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RESEARCH ARTICLE

Robustness in Mesh Networks Using Connected Safe Set and Applications

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ABSTRACT The robustness of the networks is their capacity to remain operational in the presence of faults and disruptions, and thereby it is an important tool to provide data transmission in telecommunications networks such as wireless networks, enterprise networks, and cloud computing networks. The connected collection of nodes in a network, excluding which results in the decomposition of a network into components such that the cardinality of each component is at most equal to the cardinality of the collection, is referred to as a *connected safe set* (CSS). The least size of CSS is known as *connected safe number* (CSN). The identification of the connected collection of nodes in the networks, capable of enduring dual load in case of faults and disruption, can be realized as CSS. Mesh networks (MNs) have become an integral part of a variety of domains, such as smart cities, disaster recovery, Internet of Things (IoT) and military defense, due to their decentralized nature and ability to reconfigure as conditions change dynamically. In this paper, the CSS and CSN for various types of MNs, such as triangular, triangular circular, double triangular circular, and quadrangular necklace mesh are computed. Finally, an application of CSS in the context of optimal router installation on certain MNs is included.

INDEX TERMS Connected safe number, connected safe set, wireless communication networks, mesh networks, IoT, robustness.

I. INTRODUCTION

Networks play a pivotal role in our modern interconnected world [9] as they enable seamless communication, facilitate internet access, connect IoT devices, and enable efficient data transfer and storage. These networks have been recognized for their adaptability and topologies, which reflect the communication patterns of various natural phenomena. Topology in the current scenario, with advanced technology and rapidly growing dependence on digital connectivity, has become a crucial factor in ensuring its effectiveness in communication devices and their mechanisms. Depending on these networks, it is essential to innovate and optimize the topology to seek uninterrupted connectivity. Mesh topologies [23] are in fashion owing to their efficiency and fast data accessibility, which rely on interconnected nodes to transmit data. These are highly reliable through their substantial redundancy, ensuring the availability of alternative routes to bypass faulty nodes. They are widely used in areas such as industrial automation, military communications, and disaster recovery.

Mesh networks [11] are either fully connected, partially connected, or hybrid. Partially connected mesh networks are less complex, cost-effective, and easy to maintain as compared to fully connected ones because some nodes are not directly connected. On the other hand, they could be more reliable and more robust on account of limited alternative paths for data transmission when faults and disruptions occur, which can lead to more significant downtime and slower data transfer rates. Therefore, increasing the robustness [2] of partially connected mesh networks is a significant challenge

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that is the ability of networks to maintain their operations despite any faults or disruptions. To overcome this challenge, we propose a technique that uses a connected collection of nodes in the MNs to distribute the load among its components in the presence of faults and disruptions. The identification of such a minimum set of components is regarded in graph theory, namely the CSS problem. The CSS problem and its related variant were initially presented in [14] with the motivation of addressing the *facility location problem* (FLP). The FLP pertains to finding the optimal locations to facilitate the clients and has many applications, including transportation, emergency response, and distribution [8], [26]. The FLP is a combinatorial optimization problem extensively studied in the literature [19].

For terminology and notation not explained in this paper, we refer readers to [6]. To provide context for our discussion on CSS, we will review some basic definitions and properties. When discussing a network, the node and edge sets are typically referred to as $V(G)$ and $E(G)$, respectively. The order of G is the number of nodes in the network. The degree of node v of G denoted by $deg(v)$ is the number of nodes attached by edges with v. Additionally, the subgraph induced by a subset X of $V(G)$ can be denoted as $G[X]$. The set of all connected components of G is denoted by $C(G)$. Two subgraphs X and Y of G are said to be adjacent if they have no nodes in common, but they share at least one edge. More formally, X and Y are adjacent if and only if $E(X, Y) \neq \emptyset$, where $E(X, Y) = \{e = uv : u \in X \text{ and } v \in Y\} \subseteq E(G)$. A subset $\mathcal{S}(G) \neq \emptyset \subset V(G)$ is called safe set (SS) if, for every connected component $X \in C(G \setminus \mathcal{S}(G))$ and every connected component $Y \in C(G[\mathcal{S}(G)])$, we have $|Y| \geq |X|$, whenever $E(X, Y) \neq \emptyset$. If $C(G[\mathcal{S}(G)]) = \{\mathcal{S}(G)\}$, then $\mathcal{S}(G)$ is known as CSS. For a connected network G, the safe number (SN) and CSN (respectively) are defined as follows:

$$s(G) = \min\{|\mathcal{S}(G)| : \mathcal{S}(G) \text{ is a SS of } G\}.$$
$$cs(G) = min\{|\mathcal{S}(G)| : \mathcal{S}(G) \text{ is a CSS of } G\}.$$

Fujita et al. in [14] investigated the existence of a general algorithm for computing the SN and CSN. They concluded that it is an NP-complete. However, the authors in [14] also demonstrate that the minimum cardinality of SS of a tree can be calculated in linear time. Additionally, Águeda et al. in [1] showed that the SN of trees can be computed in $O(n^5)$ time. The authors in [12] explored the connection between the SN and integrity in a connected graph. Iqbal et al. in [16] presented the CSS of ladder, wheel, and sunlet graphs, even though there is no known algorithm to find the minimum SS and CSS of any connected graph. Furthermore, some authors in [17] studied the SS and computed the SN and CSN of a cartesian product of two complete graphs. Belmonte et al. [5] studied the parameterized complexity of safe set problems. Different classes of graphs and their CSN have been studied and computed as shown in TABLE 1.

Bapat et al. [4] considered a large network as a community of small communities with mutual connections to gain the majority to control network consensus and introduced the concept of weighted safe sets (WSS) in graphs. They provide an efficient algorithm for calculating the WSS for a weighted path. Recently, for a weighted tree's CSN, Ehard and Rautenbach [10] presented a polynomial-time approximation solution. Fujita et al. [13] explored the potential equality of a graph's weighted safe number (WSN) and its connected weighted safe number (CWSN) for path and cycle graphs. In addition, the authors in [15] defined a graph G to have a stable structure if equality holds between its WSN and CWSN for any weight function defined on its vertices. For further research on the WSS problem, see [21], [22], and [27].

TABLE 1. CSN of certain classes of graphs.

G	cs(G)
P_n [14]	$\lceil \frac{n}{3} \rceil$
C_n [14]	$\lceil \frac{n}{2} \rceil$
W_n [16]	$\lceil \sqrt{n} \rceil$
S_n [16]	$\lfloor \frac{n}{3} \rfloor$
L_n [16]	$\begin{cases} \lceil \frac{n}{3} \rceil, & \text{if } 4 \leq n \leq 8 \\ \lfloor \frac{n}{4} \rfloor, & \text{if } n \geq 4 \text{ and } n \equiv 0, 2, 6 (\text{mod } 8) \\ \lceil \frac{n}{4} \rceil + 1, & \text{otherwise.} \end{cases}$

A. MAJOR CONTRIBUTIONS

The main results of this study are summarized as follows:

Theorem 1:

a) For $n \geq 1$,

$$cs(T_n) = \begin{cases} \lceil \dfrac{2n+1}{4} \rceil + 1, & \text{if } n \cong 1(\text{mod } 2) \\ \lceil \dfrac{2n+1}{4} \rceil, & \text{if } n \cong 0(\text{mod } 4). \end{cases}$$

b) For $n \geq 3$,

$$cs(T_n^1) = \lfloor \frac{2n}{3} \rfloor + 1.$$

c) For $n \geq 3$,

$$cs(T_n^2) = \lceil \frac{2n}{3} \rceil + 1$$

d) For $n \geq 3$,

$$cs(Q_n) = \lceil \frac{2n+2}{3} \rceil + 1.$$

where T_n, T_n^1, T_n^2, and Q_n denote the triangular, triangular circular, double triangular circular, and quadrangular necklace mesh networks respectively.

This paper is structured as follows: Section II calculates the CSS and CSN of the triangular mesh network. Section III focuses on computing the CSS and CSN of the triangular circular mesh network. In Section IV, the CSS and CSN of a double triangular circular mesh network are determined. Section V calculates the CSS and CSN of the quadrangular necklace mesh network. Section VI showcases an application of CSS for the optimal installation of routers on certain MNs. Finally, the paper concludes in Section VII.

II. CONNECTED SAFE SET OF TRIANGULAR MESH NETWORK

In this section, the CSS and CSN of the triangular mesh network are computed. A triangular mesh network T_n of order $2n+1$ is constructed by beginning with two paths P_n^1 and P_{n+1}^2. Then, connect each w_j node of P_n^1 to the nodes u_j and u_{j+1} of P_n^2 to produce a mesh of $2n-1$ triangles. For $n \geq 1$, the node set and edge set of T_n are defined as $V(T_n) = \{w_1, w_2, \ldots, w_n, u_1, u_2, \ldots, u_{n+1}\}$ and $E(T_n) = \{w_iw_{i+1}, u_ju_{j+1}, w_ju_j, w_ju_{j+1} \mid 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq n\}$ respectively. The labeling is illustrated in Figure 1. For more study, we refer the readers to [3].

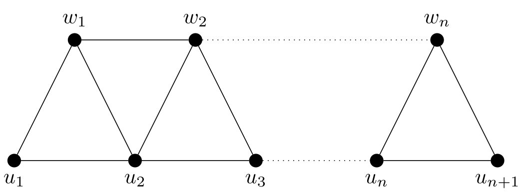

FIGURE 1. The triangular mesh network T_n.

Lemma 1: For $n \geq 1$,

$$|\mathcal{S}(T_n)| \geq \begin{cases} \lceil \frac{2n+1}{4} \rceil + 1, & \text{if } n \cong 1(\text{mod } 2) \\ \lceil \frac{2n+1}{4} \rceil, & \text{if } n \cong 0(\text{mod } 2). \end{cases}$$

Proof 1: For $n = 1, 2$, it can be verified that $\mathcal{S}(Tm_1) = \{w_1, u_2\}$ is CSS. When $n \geq 3$, the proof is divided into two cases:

Case 1: When $|\mathcal{S}(T_n)|$ is odd.

First, we consider that $n \cong 0(\text{mod } 2)$. Let $\mathcal{S}(Tm_n) = \{u_{\lceil \frac{k}{2} \rceil+1}, u_{\lceil \frac{k}{2} \rceil+2}, \ldots, u_{\lceil \frac{k}{2} \rceil+(k-3)}, w_{\lceil \frac{k}{2} \rceil}, w_{\lceil \frac{k}{2} \rceil+(k-3)}\}$ be subset of $V(T_n)$ such that $T_n[\mathcal{S}(T_n)]$ is induced subgraph, where $k = \lceil \frac{2n+1}{4} \rceil + 1$. Since the vertices u_i in $\mathcal{S}(T_n)$ form an increasing sequence, this implies that the edges u_iu_{i+1}, $u_{\lceil \frac{k}{2} \rceil+1}w_{\lceil \frac{k}{2} \rceil}$, $u_{\lceil \frac{k}{2} \rceil+(k-3)}w_{\lceil \frac{k}{2} \rceil+(k-3)} \in E(T_n)$ for all $\lceil \frac{k}{2} \rceil + 1 \leq i \leq \lceil \frac{k}{2} \rceil + (k-3)$. Therefore, $T_n[\mathcal{S}(T_n)]$ is connected. Now $C(T_n \setminus \mathcal{S}(T_n)) = \{D_1, D_2, D_3\}$, where $D_1 = \{w_1, w_2, \ldots, w_{\lceil \frac{k}{2} \rceil-1}, u_1, u_2, \ldots, u_{\lceil \frac{k}{2} \rceil}\}$, $D_2 = \{w_{\lceil \frac{k}{2} \rceil+(k-2)}, w_{\lceil \frac{k}{2} \rceil+(k-1)}, \ldots, w_n\, u_{\lceil \frac{k}{2} \rceil+(k-2)}, u_{\lceil \frac{k}{2} \rceil+(k)}, \ldots, u_{n+1}\}$, and $D_3 = \{w_{\lceil \frac{k}{2} \rceil+1}, w_{\lceil \frac{k}{2} \rceil+2}, \ldots, w_{\lceil \frac{k}{2} \rceil+(k-4)}\}$. It follows that $|\mathcal{S}(T_n)| = k-1$, $|D_1| = 2\lceil \frac{k}{2} \rceil - 1 \leq k = \lceil \frac{2n+1}{4} \rceil$, $|D_2| \leq 2n - 2\lceil \frac{k}{2} \rceil - 2k + 7 \leq 2n - 3k + 6$, and $|D_3| = k-4$, where $k = \lceil \frac{2n+1}{4} \rceil + 1$. This implies that $|D_j| \leq |\mathcal{S}(T_n)|$ for all $1 \leq j \leq 3$. In consequence, $\mathcal{S}(T_n)$ is CSS, and $|\mathcal{S}(T_n)| \geq \lceil \frac{2n+1}{4} \rceil$.

By considering $n \cong 1(\text{mod } 2)$ and choosing $\mathcal{S}(Tm_n) = \{u_{\lceil \frac{k}{2} \rceil+1}, u_{\lceil \frac{k}{2} \rceil+2}, \ldots, u_{\lceil \frac{k}{2} \rceil+(k-2)}, w_{\lceil \frac{k}{2} \rceil}, w_{\lceil \frac{k}{2} \rceil+(k-2)}\}$ for $k = \lceil \frac{2n+1}{4} \rceil + 1$, similarly, we can show that $|\mathcal{S}(T_n)| \geq \lceil \frac{2n+1}{4} \rceil + 1$.

Case 2: When $|\mathcal{S}(T_n)|$ is even.

First, we consider that $n \cong 0(\text{mod } 2)$. Let $\mathcal{S}(Tm_n) = \{w_{\lceil \frac{k}{2} \rceil+1}, w_{\lceil \frac{k}{2} \rceil+2}, \ldots, w_{\lceil \frac{k}{2} \rceil+(k-1)}, u_{\lceil \frac{k}{2} \rceil+1}, u_{\lceil \frac{k}{2} \rceil+(k)}\}$ be subset of $V(T_n)$ such that $T_n[\mathcal{S}(T_n)]$ is induced subgraph, where $k = \lceil \frac{2n+1}{4} \rceil$. Since the nodes u_i in $\mathcal{S}(T_n)$ form an increasing sequence, it follows that the edges u_iu_{i+1}, $w_{\lceil \frac{k}{2} \rceil+1}u_{\lceil \frac{k}{2} \rceil+1}$, $w_{\lceil \frac{k}{2} \rceil+(k-1)}u_{\lceil \frac{k}{2} \rceil+(k)} \in E(T_n)$ for all $\lceil \frac{k}{2} \rceil + 1 \leq i \leq \lceil \frac{k}{2} \rceil + (k-2)$. Therefore, $T_n[\mathcal{S}(T_n)]$ is connected. Now $C(T_n \setminus \mathcal{S}(T_n)) = \{D_1, D_2, D_3\}$, where $D_1 = \{w_1, w_2, \ldots, w_{\lceil \frac{k}{2} \rceil}, u_1, u_2, \ldots, u_{\lceil \frac{k}{2} \rceil}\}$, $D_2 = \{w_{\lceil \frac{k}{2} \rceil+(k)}, w_{\lceil \frac{k}{2} \rceil+(k+1)}, \ldots, w_n, u_{\lceil \frac{k}{2} \rceil+(k+1)}, u_{\lceil \frac{k}{2} \rceil+(k+2)}, \ldots, u_{n+1}\}$, and $D_3 = \{u_{\lceil \frac{k}{2} \rceil+2}, w_{\lceil \frac{k}{2} \rceil+3}, \ldots, w_{\lceil \frac{k}{2} \rceil+(k-3)}\}$. It can be verify that $|\mathcal{S}(T_n)| = k+1$. As $|\mathcal{S}(T_n)|$ is even and $|\mathcal{S}(T_n)| = k+1$ this implies that k is odd and we have $\lceil \frac{k}{2} \rceil + \lceil \frac{k}{2} \rceil \leq k+1$. Now $|D_1| \leq k+1$, $|D_2| \leq 2n - 3k + 1$, and $|D_3| = k-2$, where $k = \lceil \frac{2n+1}{4} \rceil$. This implies that $|D_j| \leq |\mathcal{S}(T_n)|$ for all $1 \leq j \leq 3$. In consequence, $\mathcal{S}(T_n)$ is CSS, and $|\mathcal{S}(T_n)| \geq \lceil \frac{2n+1}{4} \rceil$.

By considering $n \cong 1(\text{mod } 2)$ and choosing $\mathcal{S}(Tm_n) = \{w_{\lceil \frac{k}{2} \rceil+1}, w_{\lceil \frac{k}{2} \rceil+2}, \ldots, w_{\lceil \frac{k}{2} \rceil+(k-1)}, u_{\lceil \frac{k}{2} \rceil+1}, u_{\lceil \frac{k}{2} \rceil+(k)}\}$ for $k = \lceil \frac{2n+1}{4} \rceil$, similarly, we can show that $|\mathcal{S}(T_n)| \geq \lceil \frac{2n+1}{4} \rceil + 1$.

Theorem 2: For $n \geq 1$,

$$cs(T_n) = \begin{cases} \lceil \frac{2n+1}{4} \rceil + 1, & \text{if } n \cong 1(\text{mod } 2) \\ \lceil \frac{2n+1}{4} \rceil, & \text{if } n \cong 0(\text{mod } 2). \end{cases}$$

Proof 1: The proof is divided into two case:

Case 1: (When $n \cong 1(\text{mod } 2)$)

Consider a CSS $\mathcal{S}(T_n)$ with cardinality $cs(T_n)$. Let $D_1, D_2, \ldots, D_t$ be the components of $C(T_n \setminus \mathcal{S}(T_n))$, ordered in such a way that $|D_1| \leq |D_2| \leq \cdots \leq |D_t|$.

If $t = 1$, then there must exist a node x in $\mathcal{S}(T_n)$ such that:

(i) $\mathcal{S}(T_n) \setminus \{x\}$ is connected

(ii) Either $E(T_n \setminus \mathcal{S}(T_n), x)$ is empty or non-empty.

If $E(T_n \setminus \mathcal{S}(T_n), \{x\}) = \emptyset$, then there exists a vertex y in $T_n \setminus \mathcal{S}(T_n)$ such that $E(\mathcal{S}(T_n), \{y\}) \neq \emptyset$. Then by removing x from $\mathcal{S}(T_n)$ and adding y in $\mathcal{S}(T_n)$, we obtain another CSS $\mathcal{S}^*(T_n)$ such that the cardinality of greatest component of $C(T_n \setminus \mathcal{S}^*(T_n))$ is smaller than $|D_1|$, which is a contradiction.

If $E(T_n \setminus \mathcal{S}(T_n), \{x\}) \neq \emptyset$, there exists a vertex y in $T_n \setminus \mathcal{S}(T_n)$ such that $E(\mathcal{S}(T_n) \setminus \{x\}, \{y\}) \neq \emptyset$, $E(\{x\}, \{y\}) \neq \emptyset$, and $T_n[\mathcal{S}(T_n) \setminus \{x\} \cup \{y\}]$ is connected. In that case, removing x from $\mathcal{S}(T_n)$ and adding y to $\mathcal{S}(T_n)$ will result in another CSS $\mathcal{S}^*(T_n)$ such that the cardinality of greatest component of $C(T_n \setminus \mathcal{S}^*(T_n))$ is smaller than $|D_1|$, and this leads to a contradiction. It follows from the above discussion and using Lemma 1 that $t \geq 2$.

For $n \leq 4$, it is straightforward to verify that $t \leq 2$. For $n \geq 5$, we claim that $t = 3$.

Assume for the sake of argument that $t = 4$, it is only possible when $\mathcal{S}(T_n)$ have at least three end nodes s_i, s_j, and s_k such that $i < j < k$. It follows that $\mathcal{S}^*(G) = \mathcal{S}(G) \setminus \{s_j\}$ is another CSS of smaller cardinality, a contradiction. A similar reasoning applies for $t \geq 5$. In consequence, $t = 3$.

We want to prove that $cs(T_n) = \lceil \frac{2n+1}{4} \rceil + 1$. Assume for contradiction $cs(T_n) = \lceil \frac{2n+1}{4} \rceil$. Note that $\mathcal{S}(T_n)$ contains two end nodes, which means that the set of nodes in $T_n \setminus \mathcal{S}(T_n)$ between these end nodes make a component say D_1, and of course $|D_1| = |\mathcal{S}(T_n)| - 3$. From the definition of CSS

$|\mathcal{S}(T_n)| \geq |D_t|$, we have

$$\sum_{t=1}^{3} |D_t| + \mathcal{S}(T_n) = \sum_{t=2}^{3} |D_t| + 2|\mathcal{S}(T_n)| - 3 \leq 4|\mathcal{S}(G)| - 3$$
$$\leq 4\lceil \frac{2n+1}{4} \rceil - 3 < 2n + 1.$$

Consequent to this at least one component of $C(T_n \setminus \mathcal{S}(T_n))$ has cardinality greater than $|\mathcal{S}(T_n)|$, which is a contradiction. It follows from the above remarks and with the help of Lemma 1 $cs(T_n) = \lceil \frac{2n+1}{4} \rceil + 1$.

Case 2: (When $n \cong 0(\text{mod } 2)$) The same reasoning applies to this case.

III. CONNECTED SAFE SET OF TRIANGULAR CIRCULAR MESH NETWORK

In this section, the CSS and CSN of the triangular circular mesh network are computed. A triangular circular mesh T_n^1 of order $2n$ is constructed by beginning with a cycle C_n and inserting u_j nodes outside of each adjacent pair of nodes w_j and w_{j+1} in C_n. Then, connect each u_j node to the nodes w_j and w_{j+1} to form a mesh of n triangles. For $n \geq 3$, the node set and edge set of T_n^1 are defined by $V(T_n^1) = \{w_1, w_2, \ldots, w_n, u_1, u_2, \ldots, u_n\}$ and $E(T_n^1) = \{w_i w_{i+1}, w_j u_j, w_i u_{i+1}, w_n w_1, w_n u_1 \mid 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq n\}$ respectively. The labeling is illustrated in Figure 2. For a more detailed study, we refer the readers to [7].

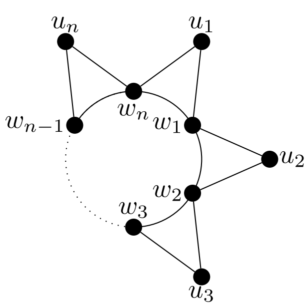

FIGURE 2. The triangular mesh network T_n^1.

Lemma 2: For $n \geq 3$,

$$|\mathcal{S}(T_n^1)| \geq \lfloor \frac{2n}{3} \rfloor + 1.$$

Proof 2: Let $S(T_n^1) = \{w_1, w_2, \ldots, w_{\lfloor \frac{2n}{3} \rfloor + 1}\}$ be the subset of node set $V(T_n^1)$. Since $w_i w_{i+1} \in E(T_n^1)$, where $1 \leq i \leq \lfloor \frac{2n}{3} \rfloor + 1$. Therefore, $T_n^1[S(T_n^1)]$ is connected. Now $C(T_n^1 \setminus S(T_n^1)) = \{D_1, D_2, \ldots, D_t\}$, where $D_1 = \{u_2\}, D_2 = \{u_3\}, \ldots, D_{t-1} = \{u_t\}$, and $D_t = \{w_{\lfloor \frac{2n}{3} \rfloor + 2}, w_{\lfloor \frac{2n}{3} \rfloor + 3}, \ldots, w_n, u_1, u_{\lfloor \frac{2n}{3} \rfloor + 2}, \ldots, u_n\}$. It follows that $t = \lfloor \frac{2n}{3} \rfloor + 1$, $|S(T_n^1)| = \lfloor \frac{2n}{3} \rfloor + 1$, $|D_i| = 1$ for $1 \leq i \leq \lfloor \frac{2n}{3} \rfloor$, and $|D_t| = 2n - |S(T_n^1)| - \sum_{i=1}^{t-1} |D_i| = 2n - 2\lfloor \frac{2n}{3} \rfloor - 1$. This implies that $|D_i| \leq |S|$ for $1 \leq i \leq t$. Hence, $S(T_n^1)$ is a connected safe set and $|S(T_n^1)| \geq \lfloor \frac{2n}{3} \rfloor + 1$.

Theorem 3: For $n \geq 3$,

$$cs(T_n^1) = \lfloor \frac{2n}{3} \rfloor + 1$$

Proof 3: Let $\mathcal{S}(T_n^1)$ be a CSS of cardinality $cs(T_n^1)$. Let $C(H_n \setminus \mathcal{S}(T_n^1)) = \{D_1, D_2, D_3, \ldots, D_t\}$ ordered such that $|D_1| \leq |D_2| \leq |D_3| \leq \cdots \leq |D_t|$.

If $t = 1$, then there exist such x and y nodes in $V(T_n^1)$ such that $x \in D_1$, $y \in \mathcal{S}(T_n^1)$, $E(\{x\}, \mathcal{S}(T_n^1)) \neq \emptyset$, $E(\{y\}, D_1) = \emptyset$, and $\mathcal{S}(T_n^1) \setminus \{y\}$ is connected. Then by removing node y from $\mathcal{S}(T_n^1)$ and adding node x in $\mathcal{S}(T_n^1)$. We get another CSS $\mathcal{S}^*(T_n^1)$, and for $\mathcal{S}^*(T_n^1)$, $|D_1| > max\{|D| \mid D \in C(T_n^1 \setminus \mathcal{S}^*(T_n^1))\}$, which is a contradiction. As a consequence, $t \geq 2$.

We claim that $t \leq |\mathcal{S}(T_n^1)|$. If $\mathcal{S}(T_n^1) \subset \{w_1, w_2, \ldots, w_n\}$ and $T_n^1[\mathcal{S}(T_n^1)]$ is a path, then $C(T_n^1 \setminus \mathcal{S}(T_n^1))$ contains maximum components. For that choice of CSS, it follows from Lemma 2 that $C(T_n^1 \setminus \mathcal{S}(T_n^1))$ have exactly $|\mathcal{S}(T_n^1)| - 1$ components of cardinality 1 and one component of cardinality greater than 1. It follows that $t = |\mathcal{S}(T_n^1)|$. If $\mathcal{S}(T_n^1) \cap \{w_1, w_2, \ldots, w_n\}$ is not equal to $\mathcal{S}(T_n^1)$, but is nonempty, then clearly $t < |\mathcal{S}(T_n^1)|$. Consequently, $t \leq |\mathcal{S}(T_n^1)|$.

Now we want to show that $cs(T_n^1) = \lfloor \frac{2n}{3} \rfloor + 1$. Suppose on contrary $cs(T_n^1) = \lfloor \frac{2n}{3} \rfloor$. From the definition of CSS, it is clear that $|\mathcal{S}(T_n^1)| \geq |D_t|$. Assume $|\mathcal{S}(T_n^1)| = |D_t|$. Note that

$$\sum_{i=1}^{t} |D_i| + |\mathcal{S}(T_n^1)| \leq 3(\lfloor \frac{2n}{3} \rfloor) - 1 < 2n.$$

Consequent to this at least one component of $C(T_n^1 \setminus \mathcal{S}(T_n^1))$ has cardinality greater than $|\mathcal{S}(T_n^1)|$, a contradictions. It follows from the above remarks and with the help of Lemma 2 $cs(T_n^1) = \lfloor \frac{2n}{3} \rfloor + 1$.

IV. CONNECTED SAFE SET OF DOUBLE TRIANGULAR CIRCULAR MESH NETWORK

In this section, the CSS and CSN of the double triangular circular mesh network are computed. A double triangular circular mesh T_n^2 of order $2n$ is constructed by beginning with two cycles C_n^1 and C_n^2. Then, connect u_j nodes of C_n^2 to the nodes w_j and w_{j+1} of C_n^1 to produce a mesh of $2n$ triangles. For $n \geq 3$, the node set and edge set are defined by $V(T_n^2) = \{w_1, w_2, \ldots, w_n, u_1, u_2, \ldots, u_n\}$ and $E(T_n^2) = \{w_i w_{i+1}, u_i u_{i+1}, w_i u_{i+1}, w_j u_j, w_1 w_n, u_1 u_n, w_n u_1, w_n u_n \mid 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq n-1\}$ respectively. The labeling is illustrated in Figure 3. For more study, we refer the readers to [25].

Lemma 3: For $n \geq 3$,

$$|\mathcal{S}(T_n^2)| \geq \lceil \frac{2n}{3} \rceil + 1$$

Proof 4: Let $\mathcal{S}(T_n^2) = \{w_1, w_2, \ldots, w_{\lceil \frac{2n}{3} \rceil - 1}, u_1, u_{\lceil \frac{2n}{3} \rceil}\}$ be the subset of $V(T_n^2)$. Since $w_i w_{i+1}, w_1 u_1, w_{\lceil \frac{2n}{3} \rceil - 1} u_{\lceil \frac{2n}{3} \rceil} \in E(T_n^2)$, where $1 \leq i \leq \lceil \frac{2n}{3} \rceil - 1$. Therefore, $T_n^2[\mathcal{S}(T_n^2)]$ is connected. Now $C(T_n^2 \setminus \mathcal{S}(T_n^2)) = \{D_1, D_2\}$, where $D_1 = \{u_2, u_3, \ldots, u_{\lceil \frac{2n}{3} \rceil - 1}\}$ and $D_2 =$

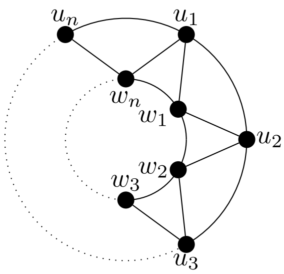

FIGURE 3. The double triangular circular mesh network T_n^2.

$\{w_{\lceil\frac{2n}{3}\rceil}, w_{\lceil\frac{2n}{3}\rceil+1}, \ldots, w_n, u_{\lceil\frac{2n}{3}\rceil+1}, u_{\lceil\frac{2n}{3}\rceil+2}, \ldots, u_n\}$. It follows that $|D_1| = \lceil\frac{2n}{3}\rceil - 2$ and $|D_2| = 2n - 2\lceil\frac{2n}{3}\rceil + 1$. This implies that $|\mathcal{S}(T_n^2)| \geq |D_i|$, where $1 \leq i \leq 2$. Consequently, $\mathcal{S}(T_n^2)$ is a CSS and $|\mathcal{S}(T_n^2)| \geq \lceil\frac{2n}{3}\rceil + 1$.

Theorem 4: For $n \geq 3$,

$$cs(T_n^2) = \lceil\frac{2n}{3}\rceil + 1$$

Proof 5: Suppose that $V(T_n^2) = A \cup B$, where $A = \{w_1, w_2, \ldots, w_n\}$ and $B = \{u_1, u_2, \ldots, u_n\}$. Let $\mathcal{S}(T_n^2) = X \cup Y$ be the CSS of cardinality $cs(T_n^2)$ such that $X \subset A$ and $Y \subset B$. Let $D_1, D_2, \ldots, D_t$ be the components of $C(T_n^2 \setminus \mathcal{S}(T_n^2))$, arranged in such a way that $|D_1| \leq |D_2| \leq \cdots \leq |D_t|$.

If $t = 1$, then there must exists x_1 and x_2 nodes in $\mathcal{S}(T_n^2)$ and y_1 and y_2 nodes in D_1 such that:

(i) $\mathcal{S}(T_n^2) \setminus \{x_1, x_2\}$ is connected.

(ii) $T_n^2 \setminus \mathcal{S}^*(T_n^2)$ is disconnected, where $\mathcal{S}^*(T_n^2) = \mathcal{S}(T_n^2) \cup \{y_1, y_2\} \setminus \{x_1, x_2\}$.

(iii) $T_n^2[\mathcal{S}^*(T_n^2)]$ is connected.

Clearly, $\mathcal{S}^*(T_n^2)$ is another CSS and for $\mathcal{S}^*(T_n^2)$, $|D_1| > max\{|D| \mid D \in C(T_n^2 \setminus \mathcal{S}^*(T_n^2))\}$, which is a contradiction. As a consequence, $t \geq 2$.

We claim that $t = 2$. Suppose for the sake of contradiction that $t = 3$, then $\mathcal{S}(T_n^2)$ must have three end nodes, namely x_i, x_j, and x_k with $i < j < k$. Then $\mathcal{S}^*(T_n^2) = \mathcal{S}(T_n^2) \setminus \{x_j\}$ is another CSS of smaller cardinality, which contradicts our initial assumption. The similar reasoning applies for $t \geq 4$. Therefore, $t = 2$.

We are going to show that $cs(T_n^2) = \lceil\frac{2n}{3}\rceil + 1$. Suppose for contradiction $cs(T_n^2) = \lceil\frac{2n}{3}\rceil$. Since $t = 2$ and there are two end nodes, x_i and x_k, in $\mathcal{S}(T_n^2)$, the set of intermediary nodes on the shortest path connecting nodes x_i and x_j forms a component D_1. Clearly, $|D_1| = \lceil\frac{2n}{3}\rceil - 3$. As D_t is a component with $t = 2$, we have $|D_2| = 2n - |D_1| - |\mathcal{S}(T_n^2)| = 2n - 2\lceil\frac{2n}{3}\rceil + 3 > |\mathcal{S}(T_n^2)|$, which is contradictory. It follows from above remarks and with the help of Lemma 3, $cs(T_n^2) = \lceil\frac{2n}{3}\rceil + 1$.

V. CONNECTED SAFE SET OF QUADRANGULAR NECKLACE MESH NETWORK

In this section, the CSS and CSN of the quadrangular necklace mesh network are computed. A quadrangular necklace mesh Q_n is constructed by starting with a path P_n of n nodes and a cycle C_{n+2} of $n+2$ nodes. The nodes of P_n are labeled $u_1, u_2, \ldots, u_n$, and the nodes of C_{n+2} are labeled $w_0, w_1, \ldots, w_{n+1}$. Then, connect the nodes u_j to w_j, w_0 to u_1, and u_n to w_{n+1} for $j = 1, 2, \ldots, n$ to form a mesh of $n-1$ quadrilateral faces, which are arranged in a necklace-like fashion. For $n \geq 3$, the node set and edge set of Q_n are defined as $V(Q_n) = \{w_0, w_1, w_2, \ldots, w_n, w_{n+1}, u_1, u_2, \ldots, u_n\}$ and $E(Q_n) = \{w_iw_{i+1}, u_ju_{j+1}, w_ju_j, w_0w_{n+1}, w_0u_1, w_nu_n, u_nw_{n+1} \mid 0 \leq i \leq n \text{ and } 1 \leq j \leq n-1\}$ respectively. The labeling is illustrated in Figure 4. For more study, we refer the readers to [28].

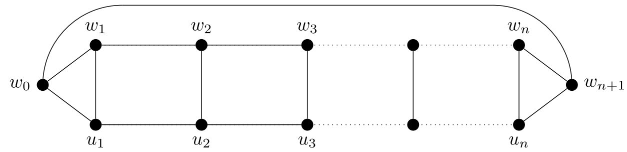

FIGURE 4. The quadrangular necklace mesh network Q_n.

Lemma 4: For $n \geq 2$,

$$|\mathcal{S}(Q_n)| \geq \lceil\frac{2n+2}{3}\rceil + 1$$

Proof 6: Let $\mathcal{S}(Q_n) = \{w_0, w_1, \ldots, w_{\lceil\frac{2n+2}{3}\rceil-1}, u_{\lceil\frac{2n+2}{3}\rceil-1}\}$ be the subset of $V(Q_n)$ such that $Q_n[\mathcal{S}(Q_n)]$ is induced subgraph. Since $w_iw_{i+1}, w_{\lceil\frac{2n+2}{3}\rceil-1}u_{\lceil\frac{2n+2}{3}\rceil-1} \in E(Q_n)$, where $0 \leq i \leq \lceil\frac{2n+2}{3}\rceil - 1$. From Figure 2, one can verify that $Q_n[\mathcal{S}(Q_n)]$ is path. Therefore, $Q_n[\mathcal{S}(Q_n)]$ is connected. Now $C(Q_n \setminus \mathcal{S}(Q_n)) = \{D_1, D_2\}$, where $D_1 = \{u_1, u_2, \ldots, u_{\lceil\frac{2n+2}{3}\rceil-2}\}$ and $D_2 = \{w_{\lceil\frac{2n+2}{3}\rceil}, w_{\lceil\frac{2n+2}{3}\rceil+1}, \ldots, w_{n+1}, u_{\lceil\frac{2n+2}{3}\rceil}, u_{\lceil\frac{2n+2}{3}\rceil+1}, \ldots, u_n\}$. It follows that $|\mathcal{S}(Q_n)| = \lceil\frac{2n+2}{3}\rceil + 1$, $|D_1| = \lceil\frac{2n+2}{3}\rceil - 2$, and $|D_2| = 2n - 2\lceil\frac{2n+2}{3}\rceil + 3$. This implies that $|\mathcal{S}(Q_n)| \geq |D_i|$, for all $1 \leq i \leq 2$. As a consequent, $\mathcal{S}(Q_n)$ is CSS and $|\mathcal{S}(Q_n)| \geq \lceil\frac{2n+2}{3}\rceil + 1$.

Theorem 5: For $n \geq 2$,

$$cs(Q_n) = \lceil\frac{2n+2}{3}\rceil + 1$$

Proof 7: Let $\mathcal{S}(Q_n)$ be a CSS of cardinality $cs(Q_n)$. Let $C(Q_n \setminus \mathcal{S}(Q_n)) = \{D_1, D_2, \ldots, D_t\}$ ordered such that $|D_1| \leq D_2 \leq \cdots \leq |D_t|$.

If $t = 1$, then there exist x and y nodes in $V(G)$ such that:

(i) $x \in D_1$ and $y \in \mathcal{S}(Q_n)$.

(ii) $E(\{x\}, \mathcal{S}(Q_n)) \neq \emptyset$ and $E(\{x\}, \{y\}) = \emptyset$.

(iii) $\mathcal{S}(Q_n) \setminus \{y\}$ is connected.

(iv) $Q_n \setminus \mathcal{S}^*(Q_n)$ is connected, where $\mathcal{S}^*(Q_n) = (\mathcal{S}(Q_n) \setminus \{y\}) \cup \{x\}$.

Then by removing node y from $\mathcal{S}(Q_n)$ and adding node x in $\mathcal{S}(Q_n)$. We get another CSS $\mathcal{S}^*(Q_n)$, and for $\mathcal{S}^*(Q_n)$, $|D_1| > max\{|D| \mid D \in C(Q_n \setminus \mathcal{S}^*(Q_n))\}$, which is a contradiction. As a consequence, $t \geq 2$.

Since Q_n have two paths $P_1(w_1 \to w_n)$ and $P_2(u_1 \to u_n)$. Let $v \in P_1$ and $u \in P_2$ such that $vu \in E(Q_n)$. Then $\{w_0, v, u\}$ is a cut-node set. We assume that $\{w_0, v, u\} \subset \mathcal{S}(Q_n)$ such that either v or u is end node. Without loss of generality, we suppose that u is an end node. We claim that there does not exist another end node w in $\mathcal{S}(Q_n)$ such that $w \notin \{w_0, u\}$. We assume for contradiction that w another end node in $\mathcal{S}(Q_n)$ then $\mathcal{S}(Q_n)' = \mathcal{S}(Q_n) \setminus \{w\}$ is another CSS of cardinality smaller than $\mathcal{S}(Q_n)$, a contradiction. Therefore, by using Lemma 1, it follows easily that $Q_n[\mathcal{S}(Q_n)]$ is a path and $t = 2$.

Suppose for contradiction $cs(Q_n) = \lceil\frac{2n+2}{3}\rceil$. Since $Q_n[\mathcal{S}(Q_n)]$ is path, $t = 2$, and $deg(u) = 1$ then the nodes set $\{u_1, u_2, \ldots, u_{i-1}\}$ makes a component say D_1. Since $|\mathcal{S}(Q_n)| = \lceil\frac{2n+2}{3}\rceil$ then clearly $i = \lceil\frac{2n+2}{3}\rceil - 2$. So, $|D_1| = \lceil\frac{2n+2}{3}\rceil - 3$ and $D_2 = |V(Q_n) - \mathcal{S}(Q_n) - D_1| = 2n + 2 - 2\lceil\frac{2n+2}{3}\rceil + 3 \geq |\mathcal{S}(Q_n)|$, a contradiction. It follows from the above remarks and with the help of Lemma 1 $cs(Q_n) = \lceil\frac{2n+2}{3}\rceil + 1$.

VI. APPLICATION

The applications of CSS and its related variants can be seen in network design [14] and analysis [4]. In particular, we consider an application of optimal router installation in certain MNs in case of faults and disruptions. Routers are networking devices that connect multiple networks and forward data packets between them. They can perform various functions like routing, filtering, firewalling, and network address translation. As routers have to receive and forward data packets, their disruption or malfunctioning may degrade the network performance and security due to the following problems:

- (i) The devices that rely on them for accessing other networks will lose their connectivity.
- (ii) If routers are overloaded or misconfigured, they may experience delays, errors, or packet loss in forwarding packets.

The issues discussed in the given scenario can be resolved by providing backup routers capable of enduring dual loads to ensure that data is transferred safely and efficiently. It can be sought by considering as follows:

- (i) Some nodes of a mesh network (MN) are to be identified as backup nodes that can take up the dual loads.
- (ii) The loads taken by the faulty nodes can be taken by the backup nodes.
- (iii) The removal of the backup nodes should result in connected components of MN.
- (iv) The count of backup nodes should be greater or equal to the size of each connected component.

Selecting such required backup components in the MN of minimum size is related to CSS problems. As an illustrative case, consider a double triangular circular mesh network T_6^2. The aim is to install routers on its nodes and specify certain nodes with dual capacities. In the view of Lemma 3, w_1, u_1, u_2, u_3 and w_4 are nodes of T_6^2 with such capacity. The robustness of MN can be enhanced by equipping these nodes to endure dual loads so that the MN remains operational in the presence of malfunctioning routers. Consider a double triangular circular mesh network T_6^2 as shown in Figure 5, then in the view of Lemma 3, $\mathcal{S}(T_6^2) = \{u_1, w_1, w_2, w_3, u_4\}$ is a CSS, and $D_1 = \{u_2, u_3\}$, $D_2 = \{w_4, w_5, w_6, u_5, u_6, \}$ are components of $C(T_6^2 \setminus \mathcal{S}(T_6^2))$. It is clear from Figure 5 that

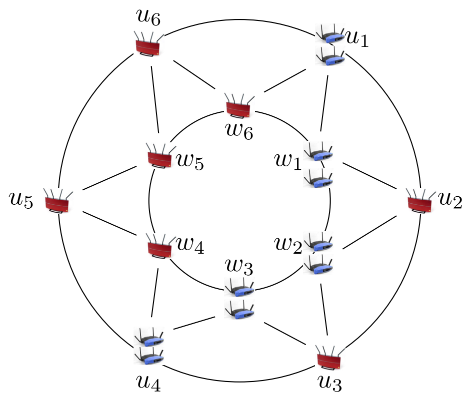

FIGURE 5. Optimal router installation on T_6^2 using CSS.

disruptions or malfunctions in either D_1 or D_2 routers can potentially degrade the network's performance and security. This issue can be handled by shifting faulty components' loads to backup components $\mathcal{S}(T_6^2)$. This illustrates the applicability of CSS for optimal installation of routers on certain MNs.

VII. CONCLUSION

This article emphasizes the computation of CSS and CSN for various triangular and quadrangular mesh networks. Additionally, we demonstrate how CSS can be used to optimize router installation on specific mesh network nodes.

Computing the minimum value of CSS is an NP-complete problem [14]. Therefore, our results explain the complexities associated with these variants and their significance. Our findings reveal that both variants are order-dependent, which might limit their applicability in larger wireless communication networks (WCN). However, CSS decomposes the larger WCN into subnetworks, which allows more efficient processing and analysis, and it is also useful to handle larger networks in a convenient way for uninterrupted and seamless connectivity. For future studies, it is worth exploring the CSS and CSN of other types of mesh networks related to IoT.

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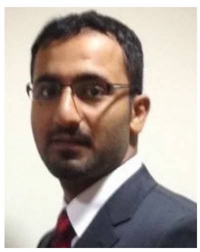

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