IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received 16 December 2023, accepted 11 January 2024, date of publication 16 January 2024, date of current version 29 January 2024.

Digital Object Identifier 10.1109/ACCESS.2024.3354823

RESEARCH ARTICLE

Circular Pythagorean Fuzzy Hamacher Aggregation Operators With Application in the Assessment of Goldmines

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This work was supported in part by the National Science and Technology Council, Taiwan, under Grant NSTC 112-2118-M-033-004.

ABSTRACT The circular Pythagorean fuzzy (Cir-PyF) sets (Cir-PyFSs) not only contain the membership and non-membership grades, but also have the radius around the circle of each element. Cir-PyFSs can cope with many real-life problems. In this paper, we consider the Hamacher t-norm and t-conorm operational laws for any two Cir-PyF numbers (Cir-PyFNs) and describe their exceptional cases such as algebraic and Einstein operational laws. Furthermore, the Cir-PyF Hamacher averaging (Cir-PyFHA) operator, Cir-PyF Hamacher ordered averaging (Cir-PyFHOA) operator, Cir-PyF Hamacher geometric (Cir-PyFHG) operator, and Cir-PyF Hamacher ordered geometric (Cir-PyFHOG) operator are proposed. Some properties and theorems for the above operators are also discussed in detail. Moreover, to select the simplest and best procedure for evaluating the source of gold in mines, we illustrate the application of the multi-attribute decision-making (MADM) technique based on the derived operators. Finally, we demonstrate some examples for comparing the proposed operators with some existing methods to expand the attraction of the proposed way.

INDEX TERMS Fuzzy sets, circular Pythagorean fuzzy sets, Hamacher aggregation operators, decision-making problem, assessment of goldmines.

I. INTRODUCTION

In our genuine life problems, we continuously face distinct types of decision-making problems, where our basic and initial focus is to learn how to make a better decision. So multi-attribute decision-making (MADM) becomes an important procedure in the decision sciences. However, there always exist ambiguity and uncertainty in our life issues in which we cannot make decisions by only using crisp data. To treat such issues, Zadeh [1] developed a novel theory of fuzzy sets (FSs) where FSs have a truth grade whose range is a unit interval, and had been applied in various areas for handling uncertainty arising from ambiguity and partial belongingness [2], [3], [4], [5]. During an election in any country, many people have noticed that some people have put their vote in favor of some candidates, some people have cast their vote against some candidates, some people have refused their vote, or some people have not shown has presence in the election because of some problems. In this case, many people have observed that the FSs theory is not enough for evaluating some awkward and unreliable problems, because FSs have only one grade. However, for most of these problems, we face positive, negative, and refusal responses, which is a very complicated problem and FSs cannot cope them. For this, Atanassov [5], [6] discovered the intuitionistic FSs (IFSs). IFSs fully fit the criteria of the above problems and the theory of IFSs is very suitable for evaluating or addressing the above dilemma, because IFSs have two grades, called truth "$\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}(x)$" and falsity "$\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}(x)$" with $0 \leq \mathbb{T}_{\mathcal{C}_{\mathcal{F}}}(x) + \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}(x) \leq 1$. IFSs have various applications because the mathematical form of IFSs is strong and reliable due to their condition [7], [8], [9], [10].

Although IFSs have many applications, but the condition of IFSs with $0 \leq \mathbb{T}_{\mathcal{C}_{\mathcal{F}}}(x) + \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}(x) \leq 1$ is failed when we have $\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}(x) = 0.6$ and $\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}(x) = 0.7$, i.e. $0.6 + 0.7 = 1.3 \notin [0, 1]$. Because of the above problem, many scholars

The associate editor coordinating the review of this manuscript and approving it for publication was Giovanni Pau.

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had avoided using it for addressing awkward and unreliable problems. Yager and Abbasov [11] exposed Pythagorean FSs (PyFSs), where PyFSs have the same structure as IFSs, but changed its condition with $0\leq\mathbb{T}^2_{\mathcal{C}_\mathcal{F}}(x)+\mathbb{F}^2_{\mathcal{C}_\mathcal{F}}(x)\leq 1$. PyFSs become the extensions of FSs and IFSs with various applications, such as [12], [13], and [14]. These different types of FSs extensions had taken the truth and falsity grades, but they avoided the angle between truth and falsity grades. For this, Atanassov [15] exposed a novel extension of IFSs, called circular IFSs (Cir-IFSs) in which Cir-IFSs not only cover the truth and falsity grades, but also a radius around the circle of each element. These Cir-IFSs had gotten more applications, such as [16], [17], and [18]. Additionally, Bozyigit et al. [19] gave the extension of Cir-IFSs to circular PyFSs (Cir-PyFSs) in which Cir-PyFSs contain the truth and falsity grades with radius, but replace the condition of $0\leq\mathbb{T}_{\mathcal{C}_\mathcal{F}}(x)+\mathbb{F}_{\mathcal{C}_\mathcal{F}}(x)\leq 1$ with the condition of $0\leq\mathbb{T}^2_{\mathcal{C}_\mathcal{F}}(x)+\mathbb{F}^2_{\mathcal{C}_\mathcal{F}}(x)\leq 1$.

Hamacher [20] exposed the novel theory of Hamacher norms by including a parameter in the modified version of algebraic norms, where the algebraic norms and Einstein norms are the subparts of the Hamacher norms. Huang [21] considered aggregation operators (AOs) for IFSs based on Hamacher norms. Garg [22] gave Hamacher AOs for IFSs with entropy weight, Wu and Wei [23] addressed the Hamacher AOs for PyFSs, and Ozer [24] gave Hamacher prioritized AOs for complex picture FSs. The idea of Cir-PyFSs was recently developed by Bozyigit et al. [19] in 2023, and no one had evaluated any kind of operators, methods, or measures based on Cir-PyFSs, but only was considered by Alsattar et al. [25] in developing sustainable smart living framework by the three-way decision-based conditional probabilities in Cir-PyFSs. In this paper, we focus on this new developed Cir-PyFSs, and give their Hamacher AOs with properties and applied them in the assessment of goldmines. Thus, the major contributions of this paper are listed as follows:

1) To determine the Hamacher operational laws for any two Cir-PyFSs and describe their exceptional cases like algebraic and Einstein operational laws.
2) To derive the Cir-PyF Hamacher averaging (Cir-PyFHA), Cir-PyF Hamacher ordered averaging (Cir-PyFHOA), Cir-PyF Hamacher geometric (Cir-PyFHG), and Cir-PyF Hamacher ordered geometric (Cir-PyFHOG) operators the Cir-PyFHA operator.
3) To discuss the properties of the proposed operators on Cir-PyFSs.
4) To select the best and simplest procedure for evaluating the source of gold in mines, we construct the MADM technique based on the derived operators.
5) To demonstrate some examples for comparing the derived operators with some existing operators.

This paper is arranged as follows. In Section II, we discuss the Cir-PyFSs and their operational laws with Hamacher t-norm and t-conorm. In Section III, we give the Hamacher operational laws for any two Cir-PyFNs and describe their exceptional cases like algebraic and Einstein operational laws. Furthermore, the Cir-PyFHA, Cir-PyFHOA, Cir-PyFHG, and Cir-PyFHOG operators are proposed. Additionally, some properties and unique cases for the above operators are also discussed in detail. In Section IV, we illustrate the MADM technique based on the derived operators and then applied it in selecting the best and simplest procedure for evaluating the source of gold in mines. In Section V, we demonstrate some examples for comparing the invented operators with some existing operators to expand the attraction of the proposed operators. Some concluding remarks are stated in Section VI.

II. PRELIMANARIES

This section briefly explains one of important ideas, such as circular Pythagorean fuzzysets (Cir-PyFSs) and their operational laws. Further, we state the basic idea of Hamacher t-norm and t-conorm.

Definition 1: [19] Let $x$be a universal set. The Cir-PyFS $\mathcal{C}_\mathcal{F}$ is defined as:

$$\mathcal{C}_\mathcal{F}=\left\{\left(\mathbb{T}_{\mathcal{C}_\mathcal{F}}(x),\mathbb{F}_{\mathcal{C}_\mathcal{F}}(x),\mathbb{R}_{\mathcal{C}_\mathcal{F}}(x)\right):x\in\mathcal{X}\right\}$$

where $0\leq\mathbb{T}^2_{\mathcal{C}_\mathcal{F}}(x)+\mathbb{F}^2_{\mathcal{C}_\mathcal{F}}(x)\leq 1$ in which the term $\mathbb{T}_{\mathcal{C}_\mathcal{F}}(x)$ represents the truth grade, $\mathbb{F}_{\mathcal{C}_\mathcal{F}}(x)$ represents the falsity grade, and $\mathbb{R}_{\mathcal{C}_\mathcal{F}}(x)$ states the radius around the circle of each element. Furthermore, the refusal grade is defined as $\mathfrak{P}=\left(1-\left(\mathbb{T}^2_{\mathcal{C}_\mathcal{F}}(x)+\mathbb{F}^2_{\mathcal{C}_\mathcal{F}}(x)\right)\right)^{\frac{1}{2}}$ and a circular Pythagorean fuzzy number (Cir-PyFN) is expressed with $\mathcal{C}^j_\mathcal{F}=\left(\mathbb{T}^j_{\mathcal{C}_\mathcal{F}},\mathbb{F}^j_{\mathcal{C}_\mathcal{F}},\mathbb{R}^j_{\mathcal{C}_\mathcal{F}}\right),j=1,2,\ldots,z.$

Definition 2: [19] Let $\mathcal{C}^j_\mathcal{F}=\left(\mathbb{T}^j_{\mathcal{C}_\mathcal{F}},\mathbb{F}^j_{\mathcal{C}_\mathcal{F}},\mathbb{R}^j_{\mathcal{C}_\mathcal{F}}\right)$, $j=1,2$ be any two Cir-PyFNs. Then, the following operators are defined as shown in the equation at the bottom of the next page.

Definition 3: [19] Let $\mathcal{C}^j_\mathcal{F}=\left(\mathbb{T}^j_{\mathcal{C}_\mathcal{F}},\mathbb{F}^j_{\mathcal{C}_\mathcal{F}},\mathbb{R}^j_{\mathcal{C}_\mathcal{F}}\right)$, $j=1,2$ be any two Cir-PyFNs. Then, the score value $S\left(\mathcal{C}^j_\mathcal{F}\right)$ and accuracy value $H\left(\mathcal{C}^j_\mathcal{F}\right)$ are defined as:

$$S\left(\mathcal{C}^j_\mathcal{F}\right)=\left(\left(\mathbb{T}^j_{\mathcal{C}_\mathcal{F}}\right)^2-\left(\mathbb{F}^j_{\mathcal{C}_\mathcal{F}}\right)^2\right)*\mathbb{R}^j_{\mathcal{C}_\mathcal{F}}\in[-1,1];$$

$$H\left(\mathcal{C}^j_\mathcal{F}\right)=\left(\left(\mathbb{T}^j_{\mathcal{C}_\mathcal{F}}\right)^2+\left(\mathbb{F}^j_{\mathcal{C}_\mathcal{F}}\right)^2\right)*\mathbb{R}^j_{\mathcal{C}_\mathcal{F}}\in[0,1].$$

The score and accuracy values with great rules have that, if $S\left(\mathcal{C}^1_\mathcal{F}\right)>S\left(\mathcal{C}^2_\mathcal{F}\right)$, then $\mathcal{C}^1_\mathcal{F}>\mathcal{C}^2_\mathcal{F}$; If $H\left(\mathcal{C}^1_\mathcal{F}\right)>H\left(\mathcal{C}^2_\mathcal{F}\right)$, then $\mathcal{C}^1_\mathcal{F}>\mathcal{C}^2_\mathcal{F}$.

Definition 4: [20] Let $\mathcal{C}^j_\mathcal{F}\in[0,1],j=1,2$ be any two non-negative integers. Then, Hamacher t-norm and t-conorm for $\emptyset^s_c>0$ are defined as:

$$DTN\left(\mathcal{C}^1_\mathcal{F},\mathcal{C}^2_\mathcal{F}\right)=\frac{\mathcal{C}^1_\mathcal{F}*\mathcal{C}^2_\mathcal{F}}{\emptyset^s_c+\left(1-\emptyset^s_c\right)\left(\mathcal{C}^1_\mathcal{F}+\mathcal{C}^2_\mathcal{F}-\mathcal{C}^1_\mathcal{F}*\mathcal{C}^2_\mathcal{F}\right)};$$

$$DTCN\left(\mathcal{C}^1_\mathcal{F},\mathcal{C}^2_\mathcal{F}\right)=\frac{\mathcal{C}^1_\mathcal{F}+\mathcal{C}^2_\mathcal{F}-\mathcal{C}^1_\mathcal{F}*\mathcal{C}^2_\mathcal{F}-\left(1-\emptyset^s_c\right)*\mathcal{C}^1_\mathcal{F}*\mathcal{C}^2_\mathcal{F}}{1-\left(1-\emptyset^s_c\right)\left(\mathcal{C}^1_\mathcal{F}*\mathcal{C}^2_\mathcal{F}\right)}.$$

Thus, for $\emptyset_c^s = 1, 2$, we can easily derive the algebraic and Einstein norms, respectively.

III. HAMACHER AOs FOR CIR-PyFSs

In this section, we evaluate the following operators, Cir-PyF Hamacher averaging (Cir-PyFHA), Cir-PyF Hamacher ordered averaging (Cir-PyFHOA), Cir-PyF Hamacher geometric (Cir-PyFHG), and Cir-PyF Hamacher ordered geometric (Cir-PyFHOG) operators, based on Hamacher operational laws for Cir-PyFSs. Further, we discuss three basic properties of the above operators. From now on, we consider the collection of Cir-PyFNs with $\mathcal{C}_{\mathcal{F}}^{j} = \left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right), j = 1, 2, \ldots, z$.

Definition 5: The following Hamacher operational laws are defined, as shown in the equation at the bottom of pages 4 and 5.

Definition 6: The Cir-PyFHAoperators are defined as:

$$\begin{aligned}
&Cir_{PyFHA_{TN}}\left(\mathcal{C}_{\mathcal{F}}^{1},\mathcal{C}_{\mathcal{F}}^{2},\ldots,\mathcal{C}_{\mathcal{F}}^{z}\right)\\
&\quad=\mathcal{U}_{TN}^{1}\mathcal{C}_{\mathcal{F}}^{1}\oplus_{TN}\mathcal{U}_{TN}^{2}\mathcal{C}_{\mathcal{F}}^{2}\oplus_{TN}\ldots\oplus_{TN}\mathcal{U}_{TN}^{z}\mathcal{C}_{\mathcal{F}}^{z}\\
&\quad=\oplus_{TN}{}_{j=1}^{z}\left(\mathcal{U}_{TN}^{j}\mathcal{C}_{\mathcal{F}}^{j}\right);\\
&Cir_{PyFHA_{TCN}}\left(\mathcal{C}_{\mathcal{F}}^{1},\mathcal{C}_{\mathcal{F}}^{2},\ldots,\mathcal{C}_{\mathcal{F}}^{z}\right)\\
&\quad=\mathcal{U}_{TCN}^{1}\mathcal{C}_{\mathcal{F}}^{1}\oplus_{TCN}\mathcal{U}_{TCN}^{2}\mathcal{C}_{\mathcal{F}}^{2}\oplus_{TCN}\ldots\oplus_{TCN}\mathcal{U}_{TCN}^{z}\mathcal{C}_{\mathcal{F}}^{z}\\
&\quad=\oplus_{TCN}{}_{j=1}^{z}\left(\mathcal{U}_{TCN}^{j}\mathcal{C}_{\mathcal{F}}^{j}\right).
\end{aligned}$$

Further, for more simplification, we consider the weight vector $\mathcal{U}_{TN}^{j} \in [0, 1]$ with $\sum_{j=1}^{z} \mathcal{U}_{TN}^{j} = 1$.

Theorem 1: For evaluating the operators defined in Definition 6, we obtain $Cir\_PyFHA_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right)$ and $Cir\_PyFHA_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right)$ for Cir-PyFSs as follows, as shown in the equation at the bottom of page 6.

Proof: With the help of mathematical induction, we can prove it. First, we set $z = 2$. Then, as shown in the equation at the bottom of page 6. and

$$\mathcal{U}_{TN}^{2}\mathcal{C}_{\mathcal{F}}^{2} = \left(\begin{array}{c}
\left(\dfrac{\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)^{\mathcal{U}_{TN}^{2}}-\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)^{\mathcal{U}_{TN}^{2}}}{\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)^{\mathcal{U}_{TN}^{2}}+(\emptyset_c^s-1)\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)^{\mathcal{U}_{TN}^{2}}}\right)^{\frac{1}{2}},\\
\dfrac{\left(\emptyset_c^s\right)^{1/2}*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{\mathcal{U}_{TN}^{2}}}{\left(\left(1+(\emptyset_c^s-1)\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)\right)^{\mathcal{U}_{TN}^{1}}+(\emptyset_c^s-1)\left(\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)^{\mathcal{U}_{TN}^{2}}\right)^{\frac{1}{2}}},\\
\left(\dfrac{\left(1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)^{\mathcal{U}_{TN}^{2}}-\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)^{\mathcal{U}_{TN}^{2}}}{\left(1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)^{\mathcal{U}_{TN}^{2}}+(\emptyset_c^s-1)\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)^{\mathcal{U}_{TN}^{2}}}\right)^{\frac{1}{2}}
\end{array}\right).$$

Thus, as shown in the equation at the bottom of page 7.

$$\mathcal{C}_{\mathcal{F}}^{1}\oplus_{TN}\mathcal{C}_{\mathcal{F}}^{2} = \left(\left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)^{\frac{1}{2}},\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right),\left(\begin{array}{c}\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\\-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\end{array}\right)^{\frac{1}{2}}\right);$$

$$\mathcal{C}_{\mathcal{F}}^{1}\oplus_{TCN}\mathcal{C}_{\mathcal{F}}^{2} = \left(\left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)^{\frac{1}{2}},\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right),\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)\right);$$

$$\mathcal{C}_{\mathcal{F}}^{1}\otimes_{TN}\mathcal{C}_{\mathcal{F}}^{2} = \left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right),\left(\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)^{\frac{1}{2}},\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)\right);$$

$$\mathcal{C}_{\mathcal{F}}^{1}\otimes_{TCN}\mathcal{C}_{\mathcal{F}}^{2} = \left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right),\left(\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)^{\frac{1}{2}},\left(\begin{array}{c}\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\\-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\end{array}\right)^{\frac{1}{2}}\right);$$

$$\mathcal{U}_{TN}^{s}\mathcal{C}_{\mathcal{F}}^{1} = \left(\left(1-\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}\right)^{\mathcal{U}_{TN}^{s}}\right)^{\frac{1}{2}},\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{\mathcal{U}_{TN}^{s}},\left(1-\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}\right)^{\mathcal{U}_{TN}^{s}}\right)^{\frac{1}{2}}\right);$$

$$\mathcal{U}_{TCN}^{s}\mathcal{C}_{\mathcal{F}}^{1} = \left(\left(1-\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}\right)^{\mathcal{U}_{TCN}^{s}}\right)^{\frac{1}{2}},\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{\mathcal{U}_{TCN}^{s}},\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{\mathcal{U}_{TCN}^{s}}\right);$$

$$\left(\mathcal{C}_{\mathcal{F}}^{1}\right)^{\mathcal{U}_{TN}^{s}} = \left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{\mathcal{U}_{TN}^{s}},\left(1-\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}\right)^{\mathcal{U}_{TN}^{s}}\right)^{\frac{1}{2}},\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{\mathcal{U}_{TN}^{s}}\right);$$

$$\left(\mathcal{C}_{\mathcal{F}}^{1}\right)^{\mathcal{U}_{TCN}^{s}} = \left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{\mathcal{U}_{TCN}^{s}},\left(1-\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}\right)^{\mathcal{U}_{TCN}^{s}}\right)^{\frac{1}{2}},\left(1-\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}\right)^{\mathcal{U}_{TCN}^{s}}\right)^{\frac{1}{2}}\right).$$

Further, we assume that it is also held for $z = q$, thus, as shown in the equation at the bottom of page 8.

Finally, we evaluate that it should be also hold for $z = q + 1$. We have that, as shown in the equation at the bottom of page 9.

The required target is hold. After following the same procedure, we can easily derive the below theory, such as, as shown in the equation at the bottom of page 10.

Thus, this proves the theorem. □

Further, we simplify the basic properties of the above procedures.

Property 1: When $\mathcal{C}_{\mathcal{F}}^{j} = \mathcal{C}_{\mathcal{F}} = \left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}, \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}, \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right), j = 1, 2, \ldots, z$,then we have $Cir\_PyFHA_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) = \mathcal{C}_{\mathcal{F}}$ and $Cir\_PyFHA_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) = \mathcal{C}_{\mathcal{F}}$.

Proof: We know that $\mathcal{C}_{\mathcal{F}}^{j} = \mathcal{C}_{\mathcal{F}} = \left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}, \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}, \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)$, $j = 1, 2, \ldots, z$, thus, as shown in the equation at the bottom of page 11. We evaluate the same procedure and to have the following ideas, such as $Cir\_PyFHA_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) = \mathcal{C}_{\mathcal{F}}$. □

Property 2: When $\mathcal{C}_{\mathcal{F}}^{j} = \left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right) \leq \mathcal{C}_{\#}^{j} = \left(\mathbb{T}_{\mathcal{C}_{\#}}^{j}, \mathbb{F}_{\mathcal{C}_{\#}}^{j}, \mathbb{R}_{\mathcal{C}_{\#}}^{j}\right)$, then we have $Cir\_PyFHA_{TN}(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots,$

$$\mathcal{C}_{\mathcal{F}}^{1} \oplus_{TN} \mathcal{C}_{\mathcal{F}}^{2} = \left(\begin{array}{c} \left(\frac{\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-(1-\emptyset_{c}^{s})*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}}{1-(1-\emptyset_{c}^{s})\left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)}\right)^{\frac{1}{2}}, \\ \frac{\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)}{\left(\emptyset_{c}^{s}+(1-\emptyset_{c}^{s})\left(\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)\right)^{\frac{1}{2}}}, \\ \left(\frac{\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-(1-\emptyset_{c}^{s})*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}}{1-(1-\emptyset_{c}^{s})\left(\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)}\right)^{\frac{1}{2}} \end{array}\right);$$

$$\mathcal{C}_{\mathcal{F}}^{1} \oplus_{TCN} \mathcal{C}_{\mathcal{F}}^{2} = \left(\begin{array}{c} \left(\frac{\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-(1-\emptyset_{c}^{s})*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}}{1-(1-\emptyset_{c}^{s})\left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)}\right)^{\frac{1}{2}}, \\ \frac{\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)}{\left(\emptyset_{c}^{s}+(1-\emptyset_{c}^{s})\left(\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)\right)^{\frac{1}{2}}}, \\ \frac{\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)}{\left(\emptyset_{c}^{s}+(1-\emptyset_{c}^{s})\left(\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)\right)^{\frac{1}{2}}} \end{array}\right);$$

$$\mathcal{C}_{\mathcal{F}}^{1} \otimes_{TN} \mathcal{C}_{\mathcal{F}}^{2} = \left(\begin{array}{c} \frac{\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)}{\left(\emptyset_{c}^{s}+(1-\emptyset_{c}^{s})\left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)\right)^{\frac{1}{2}}}, \\ \left(\frac{\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-(1-\emptyset_{c}^{s})*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}}{1-(1-\emptyset_{c}^{s})\left(\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)}\right)^{\frac{1}{2}}, \\ \frac{\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)}{\left(\emptyset_{c}^{s}+(1-\emptyset_{c}^{s})\left(\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)\right)^{\frac{1}{2}}} \end{array}\right);$$

$$\mathcal{C}_{\mathcal{F}}^{1} \otimes_{TCN} \mathcal{C}_{\mathcal{F}}^{2} = \left(\begin{array}{c} \frac{\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)}{\left(\emptyset_{c}^{s}+(1-\emptyset_{c}^{s})\left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)\right)^{\frac{1}{2}}}, \\ \left(\frac{\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-(1-\emptyset_{c}^{s})*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}}{1-(1-\emptyset_{c}^{s})\left(\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)}\right)^{\frac{1}{2}}, \\ \left(\frac{\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}+\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}-(1-\emptyset_{c}^{s})*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}}{1-(1-\emptyset_{c}^{s})\left(\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{2}*\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{2}\right)^{2}\right)}\right)^{\frac{1}{2}} \end{array}\right);$$

$\mathcal{C}^z_{\mathcal{F}}) \leq Cir\_PyFHA_{TN}\left(\mathcal{C}^1_{\#}, \mathcal{C}^2_{\#}, \ldots, \mathcal{C}^z_{\#}\right)$ and Cir_PyFHA_{TCN} $(\mathcal{C}^1_{\mathcal{F}}, \mathcal{C}^2_{\mathcal{F}}, \ldots, \mathcal{C}^z_{\mathcal{F}}) \leq Cir\_PyFHA_{TCN}\left(\mathcal{C}^1_{\#}, \mathcal{C}^2_{\#}, \ldots, \mathcal{C}^z_{\#}\right)$.

Proof: We know that $\mathcal{C}^j_{\mathcal{F}} = \left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}, \mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}, \mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right) \leq \mathcal{C}^j_{\#} = \left(\mathbb{T}^j_{\mathcal{C}_{\#}}, \mathbb{F}^j_{\mathcal{C}_{\#}}, \mathbb{R}^j_{\mathcal{C}_{\#}}\right)$. Thus, $\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}} \leq \mathbb{T}^j_{\mathcal{C}_{\#}}$, $\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}} \geq \mathbb{F}^j_{\mathcal{C}_{\#}}$ and $\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}} \leq \mathbb{R}^j_{\mathcal{C}_{\#}}$. Then, as shown in the equation at the bottom of page 12.

Additionally, we have, as shown in the equation at the bottom of page 12.

Finally, we evaluate the radius, such as, as shown in the equation at the bottom of page 12.

After evaluating the above inequality, we can easily derive our required result based on the score values,

$$\mathcal{U}^s_{TN}\mathcal{C}^1_{\mathcal{F}} = \left(\begin{array}{c} \left(\dfrac{\left(1+(\varnothing^s_c-1)\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}} - \left(1-\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}}{\left(1+(\varnothing^s_c-1)\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}} + (\varnothing^s_c-1)\left(1-\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}}\right)^{\frac{1}{2}}, \\ \dfrac{(\varnothing^s_c)^{1/2} * \left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^s_{TN}}}{\left(\left(1+(\varnothing^s_c-1)\left(1-\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^s_{TN}} + (\varnothing^s_c-1)\left(\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}\right)^{\frac{1}{2}}}, \\ \left(\dfrac{\left(1+(\varnothing^s_c-1)\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}} - \left(1-\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}}{\left(1+(\varnothing^s_c-1)\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}} + (\varnothing^s_c-1)\left(1-\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}}\right)^{\frac{1}{2}} \end{array}\right);$$

$$\mathcal{U}^s_{TCN}\mathcal{C}^1_{\mathcal{F}} = \left(\begin{array}{c} \left(\dfrac{\left(1+(\varnothing^s_c-1)\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}} - \left(1-\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}}{\left(1+(\varnothing^s_c-1)\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}} + (\varnothing^s_c-1)\left(1-\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}}\right)^{\frac{1}{2}}, \\ \dfrac{(\varnothing^s_c)^{1/2} * \left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^s_{TN}}}{\left(\left(1+(\varnothing^s_c-1)\left(1-\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^s_{TN}} + (\varnothing^s_c-1)\left(\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}\right)^{\frac{1}{2}}}, \\ \dfrac{(\varnothing^s_c)^{1/2} * \left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^s_{TN}}}{\left(\left(1+(\varnothing^s_c-1)\left(1-\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^s_{TN}} + (\varnothing^s_c-1)\left(\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}\right)^{\frac{1}{2}}} \end{array}\right);$$

$$\left(\mathcal{C}^1_{\mathcal{F}}\right)^{\mathcal{U}^s_{TN}} = \left(\begin{array}{c} \dfrac{(\varnothing^s_c)^{1/2} * \left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^s_{TN}}}{\left(\left(1+(\varnothing^s_c-1)\left(1-\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^s_{TN}} + (\varnothing^s_c-1)\left(\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}\right)^{\frac{1}{2}}}, \\ \left(\dfrac{\left(1+(\varnothing^s_c-1)\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}} - \left(1-\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}}{\left(1+(\varnothing^s_c-1)\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}} + (\varnothing^s_c-1)\left(1-\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}}\right)^{\frac{1}{2}}, \\ \dfrac{(\varnothing^s_c)^{1/2} * \left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^s_{TN}}}{\left(\left(1+(\varnothing^s_c-1)\left(1-\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^s_{TN}} + (\varnothing^s_c-1)\left(\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}\right)^{\frac{1}{2}}} \end{array}\right);$$

$$\left(\mathcal{C}^1_{\mathcal{F}}\right)^{\mathcal{U}^s_{TCN}} = \left(\begin{array}{c} \dfrac{(\varnothing^s_c)^{1/2} * \left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^s_{TN}}}{\left(\left(1+(\varnothing^s_c-1)\left(1-\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^s_{TN}} + (\varnothing^s_c-1)\left(\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}\right)^{\frac{1}{2}}}, \\ \left(\dfrac{\left(1+(\varnothing^s_c-1)\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}} - \left(1-\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}}{\left(1+(\varnothing^s_c-1)\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}} + (\varnothing^s_c-1)\left(1-\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}}\right)^{\frac{1}{2}}, \\ \left(\dfrac{\left(1+(\varnothing^s_c-1)\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}} - \left(1-\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}}{\left(1+(\varnothing^s_c-1)\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}} + (\varnothing^s_c-1)\left(1-\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^s_{TN}}}\right)^{\frac{1}{2}} \end{array}\right).$$

such as, $Cir\_PyFHA_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) \leq Cir\_PyFHA_{TN}\left(\mathcal{C}_{\#}^{1}, \mathcal{C}_{\#}^{2}, \ldots, \mathcal{C}_{\#}^{z}\right)$. We evaluate the same procedure and try to evaluate the following ideas, such as $Cir\_PyFHA_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) \leq Cir\_PyFHA_{TCN}\left(\mathcal{C}_{\#}^{1}, \mathcal{C}_{\#}^{2}, \ldots, \mathcal{C}_{\#}^{z}\right)$. □

Property 3: When

$$\mathcal{C}_{\mathcal{F}_{TN}}^{-}=\left(\min_j \mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \max_j \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \min_j \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right),$$
$$\mathcal{C}_{\mathcal{F}_{TCN}}^{-}=\left(\min_j \mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \max_j \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \max_j \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)$$

and

$$\mathcal{C}_{\mathcal{F}_{TN}}^{+}=\left(\max_j \mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \min_j \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \max_j \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right),$$
$$\mathcal{C}_{\mathcal{F}_{TCN}}^{+}=\left(\max_j \mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \min_j \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \min_j \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right),$$

we have $\mathcal{C}_{\mathcal{F}_{TN}}^{-} \leq Cir\_PyFHA_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) \leq \mathcal{C}_{\mathcal{F}_{TN}}^{+}$ and $\mathcal{C}_{\mathcal{F}_{TCN}}^{-} \leq Cir\_PyFHA_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) \leq \mathcal{C}_{\mathcal{F}_{TCN}}^{+}$.

Proof: To choose the information in Property 1 and Property 2, we have $Cir\_PyFHA_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) \leq Cir\_PyFHA_{TN}\left(\mathcal{C}^{+1}_{\mathcal{F}}, \mathcal{C}^{+2}_{\mathcal{F}}, \ldots, \mathcal{C}^{+z}_{\mathcal{F}}\right) = \mathcal{C}_{\mathcal{F}_{TN}}^{+}$ and

$$Cir\_PyFHA_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right)=\left(\begin{array}{c}\left(\frac{\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}-\prod_{j=1}^{z}\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}}{\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}+(\emptyset_c^s-1)\prod_{j=1}^{z}\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}}\right)^{\frac{1}{2}}, \\ \frac{(\emptyset_c^s)^{1/2} * \prod_{j=1}^{z}\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^{\mathcal{U}_{TN}^{j}}}{\left(\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)\right)^{\mathcal{U}_{TN}^{j}}+(\emptyset_c^s-1)\prod_{j=1}^{z}\left(\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}\right)^{\frac{1}{2}}}, \\ \left(\frac{\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}-\prod_{j=1}^{z}\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}}{\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}+(\emptyset_c^s-1)\prod_{j=1}^{z}\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}}\right)^{\frac{1}{2}}\end{array}\right);$$

$$Cir\_PyFHA_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right)=\left(\begin{array}{c}\left(\frac{\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}-\prod_{j=1}^{z}\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}}{\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}+(\emptyset_c^s-1)\prod_{j=1}^{z}\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}}\right)^{\frac{1}{2}}, \\ \frac{(\emptyset_c^s)^{1/2} * \prod_{j=1}^{z}\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^{\mathcal{U}_{TCN}^{j}}}{\left(\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)\right)^{\mathcal{U}_{TCN}^{j}}+(\emptyset_c^s-1)\prod_{j=1}^{z}\left(\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}\right)^{\frac{1}{2}}}, \\ \frac{(\emptyset_c^s)^{1/2} * \prod_{j=1}^{z}\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^{\mathcal{U}_{TCN}^{j}}}{\left(\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)\right)^{\mathcal{U}_{TCN}^{j}}+(\emptyset_c^s-1)\prod_{j=1}^{z}\left(\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}\right)^{\frac{1}{2}}}\end{array}\right).$$

$$\mathcal{U}_{TN}^{1}\mathcal{C}_{\mathcal{F}}^{1}=\left(\begin{array}{c}\left(\frac{\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^2\right)^{\mathcal{U}_{TN}^{1}}-\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^2\right)^{\mathcal{U}_{TN}^{1}}}{\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^2\right)^{\mathcal{U}_{TN}^{1}}+(\emptyset_c^s-1)\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^2\right)^{\mathcal{U}_{TN}^{1}}}\right)^{\frac{1}{2}}, \\ \frac{(\emptyset_c^s)^{1/2} * \left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^{\mathcal{U}_{TN}^{1}}}{\left(\left(1+(\emptyset_c^s-1)\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^2\right)\right)^{\mathcal{U}_{TN}^{1}}+(\emptyset_c^s-1)\left(\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^2\right)^{\mathcal{U}_{TN}^{1}}\right)^{\frac{1}{2}}}, \\ \left(\frac{\left(1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^2\right)^{\mathcal{U}_{TN}^{1}}-\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^2\right)^{\mathcal{U}_{TN}^{1}}}{\left(1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^2\right)^{\mathcal{U}_{TN}^{1}}+(\emptyset_c^s-1)\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{1}\right)^2\right)^{\mathcal{U}_{TN}^{1}}}\right)^{\frac{1}{2}}\end{array}\right),$$

$Cir\_PyFHA_{TN}\left(\mathcal{C}^1_{\mathcal{F}},\mathcal{C}^2_{\mathcal{F}},\ldots,\mathcal{C}^z_{\mathcal{F}}\right)\geq Cir\_PyFHA_{TN}\left(\mathcal{C}^{-1}_{\mathcal{F}},\mathcal{C}^{-2}_{\mathcal{F}},\ldots,\mathcal{C}^{-z}_{\mathcal{F}}\right)=\mathcal{C}^{-}_{\mathcal{F}_{TN}}$. Thus, $\mathcal{C}^{-}_{\mathcal{F}_{TN}}\leq Cir\_PyFHA_{TN}\left(\mathcal{C}^1_{\mathcal{F}},\mathcal{C}^2_{\mathcal{F}},\ldots,\mathcal{C}^z_{\mathcal{F}}\right)\leq\mathcal{C}^{+}_{\mathcal{F}_{TN}}$. We evaluate the same procedure to evaluate the following result $\mathcal{C}^{-}_{\mathcal{F}_{TCN}}\leq Cir\_PyFHA_{TCN}\left(\mathcal{C}^1_{\mathcal{F}},\mathcal{C}^2_{\mathcal{F}},\ldots,\mathcal{C}^z_{\mathcal{F}}\right)\leq\mathcal{C}^{+}_{\mathcal{F}_{TCN}}$. □

Definition 7: The Cir-PyFHOA operators are defined as follows:

$$Cir\_PyFHOA_{TN}\left(\mathcal{C}^1_{\mathcal{F}},\mathcal{C}^2_{\mathcal{F}},\ldots,\mathcal{C}^z_{\mathcal{F}}\right)=\mathcal{U}^1_{TN}\mathcal{C}^{o(1)}_{\mathcal{F}}\oplus_{TN}\mathcal{U}^2_{TN}\mathcal{C}^{o(2)}_{\mathcal{F}}\oplus_{TN}\ldots\oplus_{TN}\mathcal{U}^z_{TN}\mathcal{C}^{o(z)}_{\mathcal{F}}=\oplus_{TN}{}_{j=1}^{z}\left(\mathcal{U}^j_{TN}\mathcal{C}^{o(j)}_{\mathcal{F}}\right);$$

$$Cir\_PyFHOA_{TCN}\left(\mathcal{C}^1_{\mathcal{F}},\mathcal{C}^2_{\mathcal{F}},\ldots,\mathcal{C}^z_{\mathcal{F}}\right)=\mathcal{U}^1_{TCN}\mathcal{C}^{o(1)}_{\mathcal{F}}\oplus_{TCN}\mathcal{U}^2_{TCN}\mathcal{C}^{o(2)}_{\mathcal{F}}\oplus_{TCN}\ldots\oplus_{TCN}\mathcal{U}^z_{TCN}\mathcal{C}^{o(z)}_{\mathcal{F}}=\oplus_{TCN}{}_{j=1}^{z}\left(\mathcal{U}^j_{TCN}\mathcal{C}^{o(j)}_{\mathcal{F}}\right).$$

Further for more simplification, we consider the weight vector $\mathcal{U}^j_{TN}\in[0,1]$ with $\sum_{j=1}^{z}\mathcal{U}^j_{TN}=1$, where $o(j)\leq o(j-1)$.

Theorem 2: For evaluating the operators defined in Definition 7, we obtain $Cir\_PyFHOA_{TN}\left(\mathcal{C}^1_{\mathcal{F}},\mathcal{C}^2_{\mathcal{F}},\ldots,\mathcal{C}^z_{\mathcal{F}}\right)$ and $Cir\_PyFHOA_{TCN}\left(\mathcal{C}^1_{\mathcal{F}},\mathcal{C}^2_{\mathcal{F}},\ldots,\mathcal{C}^z_{\mathcal{F}}\right)$ for Cir-PyFSs as follows, as shown in the equation at the bottom of the page 13.

$$Cir\_PyFHA_{TN}\left(\mathcal{C}^1_{\mathcal{F}},\mathcal{C}^2_{\mathcal{F}}\right)=\mathcal{U}^1_{TN}\mathcal{C}^1_{\mathcal{F}}\oplus_{TN}\mathcal{U}^2_{TN}\mathcal{C}^2_{\mathcal{F}}=\left(\begin{array}{c}\left(\begin{array}{c}\left(\frac{\left(1+(\emptyset^s_c-1)\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^1_{TN}}-\left(1-\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^1_{TN}}}{\left(1+(\emptyset^s_c-1)\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^1_{TN}}+(\emptyset^s_c-1)\left(1-\left(\mathbb{T}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^1_{TN}}}\right)^{\frac{1}{2}},\\ \frac{(\emptyset^s_c)^{\frac{1}{2}}*\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^1_{TN}}}{\left(\left(1+(\emptyset^s_c-1)\left(1-\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^1_{TN}}+(\emptyset^s_c-1)\left(\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^1_{TN}}\right)^{\frac{1}{2}}},\\ \left(\frac{\left(1+(\emptyset^s_c-1)\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^1_{TN}}-\left(1-\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^1_{TN}}}{\left(1+(\emptyset^s_c-1)\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^1_{TN}}+(\emptyset^s_c-1)\left(1-\left(\mathbb{R}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^1_{TN}}}\right)^{\frac{1}{2}}\end{array}\right)\oplus_{TN}\\ \left(\begin{array}{c}\left(\frac{\left(1+(\emptyset^s_c-1)\left(\mathbb{T}^2_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^2_{TN}}-\left(1-\left(\mathbb{T}^2_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^2_{TN}}}{\left(1+(\emptyset^s_c-1)\left(\mathbb{T}^2_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^2_{TN}}+(\emptyset^s_c-1)\left(1-\left(\mathbb{T}^2_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^2_{TN}}}\right)^{\frac{1}{2}},\\ \frac{(\emptyset^s_c)^{1/2}*\left(\mathbb{F}^2_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^2_{TN}}}{\left(\left(1+(\emptyset^s_c-1)\left(1-\left(\mathbb{F}^2_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^1_{TN}}+(\emptyset^s_c-1)\left(\left(\mathbb{F}^2_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^2_{TN}}\right)^{\frac{1}{2}}},\\ \left(\frac{\left(1+(\emptyset^s_c-1)\left(\mathbb{R}^2_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^2_{TN}}-\left(1-\left(\mathbb{R}^2_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^2_{TN}}}{\left(1+(\emptyset^s_c-1)\left(\mathbb{R}^2_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^2_{TN}}+(\emptyset^s_c-1)\left(1-\left(\mathbb{R}^2_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^2_{TN}}}\right)^{\frac{1}{2}}\end{array}\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\frac{\prod_{j=1}^{2}\left(1+(\emptyset^s_c-1)\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}-\prod_{j=1}^{2}\left(1-\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{2}\left(1+(\emptyset^s_c-1)\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}+(\emptyset^s_c-1)\prod_{j=1}^{2}\left(1-\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}\right)^{\frac{1}{2}},\\ \frac{(\emptyset^s_c)^{1/2}*\prod_{j=1}^{2}\left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^j_{TN}}}{\left(\prod_{j=1}^{2}\left(1+(\emptyset^s_c-1)\left(1-\left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^j_{TN}}+(\emptyset^s_c-1)\prod_{j=1}^{2}\left(\left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}\right)^{\frac{1}{2}}},\\ \left(\frac{\prod_{j=1}^{2}\left(1+(\emptyset^s_c-1)\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}-\prod_{j=1}^{2}\left(1-\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{2}\left(1+(\emptyset^s_c-1)\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}+(\emptyset^s_c-1)\prod_{j=1}^{2}\left(1-\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}\right)^{\frac{1}{2}}\end{array}\right).$$

Proof: The proof is similar as those of Theorem 1.□

Further, we simplify the basic properties of the above procedures.

Property 4: When $\mathcal{C}_{\mathcal{F}}^{j} = \mathcal{C}_{\mathcal{F}} = \left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}, \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}, \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right), j = 1, 2, \ldots, z$, thus, $Cir\_PyFHOA_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) = \mathcal{C}_{\mathcal{F}}$ and $Cir\_PyFHOA_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) = \mathcal{C}_{\mathcal{F}}$.

Proof: The proof is similar as those of Property 1.□

Property 5: When $\mathcal{C}_{\mathcal{F}}^{j} = \left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right) \leq \mathcal{C}_{\#}^{j} = \left(\mathbb{T}_{\mathcal{C}_{\#}}^{j}, \mathbb{F}_{\mathcal{C}_{\#}}^{j}, \mathbb{R}_{\mathcal{C}_{\#}}^{j}\right)$, thus,

$$Cir\_PyFHOA_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) \leq Cir\_PyFHOA_{TN}\left(\mathcal{C}_{\#}^{1}, \mathcal{C}_{\#}^{2}, \ldots, \mathcal{C}_{\#}^{z}\right)$$

and

$$Cir\_PyFHOA_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) \leq Cir\_PyFHOA_{TCN}\left(\mathcal{C}_{\#}^{1}, \mathcal{C}_{\#}^{2}, \ldots, \mathcal{C}_{\#}^{z}\right).$$

Proof: The proof is similar as those of Property 2.□

Property 6: When $\mathcal{C}_{\mathcal{F}_{TN}}^{-} = \left(\min_j \mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \max_j \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \min_j \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)$, $\mathcal{C}_{\mathcal{F}_{TCN}}^{-} = \left(\min_j \mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \max_j \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \max_j \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)$ and $\mathcal{C}_{\mathcal{F}_{TN}}^{+} = \left(\max_j \mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \min_j \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \max_j \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)$, $\mathcal{C}_{\mathcal{F}_{TCN}}^{+} = \left(\max_j \mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \min_j \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \min_j \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)$, thus,

$\mathcal{C}_{\mathcal{F}_{TN}}^{-} \leq Cir\_PyFHOA_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) \leq \mathcal{C}_{\mathcal{F}_{TN}}^{+}$ and

$$\mathcal{C}_{\mathcal{F}_{TCN}}^{-} \leq Cir\_PyFHOA_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) \leq \mathcal{C}_{\mathcal{F}_{TCN}}^{+}.$$

Proof: The proof is similar as those of Property 3.□

Definition 8: The Cir-PyFHG operators are defined as follows:

$$Cir\_PyFHG_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) = \left(\mathcal{C}_{\mathcal{F}}^{1}\right)^{\mathcal{U}_{TN}^{1}} \otimes_{TN} \left(\mathcal{C}_{\mathcal{F}}^{2}\right)^{\mathcal{U}_{TN}^{2}} \otimes_{TN} \ldots \otimes_{TN} \left(\mathcal{C}_{\mathcal{F}}^{z}\right)^{\mathcal{U}_{TN}^{z}} = \otimes_{TN}{}_{j=1}^{z} \left(\mathcal{C}_{\mathcal{F}}^{j}\right)^{\mathcal{U}_{TN}^{j}}$$

$$Cir\_PyFHG_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) = \left(\mathcal{C}_{\mathcal{F}}^{1}\right)^{\mathcal{U}_{TCN}^{1}} \otimes_{TN} \left(\mathcal{C}_{\mathcal{F}}^{2}\right)^{\mathcal{U}_{TCN}^{2}} \otimes_{TN} \ldots \otimes_{TN} \left(\mathcal{C}_{\mathcal{F}}^{z}\right)^{\mathcal{U}_{TCN}^{z}} = \otimes_{TCN}{}_{j=1}^{z} \left(\mathcal{C}_{\mathcal{F}}^{j}\right)^{\mathcal{U}_{TCN}^{j}}.$$

Further for more simplification, we consider the weight vector $\mathcal{U}_{TN}^{j} \in [0, 1]$ with $\sum_{j=1}^{z} \mathcal{U}_{TN}^{j} = 1$.

Theorem 3: For evaluating the operators defined in Definition 8, we obtain $Cir\_PyFHG_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right)$ and $Cir\_PyFHG_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right)$ for Cir-PyFSs as follows, as shown in the equation at the bottom of the page 14.

Proof: The proof is similar as those of Theorem 1.□

Further, we simplify the basic properties of the above procedures.

Property 7: When $\mathcal{C}_{\mathcal{F}}^{j} = \mathcal{C}_{\mathcal{F}} = \left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}, \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}, \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)$, $j = 1, 2, \ldots, z$, thus,

$$Cir\_PyFHG_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) = \mathcal{C}_{\mathcal{F}}$$

and

$$Cir\_PyFHG_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) = \mathcal{C}_{\mathcal{F}}.$$

Proof: The proof is similar as those of Property 1.□

Property 8: When $\mathcal{C}_{\mathcal{F}}^{j} = \left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right) \leq \mathcal{C}_{\#}^{j} = \left(\mathbb{T}_{\mathcal{C}_{\#}}^{j}, \mathbb{F}_{\mathcal{C}_{\#}}^{j}, \mathbb{R}_{\mathcal{C}_{\#}}^{j}\right)$, thus,

$$Cir\_PyFHG_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) \leq Cir\_PyFHG_{TN}\left(\mathcal{C}_{\#}^{1}, \mathcal{C}_{\#}^{2}, \ldots, \mathcal{C}_{\#}^{z}\right)$$

and

$$Cir\_PyFHG_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) \leq Cir\_PyFHG_{TCN}\left(\mathcal{C}_{\#}^{1}, \mathcal{C}_{\#}^{2}, \ldots, \mathcal{C}_{\#}^{z}\right).$$

$$Cir\_PyFHA_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{q}\right) = \mathcal{U}_{TN}^{1}\mathcal{C}_{\mathcal{F}}^{1} \oplus_{TN} \mathcal{U}_{TN}^{2}\mathcal{C}_{\mathcal{F}}^{2} \oplus_{TN} \ldots \oplus_{TN} \mathcal{U}_{TN}^{q}\mathcal{C}_{\mathcal{F}}^{q} =$$

$$\oplus_{TN}{}_{j=1}^{q}\left(\mathcal{U}_{TN}^{j}\mathcal{C}_{\mathcal{F}}^{j}\right) = \begin{pmatrix} \left(\dfrac{\prod_{j=1}^{q}\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}} - \prod_{j=1}^{q}\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}}{\prod_{j=1}^{q}\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}} + (\emptyset_c^s-1)\prod_{j=1}^{q}\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}}\right)^{\frac{1}{2}}, \\ \dfrac{\left(\emptyset_c^s\right)^{1/2} * \prod_{j=1}^{q}\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^{\mathcal{U}_{TN}^{j}}}{\left(\prod_{j=1}^{q}\left(1+(\emptyset_c^s-1)\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)\right)^{\mathcal{U}_{TN}^{j}} + (\emptyset_c^s-1)\prod_{j=1}^{q}\left(\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}\right)^{\frac{1}{2}}}, \\ \left(\dfrac{\prod_{j=1}^{q}\left(1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}} - \prod_{j=1}^{q}\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}}{\prod_{j=1}^{q}\left(1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}} + (\emptyset_c^s-1)\prod_{j=1}^{q}\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}}\right)^{\frac{1}{2}} \end{pmatrix}.$$

$$
\begin{aligned}
&Cir\_PyFHA_{TN}\left(\mathcal{C}^1_{\mathcal{F}},\mathcal{C}^2_{\mathcal{F}},\ldots,\mathcal{C}^{q+1}_{\mathcal{F}}\right)=\mathcal{U}^1_{TN}\mathcal{C}^1_{\mathcal{F}}\oplus_{TN}\mathcal{U}^2_{TN}\mathcal{C}^2_{\mathcal{F}}\oplus_{TN}\ldots\oplus_{TN}\mathcal{U}^q_{TN}\mathcal{C}^q_{\mathcal{F}}\oplus_{TN}\mathcal{U}^{q+1}_{TN}\mathcal{C}^{q+1}_{\mathcal{F}}\\
&=\oplus_{TN}{}^{q}_{j=1}\left(\mathcal{U}^j_{TN}\mathcal{C}^j_{\mathcal{F}}\right)\oplus_{TN}\mathcal{U}^{q+1}_{TN}\mathcal{C}^{q+1}_{\mathcal{F}}\\
&=\left(\begin{array}{c}
\left(\frac{\prod_{j=1}^{q}\left(1+(\varnothing_c^s-1)\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}-\prod_{j=1}^{q}\left(1-\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{q}\left(1+(\varnothing_c^s-1)\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}+(\varnothing_c^s-1)\prod_{j=1}^{q}\left(1-\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}\right)^{\frac{1}{2}},\\
\frac{(\varnothing_c^s)^{1/2}*\prod_{j=1}^{q}\left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^j_{TN}}}{\left(\prod_{j=1}^{q}\left(1+(\varnothing_c^s-1)\left(1-\left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^j_{TN}}+(\varnothing_c^s-1)\prod_{j=1}^{q}\left(\left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}\right)^{\frac{1}{2}}},\\
\left(\frac{\prod_{j=1}^{q}\left(1+(\varnothing_c^s-1)\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}-\prod_{j=1}^{q}\left(1-\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{q}\left(1+(\varnothing_c^s-1)\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}+(\varnothing_c^s-1)\prod_{j=1}^{q}\left(1-\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}\right)^{\frac{1}{2}}
\end{array}\right)\oplus_{TN}\mathcal{U}^{q+1}_{TN}\mathcal{C}^{q+1}_{\mathcal{F}}=\\
&\left(\left(\begin{array}{c}
\left(\frac{\prod_{j=1}^{q}\left(1+(\varnothing_c^s-1)\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}-\prod_{j=1}^{q}\left(1-\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{q}\left(1+(\varnothing_c^s-1)\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}+(\varnothing_c^s-1)\prod_{j=1}^{q}\left(1-\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}\right)^{\frac{1}{2}},\\
\frac{(\varnothing_c^s)^{\frac{1}{2}}*\prod_{j=1}^{q}\left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^j_{TN}}}{\left(\prod_{j=1}^{q}\left(1+(\varnothing_c^s-1)\left(1-\left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^j_{TN}}+(\varnothing_c^s-1)\prod_{j=1}^{q}\left(\left(\mathbb{F}^1_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}\right)^{\frac{1}{2}}},\\
\left(\frac{\prod_{j=1}^{q}\left(1+(\varnothing_c^s-1)\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}-\prod_{j=1}^{q}\left(1-\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{q}\left(1+(\varnothing_c^s-1)\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}+(\varnothing_c^s-1)\prod_{j=1}^{q}\left(1-\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}\right)^{\frac{1}{2}}
\end{array}\right)\oplus_{TN}\right.\\
&\left.\left(\begin{array}{c}
\left(\frac{\left(1+(\varnothing_c^s-1)\left(\mathbb{T}^{q+1}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^{q+1}_{TN}}-\left(1-\left(\mathbb{T}^{q+1}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^{q+1}_{TN}}}{\left(1+(\varnothing_c^s-1)\left(\mathbb{T}^{q+1}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^{q+1}_{TN}}+(\varnothing_c^s-1)\left(1-\left(\mathbb{T}^{q+1}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^{q+1}_{TN}}}\right)^{\frac{1}{2}},\\
\frac{(\varnothing_c^s)^{\frac{1}{2}}*\left(\mathbb{F}^{q+1}_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^{q+1}_{TN}}}{\left(\left(1+(\varnothing_c^s-1)\left(1-\left(\mathbb{F}^{q+1}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^{q+1}_{TN}}+(\varnothing_c^s-1)\left(\left(\mathbb{F}^{q+1}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^{q+1}_{TN}}\right)^{\frac{1}{2}}},\\
\left(\frac{\left(1+(\varnothing_c^s-1)\left(\mathbb{R}^{q+1}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^{q+1}_{TN}}-\left(1-\left(\mathbb{R}^{q+1}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^{q+1}_{TN}}}{\left(1+(\varnothing_c^s-1)\left(\mathbb{R}^{q+1}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^{q+1}_{TN}}+(\varnothing_c^s-1)\left(1-\left(\mathbb{R}^{q+1}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^{q+1}_{TN}}}\right)^{\frac{1}{2}}
\end{array}\right)\right)\\
&=\left(\begin{array}{c}
\left(\frac{\prod_{j=1}^{q+1}\left(1+(\varnothing_c^s-1)\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}-\prod_{j=1}^{q+1}\left(1-\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{q+1}\left(1+(\varnothing_c^s-1)\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}+(\varnothing_c^s-1)\prod_{j=1}^{q+1}\left(1-\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}\right)^{\frac{1}{2}},\\
\frac{(\varnothing_c^s)^{\frac{1}{2}}*\prod_{j=1}^{q+1}\left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^j_{TN}}}{\left(\prod_{j=1}^{q+1}\left(1+(\varnothing_c^s-1)\left(1-\left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^j_{TN}}+(\varnothing_c^s-1)\prod_{j=1}^{q+1}\left(\left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}\right)^{\frac{1}{2}}},\\
\left(\frac{\prod_{j=1}^{q+1}\left(1+(\varnothing_c^s-1)\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}-\prod_{j=1}^{q+1}\left(1-\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{q+1}\left(1+(\varnothing_c^s-1)\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}+(\varnothing_c^s-1)\prod_{j=1}^{q+1}\left(1-\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}\right)^{\frac{1}{2}}
\end{array}\right).
\end{aligned}
$$

Proof: The proof is similar as those of Property 2.□

Property 9: When $\mathcal{C}^{-}_{\mathcal{F}_{TN}} = \left(\min_j \mathbb{T}^{j}_{\mathcal{C}_{\mathcal{F}}}, \max_j \mathbb{F}^{j}_{\mathcal{C}_{\mathcal{F}}}, \min_j \mathbb{R}^{j}_{\mathcal{C}_{\mathcal{F}}}\right), \mathcal{C}^{-}_{\mathcal{F}_{TCN}} = \left(\min_j \mathbb{T}^{j}_{\mathcal{C}_{\mathcal{F}}}, \max_j \mathbb{F}^{j}_{\mathcal{C}_{\mathcal{F}}}, \max_j \mathbb{R}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)$ and

$$\mathcal{C}^{+}_{\mathcal{F}_{TN}} = \left(\max_j \mathbb{T}^{j}_{\mathcal{C}_{\mathcal{F}}}, \min_j \mathbb{F}^{j}_{\mathcal{C}_{\mathcal{F}}}, \max_j \mathbb{R}^{j}_{\mathcal{C}_{\mathcal{F}}}\right),$$
$$\mathcal{C}^{+}_{\mathcal{F}_{TCN}} = \left(\max_j \mathbb{T}^{j}_{\mathcal{C}_{\mathcal{F}}}, \min_j \mathbb{F}^{j}_{\mathcal{C}_{\mathcal{F}}}, \min_j \mathbb{R}^{j}_{\mathcal{C}_{\mathcal{F}}}\right),$$

$\mathcal{C}^{-}_{\mathcal{F}_{TN}} \leq Cir\_PyFHG_{TN}\left(\mathcal{C}^{1}_{\mathcal{F}}, \mathcal{C}^{2}_{\mathcal{F}}, \ldots, \mathcal{C}^{z}_{\mathcal{F}}\right) \leq \mathcal{C}^{+}_{\mathcal{F}_{TN}}$ and

$$\mathcal{C}^{-}_{\mathcal{F}_{TCN}} \leq Cir\_PyFHG_{TCN}\left(\mathcal{C}^{1}_{\mathcal{F}}, \mathcal{C}^{2}_{\mathcal{F}}, \ldots, \mathcal{C}^{z}_{\mathcal{F}}\right) \leq \mathcal{C}^{+}_{\mathcal{F}_{TCN}}.$$

Proof: The proof is similar as those of Property 3.□

Definition 9: The Cir-PyFHOG operators are defined as follows:

$$\begin{aligned} &Cir_{PyFHOG_{TN}\left(\mathcal{C}^{1}_{\mathcal{F}}, \mathcal{C}^{2}_{\mathcal{F}}, \ldots, \mathcal{C}^{z}_{\mathcal{F}}\right)} \\ &= \left(\mathcal{C}^{o(1)}_{\mathcal{F}}\right)^{\mathcal{U}^{1}_{TN}} \otimes_{TN} \left(\mathcal{C}^{o(2)}_{\mathcal{F}}\right)^{\mathcal{U}^{2}_{TN}} \otimes_{TN} \ldots \otimes_{TN} \left(\mathcal{C}^{o(z)}_{\mathcal{F}}\right)^{\mathcal{U}^{z}_{TN}} \\ &= \otimes_{TN}{}_{j=1}^{z} \left(\mathcal{C}^{o(j)}_{\mathcal{F}}\right)^{\mathcal{U}^{j}_{TN}} \end{aligned}$$

and

$$\begin{aligned} &Cir\_PyFHOG_{TCN}\left(\mathcal{C}^{1}_{\mathcal{F}}, \mathcal{C}^{2}_{\mathcal{F}}, \ldots, \mathcal{C}^{z}_{\mathcal{F}}\right) \\ &= \left(\mathcal{C}^{o(1)}_{\mathcal{F}}\right)^{\mathcal{U}^{1}_{TCN}} \otimes_{TN} \left(\mathcal{C}^{o(2)}_{\mathcal{F}}\right)^{\mathcal{U}^{2}_{TCN}} \otimes_{TN} \ldots \otimes_{TN} \left(\mathcal{C}^{o(z)}_{\mathcal{F}}\right)^{\mathcal{U}^{z}_{TCN}} \\ &= \otimes_{TCN}{}_{j=1}^{z} \left(\mathcal{C}^{o(j)}_{\mathcal{F}}\right)^{\mathcal{U}^{j}_{TCN}}. \end{aligned}$$

Further for more simplification, we consider the weight vector $\mathcal{U}^{j}_{TN} \in [0, 1]$ with $\sum_{j=1}^{z} \mathcal{U}^{j}_{TN} = 1$, where $o(j) \leq o(j-1)$.

Theorem 4: For evaluating the operators defined in Definition 9, we obtain $Cir\_PyFHOG_{TN}\left(\mathcal{C}^{1}_{\mathcal{F}}, \mathcal{C}^{2}_{\mathcal{F}}, \ldots, \mathcal{C}^{z}_{\mathcal{F}}\right)$ and $Cir\_PyFHOG_{TCN}\left(\mathcal{C}^{1}_{\mathcal{F}}, \mathcal{C}^{2}_{\mathcal{F}}, \ldots, \mathcal{C}^{z}_{\mathcal{F}}\right)$ for Cir-PyFSs as follows, as shown in the equation at the bottom of the page 15.

Proof: The proof is similar as those of Theorem 1.□

Further, we simplify the basic properties of the above procedures.

Property 10: When $\mathcal{C}^{j}_{\mathcal{F}} = \mathcal{C}_{\mathcal{F}} = \left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}, \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}, \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)$, $j = 1, 2, \ldots, z$,then $Cir\_PyFHOG_{TN}\left(\mathcal{C}^{1}_{\mathcal{F}}, \mathcal{C}^{2}_{\mathcal{F}}, \ldots, \mathcal{C}^{z}_{\mathcal{F}}\right) = \mathcal{C}_{\mathcal{F}}$ and $Cir\_PyFHOG_{TCN}\left(\mathcal{C}^{1}_{\mathcal{F}}, \mathcal{C}^{2}_{\mathcal{F}}, \ldots, \mathcal{C}^{z}_{\mathcal{F}}\right) = \mathcal{C}_{\mathcal{F}}$.

Proof: The proof is similar as those of Property 1.□

Property 11: When $\mathcal{C}^{j}_{\mathcal{F}} = \left(\mathbb{T}^{j}_{\mathcal{C}_{\mathcal{F}}}, \mathbb{F}^{j}_{\mathcal{C}_{\mathcal{F}}}, \mathbb{R}^{j}_{\mathcal{C}_{\mathcal{F}}}\right) \leq \mathcal{C}^{j}_{\#} = \left(\mathbb{T}^{j}_{\mathcal{C}_{\#}}, \mathbb{F}^{j}_{\mathcal{C}_{\#}}, \mathbb{R}^{j}_{\mathcal{C}_{\#}}\right)$, then

$$\begin{aligned} &Cir\_PyFHOG_{TN}\left(\mathcal{C}^{1}_{\mathcal{F}}, \mathcal{C}^{2}_{\mathcal{F}}, \ldots, \mathcal{C}^{z}_{\mathcal{F}}\right) \\ &\quad \leq Cir\_PyFHOG_{TN}\left(\mathcal{C}^{1}_{\#}, \mathcal{C}^{2}_{\#}, \ldots, \mathcal{C}^{z}_{\#}\right) \end{aligned}$$

and

$$\begin{aligned} &Cir\_PyFHOG_{TCN}\left(\mathcal{C}^{1}_{\mathcal{F}}, \mathcal{C}^{2}_{\mathcal{F}}, \ldots, \mathcal{C}^{z}_{\mathcal{F}}\right) \\ &\quad \leq Cir\_PyFHOG_{TCN}\left(\mathcal{C}^{1}_{\#}, \mathcal{C}^{2}_{\#}, \ldots, \mathcal{C}^{z}_{\#}\right). \end{aligned}$$

Proof: The proof is similar as those of Property 2.□

Property 12: When $\mathcal{C}^{-}_{\mathcal{F}_{TN}} = \left(\min_j \mathbb{T}^{j}_{\mathcal{C}_{\mathcal{F}}}, \max_j \mathbb{F}^{j}_{\mathcal{C}_{\mathcal{F}}}, \min_j \mathbb{R}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)$, $\mathcal{C}^{-}_{\mathcal{F}_{TCN}} = \left(\min_j \mathbb{T}^{j}_{\mathcal{C}_{\mathcal{F}}}, \max_j \mathbb{F}^{j}_{\mathcal{C}_{\mathcal{F}}}, \max_j \mathbb{R}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)$ and $\mathcal{C}^{+}_{\mathcal{F}_{TN}} = \left(\max_j \mathbb{T}^{j}_{\mathcal{C}_{\mathcal{F}}}, \min_j \mathbb{F}^{j}_{\mathcal{C}_{\mathcal{F}}}, \max_j \mathbb{R}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)$, $\mathcal{C}^{+}_{\mathcal{F}_{TCN}} = \left(\max_j \mathbb{T}^{j}_{\mathcal{C}_{\mathcal{F}}}, \min_j \mathbb{F}^{j}_{\mathcal{C}_{\mathcal{F}}}, \min_j \mathbb{R}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)$,then $\mathcal{C}^{-}_{\mathcal{F}_{TN}} \leq Cir\_PyFHOG_{TN}\left(\mathcal{C}^{1}_{\mathcal{F}}, \mathcal{C}^{2}_{\mathcal{F}}, \ldots, \mathcal{C}^{z}_{\mathcal{F}}\right) \leq \mathcal{C}^{+}_{\mathcal{F}_{TN}}$ and $\mathcal{C}^{-}_{\mathcal{F}_{TCN}} \leq Cir\_PyFHOG_{TCN}(\mathcal{C}^{1}_{\mathcal{F}}, \mathcal{C}^{2}_{\mathcal{F}}, \ldots, \mathcal{C}^{z}_{\mathcal{F}}) \leq \mathcal{C}^{+}_{\mathcal{F}_{TCN}}$.

Proof: The proof is similar as those of Property 3.□

Various existing operators such as algebraic operator and Einstein operator are the special cases of the proposed operators. Further, these proposed operators are very reliable and dominant in coping with vague and rational information.

$$Cir\_PyFHA_{TCN}\left(\mathcal{C}^{1}_{\mathcal{F}}, \mathcal{C}^{2}_{\mathcal{F}}, \ldots, \mathcal{C}^{z}_{\mathcal{F}}\right) = \left(\begin{array}{c} \left(\dfrac{\prod_{j=1}^{z}\left(1+(\emptyset^{s}_{c}-1)\left(\mathbb{T}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}^{j}_{TCN}} - \prod_{j=1}^{z}\left(1-\left(\mathbb{T}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}^{j}_{TCN}}}{\prod_{j=1}^{z}\left(1+(\emptyset^{s}_{c}-1)\left(\mathbb{T}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}^{j}_{TCN}} + (\emptyset^{s}_{c}-1)\prod_{j=1}^{z}\left(1-\left(\mathbb{T}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}^{j}_{TCN}}}\right)^{\frac{1}{2}}, \\ \dfrac{\left(\emptyset^{s}_{c}\right)^{1/2} * \prod_{j=1}^{z}\left(\mathbb{F}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^{j}_{TCN}}}{\left(\prod_{j=1}^{z}\left(1+(\emptyset^{s}_{c}-1)\left(1-\left(\mathbb{F}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)\right)^{\mathcal{U}^{j}_{TCN}} + (\emptyset^{s}_{c}-1)\prod_{j=1}^{z}\left(\left(\mathbb{F}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}^{j}_{TCN}}\right)^{\frac{1}{2}}}, \\ \dfrac{\left(\emptyset^{s}_{c}\right)^{1/2} * \prod_{j=1}^{z}\left(\mathbb{R}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^{j}_{TCN}}}{\left(\prod_{j=1}^{z}\left(1+(\emptyset^{s}_{c}-1)\left(1-\left(\mathbb{R}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)\right)^{\mathcal{U}^{j}_{TCN}} + (\emptyset^{s}_{c}-1)\prod_{j=1}^{z}\left(\left(\mathbb{R}^{j}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}^{j}_{TCN}}\right)^{\frac{1}{2}}} \end{array}\right).$$

IV. MADM TECHNIQUE USING THE PROPOSED OPERATORS WITH APPLICATION IN THE ASSESSMENT OF GOLDMINES

In this section, we calculate the system of MADM technique based on the proposed operators for Cir-PyFSs, such as Cir_PyFHA_{TN}, Cir_PyFHA_{TCN}, Cir_PyFHG_{TN} and Cir_PyFHG_{TCN} operators for evaluating the most preferable optimal among the collection of decisions and then applied it in the assessment of goldmines.

$$Cir\_PyFHA_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1},\mathcal{C}_{\mathcal{F}}^{2},\ldots,\mathcal{C}_{\mathcal{F}}^{z}\right)=\left(\begin{array}{c}\left(\frac{\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}-\prod_{j=1}^{z}\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}}{\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}+(\emptyset_c^s-1)\prod_{j=1}^{z}\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}}\right)^{\frac{1}{2}},\\ \frac{\left(\emptyset_c^s\right)^{1/2}*\prod_{j=1}^{z}\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}_{TN}^{j}}}{\left(\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)\right)^{\mathcal{U}_{TN}^{j}}+(\emptyset_c^s-1)\prod_{j=1}^{z}\left(\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}\right)^{\frac{1}{2}}},\\ \left(\frac{\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}-\prod_{j=1}^{z}\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}}{\prod_{j=1}^{z}\left(1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}+(\emptyset_c^s-1)\prod_{j=1}^{z}\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}}\right)^{\frac{1}{2}}\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\frac{\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\sum_{j=1}^{z}\mathcal{U}_{TN}^{j}}-\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\sum_{j=1}^{z}\mathcal{U}_{TN}^{j}}}{\left(1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\sum_{j=1}^{z}\mathcal{U}_{TN}^{j}}+(\emptyset_c^s-1)\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\sum_{j=1}^{z}\mathcal{U}_{TN}^{j}}}\right)^{\frac{1}{2}},\\ \frac{\left(\emptyset_c^s\right)^{1/2}*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)^{\sum_{j=1}^{z}\mathcal{U}_{TN}^{j}}}{\left(\left(1+(\emptyset_c^s-1)\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)\right)^{\sum_{j=1}^{z}\mathcal{U}_{TN}^{j}}+(\emptyset_c^s-1)\left(\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\sum_{j=1}^{z}\mathcal{U}_{TN}^{j}}\right)^{\frac{1}{2}}},\\ \left(\frac{\left(1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\sum_{j=1}^{z}\mathcal{U}_{TN}^{j}}-\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\sum_{j=1}^{z}\mathcal{U}_{TN}^{j}}}{\left(1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\sum_{j=1}^{z}\mathcal{U}_{TN}^{j}}+(\emptyset_c^s-1)\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\sum_{j=1}^{z}\mathcal{U}_{TN}^{j}}}\right)^{\frac{1}{2}}\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\frac{1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}-1+\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}}{1+(\emptyset_c^s-1)\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}+(\emptyset_c^s-1)\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)}\right)^{\frac{1}{2}},\\ \frac{\left(\emptyset_c^s\right)^{1/2}*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)}{\left(1+(\emptyset_c^s-1)\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)+(\emptyset_c^s-1)\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\frac{1}{2}}},\\ \left(\frac{1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}-1+\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}}{1+(\emptyset_c^s-1)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}+(\emptyset_c^s-1)\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)}\right)^{\frac{1}{2}}\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\frac{\emptyset_c^s\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}+\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}}{1+\emptyset_c^s\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}+\emptyset_c^s-\emptyset_c^s\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}-1+\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}}\right)^{\frac{1}{2}},\\ \frac{\left(\emptyset_c^s\right)^{1/2}*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)}{\left(1+\emptyset_c^s-\emptyset_c^s\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}-1+\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}+\emptyset_c^s\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}\right)^{\frac{1}{2}}},\\ \left(\frac{\emptyset_c^s\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}+\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}}{1+\emptyset_c^s\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}+\emptyset_c^s-\emptyset_c^s\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}-1+\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}}\right)^{\frac{1}{2}}\end{array}\right)=\left(\begin{array}{c}\left(\frac{\emptyset_c^s\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}}{\emptyset_c^s}\right)^{\frac{1}{2}},\\ \frac{\left(\emptyset_c^s\right)^{1/2}*\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}\right)}{\left(\emptyset_c^s\right)^{\frac{1}{2}}},\\ \left(\frac{\emptyset_c^s\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)^{2}}{\emptyset_c^s}\right)^{\frac{1}{2}}\end{array}\right)$$

$$=\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}},\mathbb{F}_{\mathcal{C}_{\mathcal{F}}},\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}\right)=\mathcal{C}_{\mathcal{F}}.$$

A. THE MADM PROCEDURES USING THE PROPOSED OPERATORS

Let a finite number of alternatives be $\{\mathcal{C}^1_{\mathcal{F}}, \mathcal{C}^2_{\mathcal{F}}, \ldots, \mathcal{C}^z_{\mathcal{F}}\}$ with their attributes of $\left\{\mathcal{C}^1_{AT}, \mathcal{C}^2_{AT}, \ldots, \mathcal{C}^y_{AT}\right\}$ in which these attributes are obtained for each alternative with weights $\mathcal{U}^j_{TCN} \in [0, 1]$ and $\sum_{j=1}^{z} \mathcal{U}^j_{TCN} = 1$. Furthermore, we compute the matrix by including Cir-PyFSs with the characteristic of Cir-PyFSs being $0 \le \mathbb{T}^2_{\mathcal{C}_{\mathcal{F}}}(x) + \mathbb{F}^2_{\mathcal{C}_{\mathcal{F}}}(x) \le 1$, where the

$$\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}} \le \mathbb{T}^j_{\mathcal{C}_{\#}} \Rightarrow \left(1 - \left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right) \ge \left(1 - \left(\mathbb{T}^j_{\mathcal{C}_{\#}}\right)^2\right) \Rightarrow \prod_{j=1}^{z} \left(1 - \left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}} \ge \prod_{j=1}^{z} \left(1 - \left(\mathbb{T}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}}$$
$$\Rightarrow \prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}} - \prod_{j=1}^{z} \left(1 - \left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}} \le \prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(\mathbb{T}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}}$$
$$- \prod_{j=1}^{z} \left(1 - \left(\mathbb{T}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}} \Rightarrow \frac{\prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}} - \prod_{j=1}^{z} \left(1 - \left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}} + (\varnothing^s_c - 1)\prod_{j=1}^{z} \left(1 - \left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}$$
$$\le \frac{\prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(\mathbb{T}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}} - \prod_{j=1}^{z} \left(1 - \left(\mathbb{T}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(\mathbb{T}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}} + (\varnothing^s_c - 1)\prod_{j=1}^{z} \left(1 - \left(\mathbb{T}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}}}.$$

$$\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}} \ge \mathbb{F}^j_{\mathcal{C}_{\#}} \Rightarrow \prod_{j=1}^{z} \left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^j_{TN}} \ge \prod_{j=1}^{z} \left(\mathbb{F}^j_{\mathcal{C}_{\#}}\right)^{\mathcal{U}^j_{TN}} \Rightarrow (\varnothing^s_c)^{\frac{1}{2}} * \prod_{j=1}^{z} \left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^j_{TN}} \ge (\varnothing^s_c)^{\frac{1}{2}} * \prod_{j=1}^{z} \left(\mathbb{F}^j_{\mathcal{C}_{\#}}\right)^{\mathcal{U}^j_{TN}}$$
$$\Rightarrow \frac{(\varnothing^s_c)^{\frac{1}{2}} * \prod_{j=1}^{z} \left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^j_{TN}}}{\left(\prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(1 - \left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^j_{TN}} + (\varnothing^s_c - 1)\prod_{j=1}^{z} \left(\left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}\right)^{\frac{1}{2}}}$$
$$\ge \frac{(\varnothing^s_c)^{\frac{1}{2}} * \prod_{j=1}^{z} \left(\mathbb{F}^j_{\mathcal{C}_{\#}}\right)^{\mathcal{U}^j_{TN}}}{\left(\prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(1 - \left(\mathbb{F}^j_{\mathcal{C}_{\#}}\right)^2\right)\right)^{\mathcal{U}^j_{TN}} + (\varnothing^s_c - 1)\prod_{j=1}^{z} \left(\left(\mathbb{F}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}}\right)^{\frac{1}{2}}}.$$

$$\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}} \le \mathbb{R}^j_{\mathcal{C}_{\#}} \Rightarrow \left(1 - \left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right) \ge \left(1 - \left(\mathbb{R}^j_{\mathcal{C}_{\#}}\right)^2\right) \Rightarrow \prod_{j=1}^{z} \left(1 - \left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}} \ge \prod_{j=1}^{z} \left(1 - \left(\mathbb{R}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}}$$
$$\Rightarrow \prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}$$
$$- \prod_{j=1}^{z} \left(1 - \left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}} \le \prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(\mathbb{R}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}} - \prod_{j=1}^{z} \left(1 - \left(\mathbb{R}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}}$$
$$\Rightarrow \frac{\prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}} - \prod_{j=1}^{z} \left(1 - \left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}} + (\varnothing^s_c - 1)\prod_{j=1}^{z} \left(1 - \left(\mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}$$
$$\le \frac{\prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(\mathbb{R}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}} - \prod_{j=1}^{z} \left(1 - \left(\mathbb{R}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{z} \left(1 + (\varnothing^s_c - 1)\left(\mathbb{R}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}} + (\varnothing^s_c - 1)\prod_{j=1}^{z} \left(1 - \left(\mathbb{R}^j_{\mathcal{C}_{\#}}\right)^2\right)^{\mathcal{U}^j_{TN}}}.$$

term $\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}(x)$ represents the truth grade, $\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}(x)$ represents the falsity grade, and $\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}(x)$ states the radius around the circle of each element. Thus, we have the refusal grade with $\mathfrak{P} = \left(1 - \left(\mathbb{T}^2_{\mathcal{C}_{\mathcal{F}}}(x) + \mathbb{F}^2_{\mathcal{C}_{\mathcal{F}}}(x)\right)\right)^{\frac{1}{2}}$ and also express the simple structure of Cir-PyFN with $\mathcal{C}^j_{\mathcal{F}} = \left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}, \mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}, \mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right), j = 1, 2, \ldots, z$. We now organize a procedure for addressing the major steps of the MADM technique for Cir-PyFNs based on the proposed operators as follows:

Step 1: Compute the matrix by including the Cir-PyFNs.

Step 2: Normalize the matrix, if the matrix has cast type of criteria, such as

$$Z = \begin{cases} \left(\mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}, \mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}, \mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right) & \textit{benefit} \\ \left(\mathbb{F}^j_{\mathcal{C}_{\mathcal{F}}}, \mathbb{T}^j_{\mathcal{C}_{\mathcal{F}}}, \mathbb{R}^j_{\mathcal{C}_{\mathcal{F}}}\right) & \textit{cost}. \end{cases}$$

But, if the matrix has benefit types of criteria, then we do not need to normalize the matrix.

Step 3: Aggregate the data in the matrix with the help of the proposed operators, such as Cir_PyFHA_{TN}, Cir_PyFHA_{TCN}, Cir_PyFHG_{TN} and Cir_PyFHG_{TCN} operators created in Theorem 1 and Theorem 3 of Section III.

Step 4: Expose the score data of every alternative.

Step 5: Rank all the alternatives according to their score data and discover the best one.

With the help of the above procedure, we next apply it for evaluating the analysis of goldmines to enhance the capability and rationality of the proposed method.

B. THE ASSESSMENT OF GOLDMINES

The major theme of this application is to find the key component of assessing goldmines under the consideration of the evaluated operators. For this, we consider the problem of assessment of goldmines, which involves many features of a gold mining operation to evaluate its safety, environmental impact, potential for profitable gold extraction, and economic viability. In this application, we select the five best kinds of key components as alternatives which are listed below:

1) Geological Survey "$\mathcal{C}^1_{\mathcal{F}}$".
2) Resource Estimation "$\mathcal{C}^2_{\mathcal{F}}$".
3) Mining Method Selection "$\mathcal{C}^3_{\mathcal{F}}$".
4) Gold Recovery Procedure "$\mathcal{C}^4_{\mathcal{F}}$".
5) Safety Assessments "$\mathcal{C}^5_{\mathcal{F}}$".

The above five alternatives are very reliable and have major features or key components, but our main theme is to select the best one among the best five. For this, we select five criteria, such as "$\mathcal{C}^1_{AT}$" (growth analysis), "$\mathcal{C}^2_{AT}$" (Political impact), "$\mathcal{C}^3_{AT}$" (environmental impact), "$\mathcal{C}^4_{AT}$"

$$Cir\_PyFHOA_{TN}\left(\mathcal{C}^1_{\mathcal{F}}, \mathcal{C}^2_{\mathcal{F}}, \ldots, \mathcal{C}^z_{\mathcal{F}}\right) = \left(\begin{array}{c} \left(\dfrac{\prod_{j=1}^{z}\left(1+(\emptyset^s_c-1)\left(\mathbb{T}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}} - \prod_{j=1}^{z}\left(1-\left(\mathbb{T}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{z}\left(1+(\emptyset^s_c-1)\left(\mathbb{T}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}} + (\emptyset^s_c-1)\prod_{j=1}^{z}\left(1-\left(\mathbb{T}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}\right)^{\frac{1}{2}}, \\ \dfrac{\left(\emptyset^s_c\right)^{1/2} * \prod_{j=1}^{z}\left(\mathbb{F}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^j_{TN}}}{\left(\prod_{j=1}^{z}\left(1+(\emptyset^s_c-1)\left(1-\left(\mathbb{F}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^j_{TN}} + (\emptyset^s_c-1)\prod_{j=1}^{z}\left(\left(\mathbb{F}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}\right)^{\frac{1}{2}}}, \\ \left(\dfrac{\prod_{j=1}^{z}\left(1+(\emptyset^s_c-1)\left(\mathbb{R}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}} - \prod_{j=1}^{z}\left(1-\left(\mathbb{R}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}{\prod_{j=1}^{z}\left(1+(\emptyset^s_c-1)\left(\mathbb{R}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}} + (\emptyset^s_c-1)\prod_{j=1}^{z}\left(1-\left(\mathbb{R}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TN}}}\right)^{\frac{1}{2}} \end{array}\right);$$

$$Cir\_PyFHOA_{TCN}\left(\mathcal{C}^1_{\mathcal{F}}, \mathcal{C}^2_{\mathcal{F}}, \ldots, \mathcal{C}^z_{\mathcal{F}}\right) = \left(\begin{array}{c} \left(\dfrac{\prod_{j=1}^{z}\left(1+(\emptyset^s_c-1)\left(\mathbb{T}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TCN}} - \prod_{j=1}^{z}\left(1-\left(\mathbb{T}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TCN}}}{\prod_{j=1}^{z}\left(1+(\emptyset^s_c-1)\left(\mathbb{T}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TCN}} + (\emptyset^s_c-1)\prod_{j=1}^{z}\left(1-\left(\mathbb{T}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TCN}}}\right)^{\frac{1}{2}}, \\ \dfrac{\left(\emptyset^s_c\right)^{1/2} * \prod_{j=1}^{z}\left(\mathbb{F}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^j_{TCN}}}{\left(\prod_{j=1}^{z}\left(1+(\emptyset^s_c-1)\left(1-\left(\mathbb{F}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^j_{TCN}} + (\emptyset^s_c-1)\prod_{j=1}^{z}\left(\left(\mathbb{F}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TCN}}\right)^{\frac{1}{2}}}, \\ \dfrac{\left(\emptyset^s_c\right)^{1/2} * \prod_{j=1}^{z}\left(\mathbb{R}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^{\mathcal{U}^j_{TCN}}}{\left(\prod_{j=1}^{z}\left(1+(\emptyset^s_c-1)\left(1-\left(\mathbb{R}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)\right)^{\mathcal{U}^j_{TCN}} + (\emptyset^s_c-1)\prod_{j=1}^{z}\left(\left(\mathbb{R}^{o(j)}_{\mathcal{C}_{\mathcal{F}}}\right)^2\right)^{\mathcal{U}^j_{TCN}}\right)^{\frac{1}{2}}} \end{array}\right).$$

TABLE 1. Cir-PyF decision matrix.

	$\mathcal{C}_{AT}^{1}$	$\mathcal{C}_{AT}^{2}$	$\mathcal{C}_{AT}^{3}$	$\mathcal{C}_{AT}^{4}$	$\mathcal{C}_{AT}^{5}$
$\mathcal{C}_{\mathcal{F}}^{1}$	(0.8,0.4,0.3)	(0.81,0.41,0.31)	(0.82,0.42,0.32)	(0.83,0.43,0.33)	(0.84,0.44,0.34)
$\mathcal{C}_{\mathcal{F}}^{2}$	(0.7,0.1,0.2)	(0.71,0.11,0.21)	(0.72,0.12,0.22)	(0.73,0.13,0.23)	(0.74,0.14,0.24)
$\mathcal{C}_{\mathcal{F}}^{3}$	(0.9,0.2,0.1)	(0.91,0.21,0.11)	(0.92,0.22,0.12)	(0.93,0.23,0.13)	(0.94,0.24,0.14)
$\mathcal{C}_{\mathcal{F}}^{4}$	(0.7,0.6,0.5)	(0.71,0.61,0.51)	(0.72,0.62,0.52)	(0.73,0.63,0.53)	(0.74,0.64,0.54)
$\mathcal{C}_{\mathcal{F}}^{5}$	(0.3,0.2,0.6)	(0.31,0.21,0.61)	(0.32,0.22,0.62)	(0.33,0.23,0.63)	(0.34,0.24,0.64)

(safety analysis), "$\mathcal{C}_{AT}^{5}$" (other factors). Based on the above criteria, we select the best one under the consideration of the following weight vector, such as $(0.1, 0.4, 0.2, 0.1, 0.2)^T$. Therefore, we organize a procedure for addressing the above problem. The major steps of the procedure of MADM technique are listed as follows:

Step 1: Compute the matrix by including the Cir-PyFNs, as shown in Table 1.

Step 2: Normalize the matrix, if the matrix has cast type of criteria, such as

$$Z = \begin{cases} \left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right) & benefit \\ \left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}, \mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}, \mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right) & cost. \end{cases}$$

But, if the matrix has benefit types of criteria, then we do not need to normalize the matrix. Anyhow, the data in Table 1 do not need to be normalized.

Step 3: Aggregate the data in matrix with the help of our proposedoperators, as shown in Table 2.

Step 4: Expose the score data of each alternative, as shown in Table 3.

Step 5: Rank all the alternatives according to their score data and discover the best one, as shown in Table 4.

To consider the theory of Cir_PyFHA_{TN}, Cir_PyFHA_{TCN}, Cir_PyFHG_{TN} and Cir_PyFHG_{TCN}, we expose the best optimal $\mathcal{C}_{\mathcal{F}}^{1}$ which represents the geological survey. Further, we simplify the operators under the presence of the Cir-IF set theory to expand the assets of the assessed operators. The Cir-IF data is listed in Table 5. Thus, to use the four types of operators, the score values of the aggregated values are stated

$$Cir\_PyFHG_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) = \left(\begin{array}{c} \frac{\left(\emptyset_c^s\right)^{1/2} * \prod_{j=1}^{z}\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^{\mathcal{U}_{TN}^{j}}}{\left(\prod_{j=1}^{z}\left(1+\left(\emptyset_c^s-1\right)\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)\right)^{\mathcal{U}_{TN}^{j}}+\left(\emptyset_c^s-1\right)\prod_{j=1}^{z}\left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}\right)^{\frac{1}{2}}}, \\ \left(\frac{\prod_{j=1}^{z}\left(1+\left(\emptyset_c^s-1\right)\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}-\prod_{j=1}^{z}\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}}{\prod_{j=1}^{z}\left(1+\left(\emptyset_c^s-1\right)\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}+\left(\emptyset_c^s-1\right)\prod_{j=1}^{z}\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}}\right)^{\frac{1}{2}}, \\ \frac{\left(\emptyset_c^s\right)^{1/2} * \prod_{j=1}^{z}\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^{\mathcal{U}_{TN}^{j}}}{\left(\prod_{j=1}^{z}\left(1+\left(\emptyset_c^s-1\right)\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)\right)^{\mathcal{U}_{TN}^{j}}+\left(\emptyset_c^s-1\right)\prod_{j=1}^{z}\left(\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TN}^{j}}\right)^{\frac{1}{2}}} \end{array}\right);$$

$$Cir\_PyFHG_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1}, \mathcal{C}_{\mathcal{F}}^{2}, \ldots, \mathcal{C}_{\mathcal{F}}^{z}\right) = \left(\begin{array}{c} \frac{\left(\emptyset_c^s\right)^{1/2} * \prod_{j=1}^{z}\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^{\mathcal{U}_{TCN}^{j}}}{\left(\prod_{j=1}^{z}\left(1+\left(\emptyset_c^s-1\right)\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)\right)^{\mathcal{U}_{TCN}^{j}}+\left(\emptyset_c^s-1\right)\prod_{j=1}^{z}\left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}\right)^{\frac{1}{2}}}, \\ \left(\frac{\prod_{j=1}^{z}\left(1+\left(\emptyset_c^s-1\right)\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}-\prod_{j=1}^{z}\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}}{\prod_{j=1}^{z}\left(1+\left(\emptyset_c^s-1\right)\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}+\left(\emptyset_c^s-1\right)\prod_{j=1}^{z}\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}}\right)^{\frac{1}{2}}, \\ \left(\frac{\prod_{j=1}^{z}\left(1+\left(\emptyset_c^s-1\right)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}-\prod_{j=1}^{z}\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}}{\prod_{j=1}^{z}\left(1+\left(\emptyset_c^s-1\right)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}+\left(\emptyset_c^s-1\right)\prod_{j=1}^{z}\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{j}\right)^2\right)^{\mathcal{U}_{TCN}^{j}}}\right)^{\frac{1}{2}} \end{array}\right).$$

TABLE 2. Cir-PyF aggregated results based on the proposed operators.

	Cir_PyFHA_{TN}	Cir_PyFHA_{TCN}	Cir_PyFHG_{TN}	Cir_PyFHG_{TCN}
$\mathcal{C}_{\mathcal{F}}^{1}$	(0.8194,0.4188,0.3192)	(0.8194,0.4188,0.3187)	(0.819,0.4192,0.3187)	(0.819,0.4192,0.3192)
$\mathcal{C}_{\mathcal{F}}^{2}$	(0.7192,0.1183,0.2193)	(0.7192,0.1183,0.2186)	(0.7189,0.1197,0.2186)	(0.7189,0.1197,0.2193)
$\mathcal{C}_{\mathcal{F}}^{3}$	(0.9200,0.2186,0.1197)	(0.9200,0.2186,0.1183)	(0.9190,0.2193,0.1183)	(0.9190,0.2193,0.1197)
$\mathcal{C}_{\mathcal{F}}^{4}$	(0.7192,0.6189,0.5192)	(0.7192,0.6189,0.5188)	(0.7189,0.6192,0.5188)	(0.7189,0.6192,0.5192)
$\mathcal{C}_{\mathcal{F}}^{5}$	(0.3192,0.2186,0.6192)	(0.3192,0.2186,0.6189)	(0.3187,0.2193,0.6189)	(0.3187,0.2193,0.6192)

TABLE 3. Representation of the score values.

	Cir_PyFHA_{TN}	Cir_PyFHA_{TCN}	Cir_PyFHG_{TN}	Cir_PyFHG_{TCN}
$\mathcal{C}_{\mathcal{F}}^{1}$	0.07919	0.07907	0.0789	0.07902
$\mathcal{C}_{\mathcal{F}}^{2}$	0.05522	0.05503	0.05494	0.05513
$\mathcal{C}_{\mathcal{F}}^{3}$	0.04781	0.04725	0.04712	0.04767
$\mathcal{C}_{\mathcal{F}}^{4}$	0.03487	0.03485	0.03463	0.03465
$\mathcal{C}_{\mathcal{F}}^{5}$	0.01676	0.01675	0.01655	0.01656

in Table 6. Finally, we rank all the alternatives according to their score data and discover the best one, as shown in Table 7.

We next compare the proposed operators with these existing methods.

V. COMPARATIVE ANALYSIS

In this section, we make some comparisons of the proposed operators with some existing techniques based on the data shown in Table 1 and Table 5. For comparisons, we select

$$Cir\_PyFHOG_{TN}\left(\mathcal{C}_{\mathcal{F}}^{1},\mathcal{C}_{\mathcal{F}}^{2},\ldots,\mathcal{C}_{\mathcal{F}}^{z}\right)=\left(\begin{array}{c}\frac{\left(\varnothing_{c}^{s}\right)^{1/2}*\prod_{j=1}^{z}\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{\mathcal{U}_{TN}^{j}}}{\left(\prod_{j=1}^{z}\left(1+\left(\varnothing_{c}^{s}-1\right)\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)\right)^{\mathcal{U}_{TN}^{j}}+\left(\varnothing_{c}^{s}-1\right)\prod_{j=1}^{z}\left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}\right)^{\frac{1}{2}}},\\ \left(\frac{\prod_{j=1}^{z}\left(1+\left(\varnothing_{c}^{s}-1\right)\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}-\prod_{j=1}^{z}\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}}{\prod_{j=1}^{z}\left(1+\left(\varnothing_{c}^{s}-1\right)\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}+\left(\varnothing_{c}^{s}-1\right)\prod_{j=1}^{z}\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}}\right)^{\frac{1}{2}},\\ \frac{\left(\varnothing_{c}^{s}\right)^{1/2}*\prod_{j=1}^{z}\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{\mathcal{U}_{TN}^{j}}}{\left(\prod_{j=1}^{z}\left(1+\left(\varnothing_{c}^{s}-1\right)\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)\right)^{\mathcal{U}_{TN}^{j}}+\left(\varnothing_{c}^{s}-1\right)\prod_{j=1}^{z}\left(\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TN}^{j}}\right)^{\frac{1}{2}}}\end{array}\right);$$

$$Cir\_PyFHOG_{TCN}\left(\mathcal{C}_{\mathcal{F}}^{1},\mathcal{C}_{\mathcal{F}}^{2},\ldots,\mathcal{C}_{\mathcal{F}}^{z}\right)=\left(\begin{array}{c}\frac{\left(\varnothing_{c}^{s}\right)^{1/2}*\prod_{j=1}^{z}\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{\mathcal{U}_{TCN}^{j}}}{\left(\prod_{j=1}^{z}\left(1+\left(\varnothing_{c}^{s}-1\right)\left(1-\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)\right)^{\mathcal{U}_{TCN}^{j}}+\left(\varnothing_{c}^{s}-1\right)\prod_{j=1}^{z}\left(\left(\mathbb{T}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TCN}^{j}}\right)^{\frac{1}{2}}},\\ \left(\frac{\prod_{j=1}^{z}\left(1+\left(\varnothing_{c}^{s}-1\right)\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TCN}^{j}}-\prod_{j=1}^{z}\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TCN}^{j}}}{\prod_{j=1}^{z}\left(1+\left(\varnothing_{c}^{s}-1\right)\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TCN}^{j}}+\left(\varnothing_{c}^{s}-1\right)\prod_{j=1}^{z}\left(1-\left(\mathbb{F}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TCN}^{j}}}\right)^{\frac{1}{2}},\\ \left(\frac{\prod_{j=1}^{z}\left(1+\left(\varnothing_{c}^{s}-1\right)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TCN}^{j}}-\prod_{j=1}^{z}\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TCN}^{j}}}{\prod_{j=1}^{z}\left(1+\left(\varnothing_{c}^{s}-1\right)\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TCN}^{j}}+\left(\varnothing_{c}^{s}-1\right)\prod_{j=1}^{z}\left(1-\left(\mathbb{R}_{\mathcal{C}_{\mathcal{F}}}^{o(j)}\right)^{2}\right)^{\mathcal{U}_{TCN}^{j}}}\right)^{\frac{1}{2}}\end{array}\right).$$

TABLE 4. Ranking results of the score values.

Methods	Ranking values	Best optimal
Cir_PyFHA_{TN}	$\mathcal{C}_{\mathcal{F}}^{1} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{5}$	$\mathcal{C}_{\mathcal{F}}^{1}$
Cir_PyFHA_{TCN}	$\mathcal{C}_{\mathcal{F}}^{1} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{5}$	$\mathcal{C}_{\mathcal{F}}^{1}$
Cir_PyFHG_{TN}	$\mathcal{C}_{\mathcal{F}}^{1} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{5}$	$\mathcal{C}_{\mathcal{F}}^{1}$
Cir_PyFHG_{TCN}	$\mathcal{C}_{\mathcal{F}}^{1} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{5}$	$\mathcal{C}_{\mathcal{F}}^{1}$

TABLE 5. Cir-IF decision matrix.

	$\mathcal{C}_{AT}^{1}$	$\mathcal{C}_{AT}^{2}$	$\mathcal{C}_{AT}^{3}$	$\mathcal{C}_{AT}^{4}$	$\mathcal{C}_{AT}^{5}$
$\mathcal{C}_{\mathcal{F}}^{1}$	(0.5,0.4,0.3)	(0.51,0.41,0.31)	(0.52,0.42,0.32)	(0.53,0.43,0.33)	(0.54,0.44,0.34)
$\mathcal{C}_{\mathcal{F}}^{2}$	(0.7,0.1,0.2)	(0.71,0.11,0.21)	(0.72,0.12,0.22)	(0.73,0.13,0.23)	(0.74,0.14,0.24)
$\mathcal{C}_{\mathcal{F}}^{3}$	(0.6,0.2,0.1)	(0.61,0.21,0.11)	(0.62,0.22,0.12)	(0.63,0.23,0.13)	(0.64,0.24,0.14)
$\mathcal{C}_{\mathcal{F}}^{4}$	(0.7,0.2,0.5)	(0.71,0.21,0.51)	(0.72,0.22,0.52)	(0.73,0.23,0.53)	(0.74,0.24,0.54)
$\mathcal{C}_{\mathcal{F}}^{5}$	(0.3,0.2,0.6)	(0.31,0.21,0.61)	(0.32,0.22,0.62)	(0.33,0.23,0.63)	(0.34,0.24,0.64)

TABLE 6. Representation of the score values (for Table 5).

	Cir_PyFHA_{TN}	Cir_PyFHA_{TCN}	Cir_PyFHG_{TN}	Cir_PyFHG_{TCN}
$\mathcal{C}_{\mathcal{F}}^{1}$	0.01503	0.01501	0.0149	0.01493
$\mathcal{C}_{\mathcal{F}}^{2}$	0.05522	0.05503	0.05494	0.05513
$\mathcal{C}_{\mathcal{F}}^{3}$	0.02009	0.01986	0.01981	0.02005
$\mathcal{C}_{\mathcal{F}}^{4}$	0.12191	0.12183	0.12162	0.1217
$\mathcal{C}_{\mathcal{F}}^{5}$	0.01676	0.01675	0.01655	0.01656

TABLE 7. Ranking results of the score values.

Methods	Ranking values	Best optimal
Cir_PyFHA_{TN}	$\mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{5} > \mathcal{C}_{\mathcal{F}}^{1}$	$\mathcal{C}_{\mathcal{F}}^{4}$
Cir_PyFHA_{TCN}	$\mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{5} > \mathcal{C}_{\mathcal{F}}^{1}$	$\mathcal{C}_{\mathcal{F}}^{4}$
Cir_PyFHG_{TN}	$\mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{5} > \mathcal{C}_{\mathcal{F}}^{1}$	$\mathcal{C}_{\mathcal{F}}^{4}$
Cir_PyFHG_{TCN}	$\mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{5} > \mathcal{C}_{\mathcal{F}}^{1}$	$\mathcal{C}_{\mathcal{F}}^{4}$

TABLE 8. Comparison matrix (using Table 1).

Methods	Score values	Ranking values
Xu [26]	$--failed\ --$	$--failed\ --$
Huang [21]	$--failed\ --$	$--failed\ --$
Garg [22]	$--failed\ --$	$--failed\ --$
Peng and Yuan [27]	$--failed\ --$	$--failed\ --$
Garg [28]	$--failed\ --$	$--failed\ --$
Wu and Wei [23]	$--failed\ --$	$--failed\ --$
Cir_PyFHA_{TN}	0.07919,0.05522,0.04781,0.03487,0.01676	$\mathcal{C}_{\mathcal{F}}^{1} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{5}$
Cir_PyFHA_{TCN}	0.07907,0.05503,0.04725,0.03485,0.01675	$\mathcal{C}_{\mathcal{F}}^{1} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{5}$
Cir_PyFHG_{TN}	0.0789,0.05494,0.04712,0.03463,0.01655	$\mathcal{C}_{\mathcal{F}}^{1} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{5}$
Cir_PyFHG_{TCN}	0.07902,0.05513,0.04767,0.03465,0.01656	$\mathcal{C}_{\mathcal{F}}^{1} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{5}$

the following existing methods: the AOs and geometric AOs for IFSs proposed by Xu [26], simple AOs for PyFSs evaluated by Peng and Yuan [27], the AOs for PyFSs based on confidence levels exposed by Grag [28], AOs based on Hamacher norms for IFSs proposed by Huang [21], the Hamacher AOs for IFSs using the entropy measures derived by Garg [22], and the Hamacher AOs for PyFSs addressed by Wu and Wei [23]. Thus, based on the

TABLE 9. Comparison matrix (using Table 5).

Methods	Score values	Ranking values
Xu [26]	$--failed--$	$--failed--$
Huang [21]	$--failed--$	$--failed--$
Garg [22]	$--failed--$	$--failed--$
Peng and Yuan [27]	$--failed--$	$--failed--$
Garg [28]	$--failed--$	$--failed--$
Wu and Wei [23]	$--failed--$	$--failed--$
Cir_PyFHA_{TN}	0.01503,0.05522,0.02009,0.12191,0.01676	$\mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{5} > \mathcal{C}_{\mathcal{F}}^{1}$
Cir_PyFHA_{TCN}	0.01501,0.05503,0.01986,0.12183,0.01675	$\mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{5} > \mathcal{C}_{\mathcal{F}}^{1}$
Cir_PyFHG_{TN}	0.0149,0.05494,0.01981,0.12162,0.01655	$\mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{5} > \mathcal{C}_{\mathcal{F}}^{1}$
Cir_PyFHG_{TCN}	0.01493,0.05513,0.02005,0.1217,0.01656	$\mathcal{C}_{\mathcal{F}}^{4} > \mathcal{C}_{\mathcal{F}}^{2} > \mathcal{C}_{\mathcal{F}}^{3} > \mathcal{C}_{\mathcal{F}}^{5} > \mathcal{C}_{\mathcal{F}}^{1}$

data shown in Table 1, the comparative analysis is listed in Table 8.

To consider Cir_PyFHA_{TN}, Cir_PyFHA_{TCN}, Cir_PyFHG_{TN} and Cir_PyFHG_{TCN}, we expose the best optimal $\mathcal{C}_{\mathcal{F}}^{1}$ which represents the geological survey. Further, to show the supremacy and effectiveness of the derived theory, we select the data in Table 5, the comparative analysis is listed in Table 9.

To consider Cir_PyFHA_{TN}, Cir_PyFHA_{TCN}, Cir_PyFHG_{TN} and Cir_PyFHG_{TCN}, we expose the best optimal $\mathcal{C}_{\mathcal{F}}^{4}$ which represents the geological survey. But we notice that the prevailing techniques have failed, because these operators are computed based on FSs, IFSs, PyFSs, and Cir-IFSs which are the subpart of the Cir-PyFSs. The existing operators have limitation due to this reason, we are not able to evaluate the data in Table 1 and Table 5 because of the above problem. Hence, the proposed operators are reliable and dominant technique to cope with uncertain and unreliable information in decision-making problems.

VI. CONCLUSION

Cir-PyFS is a good way for depicting vague and unreliable information in genuine life problems. These FSs, IFSs, PyFSs, and Cir-IFSs are the subpart of Cir-PyFSs. Since Cir-PyFS was defined by Bozyigit et al. [19] in 2023, there were still less researchers to work on it. In this paper, we continue working on Cir-PyFSs by firstly giving the Hamacher operational laws for any two Cir-PyFSs, and then derive the Cir-PyFHA, Cir-PyFHOA, Cir-PyFHG, and Cir-PyFHOG operators. We make advanced discussion the properties of these proposed operators on Cir-PyFSs. Based on the proposed operators, we construct a MADM method and then apply it to select the best and simplest procedure for evaluating the source of gold in mines. We also demonstrate some examples for comparing the proposed operators with some existing operators to expand the attraction of these evaluated operators. In the future, we will concentrate on evaluating the Hamacher and Dombi aggregation operators [29] and three-way decisions [25], [30] based on circular q-rung orthopair fuzzy sets with extensions and then try to implement them in more real-life applications, for instance, pattern recognition, machine learning, data mining, and data analysis to enhance the worth of the presented operators.

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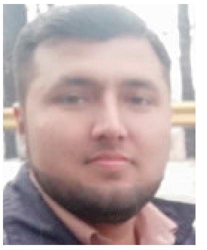

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