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RESEARCH ARTICLE

Honeycomb Rhombic Torus Vertex-Edge Based Resolvability Parameters and Its Application in Robot Navigation

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ABSTRACT In the aircraft sector, honeycomb composite materials are frequently employed. Recent research has demonstrated the benefits of honeycomb structures in applications involving nanohole arrays in anodized alumina, micro-porous arrays in polymer thin films, activated carbon honeycombs, and photonic band gap honeycomb structures. The resolvability parameter is the area of graph theory that is most commonly explored. This results in an original network reconfiguration. Occasionally in terms of atoms (metric dimension), and sometimes in terms of bounds (edge metric dimension). In this article, we examined the honeycomb rhombic torus's metric, edge metric, mixed metric, and partition dimension. The application of edge metric dimension is also discussed in it.

INDEX TERMS Edge metric dimension, honeycomb rhombic torus, metric dimension, mixed metric dimension, partition dimension, resolving set.

I. INTRODUCTION

Chemical graph theory examines all facets of how graph theory is applied to many branches of chemistry, such as computational, theoretical, and organic chemistry, as well as bioinformatics and computational biology. By referring to chemical graph theory as opposed to graph theory, it is highlighted that one is permitted to rely on intuitive knowledge of many ideas and theorems rather than on rigorous mathematical proofs. A molecule is turned into a molecular graph by transforming bonds into edges and vertices [1], [2]. The spatial configuration of atoms in a molecule is known as chemical structure. The molecular geometry of the molecule is determined by its chemical structure. A molecule's chemical structure is the physical configuration of its atoms and chemical connections. Its determination involves a chemist identifying the target molecule or other solid's molecular geometry and, when possible and essential, its electronic structure [3].

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The physical characteristics of a chemical structure have a specific mathematical shape. Each vertex or atom has a unique location or identity in the structure. In applied graph theory and metric basis in graph theory, a minimal cardinal set is selected from the vertex set. If that set occupies a singular place with the vertex set, it is referred to as a location or resolving set [4], [5].

P. J. Slater was the first to propose the concept of metric dimension in graph theory. It has found use in a variety of fields, including pattern recognition, image processing, network theory, and optimization. Many other authors have investigated the metric dimension of different types of common graphs [6].

We are talking about the NP-hardness and computational complexity resolvability parameters [7], [8], [9], [10]. The numerous practical uses of the metric dimension in daily life are investigated extensively, which motivates researchers. Computer networks [11], Sonar, Coast Guard Loran [5], combinatorial optimization [12], facility locating issues, robot navigation [13], image processing [14], [15], weighing problem [16] and pharmaceutical chemistry [8] are only a few examples of the many scientific disciplines. There are numerous real-world uses for partition dimensions, such as mastermind game cryptography [17], network discovery [18] and validation techniques, and the Djokovic-Winkler [19] resolvability parameters' association with computational complexity, to search more about it, see [20], [21], [22], [23], and [24].

This parameter and its associated variations are therefore investigated for diverse chemical networks because the metric dimension is employed in many fields, but mainly in chemistry. In this study [25], many polycyclic aromatic compound resolvability parameters, including metric, edge metric, fault metric, edge fault metric, and partition dimension, are examined, they demonstrated that none of the aforementioned parameters vary with the structure's chosen order and are all constant. In this article [26] discusses the bounded partition dimension of several convex polytope graph types. The hexagonal Möbius ladder network's partition dimension and partition resolving sets are explored in [27]. Resolving sets of Silicate stars are discussed in [28]. The partition and bounded partition dimension of $n-3$ graphs and $(4, 6)$ fullerene is proved in [29]. Upper bounds of the metric dimension of cellulose networks discussed in [30]. For farther discussion on nano-structures, see [31]. The metric and edge metric dimension of the patched network is parametric, discussed in [32]. The cocktail party and jellyfish graph are explained and their resolvability parameters are explored in [33]. Metric dimensions of nanotubes VC_5C_7 and H-naphthalene are discussed in [34]. The n-sunlit and prism graphs are explained in [35], they have constant edge metric dimensions. Computation of metric and edge metric dimension of different types of windmill graph in [36]. Some graphs having less edge metric dimension than the metric dimension are computed in [37]. The partition dimension and resolving partition set are computed in [38]. The upper and lower bound of the generalized Mobius ladder is computed in [39]. The metric basis of alpha-boron nanotubes and sheets is discussed in [40]. More results about metric dimensions are in [41], [42], [43], [44], [45], and [46]. The research's methodology is described in the section below.

Definition 1.1 [47]: If R has a unique representation for each vertex in $V(G)$, then taking a R is a resolving set. The metric dimension, $dim(G)$, stands for the number of elements in R.

Definition 1.2 [47]: If R_e has a unique representation for each edge that belongs to $E(G)$, then R_e is a edge resolving set. Edge metric dimension, or $dim_e(G)$, is the total number of elements in R_e.

Definition 1.3 [47]: If R_p has a unique representation for each vertex in $V(G)$, then that set is a resolving set.The partition dimension, indicated by $pd(G)$, is the number of elements in R_p.

Definition 1.4 [47]: If R_m has a unique representation for each vertex in $V(G)$ and for each edge $E(G)$, then that set is a mixed resolving set.The mixed metric dimension, indicated by $dim_m(G)$, is the number of elements in R_m.

In the Section II, the construction of the honeycomb rhombic torus are given and also the results are presented. In the Section III, application of the topic is described. In Section IV, conclusion is drawn along with few conjectures are also presented. At the end references are stated.

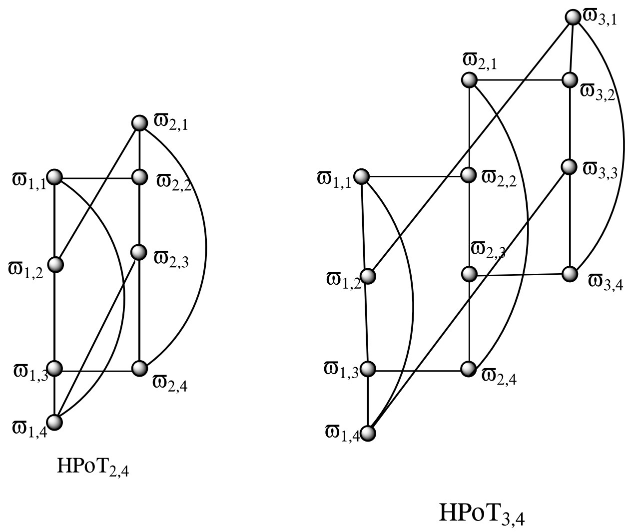

FIGURE 1. Honeycomb rhombic torus, $HRoT_{2,4}$ and $HRoT_{3,4}$.

II. CONSTRUCTION OF THE HONEYCOMB RHOMBIC TORUS

One of the key performance determinants of the corresponding parallel computer is the efficiency of an interconnection network. This has led to the design and analysis of interconnection networks becoming a primary focus of parallel computing research. Because honeycomb-type torus networks have fewer nodes than torus networks, they are more appealing as alternatives because they need less effort and money to implement. The advantages of honeycomb structures in applications include photonic band gap honeycomb structures, micro-porous arrays in polymer thin films, activated carbon honeycombs, and nanohole arrays in anodized alumina. In the Honeycomb rhombic torus, all vertices have degree 3. We turned this particular network into a graph by making some assumptions about transformations, we use the notation $HRoT_{m,n}$ to depict the graph. This graph has 8 vertices and 12 edges for the first considered graph which is $m = n = 4$, shown in Figure 1.

In general, the count of vertices is $|V(HRoT_{m,n})| = mn$, and count of edges is $|E(HRoT_{m,n})| = \frac{3mn}{2}$. Figure 2 shows the labeling that was applied to the vertices for our results. Figure 2 also provides two parameters, m, and n, to help you expand this network. With the use of the vertex and edge sets shown in Figure 2, general labeling can be applied.

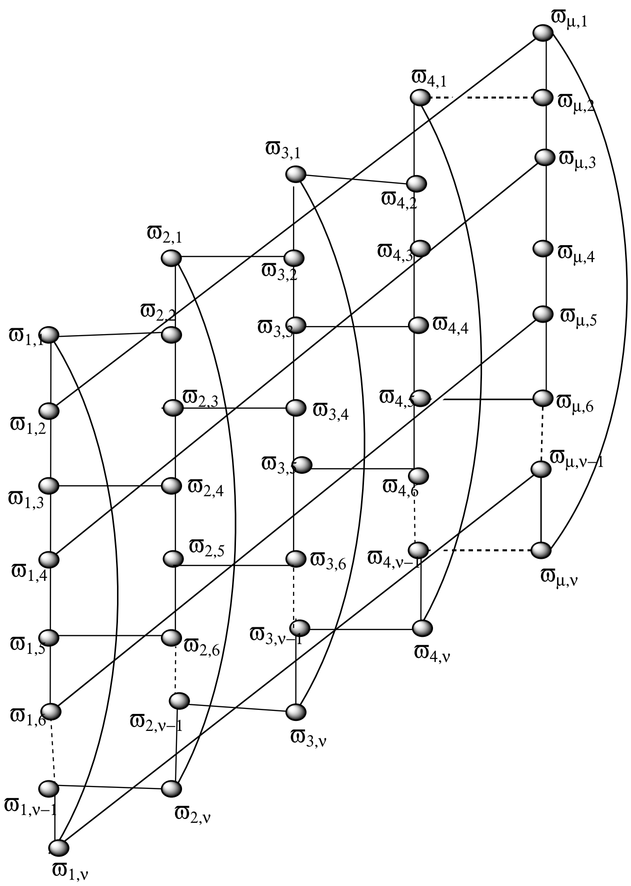

FIGURE 2. General structure of honeycomb rhombic torus.

Theorem 2.1: Let $HRoT_{m,n}$ be a structure of honeycomb rhombic torus with $m \geq 4$ and $n \geq 4$. Then $dim\left(HRoT_{m,n}\right) = m$.

Proof: We prove that the metric dimension of the honeycomb rhombic torus is m, by the following cases.

Case 1: If R, be the subset of the $V(HRoT_{m,n})$ where $R = \{\varpi_{1,1}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}\}$, then $|R| \leq m-1$, but $r(\varpi_{1,i}|\{R\}) = r(\varpi_{1,2i}|\{R\})$, where $i = 2, 4$; $r(\varpi_{j+1,2}|\{R\}) = r(\varpi_{j,4}|\{R\})$ where $j = 1, 3$ and $r(\varpi_{4,k}|\{R\}) = r(\varpi_{4,k+2}|\{R\})$, where $k = 1, 3$. Here R is not fulfilling the condition of a resolving set. So R is not the possible resolving set.

Case 2: If R, be the subset of the $V(HRoT_{m,n})$ where $R = \{\varpi_{1,2}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}\}$, then $|R| \leq m-1$, but $r(\varpi_{i,j}|\{R\}) = r(\varpi_{i,2j}|\{R\})$, where $i = 1, 2$ and $j = 2, 4$; and $r(\varpi_{5,k}|\{R\}) = r(\varpi_{5,k+2}|\{R\})$, where $k = 1, 3$. Here R is not fulfilling the condition of a resolving set. So R is not the possible resolving set.

Case 3: If R, be the subset of the $V(HRoT_{m,n})$ where $R = \{\varpi_{1,1}, \varpi_{1,2}, \varpi_{1,3}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}, \varpi_{m,1}\}$, then $|R| \leq m+2$. We demonstrate that R is the structure's resolving set. Without losing generality, we can suppose that every vertex in $V(HRoT_{m,n}) \setminus R$ having values, which is not repeated to any other vertex in regard with R. So R is a resolving set of $HRoT_{m,n}$.

Case 4: If R, be the subset of the $V(HRoT_{m,n})$ where $R = \{\varpi_{1,1}, \varpi_{1,2}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}, \varpi_{m,1}\}$, then $|R| \leq m+1$. We demonstrate that R is the structure's resolving set. Without losing generality, we can suppose that every vertex in $V(HRoT_{m,n}) \setminus R$ having values, which is not repeated to any other vertex in regard with R. So R is a resolving set of $HRoT_{m,n}$.

Case 5: If R, be the subset of the $V(HRoT_{m,n})$ where $R = \{\varpi_{1,1}, \varpi_{1,2}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}\}$, then $|R| \leq m$. We demonstrate that R is the structure's resolving set. We demonstrate that R is the structure's resolving set. Without losing generality, we can suppose that every vertex in $V(HRoT_{m,n}) \setminus R$ having values, which is not repeated to any other vertex in regard with R. So R is a resolving set of $HRoT_{m,n}$.

Hence, the outcome is

$$dim\left(HRoT_{m,n}\right) = m. \tag{1}$$

□

Theorem 2.2: Let $HRoT_{m,n}$ be a structure of honeycomb rhombic torus with $m \geq 4$ and $n \geq 4$. Then $dim_e\left(HRoT_{m,n}\right) = m$.

Proof: We prove that the metric dimension of the honeycomb rhombic torus is m, by the following cases.

Case 1: If R_e, be the subset of the $V(HRoT_{m,n})$ where $R_e = \{\varpi_{1,1}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}\}$, then $|R_e| \leq m-1$, but $r(\varpi_{1,i}|\{R_e\}) = r(\varpi_{1,2i}|\{R_e\})$, where $i = 2, 4$; $r(\varpi_{j+1,2}|\{R_e\}) = r(\varpi_{j,4}|\{R_e\})$ where $j = 1, 3$ and $r(\varpi_{4,k}|\{R_e\}) = r(\varpi_{4,k+2}|\{R_e\})$, where $k = 1, 3$. Here R_e is not fulfilling the condition of a resolving set. So R_e is not the possible resolving set.

Case 2: If R_e, be the subset of the $V(HRoT_{m,n})$ where $R_e = \{\varpi_{1,2}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}\}$, then $|R_e| \leq m-1$, but $r(\varpi_{i,j}|\{R_e\}) = r(\varpi_{i,2j}|\{R_e\})$, where $i = 1, 2$ and $j = 2, 4$; and $r(\varpi_{5,k}|\{R_e\}) = r(\varpi_{5,k+2}|\{R_e\})$, where $k = 1, 3$. Here R_e is not fulfilling the condition of a resolving set. So R_e is not the possible resolving set.

Case 3: If R_e, be the subset of the $V(HRoT_{m,n})$ where $R_e = \{\varpi_{1,1}, \varpi_{1,2}, \varpi_{1,3}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}, \varpi_{m,1}\}$, then $|R_e| \leq m+2$. We demonstrate that R_e is the structure's resolving set. Without losing generality, we can suppose that every vertex in $V(HRoT_{m,n}) \setminus R_e$ having values, which is not repeated to any other vertex in regard with R_e. So R is a resolving set of $HRoT_{m,n}$.

Case 4: If R_e, be the subset of the $V(HRoT_{m,n})$ where $R_e = \{\varpi_{1,1}, \varpi_{1,2}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}, \varpi_{m,1}\}$, then $|R_e| \leq m+1$. We demonstrate that R_e is the structure's resolving set. Without losing generality, we can suppose that every vertex in $V(HRoT_{m,n}) \setminus R_e$ having values, which is not repeated to any other vertex in regard with R_e. So R_e is a resolving set of $HRoT_{m,n}$.

Case 5: If R_e, be the subset of the $V(HRoT_{m,n})$ where $R_e = \{\varpi_{1,1}, \varpi_{1,2}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}\}$, then $|R_e| \leq m$. We demonstrate that R_e is the structure's resolving set. We demonstrate that R_e is the structure's resolving set. Without losing generality, we can suppose that every vertex in $V(HRoT_{m,n}) \setminus R_e$ having values, which is not repeated to any other vertex in regard with R_e. So R_e is a resolving set of $HRoT_{m,n}$.

Hence, the outcome is

$$dim_e\left(HRoT_{m,n}\right) = m. \tag{2}$$

□

Theorem 2.3: Let $HRoT_{m,n}$ be a structure of honeycomb rhombic torus with $m \geq 4$ and $n \geq 4$. Then $dim_m\left(HRoT_{m,n}\right) = m$.

Proof: We prove that the metric dimension of the honeycomb rhombic torus is m, by the following cases.

Case 1: If R_m, be the subset of the $V(HRoT_{m,n})$ where $R_m = \{\varpi_{1,1}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}\}$, then $|R_m| \leq m-1$, but $r(\varpi_{1,i}|\{R_m\}) = r(\varpi_{1,2i}|\{R_m\})$, where $i = 2,4$; $r(\varpi_{j+1,2}|\{R_m\}) = r(\varpi_{j,4}|\{R_m\})$, where $j = 1,3$ and $r(\varpi_{4,k}|\{R_m\}) = r(\varpi_{4,k+2}|\{R_m\})$, where $k = 1,3$. Here R_m is not fulfilling the condition of a resolving set. So R_m is not the possible resolving set.

Case 2: If R_m, be the subset of the $V(HRoT_{m,n})$ where $R_m = \{\varpi_{1,2}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}\}$, then $|R_m| \leq m-1$, but $r(\varpi_{i,j}|\{R_m\}) = r(\varpi_{i,2j}|\{R_m\})$, where $i = 1,2$ and $j = 2,4$; and $r(\varpi_{5,k}|\{R_m\}) = r(\varpi_{5,k+2}|\{R_m\})$, where $k = 1,3$. Here R_m is not fulfilling the condition of a resolving set. So R_m is not the possible resolving set.

Case 3: If R_m, be the subset of the $V(HRoT_{m,n})$ where $R_m = \{\varpi_{1,1}, \varpi_{1,2}, \varpi_{1,3}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}, \varpi_{m,1}\}$, then $|R_m| \leq m+2$. We demonstrate that R_m is the structure's resolving set. Without losing generality, we can suppose that every vertex in $V(HRoT_{m,n}) \setminus R_m$ having values, which is not repeated to any other vertex in regard with R_m. So R_m is a resolving set of $HRoT_{m,n}$.

Case 4: If R_m, be the subset of the $V(HRoT_{m,n})$ where $R_m = \{\varpi_{1,1}, \varpi_{1,2}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}, \varpi_{m,1}\}$, then $|R_m| \leq m+1$. We demonstrate that R_m is the structure's resolving set. Without losing generality, we can suppose that every vertex in $V(HRoT_{m,n}) \setminus R_m$ having values, which is not repeated to any other vertex in regard with R_m. So R_m is a resolving set of $HRoT_{m,n}$.

Case 5: If R_m, be the subset of the $V(HRoT_{m,n})$ where $R_m = \{\varpi_{1,1}, \varpi_{1,2}, \varpi_{2,1}, \varpi_{3,1}, \ldots, \varpi_{m-1,1}\}$, then $|R_m| \leq m$. We demonstrate that R_m is the structure's resolving set. We demonstrate that R_m is the structure's resolving set. Without losing generality, we can suppose that every vertex in $V(HRoT_{m,n}) \setminus R_m$ having values, which is not repeated to any other vertex in regard with R_m. So R_m is a resolving set of $HRoT_{m,n}$.

Hence, the outcome is

$$dim_m\left(HRoT_{m,n}\right) = m. \tag{3}$$

□

Theorem 2.4: Let $HRoT_{m,n}$ be a structure of honeycomb rhombic torus with $m \geq 4$ and $n \geq 4$. Then $4 \leq pd\left(HRoT_{m,n}\right) \leq m+1$.

Proof: We prove that the partition dimension of the honeycomb rhombic torus is $m+1$, by the following cases.

Case 1: If R_p, be the subset of the $V(HRoT_{m,n})$, the partition resolving set is $R_p = \{R_{p_1}, R_{p_2}, R_{p_3}, \ldots R_{p_m}\}$ where $R_{p_1} = \{\varpi_{1,1}\}, R_{p_2} = \{\varpi_{2,1}\}, R_{p_1} = \{\varpi_{3,1}\}, \ldots, R_{p_m-1} = \varpi_{m-1,1}\}, R_{p_m} = \{V(HRoT_{m,n}) \setminus R_{p_1}, R_{p_2}, \ldots, R_{p_m-1}\}$ then $|R_p| \leq m$, but $r(\varpi_{1,i} \setminus \{R_p\}) = r(\varpi_{1,2i}|\{R_p\})$, where $i = 2,4$; $r(\varpi_{j+1,2}|\{R_p\}) = r(\varpi_{j,4}|\{R_p\})$ where $j = 1,3$ and $r(\varpi_{4,k}|\{R_p\}) = r(\varpi_{4,k+2}|\{R_p\})$, where $k = 1,3$. Here R_p is not fulfilling the condition of a resolving set. So R_p is not the possible resolving set.

Case 2: If R_p, be the subset of the $V(HRoT_{m,n})$, the partition resolving set is $R_p = \{R_{p_1}, R_{p_2}, R_{p_3}, \ldots R_{p_{m-1}}\}$ where $R_{p_1} = \{\varpi_{1,1}\}, R_{p_2} = \{\varpi_{2,1}\}, R_{p_1} = \{\varpi_{3,1}\}, \ldots, R_{p_{m-2}} = \varpi_{m-2,1}\}, R_{p_{m-1}} = \{V(HRoT_{m,n}) \setminus R_{p_1}, R_{p_2}, \ldots, R_{p_{m-2}}\}$ then $|R_p| \leq m-1$, then $|R_p| \leq m-1$, but $r(\varpi_{i,j}|\{R_p\}) = r(\varpi_{i,2j}|\{R_p\})$, where $i = 1,2$ and $j = 2,4$; and $r(\varpi_{5,k}|\{R_p\}) = r(\varpi_{5,k+2}|\{R_p\})$, where $k = 1,3$. Here R_p is not fulfilling the condition of a resolving set. So R_p is not the possible resolving set.

Case 3: If R_p, be the subset of the $V(HRoT_{m,n})$, the partition resolving set is $R_p = \{R_{p_1}, R_{p_2}, R_{p_3}, \ldots R_{p_m}\}$ where $R_{p_1} = \{\varpi_{1,1}\}, R_{p_2} = \{\varpi_{2,1}\}, R_{p_1} = \{\varpi_{3,1}\}, \ldots, R_{p_m-1} = \varpi_{m-1,1}\}, R_{p_m} = \{V(HRoT_{m,n}) \setminus R_{p_1}, R_{p_2}, \ldots, R_{p_m-1}\}$ then $|R_p| \leq m+2$. We demonstrate that R_p is the structure's resolving set. Without losing generality, we can suppose that every vertex in $V(HRoT_{m,n}) \setminus R_p$ having values, which is not repeated to any other vertex in regard with R_p. So R_p is a resolving set of $HRoT_{m,n}$.

Case 4: If R_p, be the subset of the $V(HRoT_{m,n})$, the partition resolving set is $R_p = \{R_{p_1}, R_{p_2}, R_{p_3}, \ldots R_{p_m}\}$ where $R_{p_1} = \{\varpi_{1,1}\}, R_{p_2} = \{\varpi_{2,1}\}, R_{p_1} = \{\varpi_{3,1}\}, \ldots, R_{p_m-1} = \varpi_{m-1,1}\}, R_{p_m} = \{V(HRoT_{m,n}) \setminus R_{p_1}, R_{p_2}, \ldots, R_{p_m-1}\}$ then $|R_p| \leq m+1$. We demonstrate that R_p is the structure's resolving set. Without losing generality, we can suppose that every vertex in $V(HRoT_{m,n}) \setminus R_p$ having values, which is not repeated to any other vertex in regard with R_p. So R_p is a resolving set of $HRoT_{m,n}$.

Case 5: If R_p, be the subset of the $V(HRoT_{m,n})$, the partition resolving set is $R_p = \{R_{p_1}, R_{p_2}, R_{p_3}, \ldots R_{p_m}\}$ where $R_{p_1} = \{\varpi_{1,1}\}, R_{p_2} = \{\varpi_{2,1}\}, R_{p_1} = \{\varpi_{3,1}\}, \ldots, R_{p_m-1} = \varpi_{m-1,1}\}, R_{p_m} = \{V(HRoT_{m,n}) \setminus R_{p_1}, R_{p_2}, \ldots, R_{p_m-1}\}$ then $|R_p| \leq m$. We demonstrate that R_p is the structure's resolving set. We demonstrate that R_p is the structure's resolving set. Without losing generality, we can suppose that every vertex in $V(HRoT_{m,n}) \setminus R_p$ having values, which is not repeated to any other vertex in regard with R_p. So R_p is a resolving set of $HRoT_{m,n}$.

Hence, the outcome is

$$4 \leq pd\left(HRoT_{m,n}\right) \leq m+1. \tag{4}$$

□

III. APPLICATION

Robot navigation is the ability of the robot to ascertain its position inside its frame of reference and then map out a course toward a specific point. The robot or any other mobility device needs representation, such as a map of the

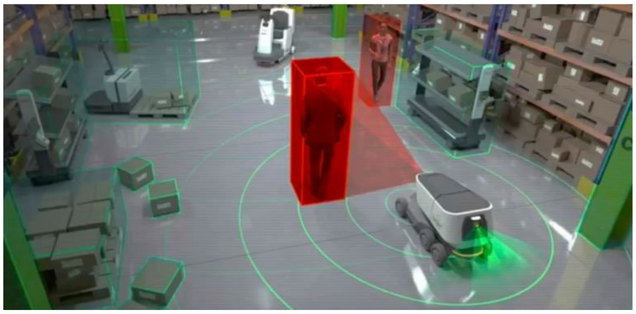

FIGURE 3. **A robot navigation is discussed.**

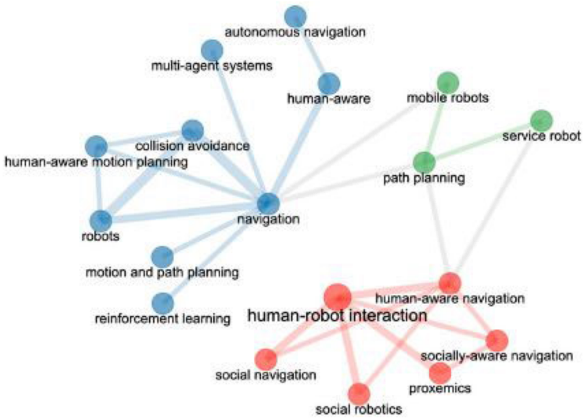

FIGURE 4. **Socially aware robot navigation.**

world, and the capacity to understand that representation in order to travel in its surroundings. The Figure 3 is taken from thr website [51]. Robot navigation's job is to quickly find our position anytime we need to know. We can identify a selection of nodes in a given network such that their position in the network is uniquely identifiable, assuming that a robot traveling in a sensor network can autonomously determine the distances to a collection of landmarks. This was accomplished through the concept of "edge landmarks in the graph," which was later expanded as "edge metric dimension," in which the networks were taken into account inside the framework of a graph as shown in Figure 4.

The Figure 4 is taken from thr website [52]. All networking elements, including clients, the internet, servers, firewalls, sensor nodes, base stations, and mobile nodes are continuously monitored and estimated as part of the analytical process of network monitoring in order to maintain and improve their availability. To check their availability, taking a network device system as shown in Figure 5.

The Figure 5 is taken from thr website [53]. According to this study we draw the graph in Figure 6 from Figure 5, having 10 vertices, which are represented by the set $V(G) = \{v_1, v_2, v_3, v_4, \ldots, v_{10}\}$ and 9 edges, which are represented by the set $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_5v_7, v_5v_8, v_5v_9, v_3v_{10}\}$, shown in Figure 6, $dim_e(G) = 5$, with edge resolving set $R = \{v_6, v_7, v_8, v_9, v_{10}\}$.

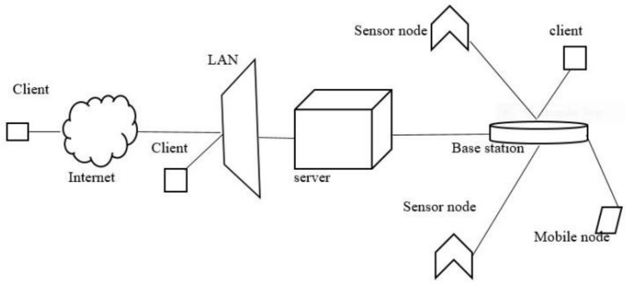

FIGURE 5. **A network device system.**

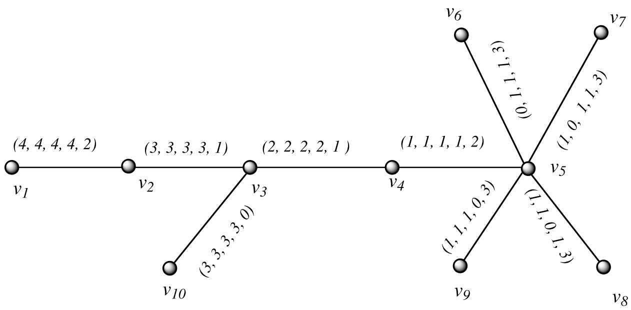

FIGURE 6. **Graphical representation of the Figure 5.**

IV. CONCLUSION

In the aircraft sector, honeycomb composite materials are frequently employed. Recent advancements have demonstrated the benefits of honeycomb structures. In this article, we examined the honeycomb rhombic torus's metric, edge metric, mixed metric, and partition dimension. The application of edge metric dimension is also discussed, in robot navigation.

A. CONJECTURES

In this subsection, few conjectures are stated. One can consider open problems are well. All these conjectures are checked thoroughly and new work can be done on proving these.

Conjecture 0.1: *Let $HRoT_{m,n}$ be a structure of honeycomb rhombic torus with $m = 3$ and $n \geq 4$. Then $dim\left(HRoT_{m,n}\right) = 4$.*

Conjecture 0.2: *Let $HRoT_{m,n}$ be a structure of honeycomb rhombic torus with $m = 3$ and $n \geq 4$. Then $dim_e\left(HRoT_{m,n}\right) = 4$.*

Conjecture 0.3: *Let $HRoT_{m,n}$ be a structure of honeycomb rhombic torus with $m = 3$ and $n \geq 4$. Then $dim_m\left(HRoT_{m,n}\right) = 4$.*

Conjecture 0.4: *Let $HRoT_{m,n}$ be a structure of honeycomb rhombic torus with $m = 3$ and $n \geq 4$. Then $4 \leq pd\left(HRoT_{m,n}\right) \leq 5$.*

Except these conjectures one can also discuss the metric resolvability in fuzzy graph theory and their extensions like given in the [48], [49], and [50].

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