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RESEARCH ARTICLE

Identification of Types of Pollution That Mostly Affect the Environment by Using Picture Fuzzy Soft Aggregation Operators

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ABSTRACT The release of harmful materials into the environment is called pollution and the harmful materials are called pollutants. There are four basic categories of pollution: land, water, noise, and air pollution. All forms of pollution often have severe consequences on human health as well as the environment and wildlife. There are certain decision-making scenarios like the phenomenon of voting where we have to utilize the third grade called abstinence grade along with membership grade and non-membership grade. Many remarkable fuzzy structures like the intuitionistic fuzzy set, Pythagorean fuzzy set and q-rung orthopair fuzzy set can never discuss abstinence grades that show their flaws. Moreover, we can observe that the parametrization tool is a remarkable instrument used in soft set theory and all above-mentioned structures fail to cover the parametrization as well. Moreover, Einstein operations comprise Einstein product and Einstein sum, which serve as excellent substitutes for algebraic product and algebraic sum. So keeping in view the characteristics of the parametrization tool, the more advanced structure of the picture fuzzy soft set and Einstein operational rules, in this article, we have established Einstein operational laws for picture fuzzy soft numbers. Moreover, we have elaborated the basic notion of Einstein-weighted average operators and Einstein-weighted geometric aggregation operators. Furthermore, we have discussed the basic properties of these introduced notions. Moreover, we have discussed the algorithm for the application of these aggregation operators in the identification of types of pollution that mostly affect the environment. We have provided a comparison of these introduced works for the superiority of these introduced conceptions.

INDEX TERMS Environmental pollution, artificial intelligence, picture fuzzy soft set, Einstein aggregation operators.

I. INTRODUCTION

The word pollution comes from the Latin word 'polluere' which simply means epidemic. The presence of hazardous substances in the land, water, and air is referred to as pollution since it can harm both the environment and living beings. (1) Air pollution (2) Water pollution (3) Noise pollution (4) Land pollution are the several types of pollution. Air pollution is the mixing of various harmful materials, such as hazardous gases and chemicals with air. Burning materials, vehicle exhaust fumes, or unfavorable industrial waste pollution could all contribute to this type of contamination. Water pollution is the poisoning of the earth's water supply. It includes the bacterial, chemical, and particle pollution of water that lowers the water's cleanliness. One of the most prevalent types of pollution is the leakage

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of oil as well as waste. The poor quality of life in the affected areas is caused by the loud noises created by human activity. It can fire from several sources, including trains, automobiles, loud music, aircraft, and more. Even hearing loss, whether permanent or temporary, as well as disturbances to wildlife, might come from this. Many scientists have made remarkable efforts and discussed the consequences of environmental pollution. Khan and Ghouri [1] reveal that various types of pollutants are substantially harming not just humans through illnesses and issues, but also animals, trees, and plants. Moreover, Martinez [2] reveals that one of the most effective medicines used in human therapy is the antibiotic. However, these antibiotics must also be regarded as significant pollutants since they might be harmful to microorganisms. Tsai et al. [3] established that toxic materials such as metals, air pollutants, and phthalates, may raise the chance of developing chronic kidney disease or accelerate its progression. Molodtsov [4]soft set $\left(S_{ft}S\right)$ idea is a new strategy for handling ambiguous data. According to Molodtsov, one of the key benefits of $S_{ft}S$ theory is that, unlike theories of fuzzy sets (FSs) [5], it is not constrained by the limitations of parameterization tools. When compared to some established mathematical methods for dealing with uncertainties, such as the theory of probability, the concept of fuzzy sets [5], and the analysis of rough sets, the benefit of $S_{ft}S$ approach is that it is free of the shortcomings of parametrization tools of those concepts.

Many new advancements based on $S_{ft}S$ and FSs have been studied and the concept of fuzzy soft set $\left(FS_{ft}S\right)$ [6], intuitionistic fuzzy $S_{ft}S$ $\left(IFS_{ft}S\right)$ [7], Pythagorean fuzzy $S_{ft}S$ $\left(P_yFS_{ft}S\right)$ [8] and q-rung orthopair fuzzy $S_{ft}S(q-ROFS_{ft}S)$ [9] have been delivered respectively. All the above structures can only deal with MG and NMG in their structure. These structures lack the property to discuss the AG in their structures. So based on this observation, Cuong [10] proposed a remarkable result in this regard and proposed the notion of a picture fuzzy set (PFS). Note that PFS is a valuable structure because it uses more advanced conditions that sum (MG, NMG, AG) must belong to unit interval [0, 1].

A. LITERATURE REVIEW

Research on $S_{ft}S$ including all above mentioned hybrid notions has been active recently, and significant advancements have been made including the use of fundamental $S_{ft}S$ theory [11], $S_{ft}S$ theory in abstract algebra [12], and $S_{ft}S$ for data analysis [13] and especially in decision-making [14]. Aktaş and Çagman [15] started the use of $S_{ft}S$ in algebra. In BCK/BCI algebra, Jun and Park [16] discussed soft ideal theory. Moreover, Ali et al. [17] introduced algebraic notions of $S_{ft}S$ based on new operations. Based on the notion of $IFS_{ft}S$, $PyFS_{ft}S$ and $q-q-ROFS_{ft}S$, many new developments have been made. Xiao et al. [18] introduced a combined forecasting approach under the environment of $FS_{ft}S$. Moreover, Agarwal et al. [19] produced generalized $IFS_{ft}S$ and provide its applications in decision-making problems. Some entropy measures based on $IFS_{ft}S$ and interval-valued $IFS_{ft}S$ has been developed by Jiang et al. [20]. Based on the conception of $PyFS_{ft}S$, some techniques like TOPSIS methods and VIKOR methods have been developed by Naeem et al. [21]. Zulqarnain et al. [22] introduced some aggregation operators and applied these notions to green supplier chain management. Also, Mahmood and Ali [23] proposed a method of MCDM approach based on the settings of complex $PyFS_{ft}Ss$. Moreover. Akram et al. [24] proposed an MCGDM model based on complex $PyFS_{ft}S$. As $q-ROFS_{ft}S$ is a more advanced structure by using the constraint that sum $(MG^q,\ NMG^q)$ must belong to [0, 1] for $q{\geq}1$, so based on the conception of $q-ROFS_{ft}S$, some average and geometric aggregation have been developed by Hussain et al. [9]. Furthermore, Riaz et al. [25] established the notion of TOPSIS and VIKOR methods for the environment of $q-ROFS_{ft}Ss$. Also, Hussain et al. [26] proposed $q-ROFS_{ft}$ operators based on Dombi t-norms and t-conorm with their application in decision-making.

B. MOTIVATION OF THE PROPOSED WORK

A lot of ambiguity, imprecision, and uncertainty exist in the real world. In many fields, including economics, engineering, environmental research, medical science, and social science, dealing with uncertainties is a significant difficulty. Recently, many authors have developed an interest in modeling ambiguity. Yang et al. [27] introduce the notion of picture fuzzy soft set $\left(P_cFS_{ft}S\right)$. In general, $P_cFS_{ft}S$ models are employed when there are multiple possible responses from humans, such as "no," "yes," "abstain," and "refusal." For example, a departmental student body might serve as a good illustration of $P_cFS_{ft}S$. There is some group of students who want to visit two places: one in the UK and the other in Canada, but there are some students who want to visit the UK (MG), not Canada (NMG). However, some students prefer to visit Canada (MG) over the UK (NMG), and some students want to visit both places the UK and Canada i.e., neutral students. But some students refuse to attend both places i.e., refused grades. The legitimacy of the overall conclusion in decision-making is primarily dependent on the information aggregation stage.

In this situation, the notion of $P_cFS_{ft}S$ is a valuable structure and all the above notions like $IFS_{ft}S$, $PyFS_{ft}S$ and $q-ROFS_{ft}S$ lacks the property to discuss the AG. Moreover, if we discuss the developed notions, then we can observe that

1. If we ignore the AG in the main definition of the developed approach of $P_cFS_{ft}EWA$ and $P_cFS_{ft}EWG$ aggregation operators then the produced work degenerates into intuitionistic fuzzy soft Einstein weighted average $(IFS_{ft}EWA)$ and intuitionistic fuzzy soft Einstein weighted geometric $(IFS_{ft}EWG)$ aggregation operators.
2. If we use only one parameter then the developed notions degenerate into picture fuzzy Einstein weighted average $(PFEWA)$ and picture fuzzy Einstein weighted geometric $(PFEWG)$ aggregation operators.

3. The developed aggregation operators provide more space to decision-makers if they want to provide their assessment in the form of PFS_{ft} data.

It means that the developed theory has many advantages over existing notions. So keeping in view the advanced structure of $P_cFS_{ft}S$ and importance of Einstein t-norm and t-conorm, here in this article we aim to study some new aggregation operators called P_cFS_{ft} Einstein's weighted average ($P_cFS_{ft}EWA$) and P_cFS_{ft} Einstein weighted geometric ($P_cFS_{ft}EWG$) aggregation operators. The study of different types of pollution is very important in real-life problems because these types of pollution not only cause issues for human beings and animals but also plants in terms of polluting the environment. Here we aim to identify types of pollution that mostly affect the environment by using the developed conceptions. For this, we have developed an algorithm for the selection of types of pollution that have severe effects on the environment and climate change.

The rest of the text is given as: We have overviewed some fundamental definitions of PFS, PFEWA aggregation operators, $P_cFS_{ft}S$ in the second section. The fundamental ideas of $P_cFS_{ft}EWA$ and $P_cFS_{ft}EWG$ aggregation operators are covered in section III. We established the DM technique and provided an algorithm along with a descriptive example in section IV to show how to apply these newly created concepts. In section V, it is discussed how these thoughts compare to different other ideas. Remarks at the end are covered in section VI.

II. PRELIMINARIES

In this section, we will go over the definitions of PFS [10]. Moreover, we will discuss the notion of PFEWA aggregation operators defined by Khan et al. [31]. Additionally, we have given the fundamental notions of $P_cFS_{ft}S$ defined by Yang et al. [27].

Definition 1 ([10]): Let $\mathbb{Q}$ denote the universal set, a PFS over $\mathbb{Q}$ is

$$PFS = \left\{ \varpi\colon \; \digamma(\varpi),\; \mho(\varpi),\; \hbar(\varpi) \mid \varpi\in\mathbb{Q} \right\}$$

where $\digamma : \mathbb{Q} \to [0,\ 1]$, $\hbar\colon \mathbb{Q} \to [0,\ 1]$ and $\mho : \mathbb{Q} \to [0,\ 1]$ and $\digamma(\varpi)$, $\mho(\varpi)$, $\hbar(\varpi)$ are called MG, AG, and NMG respectively with $0\leq\digamma(\varpi)+\mho+\hbar(\varpi)\leq 1$. Furthermore for all $\digamma\in\mathbb{Q}$, $r(\varpi) = 1 - \digamma(\varpi) - \mho(\varpi) - \hbar(\varpi)$ is called refusal grade and the triplet $(\digamma(\varpi),\ \mho(\varpi),\ \hbar(\varpi))$ is called PFN.

Definition 2 ([31]): Let $\mathbb{G}_P = (\digamma_{\mathbb{G}_P},\ \mho_{\mathbb{G}_P},\ \hbar_{\mathbb{G}_P})$ $(p = 1,\ 2,\ \ldots,\ n)$ be the family of PFNs, then PF Einstein weighted average aggregation operators are defined by

$$PFEWA(\mathbb{G}_1,\ \mathbb{G}_2,\ \mathbb{G}_3,\ \ldots,\ \mathbb{G}_n) = \left(\begin{array}{c} \frac{\prod_{p=1}^{n}\left(1+\digamma_{\mathbb{G}_P}\right)^{\varrho_p}-\prod_{p=1}^{n}\left(1-\digamma_{\mathbb{G}_P}\right)^{\varrho_p}}{\prod_{p=1}^{n}\left(1+\digamma_{\mathbb{G}_P}\right)^{\varrho_p}-\prod_{p=1}^{n}\left(1-\digamma_{\mathbb{G}_P}\right)^{\varrho_p}}, \\ \frac{2\prod_{p=1}^{n}\left(\mho_{\mathbb{G}_P}\right)^{\varrho_p}}{\prod_{p=1}^{n}\left(2-\mho_{\mathbb{G}_P}\right)^{\varrho_p}+\prod_{p=1}^{n}\left(\mho_{\mathbb{G}_P}\right)^{\varrho_p}}, \\ \frac{2\prod_{p=1}^{n}\left(\hbar_{\mathbb{G}_P}\right)^{\varrho_p}}{\prod_{p=1}^{n}\left(2-\hbar_{\mathbb{G}_P}\right)^{\varrho_p}+\prod_{p=1}^{n}\left(\hbar_{\mathbb{G}_P}\right)^{\varrho_p}} \end{array} \right)$$

where $\varrho = (\varrho_1,\ \varrho_2,\ \ldots,\ \varrho_n)$ denote the weight vectors (WVs) for $\mathbb{G}_P$ with condition that $\sum_{p=1}^{n}\varrho_p = 1$ and $\varrho_p\in[0,\ 1]$.

Definition 3 ([27]): For universal set $\mathbb{Q}$, and E being a set of parameters and $A\subseteq E$. A pair $(P,\ A)$ is said to be $P_cFS_{ft}S$ over $\mathbb{Q}$, where $P : A \to PFS^{\mathbb{Q}}$ is given by

$$\mathbb{G}_{\mathfrak{h}_j}(\varpi_{\mathfrak{i}}) = \left\{\left\langle \varpi_{\mathfrak{i}},\ \digamma_j(\varpi_{\mathfrak{i}}),\ \mho_j(\varpi_{\mathfrak{i}}),\ \hbar_j(\varpi_{\mathfrak{i}})\right\rangle \middle| \varpi_{\mathfrak{i}}\in\mathbb{Q}\right\}$$

where $PFS^{\mathbb{Q}}$ represent the family of PFS. Here $\digamma_j(\varpi_{\mathfrak{i}})$, $\mho_j(\varpi_{\mathfrak{i}})$, $\hbar_j(\varpi_{\mathfrak{i}})$ denote the MG, AG, and NMG respectively with $0\leq\digamma_j(\varpi_{\mathfrak{i}}) + \mho_j(\varpi_{\mathfrak{i}}) + \hbar_j(\varpi_{\mathfrak{i}}) \leq 1$.

III. EINSTEIN AGGREGATION OPERATORS BASED ON PICTURE FUZZY SOFT SETS

In this section, we have to study the basic operational laws for $P_cFS_{ft}Ns$ using the Einstein t-norms and t-conorm. Moreover, we develop the basic definition of picture fuzzy soft Einstein weighted average and geometric aggregation operators.

A. OPERATIONAL LAWS FOR PICTURE FUZZY SOFT NUMBERS

Definition 4: Let $\mathbb{G}_{11} = (\digamma_{11},\ \mho_{11},\ \hbar_{11})$ and $\mathbb{G}_{12} = (\digamma_{12},\ \mho_{12},\ \hbar_{12})$ be two $P_cFS_{ft}Ns$ and $\mathfrak{p}\geq 0$, then based on Einstein's norm and t-conorm we can get

1.
$$\mathbb{G}_{11}\oplus\mathbb{G}_{12} = \left(\frac{(1+\digamma_{11})-(1-\digamma_{12})}{(1+\digamma_{11})+(1-\digamma_{12})},\ \frac{2\mho_{11}}{(2-\mho_{11})+\mho_{12}},\ \frac{2\hbar_{11}}{(2-\hbar_{11})+\hbar_{12}}\right)$$

2.
$$\mathbb{G}_{11}\otimes\mathbb{G}_{12} = \left(\frac{2\digamma_{11}}{(2-\digamma_{11})+\digamma_{12}},\ \frac{(1+\mho_{11})-(1-\mho_{12})}{(1+\mho_{11})+(1-\mho_{12})},\ \frac{(1+\hbar_{11})-(1-\hbar_{12})}{(1+\hbar_{11})+(1-\hbar_{12})}\right)$$

3.
$$\mathfrak{p}\mathbb{G}_{11} = \left(\frac{(1+\digamma_{11})^{\mathfrak{p}}-(1-\digamma_{12})^{\mathfrak{p}}}{(1+\digamma_{11})^{\mathfrak{p}}+(1-\digamma_{12})^{\mathfrak{p}}},\ \frac{2(\mho_{11})^{\mathfrak{p}}}{(2-\mho_{11})^{\mathfrak{p}}+(\mho_{12})^{\mathfrak{p}}},\ \frac{2(\hbar_{11})^{\mathfrak{p}}}{(2-\hbar_{11})^{\mathfrak{p}}+(\hbar_{12})^{\mathfrak{p}}}\right)$$

4.
$$(\mathbb{G}_{11})^{\mathfrak{p}} = \left(\frac{2(\digamma_{11})^{\mathfrak{p}}}{(2-\digamma_{11})^{\mathfrak{p}}+(\digamma_{12})^{\mathfrak{p}}},\ \frac{(1+\mho_{11})^{\mathfrak{p}}-(1-\mho_{12})^{\mathfrak{p}}}{(1+\mho_{11})^{\mathfrak{p}}+(1-\mho_{12})^{\mathfrak{p}}},\ \frac{(1+\hbar_{11})^{\mathfrak{p}}-(1-\hbar_{12})^{\mathfrak{p}}}{(1+\hbar_{11})^{\mathfrak{p}}+(1-\hbar_{12})^{\mathfrak{p}}}\right)$$

Definition 5: Let $\mathbb{G}_{ij} = (\digamma_{ij}, \upsilon_{ij}, \hbar_{ij})$ be the family of $P_cFS_{ft}Ns$, the score function, and the accuracy function are defined by

$$S(\mathbb{G}_{ij}) = \digamma_{ij} - \upsilon_{ij} - \hbar_{ij}$$

And

$$A(\mathbb{G}_{ij}) = \digamma_{ij} + \upsilon_{ij} + \hbar_{ij}$$

Where $\mathbb{G}_{ij} \in [-1, 1]$ and $A(\mathbb{G}_{ij}) \in [0, 1]$.

Note that for two $P_cFS_{ft}Ns$ $\mathbb{G}_{ij}$ and $\mathbb{G}'_{ij}$, we have

1) if $S(\mathbb{G}_{ij}) > S(\mathbb{G}'_{ij})$ then $\mathbb{G}_{ij} > \mathbb{G}'_{ij}$
2) if $S(\mathbb{G}_{ij}) < S(\mathbb{G}'_{ij})$ then $\mathbb{G}_{ij} < \mathbb{G}'_{ij}$
3) if $S(\mathbb{G}_{ij}) = S(\mathbb{G}'_{ij})$ then
   (i) if $A(\mathbb{G}_{ij}) > A(\mathbb{G}'_{ij})$ then $\mathbb{G}_{ij} > \mathbb{G}'_{ij}$
   (ii) if $A(\mathbb{G}_{ij}) < A(\mathbb{G}'_{ij})$ then $\mathbb{G}_{ij} < \mathbb{G}'_{ij}$
   (iii) if $A(\mathbb{G}_{ij}) = A(\mathbb{G}'_{ij})$ then $\mathbb{G}_{ij} = \mathbb{G}'_{ij}$.

B. PICTURE FUZZY SOFT EINSTEIN WEIGHTED AVERAGE AGGREGATION OPERATORS

Definition 6: Let $\mathbb{G}_{ij} = (\digamma_{ij}, \upsilon_{ij}, \hbar_{ij})$ be the collection of $P_cFS_{ft}Ns$, then $P_cFS_{ft}EWA$ an operator is defined by

$$P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) = \oplus_{j=1}^{m} \varsigma_j \left(\oplus_{i=1}^{n} \varrho_i \mathbb{G}_{ij}\right) \tag{1}$$

where $(i = 1, 2, 3, \ldots, n)$, $(j = 1, 2, 3, \ldots, m)$ and ϱ_i, ς_j denote the WVs with the condition that $\sum_{i=1}^{n} \varrho_i = 1$ and $\sum_{j=1}^{m} \varsigma_j = 1$.

Theorem 1: Let $\mathbb{G}_{ij} = (\digamma_{ij}, \upsilon_{ij}, \hbar_{ij})$ be the collection of $P_cFS_{ft}Ns$, then the aggregated result obtained by using the equation (1) is given by

$$\begin{aligned} &P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) = \oplus_{j=1}^{m} \varsigma_j \left(\oplus_{i=1}^{n} \varrho_i \mathbb{G}_{ij}\right) \\ &= \left(\begin{array}{c} \frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j}}, \\ \frac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\upsilon_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(2-\upsilon_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\upsilon_{ij})^{\varrho_i}\right)^{\varsigma_j}}, \\ \frac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(2-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}} \end{array} \right) \end{aligned} \tag{2}$$

where $(i = 1, 2, 3, \ldots, n)$, $(j = 1, 2, 3, \ldots, m)$ and ϱ_i, ς_j denote the WVs with the condition that $\sum_{i=1}^{n} \varrho_i = 1$ and $\sum_{j=1}^{m} \varsigma_j = 1$.

Proof: We will use the mathematical induction method to prove the result

For $n = 1$ we get $\varrho_i = 1$, as shown in the equation at the bottom of the next page.

Now for $m = 1$, we get $\varsigma_j = 1$

$$\begin{aligned} &P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) \\ &= \oplus_{i=1}^{n} \varrho_i \mathbb{G}_{i1} \\ &= \left(\begin{array}{c} \frac{\prod_{i=1}^{n}(1+\digamma_{i1})^{\varrho_i} - \prod_{i=1}^{n}(1-\digamma_{i1})^{\varrho_i}}{\prod_{i=1}^{n}(1+\digamma_{i1})^{\varrho_i} + \prod_{i=1}^{n}(1-\digamma_{i1})^{\varrho_i}}, \\ \frac{2\prod_{i=1}^{n}(\upsilon_{i1})^{\varrho_i}}{\prod_{i=1}^{n}(2-\upsilon_{i1})^{\varrho_i} + \prod_{i=1}^{n}(\upsilon_{i1})^{\varrho_i}}, \\ \frac{2\prod_{i=1}^{n}(\hbar_{i1})^{\varrho_i}}{\prod_{i=1}^{n}(2-\hbar_{i1})^{\varrho_i} + \prod_{i=1}^{n}(2-\hbar_{i1})^{\varrho_i}} \end{array} \right) \\ &= \left(\begin{array}{c} \frac{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}(1+\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j} - \prod_{j=1}^{1}\left(\prod_{i=1}^{n}(1-\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}(1+\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{1}\left(\prod_{i=1}^{n}(1-\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j}}, \\ \frac{2\prod_{j=1}^{1}\left(\prod_{i=1}^{n}(\upsilon_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}(2-\upsilon_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{1}\left(\prod_{i=1}^{n}(\upsilon_{ij})^{\varrho_i}\right)^{\varsigma_j}}, \\ \frac{2\prod_{j=1}^{1}\left(\prod_{i=1}^{n}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}(2-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{1}\left(\prod_{i=1}^{n}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}} \end{array} \right) \end{aligned}$$

So equation (2) is valid for $m = 1$ and $n = 1$.

Now suppose that the above equation holds for $n = \ell_2$, $m = \ell_1 + 1$ and for $n = \ell_2 + 1$, $m = \ell_1$, then

$$\begin{aligned} &P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) \\ &= \oplus_{j=1}^{\ell_1+1} \varsigma_j \left(\oplus_{i=1}^{\ell_2} \varrho_i \mathbb{G}_{ij}\right) \\ &= \left(\begin{array}{c} \frac{\left(\begin{array}{c}\prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2}(1+\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j} - \\ \prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2}(1-\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j}\end{array}\right)}{\left(\begin{array}{c}\prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2}(1+\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j} + \\ \prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2}(1-\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j}\end{array}\right)}, \\ \frac{2\prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2}(\upsilon_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\left(\begin{array}{c}\prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2}(2-\upsilon_{ij})^{\varrho_i}\right)^{\varsigma_j} + \\ \prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2}(\upsilon_{ij})^{\varrho_i}\right)^{\varsigma_j}\end{array}\right)}, \\ \frac{2\prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\left(\begin{array}{c}\prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2}(2-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j} + \\ \prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}\end{array}\right)} \end{array} \right), \end{aligned}$$

$$= \oplus_{j=1}^{\ell_1} \varsigma_j \left(\oplus_{\ddot{i}=1}^{\ell_2+1} \varrho_{\ddot{i}} \mathbb{G}_{\ddot{i}j} \right)$$

$$= \begin{pmatrix} \dfrac{\begin{pmatrix} \prod_{j=1}^{\ell_1} \left(\prod_{\ddot{i}=1}^{\ell_2+1} (1+\digamma_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} - \\ \prod_{j=1}^{\ell_1} \left(\prod_{\ddot{i}=1}^{\ell_2+1} (1-\digamma_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} \end{pmatrix}}{\begin{pmatrix} \prod_{j=1}^{\ell_1} \left(\prod_{\ddot{i}=1}^{\ell_2+1} (1+\digamma_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} + \\ \prod_{j=1}^{\ell_1} \left(\prod_{\ddot{i}=1}^{\ell_2+1} (1-\digamma_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} \end{pmatrix}}, \\ \dfrac{2 \prod_{j=1}^{\ell_1} \left(\prod_{\ddot{i}=1}^{\ell_2+1} (\upsilon_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j}}{\begin{pmatrix} \prod_{j=1}^{\ell_1} \left(\prod_{\ddot{i}=1}^{\ell_2+1} (2-\upsilon_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} + \\ \prod_{j=1}^{\ell_1} \left(\prod_{\ddot{i}=1}^{\ell_2+1} (\upsilon_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} \end{pmatrix}}, \\ \dfrac{2 \prod_{j=1}^{\ell_1} \left(\prod_{\ddot{i}=1}^{\ell_2+1} (\hbar_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j}}{\begin{pmatrix} \prod_{j=1}^{\ell_1} \left(\prod_{\ddot{i}=1}^{\ell_2+1} (2-\hbar_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} \\ + \prod_{j=1}^{\ell_1} \left(\prod_{\ddot{i}=1}^{\ell_2+1} (\hbar_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} \end{pmatrix}} \end{pmatrix}$$

Now suppose that the above equation holds for $n = \ell_2 + 1,\ m = \ell_1 + 1$ then

$$\begin{aligned} & P_cFS_{ft}EWA\left(\mathbb{G}_{11},\ \mathbb{G}_{12},\ \mathbb{G}_{13},\ \ldots,\ \mathbb{G}_{nm}\right) \\ &= \oplus_{j=1}^{\ell_1+1} \varsigma_j \left(\oplus_{\ddot{i}=1}^{\ell_2+1} \varrho_{\ddot{i}} \mathbb{G}_{\ddot{i}j} \right) \\ &= \oplus_{j=1}^{\ell_1+1} \varsigma_j \left(\oplus_{\ddot{i}=1}^{\ell_2} \varrho_{\ddot{i}} \mathbb{G}_{\ddot{i}j} \oplus \varrho_{\ddot{i}+1} \mathbb{G}_{(\ell_2+1)j} \right) \\ &= \left(\oplus_{j=1}^{\ell_1+1} \oplus_{\ddot{i}=1}^{\ell_2} \varrho_{\ddot{i}} \varsigma_j \mathbb{G}_{\ddot{i}j} \right) \oplus \left(\oplus_{j=1}^{\ell_1+1} \varsigma_j \varrho_{\ddot{i}+1} \mathbb{G}_{(\ell_2+1)j} \right) \end{aligned}$$

$$= \begin{pmatrix} \dfrac{\begin{pmatrix} \prod_{j=1}^{\ell_1+1} \left(\prod_{\ddot{i}=1}^{\ell_2} (1+\digamma_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} - \\ \prod_{j=1}^{\ell_1+1} \left(\prod_{\ddot{i}=1}^{\ell_2} (1-\digamma_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} \end{pmatrix}}{\begin{pmatrix} \prod_{j=1}^{\ell_1+1} \left(\prod_{\ddot{i}=1}^{\ell_2} (1+\digamma_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} + \\ \prod_{j=1}^{\ell_1+1} \left(\prod_{\ddot{i}=1}^{\ell_2} (1-\digamma_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} \end{pmatrix}} \oplus \dfrac{\begin{pmatrix} \prod_{j=1}^{\ell_1+1} \left((1+\digamma_{(\ell_2+1)j})^{\varrho(\ell_2+1)} \right)^{\varsigma_j} - \\ \prod_{j=1}^{\ell_1+1} \left((1-\digamma_{(\ell_2+1)j})^{\varrho(\ell_2+1)} \right)^{\varsigma_j} \end{pmatrix}}{\begin{pmatrix} \prod_{j=1}^{\ell_1+1} \left((1+\digamma_{(\ell_2+1)j})^{\varrho(\ell_2+1)} \right)^{\varsigma_j} + \\ \prod_{j=1}^{\ell_1+1} \left((1-\digamma_{(\ell_2+1)j})^{\varrho(\ell_2+1)} \right)^{\varsigma_j} \end{pmatrix}}, \\ \dfrac{2 \prod_{j=1}^{\ell_1+1} \left(\prod_{\ddot{i}=1}^{\ell_2} (\upsilon_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j}}{\begin{pmatrix} \prod_{j=1}^{\ell_1+1} \left(\prod_{\ddot{i}=1}^{\ell_2} (2-\upsilon_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} + \\ \prod_{j=1}^{\ell_1+1} \left(\prod_{\ddot{i}=1}^{\ell_2} (\upsilon_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} \end{pmatrix}} \oplus \dfrac{2 \prod_{j=1}^{\ell_1+1} \left((\upsilon_{(\ell_2+1)j})^{\varrho(\ell_2+1)} \right)^{\varsigma_j}}{\begin{matrix} \prod_{j=1}^{\ell_1+1} \left((2-\upsilon_{(\ell_2+1)j})^{\varrho(\ell_2+1)} \right)^{\varsigma_j} + \\ \prod_{j=1}^{\ell_1+1} \left((\upsilon_{(\ell_2+1)j})^{\varrho(\ell_2+1)} \right)^{\varsigma_j} \end{matrix}}, \\ \dfrac{2 \prod_{j=1}^{\ell_1+1} \left(\prod_{\ddot{i}=1}^{\ell_2} (\hbar_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j}}{\begin{pmatrix} \prod_{j=1}^{\ell_1+1} \left(\prod_{\ddot{i}=1}^{\ell_2} (2-\hbar_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} + \\ \prod_{j=1}^{\ell_1+1} \left(\prod_{\ddot{i}=1}^{\ell_2} (\hbar_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} \end{pmatrix}} \oplus \dfrac{2 \prod_{j=1}^{\ell_1+1} \left((\hbar_{(\ell_2+1)j})^{\varrho(\ell_2+1)} \right)^{\varsigma_j}}{\begin{pmatrix} \prod_{j=1}^{\ell_1+1} \left((2-\hbar_{(\ell_2+1)j})^{\varrho(\ell_2+1)} \right)^{\varsigma_j} + \\ \prod_{j=1}^{\ell_1+1} \left((\hbar_{(\ell_2+1)j})^{\varrho(\ell_2+1)} \right)^{\varsigma_j} \end{pmatrix}} \end{pmatrix}$$

$$P_cFS_{ft}EWA\left(\mathbb{G}_{11},\ \mathbb{G}_{12},\ \mathbb{G}_{13},\ \ldots,\ \mathbb{G}_{nm}\right) = \oplus_{j=1}^{m} \varsigma_j \mathbb{G}_{1j}$$

$$= \begin{pmatrix} \dfrac{\prod_{j=1}^{m} (1+\digamma_{1j})^{\varsigma_j} - \prod_{j=1}^{m} (1-\digamma_{1j})^{\varsigma_j}}{\prod_{j=1}^{m} (1+\digamma_{1j})^{\varsigma_j} + \prod_{j=1}^{m} (1-\digamma_{1j})^{\varsigma_j}}, \\ \dfrac{2 \prod_{j=1}^{m} (\upsilon_{1j})^{\varsigma_j}}{\prod_{j=1}^{m} (2-\upsilon_{1j})^{\varsigma_j} + \prod_{j=1}^{m} (\upsilon_{1j})^{\varsigma_j}}, \\ \dfrac{2 \prod_{j=1}^{m} (\hbar_{1j})^{\varsigma_j}}{\prod_{j=1}^{m} (2-\hbar_{1j})^{\varsigma_j} + \prod_{j=1}^{m} (2-\hbar_{1j})^{\varsigma_j}} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\prod_{j=1}^{m} \left(\prod_{\ddot{i}=1}^{1} (1+\digamma_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} - \prod_{j=1}^{m} \left(\prod_{\ddot{i}=1}^{1} (1-\digamma_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j}}{\prod_{j=1}^{m} \left(\prod_{\ddot{i}=1}^{1} (1+\digamma_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} + \prod_{j=1}^{m} \left(\prod_{\ddot{i}=1}^{1} (1-\digamma_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j}}, \\ \dfrac{2 \prod_{j=1}^{m} \left(\prod_{\ddot{i}=1}^{1} (\upsilon_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j}}{\prod_{j=1}^{m} \left(\prod_{\ddot{i}=1}^{1} (2-\upsilon_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} + \prod_{j=1}^{m} \left(\prod_{\ddot{i}=1}^{1} (\upsilon_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j}}, \\ \dfrac{2 \prod_{j=1}^{m} \left(\prod_{\ddot{i}=1}^{1} (\hbar_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j}}{\prod_{j=1}^{m} \left(\prod_{\ddot{i}=1}^{1} (2-\hbar_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j} + \prod_{j=1}^{m} \left(\prod_{\ddot{i}=1}^{1} (\hbar_{\ddot{i}j})^{\varrho_{\ddot{i}}} \right)^{\varsigma_j}} \end{pmatrix}.$$

$$= \left(\begin{array}{c} \frac{\left(\prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2+1}(1+F_{ij})^{\varrho_i}\right)^{\varsigma_j} - \prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2+1}(1-F_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)}{\left(\prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2+1}(1+F_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2+1}(1-F_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)}, \\ \frac{2\prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2+1}(\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\left(\prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2+1}(2-\mho_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2+1}(\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)}, \\ \frac{2\prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2+1}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\left(\prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2+1}(2-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{\ell_1+1}\left(\prod_{i=1}^{\ell_2+1}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)} \end{array} \right)$$

$$= \oplus_{j=1}^{\ell_1+1}\varsigma_j\left(\oplus_{i=1}^{\ell_2+1}\varrho_i \mathbb{G}_{ij}\right)$$

Hence the result is true for $m = \ell_1 + 1$ and $n = \ell_2 + 1$.

Example 1: Suppose a company wants to install the best software "X" and a team of four experts $C = \{C_1, C_2, C_3, C_4\}$ is invited to give their assessment. Let $\varrho_i = (0.18, 0.24, 0.32, 0.26)$ denote the WVs for experts. Assume that the collection $\pi = \{\pi_1 = Usability, \pi_2 = Efficiency, \pi_3 = Reliability, \pi_4 = Accuracy\}$ denote the set of parameters with WVs $\varsigma_j = (0.19, 0.31, 0.22, 0.28)$. Assume that the experts present their analysis as $P_cFS_{ft}Ns$ given in Table 1.

Now we use equation (2) to get the result, as shown in the equation at the bottom of the page.

Theorem 2: Let $\mathbb{G}_{ij} = (F_{ij}, \mho_{ij}, \hbar_{ij})$ be the collection of $P_cFS_{ft}Ns$, then

$$P_cFS_{ft}WA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) \geq P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm})$$

where ϱ_i, ς_j denote the WVs such as $\varrho_i, \varsigma_j > 0$ using the constraint that $\sum_{i=1}^{n}\varrho_i = 1$ and $\sum_{j=1}^{m}\varsigma_j = 1$.

Proof: As we know that

$$\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+F_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-F_{ij})^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq \sum_{j=1}^{m}\varsigma_j\sum_{i=1}^{n}\varrho_i(1+F_{ij}) + \sum_{j=1}^{m}\varsigma_j\sum_{i=1}^{n}\varrho_i(1-F_{ij}),$$
$$\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+F_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-F_{ij})^{\varrho_i}\right)^{\varsigma_j} \leq \sqrt{2}$$
$$\frac{\prod_{j=1}^{m}(\prod_{i=1}^{n}(1+F_{ij})^{\varrho_i})^{\varsigma_j} - \prod_{j=1}^{m}(\prod_{i=1}^{n}(1-F_{ij})^{\varrho_i})^{\varsigma_j}}{\prod_{j=1}^{m}(\prod_{i=1}^{n}(1+F_{ij})^{\varrho_i})^{\varsigma_j} + \prod_{j=1}^{m}(\prod_{i=1}^{n}(1-F_{ij})^{\varrho_i})^{\varsigma_j}}$$
$$\leq 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-F_{ij})^{\varrho_i}\right)^{\varsigma_j} \quad (3)$$

Again

$$\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(2-\mho_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq \sum_{j=1}^{m}\varsigma_j\sum_{i=1}^{n}\varrho_i(2-\mho_{ij}) + \sum_{j=1}^{m}\varsigma_j\sum_{i=1}^{n}\varrho_i(\mho_{ij}),$$

$$P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) = \oplus_{j=1}^{m}\varsigma_j\left(\oplus_{i=1}^{n}\varrho_i\mathbb{G}_{ij}\right)$$

$$= \left(\begin{array}{c} \frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+F_{ij})^{\varrho_i}\right)^{\varsigma_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-F_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+F_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-F_{ij})^{\varrho_i}\right)^{\varsigma_j}}, \\ \frac{2\prod_{j=1}^{m}(\prod_{i=1}^{n}(\mho_{ij})^{\varrho_i})^{\varsigma_j}}{\prod_{j=1}^{m}(\prod_{i=1}^{n}(2-\mho_{ij})^{\varrho_i})^{\varsigma_j} + \prod_{j=1}^{m}(\prod_{i=1}^{n}(\mho_{ij})^{\varrho_i})^{\varsigma_j}}, \\ \frac{2\prod_{j=1}^{m}(\prod_{i=1}^{n}(\hbar_{ij})^{\varrho_i})^{\varsigma_j}}{\prod_{j=1}^{m}(\prod_{i=1}^{n}(2-\hbar_{ij})^{\varrho_i})^{\varsigma_j} + \prod_{j=1}^{m}(\prod_{i=1}^{n}(\hbar_{ij})^{\varrho_i})^{\varsigma_j}} \end{array} \right)$$

$$= \left(\begin{array}{c} \frac{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}(1+F_{ij})^{\varrho_i}\right)^{\varsigma_j} - \prod_{j=1}^{4}\left(\prod_{i=1}^{4}(1-F_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}(1+F_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{4}\left(\prod_{i=1}^{4}(1-F_{ij})^{\varrho_i}\right)^{\varsigma_j}}, \\ \frac{2\prod_{j=1}^{4}\left(\prod_{i=1}^{4}(\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}(2-\mho_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{4}\left(\prod_{i=1}^{4}(\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}}, \\ \frac{2\prod_{j=1}^{4}\left(\prod_{i=1}^{4}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}(2-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{4}\left(\prod_{i=1}^{4}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}} \end{array} \right)$$

$$
= \left(\begin{array}{c}
\left(\dfrac{\begin{array}{c}
\left\{(1+0.21)^{18}\times(1+0.19)^{24}\times(1+0.31)^{32}\times(1+0.11)^{26}\right\}^{19}\times \\
\left\{(1+0.41)^{18}\times(1+0.15)^{24}\times(1+0.24)^{32}\times(1+0.27)^{26}\right\}^{31}\times \\
\left\{(1+0.30)^{18}\times(1+0.11)^{24}\times(1+0.17)^{32}\times(1+0.41)^{26}\right\}^{22}\times \\
\left\{(1+0.10)^{18}\times(1+0.50)^{24}\times(1+0.11)^{32}\times(1+0.19)^{26}\right\}^{28}- \\
\left\{(1-0.21)^{18}\times(1-0.19)^{24}\times(1-0.31)^{32}\times(1-0.11)^{26}\right\}^{19}\times \\
\left\{(1-0.41)^{18}\times(1-0.15)^{24}\times(1-0.24)^{32}\times(1-0.27)^{26}\right\}^{31}\times \\
\left\{(1-0.30)^{18}\times(1-0.11)^{24}\times(1-0.17)^{32}\times(1-0.41)^{26}\right\}^{22}\times \\
\left\{(1-0.10)^{18}\times(1-0.50)^{24}\times(1-0.11)^{32}\times(1-0.19)^{26}\right\}^{28}
\end{array}}{\begin{array}{c}
\left\{(1+0.21)^{18}\times(1+0.19)^{24}\times(1+0.31)^{32}\times(1+0.11)^{26}\right\}^{19}\times \\
\left\{(1+0.41)^{18}\times(1+0.15)^{24}\times(1+0.24)^{32}\times(1+0.27)^{26}\right\}^{31}\times \\
\left\{(1+0.30)^{18}\times(1+0.11)^{24}\times(1+0.17)^{32}\times(1+0.41)^{26}\right\}^{22}\times \\
\left\{(1+0.10)^{18}\times(1+0.50)^{24}\times(1+0.11)^{32}\times(1+0.19)^{26}\right\}^{28}+ \\
\left\{(1-0.21)^{18}\times(1-0.19)^{24}\times(1-0.31)^{32}\times(1-0.11)^{26}\right\}^{19}\times \\
\left\{(1-0.41)^{18}\times(1-0.15)^{24}\times(1-0.24)^{32}\times(1-0.27)^{26}\right\}^{31}\times \\
\left\{(1-0.30)^{18}\times(1-0.11)^{24}\times(1-0.17)^{32}\times(1-0.41)^{26}\right\}^{22}\times \\
\left\{(1-0.10)^{18}\times(1-0.50)^{24}\times(1-0.11)^{32}\times(1-0.19)^{26}\right\}^{28}
\end{array}} \right), \\
\left(\dfrac{2\left(\begin{array}{c}
\left\{(0.23)^{18}\times(0.13)^{24}\times(0.33)^{32}\times(0.13)^{26}\right\}^{19}\times \\
\left\{(0.20)^{18}\times(0.16)^{24}\times(0.25)^{32}\times(0.28)^{26}\right\}^{31}\times \\
\left\{(0.29)^{18}\times(0.40)^{24}\times(0.13)^{32}\times(0.42)^{26}\right\}^{22}\times \\
\left\{(0.20)^{18}\times(0.15)^{24}\times(0.53)^{32}\times(0.20)^{26}\right\}^{28}
\end{array}\right)}{\left(\begin{array}{c}
\left\{(2-0.23)^{18}\times(2-0.13)^{24}\times(2-0.33)^{32}\times(2-0.13)^{26}\right\}^{19}\times \\
\left\{(2-0.20)^{18}\times(2-0.16)^{24}\times(2-0.25)^{32}\times(2-0.28)^{26}\right\}^{31}\times \\
\left\{(2-0.29)^{18}\times(2-0.40)^{24}\times(2-0.13)^{32}\times(2-0.42)^{26}\right\}^{22}\times \\
\left\{(2-0.20)^{18}\times(2-0.15)^{24}\times(2-0.53)^{32}\times(2-0.20)^{26}\right\}^{28}
\end{array}\right)+\left(\begin{array}{c}
\left\{(0.23)^{18}\times(0.13)^{24}\times(0.33)^{32}\times(0.13)^{26}\right\}^{19}\times \\
\left\{(0.20)^{18}\times(0.16)^{24}\times(0.25)^{32}\times(0.28)^{26}\right\}^{31}\times \\
\left\{(0.29)^{18}\times(0.40)^{24}\times(0.13)^{32}\times(0.42)^{26}\right\}^{22}\times \\
\left\{(0.20)^{18}\times(0.15)^{24}\times(0.53)^{32}\times(0.20)^{26}\right\}^{28}
\end{array}\right)} \right), \\
\left(\dfrac{2\left(\begin{array}{c}
\left\{(0.24)^{18}\times(0.14)^{24}\times(0.34)^{32}\times(0.15)^{26}\right\}^{19}\times \\
\left\{(0.17)^{18}\times(0.18)^{24}\times(0.26)^{32}\times(0.29)^{26}\right\}^{31}\times \\
\left\{(0.27)^{18}\times(0.24)^{24}\times(0.14)^{32}\times(0.14)^{26}\right\}^{22}\times \\
\left\{(0.30)^{18}\times(0.16)^{24}\times(0.13)^{32}\times(0.21)^{26}\right\}^{28}
\end{array}\right)}{\left(\begin{array}{c}
\left\{(2-0.24)^{18}\times(2-0.14)^{24}\times(2-0.34)^{32}\times(2-0.15)^{26}\right\}^{19}\times \\
\left\{(2-0.17)^{18}\times(2-0.18)^{24}\times(2-0.26)^{32}\times(2-0.29)^{26}\right\}^{31}\times \\
\left\{(2-0.27)^{18}\times(2-0.24)^{24}\times(2-0.14)^{32}\times(2-0.14)^{26}\right\}^{22}\times \\
\left\{(2-0.30)^{18}\times(2-0.16)^{24}\times(2-0.13)^{32}\times(2-0.21)^{26}\right\}^{28}
\end{array}\right)+\left(\begin{array}{c}
\left\{(0.24)^{18}\times(0.14)^{24}\times(0.34)^{32}\times(0.15)^{26}\right\}^{19}\times \\
\left\{(0.17)^{18}\times(0.18)^{24}\times(0.26)^{32}\times(0.29)^{26}\right\}^{31}\times \\
\left\{(0.27)^{18}\times(0.24)^{24}\times(0.14)^{32}\times(0.14)^{26}\right\}^{22}\times \\
\left\{(0.30)^{18}\times(0.16)^{24}\times(0.13)^{32}\times(0.21)^{26}\right\}^{28}
\end{array}\right)} \right)
\end{array} \right)
$$

$$
= (0.2393,\ 0.23777,\ 0.1998)
$$

TABLE 1. P_cFS_{ft} **information.**

(0.21, 0.23, 0.24)	(0.41, 0.20, 0.17)	(0.30, 0.29, 0.27)	(0.10, 0.20, 0.30)
(0.19, 0.13, 0.14)	(0.15, 0.16, 0.18)	(0.11, 0.40, 0.24)	(0.50, 0.15, 0.16)
(0.31, 0.33, 0.34)	(0.24, 0.25, 0.26)	(0.17, 0.13, 0.14)	(0.11, 0.53, 0.13)
(0.11, 0.13, 0.15)	(0.27, 0.28, 0.29)	(0.41, 0.42, 0.14)	(0.19, 0.20, 0.21)

$$\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(2-\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}\leq\sqrt{2}$$

$$\frac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(2-\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}}\geq\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mho_{ij})^{\varrho_i}\right)^{\varsigma_j} \quad (4)$$

Similarly,

$$\frac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(2-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}}\geq\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j} \quad (5)$$

Let $P_cFS_{ft}WA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) = \mathbb{G} = (\digamma_{\mathbb{G}}, \mho_{\mathbb{G}}, \hbar_{\mathbb{G}})$ and $P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) = \mathbb{G}^{\circ} = (\digamma_{\mathbb{G}^{\circ}}, \mho_{\mathbb{G}^{\circ}}, \hbar_{\mathbb{G}^{\circ}})$. Then, (3), (4), and (5) can be converted into the forms $\digamma_{\mathbb{G}}\geq\digamma_{\mathbb{G}^{\circ}}$, $\hbar_{\mathbb{G}}\leq\hbar_{\mathbb{G}^{\circ}}$ and $\mho_{\mathbb{G}}\leq\mho_{\mathbb{G}^{\circ}}$. Hence $S(\mathbb{G}) = \digamma_{\mathbb{G}} - \mho_{\mathbb{G}} - \hbar_{\mathbb{G}} \geq \digamma_{\mathbb{G}^{\circ}} - \mho_{\mathbb{G}^{\circ}} - \hbar_{\mathbb{G}^{\circ}} = S(\mathbb{G}^{\circ})$. So, $S(\mathbb{G}) \geq S(\mathbb{G}^{\circ})$.

If $S(\mathbb{G}) > S(\mathbb{G}^{\circ})$ then

$$P_cFS_{ft}WA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) \geq P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) \quad (6)$$

If $S(\mathbb{G}) = S(\mathbb{G}^{\circ})$ then $\digamma_{\mathbb{G}} - \mho_{\mathbb{G}} - \hbar_{\mathbb{G}} = \digamma_{\mathbb{G}^{\circ}} - \mho_{\mathbb{G}^{\circ}} - \hbar_{\mathbb{G}^{\circ}} = S(\mathbb{G}^{\circ})$. Hence, $\digamma_{\mathbb{G}} = \digamma_{\mathbb{G}^{\circ}}$, $\hbar_{\mathbb{G}} = \hbar_{\mathbb{G}^{\circ}}$ and $\mho_{\mathbb{G}} = \mho_{\mathbb{G}^{\circ}}$ then the accuracy function $A(\mathbb{G}) = \digamma_{\mathbb{G}} + \mho_{\mathbb{G}} + \hbar_{\mathbb{G}} = \digamma_{\mathbb{G}^{\circ}} + \mho_{\mathbb{G}^{\circ}} +$

$$P_cFS_{ft}WA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) = \begin{pmatrix} 1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}(1-\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j}, \\ \prod_{j=1}^{4}\left(\prod_{i=1}^{4}(\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}, \\ \prod_{j=1}^{4}\left(\prod_{i=1}^{4}(\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j} \end{pmatrix}$$

$$P_cFS_{ft}WA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \mathbb{G}_{44})$$

$$= \begin{pmatrix} 1-\begin{pmatrix} \{(1-0.21)^{18}\times(1-0.19)^{24}\times(1-0.31)^{32}\times(1-0.11)^{26}\}^{19}\times \\ \{(1-0.41)^{18}\times(1-0.15)^{24}\times(1-0.24)^{32}\times(1-0.27)^{26}\}^{31}\times \\ \{(1-0.30)^{18}\times(1-0.11)^{24}\times(1-0.17)^{32}\times(1-0.41)^{26}\}^{22}\times \\ \{(1-0.10)^{18}\times(1-0.50)^{24}\times(1-0.11)^{32}\times(1-0.19)^{26}\}^{28} \end{pmatrix}, \\ \left(\begin{pmatrix} \{(0.23)^{18}\times(0.20)^{24}\times(0.29)^{32}\times(0.20)^{26}\}^{19}\times \\ \{(0.13)^{18}\times(0.16)^{24}\times(0.40)^{32}\times(0.15)^{26}\}^{31}\times \\ \{(0.33)^{18}\times(0.25)^{24}\times(0.13)^{32}\times(0.53)^{26}\}^{22}\times \\ \{(0.13)^{18}\times(0.28)^{24}\times(0.42)^{32}\times(0.20)^{26}\}^{28} \end{pmatrix}\right), \\ \left(\begin{pmatrix} \{(0.24)^{18}\times(0.17)^{24}\times(0.27)^{32}\times(0.30)^{26}\}^{19}\times \\ \{(0.14)^{18}\times(0.18)^{24}\times(0.24)^{32}\times(0.16)^{26}\}^{31}\times \\ \{(0.34)^{18}\times(0.26)^{24}\times(0.14)^{32}\times(0.13)^{26}\}^{22}\times \\ \{(0.15)^{18}\times(0.29)^{24}\times(0.14)^{32}\times(0.21)^{26}\}^{28} \end{pmatrix}\right) \end{pmatrix}$$

$$= (0.2452,\ 0.2343,\ 0.1987).$$

$\hbar_{\mathbb{G}^{\circ}} = A\left(\mathbb{G}^{\circ}\right)$. Thus

$$P_cFS_{ft}WA\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right) = P_cFS_{ft}EWA\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right) \quad (7)$$

From (6) and (7), we get

$$P_cFS_{ft}WA\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right) \geq P_cFS_{ft}EWA\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right).$$

Example 2: Consider all data from example 1 and aggregate the given data by using as shown in the equation at the bottom of the previous page.

Since the score value for $P_cFS_{ft}WA\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \mathbb{G}_{44}\right) = -0.1877$ and $P_cFS_{ft}EWA\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \mathbb{G}_{44}\right) = -0.1981$.

Hence from examples 1 and 2, it is proven that

$$P_cFS_{ft}WA\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right) \geq P_cFS_{ft}EWA\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right).$$

C. PROPERTIES OF PICTURE FUZZY SOFT EINSTEIN WEIGHTED AVERAGE OPERATORS

In this section, we will discuss the basic properties like Idempotency, Boundedness, and Homogeneity.

1. **Idempotency:** If $\mathbb{G}_{\mathbb{i}\mathbb{j}} = \left(\digamma_{\mathbb{i}\mathbb{j}}, \mho_{\mathbb{i}\mathbb{j}}, \hbar_{\mathbb{i}\mathbb{j}}\right) = \mathbb{G} = \left(\digamma, \mho, \hbar\right)$ for all $\mathbb{i}, \mathbb{j}$, then

$$P_cFS_{ft}EWA\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right) = \mathbb{G}.$$

Proof: As we know that

$$P_cFS_{ft}EWA\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right) = \left(\begin{array}{c} \frac{\left(\left(1+\digamma_{\mathbb{i}\mathbb{j}}\right)^{\sum_{\mathbb{i}=1}^{n}\varrho_{\mathbb{i}}}\right)^{\sum_{\mathbb{j}=1}^{m}\varsigma_{\mathbb{j}}} - \left(\left(1-\digamma_{\mathbb{i}\mathbb{j}}\right)^{\sum_{\mathbb{i}=1}^{n}\varrho_{\mathbb{i}}}\right)^{\sum_{\mathbb{j}=1}^{m}\varsigma_{\mathbb{j}}}}{\left(\left(1+\digamma_{\mathbb{i}\mathbb{j}}\right)^{\sum_{\mathbb{i}=1}^{n}\varrho_{\mathbb{i}}}\right)^{\sum_{\mathbb{j}=1}^{m}\varsigma_{\mathbb{j}}} + \left(\left(1+\digamma_{\mathbb{i}\mathbb{j}}\right)^{\sum_{\mathbb{i}=1}^{n}\varrho_{\mathbb{i}}}\right)^{\sum_{\mathbb{j}=1}^{m}\varsigma_{\mathbb{j}}}}, \\ \frac{2\left(\left(\mho_{\mathbb{i}\mathbb{j}}\right)^{\sum_{\mathbb{i}=1}^{n}\varrho_{\mathbb{i}}}\right)^{\sum_{\mathbb{j}=1}^{m}\varsigma_{\mathbb{j}}}}{\left(\left(2-\mho_{\mathbb{i}\mathbb{j}}\right)^{\sum_{\mathbb{i}=1}^{n}\varrho_{\mathbb{i}}}\right)^{\sum_{\mathbb{j}=1}^{m}\varsigma_{\mathbb{j}}} + \left(\left(\mho_{\mathbb{i}\mathbb{j}}\right)^{\sum_{\mathbb{i}=1}^{n}\varrho_{\mathbb{i}}}\right)^{\sum_{\mathbb{j}=1}^{m}\varsigma_{\mathbb{j}}}}, \\ \frac{2\left(\left(\hbar_{\mathbb{i}\mathbb{j}}\right)^{\sum_{\mathbb{i}=1}^{n}\varrho_{\mathbb{i}}}\right)^{\sum_{\mathbb{j}=1}^{m}\varsigma_{\mathbb{j}}}}{\left(\left(2-\hbar_{\mathbb{i}\mathbb{j}}\right)^{\sum_{\mathbb{i}=1}^{n}\varrho_{\mathbb{i}}}\right)^{\sum_{\mathbb{j}=1}^{m}\varsigma_{\mathbb{j}}} + \left(\left(\hbar_{\mathbb{i}\mathbb{j}}\right)^{\sum_{\mathbb{i}=1}^{n}\varrho_{\mathbb{i}}}\right)^{\sum_{\mathbb{j}=1}^{m}\varsigma_{\mathbb{j}}}} \end{array}\right)$$

$$= \left(\begin{array}{c} \frac{\left(1+\digamma_{\mathbb{i}\mathbb{j}}\right)-\left(1-\digamma_{\mathbb{i}\mathbb{j}}\right)}{\left(1+\digamma_{\mathbb{i}\mathbb{j}}\right)-\left(1-\digamma_{\mathbb{i}\mathbb{j}}\right)}, \frac{2\left(\mho_{\mathbb{i}\mathbb{j}}\right)}{\left(2-\mho_{\mathbb{i}\mathbb{j}}\right)+\left(\mho_{\mathbb{i}\mathbb{j}}\right)}, \\ \frac{2\left(\hbar_{\mathbb{i}\mathbb{j}}\right)}{\left(2-\hbar_{\mathbb{i}\mathbb{j}}\right)+\left(\hbar_{\mathbb{i}\mathbb{j}}\right)} \end{array}\right)$$

$$= \left(\begin{array}{c} \frac{\left(1+\digamma\right)-\left(1-\digamma\right)}{\left(1+\digamma\right)+\left(1-\digamma\right)}, \frac{2(\mho)}{(2-\mho)+(\mho)}, \\ \frac{2\left(\hbar\right)}{\left(2-\hbar\right)+\left(\hbar\right)} \end{array}\right)$$

$$= \left(\digamma, \mho, \hbar\right) = \mathbb{G}.$$

2. **Boundedness:** Let $\mathbb{G}_{\mathbb{i}\mathbb{j}} = \left(\digamma_{\mathbb{i}\mathbb{j}}, \mho_{\mathbb{i}\mathbb{j}}, \hbar_{\mathbb{i}\mathbb{j}}\right)$ be the collection of $P_cFS_{ft}Ns$ and $\mathbb{G}_{min} = min\left(\mathbb{G}_{\mathbb{i}\mathbb{j}}\right)$ and $\mathbb{G}_{max} = max\left(\mathbb{G}_{\mathbb{i}\mathbb{j}}\right)$. Then $\mathbb{G}_{min} \leq P_cFS_{ft}EWA\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right) \leq \mathbb{G}_{max}$.

Proof: Let $f\left(\mathfrak{a}\right) = \left(1-\mathfrak{a}/1+\mathfrak{a}\right)$, $\mathfrak{a} \in [0, 1]$ then $d/d(f(\mathfrak{a})) = d/d\mathfrak{a}\left(1-\mathfrak{a}/1+\mathfrak{a}\right) = -2/(1+\mathfrak{a})^2 < 0$ that shows that $f\left(\mathfrak{a}\right)$ is a decreasing function on [0, 1]. So, $\digamma_{min} \leq \digamma_{\mathbb{i}\mathbb{j}} \leq \digamma_{max}$ for all $\mathbb{i}, \mathbb{j}$. Hence $f\left(\digamma_{min}\right) \leq f\left(\digamma_{\mathbb{i}\mathbb{j}}\right) \leq f\left(\digamma_{max}\right)$.

Assume that $\varrho_{\mathbb{i}}, \varsigma_{\mathbb{j}}$ are the WVs such that $\varrho_{\mathbb{i}}, \varsigma_{\mathbb{j}}$ and $\sum_{\mathbb{i}=1}^{n}\varrho_{\mathbb{i}} = 1$ and $\sum_{\mathbb{j}=1}^{m}\varsigma_{\mathbb{j}} = 1$. We have

$$\Longleftrightarrow \prod_{\mathbb{j}=1}^{m}\left(\prod_{\mathbb{i}=1}^{n}\left(\frac{1-\digamma_{max}}{1+\digamma_{max}}\right)^{\varrho_{\mathbb{i}}}\right)^{\varsigma_{\mathbb{j}}} \leq \prod_{\mathbb{j}=1}^{m}\left(\prod_{\mathbb{i}=1}^{n}\left(\frac{1-\digamma_{\mathbb{i}\mathbb{j}}}{1+\digamma_{\mathbb{i}\mathbb{j}}}\right)^{\varrho_{\mathbb{i}}}\right)^{\varsigma_{\mathbb{j}}} \leq \prod_{\mathbb{j}=1}^{m}\left(\prod_{\mathbb{i}=1}^{n}\left(\frac{1-\digamma_{min}}{1+\digamma_{min}}\right)^{\varrho_{\mathbb{i}}}\right)^{\varsigma_{\mathbb{j}}}$$

$$\Longleftrightarrow \left(\left(\frac{1-\digamma_{max}}{1+\digamma_{max}}\right)^{\sum_{\mathbb{i}=1}^{n}\varrho_{\mathbb{i}}}\right)^{\sum_{\mathbb{j}=1}^{m}\varsigma_{\mathbb{j}}} \leq \prod_{\mathbb{j}=1}^{m}\left(\prod_{\mathbb{i}=1}^{n}\left(\frac{1-\digamma_{\mathbb{i}\mathbb{j}}}{1+\digamma_{\mathbb{i}\mathbb{j}}}\right)^{\varrho_{\mathbb{i}}}\right)^{\varsigma_{\mathbb{j}}} \leq \left(\left(\frac{1-\digamma_{min}}{1+\digamma_{min}}\right)^{\sum_{\mathbb{i}=1}^{n}\varrho_{\mathbb{i}}}\right)^{\sum_{\mathbb{j}=1}^{m}\varsigma_{\mathbb{j}}}$$

$$\Longleftrightarrow 1+\left(\frac{1-\digamma_{max}}{1+\digamma_{max}}\right) \leq 1+\prod_{\mathbb{j}=1}^{m}\left(\prod_{\mathbb{i}=1}^{n}\left(\frac{1-\digamma_{\mathbb{i}\mathbb{j}}}{1+\digamma_{\mathbb{i}\mathbb{j}}}\right)^{\varrho_{\mathbb{i}}}\right)^{\varsigma_{\mathbb{j}}} \leq 1+\left(\frac{1-\digamma_{min}}{1+\digamma_{min}}\right)$$

$$\Longleftrightarrow \left(\frac{2}{1+\digamma_{max}}\right) \leq 1+\prod_{\mathbb{j}=1}^{m}\left(\prod_{\mathbb{i}=1}^{n}\left(\frac{1-\digamma_{\mathbb{i}\mathbb{j}}}{1+\digamma_{\mathbb{i}\mathbb{j}}}\right)^{\varrho_{\mathbb{i}}}\right)^{\varsigma_{\mathbb{j}}} \leq \left(\frac{2}{1+\digamma_{min}}\right)$$

$$\Longleftrightarrow \left(\frac{1+\digamma_{min}}{2}\right) \leq \frac{1}{1+\prod_{\mathbb{j}=1}^{m}\left(\prod_{\mathbb{i}=1}^{n}\left(\frac{1-\digamma_{\mathbb{i}\mathbb{j}}}{1+\digamma_{\mathbb{i}\mathbb{j}}}\right)^{\varrho_{\mathbb{i}}}\right)^{\varsigma_{\mathbb{j}}}} \leq \left(\frac{1+\digamma_{max}}{2}\right)$$

$$\Longleftrightarrow \left(1+\digamma_{min}\right) \leq \frac{2}{1+\prod_{\mathbb{j}=1}^{m}\left(\prod_{\mathbb{i}=1}^{n}\left(\frac{1-\digamma_{\mathbb{i}\mathbb{j}}}{1+\digamma_{\mathbb{i}\mathbb{j}}}\right)^{\varrho_{\mathbb{i}}}\right)^{\varsigma_{\mathbb{j}}}} \leq \left(1+\digamma_{max}\right)$$

$$\Longleftrightarrow \left(1+\digamma_{min}-1\right) \leq \frac{2}{1+\prod_{\mathbb{j}=1}^{m}\left(\prod_{\mathbb{i}=1}^{n}\left(\frac{1-\digamma_{\mathbb{i}\mathbb{j}}}{1+\digamma_{\mathbb{i}\mathbb{j}}}\right)^{\varrho_{\mathbb{i}}}\right)^{\varsigma_{\mathbb{j}}}} - 1 \leq \left(1+\digamma_{max}\right) - 1$$

$$\Longleftrightarrow \left(\digamma_{min}\right) \leq \frac{2}{1+\prod_{\mathbb{j}=1}^{m}\left(\prod_{\mathbb{i}=1}^{n}\left(\frac{1-\digamma_{\mathbb{i}\mathbb{j}}}{1+\digamma_{\mathbb{i}\mathbb{j}}}\right)^{\varrho_{\mathbb{i}}}\right)^{\varsigma_{\mathbb{j}}}} - 1 \leq \left(\digamma_{max}\right)$$

$$\Longleftrightarrow \left(\digamma_{min}\right)$$

$$\leq \frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\text{Ƒ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\text{Ƒ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\text{Ƒ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\text{Ƒ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}} \leq \left(\text{Ƒ}_{max}\right) \tag{8}$$

Now assume that $g(\mathfrak{a}) = \frac{(2-\mathfrak{a})}{(\mathfrak{a})}$ for $\mathfrak{a} \in [0, 1]$ then $d/d\mathfrak{a}\left(\frac{(2-\mathfrak{a})}{(\mathfrak{a})}\right) = \frac{-2}{\mathfrak{a}^2} < 0$. Hence $g(\mathfrak{a})$ is a decreasing function in [0, 1]. As $\hbar_{min} \leq \hbar_{ij} \leq \hbar_{max}$ for all i, j, then $g(\hbar_{max}) \leq g(\hbar_{ij}) \leq g(\hbar_{min})$.

So, $\frac{(2-\hbar_{max})}{(\hbar_{max})} \leq \frac{(2-\hbar_{ij})}{(\hbar_{ij})} \leq \frac{(2-\hbar_{min})}{(\hbar_{min})}$. Assume that ϱ_i, ς_j are the WVs such that ϱ_i, ς_j and $\sum_{i=1}^{n} \varrho_i = 1$ and $\sum_{j=1}^{m} \varsigma_j = 1$. We have

$$\Longleftrightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\text{ʊ}_{max}}{\text{ʊ}_{max}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\text{ʊ}_{ij}}{\text{ʊ}_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\text{ʊ}_{min}}{\text{ʊ}_{min}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\Longleftrightarrow \left(\left(\frac{2-\text{ʊ}_{max}}{\text{ʊ}_{max}}\right)^{\sum_{i=1}^{n}\varrho_i=1}\right)^{\sum_{j=1}^{m}\varsigma_j=1}$$
$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\text{ʊ}_{ij}}{\text{ʊ}_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq \left(\left(\frac{2-\text{ʊ}_{min}}{\text{ʊ}_{min}}\right)^{\sum_{i=1}^{n}\varrho_i=1}\right)^{\sum_{j=1}^{m}\varsigma_j=1}$$
$$\Longleftrightarrow 1+\left(\frac{2-\text{ʊ}_{max}}{\text{ʊ}_{max}}\right)$$
$$\leq 1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\text{ʊ}_{ij}}{\text{ʊ}_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq 1+\left(\frac{2-\text{ʊ}_{min}}{\text{ʊ}_{min}}\right)$$
$$\Longleftrightarrow \left(\frac{2}{\text{ʊ}_{max}}\right) \leq 1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\text{ʊ}_{ij}}{\text{ʊ}_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq \left(\frac{2}{\text{ʊ}_{min}}\right)$$
$$\Longleftrightarrow \left(\frac{\text{ʊ}_{min}}{2}\right) \leq \left(\frac{1}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\text{ʊ}_{ij}}{\text{ʊ}_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}}\right)$$
$$\leq \left(\frac{\text{ʊ}_{max}}{2}\right)$$
$$\Longleftrightarrow (\text{ʊ}_{min}) \leq \left(\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-\text{ʊ}_{ij}}{\text{ʊ}_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}}\right)$$
$$\leq (\text{ʊ}_{max})$$
$$\Longleftrightarrow (\text{ʊ}_{min})$$
$$\leq \frac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\text{ʊ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\text{ʊ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\text{ʊ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}} \leq (\text{ʊ}_{max}) \tag{9}$$

Similarly,

$$(\hbar_{min}) \leq \frac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\hbar_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\hbar_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\hbar_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}} \leq (\hbar_{max}) \tag{10}$$

Let $P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) = \mathbb{G}$. Then inequalities (8), (9), and (10) can be written as $\text{Ƒ}_{min} \leq \text{Ƒ} \leq \text{Ƒ}_{max}$, $\hbar_{min} \leq \hbar \leq \hbar_{max}$ and $\text{ʊ}_{min} \leq \text{ʊ} \leq \text{ʊ}_{max}$. Thus $S(\mathbb{G}) = \text{Ƒ}_{\mathbb{G}} - \text{ʊ}_{\mathbb{G}} - \hbar_{\mathbb{G}} \leq \text{Ƒ}_{max} - \text{ʊ}_{min} - \hbar_{min} = S(\mathbb{Z}_{max})$ and $S(\mathbb{G}) = \text{Ƒ}_{\mathbb{G}} - \text{ʊ}_{\mathbb{G}} - \hbar_{\mathbb{G}} \leq \text{Ƒ}_{min} - \text{ʊ}_{max} - \hbar_{max} = S(\mathbb{Z}_{min})$.

If $S(\mathbb{G}) < S(\mathbb{Z}_{max})$ and $S(\mathbb{G}) > S(\mathbb{Z}_{min})$ then

$$\mathbb{G}_{min} \leq P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) \leq \mathbb{G}_{max}$$

If $S(\mathbb{G}) = S(\mathbb{Z}_{max})$ then $\text{Ƒ} = \text{Ƒ}_{max}$, $\text{ʊ} = \text{ʊ}_{max}$ *and* $\hbar = \hbar_{max}$. Then $S(\mathbb{G}) = \text{Ƒ} - \text{ʊ} - \hbar = \text{Ƒ}_{max} - \text{ʊ}_{max} - \hbar_{max} = S(\mathbb{Z}_{max})$. Therefore,

$$P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) = \mathbb{G}_{max}$$

If $S(\mathbb{G}) = S(\mathbb{Z}_{min})$ then $\text{Ƒ} - \text{ʊ} - \hbar = \text{Ƒ}_{min} - \text{ʊ}_{min} - \hbar_{min}$ that is $\text{Ƒ} = \text{Ƒ}_{min}$, $\text{ʊ} = \text{ʊ}_{min}$ *and* $\hbar = \hbar_{min}$.

Thus $A(\mathbb{G}) = \text{Ƒ} + \text{ʊ} + \hbar = \text{Ƒ}_{min} + \text{ʊ}_{min} + \hbar_{min} = A(\mathbb{Z}_{min})$

$$P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) = \mathbb{G}_{min}$$

Thus

$$\mathbb{G}_{min} \leq P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) \leq \mathbb{G}_{max}.$$

3. **Homogeneity:** Let $\mathbb{G}_{ij} = (\text{Ƒ}_{ij}, \text{ʊ}_{ij}, \hbar_{ij})$ be the collection of $P_cFS_{ft}Ns$ and $\mathfrak{p} > 0$ then

$$P_cFS_{ft}EWA(\mathfrak{p}\mathbb{G}_{11}, \mathfrak{p}\mathbb{G}_{12}, \mathfrak{p}\mathbb{G}_{13}, \ldots, \mathfrak{p}\mathbb{G}_{nm}) = \mathfrak{p}P_cFS_{ft}EWA(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}).$$

Proof: Let $\mathbb{G}_{ij} = (\text{Ƒ}_{ij}, \text{ʊ}_{ij}, \hbar_{ij})$ be a $P_cFS_{ft}N$ and $\mathfrak{p} > 0$ then

$$\mathfrak{p}\mathbb{G}_{ij} = \left(\frac{\left(1+\text{Ƒ}_{ij}\right)^{\mathfrak{p}}-\left(1-\text{Ƒ}_{ij}\right)^{\mathfrak{p}}}{\left(1+\text{Ƒ}_{ij}\right)^{\mathfrak{p}}+\left(1-\text{Ƒ}_{ij}\right)^{\mathfrak{p}}}, \frac{2\left(\text{ʊ}_{ij}\right)^{\mathfrak{p}}}{\left(2-\text{ʊ}_{ij}\right)^{\mathfrak{p}}+\left(\text{ʊ}_{ij}\right)^{\mathfrak{p}}}, \frac{2\left(\hbar_{ij}\right)^{\mathfrak{p}}}{\left(2-\hbar_{ij}\right)^{\mathfrak{p}}+\left(\hbar_{ij}\right)^{\mathfrak{p}}}\right)$$

So,

$$P_cFS_{ft}EWA(\mathfrak{p}\mathbb{G}_{11}, \mathfrak{p}\mathbb{G}_{12}, \mathfrak{p}\mathbb{G}_{13}, \ldots, \mathfrak{p}\mathbb{G}_{nm}) = \left(\begin{array}{c}\frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\text{Ƒ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\text{Ƒ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\text{Ƒ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\text{Ƒ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}}, \\ \frac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\text{ʊ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\text{ʊ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\text{ʊ}_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}}, \\ \frac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\hbar_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\hbar_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\hbar_{ij}\right)^{\varrho_i}\right)^{\varsigma_j}}\end{array}\right)$$

$$= \left(\frac{\left(\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}\left(1+\digamma_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)^{\mathfrak{p}} - \left(\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}\left(1-\digamma_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)^{\mathfrak{p}}}{\left(\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}\left(1+\digamma_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)^{\mathfrak{p}} + \left(\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}\left(1-\digamma_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)^{\mathfrak{p}}}, \frac{2\left(\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}\left(\mho_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)^{\mathfrak{p}}}{\left(\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}\left(2-\mho_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)^{\mathfrak{p}} + \left(\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}\left(\mho_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)^{\mathfrak{p}}}, \frac{2\left(\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}\left(\hbar_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)^{\mathfrak{p}}}{\left(\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}\left(2-\hbar_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)^{\mathfrak{p}} + \left(\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}\left(\hbar_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)^{\mathfrak{p}}} \right)$$

$$= \mathfrak{p}P_cFS_{ft}EWA\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right).$$

D. PICTURE FUZZY SOFT EINSTEIN WEIGHTED GEOMETRIC AGGREGATION OPERATORS

Definition 7: Let $\mathbb{G}_{\imath j} = \left(\digamma_{\imath j}, \mho_{\imath j}, \hbar_{\imath j}\right)$ be the collection of PFSNs, then picture fuzzy soft Einstein weighted average $\left(P_cFS_{ft}EWG\right)$ an operator is defined by

$$P_cFS_{ft}EWG\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right) = \otimes_{j=1}^{m}\varsigma_{j}\left(\otimes_{\imath=1}^{n}\varrho_{\imath}\mathbb{G}_{\imath j}\right) \quad (11)$$

where $(\imath = 1, 2, 3, \ldots, n)$, $(j = 1, 2, 3, \ldots, m)$ and $\varrho_{\imath}, \varsigma_{j}$ denote the WVs with the condition that $\sum_{\imath=1}^{n}\varrho_{\imath} = 1$ and $\sum_{j=1}^{m}\varsigma_{j} = 1$.

Theorem 3: Let $\mathbb{G}_{\imath j} = \left(\digamma_{\imath j}, \mho_{\imath j}, \hbar_{\imath j}\right)$ be the collection of $P_cFS_{ft}Ns$, then the aggregated result obtained by using the equation (28) is given by

$$P_cFS_{ft}EWG\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right) = \otimes_{j=1}^{m}\varsigma_{j}\left(\otimes_{\imath=1}^{n}\varrho_{\imath}\mathbb{G}_{\imath j}\right)$$
$$= \left(\frac{2\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}\left(\digamma_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}}{\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}\left(2-\digamma_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}} + \prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}\left(\digamma_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}}, \frac{\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}(1+\mho_{\imath j})^{\varrho_{\imath}}\right)^{\varsigma_{j}} - \prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}(1-\mho_{\imath j})^{\varrho_{\imath}}\right)^{\varsigma_{j}}}{\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}(1+\mho_{\imath j})^{\varrho_{\imath}}\right)^{\varsigma_{j}} + \prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}(1-\mho_{\imath j})^{\varrho_{\imath}}\right)^{\varsigma_{j}}}, \frac{\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}(1+\hbar_{\imath j})^{\varrho_{\imath}}\right)^{\varsigma_{j}} - \prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}(1-\hbar_{\imath j})^{\varrho_{\imath}}\right)^{\varsigma_{j}}}{\prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}(1+\hbar_{\imath j})^{\varrho_{\imath}}\right)^{\varsigma_{j}} + \prod_{j=1}^{m}\left(\prod_{\imath=1}^{n}(1-\hbar_{\imath j})^{\varrho_{\imath}}\right)^{\varsigma_{j}}} \right) \quad (12)$$

where $(\imath = 1, 2, 3, \ldots, n)$, $(j = 1, 2, 3, \ldots, m)$ and $\varrho_{\imath}, \varsigma_{j}$ denote the WVs with the condition that $\sum_{\imath=1}^{n}\varrho_{\imath} = 1$ and $\sum_{j=1}^{m}\varsigma_{j} = 1$.

Proof: Here, we shall employ the mathematical induction method.

For $n = 1$ we get $\varrho_{\imath} = 1$, as shown in the equation at the bottom of the next page.

Now for $m = 1$, we get $\varsigma_{j} = 1$, as shown in the equation at the bottom of the next page.

So equation (12) is valid for $m = 1$ and $n = 1$.

Now suppose that the above equation holds for $n = \ell_2$, $m = \ell_1 + 1$ and for $n = \ell_2 + 1$, $m = \ell_1$, then

$$P_cFS_{ft}EWG\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right) = \otimes_{j=1}^{\ell_1+1}\varsigma_{j}\left(\otimes_{\imath=1}^{\ell_2}\varrho_{\imath}\mathbb{G}_{\imath j}\right)$$
$$= \left(\frac{2\prod_{j=1}^{\ell_1+1}\left(\prod_{\imath=1}^{\ell_2}\left(\digamma_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}}{\left(\prod_{j=1}^{\ell_1+1}\left(\prod_{\imath=1}^{\ell_2}\left(2-\digamma_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}} + \prod_{j=1}^{\ell_1+1}\left(\prod_{\imath=1}^{\ell_2}\left(\digamma_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)}, \frac{\left(\prod_{j=1}^{\ell_1+1}\left(\prod_{\imath=1}^{\ell_2}\left(1+\mho_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}} - \prod_{j=1}^{\ell_1+1}\left(\prod_{\imath=1}^{\ell_2}\left(1-\mho_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)}{\left(\prod_{j=1}^{\ell_1+1}\left(\prod_{\imath=1}^{\ell_2}\left(1+\mho_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}} + \prod_{j=1}^{\ell_1+1}\left(\prod_{\imath=1}^{\ell_2}\left(1-\mho_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)}, \frac{\left(\prod_{j=1}^{\ell_1+1}\left(\prod_{\imath=1}^{\ell_2}\left(1+\hbar_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}} - \prod_{j=1}^{\ell_1+1}\left(\prod_{\imath=1}^{\ell_2}\left(1-\hbar_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)}{\left(\prod_{j=1}^{\ell_1+1}\left(\prod_{\imath=1}^{\ell_2}\left(1+\hbar_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}} + \prod_{j=1}^{\ell_1+1}\left(\prod_{\imath=1}^{\ell_2}\left(1-\hbar_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}\right)} \right),$$

$$\otimes_{j=1}^{\ell_1}\varsigma_{j}\left(\otimes_{\imath=1}^{\ell_2+1}\varrho_{\imath}\mathbb{G}_{\imath j}\right)$$
$$= \left(\frac{2\prod_{j=1}^{\ell_1}\left(\prod_{\imath=1}^{\ell_2+1}\left(\digamma_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}}{\prod_{j=1}^{\ell_1}\left(\prod_{\imath=1}^{\ell_2+1}\left(2-\digamma_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}} + \prod_{j=1}^{\ell_1}\left(\prod_{\imath=1}^{\ell_2+1}\left(\digamma_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}}, \frac{\prod_{j=1}^{\ell_1}\left(\prod_{\imath=1}^{\ell_2+1}\left(1+\mho_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}} - \prod_{j=1}^{\ell_1}\left(\prod_{\imath=1}^{\ell_2+1}\left(1-\mho_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}}{\prod_{j=1}^{\ell_1}\left(\prod_{\imath=1}^{\ell_2+1}\left(1+\mho_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}} + \prod_{j=1}^{\ell_1}\left(\prod_{\imath=1}^{\ell_2+1}\left(1-\mho_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}}, \frac{\prod_{j=1}^{\ell_1}\left(\prod_{\imath=1}^{\ell_2+1}\left(1+\hbar_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}} - \prod_{j=1}^{\ell_1}\left(\prod_{\imath=1}^{\ell_2+1}\left(1-\hbar_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}}{\prod_{j=1}^{\ell_1}\left(\prod_{\imath=1}^{\ell_2+1}\left(1+\hbar_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}} + \prod_{j=1}^{\ell_1}\left(\prod_{\imath=1}^{\ell_2+1}\left(1-\hbar_{\imath j}\right)^{\varrho_{\imath}}\right)^{\varsigma_{j}}} \right)$$

Now suppose that the above equation holds for $n = \ell_2 + 1$, $m = \ell_1 + 1$ then

$$P_cFS_{ft}EWG\left(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}\right) = \otimes_{j=1}^{\ell_1+1}\varsigma_{j}\left(\otimes_{\imath=1}^{\ell_2+1}\varrho_{\imath}\mathbb{G}_{\imath j}\right)$$
$$= \otimes_{j=1}^{\ell_1+1}\varsigma_{j}\left(\otimes_{\imath=1}^{\ell_2}\varrho_{\imath}\mathbb{G}_{\imath j} \otimes \varrho_{\imath+1}\mathbb{G}_{(\ell_2+1)j}\right)$$
$$= \left(\otimes_{j=1}^{\ell_1+1}\otimes_{\imath=1}^{\ell_2}\varrho_{\imath}\varsigma_{j}\mathbb{G}_{\imath j}\right) \otimes \left(\otimes_{j=1}^{\ell_1+1}\varsigma_{j}\varrho_{\imath+1}\mathbb{G}_{(\ell_2+1)j}\right)$$

$$
=\begin{pmatrix}
\dfrac{2\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2}\left(\mathcal{F}_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2}\left(2-\mathcal{F}_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2}\left(\mathcal{F}_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}\otimes
\dfrac{2\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\left(\mathcal{F}_{(\ell_2+1)\mathrm{j}}\right)^{\varrho(\ell_2+1)}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\left(2-\mathcal{F}_{(\ell_2+1)\mathrm{j}}\right)^{\varrho(\ell_2+1)}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\left(\mathcal{F}_{(\ell_2+1)\mathrm{j}}\right)^{\varrho(\ell_2+1)}\right)^{\varsigma_{\mathrm{j}}}}, \\
\dfrac{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2}\left(1+\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}-\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2}\left(1-\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2}\left(1+\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2}\left(1-\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}\otimes
\dfrac{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\left(1+\mho_{(\ell_2+1)\mathrm{j}}\right)^{\varrho(\ell_2+1)}\right)^{\varsigma_{\mathrm{j}}}-\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\left(1-\mho_{(\ell_2+1)\mathrm{j}}\right)^{\varrho(\ell_2+1)}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\left(1+\mho_{(\ell_2+1)\mathrm{j}}\right)^{\varrho(\ell_2+1)}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\left(1-\mho_{(\ell_2+1)\mathrm{j}}\right)^{\varrho(\ell_2+1)}\right)^{\varsigma_{\mathrm{j}}}}, \\
\dfrac{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2}\left(1+\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}-\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2}\left(1-\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2}\left(1+\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2}\left(1-\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}\otimes
\dfrac{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\left(1+\hbar_{(\ell_2+1)\mathrm{j}}\right)^{\varrho(\ell_2+1)}\right)^{\varsigma_{\mathrm{j}}}-\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\left(1-\hbar_{(\ell_2+1)\mathrm{j}}\right)^{\varrho(\ell_2+1)}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\left(1+\hbar_{(\ell_2+1)\mathrm{j}}\right)^{\varrho(\ell_2+1)}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\left(1-\hbar_{(\ell_2+1)\mathrm{j}}\right)^{\varrho(\ell_2+1)}\right)^{\varsigma_{\mathrm{j}}}}
\end{pmatrix}
$$

$$
=\begin{pmatrix}
\dfrac{2\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2+1}\left(\mathcal{F}_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2+1}\left(2-\mathcal{F}_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2+1}\left(\mathcal{F}_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}, \\
\dfrac{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2+1}\left(1+\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}-\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2+1}\left(1-\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2+1}\left(1+\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2+1}\left(1-\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}, \\
\dfrac{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2+1}\left(1+\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}-\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2+1}\left(1-\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2+1}\left(1+\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{\ell_1+1}\left(\prod_{\mathfrak{i}=1}^{\ell_2+1}\left(1-\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}
\end{pmatrix}
$$

$$
=\otimes_{\mathrm{j}=1}^{\ell_1+1}\varsigma_{\mathrm{j}}\left(\otimes_{\mathfrak{i}=1}^{\ell_2+1}\varrho_{\mathfrak{i}}\mathbb{G}_{\mathfrak{i}\mathrm{j}}\right)
$$

Hence the result is true for $m=\ell_1+1$ and $n=\ell_2+1$.

Example 3: Consider the data of example 1 and apply the notion of $P_cFS_{ft}EWG$ aggregation operator, we get, as shown in the equation at the bottom of the next page.

E. PROPERTIES OF PICTURE FUZZY SOFT EINSTEIN WEIGHTED GEOMETRIC AGGREGATION OPERATORS

Here in this phase of the article, we have to discuss some fundamental characteristics of $P_cFS_{ft}EWG$ aggregation operators.

$$
P_cFS_{ft}EWG\left(\mathbb{G}_{11},\ \mathbb{G}_{12},\ \mathbb{G}_{13},\ \ldots,\ \mathbb{G}_{nm}\right)=\otimes_{\mathrm{j}=1}^{m}\varsigma_{\mathrm{j}}\mathbb{G}_{1\mathrm{j}}
$$

$$
=\begin{pmatrix}
\dfrac{2\prod_{\mathrm{j}=1}^{m}\left(\mathcal{F}_{1\mathrm{j}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{m}\left(2-\mathcal{F}_{1\mathrm{j}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{m}\left(\mathcal{F}_{1\mathrm{j}}\right)^{\varsigma_{\mathrm{j}}}}, \\
\dfrac{\prod_{\mathrm{j}=1}^{m}\left(1+\mho_{1\mathrm{j}}\right)^{\varsigma_{\mathrm{j}}}-\prod_{\mathrm{j}=1}^{m}\left(1-\mho_{1\mathrm{j}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{m}\left(1+\mho_{1\mathrm{j}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{m}\left(1-\mho_{1\mathrm{j}}\right)^{\varsigma_{\mathrm{j}}}}, \\
\dfrac{\prod_{\mathrm{j}=1}^{m}\left(1+\hbar_{1\mathrm{j}}\right)^{\varsigma_{\mathrm{j}}}-\prod_{\mathrm{j}=1}^{m}\left(1-\hbar_{1\mathrm{j}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{m}\left(1+\hbar_{1\mathrm{j}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{m}\left(1-\hbar_{1\mathrm{j}}\right)^{\varsigma_{\mathrm{j}}}}
\end{pmatrix}
=\begin{pmatrix}
\dfrac{2\prod_{\mathrm{j}=1}^{m}\left(\prod_{\mathfrak{i}=1}^{1}\left(\mathcal{F}_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{m}\left(\prod_{\mathfrak{i}=1}^{1}\left(2-\mathcal{F}_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{m}\left(\prod_{\mathfrak{i}=1}^{1}\left(\mathcal{F}_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}, \\
\dfrac{\prod_{\mathrm{j}=1}^{m}\left(\prod_{\mathfrak{i}=1}^{1}\left(1+\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}-\prod_{\mathrm{j}=1}^{m}\left(\prod_{\mathfrak{i}=1}^{1}\left(1-\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{m}\left(\prod_{\mathfrak{i}=1}^{1}\left(1+\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{m}\left(\prod_{\mathfrak{i}=1}^{1}\left(1-\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}, \\
\dfrac{\prod_{\mathrm{j}=1}^{m}\left(\prod_{\mathfrak{i}=1}^{1}\left(1+\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}-\prod_{\mathrm{j}=1}^{m}\left(\prod_{\mathfrak{i}=1}^{1}\left(1-\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{m}\left(\prod_{\mathfrak{i}=1}^{1}\left(1+\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{m}\left(\prod_{\mathfrak{i}=1}^{1}\left(1-\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}
\end{pmatrix}.
$$

$$
P_cFS_{ft}EWG\left(\mathbb{G}_{11},\ \mathbb{G}_{12},\ \mathbb{G}_{13},\ \ldots,\ \mathbb{G}_{nm}\right)=\otimes_{\mathfrak{i}=1}^{n}\varrho_{\mathfrak{i}}\mathbb{G}_{\mathfrak{i}1}
$$

$$
=\begin{pmatrix}
\dfrac{2\prod_{\mathfrak{i}=1}^{n}\left(\mathcal{F}_{\mathfrak{i}1}\right)^{\varrho_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{n}\left(2-\mathcal{F}_{\mathfrak{i}1}\right)^{\varrho_{\mathfrak{i}}}+\prod_{\mathfrak{i}=1}^{n}\left(\mathcal{F}_{\mathfrak{i}1}\right)^{\varrho_{\mathfrak{i}}}}, \\
\dfrac{\prod_{\mathfrak{i}=1}^{n}\left(1+\mho_{\mathfrak{i}1}\right)^{\varrho_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{n}\left(1-\mho_{\mathfrak{i}1}\right)^{\varrho_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{n}\left(1+\mho_{\mathfrak{i}1}\right)^{\varrho_{\mathfrak{i}}}+\prod_{\mathfrak{i}=1}^{n}\left(1-\mho_{\mathfrak{i}1}\right)^{\varrho_{\mathfrak{i}}}}, \\
\dfrac{\prod_{\mathfrak{i}=1}^{n}\left(1+\hbar_{\mathfrak{i}1}\right)^{\varrho_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{n}\left(1-\hbar_{\mathfrak{i}1}\right)^{\varrho_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{n}\left(1+\hbar_{\mathfrak{i}1}\right)^{\varrho_{\mathfrak{i}}}+\prod_{\mathfrak{i}=1}^{n}\left(1-\hbar_{\mathfrak{i}1}\right)^{\varrho_{\mathfrak{i}}}}
\end{pmatrix}
=\begin{pmatrix}
\dfrac{2\prod_{\mathrm{j}=1}^{1}\left(\prod_{\mathfrak{i}=1}^{n}\left(\mathcal{F}_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{1}\left(\prod_{\mathfrak{i}=1}^{n}\left(2-\mathcal{F}_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{1}\left(\prod_{\mathfrak{i}=1}^{n}\left(\mathcal{F}_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}, \\
\dfrac{\prod_{\mathrm{j}=1}^{1}\left(\prod_{\mathfrak{i}=1}^{n}\left(1+\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}-\prod_{\mathrm{j}=1}^{1}\left(\prod_{\mathfrak{i}=1}^{n}\left(1-\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{1}\left(\prod_{\mathfrak{i}=1}^{n}\left(1+\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{1}\left(\prod_{\mathfrak{i}=1}^{n}\left(1-\mho_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}} \\
\dfrac{\prod_{\mathrm{j}=1}^{1}\left(\prod_{\mathfrak{i}=1}^{n}\left(1+\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}-\prod_{\mathrm{j}=1}^{1}\left(\prod_{\mathfrak{i}=1}^{n}\left(1-\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{1}\left(\prod_{\mathfrak{i}=1}^{n}\left(1+\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}+\prod_{\mathrm{j}=1}^{1}\left(\prod_{\mathfrak{i}=1}^{n}\left(1-\hbar_{\mathfrak{i}\mathrm{j}}\right)^{\varrho_{\mathfrak{i}}}\right)^{\varsigma_{\mathrm{j}}}}
\end{pmatrix}
$$

(1) **Idempotency:** If $\mathbb{G}_{ij} = (\mathfrak{f}_{ij}, \mathfrak{t}_{ij}, \hbar_{ij}) = \mathbb{G} = (\mathfrak{f}, \mathfrak{t}, \hbar)$ for all i, j, then

$$P_cFS_{ft}EWG(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) = \mathbb{G}.$$

Proof: As we know that

$$P_cFS_{ft}EWG(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm})$$

$$= \left(\begin{array}{c} \frac{2\left(\left(\mathfrak{f}_{ij}\right)^{\sum_{i=1}^n \varrho_i}\right)^{\sum_{j=1}^m \varsigma_j}}{\left(\left(2-\mathfrak{f}_{ij}\right)^{\sum_{i=1}^n \varrho_i}\right)^{\sum_{j=1}^m \varsigma_j} + \left(\left(\mathfrak{f}_{ij}\right)^{\sum_{i=1}^n \varrho_i}\right)^{\sum_{j=1}^m \varsigma_j}}, \\ \frac{\left((1+\mathfrak{t}_{ij})^{\sum_{i=1}^n \varrho_i}\right)^{\sum_{j=1}^m \varsigma_j} - \left((1-\mathfrak{t}_{ij})^{\sum_{i=1}^n \varrho_i}\right)^{\sum_{j=1}^m \varsigma_j}}{\left((1+\mathfrak{t}_{ij})^{\sum_{i=1}^n \varrho_i}\right)^{\sum_{j=1}^m \varsigma_j} + \left((1+\mathfrak{t}_{ij})^{\sum_{i=1}^n \varrho_i}\right)^{\sum_{j=1}^m \varsigma_j}} \\ \frac{\left((1+\hbar_{ij})^{\sum_{i=1}^n \varrho_i}\right)^{\sum_{j=1}^m \varsigma_j} - \left((1-\hbar_{ij})^{\sum_{i=1}^n \varrho_i}\right)^{\sum_{j=1}^m \varsigma_j}}{\left((1+\hbar_{ij})^{\sum_{i=1}^n \varrho_i}\right)^{\sum_{j=1}^m \varsigma_j} + \left((1+\hbar_{ij})^{\sum_{i=1}^n \varrho_i}\right)^{\sum_{j=1}^m \varsigma_j}} \end{array} \right)$$

$$= \left(\begin{array}{c} \frac{2(\mathfrak{f}_{ij})}{(2-\mathfrak{f}_{ij})+(\mathfrak{f}_{ij})}, \frac{(1+\mathfrak{t}_{ij})-(1-\mathfrak{t}_{ij})}{(1+\mathfrak{t}_{ij})+(1-\mathfrak{t}_{ij})} \\ \frac{(1+\hbar_{ij})-(1-\hbar_{ij})}{(1+\hbar_{ij})+(1-\hbar_{ij})} \end{array} \right)$$

$$= \left(\begin{array}{c} \frac{2(\mathfrak{f})}{(2-\mathfrak{t})+(\mathfrak{t})}, \frac{(1+\mathfrak{t})-(1-\mathfrak{t})}{(1+\mathfrak{t})+(1-\mathfrak{t})} \\ \frac{(1+\hbar)-(1-\hbar)}{(1+\hbar)+(1-\hbar)} \end{array} \right)$$

$$= (\mathfrak{f}, \mathfrak{t}, \hbar) = \mathbb{G}$$

(2) **Boundedness:** Let $\mathbb{G}_{ij} = (\mathfrak{f}_{ij}, \mathfrak{t}_{ij}, \hbar_{ij})$ be the collection of $P_cFS_{ft}Ns$ and $\mathfrak{p} > 0$ then

$$P_cFS_{ft}EWG(\mathfrak{p}\mathbb{G}_{11}, \mathfrak{p}\mathbb{G}_{12}, \mathfrak{p}\mathbb{G}_{13}, \ldots, \mathfrak{p}\mathbb{G}_{nm})$$

$$= \mathfrak{p}P_cFS_{ft}EWG(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}).$$

Proof: Let $f(\mathfrak{a}) = \frac{(2-\mathfrak{a})}{(\mathfrak{a})}$ for $\mathfrak{a} \in [0, 1]$ then $d/d\mathfrak{a}\left(\frac{(2-\mathfrak{a})}{(\mathfrak{a})}\right) = \frac{-2}{\mathfrak{a}^2} < 0$. Hence $f(\mathfrak{a})$ is a decreasing function in $[0, 1]$. As $\mathfrak{f}_{min} \leq \mathfrak{f}_{ij} \leq \mathfrak{f}_{max}$ for all i, j, then $f(\mathfrak{f}_{max}) \leq f(\mathfrak{f}_{ij}) \leq f(\mathfrak{f}_{min})$.

So, $\frac{(2-\mathfrak{f}_{max})}{(\mathfrak{f}_{max})} \leq \frac{(2-\mathfrak{f}_{ij})}{(\mathfrak{f}_{ij})} \leq \frac{(2-\mathfrak{f}_{min})}{(\mathfrak{f}_{min})}$. Assume that ϱ_i, ς_j are the WVs such that ϱ_i, ς_j and $\sum_{i=1}^n \varrho_i = 1$ and $\sum_{j=1}^m \varsigma_j = 1$. We have

$$\Longleftrightarrow \prod_{j=1}^m \left(\prod_{i=1}^n \left(\frac{2-\mathfrak{f}_{max}}{\mathfrak{f}_{max}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq \prod_{j=1}^m \left(\prod_{i=1}^n \left(\frac{2-\mathfrak{f}_{ij}}{\mathfrak{f}_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq \prod_{j=1}^m \left(\prod_{i=1}^n \left(\frac{2-\mathfrak{f}_{min}}{\mathfrak{f}_{min}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\Longleftrightarrow 1 + \left(\frac{2-\mathfrak{f}_{max}}{\mathfrak{f}_{max}}\right)$$
$$\leq 1 + \prod_{j=1}^m \left(\prod_{i=1}^n \left(\frac{2-\mathfrak{f}_{ij}}{\mathfrak{f}_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq 1 + \left(\frac{2-\mathfrak{f}_{min}}{\mathfrak{f}_{min}}\right) \Longleftrightarrow \left(\frac{2}{\mathfrak{f}_{max}}\right)$$
$$\leq 1 + \prod_{j=1}^m \left(\prod_{i=1}^n \left(\frac{2-\mathfrak{f}_{ij}}{\mathfrak{f}_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq \left(\frac{2}{\mathfrak{f}_{min}}\right) \Longleftrightarrow \left(\frac{\mathfrak{f}_{min}}{2}\right)$$

$$P_cFS_{ft}EWG(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm})$$
$$= \otimes_{j=1}^m \varsigma_j \left(\otimes_{i=1}^n \varrho_i \mathbb{G}_{ij}\right)$$

$$= \left(\begin{array}{c} \frac{2\prod_{j=1}^m \left(\prod_{i=1}^n (\mathfrak{f}_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^m \left(\prod_{i=1}^n (2-\mathfrak{f}_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^m \left(\prod_{i=1}^n (\mathfrak{f}_{ij})^{\varrho_i}\right)^{\varsigma_j}}, \\ \frac{\prod_{j=1}^m \left(\prod_{i=1}^n (1+\mathfrak{t}_{ij})^{\varrho_i}\right)^{\varsigma_j} - \prod_{j=1}^m \left(\prod_{i=1}^n (1-\mathfrak{t}_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^m \left(\prod_{i=1}^n (1+\mathfrak{t}_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^m \left(\prod_{i=1}^n (1-\mathfrak{t}_{ij})^{\varrho_i}\right)^{\varsigma_j}} \\ \frac{\prod_{j=1}^m \left(\prod_{i=1}^n (1+\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j} - \prod_{j=1}^m \left(\prod_{i=1}^n (1-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^m \left(\prod_{i=1}^n (1+\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^m \left(\prod_{i=1}^n (1-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}} \end{array} \right)$$

$$= \left(\begin{array}{c} \frac{2\prod_{j=1}^4 \left(\prod_{i=1}^4 (\mathfrak{f}_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^4 \left(\prod_{i=1}^4 (2-\mathfrak{f}_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^4 \left(\prod_{i=1}^4 (\mathfrak{f}_{ij})^{\varrho_i}\right)^{\varsigma_j}}, \\ \frac{\prod_{j=1}^4 \left(\prod_{i=1}^4 (1+\mathfrak{t}_{ij})^{\varrho_i}\right)^{\varsigma_j} - \prod_{j=1}^4 \left(\prod_{i=1}^4 (1-\mathfrak{t}_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^4 \left(\prod_{i=1}^4 (1+\mathfrak{t}_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^4 \left(\prod_{i=1}^4 (1-\mathfrak{t}_{ij})^{\varrho_i}\right)^{\varsigma_j}} \\ \frac{\prod_{j=1}^4 \left(\prod_{i=1}^4 (1+\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j} - \prod_{j=1}^4 \left(\prod_{i=1}^4 (1-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^4 \left(\prod_{i=1}^4 (1+\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^4 \left(\prod_{i=1}^4 (1-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}} \end{array} \right)$$

$$
= \left(\left(\frac{2\left(\begin{array}{l}\left\{(0.21)^{18}\times(0.19)^{24}\times(0.31)^{32}\times(0.11)^{26}\right\}^{19}\times\\ \left\{(0.41)^{18}\times(0.15)^{24}\times(0.24)^{32}\times(0.27)^{26}\right\}^{31}\times\\ \left\{(0.30)^{18}\times(0.11)^{24}\times(0.17)^{32}\times(0.41)^{26}\right\}^{22}\times\\ \left\{(0.10)^{18}\times(0.50)^{24}\times(0.11)^{32}\times(0.19)^{26}\right\}^{28}\end{array}\right)}{\left(\begin{array}{l}\left\{(2-0.21)^{18}\times(2-0.19)^{24}\times(2-0.31)^{32}\times(2-0.11)^{26}\right\}^{19}\times\\ \left\{(2-0.41)^{18}\times(2-0.15)^{24}\times(2-0.24)^{32}\times(2-0.27)^{26}\right\}^{31}\times\\ \left\{(2-0.30)^{18}\times(2-0.11)^{24}\times(2-0.17)^{32}\times(2-0.41)^{26}\right\}^{22}\times\\ \left\{(2-0.10)^{18}\times(2-0.50)^{24}\times(2-0.11)^{32}\times(2-0.19)^{26}\right\}^{28}\end{array}\right) c \left(\begin{array}{l}\left\{(0.21)^{18}\times(0.19)^{24}\times(0.31)^{32}\times(0.11)^{26}\right\}^{19}\times\\ \left\{(0.41)^{18}\times(0.15)^{24}\times(0.24)^{32}\times(0.27)^{26}\right\}^{31}\times\\ \left\{(0.30)^{18}\times(0.11)^{24}\times(0.17)^{32}\times(0.41)^{26}\right\}^{22}\times\\ \left\{(0.10)^{18}\times(0.50)^{24}\times(0.11)^{32}\times(0.19)^{26}\right\}^{28}\end{array}\right)} \right), \right.
$$

$$
\left(\frac{\left(\begin{array}{l}\left\{(1+0.23)^{18}\times(1+0.13)^{24}\times(1+0.33)^{32}\times(1+0.13)^{26}\right\}^{19}\times\\ \left\{(1+0.20)^{18}\times(1+0.16)^{24}\times(1+0.25)^{32}\times(1+0.28)^{26}\right\}^{31}\times\\ \left\{(1+0.29)^{18}\times(1+0.40)^{24}\times(1+0.13)^{32}\times(1+0.42)^{26}\right\}^{22}\times\\ \left\{(1+0.20)^{18}\times(1+0.15)^{24}\times(1+0.53)^{32}\times(1+0.20)^{26}\right\}^{28}\end{array}\right) - \left(\begin{array}{l}\left\{(1-0.23)^{18}\times(1-0.13)^{24}\times(1-0.33)^{32}\times(1-0.13)^{26}\right\}^{19}\times\\ \left\{(1-0.20)^{18}\times(1-0.16)^{24}\times(1-0.25)^{32}\times(1-0.28)^{26}\right\}^{31}\times\\ \left\{(1+0.29)^{18}\times(1-0.40)^{24}\times(1-0.13)^{32}\times(1-0.42)^{26}\right\}^{22}\times\\ \left\{(1+0.20)^{18}\times(1-0.15)^{24}\times(1-0.53)^{32}\times(1-0.20)^{26}\right\}^{28}\end{array}\right)}{\left(\begin{array}{l}\left\{(1+0.23)^{18}\times(1+0.13)^{24}\times(1+0.33)^{32}\times(1+0.13)^{26}\right\}^{19}\times\\ \left\{(1+0.20)^{18}\times(1+0.16)^{24}\times(1+0.25)^{32}\times(1+0.28)^{26}\right\}^{31}\times\\ \left\{(1+0.29)^{18}\times(1+0.40)^{24}\times(1+0.13)^{32}\times(1+0.42)^{26}\right\}^{22}\times\\ \left\{(1+0.20)^{18}\times(1+0.15)^{24}\times(1+0.53)^{32}\times(1+0.20)^{26}\right\}^{28}\end{array}\right) + \left(\begin{array}{l}\left\{(1-0.23)^{18}\times(1-0.13)^{24}\times(1-0.33)^{32}\times(1-0.13)^{26}\right\}^{19}\times\\ \left\{(1-0.20)^{18}\times(1-0.16)^{24}\times(1-0.25)^{32}\times(1-0.28)^{26}\right\}^{31}\times\\ \left\{(1+0.29)^{18}\times(1-0.40)^{24}\times(1-0.13)^{32}\times(1-0.42)^{26}\right\}^{22}\times\\ \left\{(1-0.20)^{18}\times(1-0.15)^{24}\times(1-0.53)^{32}\times(1-0.20)^{26}\right\}^{28}\end{array}\right)} \right),
$$

$$
\left.\left(\frac{\left(\begin{array}{l}\left\{(1+0.24)^{18}\times(1+0.14)^{24}\times(1+0.34)^{32}\times(1+0.15)^{26}\right\}^{19}\times\\ \left\{(1+0.17)^{18}\times(1+0.18)^{24}\times(1+0.26)^{32}\times(1+0.29)^{26}\right\}^{31}\times\\ \left\{(1+0.27)^{18}\times(1+0.24)^{24}\times(1+0.14)^{32}\times(1+0.14)^{26}\right\}^{22}\times\\ \left\{(1+0.30)^{18}\times(1+0.16)^{24}\times(1+0.13)^{32}\times(1+0.21)^{26}\right\}^{28}\end{array}\right) - \left(\begin{array}{l}\left\{(1-0.24)^{18}\times(1-0.14)^{24}\times(1-0.34)^{32}\times(1-0.15)^{26}\right\}^{19}\times\\ \left\{(1-0.17)^{18}\times(1-0.18)^{24}\times(1-0.26)^{32}\times(1-0.29)^{26}\right\}^{31}\times\\ \left\{(1-0.27)^{18}\times(1-0.24)^{24}\times(1-0.14)^{32}\times(1-0.14)^{26}\right\}^{22}\times\\ \left\{(1-0.30)^{18}\times(1-0.16)^{24}\times(1-0.13)^{32}\times(1-0.21)^{26}\right\}^{28}\end{array}\right)}{\left(\begin{array}{l}\left\{(1+0.24)^{18}\times(1+0.14)^{24}\times(1+0.34)^{32}\times(1+0.15)^{26}\right\}^{19}\times\\ \left\{(1+0.17)^{18}\times(1+0.18)^{24}\times(1+0.26)^{32}\times(1+0.29)^{26}\right\}^{31}\times\\ \left\{(1+0.27)^{18}\times(1+0.24)^{24}\times(1+0.14)^{32}\times(1+0.14)^{26}\right\}^{22}\times\\ \left\{(1+0.30)^{18}\times(1+0.16)^{24}\times(1+0.13)^{32}\times(1+0.21)^{26}\right\}^{28}\end{array}\right) + \left(\begin{array}{l}\left\{(1-0.24)^{18}\times(1-0.14)^{24}\times(1-0.34)^{32}\times(1-0.15)^{26}\right\}^{19}\times\\ \left\{(1-0.17)^{18}\times(1-0.18)^{24}\times(1-0.26)^{32}\times(1-0.29)^{26}\right\}^{31}\times\\ \left\{(1-0.27)^{18}\times(1-0.24)^{24}\times(1-0.14)^{32}\times(1-0.14)^{26}\right\}^{22}\times\\ \left\{(1-0.30)^{18}\times(1-0.16)^{24}\times(1-0.13)^{32}\times(1-0.21)^{26}\right\}^{28}\end{array}\right)} \right) \right)
$$

$$
= (0.2119,\ 0.2637,\ 0.2097).
$$

$$\leq \left(\frac{1}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-F_{ij}}{F_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}} \right) \leq \left(\frac{F_{max}}{2}\right)$$
$$\Longleftrightarrow (F_{min})$$
$$\leq \left(\frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-F_{ij}}{F_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}} \right)$$
$$\leq (F_{max})$$
$$\Longleftrightarrow (F_{min})$$
$$\leq \frac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(F_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(2-F_{ij})^{\varrho_i}\right)^{\varsigma_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(F_{ij})^{\varrho_i}\right)^{\varsigma_j}}$$
$$\leq (F_{max}) \tag{13}$$

Now assume that $g(\mathfrak{a}) = (1-\mathfrak{a}/1+\mathfrak{a})$, $\mathfrak{a}\in[0,1]$ then $d/d(g(\mathfrak{a})) = d/d\mathfrak{a}\,(1-\mathfrak{a}/1+\mathfrak{a}) = -2/(1+\mathfrak{a})^2 < 0$ that shows that $g(\mathfrak{a})$ is a decreasing function on [0, 1]. So, $\hbar_{min}\leq\hbar_{ij}\leq\hbar_{max}$ for all i, j. Hence $g(\hbar_{min})\leq g(\hbar_{ij})\leq g(\hbar_{max})$.

Assume that ϱ_i, ς_j are the WVs such that ϱ_i, ς_j and $\sum_{i=1}^{n}\varrho_i = 1$ and $\sum_{j=1}^{m}\varsigma_j = 1$. We have

$$\Longleftrightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\hbar_{max}}{1+\hbar_{max}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\hbar_{ij}}{1+\hbar_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\hbar_{min}}{1+\hbar_{min}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\Longleftrightarrow \left(\left(\frac{1-\hbar_{max}}{1+\hbar_{max}}\right)^{\sum_{i=1}^{n}\varrho_i}\right)^{\sum_{j=1}^{m}\varsigma_j}$$
$$\leq \left(\left(\frac{1-\hbar_{ij}}{1+\hbar_{ij}}\right)^{\sum_{i=1}^{n}\varrho_i}\right)^{\sum_{j=1}^{m}\varsigma_j}$$
$$\leq \left(\left(\frac{1-\hbar_{min}}{1+\hbar_{min}}\right)^{\sum_{i=1}^{n}\varrho_i}\right)^{\sum_{j=1}^{m}\varsigma_j} \Longleftrightarrow 1+\left(\frac{1-\hbar_{max}}{1+\hbar_{max}}\right)$$
$$\leq 1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\hbar_{ij}}{1+\hbar_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq 1+\left(\frac{1-\hbar_{min}}{1+\hbar_{min}}\right) \Longleftrightarrow \left(\frac{2}{1+\hbar_{max}}\right)$$
$$\leq 1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\hbar_{ij}}{1+\hbar_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}$$
$$\leq \left(\frac{2}{1+\hbar_{min}}\right) \Longleftrightarrow \left(\frac{1+\hbar_{min}}{2}\right)$$
$$\leq \frac{1}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\hbar_{ij}}{1+\hbar_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}}$$
$$\leq \left(\frac{1+\hbar_{max}}{2}\right) \Longleftrightarrow (1+\hbar_{min})$$
$$\leq \frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\hbar_{ij}}{1+\hbar_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}}$$
$$\leq (1+\hbar_{max}) \Longleftrightarrow (1+\hbar_{min}-1)$$
$$\leq \frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\hbar_{ij}}{1+\hbar_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}} - 1$$
$$\leq (1+\hbar_{max}) - 1 \Longleftrightarrow (\hbar_{min})$$
$$\leq \frac{2}{1+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1-\hbar_{ij}}{1+\hbar_{ij}}\right)^{\varrho_i}\right)^{\varsigma_j}} - 1$$
$$\leq (\hbar_{max}) \Longleftrightarrow (\hbar_{min})$$
$$\leq \frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}}$$
$$\leq (\hbar_{max}) \tag{14}$$

Similarly, we have

$$(\mho_{min})$$
$$\leq \frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}}$$
$$\leq (\hbar_{max}) \tag{15}$$

Let $P_cFS_{ft}EWG(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) = \mathbb{G}$. Then inequalities (13), (14), and (15) can be written as $F_{min}\leq F_{ij}\leq F_{max}$, $\hbar_{max}\leq\hbar_{ij}\leq\hbar_{min}$ and $\mho_{max}\leq\mho_{ij}\leq\mho_{min}$. Thus $S(\mathbb{G}) = F_{\mathbb{G}} - \mho_{\mathbb{G}} - \hbar_{\mathbb{G}} \leq F_{max} - \mho_{min} - \hbar_{min} = S(\mathbb{Z}_{max})$ and $S(\mathbb{G}) = F_{\mathbb{G}} - \mho_{\mathbb{G}} - \hbar_{\mathbb{G}} \geq F_{min} - \mho_{max} - \hbar_{max} = S(\mathbb{Z}_{min})$.

If $S(\mathbb{G}) < S(\mathbb{Z}_{max})$ and $S(\mathbb{G}) > S(\mathbb{Z}_{min})$ then

$$\mathbb{G}_{min}\leq P_cFS_{ft}EWG(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm})\leq\mathbb{G}_{max}$$

If $S(\mathbb{G}) = S(\mathbb{G}_{max})$ then $F = F_{max}$, $\mho = \mho_{max}$ *and* $\hbar = \hbar_{max}$. Then $S(\mathbb{G}) = F - \mho - \hbar = F_{max} - \mho_{max} - \hbar_{max} = S(\mathbb{Z}_{max})$. Therefore,

$$P_cFS_{ft}EWG(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) = \mathbb{G}_{max}$$

If $S(\mathbb{G}) = S(\mathbb{G}_{min})$ then $F - \mho - \hbar = F_{min} - \mho_{min} - \hbar_{min}$ that is $F = F_{min}$, $\mho = \mho_{min}$ *and* $\hbar = \hbar_{min}$.

Thus $A(\mathbb{G}) = F + \mho + \hbar = F_{min} + \mho_{min} + \hbar_{min} = A(\mathbb{Z}_{min})$

$$P_cFS_{ft}EWG(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}) = \mathbb{G}_{min}$$

Thus

$$\mathbb{G}_{min}\leq P_cFS_{ft}EWG(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm})\leq\mathbb{G}_{max}.$$

(3) **Homogeneity:** Let $\mathbb{G}_{ij} = (F_{ij}, \mho_{ij}, \hbar_{ij})$ be the collection of $P_cFS_{ft}Ns$ and $\mathfrak{p} > 0$ then

$$P_cFS_{ft}EWG(\mathfrak{p}\mathbb{G}_{11}, \mathfrak{p}\mathbb{G}_{12}, \mathfrak{p}\mathbb{G}_{13}, \ldots, \mathfrak{p}\mathbb{G}_{nm}) = \mathfrak{p}P_cFS_{ft}EWG(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}).$$

Proof: Let $\mathbb{G}_{ij} = (\digamma_{ij}, \mho_{ij}, \hbar_{ij})$ be a $P_cFS_{ft}N$ and $\mathfrak{p} > 0$ then

$$\mathfrak{p}\mathbb{G}_{ij} = \left(\frac{2(\digamma_{ij})^{\mathfrak{p}}}{(2-\digamma_{ij})^{\mathfrak{p}} + (\digamma_{ij})^{\mathfrak{p}}}, \frac{(1+\mho_{ij})^{\mathfrak{p}} - (1-\mho_{ij})^{\mathfrak{p}}}{(1+\mho_{ij})^{\mathfrak{p}} + (1-\mho_{ij})^{\mathfrak{p}}}, \frac{(1+\hbar_{ij})^{\mathfrak{p}} - (1-\hbar_{ij})^{\mathfrak{p}}}{(1+\hbar_{ij})^{\mathfrak{p}} + (1-\hbar_{ij})^{\mathfrak{p}}} \right)$$

So,

$$P_cFS_{ft}EWG(\mathfrak{p}\mathbb{G}_{11}, \mathfrak{p}\mathbb{G}_{12}, \mathfrak{p}\mathbb{G}_{13}, \ldots, \mathfrak{p}\mathbb{G}_{nm})$$

$$= \left(\frac{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(2-\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j}}, \frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\mho_{ij})^{\varrho_i}\right)^{\varsigma_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\mho_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}}, \frac{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}}{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}} \right)$$

$$= \left(\frac{2\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)^{\mathfrak{p}}}{\left(\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(2-\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)^{\mathfrak{p}} + \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\digamma_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)^{\mathfrak{p}}\right)}, \frac{\left(\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)^{\mathfrak{p}} - \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)^{\mathfrak{p}}\right)}{\left(\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)^{\mathfrak{p}} + \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\mho_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)^{\mathfrak{p}}\right)}, \frac{\left(\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)^{\mathfrak{p}} - \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)^{\mathfrak{p}}\right)}{\left(\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1+\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)^{\mathfrak{p}} + \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\hbar_{ij})^{\varrho_i}\right)^{\varsigma_j}\right)^{\mathfrak{p}}\right)} \right)$$

$$= \mathfrak{p}P_cFS_{ft}EWG(\mathbb{G}_{11}, \mathbb{G}_{12}, \mathbb{G}_{13}, \ldots, \mathbb{G}_{nm}).$$

IV. DECISION-MAKING STRATEGY

In this part of the article, we will provide the decision-making strategy for the selection of real-life problems. We will provide an algorithm for selecting the best alternative among the given possibilities.

A. ALGORITHM

Let $\mathfrak{Q}^{\rightsquigarrow} = \{\mathfrak{Q}_1^{\rightsquigarrow}, \mathfrak{Q}_2^{\rightsquigarrow}, \mathfrak{Q}_3^{\rightsquigarrow}, \ldots, \mathfrak{Q}_{\mathfrak{C}}^{\rightsquigarrow}\}$ denote a set of $\mathfrak{C}$ alternatives, $\mathfrak{R}^{\frown} = \{o_1^{\frown}, o_2^{\frown}, o_3^{\frown}, \ldots, o_n^{\frown}\}$ denote the set of experts and $\mathfrak{h} = \{\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_m\}$ denote the set of parameters. Assume that $\varrho_i, \varsigma_j > 0$ are WVs corresponding to experts and parameters respectively with a condition that $\sum_{j=1}^{m}\varsigma_j = 1$ and $\sum_{i=1}^{n}\varrho_i = 1$. Assume that decision-makers provide their assessment in the form of $P_cFS_{ft}Ns$ $\mathbb{G}_{ij} = (\digamma_{ij}, \mho_{ij}, \hbar_{ij})$. The stepwise algorithm is given below

Step 1: Get the decision matrices against each alternative $\mathfrak{Q}^{\rightsquigarrow} = (\mathbb{G}_{ij})_{n\times m}$ in the form of $P_cFS_{ft}Ns$.

Step 2: Normalize the collective data by using the formula given by

$$J_{ij} = \begin{cases} (\mathbb{G}_{ij})^c; & \text{for cost}-\text{type parameters} \\ \mathbb{G}_{ij}; & \text{for benefit}-\text{type parameters} \end{cases}$$

where $(\mathbb{G}_{ij})^c = (\hbar_{ij}, \mho_{ij}, \digamma_{ij})$

Step 3: Utilize the proposed $P_cFS_{ft}EWA$ and $P_cFS_{ft}EWG$ operators to aggregate $P_cFS_{ft}Ns$ for each alternative.

Step 4: Use definition (5) to find the score value of each alternative.

Step 5: Rank the alternatives and find out the best alternative.

Moreover, the flow chart of the proposed algorithm is given in Figure 1.

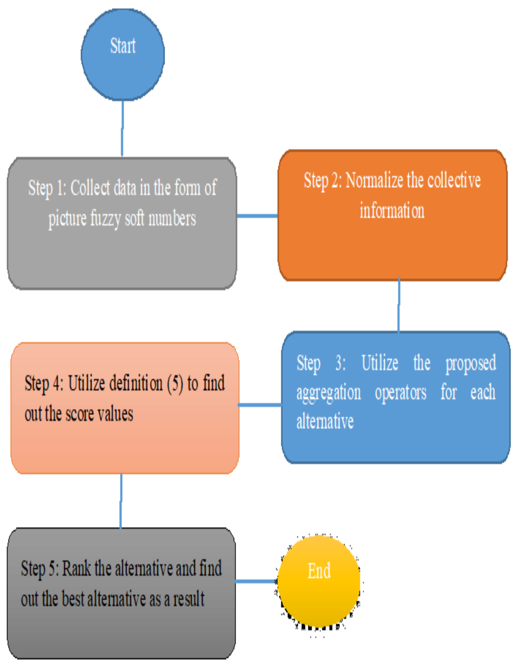

FIGURE 1. Flow chart of the proposed algorithm.

B. NUMERICAL EXAMPLE

The release of harmful materials into the environment is called pollution and the harmful materials are called pollutants. By rendering the air, water, or other aspects of the environment dirty, pollution is the process of posing a threat to public safety. Even seemingly inconsequential elements like light, sound, and temperature could be viewed as pollutants when intentionally added to an area. All forms of pollution often have severe consequences on human health as well as the environment and wildlife. Here we aim to identify the type of pollution that mostly affects our environment and due to which not only human beings but also animals and plants are affected directly based on introduced notions of $P_cFS_{ft}EWA$ and $P_cFS_{ft}EWG$ aggregation operators.

Four types of pollution damage the environment and cause climate change and complexity in disease day by day. These types are

1) WATER POLLUTION

Contamination of water happens when chemicals or potentially dangerous foreign substances—such as sewage, pesticides, fertilizer from agricultural runoff, or metals like lead or mercury—are added to the water. Water pollution badly affects the environment. According to the findings of the United States, 783 million people do not have any access to clean water. Sewage and other impurities can be prevented from getting into the water supply with proper sanitation.

2) AIR POLLUTION

Air pollution is the main cause that makes disturbances and it is an environmental risk to public health on a global scale. We breathe in tiny particles that can cause several health problems, including damage to our lungs, hearts, and brains. Despite being a global problem, air pollution disproportionately affects people in developing nations, particularly the most vulnerable sections of society, such as women, children, and the elderly.

3) NOISE POLLUTIONS

The World Health Organization (WHO) defines noise pollution as noise that is louder than 65 decibels (dB). More specifically, sound becomes hazardous over 75 dB and unpleasant at 120dB. Unwanted or excessive noise can be harmful to humans, the environment, and wildlife. Noise pollution is what we call this. Noise pollution is a common problem in many industrial settings and other industries, but it is also brought on by airplane, train, and automobile traffic as well as by outdoor building projects.

4) LAND POLLUTION

Land pollution is the term used to describe the degradation of the earth's land surfaces, both above and below the surface. The cause is the accumulation of liquid and solid wastes that contaminate groundwater and soil. The term "municipal solid waste" is frequently used to refer to both hazardous and non-hazardous trash. When waste is placed onto a piece of land, the permeability of the soil formations underlying it might increase or lessen the risk of land contamination. The likelihood of land pollution is directly correlated with the permeability of the soil.

Here we aim to study these types of pollution that mostly affect our environment and due to which not only human beings but also animals and plants are affected directly. The main cause of complexities in human diseases is these kinds of environmental pollution. So, we use the developed notions of $P_cFS_{ft}EWA$ and $P_cFS_{ft}EWG$ aggregation operators to study the worst type of pollution.

Suppose four alternatives are $\mathfrak{Q}_1^{\leadsto} = \textit{Water pollution}$, $\mathfrak{Q}_2^{\leadsto} = \textit{Air pollution}$, $\mathfrak{Q}_3^{\leadsto} = \textit{Noise pollution}$ and $\mathfrak{Q}_4^{\leadsto} = \textit{Land pollution}$. We want to identify the type of pollution that affects the climate from the given four alternatives. Let a team of four experts be invited to give their assessment. Let WVs for experts are $(0.18, 0.24, 0.32, 0.26)$. Also, assume that experts analyze these alternatives based on four parameters that are $\mathfrak{h}_1 = \textit{Increase of diseases}$, $\mathfrak{h}_2 = \textit{Climate change}$, $\mathfrak{h}_3 = \textit{Affetcs on human beings and plants}$, $\mathfrak{h}_4 = \textit{Demage of ozone layer}$ and WVs for these parameters are $(0.19, 0.31, 0.22, 0.28)$. Now use the proposed algorithm for the analysis of types of pollution.

By using the picture fuzzy soft Einstein weighted average aggregation operators

Step 1: Assume that the decision analyst proposed their assessment for each alternative in the form of P_cFS_{ft} data are given in Table 2 -5.

Step 2: No need to normalize the given data.

Step 3: Utilize the proposed $P_cFS_{ft}EWA$ aggregation operators to aggregate $P_cFS_{ft}Ns$ for each alternative. We will get

$$\mathbb{G}_1 = (0.1923, 0.1853, 0.1849),$$
$$\mathbb{G}_2 = (0.2241, 0.1824, 0.1698)$$
$$\mathbb{G}_3 = (0.1878, 0.2158, 0.1848),$$
$$\mathbb{G}_4 = (0.2310, 0.2212, 0.1818)$$

Step 4: Use definition (5) to find out the score value for each alternative given by

$$Sc(\mathbb{G}_1) = -0.1779, \ Sc(\mathbb{G}_2) = -0.1281,$$
$$Sc(\mathbb{G}_3) = -0.2128, \ Sc(\mathbb{G}_4) = -0.1720$$

Step 5: Ranking results for alternatives is given by

$$\mathbb{G}_2 > \mathbb{G}_4 > \mathbb{G}_1 > \mathbb{G}_3$$

Hence we can see that $\mathfrak{Q}_2^{\leadsto} = \textit{Air pollution}$ that is badly affecting the environment.

V. COMPARATIVE ANALYSIS

This part of the article contains the comparative study of established work with some existing notions to reveal the reliability and dominance of the introduced work.

We compare our work with Wang and Liu's [28] method, Rahman et al. [29] method, Riaz et al. [30] method, and Khan et al. [31] method.

Example 4: Suppose a man wants to get his heart treatment and he assumes three hospitals as an alternatives $\mathfrak{Q}_1^{\leadsto}$, $\mathfrak{Q}_2^{\leadsto}$ and $\mathfrak{Q}_3^{\leadsto}$. Assume that four parameters are

$$\mathfrak{h}_1 = \textit{Doctors skills},$$
$$\mathfrak{h}_1 = \textit{Caring Staff},$$
$$\mathfrak{h}_3 = \textit{Very kind hospital management},$$
$$\mathfrak{h}_4 == \textit{Affordable Hospital Charges}$$

TABLE 2. P_cFS_{ft} data for $\Omega_1^{\rightsquigarrow}$.

	$\mathfrak{h}_1$	$\mathfrak{h}_2$	$\mathfrak{h}_3$	$\mathfrak{h}_4$
$\sigma_1^{\frown}$	$(0.22, 0.21, 0.19)$	$(0.11, 0.20, 0.29)$	$(0.32, 0.11, 0.10)$	$(0.12, 0.13, 0.14)$
$\sigma_2^{\frown}$	$(0.14, 0.25, 0.17)$	$(0.21, 0.24, 0.28)$	$(0.20, 0.11, 0.18)$	$(0.17, 0.18, 0.19)$
$\sigma_3^{\frown}$	$(0.18, 0.11, 0.43)$	$(0.20, 0.19, 0.17)$	$(0.21, 0.23, 0.17)$	$(0.16, 0.15, 0.14)$
$\sigma_4^{\frown}$	$(0.20, 0.21, 0.12)$	$(0.20, 0.22, 0.16)$	$(0.24, 0.25, 0.19)$	$(0.23, 0.24, 0.18)$

TABLE 3. P_cFS_{ft} data for $\Omega_2^{\rightsquigarrow}$.

	$\mathfrak{h}_1$	$\mathfrak{h}_2$	$\mathfrak{h}_3$	$\mathfrak{h}_4$
$\sigma_1^{\frown}$	$(0.21, 0.11, 0.16)$	$(0.41, 0.30, 0.20)$	$(0.30, 0.10, 0.20)$	$(0.13, 0.14, 0.17)$
$\sigma_2^{\frown}$	$(0.24, 0.22, 0.19)$	$(0.25, 0.26, 0.18)$	$(0.24, 0.11, 0.16)$	$(0.27, 0.28, 0.13)$
$\sigma_3^{\frown}$	$(0.17, 0.10, 0.13)$	$(0.10, 0.17, 0.16)$	$(0.21, 0.22, 0.15)$	$(0.26, 0.25, 0.14)$
$\sigma_4^{\frown}$	$(0.23, 0.25, 0.42)$	$(0.24, 0.12, 0.15)$	$(0.27, 0.11, 0.17)$	$(0.13, 0.28, 0.19)$

TABLE 4. P_cFS_{ft} data for $\Omega_3^{\rightsquigarrow}$.

	$\mathfrak{h}_1$	$\mathfrak{h}_2$	$\mathfrak{h}_3$	$\mathfrak{h}_4$
$\sigma_1^{\frown}$	$(0.23, 0.27, 0.26)$	$(0.16, 0.10, 0.19)$	$(0.12, 0.31, 0.17)$	$(0.13, 0.16, 0.13)$
$\sigma_2^{\frown}$	$(0.11, 0.21, 0.19)$	$(0.11, 0.23, 0.27)$	$(0.24, 0.35, 0.10)$	$(0.11, 0.13, 0.29)$
$\sigma_3^{\frown}$	$(0.17, 0.31, 0.33)$	$(0.22, 0.16, 0.13)$	$(0.21, 0.26, 0.12)$	$(0.26, 0.35, 0.24)$
$\sigma_4^{\frown}$	$(0.30, 0.12, 0.18)$	$(0.24, 0.25, 0.26)$	$(0.12, 0.23, 0.13)$	$(0.20, 0.21, 0.13)$

TABLE 5. P_cFS_{ft} data for $\Omega_4^{\rightsquigarrow}$.

	$\mathfrak{h}_1$	$\mathfrak{h}_2$	$\mathfrak{h}_3$	$\mathfrak{h}_4$
$\sigma_1^{\frown}$	$(0.12, 0.11, 0.15)$	$(0.21, 0.10, 0.19)$	$(0.31, 0.14, 0.13)$	$(0.13, 0.15, 0.16)$
$\sigma_2^{\frown}$	$(0.16, 0.27, 0.27)$	$(0.21, 0.34, 0.18)$	$(0.12, 0.31, 0.29)$	$(0.27, 0.28, 0.29)$
$\sigma_3^{\frown}$	$(0.28, 0.21, 0.33)$	$(0.30, 0.29, 0.15)$	$(0.20, 0.23, 0.15)$	$(0.12, 0.13, 0.14)$
$\sigma_4^{\frown}$	$(0.28, 0.24, 0.22)$	$(0.40, 0.32, 0.15)$	$(0.25, 0.27, 0.17)$	$(0.21, 0.22, 0.13)$

TABLE 6. P_cFS_{ft} data for $\Omega_1^{\rightsquigarrow}$.

	$\mathfrak{h}_1$	$\mathfrak{h}_2$	$\mathfrak{h}_3$	$\mathfrak{h}_4$
$\sigma_1^{\frown}$	$(0.21, 0.11, 0.17)$	$(0.13, 0.21, 0.23)$	$(0.12, 0.13, 0.18)$	$(0.15, 0.17, 0.18)$
$\sigma_2^{\frown}$	$(0.24, 0.20, 0.18)$	$(0.23, 0.24, 0.25)$	$(0.23, 0.25, 0.29)$	$(0.27, 0.28, 0.33)$
$\sigma_3^{\frown}$	$(0.15, 0.14, 0.13)$	$(0.22, 0.16, 0.13)$	$(0.27, 0.28, 0.13)$	$(0.36, 0.10, 0.12)$
$\sigma_4^{\frown}$	$(0.27, 0.26, 0.22)$	$(0.23, 0.26, 0.17)$	$(0.20, 0.23, 0.29)$	$(0.33, 0.20, 0.13)$

TABLE 7. P_cFS_{ft} data for $\Omega_2^{\rightsquigarrow}$.

	$\mathfrak{h}_1$	$\mathfrak{h}_2$	$\mathfrak{h}_3$	$\mathfrak{h}_4$
$\sigma_1^{\frown}$	$(0.13, 0.24, 0.24)$	$(0.26, 0.13, 0.29)$	$(0.13, 0.11, 0.27)$	$(0.15, 0.16, 0.17)$
$\sigma_2^{\frown}$	$(0.21, 0.22, 0.29)$	$(0.14, 0.43, 0.17)$	$(0.20, 0.15, 0.13)$	$(0.21, 0.33, 0.39)$
$\sigma_3^{\frown}$	$(0.16, 0.11, 0.13)$	$(0.21, 0.26, 0.33)$	$(0.24, 0.25, 0.22)$	$(0.36, 0.15, 0.14)$
$\sigma_4^{\frown}$	$(0.32, 0.34, 0.12)$	$(0.14, 0.15, 0.16)$	$(0.12, 0.13, 0.16)$	$(0.10, 0.51, 0.12)$

Let WVs for experts are $(0.18,\ 0.24,\ 0.32,\ 0.26)$ and that the parameters are $(0.19,\ 0.31,\ 0.22,\ 0.28)$. We will utilize the data given in Table 6 -8 and the overall results for comparative analysis are given in Table 9.

The overall discussion of the comparative analysis is given by

1. As data given by the experts consists of picture fuzzy soft numbers. We can see that the picture fuzzy soft structure can discuss the parametrization tool as well as it can discuss the AG along with MG and NMG with the condition that the sum (MG, AG, NMG) must belong to [0, 1]. Now notice that Wang and Liu's [28] method,

TABLE 8. P_cFS_{ft} **data for** $\mathfrak{Q}_3^{\sim}$.

	$\mathfrak{h}_1$	$\mathfrak{h}_2$	$\mathfrak{h}_3$	$\mathfrak{h}_4$
$\sigma_1^{\frown}$	$(0.17, 0.14, 0.15)$	$(0.11, 0.18, 0.17)$	$(0.21, 0.15, 0.17)$	$(0.16, 0.17, 0.19)$
$\sigma_2^{\frown}$	$(0.13, 0.25, 0.27)$	$(0.29, 0.14, 0.17)$	$(0.14, 0.21, 0.23)$	$(0.17, 0.28, 0.19)$
$\sigma_3^{\frown}$	$(0.22, 0.31, 0.33)$	$(0.32, 0.39, 0.10)$	$(0.22, 0.24, 0.25)$	$(0.14, 0.15, 0.16)$
$\sigma_4^{\frown}$	$(0.21, 0.34, 0.22)$	$(0.30, 0.31, 0.13)$	$(0.21, 0.24, 0.27)$	$(0.11, 0.12, 0.13)$

TABLE 9. **Results for comparative analysis.**

Methods	Score values	Ranking
Wang and Liu's [28] method	Cannot work	No result
Rahman et al. [29] method	Cannot work	No result
Riaz et al. [30] method	Cannot work	No result
PFEWA operator Khan et al. [31]	$Sc(\mathfrak{Q}_1^{\sim}) = -0.1472, Sc(\mathfrak{Q}_2^{\sim}) = -0.1668, Sc(\mathfrak{Q}_3^{\sim}) = -0.3228$	$\mathfrak{Q}_1^{\sim} > \mathfrak{Q}_2^{\sim} > \mathfrak{Q}_3^{\sim}$
PFEWG operator Khan et al. [31] method	$Sc(\mathfrak{Q}_1^{\sim}) = -0.1696, Sc(\mathfrak{Q}_2^{\sim}) = -0.2098, Sc(\mathfrak{Q}_3^{\sim}) = -0.3465$	$\mathfrak{Q}_1^{\sim} > \mathfrak{Q}_2^{\sim} > \mathfrak{Q}_3^{\sim}$
$P_cFS_{ft}WA$	$Sc(\mathfrak{Q}_1^{\sim}) = -0.1355, Sc(\mathfrak{Q}_2^{\sim}) = -0.2026, Sc(\mathfrak{Q}_3^{\sim}) = -0.1917$	$\mathfrak{Q}_1^{\sim} > \mathfrak{Q}_3^{\sim} > \mathfrak{Q}_2^{\sim}$
$P_cFS_{ft}WG$	$Sc(\mathfrak{Q}_1^{\sim}) = -0.1732, Sc(\mathfrak{Q}_2^{\sim}) = -0.2759, Sc(\mathfrak{Q}_3^{\sim}) = -0.2388$	$\mathfrak{Q}_1^{\sim} > \mathfrak{Q}_3^{\sim} > \mathfrak{Q}_2^{\sim}$
$P_cFS_{ft}EWA$ operators (Proposed)	$Sc(\mathfrak{Q}_1^{\sim}) = -0.1393, Sc(\mathfrak{Q}_2^{\sim}) = -0.2099, Sc(\mathfrak{Q}_3^{\sim}) = -0.1966$	$\mathfrak{Q}_1^{\sim} > \mathfrak{Q}_3^{\sim} > \mathfrak{Q}_2^{\sim}$
$P_cFS_{ft}EWAG$ operators (Proposed)	$Sc(\mathfrak{Q}_1^{\sim}) = -0.1684, Sc(\mathfrak{Q}_3^{\sim}) = -0.2652, Sc(\mathfrak{Q}_3^{\sim}) = -0.2326$	$\mathfrak{Q}_1^{\sim} > \mathfrak{Q}_3^{\sim} > \mathfrak{Q}_2^{\sim}$

Rehman et al. [29] method and Riaz et al. [30] can only deal with MG and NMG. Also, all these above-given methods lack the property to discuss the parametrization tool as well. It means that the existing methods have some drawbacks. Also, we can see that if the decision makers tried to construct their data in the form of picture fuzzy soft numbers then the existing method can never tackle that kind of information. On the other hand, if we discuss the proposed aggregation operator, we can see that initiated aggregation operators have both characteristics. The developed aggregation operators not only discuss the parametrization tool but also handle the AG in their

TABLE 10. **Characteristic analysis of proposed work with existing approaches.**

Methods	Consider parametrization tool	Consider fuzzy structure
Wang and Liu's [28] method	No	Yes
Rahman et al. [29] method	No	Yes
Riaz et al. [30] method	No	Yes
PFEWA operator Khan et al. [31]	No	Yes
PFEWG operator Khan et al. [31] method	No	Yes
$P_cFS_{ft}WA$	Yes	Yes
$P_cFS_{ft}WG$	Yes	Yes
$P_cFS_{ft}EWA$ operators (Proposed)	Yes	Yes
$P_cFS_{ft}EWAG$ operators (Proposed)	Yes	Yes

structure. It means that the introduced work has both characteristics in one structure.

2. Also as far as data analysis we can see that Wang and Liu's [28] method, Rehman et al. [29] method and Riaz et al. [30] methods are restricted notions due to their condition that *sum* $(MG, NMG) \in [0, 1]$ for Wang and Liu method [28], *sum* $(MG^2, NMG^2) \in [0, 1]$ for Rehman et al. [29] method and *sum* $(MG^q, NMG^q) \in [0, 1]$ for $q \geq 1$. In these situations, the experts are bound to take their data in the form of MG and NMG. While proposed approach provides more space for decision makers to take their data in the form of picture fuzzy soft numbers that have the extra feature to discuss the AG along MG and NMG. This unique property makes the delivered approach more dominant to existing notions.
3. Now if we compare our work with the Khan et al. [31] method then we can see that although the Khan et al. [31] method can discover the AG but this structure lacks the property to discuss the parametrization tool. If we use only one parameter in the developed aggregation operators of $P_cFS_{ft}EWA$ *and* $P_cFS_{ft}EWG$ then we can observe that these developed notions degenerate into PFEWA and PFEWG aggregation operators that are developed in Khan et al. [31] approach. It means that the approaches developed by Khan et al. [31] are all special cases for the

introduced work, so the delivered work is again dominant to the existing notion.

4. Also, note that the best alternative in both cases when we apply the proposed aggregation operators and aggregation operators given by Khan et al. [31] is the same that is $\mathfrak{Q}_1^{\leadsto}$. This shows the reliability of the developed work.
5. Moreover, to show the characteristic analysis of the delivered approach with the existing notion we have provided the data in Table 10.

VI. CONCLUSION

When researchers face some issues regarding any structure in existing literature they try to develop a theory that must fit according to the situation and that theory can cover all previous drawbacks of the literature. If we discuss the structure of the picture fuzzy soft set then we can observe that the picture fuzzy soft set is a full package of different characteristics. For example, the picture fuzzy soft set can discuss the parametrization tool. Moreover, this structure can discuss the AG in its structure which is a remarkable characteristic. Because when decision-makers provide their assessment in the form of a picture fuzzy soft set. many hybrid structures like $IFS_{ft}S$, $PyFS_{ft}S$ and $q - ROFFS_{ft}S$ can never discuss such kind of data. That basic property ranks the notion of picture fuzzy soft set more dominant than that of the existing theory. Also, Einstein's t-norm and t-conorm are great substitutes for algebraic sum and product. So based on a more advanced structure of picture fuzzy soft and Einstein t-norm and t-conorm, we have established first of all operational laws rules. Then based on these newly developed operational laws we have delivered the notion of picture fuzzy soft Einstein weighted average and geometric aggregation operators. Moreover, we have discussed the properties of these delivered aggregation operators. Keeping in view the utilization perspectives of the developed approach, we have provided an algorithm for the introduced notions and illustrated an example to show the working of the initiated work. We have applied the developed approach to study and make an analysis of the types of pollution that mostly affect the environment. Furthermore, we have delivered a comparative analysis of the initiated work to show the advancement of introduced notions.

In the future, we can extend this work to the T-spherical fuzzy set [32]. Moreover, we can extend these notions to spherical fuzzy soft rough sets [33] and interval-valued T-spherical fuzzy soft sets [34]. Also, we can introduce some new terminologies like bipolar complex fuzzy set based on this developed work as given in [35].

DATA AVAILABILITY

No data were used to support this study.

CONFLICTS OF INTEREST

The authors declare that they have no conflict of interest.

REFERENCES

[1] M. A. Khan and A. M. Ghouri, "Environmental pollution: Its effects on life and its remedies," *Researcher World, J. Arts Sci. Commer.*, vol. 2, pp. 276–285, Jul. 2011.

[2] J. L. Martinez, "Environmental pollution by antibiotics and by antibiotic resistance determinants," *Environ. Pollut.*, vol. 157, no. 11, pp. 2893–2902, Nov. 2009, doi: 10.1016/j.envpol.2009.05.051.

[3] H.-J. Tsai, P.-Y. Wu, J.-C. Huang, and S.-C. Chen, "Environmental pollution and chronic kidney disease," *Int. J. Med. Sci.*, vol. 18, no. 5, pp. 1121–1129, 2021, doi: 10.7150/ijms.51594.

[4] D. Molodtsov, "Soft set theory—First results," *Comput. Math. Appl.*, vol. 37, pp. 19–31, Feb. 1999, doi: 10.1016/S0898-1221(99)00056-5.

[5] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, Jun. 1965, doi: 10.1016/S0019-9958(65)90241-X.

[6] A. R. Roy and P. K. Maji, "A fuzzy soft set theoretic approach to decision making problems," *J. Comput. Appl. Math.*, vol. 203, no. 2, pp. 412–418, Jun. 2007.

[7] P. K. Maji, A. R. Roy, and R. Biswas, "On intuitionistic fuzzy soft sets," *J. Fuzzy Math.*, vol. 12, no. 3, pp. 669–684, 2004.

[8] D. Jia-Hua, H. Zhang, and Y. He, "Possibility Pythagorean fuzzy soft set and its application," *J. Intell. Fuzzy Syst.*, vol. 36, no. 1, pp. 413–421, Feb. 2019.

[9] A. Hussain, M. I. Ali, T. Mahmood, and M. Munir, "Q-rung orthopair fuzzy soft average aggregation operators and their application in multicriteria decision-making," *Int. J. Intell. Syst.*, vol. 35, no. 4, pp. 571–599, Apr. 2020, doi: 10.1002/int.22217.

[10] B. C. Cuong and V. H. Pham, "Picture fuzzy sets: First results, Part 1. In seminar neuro-fuzzy systems with applications," in *Proc. 7th Int. Conf. Knowl. Syst. Eng. (KSE)*, Oct. 2015, pp. 132–137.

[11] T. Herawan, R. Ghazali, and M. M. Deris, "Soft set theoretic approach for dimensionality reduction," *Int. J. Database Theory Appl.*, vol. 3, pp. 4–60, Jun. 2010, doi: 10.1007/978-3-642-10583-8_20.

[12] Z. Kong, L. Gao, L. Wang, and S. Li, "The normal parameter reduction of soft sets and its algorithm," *Comput. Math. Appl.*, vol. 56, no. 12, pp. 3029–3037, Dec. 2008, doi: 10.1016/j.camwa.2008.07.013.

[13] Y. Zou and Z. Xiao, "Data analysis approaches of soft sets under incomplete information," *Knowl.-Based Syst.*, vol. 21, no. 8, pp. 941–945, Dec. 2008, doi: 10.1016/j.knosys.2008.04.004.

[14] P. K. Maji, A. R. Roy, and R. Biswas, "An application of soft sets in a decision making problem," *Comput. Math. Appl.*, vol. 44, nos. 8–9, pp. 1077–1083, Oct. 2002, doi: 10.1016/S0898-1221(02)00216-X.

[15] H. Aktas and N. Çagman, "Soft sets and soft groups," *Inf. Sci.*, vol. 177, no. 13, pp. 2726–2735, Jul. 2007, doi: 10.1016/j.ins.2006.12.008.

[16] Y. Jun and C. Park, "Applications of soft sets in ideal theory of BCK/BCI-algebras," *Inf. Sci.*, vol. 178, no. 11, pp. 2466–2475, 2008, doi: 10.1016/j.ins.2008.01.017.

[17] M. I. Ali, M. Shabir, and M. Naz, "Algebraic structures of soft sets associated with new operations," *Comput. Math. Appl.*, vol. 61, no. 9, pp. 2647–2654, May 2011, doi: 10.1016/j.camwa.2011.03.011.

[18] Z. Xiao, K. Gong, and Y. Zou, "A combined forecasting approach based on fuzzy soft sets," *J. Comput. Appl. Math.*, vol. 228, no. 1, pp. 326–333, Jun. 2009, doi: 10.1016/j.cam.2008.09.033.

[19] M. Agarwal, K. K. Biswas, and M. Hanmandlu, "Generalized intuitionistic fuzzy soft sets with applications in decision-making," *Appl. Soft Comput.*, vol. 13, no. 8, pp. 3552–3566, Aug. 2013, doi: 10.1016/j.asoc.2013.03.015.

[20] Y. Jiang, Y. Tang, H. Liu, and Z. Chen, "Entropy on intuitionistic fuzzy soft sets and on interval-valued fuzzy soft sets," *Inf. Sci.*, vol. 240, pp. 95–114, Aug. 2013, doi: 10.1016/j.ins.2013.03.052.

[21] K. Naeem, M. Riaz, X. Peng, and D. Afzal, "Pythagorean fuzzy soft MCGDM methods based on TOPSIS, VIKOR and aggregation operators," *J. Intell. Fuzzy Syst.*, vol. 37, no. 5, pp. 6937–6957, Nov. 2019, doi: 10.3233/JIFS-190905.

[22] R. M. Zulqarnain, X. L. Xin, H. Garg, and W. A. Khan, "Aggregation operators of Pythagorean fuzzy soft sets with their application for green supplier chain management," *J. Intell. Fuzzy Syst.*, vol. 40, no. 3, pp. 5545–5563, Mar. 2021, doi: 10.3233/JIFS-202781.

[23] T. Mahmood and Z. Ali, "A method to multiattribute decision making problems under interaction aggregation operators based on complex Pythagorean fuzzy soft settings and their applications," *Comput. Appl. Math.*, vol. 41, no. 6, Sep. 2022, doi: 10.1007/s40314-022-01888-1.

[24] M. Akram, F. Wasim, and F. Karaaslan, "MCGDM with complex Pythagorean fuzzy -soft model," *Exp. Syst.*, vol. 38, no. 8, Dec. 2021, Art. no. e12783, doi: 10.1111/exsy.12783.

[25] M. Riaz, M. T. Hamid, H. M. Athar Farid, and D. Afzal, ''TOPSIS, VIKOR and aggregation operators based on q-rung orthopair fuzzy soft sets and their applications,'' *J. Intell. Fuzzy Syst.*, vol. 39, no. 5, pp. 6903–6917, Nov. 2020, doi: 10.3233/JIFS-192175.

[26] A. Hussain, T. Mahmood, M. I. Ali, and A. Iampan, ''Q-rung orthopair fuzzy soft aggregation operators based on dombi t-norm and t-conorm with their applications in decision making,'' *J. Intell. Fuzzy Syst.*, vol. 43, no. 5, pp. 5685–5702, Sep. 2022, doi: 10.3233/JIFS-212921.

[27] Y. Yang, C. Liang, S. Ji, and T. Liu, ''Adjustable soft discernibility matrix based on picture fuzzy soft sets and its applications in decision making,'' *J. Intell. Fuzzy Syst.*, vol. 29, no. 4, pp. 1711–1722, Oct. 2015, doi: 10.3233/IFS-151648.

[28] W. Wang and X. Liu, ''Intuitionistic fuzzy information aggregation using Einstein operations,'' *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 5, pp. 923–938, Oct. 2012, doi: 10.1109/TFUZZ.2012.2189405.

[29] K. Rahman, S. Abdullah, R. Ahmed, and M. Ullah, ''Pythagorean fuzzy Einstein weighted geometric aggregation operator and their application to multiple attribute group decision making,'' *J. Intell. Fuzzy Syst.*, vol. 33, no. 1, pp. 635–647, Jun. 2017, doi: 10.3233/JIFS-16797.

[30] M. Riaz, W. Salabun, H. M. Athar Farid, N. Ali, and J. Watróbski, ''A robust q-rung orthopair fuzzy information aggregation using Einstein operations with application to sustainable energy planning decision management,'' *Energies*, vol. 13, no. 9, p. 2155, May 2020, doi: 10.3390/en13092155.

[31] S. Khan, S. Abdullah, and S. Ashraf, ''Picture fuzzy aggregation information based on Einstein operations and their application in decision making,'' *Math. Sci.*, vol. 13, no. 3, pp. 213–229, Sep. 2019, doi: 10.1007/s40096-019-0291-7.

[32] T. Mahmood, K. Ullah, Q. Khan, and N. Jan, ''An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets,'' *Neural Comput. Appl.*, vol. 31, no. 11, pp. 7041–7053, Nov. 2019, doi: 10.1007/s00521-018-3521-2.

[33] L. Zheng, T. Mahmood, J. Ahmmad, U. U. Rehman, and S. Zeng, ''Spherical fuzzy soft rough average aggregation operators and their applications to multi-criteria decision making,'' *IEEE Access*, vol. 10, pp. 27832–27852, 2022, doi: 10.1109/ACCESS.2022.3150858.

[34] T. Mahmood, J. Ahmmad, Z. Ali, D. Pamucar, and D. Marinkovic, ''Interval valued T-spherical fuzzy soft average aggregation operators and their applications in multiple-criteria decision making,'' *Symmetry*, vol. 13, no. 5, p. 829, May 2021, doi: 10.3390/sym13050829.

[35] T. Mahmood and U. Rehman, ''A novel approach towards bipolar complex fuzzy sets and their applications in generalized similarity measures,'' *Int. J. Intell. Syst.*, vol. 37, no. 1, pp. 535–567, Jan. 2022, doi: 10.1002/int.22639.

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