IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received 6 December 2023, accepted 24 December 2023, date of publication 26 December 2023, date of current version 25 January 2024.

Digital Object Identifier 10.1109/ACCESS.2023.3347613

RESEARCH ARTICLE

Finite-Time Boundedness of Switched Time-Varying Delay Systems With Actuator Saturation: Applications in Water Pollution Control

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The work of Saravanan Shanmugam was supported in part by the Centre for Nonlinear Systems, Chennai Institute of Technology (CIT), India, under Grant CIT/CNS/2023/RP-005. The work of Mohamed Rhaima was supported by King Saud University, Riyadh, Saudi Arabia, through the Researchers Supporting Project under Grant RSPD2024R683.

ABSTRACT This paper addresses the challenge of ensuring finite-time boundedness in switched time-varying delay systems with actuator saturation. Utilizing Lyapunov–Krasovskii functionals, we establish delay-dependent conditions through linear matrix inequalities, ensuring that switched systems with time-varying delays remain finite-time bounded. The paper also introduces the concept of average dwell time for switching signals, providing additional conditions for finite-time boundedness. Furthermore, the finite-time L_2-L_∞ performance of switched systems with time-varying delays is investigated as a measure of disturbance capability within a finite-time interval. The estimator gain matrix can be determined by solving the linear matrix inequalities. The effectiveness of the proposed approach is illustrated through numerical examples.

INDEX TERMS Actuator saturation, average dwell time, finite-time boundedness, $L_2 - L_\infty$ performance, Lyapunov-Krasovskii method, time-varying delay.

I. INTRODUCTION

Switched systems have received much attention recently, with studies exploring stability, controllability, and performance. The notable topics include finite-time stabilization, robust filtering, L_2 gain, Finite-time boundedness, stochastic, and H_∞ control in switched systems [1], [2], [3], [4], [6], [7], [8]. These collective studies significantly contribute to understanding control and stability in switched time-delay systems. Switched systems have wide applications in chemical processes, mechanical systems, automotive industry, aircraft and air traffic control, and so on. Such a class of systems is composed of a finite number of subsystems and a logical rule orchestrating the switching between the subsystems. Basically, the switching rule in most existing literatures can be classified into three categories: arbitrary switching is investigated in [8], dwell time switching is given in [7], and state dependent switching is presented in [9]. It is well known that the first two categories of switching rules require that each subsystem of a switched system is stable or stabilized. In particular, it is generally admited that dwell time switching regime is more pliant than arbitrary switching rule to some extent.

Recently, researchers have employed two methods for dealing with slow switching: dwell time and average dwell time. However, these results are somewhat conservative.

The associate editor coordinating the review of this manuscript and approving it for publication was Feiqi Deng.

In the majority of the literature, the average dwell time scheme is preferred because it yields more general results compared to dwell time. In the average dwell time approach, the number of switches within a finite interval is bounded, and the average time between consecutive switchings is not less than a constant. This method has been demonstrated to be a successful and effective technique for analyzing the stability of switched systems and designing controllers. For example, refer to [3], [10], and [11].

Actuator saturation are the key of control which is applicable to all areas of engineering and science. However, majority of actuators are not strictly accord with linearity, most of them subject to saturation in real physical systems. On the other hand, as a physical phenomenon, actuator saturation often occurs in practical systems due to physical constraints. That can severely degrade the performance of closed-loop system and sometimes even make a stable closed-loop system unstable if the controller is designed without considering this kind of nonlinearity. During the past several decades, control systems with actuator saturation have received much attention, (see for examples [12], [13], [14], [15], [16], and the references therein). The analysis and synthesis of T-S fuzzy systems with actuator saturation nonlinearities is given in [17]. A problem of robust observer-based passive control for uncertain singular time-delay systems subject to actuator saturation has been investigated in [18]. In [19] passivity controller design for singular time-delay system and actuator saturation with nonlinear disturbance are employed. H_∞ observer design for stochastic time-delayed systems with Markovian switching under partly known transition rates and actuator saturations has been investigated in [20]. The problem of exponential stabilization for a class of singularly perturbed switched systems subject to actuator saturation is studied in [21]. Based on finite-time H_∞ control problem for a class of discrete-time switched singular time-delay with actuator saturation have been investigated in [22].

Finite-time stability addresses the stability of a system over a finite-time interval and holds significant importance. It's essential to note that finite-time stability and Lyapunov asymptotic stability are distinct concepts, and they are independent of each other. Therefore, it is important to emphasize the distinction between classical Lyapunov stability and finite-time stability. However, in many practical systems, people increasingly prefer to consider the behavior of system in a finite interval [23], [24], [25]. And in recent years, finite-time stability and finite-time boundedness problems have been widely spread and used in various systems [26]. The finite-time filtering and state observer design problems have been solved respectively in [27]. In the field of control systems, recent research has focused on achieving finite-time stability and control for various dynamic systems. Reference [28] introduced a novel approach to fuzzy adaptive finite-time consensus control for high-order nonlinear multiagent systems based on event-triggered mechanisms, providing robustness against uncertainties and disturbances. Reference [29] addressed finite-time event-triggered stabilization for discrete-time fuzzy Markov jump singularly perturbed systems, offering insights into stochastic systems with singularly perturbed dynamics. Reference [30] contributed to the design of event-based finite-time control strategies for nonlinear multiagent systems with asymptotic tracking objectives, emphasizing finite-time convergence and asymptotic tracking behavior. Additionally, interval type-2 fuzzy systems with time delay and actuator faults were explored in an unidentified article, focusing on the finite-time boundedness of such systems in [31]. This growing body of research underscores the significance of finite-time control in addressing complex system dynamics.

To the best of our knowledge, the concept of finite-time boundedness represents a crucial aspect of control theory, ensuring that a system's state variables remain within predetermined bounds within a finite time frame. This property is particularly significant in real-world applications where the system's behavior must be tightly controlled and constrained to meet performance and safety requirements. Additionally, the finite-time control of switched system with $L_2 - L_\infty$ performance criteria plays a major role in this paper. The main contribution of this work as given as follows:

1) We establish sufficient conditions for ensuring finite-time boundedness through the Lyapunov-Krasovskii functional, Jensen's inequality, Wirtinger's integral inequality, and a novel integral inequality, utilizing linear matrix inequalities (LMIs).
2) Exploring the finite-time $L_2 - L_\infty$ performance of switched systems with time-varying delays as a measure of disturbance capability within a finite-time interval.
3) We address the challenging problem of finite-time boundedness in switched time-delay systems while incorporating an actuator saturation controller gain. The design of the controller gain considers factors such as attenuation levels and the average dwell time, ensuring effective control under saturation constraints. Consequently, we determine the estimator gains required for the proposed control strategy's implementation.
4) Furthermore, we demonstrate the practical relevance of our approach by applying it to real-world scenarios, specifically addressing the water pollution control problem. Our approach's effectiveness and applicability are demonstrated through numerical examples, showcasing its potential for providing sustainable solutions to control problems.

Notation: The notation used in this paper is standard. $\mathcal{R}^n$ denotes n-dimensional Euclidean space, the superscript "T" denotes the transpose and the notation $P > 0\ (\geq 0)$ means P is real symmetric positive definite matrix, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalues of matrix P, respectively. I is an identity matrix with appropriate dimension. The asterisk $*$ in a matrix is used to denote a term that is induced by symmetry.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following switched system with time varying delays as follows:

$$\left.\begin{aligned}\dot{x}(t) &= A_{p(t)}x(t) + A_{dp(t)}x(t-h(t)) + B_{p(t)}sat(u(t))\\ &\quad +B_{wp(t)}w(t),\\ z(t) &= C_{p(t)}x(t) + C_{dp(t)}x(t-h(t)) + D_{wp(t)}w(t),\\ x(t) &= \phi(t), t \in [-h_M, 0],\end{aligned}\right\} \tag{1}$$

where $x(\cdot) = [x_1(\cdot), x_2(\cdot), \dots, x_n(\cdot)]^T \in \mathcal{R}^n$ is the state vector, $u(t) \in \mathcal{R}^p$ is the control input $z(t) \in \mathcal{R}^m$ is the control output vector; $p(t) : [0, \infty) \to N = \{1, 2, \dots, n\}$ is the switching signal that is a piecewise constant function depending on time t or state $x(t)$, and n is the number of subsystems; $w(t) \in \mathcal{R}^q$ is the disturbance and satisfies $\int_0^T w^T(t)w(t) \le d, d \ge 0$, $h(t)$ is time-varying delay satisfies $0 \le h(t) \le h, \dot{h}(t) \le h_M$. $sat(u(t)) : \mathcal{R}^p \to \mathcal{R}^p$ is the control input, $sat(\cdot)$ is the saturation nonlinearity function.

A. CONTROL FORMULATION

The saturation function $sat(u(\cdot)) : \mathcal{R}^p \to \mathcal{R}^p$ is defined as follows:

$$sat(u) := [sat(u_1), sat(u_2), \dots, sat(u_p)]^T,$$

where $sat(u_l) = sig(u_l)min\{\vartheta_l, |u_l|\}$, or we can write as follows

$$\begin{cases}\vartheta_l & u_l > \vartheta_l,\\ u_l, & -\vartheta_l \le u_l \le \vartheta_l, \quad l = 1, 2, \dots, p,\\ -\vartheta_l, & u_l < -\vartheta_l.\end{cases}$$

The saturation function $sat(u(t))$ may be decomposed into a linear and a nonlinear segment, which helps elucidate its behavior and implications for the model at hand

$$sat(u(t)) = u(t) - \phi(u(t)), \tag{2}$$

where $\phi(u(t)) = [\phi_1(u(t)), \phi_2(u(t)), \dots, \phi_p(u(t))]^T \in \mathbb{R}^p$, and $\phi_l(u(t)) = u_l(t) - sat(u_l(t))$, $(l = 1, 2, \dots, p)$. Subsequently, a scalar value $0 < \epsilon < 1$ exists, satisfying the condition that

$$\epsilon u^T(t)u(t) \ge \phi^T(u(t))\phi(u(t)).$$

Therefore we design the controller as the following form:

$$u(t) = K_i x(t) \tag{3}$$

where $K_i \in \mathcal{R}^p$ is the gain matrix to be designed. Assuming that only a finite part of the non-linearity is considered during the actual system operation, i.e. the operation of the saturation is inside the sector $[\epsilon, 1]$, $0 < \epsilon < 1$. Corresponding to the switching signal $p(t)$, we have the following switching sequence, $\{x_0 : (i_0, t_0), \dots, (i_k, t_k), \dots, |i_k \in \mathbb{N}, k = 0, 1, \dots\}$. Moreover, $p(t) = i$ which means that i_kth subsystem is activated when $t \in [t_k, t_{k+1})$.

From (1) the switched time-varying delayed system and replaced with $p(t) = i$ is written as follows,

$$\left.\begin{aligned}\dot{x}(t) &= A_i x(t) + A_{di}x(t-h(t)) + B_i sat(u(t)) + B_{wi}w(t),\\ z(t) &= C_i x(t) + C_{di}x(t-h(t)) + D_{wi}w(t),\\ x(t) &= \phi(t), t \in [-h_M, 0],\end{aligned}\right\} \tag{4}$$

Definition 1 ([24] Finite-Time Boundedness): For a given time constant $c_1 > 0, c_2 > 0, T$ and symmetric matrix $R > 0$, the system (1) is said to be finite-time bounded with respect to (c_1, c_2, T, R) if there exist constants $c_2 > c_1 > 0$, such that

$$x^T(t_0)Rx(t_0) \le c_1 \Rightarrow x^T(t)Rx(t) \le c_2, \quad \forall\, t \in [0, T].$$

Definition 2 ([38] $L_2 - L_\infty$ Performance): The time-varying delay switched system (1) is said to be finite-time bounded with respect to (c_1, c_2, T, R, d) in the sense of Definition 1 and disturbance attenuation $\gamma > 0$ such that

$$\| z(t) \|_\infty^2 \le \gamma^2 \| w(t) \|_2^2,$$

where $\| z(t) \|_\infty^2 = \sup_{t>0}[z^T(t)z(t)]$,
$\| w(t) \|_2^2 = \int_0^T w^T(t)w(t)dt$.

Definition 3 ([33]): For any switching signal $p(t)$ and $t_2 \ge t_1 \ge 0$, let $N_{p(t)}(t_2, t_1)$ denote the switching number of $p(t)$ on an interval (t_1, t_2). We say that $p(t)$ has an average dwell time τ_a if

$$N_{p(t)}(t_1, t_2) \le N_0 + \frac{t_2 - t_1}{\tau_a}$$

holds for given $N_0 \ge 0$, $\tau_a > 0$, N_0 is the chatter bound. Without loss of generality, we choose $N_0 = 0$ throughout this paper.

Lemma 4: [37] For any real vectors α, β and any matrix $Q > 0$ with appropriate dimensions, it follows that

$$2\alpha^T\beta \le \alpha^T Q\alpha + \beta^T Q^{-1}\beta.$$

Lemma 5: For any positive matrices $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{R}^{n\times n}$ $\mathcal{L} \in \mathcal{R}^{n\times q}$, positive definite symmetric matrix $Q_2 \in \mathcal{R}^{n\times n}$ and any time varying delays $h(t)$, we have

$$-\int_{t-d(t)}^{t} \dot{x}^T(s)Q_2\dot{x}(s)ds \le \xi^T(t)\Big[\Psi + h\Pi^T Q_2^{-1}\Pi\Big]\xi(t) \tag{5}$$

where

$$\begin{aligned}\Pi &= \big[\,\mathcal{M}_1\ \mathcal{M}_2\ \mathcal{L}\,\big],\\ \xi^T &= \big[\,x^T(t)\ x^T(t-h(t))\ w^T(t)\,\big],\\ \Psi &= \begin{bmatrix}\mathcal{M}_1^T + \mathcal{M}_1 & -\mathcal{M}_1^T + \mathcal{M}_2 & \mathcal{L}\\ * & -\mathcal{M}_2^T - \mathcal{M}_2 & -\mathcal{L}\\ * & * & 0\end{bmatrix}\end{aligned}$$

Proof From Lemma 2.4 we have

$$-\int_{t-h(t)}^{t} \dot{x}^T(s)Q_2\dot{x}(s)ds$$

$$
\begin{aligned}
&\leq 2\left(\int_{t-h(t)}^{t}\dot{x}(s)ds\right)^{T}\Pi\xi(t)\\
&\quad+\int_{t-h(t)}^{t}\xi^{T}(t)\Pi^{T}Q_2^{-1}\Pi\xi(t)ds,\\
&\leq 2\xi^{T}(t)\begin{bmatrix} I\\ -I\\ 0\end{bmatrix}\Pi\xi(t)\\
&\quad+h\xi^{T}(t)\Pi^{T}Q_2^{-1}\Pi\xi(t),\\
&=\xi^{T}(t)\Psi\xi(t)+h\xi^{T}(t)\Pi^{T}Q_2^{-1}\Pi\xi(t). \qquad (6)
\end{aligned}
$$

Hence we conclude (5). □

Lemma 6 ([32]): For any constant matrix $M > 0$, the following inequality holds for all continuously differentiable function φ on $[a, b] \to \mathcal{R}^{n\times n}$:

$$
(b-a)\int_a^b \varphi^T(s)M\varphi(s)ds \geq \left(\int_a^b \varphi(s)ds\right)^T M\left(\int_a^b\varphi(s)ds\right) + 3\Omega^T M\Omega,
$$

where

$$
\Omega=\int_a^b\varphi(s)ds-\frac{2}{b-a}\int_a^b\int_a^s\varphi(\theta)d\theta ds.
$$

III. MAIN RESULTS

A. FINITE-TIME BOUNDEDNESS

In this section, we first derive the finite-time boundedness condition for the switched system from (4) with $u(t) = K_i x(t)$, where K_i is known constant and $z(t) = 0$:

$$
\left.\begin{aligned}
&\dot{x}(t)=A_ix(t)+A_{di}x(t-h(t))+B_i sat(K_ix(t))+B_{wi}w(t),\\
&x(t)=\phi(t),\ t\in[-h_M,0],
\end{aligned}\right\} \qquad (7)
$$

Theorem 7: For given positive scalars $T, c_1, c_2, d, h, h_M, K_i, \epsilon, \epsilon_a$ and α the system (7) is finite-time boundedness if there exist symmetric positive definite matrices $P_i > 0, Q_{1i} > 0, Q_{2i} > 0, Z_i > 0, S_i > 0$ and the appropriate dimensional matrices $M_{1i} > 0, M_{2i} > 0$ and $L_i > 0$ such that the following LMIs holds:

$$
\widetilde{\sum}=\begin{bmatrix}\sum & \Theta^T & h\Pi^T\\ * & -\frac{1}{h}Q_{2i} & 0\\ * & * & -Q_{2i}\end{bmatrix}<0, \qquad (8)
$$

$$
e^{(\alpha+\frac{\ln\mu}{\tau_a})T}\left[\left(\lambda_2+h\lambda_3+\frac{h^2}{2}\lambda_4+\frac{h^2}{2}\lambda_5\right)c_1+d\lambda_6\right]<\lambda_1c_2. \qquad (9)
$$

Then, under the following average dwell time scheme

$$
\tau_a>\tau_a^*=\frac{T\ln\mu}{\ln(c_2e^{-\alpha T})-\ln\left[\beta c_1+d\lambda_6\right]}, \qquad (10)
$$

where $\beta = \left(\lambda_2 + h\lambda_3 + \frac{h^2}{2}\lambda_4 + \frac{h^2}{2}\lambda_5\right)$, the system is finite-time bounded with respect to $(c_1, c_2, T, R, p(t))$, where $\mu>1$ satisfying

$$
P_s<\mu P_i,\ Q_{1s}<\mu Q_{1i},\ Q_{2s}<\mu Q_{2i},\ Z_s<\mu Z_i,\quad \forall i,s\in\mathbb{N}. \qquad (11)
$$

where

$$
\Sigma=\begin{bmatrix}
\Phi_{11} & \Phi_{12} & 0 & 0 & 0 & 0 & B_{wi}L_i & -B_i\\
* & \Phi_{22} & 0 & 0 & 0 & 0 & -L_i & 0\\
* & * & \Phi_{33} & 0 & 0 & \Phi_{36} & 0 & 0\\
* & * & * & \Phi_{44} & \Phi_{45} & 0 & 0 & 0\\
* & * & * & * & \Phi_{55} & 0 & 0 & 0\\
* & * & * & * & * & \Phi_{66} & 0 & 0\\
* & * & * & * & * & * & -S_i & -\epsilon_aB_i
\end{bmatrix}.
$$

$\Phi_{11} = 2P_iA_i + 2B_iK_i + Q_{1i} + dQ_{3i} + M_{1i} + M_{1i}^T - \alpha P_i$,
$\Phi_{12} = A_{di}P_i - M_{1i}^T + M_{2i}$, $\Phi_{22} = e^{-\alpha h(t)}Q_1 - M_{2i}^T - M_{2i}$, $\Phi_{33} = \frac{e^{-\alpha h}}{h}Z_i - \frac{3e^{-\alpha h}}{h}Z_i$, $\Phi_{36} = \frac{6e^{-\alpha h}}{h^2}Z_i$,
$\Phi_{44} = \frac{e^{-\alpha h}}{h}Z_i - \frac{3e^{-\alpha h}}{h}Z_i$, $\Phi_{45} = \frac{6e^{-\alpha h}}{h^2}Z_i$,
$\Phi_{55} = -\frac{12e^{-\alpha h}}{h^3}Z_i$, $\Phi_{66} = -\frac{12e^{-\alpha h}}{h^3}Z_i$,
$\Theta = \left[A_iQ_{2i}\ A_{di}Q_{2i}\ 0\ 0\ 0\ 0\ B_{wi}Q_{2i}\ B_iQ_{2i}\right]$,
$\Pi = \left[M_{1i}\ M_{2i}\ 0\ 0\ 0\ 0\ L_i\ 0\right]$.
$\lambda_1 = \lambda_{\min}(P_i)$, $\lambda_2 = \lambda_{\max}(P_i)$, $\lambda_3 = \lambda_{\max}(Q_{1i})$,
$\lambda_4 = \lambda_{\max}(Q_{2i})$, $\lambda_5 = \lambda_{\max}(Z_i)$, $\lambda_6 = \lambda_{\max}(S_i)$.

Proof Choose the following Lyapunov functional for the system (7) as:

$$
V(x(t),t)=\sum_{i=1}^{4}V_i(x(t)), \qquad (12)
$$

where

$$
\begin{aligned}
V_1(x(t),t)&=x^T(t)P_ix(t),\\
V_2(x(t),t)&=\int_{t-h(t)}^{t}e^{\alpha(t-s)}x^T(s)Q_{1i}x(s)ds,\\
V_3(x(t),t)&=\int_{-h}^{0}\int_{t+\theta}^{t}e^{\alpha(t-s)}\dot{x}^T(s)Q_{2i}\dot{x}(s)dsd\theta,\\
V_4(x(t),t)&=\int_{-h}^{0}\int_{t+\theta}^{t}e^{\alpha(t-s)}x^T(t)Z_ix(s)dsd\theta.
\end{aligned}
$$

Calculating the time derivative of $V(x(t), t)$ along the trajectories of the system (7), we have

$$
\dot{V}_1=2x^T(t)P_i\dot{x}(t), \qquad (13)
$$

$$
\begin{aligned}
\dot{V}_2&=x^T(t)Q_{1i}x(t)\\
&\quad-(1-h_M)e^{-\alpha h(t)}x^T(t-h(t))Q_{1i}x(t-h(t)), \qquad (14)
\end{aligned}
$$

$$
\dot{V}_3=h\dot{x}^T(t)Q_2\dot{x}(t)-e^{-\alpha h}\int_{t-h}^{t}\dot{x}^T(s)Q_{2i}\dot{x}(s)ds, \qquad (15)
$$

$$
\dot{V}_4=hx^T(t)Z_ix(t)-e^{-\alpha h}\int_{t-h}^{t}x^T(s)Z_ix(s)ds. \qquad (16)
$$

By applying Lemma 5 in the integral term in (15), we can get,

$$-\int_{t-h}^{t} \dot{x}^T(s)Q_{2i}\dot{x}(s)ds \leq \xi^T(t)\Big\{\Psi + h\Pi^T Q_{2i}^{-1}\Pi\Big\}\xi(t). \tag{17}$$

The integral term in (16) can be written as

$$-\int_{t-h}^{t} x^T(s)Z_i x(s)ds = -\int_{t-h}^{t-h(t)} x^T(s)Z_i x(s)ds - \int_{t-h(t)}^{t} x^T(s)Z_i x(s)ds. \tag{18}$$

By using Lemma 6, we have

$$\begin{aligned}-\int_{t-h}^{t-h(t)} x^T(s)Z_i x(s)ds &= -\frac{1}{h}\Big(\int_{t-h}^{t-h(t)} x(s)ds\Big)^T \\ &\times Z_i\Big(\int_{t-h}^{t-h(t)} x(s)ds\Big) - \frac{3}{h}\Omega_1^T Z_i \Omega_1,\end{aligned} \tag{19}$$

$$\begin{aligned}-\int_{t-h(t)}^{t} x^T(s)Z_i x(s)ds &= -\frac{1}{h}\Big(\int_{t-h(t)}^{t} x(s)ds\Big)^T \\ &\times Z_i\Big(\int_{t-h(t)}^{t} x(s)ds\Big) - \frac{3}{h}\Omega_2^T Z_i \Omega_2.\end{aligned} \tag{20}$$

where $\Omega_1 = \int_{t-h}^{t-h(t)} x(s)ds - \frac{2}{h}\int_{-h}^{-h(t)}\int_{t+\theta}^{t} x(s)dsd\theta$, $\Omega_2 = \int_{h(t)}^{t} x(s)ds - \frac{2}{h}\int_{-h(t)}^{0}\int_{t+\theta}^{t} x(s)dsd\theta$.
The saturation effect of the actuator considered in (2), we can get

$$sat(u(t)) = u(t) - \phi(u(t)), \tag{21}$$

Replacing $u(t)$ in (21) for the right hand side of (3)

$$sat(u(t)) = K_i x(t) - \phi(u(t)), \tag{22}$$

Moreover, there exists a scalar $0 < \epsilon < 1$ satisfying

$$\epsilon u^T(t)u(t) - \phi^T(u(t))\phi(u(t)) \geq 0, \tag{23}$$

Substitute (3) in (23) we have,

$$\epsilon x^T(t)K_i^T K_i x(t) - \phi^T(u(t))\phi(u(t)) \geq 0, \tag{24}$$

Therefore, for any constant $\epsilon_a > 0$, we can derive

$$\epsilon\epsilon_a x^T(t)K_i^T K_i x(t) - \epsilon_a\phi^T(u(t))\phi(u(t)) \geq 0, , \tag{25}$$

Combining from (13) to (25), we have that

$$\begin{aligned}&\dot{V}(x(t)) - \alpha V(x(t)) - w^T(t)S_i w(t) \\ &\quad = \Xi^T(t)\Big\{\Sigma + \Theta^T Q_{2i}^{-1}\Theta + h\Pi^T Q_{2i}^{-1}\Pi\Big\}\Xi(t),\end{aligned} \tag{26}$$

where $\Xi^T(t) = \Big[x^T(t)\ x^T(t-h(t))\ \int_{t-h}^{t-h(t)} x^T(s)ds\ \int_{t-h(t)}^{t} x^T(s)ds\ \int_{-h(t)}^{0}\int_{t+\theta}^{t} x^T(s)dsd\theta\ \int_{-h}^{-h(t)}\int_{t+\theta}^{t} x^T(s)dsd\theta\ w^T(t)\ \phi(u(t))\Big]$
By applying Schur complement Lemma, in (26) we get,

$$\dot{V}(x(t)) - \alpha V(x(t)) - w^T(t)S_i w(t) < 0, \tag{27}$$

It can be obtained from (27), for $t \in [t_k, t_{k+1})$,

$$\begin{aligned}V(t) &< e^{\alpha(t-t_k)}V(t_k) + \int_{t_k}^{t} e^{\alpha(t-s)}w^T(s)S_i w(s)ds, \\ &< e^{\alpha(t-t_k)}\mu V(t_{k^-}) + \int_{t_k}^{t} e^{\alpha(t-s)}w^T(s)S_i w(s)ds, \\ &< e^{\alpha(t-t_k)}\mu[e^{\alpha(t-t_{k-1})}V(t_{k-1}) \\ &\quad + \int_{t_{k-1}}^{t_k} e^{\alpha(t_k-s)}w^T(s)S_i w(s)ds] \\ &\quad + \int_{t_k}^{t} e^{\alpha(t-s)}w^T(s)S_i w(s)ds, \\ &= e^{\alpha(t-t_{k-1})}\mu V(t_{k-1}) + \mu\int_{t_{k-1}}^{t_k} e^{\alpha(t-s)}w^T(s)S_i w(s)ds \\ &\quad + \int_{t_k}^{t} e^{\alpha(t-s)}w^T(s)S_i w(s)ds < \ldots \\ &\quad \ldots < e^{\alpha(t-0)}\mu^{N_p(0,t)}V(0) \\ &\quad + \mu^{N_p(0,t)}\int_{0}^{t_1} e^{\alpha(t-s)}w^T(s)S_i w(s)ds \\ &\quad + \mu^{N_p(t_1,t)}\int_{t_1}^{t_2} e^{\alpha(t-s)}w^T(s)S_i w(s)ds + \ldots \\ &\quad + \mu\int_{t_k-1}^{t_k} e^{\alpha(t-s)}w^T(s)S_i w(s)ds \\ &\quad + \int_{t_k}^{t} e^{\alpha(t-s)}w^T(s)S_i w(s)ds, \\ &= e^{\alpha(t-0)}\mu^{N_p(0,t)}V(0) \\ &\quad + \int_{0}^{t} e^{\alpha(t-s)}\mu^{N_p(0,t)}w^T(s)S_i w(s)ds, \\ &< e^{\alpha t}\mu^{N_p(0,t)}V(0) \\ &\quad + \mu^{N_p(0,t)}e^{\alpha t}\int_{0}^{t} w^T(s)S_i w(s)ds, \\ &< e^{\alpha T}\mu^{N_p(0,T)}[V(0) + \int_{0}^{T} w^T(s)S_i w(s)ds] \\ &< e^{\alpha T}\mu^{N_p(0,T)}[V(0) + \lambda_{max}(S_i)d].\end{aligned} \tag{28}$$

From Definition 2.4, we know $N_p(0,t) < \frac{T}{\tau_a}$. Noting that $S_i < \lambda_6 I$, we have

$$V(t) < e^{(\alpha + \frac{\ln\mu}{\tau_a})T}[V(0) + \lambda_6 d]. \tag{29}$$

Then

$$\begin{aligned}V(t) = \quad V_i(t) &\geq x^T(t)\tilde{P}_i^{-1}x(t) = x^T(t)R^{\frac{1}{2}}P_i^{-1}R^{\frac{1}{2}}x(t) \\ &\geq \frac{1}{\lambda_{max}(P_i)}x^T(t)Rx(t).\end{aligned}$$

Noting that, $\lambda_1 R^{-1} < \tilde{P}_i < R^{-1}$ we have $\lambda_{max}(P_i) < 1$, then

$$V(t) > V_{1i}(t) > x^T(t)Rx(t). \tag{30}$$

On other hand

$$\begin{aligned} V(x(0)) &= x^T(0)P_i x(0) + \int_{-h(0)}^{0} e^{\alpha(-s)} x^T(s)Q_{1i}x(s)ds \\ &\quad + \int_{-h}^{0}\int_{\theta}^{0} e^{\alpha(-s)} \dot{x}^T(s)Q_{2i}\dot{x}(s)dsd\theta \\ &\quad + \int_{-h}^{0}\int_{\theta}^{0} e^{\alpha(-s)} x^T(0)Z_i x(s)dsd\theta. \\ &\leq \left(\lambda_2 + h\lambda_3 + \frac{h^2}{2}\lambda_4 + \frac{h^2}{2}\lambda_5\right) \\ &\quad \sup_{-\bar{\tau}\leq\theta\leq 0} \{x^T(\theta)Rx(\theta), \dot{x}^T(\theta)R\dot{x}(\theta)\}. \end{aligned}$$

$$\begin{aligned} &V(x(t)) \\ &\leq e^{(\alpha+\frac{\ln\mu}{\tau_a})T}\left[\left(\lambda_2 + h\lambda_3 + \frac{h^2}{2}\lambda_4 + \frac{h^2}{2}\lambda_5\right)c_1 + d\lambda_6\right]. \end{aligned} \tag{31}$$

From (9) we have

$$x^T(t)Rx(t) < c_2. \tag{32}$$

By Definition 1, the system (7) is finite-time boundedness. This completes the proof. □

Remark 8: Based on the theorem presented above, we can conclude that the system described in equation (7) shows finite-time boundedness. If we set the term $w(t) = 0$ in equation (7), we can conclude the theorem as representing finite-time stable.

Next, we focus on the finite-time boundedness of the (4).

B. FINITE-TIME $L_2 - L_\infty$ PERFORMANCE

Theorem 9: For given positive scalars T, c_1, c_2, d, h, h_M, K_i, ϵ, ϵ_a and α the system (4) with $B_i = 0$ is finite-time boundedness with a prescribed level of noise attenuation $\gamma > 0$ if there exist symmetric positive definite matrices $P_i > 0$, $Q_{1i} > 0$, $Q_{2i} > 0$, $Z_i > 0$ and any appropriate dimensional matrices $M_{1i} > 0$, $M_{2i} > 0$ and $L_i > 0$ such that the following LMIs holds:

$$\widetilde{\sum}_2 = \begin{bmatrix} \Sigma_2 & \Theta^T & h\Pi^T & \widehat{A} \\ * & -\frac{1}{h}Q_{2i} & 0 & 0 \\ * & * & -Q_{2i} & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{33}$$

$$e^{(\alpha+\frac{\ln\mu}{\tau_a})T}\left[(\beta)c_1\right] < \lambda_1 c_2. \tag{34}$$

Then, under the following average dwell time scheme

$$\tau_a > \tau_a^* = \frac{T\ln\mu}{\ln(c_2 e^{-\alpha T}) - \ln(\beta)c_1}, \tag{35}$$

the system is finite-time bounded with respect to $(c_1, c_2, T, R, p(t))$, where $\mu>1$ satisfying

$$\begin{aligned} P_s < \mu P_i, Q_{1s} < \mu Q_{1i}, Q_{2s} < \mu Q_{2i}, Z_s \\ < \mu Z_i, \quad \forall i, s \in \mathbb{N}. \end{aligned} \tag{36}$$

and

$$\Sigma_2 = \begin{bmatrix} \Phi_{11} & \Phi_{12} & 0 & 0 & 0 & 0 & L_i & -B_i \\ * & \Phi_{22} & 0 & 0 & 0 & 0 & -L_i & 0 \\ * & * & \Phi_{33} & 0 & 0 & \Phi_{36} & 0 & 0 \\ * & * & * & \Phi_{44} & \Phi_{45} & 0 & 0 & 0 \\ * & * & * & * & \Phi_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \Phi_{66} & 0 & 0 \\ * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & 0 & -\epsilon_a B_i \end{bmatrix},$$

$$\widehat{A} = \left[C_i \ C_{di} \ 0 \ 0 \ 0 \ 0 \ D_{wi} \ 0\right].$$

Φ_{11}, Φ_{12}, Φ_{22}, Φ_{33}, Φ_{36}, Φ_{44} Φ_{45}, Φ_{55}, Φ_{66} are defined in theorem 7.

Proof By following similar lines in the proof of Theorem 7, we have,

$$\dot{V}(t, e(t)) - \alpha V(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0. \tag{37}$$

Define

$$J = \gamma^2 w^T(t)w(t) - z^T(t)z(t). \tag{38}$$

Multiplying (37) by $e^{-\delta t}$, we have,

$$\frac{d}{dt}\{e^{-\delta t}V(t)\} < e^{-\delta t}J(t). \tag{39}$$

Integrating this inequality on $[0, T]$ yields

$$0 \leq e^{-\delta T}V(t) < \int_0^T e^{-\delta t}J(t)dt. \tag{40}$$

We have

$$\begin{aligned} e^{-\delta T}\int_0^T z^T(t)z(t)dt &< \int_0^T e^{-\delta t}z^T(t)z(t)dt \\ &< \gamma^2\int_0^T e^{-\delta t}w^T(t)w(t)dt \\ &< \gamma^2\int_0^T w^T(t)w(t)dt. \end{aligned} \tag{41}$$

By Definition 2 the system (4) is finite-time bounded with respect to (c_1, c_2, T, R, d) and with a prescribed level of noise attenuation $\gamma > 0$. This completes the proof. □

C. FINITE-TIME $L_2 - L_\infty$ ACTUATOR CONTROL

In this subsection, we will present a detailed procedure for actuator controller design, i.e., to find the controller gains K_i for each subsystem. The following theorem gives sufficient conditions for finite-time boundedness of the closed-loop system (4)

Theorem 10: For given positive scalars T, c_1, c_2, d, h, h_M, ϵ, ϵ_a and α the system (4) is finite-time boundedness with a prescribed level of noise attenuation $\gamma > 0$ if there exist symmetric positive definite matrices $P_i > 0$, $Q_{1i} > 0$, $Q_{2i} > 0$, $Z_i > 0$ and the appropriate matrices $M_{1i} > 0$, $M_{2i} > 0$ and $L_i > 0$ such that the following LMIs holds:

$$\widetilde{\sum}_{3} = \begin{bmatrix} \Sigma_3 & \Theta^T & h\Pi_1^T & \widehat{A}_1 & B_i^T K_i^T \\ * & -\frac{1}{h}Q_{2i}^{-1} & 0 & 0 & 0 \\ * & * & -Q_{2i} & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & 0 & -I \end{bmatrix} < 0, \tag{42}$$

$$e^{(\alpha+\frac{\ln\mu}{\tau_a})T}(\beta)c_1 < \lambda_1 c_2. \tag{43}$$

Then, under the following average dwell time scheme

$$\tau_a > \tau_a^* = \frac{T\ln\mu}{\ln(c_2 e^{-\alpha T}) - \ln(\beta)c_1}, \tag{44}$$

the system is finite-time bounded with respect to $(c_1, c_2, T, R, p(t))$, where $\mu > 1$ satisfying

$$P_s < \mu P_i, Q_{1s} < \mu Q_{1i}, Q_{2s} < \mu Q_{2i}, Z_s < \mu Z_i, \quad \forall i, s \in \mathbb{N}. \tag{45}$$

and

$$\Sigma_3 = \begin{bmatrix} \widehat{\Phi}_{11} & \widehat{\Phi}_{12} & 0 & 0 & 0 & 0 & L_i & -B_i \\ * & \widehat{\Phi}_{22} & 0 & 0 & 0 & 0 & -L_i & 0 \\ * & * & \widehat{\Phi}_{33} & 0 & 0 & \widehat{\Phi}_{36} & 0 & 0 \\ * & * & * & \widehat{\Phi}_{44} & \widehat{\Phi}_{45} & 0 & 0 & 0 \\ * & * & * & * & \widehat{\Phi}_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \widehat{\Phi}_{66} & 0 & 0 \\ * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & 0 & -\epsilon_a B_i \end{bmatrix},$$

$\widehat{\Phi}_{11} = 2P_iA_i + Q_{1i} + dQ_{3i} + M_{1i} + M_{1i}^T - \alpha P_i$,
$\widehat{\Phi}_{12} = -A_{di}P_i - M_{1i}^T + M_{2i}$, , $\widehat{\Phi}_{22} = e^{-\alpha h(t)}Q_1 - M_{2i}^T - M_{2i}$, $\widehat{\Phi}_{33} = \frac{e^{-\alpha h}}{h}Z_i - \frac{3e^{-\alpha h}}{h}Z_i$, $\widehat{\Phi}_{36} = \frac{6e^{-\alpha h}}{h^2}Z_i$,
$\widehat{\Phi}_{44} = \frac{e^{-\alpha h}}{h}Z_i - \frac{3e^{-\alpha h}}{h}Z_i$, $\widehat{\Phi}_{45} = \frac{6e^{-\alpha h}}{h^2}Z_i$, $\widehat{\Phi}_{55} = -\frac{12e^{-\alpha h}}{h^3}Z_i$,
$\widehat{\Phi}_{66} = -\frac{12e^{-\alpha h}}{h^3}Z_i$,
$\Theta = \left[A_iQ_{2i} \; A_{di}Q_{2i} \; 0 \; 0 \; 0 \; 0 \; B_{wi}Q_{2i} \; B_iQ_{2i} \right]$,
$\widehat{\Pi} = \left[M_{1i} \; M_{2i} \; 0 \; 0 \; 0 \; 0 \; L_i \; 0 \right]$,
$\widehat{A} = \left[C_i \; C_{di} \; 0 \; 0 \; 0 \; 0 \; D_{wi} \; 0 \right]$.

Proof: Following the same line of proof as presented in the Theorem 10 and applying the Schur complement lemma to equation (33), we can draw a conclusion regarding the behavior of (42). Consequently, we establish that the switching system described in (4) shows finite-time boundedness. As a result, this completes the proof. □

Remark 11: To obtain an optimal finite-time $L_2 - L_\infty$ performance against unknown inputs, the attenuation level γ^2 can be reduced to the minimum possible value such that LMIs (33)-(36) are satisfied with a fixed α. The optimization problem can be described as follows: min s:t: LMIs (33)-(36) with γ^2

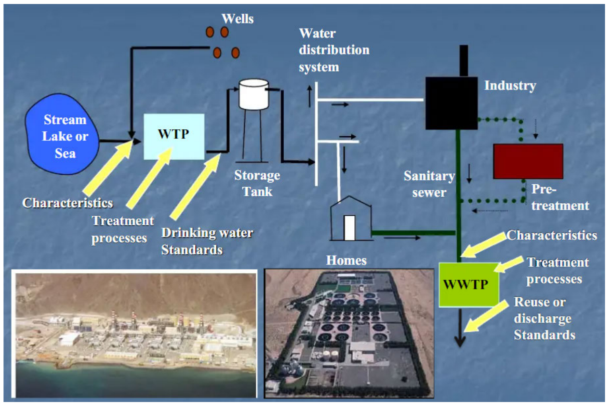

FIGURE 1. Water pollution control problem.

IV. NUMERICAL EXAMPLES

In this section we have given numerical examples to verify the effectiveness of the presented method.

Example 1: **Water Pollution Control Problem** We present the simulation results in this section, based on [39] and [40] first simplifying this water pollution system into a switched linear time-delay one, then providing a switching control method, and finally presenting the simulation results for the system (7) without disturbance. **System Description:** As an example, let us consider an area along a river that has a waste treatment facility at its beginning. At time t, $y(t)$ denotes biochemical oxygen demand (BOD) and $q(t)$ denote dissolved oxygen content in the reach. In this case, we assume constant flow rates and well-mixed water in the reach, and that when the flow enters the reach at instant $t - \tau$, BOD and DO are equal to their previous states [39]. The system dynamics are first described by defining two parameters $\wp_1 = \frac{Q_E}{v}$ and $\wp_2 = \frac{Q}{v}$, where Q_E and Q are the effluent flow and stream flow, respectively, and v defines the flow in the reach. By mass balance concentrations, we can obtain the following switched delay-differential equations that govern BOD and DO dynamics (where $\wp_1$ and $\wp_2$ are $0.1 : 0.9$ and $0.2 : 0.8$, respectively):

$$\dot{x}(t) = A_i x(t) + A_{di}x(t - h(t)) + B_i sat(u(t))$$
$$x(t) = \phi(t), \quad t \in [-h_M, 0], \tag{46}$$

where $x = [y(t) - y^*, q(t) - q^*]$, therein z^* and q^* are desired steady-state values of BOD and DO, respectively, and other parameters are listed as follows:

$$A_1 = \begin{bmatrix} -\kappa_{10} - \wp_1^1 - \wp_2^1 & 0 \\ -\kappa_{30} & -\kappa_{20} - \wp_1^1 - \wp_2^1 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -\kappa_{10} - \wp_1^2 - \wp_2^2 & 0 \\ -\kappa_{30} & -\kappa_{20} - \wp_1^2 - \wp_2^2 \end{bmatrix},$$
$$A_{d1} = \begin{bmatrix} \wp_2^1 & 0 \\ 0 & \wp_2^1 \end{bmatrix}, A_{d2} = \begin{bmatrix} \wp_2^2 & 0 \\ 0 & \wp_2^2 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} \wp_1^1 & 0 \\ 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} \wp_1^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Based on [39], the following values are chosen for the parameter values: $\wp_1^1 = 0.1, \wp_1^2 = 0.2, \wp_2^1 = 0.9,$

$\wp_2^2 = 0.8, \kappa_{10} = 1.6, \kappa_{20} = 1.0, \kappa_{30} = 0.6, z^* = 1.3750, q^{ast} = 6.0, h = 5.2, h_M = 0.5, d = 0.2, T = 5, c_1 = 2.2, \mu = 1.2, \alpha = 0.005$. By solving the LMIs in Theorem 7, that the optimal value of c_2 depends on parameter α. By solving the matrix inequalities (8)-(11), we can get the optimal bound of c_2 with different value of α in each subsystems.The smallest bound can be obtained as $c_2 = 4.5651$ when $\alpha = 0.005$ and we obtain feasible solutions as follows:

$$P_1 = \begin{bmatrix} 94.7613 & -9.7732 \\ -9.7732 & 79.0008 \end{bmatrix}, \; Q_{11} = \begin{bmatrix} 5.5558 & -0.5631 \\ -0.5631 & 4.5208 \end{bmatrix},$$
$$Q_{21} = \begin{bmatrix} 1.4102 & -0.1617 \\ -0.1617 & 1.0922 \end{bmatrix}, \; Z_1 = \begin{bmatrix} 2.6297 & -0.2707 \\ -0.2707 & 2.1399 \end{bmatrix},$$
$$S_1 = \begin{bmatrix} 29.8679 & -2.9594 \\ -2.9594 & 49.4804 \end{bmatrix}, \; P_2 = \begin{bmatrix} 78.8410 & -3.6993 \\ -3.6993 & 65.5524 \end{bmatrix},$$
$$Q_{12} = \begin{bmatrix} 49.4027 & 0.3073 \\ 0.3073 & 40.7624 \end{bmatrix}, \; Q_{22} = \begin{bmatrix} 7.3744 & -0.6521 \\ -0.6521 & 9.9667 \end{bmatrix},$$
$$Z_2 = \begin{bmatrix} 48.2782 & 2.3655 \\ 2.3655 & 21.1940 \end{bmatrix}, \; S_2 = \begin{bmatrix} 38.7956 & -2.8749 \\ -2.8749 & 75.4845 \end{bmatrix}.$$

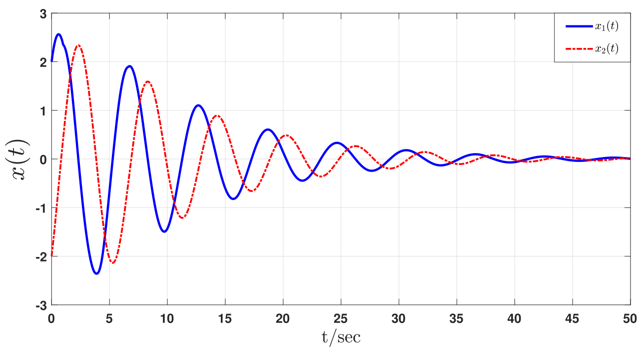

FIGURE 2. State trajectories of the considered model (46) without controller.

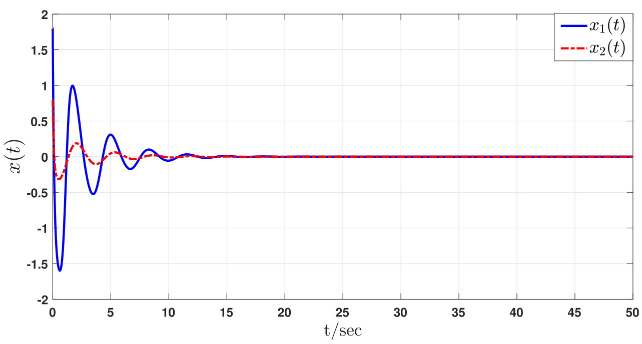

FIGURE 3. State trajectories of the considered model (46) with controller.

As a result of the above control gain matrices and $x(0) = [1.4 \;\; 2.7]^T$ as the initial value, the simulation results of the state response and control trajectory of the proposed model (46) are plotted in Figs. 1-4. Fig. 1 illustrates the state responses of the system (46) with uncontrolled. As shown in Fig. 2, the proposed control strategy can ensure the finite-time stability of the considered system (46). Fig. 3 shows the trajectory of the saturated control input. Fig. 4 illustrates the estimate of the switching signal and its water pollution control with switched system. Compared to the existing results [41], Fig. 2 shows that our method provides better robust stability than [41] even when the saturation inputs, and it is obvious that our controller (3) consumes less control energy than the controller in [39] and [41]. The asymptotic stability of [41] has been considered in addition to the comparison of the arbitrary switching signal strictness. When switching harshness varies arbitrary, the switched system may become unstable, while the same system can maintain stability when switching harshness is based on dwell time. Despite its random nature and variations in response to industrial discharges, the water pollution model does not have a high-frequency switching signal. For stream water quality control problems, slow switching signals that enforce remaining in subsystems are feasible, and saturation control inputs are practical and progressive. This confirms the superiority of the proposed method.

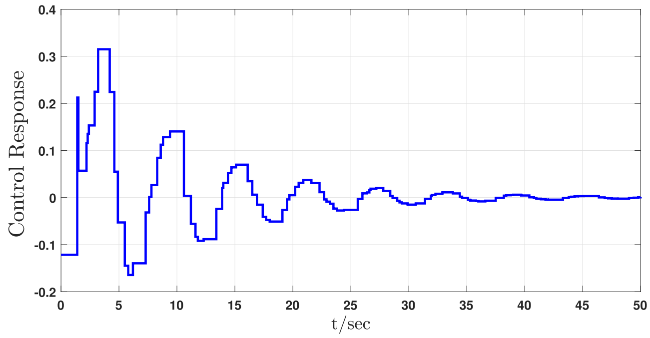

FIGURE 4. Control input response.

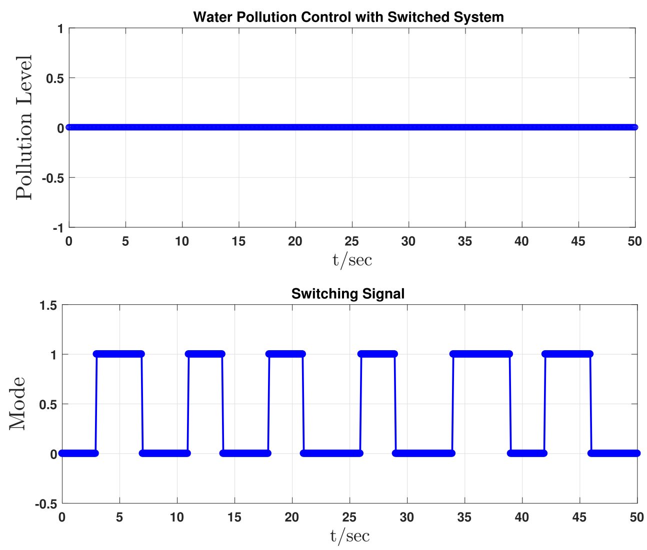

FIGURE 5. Evolution of switching signal.

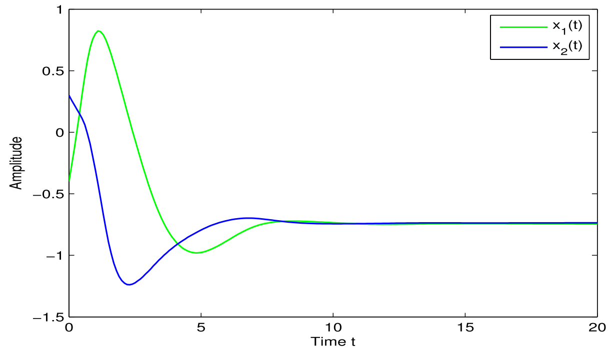

FIGURE 6. State responses of the closed-loop system (1) in Example 4.2.

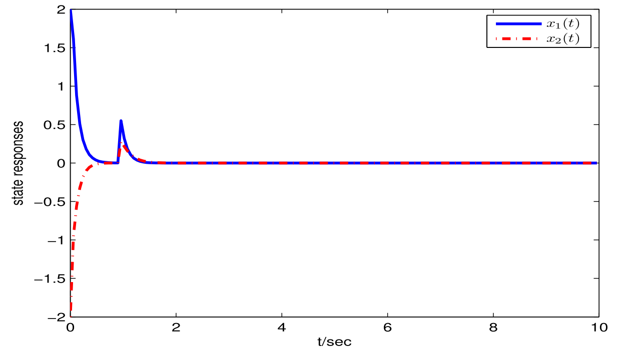

FIGURE 7. State responses of the closed-loop system (1) in Example 4.2.

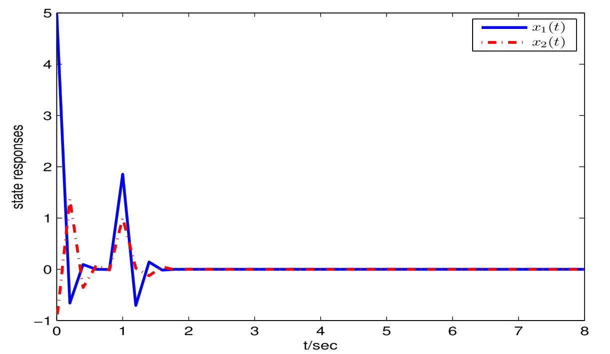

FIGURE 8. State responses of the closed-loop system (1) in Example 4.2.

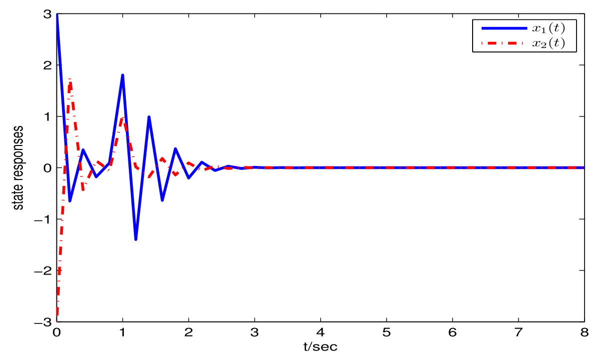

FIGURE 9. State responses of the closed-loop system (1) in Example 4.2.

Remark 12: Note that the conditions (9) are dependent on the size of c_2, then we can also get the optimal lower bound of c_2 to guarantee the finite-time stability by solving a simple optimal problem. For example, we can obtain the optimal lower bound of c_2 is 4.5651.

Example 2: Consider the actuator saturation with switched time-delay system (1) and the following parameters: $A_1 = \begin{bmatrix} 2.5 & 0 \\ 0 & 3.5 \end{bmatrix}$, $A_{d1} = \begin{bmatrix} 0.2 & 1.2 \\ -0.5 & 1.2 \end{bmatrix}$, $B_1 = \begin{bmatrix} -0.1 \\ -0.4 \end{bmatrix}$, $B_{w1} = \begin{bmatrix} 2 \\ 0.3 \end{bmatrix}$,
$C_1 = \begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.03 \end{bmatrix}$, $C_{d1} = \begin{bmatrix} -0.6 & 0.5 \\ 2.1 & 0.1 \end{bmatrix}$,
$D_{w1} = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, $A_{d2} = \begin{bmatrix} 0.1 & -1.1 \\ 0.3 & -0.8 \end{bmatrix}$,
$B_2 = \begin{bmatrix} 0.3 \\ -0.1 \end{bmatrix}$, $B_{w2} = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}$,
$C_2 = \begin{bmatrix} -0.4 & 0.1 \\ 0.2 & -0.3 \end{bmatrix}$, $C_{d2} = \begin{bmatrix} -0.8 & 0.4 \\ -2.2 & 0.2 \end{bmatrix}$,
$D_{w2} = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}$
$h = 0.7$, $h_M = 1.6$, $d = 0.003$, $T = 5$, $c_1 = 1.4$, $c_2 = 7.9$ $\mu = 1.3$, $\alpha = 0.02$. Solve the LMIs in Theorem 3.4, we obtain the feasible solutions as follows:

$$P_1 = \begin{bmatrix} 91.5023 & 59.5738 \\ 59.5738 & 59.3507 \end{bmatrix}, \; Q_{11} = \begin{bmatrix} 0.3870 & 0.2811 \\ 0.2811 & 0.2160 \end{bmatrix},$$
$$Q_{21} = \begin{bmatrix} 0.9370 & 0.6525 \\ 0.6525 & 0.5037 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} 17.6717 & 12.4013 \\ 12.4013 & 10.5504 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 7.9658 & 5.5558 \\ 5.5558 & 4.9213 \end{bmatrix}, \; Q_{21} = \begin{bmatrix} 29.0450 & 19.9782 \\ 19.9782 & 16.1920 \end{bmatrix},$$
$$Q_{22} = \begin{bmatrix} 0.3624 & 0.2719 \\ 0.2719 & 0.2129 \end{bmatrix}, \; Z_2 = \begin{bmatrix} 43.0322 & 27.1583 \\ 27.1583 & 28.3567 \end{bmatrix}.$$

We obtained the saturation gain matrices as,

$$K_1 = \begin{bmatrix} -0.1621 & -0.5423 \end{bmatrix}, \; K_2 = \begin{bmatrix} -1.7542 - 0.2486 \end{bmatrix}.$$

The system is finite time stabilizable with the prescribed $L_2 - L_\infty$ performance $\gamma^2 = 0.6$. Figure 6, Figure 7, Figure 8, and Figure 9 shows the state responses of the system with different initial values.

TABLE 1. Comparison with other works.

	[2]	[4]	[27], [22]	[34]	Our Paper
Finite-time	√	√	√	×	√
Switched	√	√	√	×	√
$L_2 - L_\infty$	×	×	×	√	√
Actuator saturation	×	√	√	×	√

Remark 13: This work addresses the finite-time boundedness of switched time-varying systems with actuator saturation. In contrast to research results on finite-time boundedness in ([2], [4], [22], and [27]), which investigated switched, filtering, and discrete-time systems respectively, we explore finite-time switched systems with actuator saturation using LMIs and introduce new integral inequalities in this paper. Consequently, the approach presented in this

work proves to be more effective while managing the system's complexity. In Table 1, a comparison table with previously published results is presented below.

V. CONCLUSION

In this study, we have investigated the intricate problem of finite-time $L_2 - L_\infty$ control for switched systems with time-varying delays and actuator saturation. Our primary contributions encompass the derivation of a sufficient condition for ensuring the finite-time boundedness of the closed-loop system, achieved through the application of the Lyapunov functional approach. We have demonstrated that the resulting controller can be efficiently obtained using cutting-edge Linear Matrix Inequality techniques. Extensive numerical examples and simulations have validated the practical effectiveness of our proposed methodology by MATLAB. Looking forward, our research opens avenues for future work, including the extension of our approach to tackle fractional-order systems with distributed delays and the exploration of solutions to address uncertainty mining and control challenges, promising advancements in the field of finite-time control.

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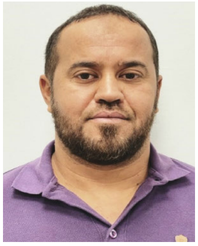

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