

where $P_{11} = (A+\hat{B}\hat{K})^T P + P(A+\hat{B}\hat{K}) + diag\{0, a\}$ and $\hat{K}_{12} = \left[\hat{k}_1\hat{k}_2\ \ \hat{k}_1 + \hat{k}_2(\hat{k}_2 - b)\right]^T$. The condition (12) can now be specified as

$$0 < \underline{\Delta t} \le \overline{\Delta t} < -\ln\tilde{\kappa}/\bar{\alpha}_2. \quad (85)$$

It follows from Theorem 1 that the CPS (81) and hence the sampled-data control system (80) are exponentially stable provided that inequalities (82)-(85) are satisfied.

Note that, given $\alpha_1 \in (0, 1/\lambda)$, there is a matrix $\bar{P} > 0$ that satisfies the LMI (82a) and so does the matrix $P = \bar{a}\bar{P} > 0$

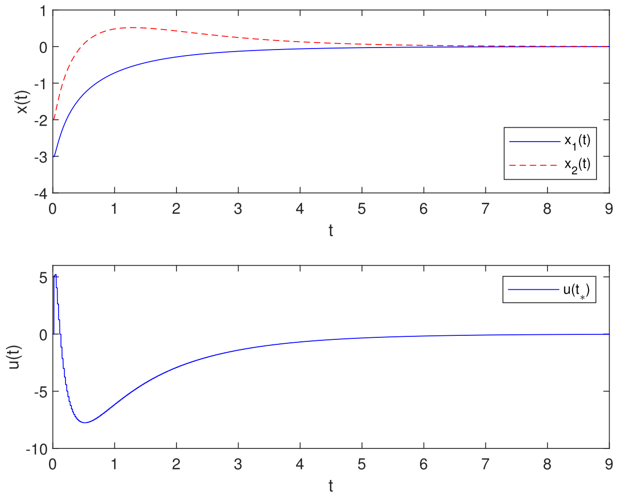

Figure 3. **The state trajectories (the higher) and control input (the lower) of sampled-data control system (80) of pendulum with $\delta = 0$, $a = 9.8$, $b = 0$, $c = 1$, $\Delta t = 0.0188$ and $x(0) = [-3\ \ -2]^T$.**

for any $\bar{a} > 0$. It is also observed that, given $\tilde{\kappa} \in (0,1)$, the matrix $P = \bar{a}\bar{P} > 0$ with sufficiently small $\bar{a} > 0$ satisfies the LMI (82b) as well. This means that, given any pair of parameters $\alpha_1 \in (0, 1/\lambda)$ and $\tilde{\kappa} \in (0,1)$, there is a solution $P > 0$ to LMI (82). Moreover, $\bar{\alpha}_2 > 0$ can be chosen sufficiently large for condition (84). Now that the vectors $\hat{K}, \hat{K}_{12}$ and $[\lambda, \alpha_1, \tilde{\kappa}]$ are known, one may improve the maximum allowable sampling interval (85) as in Section IV by reformulating the inequalities (82) and (84) into a GEP

$$\min \bar{\alpha}_2 \quad s.t.\ \ P > 0,\ LMI\ (82), (84) \quad (86)$$

where matrix P is a decision variable and positive numbers $\tilde{\kappa} \in (0,1)$, $\alpha_1 \in (0, 1/\lambda)$ are the prescribed parameters for which one can use some toolboxes such as [16] and [17] to search with some starting point.

To put it into practice, let us consider system (77) with parameters $\delta = 0$, $a = 9.8$, $c = 1$ and $b \ge 0$, of which a special case $b = 0$ is studied as an example of energy control (with sampling period $\Delta t = 0.0002$ and $\Delta t = 0.0003$) in [61]. Here the state feedback control law (78) is employed and the feedback gain $\hat{K}$ is typically designed by the pole placement, say, $[-0.6+1i,\ -0.6-1i]$ in closed-loop system (79). For the case $b = 0$,

$$\hat{K} = [\hat{k}_1\ \hat{k}_2] = [-1.36\ -1.20],$$
$$\hat{K}_{12} = \left[\hat{k}_1\hat{k}_2\ \ \hat{k}_1 + \hat{k}_2(\hat{k}_2 - b)\right]^T = [1.6320\ 0.0800]^T$$

and the solution to GEP (83) gives $1/\lambda = 1.20$. Solving the GEP (86) with prescribed parameters $\alpha_1 = 1.1480$ and $\tilde{\kappa} = 0.5440$ yields

$$\bar{\alpha}_2 = 32.1870 \ \ and \ \ P = \begin{bmatrix} 0.5311 & 0.2253 \\ 0.2253 & 0.3899 \end{bmatrix} > 0.$$

If the sampling sequence satisfies

$$0 < \underline{\Delta t} \le \overline{\Delta t} < \frac{-\ln\tilde{\kappa}}{\bar{\alpha}_2} = \frac{-\ln(0.5440)}{32.1870} = 0.0189,$$

TABLE 1. Feedback gain $\hat{K}$ and allowable $\overline{\Delta t}$ for CPS (81) with various b.

b	0.0	0.2	0.4	0.6	0.8
$\hat{K}^T$	$\begin{bmatrix}-1.36\\-1.20\end{bmatrix}$	$\begin{bmatrix}-1.36\\-1.00\end{bmatrix}$	$\begin{bmatrix}-1.36\\-0.80\end{bmatrix}$	$\begin{bmatrix}-1.36\\-0.60\end{bmatrix}$	$\begin{bmatrix}-1.36\\-0.40\end{bmatrix}$
$\overline{\Delta t}$	0.0188	0.0184	0.0188	0.0191	0.0196

the CPS (81) and hence the sampled-data control system (80) of pendulum (77) with $\delta = 0$, $a = 9.8$, $c = 1$, $b = 0$ are exponentially stable. The state trajectory and control input of sampled-data system (80) are shown in Figure 3, from which one can infer the state trajectories $z(t) = [x^T(t)\ y(t)]^T$ of CPS (81), where the sampling period $\Delta t = \overline{\Delta t} = 0.0188$, see Table 1.

We list in Table 1 the feedback gain $\hat{K}$ and the maximum allowable sampling interval $\overline{\Delta t}$ for stabilization of CPS (81) of pendulum with $\delta = 0$, $a = 9.8$, $c = 1$ and various $b \geq 0$.

VI. CONCLUSION

In light of the CPS theory of numerical methods for SDE, we have presented a CPS model of sampled-data control systems that regards the intersection of the physical and cyber, the key feature of CPS. As a theoretic foundation, we have developed the Lyapunov stability theory for the general class (1) of SiDE that is formulated to serve as a canonical form for synthesized CPS that may work in feedback loop such as those of sampled-data systems. Applying the established Lyapunov stability theory, we have proposed the stability criteria for the CPS of sampled-data control systems, have revealed the equivalence and intrinsic relationship between the two main approaches and have presented the control design method for feedback stabilization of the CPS of linear sampled-data systems. In practice, feedback control is usually based on an observer that is designed to reconstruct the state using measurements of the input and the output of the system [19], [53]. Our canonical form (1) for synthetic CPS is able to include the dynamics of observers as well as impluse effects such as those in a robot model [19], [30]. This is important for nonlinear control systems in which the so-called separation principle may not hold [33], [55]. As illustated in the classical example of pendulum, our theory applies to sampled-data systems with nonlinear feedback control satisfying the global Lipschitz condition/linear growth condition. For those with highly nonlinear control, one may improve/extend our results so that they (e.g., LaSalle-type theorems) apply to practical nonlinear control systems such as tunnel-diode circuit, inverted pendulum and the robotic manipulator with flexible joints [33], [43], [73]. Our CPS theory can be further developed by various techniques including those of Lyapunov functions/functionals [15], [33], [39], [40], [44], [48].

We have constructed a foundational theory for CPS of sampled-data control. This provokes many interesting and challenging problems. For example, one can naturally generalize the time-triggered mechanism to an event-triggered mechanism [31] and the SiDE to stochastic impulsive differential-algebraic equations (SiDAE) [9], [26] so that the canonical form for synthesized CPS can encompass event-triggered sampling/control [20], [31], [61] and equality constraints [53] on both the physical and the cyber sides. For instance, a generalization of SiDE (1) may be of the form

$$E_x \mathrm{d}x(t) = f(x(t), y(t), t)\mathrm{d}t + g(x(t), y(t), t)\mathrm{d}B(t) \quad t \in [0, \infty) \setminus \{t_k\}_{k\in\mathbb{N}} \tag{87a}$$

$$E_y \mathrm{d}y(t) = \tilde{f}(x(t), y(t), t)\mathrm{d}t + \tilde{g}(x(t), y(t), t)\mathrm{d}B(t) \quad t \in [0, \infty) \setminus \{t_k\}_{k\in\mathbb{N}} \tag{87b}$$

$$\Delta(x_{t_k^-}, y_{t_k^-}, k) := x(t_k) - x(t_k^-) = \begin{cases} h(x_{t_k^-}, y_{t_k^-}, \bar{\xi}(k), k), & \kappa_x(x_{t_k^-}, y_{t_k^-}, k) > 0 \\ 0, & \kappa_x(x_{t_k^-}, y_{t_k^-}, k) \leq 0 \end{cases} \tag{87c}$$

$$\tilde{\Delta}(x_{t_k^-}, y_{t_k^-}, k) := y(t_k) - y(t_k^-) = \begin{cases} \tilde{h}(x_{t_k^-}, y_{t_k^-}, \bar{\xi}(k), k), & \kappa_y(x_{t_k^-}, y_{t_k^-}, k) > 0 \\ 0, & \kappa_y(x_{t_k^-}, y_{t_k^-}, k) \leq 0 \end{cases} \tag{87d}$$

for all $k \in \mathbb{N}$, where $E_x \in \mathbb{R}^{n\times n}$ and $E_y \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ are constant matrices with $0 < rank(E_x) \leq n$ and $0 < rank(E_y) \leq \tilde{n}$, respectively; $h : C([t_{k-1}, t_k); \mathbb{R}^n) \times C([t_{k-1}, t_k); \mathbb{R}^{\tilde{n}}) \times \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^n$, $\tilde{h} : C([t_{k-1}, t_k); \mathbb{R}^n) \times C([t_{k-1}, t_k); \mathbb{R}^{\tilde{n}}) \times \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^{\tilde{n}}$, $\kappa_x : C([t_{k-1}, t_k); \mathbb{R}^n) \times C([t_{k-1}, t_k); \mathbb{R}^{\tilde{n}}) \times \mathbb{N} \to \mathbb{R}$ and $\kappa_y : C([t_{k-1}, t_k); \mathbb{R}^n) \times C([t_{k-1}, t_k); \mathbb{R}^{\tilde{n}}) \times \mathbb{N} \to \mathbb{R}$ are measurable functions. The functions κ_x, κ_y could involve some optimization problems such as those for model predictive control. Clearly, the generalized canonical form (87) has a much wider range of applications, which, for example, can be a dynamic model at the core of smart manufacturing/digital twins when the functions κ_x, κ_y perform simulation/prediction and real-time optimization. The proposed CPS theory may be adapted for various control strategies such as saturated control [13], adaptive control [5] and model predictive control [18]. Particularly, over past a few decades, many investigations have been conducted into fault-tolerant control systems, fault detection and diagnosis, and reconfigurable control [23], [56], [59], [64], [67], [70]. Safety-critical systems such as power networks, aircrafts, nuclear power plants and chemical plants must be resilient to faults and cyberattacks [1], [11], [35], [67]. The physical aspects of CPS will create new and difficult privacy problems. Thus the design of fault-tolerant CPS in canonical form (87) is of major importance. As an example of this research topic, a fault-tolerant control strategy could be developed for some CPS in canonical form (1) like the study [67] conducted on a plant described by linear SDE. Moreover, CPS are often implemented and operated over large-scale complex networked infrastructures such as building automation systems, power plants and transportation systems [11], [35], [38], [60], [67]. It is intereting/challenging to develop the CPS theory for complex systems including stochastic hybrid systems [66], infinite-dimensional systems [24], [37], large-scale systems [39], [40] and multi-scale dynamic systems [29].

We also recognize that the joint dynamics of a classical stochastic approximation algorithm [57]

$$X_k = X_{k-1} + \gamma_k[f(X_{k-1}) + \bar{\xi}(k)] \tag{88}$$

and the limiting ODE (ordinary differential equation) [41], [42]

$$\dot{x}(t) = f(x(t)) \tag{89}$$

can be represented by the associated CPS with $X(t) = \sum_{k=0}^{\infty} X_k \mathbb{1}_{[t_k,t_{k+1})}(t)$ and $y(t) = x(t) - X(t)$ as (see also [30, Section IV] for more details)

$$\dot{x}(t) = f(x(t)), \quad t \in [0, \infty) \tag{90a}$$
$$\dot{y}(t) = f(x(t)), \quad t \in [t_{k-1}, t_k) \tag{90b}$$
$$\begin{aligned}\tilde{\Delta}(x(t_k^-), y(t_k^-), k) &:= y(t_k) - y(t_k^-)\\ &= (t_k - t_{k-1})\big[f(x(t_k^-) - y(t_k^-)) + \bar{\xi}(k)\big],\\ &\quad k \in \mathbb{N}\end{aligned} \tag{90c}$$

in the canonical form [30, SiDE (1)] which is a particular class of SiDE (1), where $\{t_k\}_{k\in\mathbb{N}}$ with $t_1 > t_0 = 0$ is a strictly increasing sequence such that $t_k - t_{k-1} = \gamma_k > 0$ for all $k \in \mathbb{N}$ and $t_k = \sum_{j=1}^{k} \gamma_j \to \infty$ as $k \to \infty$. It is among future work to develop the CPS theory of computational methods invented in [30] for algorithms such as stochastic approximation. As an example, to develop LaSalle-type theorems for [30, SiDE (1)] with $t_k = \sum_{j=1}^{k} \gamma_j$ may offer a theoretic foundation for the CPS approach (90) to stochastic approximation algorithm (88) with limiting ODE (89). It is well known that the use of stochastic approximation is very widespread across varied applications such as systems identification, adaptive control, neural networks, adaptive signal processing and pattern recognition [4], [6], [7], [27], [36], [41], [42]. For example, stochastic approximation is a family of adaptive algorithms underlying reinforcement learning/Q learning [6], [7], [36]. The CPS theory of stochastic approximation may be helpful to describe/understand the dynamics of neural networks. It would also privide a solid base for the development of CPS theory for sampled-data (fault-tolerant) control combined with learning algorithms in the future. It is of great theoretic and practical importance to develop CPS theory for computer control systems by data-driven approaches [59], [73] and also for those by combined model-based and data-driven approaches [5]. Just name a few among future work to develop the systems science of design for CPS.

APPENDIX A
THE PROOF OF PROPOSITION 1

Proof: For every integer $\hat{n} \geq 1$, define the stopping time $\tau_{\hat{n}} = \inf\{t > 0 : |z(t)| \geq \hat{n}\}$, where we set $\inf \emptyset = \infty$ as usual. By virtue of the local Lipschitz condition (3), it is easy to see that $\tau_{\hat{n}} > 0$ *a.s.* for a sufficiently large integer $\hat{n} > |z(0)|$. It is also observed that there is a unique (right-continuous) adapted process such that

$$\begin{aligned}z(t \wedge \tau_{\hat{n}})\\ &= z(0) + \int_0^{t\wedge\tau_{\hat{n}}} F(z(s), s)\mathrm{d}s\\ &\quad + \int_0^{t\wedge\tau_{\hat{n}}} G(z(s), s)\mathrm{d}B(s)\\ &\quad + \int_0^{t\wedge\tau_{\hat{n}}} \sum_{k\geq 1} \delta(s - t_k)\Big[H_F(z(t_k^-), z(t_{k-1}), k)\\ &\qquad + H_G(z(t_k^-), z(t_{k-1}), k)\bar{\xi}(k)\Big]\mathrm{d}s\\ &= z(0) + \int_0^{t\wedge\tau_{\hat{n}}} F(z(s), s)\mathrm{d}s\\ &\quad + \int_0^{t\wedge\tau_{\hat{n}}} G(z(s), s)\mathrm{d}B(s)\\ &\quad + \int_0^{t} \sum_{k\geq 1} \delta(s - t_k)\mathbb{1}_{\{t_k\leq\tau_{\hat{n}}\}}\Big[H_F(z(t_k^-), z(t_{k-1}), k)\\ &\qquad + H_G(z(t_k^-), z(t_{k-1}), k)\bar{\xi}(k)\Big]\mathrm{d}s \quad a.s.\end{aligned} \tag{91}$$

for all $t \geq 0$, where $\delta(\cdot)$ is the Dirac delta function and $\mathbb{1}_T$ is the indicator of set T. Therefore, SiDE (2) has a unique local (right-continuous) solution $z(t)$ on $[0, \tau_e)$, where τ_e is the explosion time defined by $\tau_e = \lim_{\hat{n}\to\infty} \tau_{\hat{n}} = \inf\{t > 0 : |z(t)| \notin [0, \infty)\}$ and thus $\tau_e > 0$ *a.s.* So we only need to prove that the explosion time obeys $\tau_e = \infty$ *a.s.*

Let us consider a pair of auxiliary systems

$$\begin{cases}\dot{U}_1(t) = a + 2cU_1(t) & t \neq t_k\\ U_1(t_k) - U_1(t_k^-) = c\underline{\Delta t}\big(U_1(t_k^-) + U_1(t_{k-1})\big)/2 & k \in \mathbb{N}\end{cases}$$
$$\textit{and} \quad \dot{U}_2(t) = a + 4cU_2(t) \quad t \geq 0 \tag{92}$$

with $U_2(0) \geq U_1(0) \geq 0$, where $a \geq 0$ and $c > 0$ are both constants. The well-known comparison principle gives $U_2(t) \geq U_1(t)$ on $[t_0, t_1)$. Since $a \geq 0$ and $c > 0$, $U_1(t_1^-) \geq U_1(t_0)$ and thus

$$\begin{aligned}U_1(t_1) &= U_1(t_1^-) + c\underline{\Delta t}\big(U_1(t_1^-) + U_1(t_0)\big)/2\\ &\leq (1 + c\underline{\Delta t})U_1(t_1^-)\\ &= (1 + c\underline{\Delta t})\Big[U_1(0)e^{2ct_1} + \frac{a}{2c}(e^{2ct_1} - 1)\Big]\\ &\leq U_1(0)e^{3ct_1} + \frac{a}{2c}e^{ct_1}(e^{2ct_1} - 1)\\ &\leq U_2(0)e^{4ct_1} + \frac{a}{2c}\frac{e^{2ct_1} + 1}{2}(e^{2ct_1} - 1)\\ &= U_2(0)e^{4ct_1} + \frac{a}{4c}(e^{4ct_1} - 1) = U_2(t_1).\end{aligned}$$

By induction, it follows that $U_1(t) \leq U_2(t)$ on $[t_{k-1}, t_k]$ for all $k \geq 1$ and thus $U_1(t) \leq U_2(t)$ for all $t \geq 0$. The pair (92) of auxiliary systems implies that

$$\begin{aligned}U_1(t) &= U_1(0) + at + \int_0^t 2\,cU_1(s)\mathrm{d}s\\ &\quad + \int_0^t \sum_{k\geq 1} \delta(s - t_k)c\underline{\Delta t}\big[\big(U_1(t_k^-) + U_1(t_{k-1})\big)/2\big]\mathrm{d}s\\ &\leq U_2(t) = U_2(0) + at + \int_0^t 4\,cU_2(s)\mathrm{d}s\end{aligned} \tag{93}$$

for all $t \geq 0$. Using the Itô formula and also (91), one has

$$\begin{aligned}
&\mathbb{E}\bar{V}(z(t \wedge \tau_{\hat{n}}), t \wedge \tau_{\hat{n}}) \\
&= \bar{V}(z(0), 0))\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^{t\wedge\tau_{\hat{n}}} \overline{\mathscr{L}}\bar{V}(z(s), s + \int_0^t \sum_{k\geq 1} \delta(s-t_k) \\
&\quad \cdot \mathbb{E}\Big[\mathbb{1}_{\{t_k \leq \tau_{\hat{n}}\}}\big(\bar{V}(z(t_k^-) + \bar{\Delta}(z(t_k^-), z(t_{k-1}), k), t_k) \\
&\quad - \bar{V}(z(t_k^-), t_k)\big)\Big]\mathrm{d}s \qquad (94)
\end{aligned}$$

for any $\hat{n} \geq 1$ and all $t \geq 0$. At each $t = t_k$, by the tower property and condition (5c),

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{1}_{\{t_k \leq \tau_{\hat{n}}\}}\big(\bar{V}(z(t_k^-) + \bar{\Delta}(z(t_k^-), k), t_k) - \bar{V}(z(t_k^-), t_k)\big)\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\mathbb{1}_{\{t_k \leq \tau_{\hat{n}}\}}\big(\bar{V}(z(t_k^-) + \bar{\Delta}(z(t_k^-), z(t_{k-1}), k), t_k) \\
&\quad - \bar{V}(z(t_k^-), t_k)\big)\big|\mathcal{F}_{t_k^-}\Big]\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_{\{t_k \leq \tau_{\hat{n}}\}}\mathbb{E}\Big[\big(\bar{V}(z(t_k^-) + \bar{\Delta}(z(t_k^-), z(t_{k-1}), t_k)) \\
&\quad - \bar{V}(z(t_k^-), t_k)\big)\big|\mathcal{F}_{t_k^-}\Big]\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_{\{t_k \leq \tau_{\hat{n}}\}}\Big(\mathbb{E}\big[\bar{V}(z(t_k^-) \\
&\quad + \bar{\Delta}(z(t_k^-), z(t_{k-1}), t_k))\big|z(t_k^-)\big] - \bar{V}(z(t_k^-), t_k)\Big)\Big] \\
&\leq \mathbb{E}\Big[\mathbb{1}_{\{t_k \leq \tau_{\hat{n}}\}}\bar{K}\underline{\Delta t} \\
&\quad \cdot \big[2 + \big(\bar{V}(z(t_k^-), t_k) + \bar{V}(z(t_{k-1}, t_{k-1})\big)/2\big]\Big] \qquad (95)
\end{aligned}$$

for any $\hat{n} \geq 1$. Substituting (5b) and (95) into (94) produces

$$\begin{aligned}
&\mathbb{E}\bar{V}(z(t \wedge \tau_{\hat{n}}), t \wedge \tau_{\hat{n}}) \\
&\leq \bar{V}(z(0), 0) + \mathbb{E}\int_0^{t\wedge\tau_{\hat{n}}} 2\bar{K}\big[1 + \bar{V}(z(s), s)\big]\mathrm{d}s \\
&\quad + \int_0^t \sum_{k\geq 1}\delta(s-t_k)\mathbb{E}\Big[\mathbb{1}_{\{t_k \leq \tau_{\hat{n}}\}}\bar{K}\underline{\Delta t} \\
&\quad \cdot \big[2 + \big(\bar{V}(z(t_k^-), t_k) + \bar{V}(z(t_{k-1}, t_{k-1})\big)/2\big]\Big]\mathrm{d}s \\
&\leq \bar{V}(z(0), 0) + \mathbb{E}\int_0^{t\wedge\tau_{\hat{n}}} 2\bar{K}\big[2 + \bar{V}(z(s), s)\big]\mathrm{d}s \\
&\quad + \int_0^t \sum_{k\geq 1}\delta(s-t_k)\mathbb{E}\Big[\mathbb{1}_{\{t_k \leq \tau_{\hat{n}}\}}\bar{K}\underline{\Delta t} \\
&\quad \cdot \big(\bar{V}(z(t_k^-), t_k) + \bar{V}(z(t_{k-1}, t_{k-1})\big)/2\Big]\mathrm{d}s \\
&\leq \bar{V}(z(0), 0) + 4\bar{K}t + \mathbb{E}\int_0^{t\wedge\tau_{\hat{n}}} 2\bar{K}\bar{V}(z(s), s)\mathrm{d}s \\
&\quad + \int_0^t \sum_{k\geq 1}\delta(s-t_k)\mathbb{E}\Big[\mathbb{1}_{\{t_k \leq \tau_{\hat{n}}\}}\bar{K}\underline{\Delta t} \\
&\quad \cdot \big(\bar{V}(z(t_k^-), t_k) + \bar{V}(z(t_{k-1}, t_{k-1})\big)/2\Big]\mathrm{d}s \\
&= \bar{V}(z(0), 0) + 4\bar{K}t + \mathbb{E}\int_0^{t\wedge\tau_{\hat{n}}} 2\bar{K}\bar{V}(z(s), s)\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^{t\wedge\tau_{\hat{n}}} \sum_{k\geq 1}\delta(s-t_k)\bar{K}\underline{\Delta t} \\
&\quad \cdot \big[\big(\bar{V}(z(t_k^-), t_k) + \bar{V}(z(t_{k-1}, t_{k-1})\big)/2\big]\mathrm{d}s \qquad (96)
\end{aligned}$$

for any $\hat{n} \geq 1$ and all $t \geq 0$. By the comparison principle (93), it follows from (96) that

$$\begin{aligned}
&\mathbb{E}\bar{V}(z(t \wedge \tau_{\hat{n}}), t \wedge \tau_{\hat{n}}) \\
&\leq \bar{V}(z(0), 0) + 4\bar{K}t + \mathbb{E}\int_0^{t\wedge\tau_{\hat{n}}} 4\bar{K}\bar{V}(z(s), s)\mathrm{d}s \\
&\leq \bar{V}(z(0), 0) + 4\bar{K}t \\
&\quad + \mathbb{E}\int_0^t 4\bar{K}\bar{V}(z(s \wedge \tau_{\hat{n}}), s \wedge \tau_{\hat{n}})\mathrm{d}s \\
&\leq \bar{V}(z(0), 0) + 4\bar{K}t \\
&\quad + 4\bar{K}\int_0^t \Big[\sup_{0\leq r\leq s}\mathbb{E}\bar{V}(z(r \wedge \tau_{\hat{n}}), r \wedge \tau_{\hat{n}})\Big]\mathrm{d}s \qquad (97)
\end{aligned}$$

for all $t \geq 0$. Since the right-hand side of (97) is increasing in t, this implies that

$$\begin{aligned}
&\sup_{0\leq s\leq t}\mathbb{E}\bar{V}(z(s \wedge \tau_{\hat{n}}), s \wedge \tau_{\hat{n}}) \leq \bar{V}(z(0), 0) + 4\bar{K}t \\
&\quad + 4\bar{K}\int_0^t \Big[\sup_{0\leq r\leq s}\mathbb{E}\bar{V}(z(r \wedge \tau_{\hat{n}}), r \wedge \tau_{\hat{n}})\Big]\mathrm{d}s
\end{aligned}$$

for all $t \geq 0$. The Gronwall inequality produces

$$\begin{aligned}
&\sup_{0\leq s\leq t}\mathbb{E}\bar{V}(z(s \wedge \tau_{\hat{n}}), s \wedge \tau_{\hat{n}}) \\
&\quad \leq \Big[\bar{V}(z(0), 0) + 4\bar{K}t\Big]e^{4\bar{K}t} =: C_{\bar{V}}(t) \qquad (98)
\end{aligned}$$

for all $t \geq 0$. Note that $C_{\bar{V}}(t)$ is independent of $\hat{n}$ and $\bar{V}(z(\tau_{\hat{n}}), \tau_{\hat{n}}) \geq \bar{c}_1\hat{n}^p$ for all $\hat{n} \geq 1$. But (5a) and (98) imply that, given any $t > 0$,

$$\begin{aligned}
&\bar{c}_1\hat{n}^p\mathbb{P}\{\tau_{\hat{n}} < t\} \leq \bar{c}_1\mathbb{E}|z(t \wedge \tau_{\hat{n}})|^p \\
&\quad \leq \mathbb{E}\bar{V}(z(t \wedge \tau_{\hat{n}}), t \wedge \tau_{\hat{n}}) \leq C_{\bar{V}}(t). \qquad (99)
\end{aligned}$$

Letting $\hat{n} \to \infty$ and then $t \to \infty$ in (99) yields $\mathbb{P}\{\tau_e < \infty\} = 0$. That is, $\tau_e = \infty$ *a.s.*, which completes the proof. □

APPENDIX B
THE PROOF OF THEOREM 1

Proof: Let $\bar{K} = 1 + (\bar{\alpha}_2/2) \vee \big[2\big((\beta_0+\beta_1)\bar{c}+\bar{\beta}_0+\bar{\beta}_1\big)/\underline{\Delta t}\big]$. It follows from (8b), (8c), (9b) and (10) that the conditions (5) hold and thus, by Proposition 1, the SiDE (2) has a unique (right-continuous) solution on $[0, \infty)$.

Note that if condition (9a) holds with $\alpha_2 = 0$ (cf. the inequality (7a) in [30, Theorem 3.1]), so does it for any $\alpha_2 > 0$. Without loss of generality, we consider only the case $\alpha_2 > 0$ in the proof, in which the cyber and the physical subsystems interact with each other (cf. [30, SiDE (1)]). Some ideas and techniques in this proof are derived from our results [24, Theorem 3.1 and Remark 3.1] on pth moment input-to-state stability (ISS) of stochastic systems as well as [30] and [31]. The proof is so technical that we devide it into four steps, in which we shall: 1) show the ISS of $x(t)$ with $z(t)$ as input; 2) construct a candidate Lyapunov function for the exponential stability of SiDE (2) by combining the ones $V(t)$ and $\bar{V}(t)$; 3) define a function that breaks the time interval $[0, \infty)$ into a disjoint union of subsets on which the system has some different properties; 4) prove the exponential stability of SiDE (2). Recall that $x(t) = Cz(t)$ and $y(t) = Dz(t)$ for all $t \geq 0$. For simplicity of the notation, we write $U(t) = \mathbb{E}V(x(t), t)$ and $W(t) = \mathbb{E}\bar{V}(z(t), t)$ for all $t \geq 0$. So $U(t)$ is continous on $[0, \infty)$ while $W(t)$ is right-continuous on $[0, \infty)$ and could only jump at $\{t_k\}_{k\in\mathbb{N}}$.

Step 1: By the Itô formula and condition (9a),

$$U(t) = U(\bar{t}) + \int_{\bar{t}}^{t} \mathbb{E}\overline{\mathscr{L}}V(x(s), Dz(s), s)\mathrm{d}s$$
$$\leq U(\bar{t}) + \int_{\bar{t}}^{t} \big[-\alpha_1 U(s) + \alpha_2 W(s)\big]\mathrm{d}s \quad \forall t \geq \bar{t} \geq 0$$

and hence the upper right Dini derivative

$$\mathscr{D}^+ U(t) = \mathbb{E}\mathscr{L}V(x(t), y(t), t) \leq -\alpha_1 U(t) + \alpha_2 W(t) \tag{100}$$

for all $t \geq 0$, which implies

$$\mathscr{D}^+(t) \leq -(1-\theta)\alpha_1 U(t) \quad \text{if} \quad U(t) \geq \frac{\alpha_2}{\theta\alpha_1} \sup_{0\leq s\leq t} W(s) \tag{101}$$

where θ can be any positive on $(0, 1)$. By [14, Lemma 1] and [33, Theorem 4.18, p172], inequalities (8a) and (101) imply

$$U(t) \leq \Big(U(0)e^{-(1-\theta)\alpha_1 t}\Big) \vee \Big(\frac{\alpha_2}{\theta\alpha_1} \sup_{0\leq s\leq t} W(s)\Big) \tag{102}$$

for all $t \geq 0$. So $U(t)$ is exponentially stable if $\alpha_2 = 0$; otherwise (i.e., $\alpha_2 > 0$), $U(t)$ is ISS with $W(t)$ as input, which means that $x(t)$ is pth moment ISS with $z(t)$ as input [24]. Specifically, there is a $t^U \geq 0$ (dependent on $U(0)$ and $(\theta\alpha_1\bar{c})^{-1}\alpha_2 \sup_{0\leq s\leq t} W(s)$, see also [14], [33]) such that

$$U(t) \leq U(0)e^{-(1-\theta)\alpha_1 t}, \quad \forall 0 \leq t \leq t^U$$
$$U(t) \leq (\theta\alpha_1)^{-1}\alpha_2 \sup_{0\leq s\leq t^U} W(s), \quad \forall t \geq t^U.$$

Moreover, $U(t)$ is (expoentially) stable if $W(t)$ (exponentially) converges to zero as $t \to \infty$. If $z(t)$ is pth moment exponentially stable, so is $x(t)$ [24, Theorem 3.1 and Remark 3.1], which is also implied by $|x(t)| = |Cz(t)| \leq |z(t)|$ for all $t \geq 0$.

Step 2: By conditions (11) and (12), there exists a number $q \in \big(\alpha_1^{-1}\alpha_2(\beta_0 + \beta_1) + \bar{\beta}_1 + \bar{\beta}_2, 1\big)$ for

$$\Big(\frac{\alpha_2\bar{\alpha}_1}{\alpha_1 q} + \bar{\alpha}_2\Big)\overline{\Delta t} < -\ln q < -\ln\Big[\frac{\alpha_2}{\alpha_1}(\beta_0 + \beta_1) + \bar{\beta}_0 + \bar{\beta}_1\Big].$$

Therefore, one can find a pair of positive numbers $\theta \in (0, 1)$ sufficiently close to 1 for

$$\Big(\frac{\alpha_2\bar{\alpha}_1}{\theta\alpha_1 q} + \bar{\alpha}_2\Big)\overline{\Delta t} < -\ln q < -\ln\Big[\frac{\alpha_2}{\theta\alpha_1}(\beta_0 + \beta_1) + \bar{\beta}_0 + \bar{\beta}_1\Big] \tag{103}$$

and then $\mu \in (0, (1-\theta)\alpha_1 \underline{\Delta t}/\overline{\Delta t})$ sufficiently small for

$$\Big(\frac{\alpha_2\bar{\alpha}_1}{\theta\alpha_1 q} + \bar{\alpha}_2 + \mu\Big)\overline{\Delta t} < -\ln q < -\ln\Big[\frac{\alpha_2}{\theta\alpha_1}\beta_0 + \bar{\beta}_0 + \Big(\frac{\alpha_2}{\theta\alpha_1}\beta_1 + \bar{\beta}_1\Big)e^{\mu\overline{\Delta t}}\Big]. \tag{104}$$

Given $\mu \in (0, (1-\theta)\alpha_1 \underline{\Delta t}/\overline{\Delta t})$ by (104), let

$$\tilde{U}(t) = e^{\mu t}U(t) \quad \textit{and} \quad \tilde{W}^\mu(t) = e^{\mu t}W(t) \tag{105}$$

for all $t \geq 0$. By the Itô formula, (100) and (9b),

$$\tilde{U}(t) = \tilde{U}(\bar{t}) + \int_{\bar{t}}^{t} e^{\mu s}\big[\mu U(s) + \mathscr{D}^+ U(s)\big]\mathrm{d}s$$
$$\leq \tilde{U}(\bar{t}) + \int_{\bar{t}}^{t} e^{\mu s}\big[(\mu - \alpha_1)U(s) + \alpha_2 W(s)\big]\mathrm{d}s$$
$$= \tilde{U}(\bar{t}) + \int_{\bar{t}}^{t} \big[-(\alpha_1 - \mu)\tilde{U}(s) + \alpha_2\tilde{W}^\mu(s)\big]\mathrm{d}s \tag{106}$$

for all $t \geq \bar{t} \geq 0$ and

$$\tilde{W}^\mu(t) = \tilde{W}^\mu(\tilde{t}) + \int_{\tilde{t}}^{t} e^{\mu s}\big[\mu W(s) + \mathbb{E}\overline{\mathscr{L}}\bar{V}(z(s), s)\big]\mathrm{d}s$$
$$\leq \tilde{W}^\mu(\tilde{t}) + \int_{\tilde{t}}^{t} e^{\mu s}\big[\bar{\alpha}_1 U(s) + (\bar{\alpha}_2 + \mu)W(s)\big]\mathrm{d}s$$
$$= \tilde{W}^\mu(\tilde{t}) + \int_{\tilde{t}}^{t} \big[\bar{\alpha}_1\tilde{U}(s) + (\bar{\alpha}_2 + \mu)\tilde{W}^\mu(s)\big]\mathrm{d}s \tag{107}$$

for all $t_{k-1} \leq \tilde{t} \leq t < t_k$ and $k \in \mathbb{N}$. For convenience, let

$$\tilde{W}(t) = \frac{\alpha_2}{\theta\alpha_1}\tilde{W}^\mu(t) = \frac{\alpha_2}{\theta\alpha_1}e^{\mu t}W(t) \tag{108}$$

for all $t \geq 0$, where $\theta \in (0, 1)$ is given by (103). Define

$$\overline{W}(t) = \tilde{U}(t) \vee \tilde{W}(t) \quad \forall t \in [0, \infty). \tag{109}$$

Due to the continuity of $U(t)$ and the right-continuity of $W(t)$, $\overline{W}(t)$ is right-continuous on $[0, \infty)$ and could only jump at the impulse instants $\{t_k\}_{k\in\mathbb{N}}$. Clearly, $\overline{W}(t) \geq \tilde{U}(t)$ and $\overline{W}(t) \geq \frac{\alpha_2}{\theta\alpha_1}\tilde{W}^\mu(t)$ for all $t \geq 0$. So both $U(t)$ and $W(t)$ will be exponentially stable if there is a positive constant K such that

$$\overline{W}(t) < K \tag{110}$$

for all $t \geq t_0 = 0$. For instance, let

$$K = 1 + \frac{\alpha_1 + \alpha_2}{\theta\alpha_1 q}\big[U(t_0) + W(t_0)\big] > 0 \tag{111}$$

and hence $\overline{W}(t_0) \leq U(t_0) + \frac{\alpha_2}{\theta\alpha_1}W(t_0) < qK$.

Step 3: Define a function $\bar{v}: \mathbb{R}_+ \to \mathbb{R}$ by

$$\bar{v}(t) = \tilde{W}(t) - \tilde{U}(t) \quad \forall t \in [0, \infty) \tag{112}$$

with initial value $\bar{v}(0) = \frac{\alpha_2}{\theta\alpha_1}W(0) - U(0)$, where the functions $\tilde{U}(t)$ and $\tilde{W}(t)$ are defined by (105) and (108), respectively, as $\theta \in (0, 1)$ given by (103). Since $\tilde{U}(t)$ is continuous on $[0, \infty)$ while $\tilde{W}(t)$ is right-continuous on $[0, \infty)$ and could only jump at $\{t_k\}_{k\in\mathbb{N}}$, function $\bar{v}(t)$ is right-continuous on

$[0, \infty)$ and could only jump at the impulse instants $\{t_k\}_{k\in\mathbb{N}}$. Given any $t \geq 0$, either $\bar{v}(t) > 0$ or $\bar{v}(t) \leq 0$. So the time interval $[0, \infty)$ is broken into a disjoint union of subsets $T_+ \cup T_-$, where

$$T_+ = \{t \geq 0 : \bar{v}(t) > 0\}, T_- = \{t \geq 0 : \bar{v}(t) \leq 0\}. \quad (113)$$

From (109), (112) and (113),

$$\overline{W}(t) = \begin{cases} \tilde{W}(t), & t \in T_+ \\ \tilde{U}(t), & t \in T_- \end{cases} \quad (114)$$

and, by (106) and (113),

$$\mathscr{D}^+\tilde{U}(t) \leq -\varepsilon\,\tilde{U}(t) \quad \forall\, t \in T_- \quad (115)$$

where $\varepsilon \in (0, (1-\theta)\alpha_1 - \mu)$ is some postive number, e.g., $\varepsilon = [(1-\theta)\alpha_1 - \mu]/2$. That is, $\mathscr{D}^+\tilde{U}(t)$ is negative definite (with respect to x) and is strictly decreasing on the set T_- if $T_- \neq \emptyset$. In the case $T_+ = \emptyset$, namely, $T_- = [0, \infty)$, $\mathscr{D}^+\tilde{U}(t) \leq -c\,\tilde{U}(t)$ for all $t \geq 0$ and hence $U(t)$ is exponentially stable. In this case, due to $\tilde{W}(t) \leq \tilde{U}(t)$ on $T_- = [0, \infty)$, both $U(t)$ and $\tilde{W}(t)$ are exponentially stable.

Let us consider the other case $T_+ \neq \emptyset$. Given any $t \in T_+$, due to the right-continuity of $\bar{v}(t)$ on $[0, \infty)$, there exists an interval $[\tau_1^+(t), \tau_2^+(t))$ with $\tau_1^+(t) < \tau_2^+(t)$ such that $(\tau_1^+(t), \tau_2^+(t)) \subset T_+$, where

$$c\tau_1^+(t) = \inf\{\,\bar{\tau} \leq t :\ \bar{v}(\tau) > 0,\ \forall\tau \in [\,\bar{\tau}, t\,]\},$$
$$\tau_2^+(t) = \sup\{\,\bar{\tau} > t :\ \bar{v}(\tau) > 0,\ \forall\tau \in [\,t, \bar{\tau})\}. \quad (116)$$

Similarly, given any $\bar{t} \in T_-$, there is an ordered pair $\tau_1^-(\bar{t}) \leq \tau_2^-(\bar{t})$ such that $[\tau_1^-(\bar{t}), \tau_2^-(\bar{t})) \subset T_-$, where

$$c\tau_1^-(\bar{t}) = \inf\{\,\tilde{\tau} \leq t :\ \bar{v}(\tau) \leq 0,\ \forall\tau \in [\,\tilde{\tau}, t\,]\},$$
$$\tau_2^-(\bar{t}) = \sup\{\,\bar{\tau} \geq t :\ \bar{v}(\tau) \leq 0,\ \forall\tau \in [\,t, \bar{\tau})\}, \quad (117)$$

and $[\tau_1^-(\bar{t}), \tau_2^-(\bar{t})) = \emptyset$ if $\tau_1^-(\bar{t}) = \tau_2^-(\bar{t}) = \bar{t}$.

For convenience, we also write $\tau_1^+ = \tau_1^+(t)$, $\tau_2^+ = \tau_2^+(t)$, $\tau_1^- = \tau_1^-(\bar{t})$ and $\tau_2^- = \tau_2^-(\bar{t})$ when there is no confusion.

Step 4: Let us show (110) for all $t \geq t_0 = 0$. Define

$$\overline{\tau}_K = \inf\{t \geq t_0 : \overline{W}(t) \geq K\}, \quad (118)$$

By the choice (111), $\overline{\tau}_K > t_0 = 0$. If $\overline{\tau}_K > t_k$ for all $k \in \mathbb{N}$, then (110) holds for all $t \geq 0$ because $\underline{\Delta t} = \inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\} > 0$ and $t_k \to \infty$ as $k \to \infty$. Otherwise, there is some $k \in \mathbb{N}$ such that $t_k = \inf\{t_j : t_j \geq \overline{\tau}_K, j \in \mathbb{N}\}$. This means that either $\overline{\tau}_K = t_k$ or $t_{k-1} < \overline{\tau}_K < t_k$. If $\overline{\tau}_K = t_k$, then (110) holds for all $t \in [0, t_k)$. Particularly,

$$\overline{W}(t_{k-1}) \vee \overline{W}(t_k^-) \leq \sup_{t_{k-1}\leq t<t_k} \overline{W}(t) < K. \quad (119)$$

Moreover, either $\overline{\tau}_K = t_k \in T_+$ or $\overline{\tau}_K = t_k \in T_-$ when $\overline{\tau}_K = t_k$. If $\overline{\tau}_K = t_k \in T_+$, then $\overline{W}(t_k) = \tilde{W}(t_k) \geq K$. By condition (iii) with (104) and (119), at each $t_k \leq \overline{\tau}_K$,

$$\begin{aligned}\tilde{W}(t_k) &= \frac{\alpha_2}{\theta\alpha_1}e^{\mu t_k}W(t_k)\\ &\leq \frac{\alpha_2}{\theta\alpha_1}e^{\mu t_k}\\ &\quad\cdot\big[\beta_0 U(t_k^-) + \beta_1 U(t_{k-1}) + \bar{\beta}_0 W(t_k^-) + \bar{\beta}_1 W(t_{k-1})\big]\\ &\leq \frac{\alpha_2}{\theta\alpha_1}\beta_0\tilde{U}(t_k^-) + \bar{\beta}_0\tilde{W}(t_k^-)\\ &\quad + \Big[\frac{\alpha_2}{\theta\alpha_1}\beta_1\tilde{U}(t_{k-1}) + \bar{\beta}_1\tilde{W}(t_{k-1})\Big]e^{\mu\overline{\Delta t}}\\ &\leq \Big[\frac{\alpha_2}{\theta\alpha_1}\beta_0 + \bar{\beta}_0 + \Big(\frac{\alpha_2}{\theta\alpha_1}\beta_1 + \bar{\beta}_1\Big)e^{\mu\overline{\Delta t}}\Big]K\\ &< qK < K,\end{aligned} \quad (120)$$

which is a contradiction. So $t_k \notin T_+$ if $\overline{\tau}_K = t_k$.

If $\overline{\tau}_K = t_k \in T_-$, then there are two cases: $t_k^- \in T_-$ with $\overline{\tau}_K = t_k \in T_-$ and $t_k^- \in T_+$ with $\overline{\tau}_K = t_k \in T_-$.

Recall that $U(t)$ and hence $\tilde{U}(t)$ are continuous on $[t_0, \infty)$. If $t_k^- \in T_-$ with $\overline{\tau}_K = t_k \in T_-$, then there is a $\tau_1^- = \tau_1^-(t_k) < t_k$ in (117) such that $[\tau_1^-, t_k] \subset T_-$. By (115), $\tilde{U}(\tau_1^-) \geq \tilde{U}(t_k)e^{\varepsilon(t_k-\tau_1^-)}$. This with $\overline{\tau}_K = t_k$ produces

$$\tilde{U}(\tau_1^-) \geq \tilde{U}(t_k)e^{\varepsilon(t_k-\tau_1^-)} \geq Ke^{\varepsilon(t_k-\tau_1^-)} > K.$$

But $\overline{\tau}_K = t_k > \tau_1^-$ means that $\tilde{U}(\tau_1^-) < K$, which leads to a contradiction. Therefore, $t_k^- \notin T_-$ if $\overline{\tau}_K = t_k \in T_-$.

If $t_k^- \in T_+$ with $\overline{\tau}_K = t_k \in T_-$, then, due to the fact that $\tilde{U}(t)$ is continuous $[t_0, \infty)$,

$$\overline{W}(t_k^-) = \tilde{W}(t_k^-) > \tilde{U}(t_k^-) = \tilde{U}(t_k) \geq K. \quad (121)$$

That $t_k^- \in T_+$ implies that there is a $\tau < t_k$ so close to t_k that $\tau \in T_+$ and, hence, $\tau_1^+ = \tau_1^+(\tau) < \tau_2^+ = \tau_2^+(\tau) = t_k$ in (116) with $\tau \in (\tau_1^+, t_k) \subset T_+$. Recall that $\tilde{W}(t)$ and $\overline{W}(t)$ are continuous on (t_{k-1}, t_k). By (121), one can find some $\tau \in (\tau_1^+, t_k) \subset T_+$ sufficiently close to t_k such that $\overline{W}(\tau) = \tilde{W}(\tau) > U(t_k) \geq K$. But this is in contradiction with $\overline{\tau}_K = t_k > \tau$. Hence $t_k^- \notin T_+$ if $\overline{\tau}_K = t_k \in T_-$.

So $\overline{\tau}_K = t_k$ cannot be true. Let us proceed to check whether $t_{k-1} < \overline{\tau}_K < t_k$ could be true or not. Recall that both $\tilde{U}(t)$ and $\tilde{W}(t)$ are continuous on (t_{k-1}, t_k), which means that both $\overline{W}(t)$ and $\bar{v}(t)$ are continuous on (t_{k-1}, t_k). If $t_{k-1} < \overline{\tau}_K < t_k$, then there are two cases: **c1)** $\bar{v}(\overline{\tau}_K) < 0$, viz., $\overline{W}(\overline{\tau}_K) = \tilde{U}(\overline{\tau}_K) \geq K$ and **c2)** $\bar{v}(\overline{\tau}_K) \geq 0$, viz., $\overline{W}(\overline{\tau}_K) = \tilde{W}(\overline{\tau}_K) \geq K$ including the special case $\bar{v}(\overline{\tau}_K) = 0$, namely, $\overline{W}(\overline{\tau}_K) = \tilde{W}(\overline{\tau}_K) = \tilde{U}(\overline{\tau}_K) \geq K$.

c1) Due to the continuity of $\bar{v}(t)$ on (t_{k-1}, t_k) as well as (117), that $\bar{v}(\overline{\tau}_K) < 0$ implies that $\overline{\tau}_K \in T_-$ with $\tau_1^-(\overline{\tau}_K) < \overline{\tau}_K < \tau_2^-(\overline{\tau}_K)$ and hence, by $t_{k-1} < \overline{\tau}_K < t_k$, there is a $\tau \in (t_{k-1} \vee \tau_1^-(\overline{\tau}_K), \overline{\tau}_K)$ such that $[\tau, \overline{\tau}_K] \subset T_-$ and therefore (115) holds on $[\tau, \overline{\tau}_K]$. But this yields

$$\tilde{U}(\tau) \geq \tilde{U}(\overline{\tau}_K)e^{\varepsilon(\overline{\tau}_K-\tau)} > \tilde{U}(\overline{\tau}_K) \geq K,$$

while $\overline{\tau}_K > \tau$ gives $\tilde{U}(\tau) < K$. The contradiction means that $\bar{v}(\overline{\tau}_K) < 0$, or say, $\overline{W}(\overline{\tau}_K) = \tilde{U}(\overline{\tau}_K) \geq K > \tilde{W}(\overline{\tau}_K)$ cannot be true if $t_{k-1} < \overline{\tau}_K < t_k$.

c2) Note that $\tilde{W}(t_{k-1}) < qK$ due to (120). Define

$$\tilde{v}(t) = \tilde{W}(t) - q\tilde{U}(t) \quad \forall\, t \in [0, \infty) \quad (122)$$

with $q \in (0, 1)$ given by (12). Similarly, $\tilde{v}(t)$ is continuous on (t_{k-1}, t_k) for all $k \in \mathbb{N}$ and the interval $[0, \infty)$ is broken into a disjoint union of subsets $\widetilde{T}_+ \cup \widetilde{T}_-$, where

$$\widetilde{T}_+ = \{t \geq 0 : \tilde{v}(t) > 0\},\ \widetilde{T}_- = \{t \geq 0 : \tilde{v}(t) \leq 0\}.$$

From (112), (113) and (122), it is observed that $T_+ \subset \widetilde{T}_+$, $\widetilde{T}_- \subset T_-$ and, therefore, (115) holds on $\widetilde{T}_- \subset T_-$. Notice that $t_{k-1} < \overline{\tau}_K < t_k$ and $\bar{v}(\overline{\tau}_K) \geq 0$ (namely, $\overline{W}(\overline{\tau}_K) = \tilde{W}(\overline{\tau}_K) \geq K$) imply that $\widetilde{v}(\overline{\tau}_K) = \tilde{W}(\overline{\tau}_K) - q\tilde{U}(\overline{\tau}_K) \geq \bar{v}(\overline{\tau}_K) = \tilde{W}(\overline{\tau}_K) - \tilde{U}(\overline{\tau}_K) \geq 0$ and, hence, $\overline{\tau}_K \in T_+ \subset \widetilde{T}_+$. As in (116), there is an ordered pair $\widetilde{\tau}_1^+ = \widetilde{\tau}_1^+(\overline{\tau}_K) < \widetilde{\tau}_2^+ = \widetilde{\tau}_2^+(\overline{\tau}_K)$ such that $\overline{\tau}_K \in (\widetilde{\tau}_1^+, \widetilde{\tau}_2^+) \subset \widetilde{T}_+$. There are also two cases: i) $\widetilde{\tau}_1^+ \leq t_{k-1}$ and ii) $\widetilde{\tau}_1^+ > t_{k-1}$.

i) That $\widetilde{\tau}_1^+ \leq t_{k-1}$ means $[t_{k-1}, t_k \wedge \widetilde{\tau}_2^+) \subset \widetilde{T}_+$. Recall that, by (120), $\tilde{W}(t_{k-1}) < qK$.

ii) That $\widetilde{\tau}_1^+ > t_{k-1}$ implies $\widetilde{v}(\widetilde{\tau}_1^+) = 0$ due to the continuity of $\widetilde{v}(t)$ on (t_{k-1}, t_k). Therefore, $\tilde{W}(\widetilde{\tau}_1^+) = q\tilde{U}(\widetilde{\tau}_1^+) < qK$ since $\tilde{U}(t) < K$ for all $t < \overline{\tau}_K$.

Let $\widetilde{\tau} = t_{k-1} \vee \widetilde{\tau}_1^+$, then $\tilde{W}(\widetilde{\tau}) < qK$ and $\tilde{U}(t) \leq \tilde{W}(t)/q$ on $[\widetilde{\tau}, t_k \wedge \widetilde{\tau}_2^+) \subset \widetilde{T}_+$ It immediately follows from (107), (104) and the Gronwall inequality that

$$\begin{aligned}
\tilde{W}(t) &\leq \tilde{W}(\widetilde{\tau}) + \int_{\widetilde{\tau}}^{t} \big[\frac{\alpha_2\bar{\alpha}_1}{\theta\alpha_1}\tilde{U}(s) + (\bar{\alpha}_2 + \mu)\tilde{W}(s)\big]\mathrm{d}s \\
&\leq \tilde{W}(\widetilde{\tau}) + \int_{\widetilde{\tau}}^{t} \big(\frac{\alpha_2\bar{\alpha}_1}{\theta\alpha_1 q} + \bar{\alpha}_2 + \mu\big)\tilde{W}(s)\mathrm{d}s \\
&\leq \tilde{W}(\widetilde{\tau})e^{[(\theta\alpha_1 q)^{-1}\alpha_2\bar{\alpha}_1 + \bar{\alpha}_2 + \mu](t-\widetilde{\tau})} \\
&< qKe^{[(\theta\alpha_1 q)^{-1}\alpha_2\bar{\alpha}_1 + \bar{\alpha}_2 + \mu](t_k - t_{k-1})} \\
&\leq qKe^{[(\theta\alpha_1 q)^{-1}\alpha_2\bar{\alpha}_1 + \bar{\alpha}_2 + \mu]\overline{\Delta t}} < K
\end{aligned}$$

for all $t \in (\widetilde{\tau}, t_k \wedge \widetilde{\tau}_2^+)$, which is in contradiction with $\bar{v}(\overline{\tau}_K) \geq 0$ for $t_{k-1} < \overline{\tau}_K < t_k$.

Therefore, neither $\overline{\tau}_K = t_k$ nor $t_{k-1} < \overline{\tau}_K < t_k$ could be true for any $k \in \mathbb{N}$. So $\overline{\tau}_K > t_k$ for all $k \in \mathbb{N}$ and, hence, (110) holds for all $t \geq 0$. By condition (8b), this implies that

$$\mathbb{E}|z(t)|^p \leq \frac{\theta\alpha_1}{\alpha_2\bar{c}_1}Ke^{-\mu t}$$

for all $t \geq 0$, where $\mu > 0$ and $K > 0$ are given by (104) and (111), respectively. This means that SiDE (2) is pth moment exponentially stable (with Lyapunov exponent no larger than $-\mu$ and $\mu > 0$ given by (104)). The proof is complete. □

References

[1] R. Alur, *Principles of Cyber-Physical Systems*. Cambridge, MA, USA: MIT Press, 2015.
[2] K. J. Åström and B. Wittenmark, *Computer-Controlled Systems: Theory Design*, 3rd ed. Hoboken, NJ, USA: Prentice-Hall, 1997.
[3] K. J. Åström and P. R. Kumar, "Control: A perspective," *Automatica*, vol. 50, no. 1, pp. 3–43, Jan. 2014.
[4] M. Benaim, "A dynamical system approach to stochastic approximations," *SIAM J. Control Optim.*, vol. 34, no. 2, pp. 437–472, Mar. 1996.
[5] M. Benosman, "Model-based vs data-driven adaptive control: An overview," *Int. J. Adapt. Control Signal Process.*, vol. 32, no. 5, pp. 753–776, May 2018.
[6] A. Benveniste, M. Métivier, and P. Priouret, *Adaptive Algorithms Stochastic Approximations*. Berlin, Germany: Spring-Verlag, 1990.
[7] V. S. Borkar, *Stochastic Approximation: A Dynamical System's Viewpoint*. Cambridge, U.K.: Cambridge Univ. Press, 2008.
[8] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*. Philadelphia, PA, USA: Soc. Ind. Appl. Math., 1994.
[9] K. E. Brenan, S. L. Campbell, and L. R. Petzold, *Numerical Solution of Initial-Value Problems in Differentialalgebraic Equations*. Philadelphia, PA, USA: SIAM, 1996.
[10] Control Panel members. (2022). *Control in an Information Rich World: Report of the Panel on Future Directions in Control, Dynamics, and Systems*. Accessed: Apr. 11, 2022. [Online]. Available: https://www.cds.caltech.edu/murray/cdspanel/report/latest.pdf
[11] CPS Summit Rep. members. (2022). *Report: Cyber-physical Systems Summit*. Accessed: Apr. 11, 2022. [Online]. Available: https://iccps2012.cse.wustl.edu/_doc/CPS_summit_report.pdf
[12] C. Fei, W. Fei, X. Mao, D. Xia, and L. Yan, "Stabilisation of highly nonlinear hybrid systems by feedback control based on discrete-time state observations," *IEEE Trans. Autom. Control*, vol. 65, no. 7, pp. 2899–2912, 2020.
[13] E. Fridman, A. Seuret, and J. P. Richard, "Robust sampled-data stabilization of linear systems: An input delay approach," *Automatica*, vol. 40, no. 8, pp. 1441–1446, 2004.
[14] E. Fridman, M. Dambrine, and N. Yeganefar, "On input-to-state stability of systems with time-delay: A matrix inequalities approach," *Automatica*, vol. 44, no. 9, pp. 2364–2369, Sep. 2008.
[15] E. Fridman, "Tutorial on Lyapunov-based methods for time-delay systems," *Eur. J. Control*, vol. 20, no. 6, pp. 271–283, 2014.
[16] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox*. Natick, MA, USA: The MathWorks Inc, 1995.
[17] *Global Optim. Toolbox User's Guide*. Natick, MA, USA: MathWorks Inc, 2011.
[18] J. C. Geromel, "Sampled-data model predictive control," *IEEE Trans. Autom. Control*, vol. 67, no. 5, pp. 2466–2472, May 2022.
[19] J. W. Grizzle, J.-H. Choi, H. Hammouri, and B. Morris, "On observer-based feedback stabilization of periodic orbits in bipedal locomotion," in *Proc. Methods Models Automat. Robot. (MMAR)*, Szczecin, Poland, Aug. 2007. Accessed: Feb. 8, 2024. [Online]. Available: https://web.eecs.umich.edu/~grizzle/papers/MMAR_Grizzle_2007.pdf
[20] W. P. M. H. Heemels, K. H. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," in *Proc. IEEE 51st IEEE Conf. Decis. Control (CDC)*, Hawaii, USA, Dec. 2012, pp. 3270–3285.
[21] J. P. Hespanha, D. Liberzon, and A. R. Teel, "Lyapunov conditions for input-to-state stability of impulsive systems," *Automatica*, vol. 44, no. 11, pp. 2735–2744, Nov. 2008.
[22] D. J. Higham, "Mean-square and asymptotic stability of the stochastic theta method," *SIAM J. Numer. Anal.*, vol. 38, no. 3, pp. 753–769, Jan. 2000.
[23] Q. Hou and J. Dong, "Robust adaptive event-triggered fault-tolerant consensus control of multiagent systems with a positive minimum interevent time," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 53, no. 7, pp. 4003–4014, 2023.
[24] L. Huang and X. Mao, "On input-to-state stability of stochastic retarded systems with Markovian switching," *IEEE Trans. Autom. Control*, vol. 54, no. 8, pp. 1898–1902, Aug. 2009.
[25] L. Huang, "Stability and stabilisation of stochastic delay systems," Ph.D. thesis, Dept. Math. Statist., Univ. Strathclyde, Glasgow, U.K., 2010.
[26] L. Huang and X. Mao, "Stability of singular stochastic systems with Markovian switching," *IEEE Trans. Autom. Control*, vol. 56, no. 2, pp. 424–429, Feb. 2011.
[27] L. Huang and H. Hjalmarsson, "Recursive estimators with Markovian jumps," *Syst. Control Lett.*, vol. 61, no. 10, pp. 1009–1016, Oct. 2012.
[28] L. Huang, H. Hjalmarsson, and H. Koeppl, "Almost sure stability and stabilization of discrete-time stochastic systems," *Syst. Control Lett.*, vol. 82, pp. 26–32, Aug. 2015.
[29] L. Huang, L. Pauleve, C. Zechner, M. Unger, A. S. Hansen, and H. Koeppl, "Reconstructing dynamic molecular states from single-cell time series," *J. Roy. Soc. Interface*, vol. 13, no. 122, Sep. 2016, Art. no. 20160533.
[30] L. Huang, "Stability of cyber-physical systems of numerical methods for stochastic differential equations: Integrating the cyber and the physical of stochastic systems," *IEEE Access*, vol. 10, pp. 99479–99497, 2022.
[31] L. Huang and S. Xu, "Impulsive stabilization of systems with control delay," *IEEE Trans. Autom. Control*, vol. 68, no. 3, pp. 1784–1791, Mar. 2023.
[32] T. Kailath, *Linear System*. Hoboken, NJ, USA: Prentice-Hall, 1980.
[33] H. K. Khalil, *Nonlinear System*, 3rd ed. Hoboken, NJ, USA: Prentice-Hall, 2002.

[34] R. Khasminskii, *Stochastic Stability of Differential Equations*, 2nd ed. Berlin, Gernany: Springer-Verlag, 2012.

[35] X. Koutsoukos and P. Antsaklis, "Design science for cyberphysical systems," in *The Impact of Control Technology*, 2nd ed., T. Samad and A. M. Annaswamym, Eds. Gainesville, FL, USA: IEEE Control Systems Society, 2014. Accessed: Apr. 11, 2022. [Online]. Available: https://ieeecss.org/sites/ieeecss/files

[36] H. J. Kushner and G. Yin, *Stochastic Approximation Algorithms Applications*, 2nd ed. Cham, Switzerland: Springer, 2003.

[37] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs: A Course on Backstepping Designs*. Philadelphia, PA, USA: Soc. Ind. Appl. Math., 2008.

[38] E. A. Lee, "CPS foundations," in *Proc. 47th Design Autom. Conf.*, Jun. 2010, pp. 1–6.

[39] Y. Liu and Z. Song, *Theory Application Large-Scale Dynamic Systems: Decomposition, Stability Structure*, vol. 1. Guangzhou, China: South China Univ. Technol. Press, 1988.

[40] Y. Liu and Z. Feng, *Theory and Application of Large-Scale Dynamic Systems: Stochastic Stability Control*, vol. 4. Guangzhou, China: South China Univ. Technol. Press, 1992.

[41] L. Ljung, "Analysis of recursive stochastic algorithms," *IEEE Trans. Autom. Control*, vol. AC-22, no. 4, pp. 551–575, Aug. 1977.

[42] L. Ljung and T Söderström, *Theory and Practice of Recursive Identification*. Cambridge, MA, USA: MIT Press, 1983.

[43] F. M. Malik, M. B. Malik, and K. Munawar, "Sampled-data state feedback stabilization of a class of nonlinear systems based on Euler approximation," *Asian J. Control*, vol. 13, no. 1, pp. 186–197, Jan. 2011.

[44] X. Mao, *Stochastic Differential Equations and Applications* Chichester, U.K.: Horwood Publishing, 2007.

[45] X. Mao, "Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control," *Automatica*, vol. 49, no. 12, pp. 3677–3681, 2013.

[46] X. Mao, W. Liu, L. Hu, Q. Luo, and J. Lu, "Stabilization of hybrid stochastic differential equations by feedback control based on discrete-time state observations," *Syst. Control Lett.*, vol. 73, pp. 88–95, Nov. 2014.

[47] J. C. Maxwell, "On governors," *Proc. Roy. Soc. London*, vol. 31, no. 16, pp. 270–283, Dec. 1868.

[48] P. Naghshtabrizi, J. P. Hespanha, and A. R. Teel, "Exponential stability of impulsive systems with application to uncertain sampled-data systems," *Syst. Control Lett.*, vol. 57, no. 5, pp. 378–385, May 2008.

[49] D. Nešić and A. R. Teel, "Sampled-data control of nonlinear systems: An overview of recent results," in *Perspectives in Robust Control and Information Sciences*, vol. 268, S. R. Moheimani, Ed. Cham, Switzerland: Springer, 2001, pp. 221–239.

[50] D. Nesic and A. R. Teel, "A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1103–1122, Jul. 2004.

[51] D. Nešić and A. R. Teel, "Stabilization of sampled-data nonlinear systems via backstepping on their Euler approximate model," *Automatica*, vol. 42, no. 10, pp. 1801–1808, Oct. 2006.

[52] T. Nghiem, G. J. Pappas, R. Alur, and A. Girard, "Time-triggered implementation of dynamic controllers," *ACM Trans. Embed. Comput. Syst.*, vol. 11, no. S2, pp. 1–24, Aug. 2012.

[53] R. Nikoukhah, "A new methodology for observer design and implementation," *IEEE Trans. Autom. Control*, vol. 43, no. 2, pp. 229–234, Feb. 1998.

[54] Y. Oishi and H. Fujioka, "Stability and stabilization of aperiodic sampled-data control systems using robust linear matrix inequalities," *Automatica*, vol. 46, no. 8, pp. 1327–1333, Aug. 2010.

[55] C. Qian and W. Lin, "Output feedback control of a class of nonlinear systems: A nonseparation principle paradigm," *IEEE Trans. Autom. Control*, vol. 47, no. 10, pp. 1710–1715, Oct. 2002.

[56] L. Qin, X. He, and R. Yan, "Active fault-tolerant control for a quadrotor with sensor faults," *J. Intell. Robot. Syst.*, vol. 88, no. 2, pp. 449–467, Jan. 2017.

[57] H. Robbins and S. Monro, "A stochastic approximation method," *Ann. Math. Statist.*, vol. 22, no. 3, pp. 400–407, Sep. 1951.

[58] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*. Singapore: World Scientific, 1995.

[59] A. Sardashti and J. Nazari, "A learning-based approach to fault detection and fault-tolerant control of permanent magnet DC motors," *J. Eng. Appl. Sci.*, vol. 70, no. 1, p. 109, Dec. 2023.

[60] F. Scotton, L. Huang, S. A. Ahmadi, and B. Wahlberg, "Physics-based modeling and identification for HVAC systems?" in *Proc. Eur. Control Conf. (ECC)*, Zurich, Switzerland, Jul. 2013, pp. 1404–1409.

[61] R. Seifullaev, A. Fradkov, and E. Fridman, "Event-triggered sampled-data energy control of a pendulum," *IFAC PapersOnLine*, vols. 1–50, no. 1, pp. 15295–15300, Jul. 2017.

[62] A. Seuret, "A novel stability analysis of linear systems under asynchronous samplings," *Automatica*, vol. 48, no. 1, pp. 177–182, Jan. 2012.

[63] N. Sivashankar and P. P. Khargonekar, "Characterization of the $\mathcal{L}_2$-induced norm for linear systems with jumps with applications to sampled-data systems," *SIAM J. Control Optim.*, vol. 32, no. 4, pp. 1128–1150, Jul. 1994.

[64] K. C. Sou, H. Sandberg, and K. H. Johansson, "On the exact solution to a smart grid cyber-security analysis problem," *IEEE Trans. Smart Grid*, vol. 4, no. 2, pp. 856–865, Jun. 2013.

[65] G. Stein, "Respect the unstable," *IEEE Contr. Syst. Mag.*, vol. 23, no. 4, pp. 12–25, Aug. 2003.

[66] A. R. Teel, A. Subbaraman, and A. Sferlazza, "Stability analysis for stochastic hybrid systems: A survey," *Automatica*, vol. 50, no. 10, pp. 2435–2456, Oct. 2014.

[67] A. M. H. Teixeira, J. Araújo, H. Sandberg, and K. H. Johansson, "Distributed sensor and actuator reconfiguration for fault-tolerant networked control systems," *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 4, pp. 1517–1528, Dec. 2018.

[68] J. Tsinias, "A theorem on global stabilization of nonlinear systems by linear feedback," *Syst. Control Lett.*, vol. 17, no. 5, pp. 357–362, Nov. 1991.

[69] M. S. Valle, A. Casabona, R. Sgarlata, R. Garozzo, M. Vinci, and M. Cioni, "The pendulum test as a tool to evaluate passive knee stiffness and viscosity of patients with rheumatoid arthritis," *BMC Musculoskeletal Disorders*, vol. 7, no. 1, pp. 1–12, Dec. 2006.

[70] B. Wang, Y. Shen, and Y. Zhang, "Active fault-tolerant control for a quadrotor helicopter against actuator faults and model uncertainties," *Aerosp. Sci. Technol.*, vol. 99, Apr. 2020, Art. no. 105745.

[71] J. Wang, Q. Xue, L. Li, B. Liu, L. Huang, and Y. Chen, "Dynamic analysis of simple pendulum model under variable damping," *Alexandria Eng. J.*, vol. 61, no. 12, pp. 10563–10575, Dec. 2022.

[72] N. Wiener, *Cybernetics or Control and Communication in the Animal and the Machine*, 2nd ed. Cambridge, MA, USA: MIT Press, 1961.

[73] B. Xiao, H. K. Lam, X. Su, Z. Wang, F. P. W. Lo, and S. Chen, "Sampled-data control through model-free reinforcement learning with effective experience replay," *J. Autom. Intell.*, vol. 2, no. 1, pp. 20–30, Feb. 2023.

[74] T. Yang, *Impulsive Control Theory*. Berlin, Gernany: Springer-Verlag, 2001.

[75] S. You, W. Liu, J. Lu, X. Mao, and Q. Qiu, "Stabilization of hybrid systems by feedback control based on discrete-time state observations," *SIAM J. Control Optim.*, vol. 53, no. 2, pp. 905–925, 2015.

[76] J. I. Yuz and G. C. Goodwin, "Sampled-data models for stochastic nonlinear systems," in *Proc. 14th IFAC Symp. Syst. Identificat.*, Jan. 2006, vol. 39, no. 1, pp. 434–439.

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