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RESEARCH ARTICLE

Aggregation Operators on Pythagorean Fuzzy Hypersoft Matrices With Application in the Selection of Wastewater Treatment Plants

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ABSTRACT Pythagorean fuzzy hypersoft sets (PFHSSs) are a novel model that is projected to address the limitations of Pythagorean fuzzy soft sets (PFSSs) regarding the entitlement of a multi-argument domain for the approximation of parameters under consideration. It is more flexible and reliable as it considers the further classification of parameters into their relevant parametric valued sets. This article aims to be multi-faced. Firstly, several axiomatic properties, operational results, and aggregation operations on PFHSSs will be developed. Secondly, matrices are developed for PFHSSs, called Pythagorean fuzzy hypersoft matrices (PFHSMs). The essential basic properties and aggregation operations of PFHSMs are then characterized with the support of numerical examples. Thirdly, the matrix theory of PFHSSs is implemented in real-world decision-making scenarios for Mobile selection using the proposed choice matrix theory. At the end of the article, we go on a real-life problem for wastewater treatment. Wastewater treatment is crucial for preserving the environment and public health. It comprises purifying wastewater of contaminants and pollutants so that it may be utilized for other things or discharged safely into the environment. It is essential to protect the environment and the public health by removing toxins from domestic, industrial, and commercial sewages. We finally apply our proposed algorithm in the selection of wastewater treatment plants by employing the proposed algorithm based on PFHSMs. In fact, PFHSMs are flexible enough to be used in a wide range of fields, including image processing, expert systems, pattern recognition, and medical diagnosis. The future directions are discussed with these PFHSMs to develop MCDM techniques such as TOPSIS, VIKOR, and SAW so that they can be applied in a wider range of fields.

INDEX TERMS Fuzzy sets, pythagorean fuzzy sets (PFSs), pythagorean fuzzy soft sets (PFSSs), pythagorean fuzzy hypersoft sets (PFHSSs), pythagorean fuzzy hypersoft matrices (PFHSMs).

I. INTRODUCTION

The process of making decisions with several potential outcomes may induce uncertainty and vagueness. These uncertainties and vagueness can be brought on by unclear facts, inadequate knowledge, or uncontrollable variables. Zadeh [1] developed the mathematical framework known as fuzzy sets in 1965 to express and manage ambiguity and uncertainty in data and information. Fuzzy sets allow for partial membership, where an element may partially belong to a set, in contrast to classical sets, which are binary and define membership as either true or false. Fuzzy sets have been successfully applied in fuzzy clustering [2], [3], [4] and validity indexes [5]. Another sort of fuzzy set, known as interval-valued fuzzy sets (IVFSs) [6], enables the representation of uncertain or inaccurate information using intervals rather than precise values in which each element in IVFSs has a membership value that spans a range of potential degrees of membership. Fuzzy sets may be extended to indicate both the degree of membership and the degree of non-membership of

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an element in a set using intuitionistic fuzzy sets (IFSs) [7]. In contrast to fuzzy sets, which assign each element a membership value between 0 and 1, IFSs include a third parameter termed the hesitation degree that quantifies the level of ambiguity or hesitation in the membership assignment. These IFSs have been applied in various areas [8], [9]. Yager [10] and Yager and Abbasov [11] proposed Pythagorean fuzzy sets (PFSs) that are an extension of IFSs with a condition that the sum of the squares of both membership and non-membership grades is not exceeded from a unit interval. A more adaptable framework for dealing with ambiguity and uncertainty in decision-making processes is offered by PFSs. They make it possible to depict membership and non-membership degrees in a more sophisticated manner, by taking both the advantages and disadvantages of ambiguity. They are therefore suited for scenarios like decision-making under ambiguity or vagueness when both membership and non-membership information are pertinent. Furthermore, Smarandache [12] developed a mathematical framework of neutrosophic sets (NSs) to address issues with incomplete or conflicting information in which IFSs were extended to handle uncertain, ambiguous, and inconsistent information in the use of NSs.

In 1999, Molodtsove [13] presented soft sets (SSs) theory to handle a parameterized family of characteristics. SSs offer a flexible technique to deal with ambiguity and uncertainty in information processing and decision-making. They are used in a variety of industries, including artificial intelligence, data mining, expert systems, pattern recognition, and decision analysis. The SS theory may be integrated with various mathematical frameworks to solve more difficult issues and be used to describe imperfect or partial information. Peng et al. [14] considered Pythagorean fuzzy soft sets (PFSSs) by combining fuzzy SSs and PFSs. They did this by including interval-valued membership degrees into soft sets, which provide a framework for dealing with uncertainty and ambiguity in decision-making challenges. Naeem et al. [15] proposed some decision-making techniques using PFSSs, and Riaz et al. [16] developed Pythagorean m-polar fuzzy sets. Guleria and Bajaj [17] first presented Pythagorean fuzzy soft matrices (PFSMs) and applied them to medical diagnosis. When there is conflicting, confusing, or incomplete information available, these matrices are utilized to aid in decision-making. When making decisions, PFSMs can be used to accommodate hazy and insufficient information and allow a more flexible depiction of uncertainty. Afterward, Bajaj and Guleria [18], [19] gave advanced extension and applications, and Jafar et al. [20], [21] considered neutrosophic soft matrices with applications in agriculture and medical diagnosis.

Samandrache [22] presented hypersoft sets (HSSs) by converting SSs into multi-argument domains by splitting the attributes into further disjoint attributions, and Jafar and Saeed [23] considered the aggregation operations of HSSs. Saqlain et al. [24] proposed neutrosophic HSSs (NHSSs) with their similarity measures, and Saeed et al. [25] applied these to diagnose hepatitis with treatments. Jafar and Saeed [26] proposed a matrix theory of NHSSs and Jafar et al. [27] proposed similarity measures of NHSSs with application in renewable energy source selection. Zulqarnain et al. [28] proposed Pythagorean fuzzy HSSs (PFHSSs) and their aggregation operations. Rehman et al. [29] developed parameterized fuzzy HSSs and applied them in the diagnosis of heart diseases. In fact, PFHSSs are a novel model that is projected to address the limitations of PFSSs regarding the entitlement of a multi-argument domain for the approximation of parameters under consideration. It is more flexible and reliable as it considers the further classification of parameters into their relevant parametric valued sets. On the other hand, there is no one to consider Pythagorean fuzzy hypersoft matrices (PFHSMs) in the literature. In this paper, we should work on these PFHSMs with their application in the selection of wastewater treatment plants.

The ideas of PFSs and HSSs are combined to form PFHSSs. Before understanding how each of these ideas works together, let us first grasp them individually.

1. PFSs: By allowing for multiple membership degrees, PFSs expand on traditional fuzzy sets. PFSs employ two membership values, the degree of membership and the degree of non-membership, both of which range from 0 to 1, as opposed to a single membership value between 0 and 1. These two numbers added together may be more than 1, indicating hesitation or doubt in membership assignment.
2. HSSs: By allowing for more freedom in membership assignments, HSSs generalize fuzzy sets. Instead of an exact numerical value, membership degrees in HSSs are determined by language phrases or gradations. This makes it possible to depict uncertainty or imprecision in a more sophisticated manner.
3. PFHSMs: PFHSMs combine both ideas of PFSs and HSSs to improve the representation of uncertainty and ambiguity in a particular domain. They offer a framework for handling membership degrees that may be stated in both language and numerical terms, allowing for more accurate and adaptable modelling of fuzzy data.
4. Pythagorean fuzzy numbers, which are made up of two membership values and a degree of hesitation, are used to represent the membership degrees in PFHSMs. The amount of hesitation reflects how ambiguous or dubious the membership designation is.

Wastewater treatment is crucial for preserving the environment and the public health. It comprises purifying wastewater of contaminants and pollutants so that it may be utilized for other things or discharged safely into the environment. Facilities designed to clean and filter wastewater before it is released back into the environment are known as water reclamation facilities or wastewater treatment plants. They are essential to protect the environment and public health by removing toxins from domestic, industrial, and commercial sewages. There are different scientists and researchers who worked on wastewater treatment plants (WWTPs) by using different MCDM techniques. Ali et al. [30] developed a technique to find the most feasible WWTPs. In [31], [32], [33], [34], and [35], they were the different researchers who worked on site selection or WWTP selection under different criteria. In this article, we are going to apply our proposed algorithm in the selection of WWTPs. For better understanding, here we discuss five WWTPs which are Activated Sludge (AS-A_1), Sequential Batch Reactor (SBR- A_2), Constructed Wetland (CW-A_3), Anaerobic Lagoon (AL-A_4), Membrane Filtration (MF-A_5). There are many criteria relating to the above-discussed alternatives, but we will suppose some criteria which are Chemical Consumption (CC)-(C_1), Sludge Production (SP)-(C_2), Environmental Impact (EI)-(C_3), Energy Consumption (EC)-(C_4), Efficiency (E)-(C_5). These all seven criteria further sub-divided into many sub-criteria as shown in fig.02. Our goal is to apply the proposed algorithm on this real application and then select the best WWTP under defined criteria. The rest of this article is organized as follows. Section II is preliminaries. In Section III, PFHSMs and their different forms are considered. Section IV is about the aggregation operations of PFHSMs with their propositions. In Section V, we propose the two decision-making algorithms based on PFHSMs. Algorithm 1 has the application in real-life examples of mobile selections, and algorithm 2 will especially be applied in the WWTP selection. Finally, we make conclusions with some future directions in Section VI.

A. PRELIMANARIES

In this section we discuss some basic definitions from the literature for the better understanding of the proposed study.

Definition 1 (Zadeh [1]): A fuzzy set F is given by

$$F = \{\langle \psi, Ł(\psi), \psi \in \Psi \rangle\}$$

where $Ł : \Psi \to [0, 1]$ and Ψ is a universal set and $Ł(\psi)$ is the degree of belongingness.

Definition 2 (Atanassov [7]): An intuitionistic fuzzy set (IFS) I is given by

$$I = \{\langle \psi, (Ł(\psi), \eta(\psi)), \psi \epsilon \Psi \rangle\} \text{ s.t. } 0 \leq Ł(\psi) + \eta(\psi) \leq 1$$

where $\mu, \eta : \Psi \to [0, 1]$ and Ψ is a universal set. $Ł(\psi)$ is the degree of belongingness, and $\eta(\psi)$ is the degree of non-belongingness. Atanassov [6] discussed another factor, called degree of hesitancy, defined by $\pi(\psi) = 1 - Ł(\psi) - \eta(\psi)$.

Definition 3 (Yager [10]): A Pythagorean fuzzy set (PFS) is given by

$$P = \{\langle \psi, (Ł(\psi), \eta(\psi)) \psi \epsilon X \rangle\} \text{ s.t. } 0 \leq Ł^2(\psi) + \eta^2(\psi) \leq 1$$

where $Ł, \eta : \Psi \to [0, 1]$ and Ψ is a universal set. $Ł(\psi)$ is the degree of belongingness and $\eta(\psi)$ is the degree of non-belongingness. PFS is a flexible mode of IFS, and the degree of hesitancy is defined by $\pi(\psi) = \sqrt{1 - Ł^2(\psi) - \eta^2(\psi)}$. Fig. 1 demonstrates the graph of areas in IFSs vs that of PFSs.

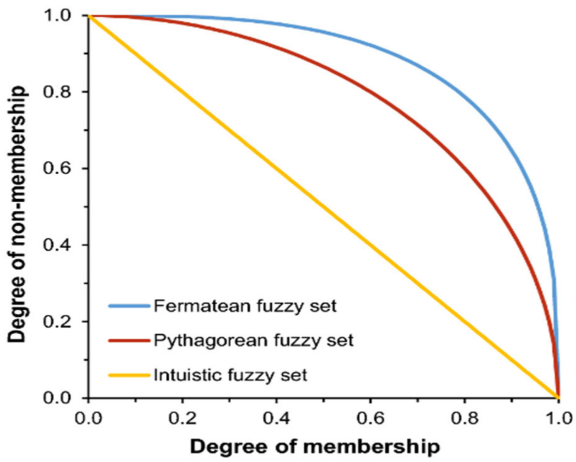

FIGURE 1. IFSs vs PFSs.

Definition 4 (Molodtsove [13]): Let Ψ be a universal set and let $Z = \{z_1, \ldots, z_s\}$ be a finite set of parameters. Let$P(\Psi)$ denote the collection of all subsets of Ψ. For any $A \subset Z$, a pair $(\wp, A)$ is called a soft set (SS) over Ψ, where the mapping $\wp$ is given by $\wp : A \to P(\Psi)$.

Definition 5 (Peng et al. [14]): Let Ψ be a universal set and let $Z = \{z_1, \ldots, z_s\}$ be a finite set of parameters. Let$\mathcal{P}(\Psi)$ denote the collection of all Pythagorean fuzzysets (PFSs) of Ψ. For any $A \subset Z$, a pair $(\wp, A)$ is called a Pythagorean fuzzy soft set (PFSS) overΨ, where the mapping $\wp$ is given by $\wp : A \to \mathcal{P}(\Psi)$.

Definition 6 (Smarandache [22]): Let Ψ be a universal set, and let $\mathfrak{Q} = \{\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3, \ldots \mathfrak{Q}_\beta\}$ be a finite set of parameters (or disjoint β attributes) with their corresponding attributive values of $\mathfrak{Q}_1^a, \mathfrak{Q}_2^b, \mathfrak{Q}_3^c, \ldots \mathfrak{Q}_\beta^z$ where a, b, c, … z = 1, 2, 3, …, n. Let $P(\Psi)$ denote the collection of all subsets of Ψ. A hypersoft set (HSS) is defined as $\left(\acute{K}, \mathfrak{Q}_1^a \times \mathfrak{Q}_2^b \times \mathfrak{Q}_3^c \times \ldots \times \mathfrak{Q}_\beta^z\right)$ over Ψ such that $\acute{K}, : \mathfrak{Q}_1^a \times \mathfrak{Q}_2^b \times \mathfrak{Q}_3^c \times \ldots \mathfrak{Q}_\beta^z \to P(\Psi)$.

Definition 7 (Zulqarnain [28]): Let $\Psi = \{\psi_1, \psi_2, \psi_3, \ldots \psi_\alpha\}$ be a universal set with α options, and let $\mathfrak{Q} = \{\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3, \ldots \mathfrak{Q}_\beta\}$ a finite set of parameters (or disjoint β attributes) with their corresponding attributive values of $\mathfrak{Q}_1^a, \mathfrak{Q}_2^b, \mathfrak{Q}_3^c, \ldots \mathfrak{Q}_\beta^z$ wherea, b, c, … z = 1, 2, 3, …, n. Let $\mathcal{P}(\Psi)$ denote the collection of all PFSs of Ψ. A Pythagorean fuzzy hypersoft set (PFHSS) over Ψ is defined as $\left(\acute{K}, \mathfrak{Q}_1^a \times \mathfrak{Q}_2^b \times \mathfrak{Q}_3^c \times \ldots \times \mathfrak{Q}_\beta^z\right)$ such that

$$\acute{K}, : \mathfrak{Q}_1^a \times \mathfrak{Q}_2^b \times \mathfrak{Q}_3^c \times \ldots \times \mathfrak{Q}_\beta^z \to \mathcal{P}(\Psi)$$

defined by $\acute{K}\left(\mathfrak{Q}_1^a \times \mathfrak{Q}_2^b \times \mathfrak{Q}_3^c \times \ldots \times \mathfrak{Q}_\beta^z\right) = \left\{\left\langle Ꝑ^j, \left(Ł_{Ꝑ}\left(Ꝑ^j, \Psi_\tau\right), \eta_{Ꝑ}\left(Ꝑ^j, \Psi_\tau\right)\right)\right\rangle, \Psi_\tau \in \Psi, Ꝑ \in \mathfrak{Q}_1^a \times \mathfrak{Q}_2^b \times \mathfrak{Q}_3^c \times \ldots \times \mathfrak{Q}_\beta^z\right\}$ where $0 \leq Ł_{Ꝑ}^2(\Psi_\tau) + \eta_{Ꝑ}^2(\Psi_\tau) \leq 1$ with the degree of hesitancy $\pi(\Psi) = \sqrt{1 - Ł^2(\Psi) - \eta^2(\Psi)}$.

II. PYTHAGOREAN FUZZY HYPERSOFT MATRICES (PFHSMs) AND THEIR DIFFERENT FORMS

Across a wide range of disciplines and sectors, matrices are essential to decision-making. They offer a methodical, systematic technique to assess several options based on a number of criteria or aspects. In this section, we are going to extend the concept of PFHSSs to PFHSMs, and we advance to present their operations. As PFHSSs give more accurate scenario for decision-making, their matrix form PFHSMs should give more quick solutions. We next define PFHSMs.

Definition 8: Based on Definition 7, let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_\alpha\}$ be a universal set with α options, and let $\mathfrak{Q} = \{\mathfrak{Q}_1, \mathfrak{Q}_2, \ldots, \mathfrak{Q}_\beta\}$ be a set of disjoint β attributes with their corresponding attributive values of $\mathfrak{Q}_1^a, \mathfrak{Q}_2^b, \ldots, \mathfrak{Q}_\beta^z$ where $a, b, c, \ldots z = 1, 2, 3, \ldots, n$. Let $\mathcal{P}(\Psi)$ denote the collection of all PFSs of Ψ. A PFHSS over Ψ is defined as $(\acute{K}, \mathfrak{Q}_1^a \times \mathfrak{Q}_2^b \times \ldots \times \mathfrak{Q}_\beta^z)$ such that $\acute{K}, :\mathfrak{Q}_1^a \times \mathfrak{Q}_2^b \times \ldots \times \mathfrak{Q}_\beta^z \rightarrow \mathcal{P}(\Psi)$ defined by $\acute{K}(\mathfrak{Q}_1^a \times \mathfrak{Q}_2^b \times \ldots \times \mathfrak{Q}_\beta^z) = \{\langle \text{\rlap{-}P}^j, (\text{\rlap{-}t}_{\text{\rlap{-}P}}(\text{\rlap{-}P}^j, \Psi_\tau), \eta_{\text{\rlap{-}P}}(\text{\rlap{-}P}^j, \Psi_\tau)) \rangle, \Psi_\tau \in \Psi, \text{\rlap{-}P} \in \mathfrak{Q}_1^a \times \mathfrak{Q}_2^b \times \mathfrak{Q}_3^c \times \ldots \times \mathfrak{Q}_\beta^z\}$ where $0 \leq \text{\rlap{-}t}_{\text{\rlap{-}P}}^2(\Psi_\tau) + \eta_{\text{\rlap{-}P}}^2(\Psi_\tau) \leq 1$. Thus, a PFHSM is defined in Table 1 with a matrix form as follows:

TABLE 1. The PFHSM of the PFHSS $(\acute{K}, \mathfrak{Q}_1^a \times \mathfrak{Q}_2^b \times \ldots \times \mathfrak{Q}_\beta^z)$.

	$\mathfrak{Q}_1^a$	$\mathfrak{Q}_2^b$	...	$\mathfrak{Q}_\beta^z$
Ψ^1	$\mathcal{X}_{p_{\text{P}}}(\Psi^1, \mathfrak{Q}_1^a)$	$\mathcal{X}_{p_{\text{P}}}(\Psi^1, \mathfrak{Q}_2^b)$	...	$\mathcal{X}_{p_{\text{P}}}(\Psi^1, \mathfrak{Q}_\beta^z)$
Ψ^2	$\mathcal{X}_{p_{\text{P}}}(\Psi^2, \mathfrak{Q}_1^a)$	$\mathcal{X}_{p_{\text{P}}}(\Psi^2, \mathfrak{Q}_2^b)$	...	$\mathcal{X}_{p_{\text{P}}}(\Psi^2, \mathfrak{Q}_\beta^z)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
Ψ^α	$\mathcal{X}_{p_{\text{P}}}(\Psi^\alpha, \mathfrak{Q}_1^a)$	$\mathcal{X}_{p_{\text{P}}}(\Psi^\alpha, \mathfrak{Q}_2^b)$	...	$\mathcal{X}_{p_{\text{P}}}(\Psi^\alpha, \mathfrak{Q}_\beta^z)$

If $\zeta_{ij} = \mathcal{X}_{p_{\text{P}}}(\Psi^i, \mathfrak{Q}_j^k)$, where $i = 1, 2, 3 \ldots \alpha, j = 1, 2, 3, \ldots \beta, k = a, b, c, \ldots z$, then a matrix is defined as

$$[\zeta_{ij}]_{\alpha\times\beta} = \begin{pmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1\beta} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2\beta} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{\alpha 1} & \zeta_{\alpha 2} & \cdots & \zeta_{\alpha\beta} \end{pmatrix}$$

where

$$\zeta_{ij} = \left(\left(\text{\rlap{-}t}_{\mathfrak{Q}_j^k}(\psi_i), \eta_{\mathfrak{Q}_j^k}(\psi_i) \right), \psi_i \in \Psi , \mathfrak{Q}_j^k \in \left(\mathfrak{Q}_1^a \times \mathfrak{Q}_2^b \times \ldots \times \mathfrak{Q}_\beta^z \right) \right) = \left(\text{\rlap{-}t}_{\mathfrak{Q}_j^k}(\psi_i), \eta_{\mathfrak{Q}_j^k}(\psi_i) \right).$$

For simplicity, we may assume that $\text{\rlap{-}t}_{\mathfrak{Q}_j^k}(\psi_i) = \text{\rlap{-}t}_{ij}$ and $\eta_{\mathfrak{Q}_j^k}(\psi_i) = \eta_{ij}$, where i represents the position of alternatives, j tells us about the attributes, hidden k tells us about sub-attributive value of the corresponding attribute, and $\mathfrak{Q}$ is the subset of the PFHSS. Thus, the matrix representation is as

$$M_{\alpha\times\beta} = \begin{bmatrix} (\text{\rlap{-}t}_{11}, \eta_{11}) & (\text{\rlap{-}t}_{12}, \eta_{12}) & \ldots & (\text{\rlap{-}t}_{1\beta}, \eta_{1\beta}) \\ (\text{\rlap{-}t}_{21}, \eta_{21}) & (\text{\rlap{-}t}_{22}, \eta_{22}) & \ldots & (\text{\rlap{-}t}_{2\beta}, \eta_{2\beta}) \\ \vdots & \vdots & \vdots & \vdots \\ (\text{\rlap{-}t}_{\alpha 1}, \eta_{\alpha 1}) & (\text{\rlap{-}t}_{\alpha 2}, \eta_{\alpha 2}) & \cdots & (\text{\rlap{-}t}_{\alpha\beta}, \eta_{\alpha\beta}) \end{bmatrix}$$

which is called a PFHSM of order $\alpha \times \beta$ over Ψ.

Example 1: Let $\Psi = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$ be the set of five alternatives (mobiles) and $\mathfrak{Q} = \{\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3\}$ be the set of attributes with $\mathfrak{Q}_1$ = Battery = {4000mah, 5000mah}, $\mathfrak{Q}_2$ = Ram = {6GB, 8GB, 10GB}, $\mathfrak{Q}_3$ = DisplaySize = {5, 6″, 7″}. Then, $\text{\rlap{-}P}$(4000mah, 8GB, 6″) = $\{\psi_1, \psi_2, \psi_3\}$, where {($\psi_1$, 4000mah(0.5, 0.6)), 8GB(0.4, 0.5), 6″(0.8, 0.6)},{(ψ_2, 4000mah(0.3, 0.7)), 8GB(0.4, 0.4), 6″(0.6, 0.7)}, and {(ψ_3, 4000mah(0.7, 0.4)), 8GB(0.6, 0.5), 6″(0.6, 0.6)}. Thus, we have that

$$M = \begin{array}{c} \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{array} \begin{array}{ccc} \text{4000mah} & \text{8GB} & 6'' \\ (0.5, 0.6) & (0.4, 0.5) & (0.8, 0.6) \\ (0.3, 0.7) & (0.4, 0.4) & (0.6, 0.7) \\ (0.7, 0.4) & (0.6, 0.5) & (0.6, 0.6) \end{array}$$

Each sum of square of the order pair of truthiness and falseness is always lying in the unit interval [0, 1]. The above-mentioned example showing the result of PFHSSs. Now suppose a collection of PFHSM of order $\alpha \times \beta$ over the set of alternatives Ψ. Then, we are proposing some types like, a PFHSM $M = [\text{\rlap{-}t}_{ij}^M, \eta_{ij}^M] \in \text{PFHSM}_{\alpha\times\beta}$ with the followings:

a. **Pythagorean fuzzy hypersoft Null Matrix**
The $\text{\rlap{-}t}_{ij}^M = 0$ and $\eta_{ij}^M = 0$, $\forall i, j$ and is denoted by O
b. **Pythagorean Fuzzy hypersoft Row Matrix**
The the number of rows of PFHSM is one, i.e $\alpha = 1$
c. **Pythagorean Fuzzy hypersoft Column Matrix**
The number of columns of PFHSM is one, i.e $\beta = 1$
d. **Pythagorean fuzzy hypersoft Rectangular Matrix**
The number of rows and columns of PFHSM are different, i.e $\alpha \neq \beta$
e. **Pythagorean Fuzzy hypersoft Square Matrix**
The number of rows and columns of PFHSM are same, i.e $\alpha = \beta$
f. **Pythagorean Fuzzy hypersoft Diagonal Matrix**
A square PFHSM is said to be diagonal PFHSM if at least one of the diagonal element is non-zero and remainings are zero, i.e $(\text{\rlap{-}t}_{ij}^M, \eta_{ij}^M) \neq (0, 0)$ for some $i = j$ and $(\text{\rlap{-}t}_{ij}^M, \eta_{ij}^M) = (0, 0)$, $\forall i = j$. It means there are only some values or all values in diagonal where some membership and non-membership exists, otherwise they all vanish everywhere other than diagonal.
g. **Pythagorean Fuzzy hypersoft Absolute Matrix**
The $\text{\rlap{-}t}_{ij}^M = 1$ and $\eta_{ij}^M = 0$, $\forall i$ and j, i.e. whole matrix truthiness are always existing and complete unit.

h. Scalar Multiplication of Pythagorean Fuzzy hypersoft Matrix

In scalar multiplication, let k be any scalar real number, and $k\mathrm{M} = \left[k\L_{ij}^{\mathrm{M}}, k\eta_{ij}^{\mathrm{M}}\right]$, $\forall i$ and j

Furthermore, we define some relations between two PFHSMs. For this, let $\mathrm{M} = [\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}]$ and $\mathrm{N} = [\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{N}}] \in \mathrm{PFHSM}_{\alpha\times\beta}$, then we have that $\mathrm{M} \subseteq \mathrm{N}$ if $\L_{ij}^{\mathrm{M}} \leq \L_{ij}^{\mathrm{N}}$ and $\eta_{ij}^{\mathrm{M}} \geq \eta_{ij}^{\mathrm{N}}$ $\forall i$ and j, and $\mathrm{M} = \mathrm{N}$ if $\L_{ij}^{\mathrm{M}} = \L_{ij}^{\mathrm{N}}$ and $\eta_{ij}^{\mathrm{M}} = \eta_{ij}^{\mathrm{N}}$ $\forall i$ and j.

III. PFHSMSs WITH AGGREGATION OPERATIONS AND THEIR PROPOSITIONS

In this section we discuss some basic operations of PFHSMs. For this, let $\mathrm{M} = [\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}]$, and$\mathrm{N} = [\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{N}}] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the two FHSMs of order $\alpha \times \beta$. Then, with the condition $0 \leq \L^{2}{}_{ij}^{\mathrm{M}} + \eta^{2}{}_{ij}^{\mathrm{M}} \leq 1$, $0 \leq \L^{2}{}_{ij}^{\mathrm{N}} + \eta^{2}{}_{ij}^{\mathrm{N}} \leq 1$, we have the followings:

a. Complement of PFHSM

If $\mathrm{M} = \left[\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}\right]$ then $\mathrm{M}^{c} = \left[\eta_{ij}^{\mathrm{M}}, \L_{ij}^{\mathrm{M}}\right]$, $\forall i$ and j.

b. Union of two PFHSMs

Let $\mathrm{M} = \left[\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}\right]$, $\mathrm{N} = \left[\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{N}}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha \times \beta$, then union of M and N is defined as

$$\mathrm{M}\cup\mathrm{N} = \left[\max\left(\L_{ij}^{\mathrm{M}}, \L_{ij}^{\mathrm{N}}\right), \min\left(\eta_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{N}}\right)\right], \forall i \text{ and } j.$$

c. Intersection of two PFHSMs

Let $\mathrm{M} = \left[\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}\right]$, $\mathrm{N} = \left[\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{N}}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha \times \beta$, then intersection of M and N is defined as

$$\mathrm{M}\cap\mathrm{N} = \left[\min\left(\L_{ij}^{\mathrm{M}}, \L_{ij}^{\mathrm{N}}\right), \max\left(\eta_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{N}}\right)\right], \forall i \text{ and } j.$$

d. Product of two PFHSMs

Let $\mathrm{M} = \left[\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}\right]$, $\mathrm{N} = \left[\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{N}}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha \times \beta$, then product of M and N is defined as

$$\mathrm{M.N} = \left[\left(\L_{ij}^{\mathrm{M}}.\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{M}} + \eta_{ij}^{\mathrm{N}} - \eta_{ij}^{\mathrm{M}}.\eta_{ij}^{\mathrm{N}}\right)\right], \quad \forall i \text{ and } j.$$

e. Addition of two PFHSMs

Let $\mathrm{M} = \left[\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}\right]$, $\mathrm{N} = \left[\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{N}}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha \times \beta$. The addition of M and N is defined as $\mathrm{M}+\mathrm{N} = \left[\left(\L_{ij}^{\mathrm{M}} + \L_{ij}^{\mathrm{N}} - \L_{ij}^{\mathrm{M}}.\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{M}}.\eta_{ij}^{\mathrm{N}}\right)\right]$, $\forall i$ and j.

f. Closed product of two PFHSMs

Let $\mathrm{M} = \left[\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}\right]$, $\mathrm{N} = \left[\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{N}}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha \times \beta$, then closed product of M and N is defined as

$$\mathrm{M}\otimes\mathrm{N} = \left[\left(\L_{ij}^{\mathrm{M}}.\L_{ij}^{\mathrm{N}}, \sqrt{\eta^{2}{}_{ij}^{\mathrm{M}} + \eta^{2}{}_{ij}^{\mathrm{N}} - \eta^{2}{}_{ij}^{\mathrm{M}}.\eta^{2}{}_{ij}^{\mathrm{N}}}\right)\right], \forall i \text{ and } j.$$

g. Closed addition of two PFHSMs

Let $\mathrm{M} = \left[\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}\right]$, $\mathrm{N} = \left[\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{N}}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha \times \beta$, then closed addition of M and N is defined as

$$\mathrm{M}\oplus\mathrm{N} = \left[\left(\sqrt{\L^{2}{}_{ij}^{\mathrm{M}} + \L^{2}{}_{ij}^{\mathrm{N}} - \L^{2}{}_{ij}^{\mathrm{M}}.\L^{2}{}_{ij}^{\mathrm{N}}}, \eta_{ij}^{\mathrm{M}}.\eta_{ij}^{\mathrm{N}}\right)\right], \forall i \text{ and } j.$$

h. Arithmetic mean of two PFHSMs

Let $\mathrm{M} = \left[\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}\right]$, $\mathrm{N} = \left[\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{N}}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha \times \beta$, then arithmetic mean of M and N is defined as

$$\mathrm{M}\oplus_{m}\mathrm{N} = \left[\left(\frac{\L_{ij}^{\mathrm{M}} + \L_{ij}^{\mathrm{N}}}{2}, \frac{\eta_{ij}^{\mathrm{M}} + \eta_{ij}^{\mathrm{N}}}{2}\right)\right], \quad \forall i \text{ and } j.$$

i. Weighted arithmetic mean of two PFHSMs

Let $\mathrm{M} = \left[\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}\right]$, $\mathrm{N} = \left[\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{N}}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha \times \beta$ then weighted arithmetic mean of M and N is defined as

$$\mathrm{M}\oplus_{\omega m}\mathrm{N} = \left[\left(\frac{\omega_1\L_{ij}^{\mathrm{M}} + \omega_2\L_{ij}^{\mathrm{N}}}{\omega_1 + \omega_2}, \frac{\omega_1\eta_{ij}^{\mathrm{M}} + \omega_2\eta_{ij}^{\mathrm{N}}}{\omega_1 + \omega_2}\right)\right], \quad \forall i \text{ and } j,$$

where ω_1, ω_2 are the weights.

j. Geometric mean of two PFHSMs

Let $\mathrm{M} = \left[\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}\right]$, $\mathrm{N} = \left[\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{N}}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha \times \beta$, then geometric mean of M and N is defined as

$$\mathrm{M}\otimes_{m}\mathrm{N} = \left[\left(\sqrt{\L_{ij}^{\mathrm{M}}.\L_{ij}^{\mathrm{N}}}, \sqrt{\eta_{ij}^{\mathrm{M}}.\eta_{ij}^{\mathrm{N}}}\right)\right], \quad \forall i \text{ and } j.$$

k. Weighted geometric mean of two PFHSMs

Let $\mathrm{M} = \left[\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}\right]$, $\mathrm{N} = \left[\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{N}}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha \times \beta$, then weighted geometric mean of M and N is defined as

$$\mathrm{M}\otimes_{\omega m}\mathrm{N} = \left[\left(\left(\L_{ij}^{\mathrm{M}}\right)^{\omega_1}.\left(\L_{ij}^{\mathrm{N}}\right)^{\omega_2}\right)^{\frac{1}{\omega_1+\omega_2}}, \left(\left(\eta_{ij}^{\mathrm{M}}\right)^{\omega_1}.\left(\eta_{ij}^{\mathrm{N}}\right)^{\omega_2}\right)^{\frac{1}{\omega_1+\omega_2}}\right],$$

ω_1, ω_2 are the weights.

l. Harmonic mean of two PFHSMs

Let $\mathrm{M} = \left[\L_{ij}^{\mathrm{M}}, \eta_{ij}^{\mathrm{M}}\right]$, $\mathrm{N} = \left[\L_{ij}^{\mathrm{N}}, \eta_{ij}^{\mathrm{N}}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha \times \beta$, then harmonic mean of M and N is defined as

$$\mathrm{M}\ominus\mathrm{N} = \left[\left(\frac{2\L_{ij}^{\mathrm{M}}.\L_{ij}^{\mathrm{N}}}{\L_{ij}^{\mathrm{M}} + \L_{ij}^{\mathrm{N}}}, \frac{2\eta_{ij}^{\mathrm{M}}.\eta_{ij}^{\mathrm{N}}}{\eta_{ij}^{\mathrm{M}} + \eta_{ij}^{\mathrm{N}}}\right)\right], \quad \forall i \text{ and } j.$$

m. Weighted harmonic mean of two PFHSMs

Let $M = \left[\text{Ł}_{ij}^{M}, \eta_{ij}^{M}\right], N = \left[\text{Ł}_{ij}^{N}, \eta_{ij}^{N}\right] \in PFHSM_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha\times\beta$, then weighted harmonic mean of M and N is denoted as Θ and defined as

$$M\Theta_{\omega}N = \left[\left(\frac{\omega_1+\omega_2}{\frac{\omega_1}{\text{Ł}_{ij}^{M}}+\frac{\omega_2}{\text{Ł}_{ij}^{N}}}, \frac{\omega_1+\omega_2}{\frac{\omega_1}{\eta_{ij}^{M}}+\frac{\omega_2}{\eta_{ij}^{N}}}\right)\right], \ \forall i, j \text{ and } \omega_1, \omega_2 > 0$$

are the weights.

Example 2: Let $M = \left[\text{Ł}_{ij}^{M}, \eta_{ij}^{M}\right], N = \left[\text{Ł}_{ij}^{N}, \eta_{ij}^{N}\right] \in PFHSM_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha \times \beta$. Then the above results can be elaborated as

$$M = \begin{bmatrix} (0.5, 0.6) & (0.4, 0.5) & (0.8, 0.6) \\ (0.3, 0.7) & (0.4, 0.4) & (0.6, 0.7) \\ (0.7, 0.4) & (0.6, 0.5) & (0.6, 0.6) \end{bmatrix}$$

$$N = \begin{bmatrix} (0.2, 0.7) & (0.6, 0.5) & (0.5, 0.5) \\ (0.4, 0.4) & (0.4, 0.8) & (0.4, 0.5) \\ (0.3, 0.6) & (0.4, 0.5) & (0.6, 0.6) \end{bmatrix}$$

Then, the above all mentioned are as below:

$$M^c = \begin{bmatrix} (0.6, 0.5) & (0.5, 0.4) & (0.6, 0.8) \\ (0.7, 0.3) & (0.4, 0.4) & (0.7, 0.6) \\ (0.4, 0.7) & (0.5, 0.6) & (0.6, 0.6) \end{bmatrix}$$

$$M \cup N = \begin{bmatrix} (0.5, 0.7) & (0.6, 0.5) & (0.8, 0.6) \\ (0.4, 0.7) & (0.4, 0.8) & (0.6, 0.7) \\ (0.7, 0.6) & (0.6, 0.5) & (0.6, 0.6) \end{bmatrix}$$

$$M \cap N = \begin{bmatrix} (0.2, 0.6) & (0.4, 0.5) & (0.5, 0.5) \\ (0.4, 0.7) & (0.4, 0.8) & (0.6, 0.7) \\ (0.7, 0.6) & (0.6, 0.5) & (0.6, 0.6) \end{bmatrix}$$

$$M.N = \begin{bmatrix} (0.2, 0.6) & (0.4, 0.5) & (0.5, 0.5) \\ (0.4, 0.7) & (0.4, 0.8) & (0.6, 0.7) \\ (0.7, 0.6) & (0.6, 0.5) & (0.6, 0.6) \end{bmatrix}$$

$$M + N = \begin{bmatrix} (0.2, 0.6) & (0.4, 0.5) & (0.5, 0.5) \\ (0.4, 0.7) & (0.4, 0.8) & (0.6, 0.7) \\ (0.7, 0.6) & (0.6, 0.5) & (0.6, 0.6) \end{bmatrix}$$

Proposition 1: Let M and N $\in PFHSM_{\alpha\times\beta}$ be the two PFHSMs. The following axioms hold:

(i) $M \cup N = N \cup M$
(ii) $M \cap N = N \cap M$
(iii) $M + N = N + M$
(iv) $M.N = N.M$
(v) $(M^c + N^c)^c = M.N$
(vi) $(M^c.N^c)^c = M + N$

Proof: We prove (i)∼ (vi) as follows.

(i) $M \cup N = \left[\max\left(\text{Ł}_{ij}^{M}, \text{Ł}_{ij}^{N}\right), \min\left(\eta_{ij}^{M}, \eta_{ij}^{N}\right)\right] = \left[\max\left(\text{Ł}_{ij}^{N}, \text{Ł}_{ij}^{M}\right), \min\left(\eta_{ij}^{N}, \eta_{ij}^{M}\right)\right] = M \cup N.$

(ii) $M \cap N = \left[\min\left(\text{Ł}_{ij}^{M}, \text{Ł}_{ij}^{N}\right), \max\left(\eta_{ij}^{M}, \eta_{ij}^{N}\right)\right] = \left[\min\left(\text{Ł}_{ij}^{N}, \text{Ł}_{ij}^{M}\right), \max\left(\eta_{ij}^{N}, \eta_{ij}^{M}\right)\right] = N \cap M.$

(iii) $M + N = \left[\left(\text{Ł}_{ij}^{M} + \text{Ł}_{ij}^{N} - \text{Ł}_{ij}^{M}.\text{Ł}_{ij}^{N}, \eta_{ij}^{M}.\eta_{ij}^{N}\right)\right] = \left[\left(\text{Ł}_{ij}^{N} + \text{Ł}_{ij}^{M} - \text{Ł}_{ij}^{N}.\text{Ł}_{ij}^{M}, \eta_{ij}^{N}.\eta_{ij}^{M}\right)\right] = N + M.$

(iv) $M.N = \left[\left(\text{Ł}_{ij}^{M}.\text{Ł}_{ij}^{N}, \eta_{ij}^{M} + \eta_{ij}^{N} - \eta_{ij}^{M}.\eta_{ij}^{N}\right)\right] = \left[\left(\text{Ł}_{ij}^{N}.\text{Ł}_{ij}^{M}, \eta_{ij}^{N} + \eta_{ij}^{M} - \eta_{ij}^{N}.\eta_{ij}^{M}\right)\right] = N.M.$

(v) Since $M = \left[\text{Ł}_{ij}^{M}, \eta_{ij}^{M}\right], N = \left[\text{Ł}_{ij}^{N}, \eta_{ij}^{N}\right] \in PFHSM_{\alpha\times\beta}$, $M^c = \left[\eta_{ij}^{M}, \text{Ł}_{ij}^{M}\right], N^c = \left[\eta_{ij}^{N}, \text{Ł}_{ij}^{N}\right]$, and $M^c + N^c = \left[\eta_{ij}^{M} + \eta_{ij}^{N} - \eta_{ij}^{M}.\eta_{ij}^{N}, \text{Ł}_{ij}^{M}.\text{Ł}_{ij}^{N}\right]$. Then, we have that

$$\left(M^c + N^c\right)^c = \left[\text{Ł}_{ij}^{M}.\text{Ł}_{ij}^{N}, \eta_{ij}^{M} + \eta_{ij}^{N} - \eta_{ij}^{M}.\eta_{ij}^{N}\right] = M.N.$$

(vi) Since $M^c = \left[\eta_{ij}^{M}, \text{Ł}_{ij}^{M}\right], N^c = \left[\eta_{ij}^{N}, \text{Ł}_{ij}^{N}\right]$, so $M^c.N^c = \left[\eta_{ij}^{M}.\eta_{ij}^{N}, \text{Ł}_{ij}^{M} + \text{Ł}_{ij}^{N} - \text{Ł}_{ij}^{M}.\text{Ł}_{ij}^{N}\right]$. Thus, $(M^c.N^c)^c = \left[\text{Ł}_{ij}^{M} + \text{Ł}_{ij}^{N} - \text{Ł}_{ij}^{M}.\text{Ł}_{ij}^{N}, \eta_{ij}^{M}.\eta_{ij}^{N}\right] = M + N.$ ■

Proposition 2: Let M and N $\in PFHSM_{\alpha\times\beta}$ be the two PFHSMs. The D′ Morgan Law holds as follows:

(i) $(M \cap N)^c = M^c \cup N^c$.
(ii) $(M \cup N)^c = M^c \cap N^c$.

Proof: We prove (i)∼ (ii) as follows.

(i) $M \cap N = ([(\text{Ł}_{ij}^{M}, \eta_{ij}^{M})] \cap [(\text{Ł}_{ij}^{N}, \eta_{ij}^{N})]) = [\min(\text{Ł}_{ij}^{M}, \text{Ł}_{ij}^{N}), \max(\eta_{ij}^{M}, \eta_{ij}^{N})]$. Then, $(M \cap N)^c = [\max(\eta_{ij}^{M}, \eta_{ij}^{N}), \min(\text{Ł}_{ij}^{M}, \text{Ł}_{ij}^{N})] = ([(\eta_{ij}^{M}, \text{Ł}_{ij}^{M})] \cup [(\eta_{ij}^{N}, \text{Ł}_{ij}^{N})]) = M^c \cup N^c$.

(ii) $M \cup N = ([(\text{Ł}_{ij}^{M}, \eta_{ij}^{M})] \cup [(\text{Ł}_{ij}^{N}, \eta_{ij}^{N})]) = [\max(\text{Ł}_{ij}^{M}, \text{Ł}_{ij}^{N}), \min(\eta_{ij}^{M}, \eta_{ij}^{N})]$. Then, $(M \cap N)^c = [\min(\eta_{ij}^{M}, \eta_{ij}^{N}), \max(\text{Ł}_{ij}^{M}, \text{Ł}_{ij}^{N})] = ([(\eta_{ij}^{M}, \text{Ł}_{ij}^{M})] \cap [(\eta_{ij}^{N}, \text{Ł}_{ij}^{N})]) = M^c \cap N^c$. ■

Proposition 3: Let M and N $\in PFHSM_{\alpha\times\beta}$ be the two PFHSMs, then the following results are satisfied.

(i) $(M^c)^c = M$
(ii) $M \cup M = M$
(iii) $M \cap M = M$

Proof: All results of (i)∼ (iii) are easily to be verified. ■

Proposition 4: Let M and N $\in PFHSM_{\alpha\times\beta}$ be the two PFHSMs, then the following results holds for arithmetic, geometric and harmonic means, respectively, and also for their weighted versions.

(i) $M \oplus_m N = N \oplus_m M$
(ii) $M \otimes_m N = N \otimes_m M$
(iii) $M\Theta N = N\Theta M$
(iv) $M \oplus_{\omega m} N = N \oplus_{\omega m} M$
(v) $M \otimes_{\omega m} N = N \otimes_{\omega m} M$
(vi) $M\Theta_{\omega} N = N\Theta_{\omega} M$

Proof: The proof of the six properties are straightforward. ■

Proposition 5: Let M and N $\in PFHSM_{\alpha\times\beta}$ be the two PFHSMs, then the following results relating to complements holds for arithmetic, geometric and harmonic means, respectively, and also for their weighted **versions.**

(i) $(M^c \oplus_m N^c)^c = M \oplus_m N$.
(ii) $(M^c \otimes_m N^c)^c = M \otimes_m N$.
(iii) $(M^c \Theta N^c)^c = N\Theta M$.
(iv) $(M^c \oplus_{\omega m} N^c)^c = N \oplus_{\omega m} M$.

(v) $(M^c \otimes_{\omega m} N^c =)^c N \otimes_{\omega m} M$.
(vi) $(M^c \Theta_\omega N^c)^c = N\Theta_\omega M$.

Proof: For (i) with $(M^c \oplus_m N^c)^c = M \oplus_m N$. Let $M = \left[Ⱡ_{ij}^M, \eta_{ij}^M \right], N = \left[Ⱡ_{ij}^N, \eta_{ij}^N \right] \in PFHSM_{\alpha\times\beta}$, then $M^c = \left[\eta_{ij}^M, Ⱡ_{ij}^M \right], N^c = \left[\eta_{ij}^N, Ⱡ_{ij}^N \right]$. So $M^c \oplus_m N^c = \left[\left(\frac{\eta_{ij}^M+\eta_{ij}^N}{2}, \frac{Ⱡ_{ij}^M+Ⱡ_{ij}^N}{2} \right) \right]$ and $(M^c \oplus N^c)^c = \left[\left(\frac{Ⱡ_{ij}^M+Ⱡ_{ij}^N}{2}, \frac{\eta_{ij}^M+\eta_{ij}^N}{2} \right) \right] = M \oplus_m N$. Similarly, the remaining propositions can be proved straightforward. ■

Definition 9: Let $M = \left[Ⱡ_{ij}^M, \eta_{ij}^M \right], N = \left[Ⱡ_{ij}^N, \eta_{ij}^N \right] \in PFHSM_{\alpha\times\beta}$ be the two PFHSMs of order $\alpha \times \beta$. Then Max-Min product of PFHSMs is defined as

$$\begin{aligned} &M * N \\ &= [Ꞇ_{ik}]_{m\times p} \\ &= \left[\left\{ \max\left(\min_j \left(Ⱡ_{ij}^M, Ⱡ_{ij}^N \right) \right), \min\left(\max_j \left(\eta_{ij}^M, \eta_{ij}^N \right) \right) \right\} \right], \\ &\forall i, j \text{ and } k. \end{aligned}$$

Proposition 6: Let M, N and $P \in PFHSM_{\alpha\times\beta}$ be the three PFHSMs. Then, associative laws under union, intersection, plus, multiplication, arithmetic, geometric and harmonic means hold.

(i) $(M \cup N) \cup P = M \cup (N \cup P)$
(ii) $(M \cap N) \cap P = M \cap (N \cap P)$
(iii) $(M + N) + P = M + (N + P)$
(iv) $(M.N).P = M.(N.P)$
(v) $(M \oplus N) \oplus P = M \oplus (N \oplus P)$
(vi) $(M \otimes N) \otimes P = M \otimes (N \otimes P)$
(vii) $(M\Theta N)\Theta P = M\Theta(N\Theta P)$

Proof: Let $M = \left[Ⱡ_{ij}^M, \eta_{ij}^M \right], N = \left[Ⱡ_{ij}^N, \eta_{ij}^N \right]$, and $P = \left[Ⱡ_{ij}^P, \eta_{ij}^P \right] \in PFHSM_{\alpha\times\beta}$. Then,

For (i) with $(M \cup N) \cup P = M \cup (N \cup P)$, we have $M \cup N = \left[\max\left(Ⱡ_{ij}^M, Ⱡ_{ij}^N\right), \min\left(\eta_{ij}^M, \eta_{ij}^N\right) \right]$ and $(M \cup N) \cup P = \left[\max\left(Ⱡ_{ij}^M, Ⱡ_{ij}^N\right), \min\left(\eta_{ij}^M, \eta_{ij}^N\right) \right] \cup \left[\left(Ⱡ_{ij}^P, \eta_{ij}^P\right) \right] = \left[\max \left(Ⱡ_{ij}^M, Ⱡ_{ij}^N, Ⱡ_{ij}^P\right), \min\left(\eta_{ij}^M, \eta_{ij}^N, \eta_{ij}^P\right) \right] = \left[\max\left(Ⱡ_{ij}^M, \left(Ⱡ_{ij}^N, Ⱡ_{ij}^P\right)\right), \min \left(\eta_{ij}^M, \left(\eta_{ij}^N, \eta_{ij}^P\right)\right) \right] = \left[\left(Ⱡ_{ij}^M, \eta_{ij}^M\right) \right] \cup \left[\max\left(Ⱡ_{ij}^N, Ⱡ_{ij}^P\right), \min \left(\eta_{ij}^N, \eta_{ij}^P\right) \right] = M \cup (N \cup P)$.

For (ii) with $(M \cap N) \cap P = M \cap (N \cap P)$, we have $M \cap N = \left[\min\left(Ⱡ_{ij}^M, Ⱡ_{ij}^N\right), \max\left(\eta_{ij}^M, \eta_{ij}^N\right) \right]$ and

$$\begin{aligned} &(M \cap N) \cap P \\ &= \left[\min\left(\left(Ⱡ_{ij}^M, Ⱡ_{ij}^N\right), Ⱡ_{ij}^P \right), \max\left(\left(\eta_{ij}^M, \eta_{ij}^N\right), \eta_{ij}^P \right) \right] \\ &= \left[\min\left(Ⱡ_{ij}^M, Ⱡ_{ij}^N, Ⱡ_{ij}^P\right), \max\left(\eta_{ij}^M, \eta_{ij}^N, \eta_{ij}^P\right) \right] \\ &= \left[\min\left(Ⱡ_{ij}^M, \left(Ⱡ_{ij}^N, Ⱡ_{ij}^P\right)\right), \max\left(\eta_{ij}^M, \left(\eta_{ij}^N, \eta_{ij}^P\right)\right) \right] \\ &= \left[\left(Ⱡ_{ij}^M, \eta_{ij}^M\right) \right] \cap \left[\min\left(Ⱡ_{ij}^N, Ⱡ_{ij}^P\right), \max\left(\eta_{ij}^N, \eta_{ij}^P\right) \right] \\ &= M \cap (N \cap P). \end{aligned}$$

For (iii) with $(M + N) + P = M + (N + P)$, we have $(M + N) = \left[\left(\left(Ⱡ_{ij}^M + Ⱡ_{ij}^N - Ⱡ_{ij}^M.Ⱡ_{ij}^N\right), \left(\eta_{ij}^M.\eta_{ij}^N\right) \right) \right]$ and $(M + N) + P = \left[\left(\left(Ⱡ_{ij}^M + Ⱡ_{ij}^N - Ⱡ_{ij}^M.Ⱡ_{ij}^N\right), \left(\eta_{ij}^M.\eta_{ij}^N\right) \right) \right] + \left[Ⱡ_{ij}^P, \eta_{ij}^P \right] = \left[\left(\left(\left(Ⱡ_{ij}^M + Ⱡ_{ij}^N\right) + Ⱡ_{ij}^P - \left(Ⱡ_{ij}^M.Ⱡ_{ij}^N\right).Ⱡ_{ij}^P \right), \left(\left(\eta_{ij}^M.\eta_{ij}^N\right).\eta_{ij}^P \right) \right) \right] = \left[\left(\left(Ⱡ_{ij}^M + \left(Ⱡ_{ij}^N + Ⱡ_{ij}^P\right) - Ⱡ_{ij}^M.\left(Ⱡ_{ij}^N.Ⱡ_{ij}^P\right) \right), \left(\eta_{ij}^M.\left(\eta_{ij}^N.\eta_{ij}^P\right) \right) \right) \right] = \left[\left(\left(Ⱡ_{ij}^M + \left(Ⱡ_{ij}^N + Ⱡ_{ij}^P\right) - Ⱡ_{ij}^M.\left(Ⱡ_{ij}^N.Ⱡ_{ij}^P\right) \right), \left(\eta_{ij}^M.\left(\eta_{ij}^N.\eta_{ij}^P\right) \right) \right) \right] = \left[Ⱡ_{ij}^M, \eta_{ij}^M \right] + \left[\left(\left(Ⱡ_{ij}^N + Ⱡ_{ij}^P - Ⱡ_{ij}^N.Ⱡ_{ij}^P\right), \left(\eta_{ij}^N.\eta_{ij}^P\right) \right) \right] = M + (N + P)$.

For (iv) $\sim$ (vii), similar proofs are followed. ■

Proposition 7: Let M, N and $R \in PFHSM_{\alpha\times\beta}$ be the three PFHSMs. The distributive laws under union, intersection, plus, multiplication, arithmetic, geometric and harmonic means hold.

(i) $M \cap (N \cup P) = (M \cap N) \cup (M \cap P)$
(ii) $M \cup (N \cap P) = (M \cup N) \cap (M \cup P)$
(iii) $(M \cap N) \cup P = (M \cup P) \cap (M \cup P)$
(iv) $(M \cup N) \cap P = (M \cap P) \cup (M \cap P)$

Proof: We prove some of the above results as follows. Let $M = \left[Ⱡ_{ij}^M, \eta_{ij}^M \right], N = \left[Ⱡ_{ij}^N, \eta_{ij}^N \right]$, and $P = \left[Ⱡ_{ij}^P, \eta_{ij}^P \right] \in PFHSM_{\alpha\times\beta}$.

For (i), we have $M \cap N = \left[\min\left(Ⱡ_{ij}^M, Ⱡ_{ij}^N\right), \max\left(\eta_{ij}^M, \eta_{ij}^N\right) \right]$, and $(M \cap N) \cup P = \left[\min\left(Ⱡ_{ij}^M, Ⱡ_{ij}^N\right), \max\left(\eta_{ij}^M, \eta_{ij}^N\right) \right] \cup \left[\left(Ⱡ_{ij}^P, \eta_{ij}^P\right) \right] = \left[\max\left(\min\left(Ⱡ_{ij}^M, Ⱡ_{ij}^N\right), Ⱡ_{ij}^P \right), \min\left(\max\left(\eta_{ij}^M, \eta_{ij}^N\right), \eta_{ij}^P \right) \right]$. Now, $(M \cap P) \cup (N \cap P) = \left[\min\left(Ⱡ_{ij}^M, Ⱡ_{ij}^P\right), \max\left(\eta_{ij}^M, \eta_{ij}^P\right) \right] \cup \left[\min\left(Ⱡ_{ij}^N, Ⱡ_{ij}^P\right), \max\left(\eta_{ij}^N, \eta_{ij}^P\right) \right] = \left[\max\left\{ \min\left(Ⱡ_{ij}^M, Ⱡ_{ij}^P\right), \min\left(Ⱡ_{ij}^N, Ⱡ_{ij}^P\right) \right\}, \min\left\{ \max\left(\eta_{ij}^M, \eta_{ij}^P\right), \max\left(\eta_{ij}^N, \eta_{ij}^P\right) \right\} \right] = \left[\max\left\{ \min\left(Ⱡ_{ij}^M, Ⱡ_{ij}^N\right), Ⱡ_{ij}^P \right\}, \min\left\{ \max\left(\eta_{ij}^M, \eta_{ij}^N\right), \eta_{ij}^P \right\} \right] = (M \cap N) \cup P$.

For (ii), we have $M \cup N = \left[\max\left(Ⱡ_{ij}^M, Ⱡ_{ij}^N\right), \min\left(\eta_{ij}^M, \eta_{ij}^N\right) \right]$ and $(M \cup N) \cap P = \left[\max\left(Ⱡ_{ij}^M, Ⱡ_{ij}^N\right), \min\left(\eta_{ij}^M, \eta_{ij}^N\right) \right] \cap \left[\left(Ⱡ_{ij}^P, \eta_{ij}^P\right) \right] = \left[\min\left(\max\left(Ⱡ_{ij}^M, Ⱡ_{ij}^N\right), Ⱡ_{ij}^P \right), \max\left(\min\left(\eta_{ij}^M, \eta_{ij}^N\right), \eta_{ij}^P \right) \right]$. Now, $(M \cup P) \cap (N \cup P) = \left[\left[\max\left(Ⱡ_{ij}^M, Ⱡ_{ij}^P\right), \min\left(\eta_{ij}^M, \eta_{ij}^P\right) \right] \cap \left[\max\left(Ⱡ_{ij}^N, Ⱡ_{ij}^P\right), \min\left(\eta_{ij}^N, \eta_{ij}^P\right) \right] \right] = \left[\min\left\{ \max\left(Ⱡ_{ij}^M, Ⱡ_{ij}^P\right), \max\left(Ⱡ_{ij}^N, Ⱡ_{ij}^P\right) \right\}, \max\left\{ \min\left(\eta_{ij}^M, \eta_{ij}^P\right), \min\left(\eta_{ij}^N, \eta_{ij}^P\right) \right\} \right] = \left[\min\left\{ \max\left(Ⱡ_{ij}^M, Ⱡ_{ij}^N\right), Ⱡ_{ij}^P \right\}, \max\left\{ \min\left(\eta_{ij}^M, \eta_{ij}^N\right), \eta_{ij}^P \right\} \right] = (M \cup N) \cap P$.

For (iii) and (vi), similar proofs are followed. ■

Proposition 8: Let M, N and $P \in PFHSM_{\alpha\times\beta}$ be the three PFHSMs. The distributive laws under union, intersection, plus, multiplication, arithmetic, geometric and harmonic means hold.

(i) $(M \cap N) + P = (M + P) \cap (M + P)$.
(ii) $(M \cap N).P = (M.P) \cap (M.P)$.
(iii) $(M \cap N) \oplus P = (M \oplus P) \cap (M \oplus P)$.
(iv) $(M \cap N) \otimes P = (M \otimes P) \cap (M \otimes P)$.
(v) $(M \cup N) + P = (M + P) \cup (M + P)$.
(vi) $(M \cup N).P = (M.N) \cup (M.P)$.
(vii) $(M \cup N) \oplus P = (M \oplus P) \cup (M \oplus P)$.
(viii) $(M \cup N) \otimes P = (M \otimes P) \cup (M \otimes P)$.
(ix) $(M \cap N)\Theta P = (M\Theta P) \cap (M\Theta P)$.
(x) $(M \cup N)\Theta P = (M\Theta P) \cup (M\Theta P)$.
(xi) $M\Theta(N \cup P) = (M\Theta N) \cup (M\Theta P)$.
(xii) $M\Theta(N \cap P) = (M\Theta N) \cap (M\Theta P)$.

Proof: These above results can be verified by using different order conditions on truthiness and falseness. ∎

Next, we are going to propose some new algorithms in the environment of PFHSMs. We present the two algorithms and then apply the proposed algorithms in real life problems. We also give the comparison with existing techniques.

IV. THE PROPOSED DECISION-MAKING ALGORITHMS WITH APPLICATION TO THE SELECTION OF WASTEWATER TREATMENT PLANTS

In this section, we propose two decision-making algorithms. We then apply them to two real applications, the selection of mobile phones and wastewater treatment plants.

A. THE PROPOSED DECISION-MAKING ALGORITHMS BASED ON PFHSMs

First, with the help of the following defined choice and weighted choices matrices, we propose the PFHSM Algorithm 1 with its flowchart, as shown in Fig. 2.

Definition 10: Let $\mathrm{M} = \left[\text{Ł}^{\mathrm{M}}_{ij}, \eta^{\mathrm{M}}_{ij}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$. The choice matrix can be defined as

$$\mathbb{C}\left(\mathrm{M}\right) = \left[\left(\frac{\sum_{j=1}^{n}\left(\text{Ł}^{\mathrm{M}}_{ij}\right)^{2}}{n}, \frac{\sum_{j=1}^{n}\left(\eta^{\mathrm{M}}_{ij}\right)^{2}}{n}\right)\right]_{m\times 1} \quad \text{for all } i.$$

Definition 11: Let $\mathrm{M} = \left[\text{Ł}^{\mathrm{M}}_{ij}, \eta^{\mathrm{M}}_{ij}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$. The weighted choice matrix is defined as

$$\mathbb{C}\left(\mathrm{M}\right) = \left[\left(\frac{\sum_{j=1}^{n}\omega_j\left(\text{Ł}^{\mathrm{M}}_{ij}\right)^{2}}{\sum\omega_j}, \frac{\sum_{j=1}^{n}\omega_j\left(\eta^{\mathrm{M}}_{ij}\right)^{2}}{\sum\omega_j}\right)\right]_{m\times 1}$$
$$\text{forall } i, \text{ and } \omega_j > 0.$$

Thus, the proposed PFHSM algorithm 1 is summarized as follows:

PFHSM Algorithm 1

Step 1: Construct the PFHSM from PFHSS.
Step 2: Choose one of the following cases.
- **Case 1:** Equal weights- - Compute the choice matrix of membership and non-membership values PFHSM according to Definition 12.
- **Case 2:** Unequal weights- - Compute the weighted choice matrix of membership and non-membership values PFHSM according to Definition 11.

Step 3: Select the alternatives with the highest value.
Step 4: End– Output the best alternative.

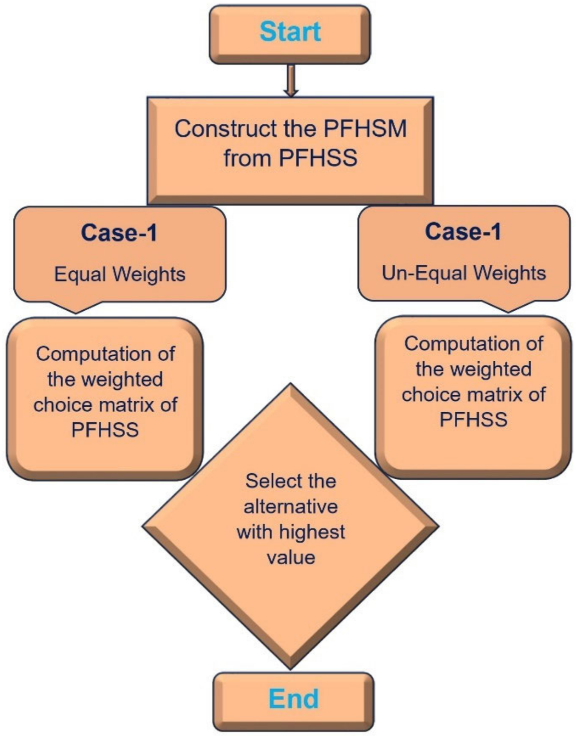

FIGURE 2. The flowchart of the proposed algorithm with choice and weighted choice matrices.

Example 3: Zaviyan wants to buy a mobile for her sister Zahra and he has an option to select a mobile out the set of five mobiles(alternatives), say $\mathcal{Z} = \{z_1, z_2, z_3, z_4, z_5\}$. Consider the five attributes of RAM, ROM, display size, and battery which further sub-divided into the attributive values with $\mathfrak{Q}^{a}_{1} = \mathrm{RAM} = \{6\mathrm{GB}, 8\mathrm{GB}\}$, $\mathfrak{Q}^{b}_{2} = \mathrm{ROM} = \{64\mathrm{GB}, 128\mathrm{GB}\}$, $\mathfrak{Q}^{c}_{3} = \mathrm{DisplaySize} = \left\{6'', 6.5'', 7''\right\}$, $\mathfrak{Q}^{d}_{4} = \mathrm{Battery} = \{4000\mathrm{mh}, 5000\mathrm{mh}, 6000\mathrm{mh}\}$. The decision making from Zaviyan for purchasing the best mobile made through hypersoft structure with the mapping $\acute{K} : \mathfrak{Q}^{a}_{1} \times \mathfrak{Q}^{b}_{2} \times \mathfrak{Q}^{c}_{3} \times \mathfrak{Q}^{d}_{4} \rightarrow \mathcal{P}(\Psi)$. Then, $\left(\acute{K}, \mathfrak{Q}^{a}_{1} \times \mathfrak{Q}^{b}_{2} \times \mathfrak{Q}^{c}_{3} \times \mathfrak{Q}^{d}_{4}\right) = \{z_3, z_4, z_5\}$ with $\acute{K}\left(8\mathrm{GB}, 128\mathrm{GB}, 6'', 5000\mathrm{mh}\right) = \{z_3, z_4, z_5\}$. By Pythagorean fuzzy hypersoft structure, we have $\acute{K}(8\mathrm{GB}, 128\mathrm{GB}, 6'', 5000\mathrm{mh}) = \{z_3, z_4, z_5\}$ for PFHSS to PFHSM as follows:

Step 1: PFHSM based on the Zaviyan's selected criterion with

$$\mathrm{M} = \begin{array}{c} \\ z_3 \\ z_4 \\ z_5 \end{array}\begin{array}{cccc} 8\mathrm{GB} & 128\mathrm{GB} & 6'' & 5000mh \\ \end{array}\left[\begin{array}{cccc} (0.5, 0.6) & (0.4, 0.5) & (0.8, 0.6) & (0.4, 0.6) \\ (0.3, 0.7) & (0.4, 0.4) & (0.6, 0.7) & (0.9, 0.6) \\ (0.7, 0.4) & (0.6, 0.5) & (0.6, 0.6) & (0.7, 0.3) \end{array}\right].$$

Case 1: Choice matrix of PFHSM with

$$\mathbb{C}\left(\mathrm{M}\right) = \left[\begin{array}{c} (0.3025, 0.3325) \\ (0.3550, 0.3750) \\ (0.4250, 0.2150) \end{array}\right].$$

Here, we can see that z_5 has a max value of truthiness and the least value of falseness, and so Zaviyan picked the optimal valuable mobile z_5.

Case 2: Weighted choice matrix of PFHSM

If the decision maker thinks that something is more important than others, then we have to use the weights whose sum is always equal to 1. So, let us introduce the weights to be $\omega_1 = 0.2, \omega_2 = 0.3, \omega_3 = 0.1, \omega_4 = 0.4$ with $\sum_1^4 \omega_j = 1$. Then,

$$\mathrm{M} = \begin{matrix} & \text{8GB} & \text{128GB} & 6'' & 5000mh \\ z_3 & (0.5, 0.6) & (0.4, 0.5) & (0.8, 0.6) & (0.4, 0.6) \\ z_4 & (0.3, 0.7) & (0.4, 0.4) & (0.6, 0.7) & (0.9, 0.6) \\ z_5 & (0.7, 0.4) & (0.6, 0.5) & (0.6, 0.6) & (0.7, 0.3) \end{matrix},$$

$$\text{and } \mathbb{C}(\mathrm{M}) = \begin{bmatrix} (0.2260, 0.3270) \\ (0.4260, 0.3390) \\ (0.4380, 0.1790) \end{bmatrix}.$$

In the above weighted choice matrix, we can observe that the preference again settles on z_5, and so according to both choice matrices, our calculated result is the selection of z_5 as the best selection for Zaviyan.

We next propose the PFHSM Algorithm 2 and then we employee the algorithm in the selection of wastewater treatment plants (WWTPs) that can be well solved by using the proposed algorithm. For this, we need to propose two definitions to build the structure of algorithm.

Definition 12: Let $\mathrm{M} = \left[Ł^{\mathrm{M}}_{ij}, \eta^{\mathrm{M}}_{ij}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the PFHSM. The value matrix is defined as

$$\zeta(\mathrm{M}) = \left[\left(\left(Ł^{\mathrm{M}}{}_{ij}\right)^2 - \left(\eta^{\mathrm{M}}{}_{ij}\right)^2\right)\right], \quad \forall i \text{ and } j.$$

Furthermore, it should be highlighted that the (i, j)th element of the value matrix serves as yet another crucial index for assessing the relative belongingness and non-belongingness.

Definition 13: Let $\mathrm{M} = \left[Ł^{\mathrm{M}}_{ij}, \eta^{\mathrm{M}}_{ij}\right], \mathsf{N} = \left[Ł^{\mathsf{N}}_{ij}, \eta^{\mathsf{N}}_{ij}\right] \in \mathrm{PFHSM}_{\alpha\times\beta}$ be the two PFHSM. The utility matrix is defined as

$$\mu(\mathsf{M}, \mathsf{N}) = \left[\mathsf{u}_{ij}\right]_{m\times n} = \boldsymbol{\zeta}(\mathrm{M}) + \zeta(\mathsf{N}), \quad \forall i \text{ and } j.$$

Furthermore, it should be highlighted that the (i, j)th element of the utility matrix serves as yet another crucial index for assessing the relative belongingness and non-belongingness.

Definition 14: The score matrix is a column matrix defined by $S(M) = \left|\sum_{j=1}^{n} \mathsf{u}_{ij}\right|$, and the highest value against alternatives is the optimal solution.

Next, we are going to propose the algorithm by using the above definitions of score and utility matrices, and then apply it in the selection of WWTPs. The proposed PFHSM algorithm 2 is summarized as follows:

Our next goal is to apply the proposed algorithm 2 on selecting the best WWTP under defined criteria. First of all, we should develop two PFHSSs that have the relation between criteria and alternatives and then follow the algorithm. The flowchart of the proposed algorithmis shown in Fig. 3.

The PFHSM Algorithm 2

Step 1: Construction PFHSSs from decision makers.
Step 2: Convert PFHSSs to PFHSMs.
Step 3: Calculate the score matrix by Definition 11.
Step 4: Calculate the utility matrix by Definition 12.
Step 5: Select the optimal solution with the highest value.
Step 6: End- Output the best one.

B. APPLICATION TO THE SELECTION OF WASTEWATER TREATMENT PLANTS

Wastewater treatment is crucial for preserving the environment and the public health. It comprises purifying wastewater of contaminants and pollutants so that it may be utilized for other things or discharged safely into the environment. Facilities designed to clean and filter wastewater before it is released back into the environment are known as water reclamation facilities or wastewater treatment plants (WWTPs). They are essential to protecting the environment and public health by removing toxins from domestic, industrial, and commercial sewage. In this section, we are going to apply our proposed algorithms in the selection of WWTPs. For better understanding, we first discuss about five WWTPs as below.

FIGURE 3. The flowchart of the proposed algorithm 2.

1) ACTIVATED SLUDGE

In many different businesses, industrial effluent is cleaned utilizing a biological procedure. Carbon, nitrogen, ammonium, and phosphorus are taken out as living components.

It uses a biological floc settling tank and an aeration tank. To create biological floc, air is introduced to primary treated industrial wastewater that contains organic organisms. Industrial water and biological substances are combined to make mixed spirits. The amount of biological waste is decreased because of a chemical and biological interaction that occurs in the aeration tank. Waste removal is influenced by a wide range of variables, including time, influent load, temperature, and oxygen availability [36].

2) SEQUENTIAL BATCH REACTOR

Although it assumes many forms, its basic objective and guiding principles never change. One or more tanks could be presented. Untreated industrial water can enter the tanks through an inlet valve, and treated water can escape through an output valve. The process includes multiple phases. To begin the process, untreated industrial effluent is first added to the tank. In the initial phase, raw wastewater is mechanically mixed without air. Tiny bubble diffusers at the bottom of the tank let air through for aeration. The next step is to settle the suspended solids. At last, the outflow valve of the tank opens, enabling the cleaned water to drain [37], [38].

3) CONSTRUCTED WETLAND

It is an environmentally friendly method of treating industrial effluent. Prior to utilizing this method, industrial wastewater should occasionally—but not always—be primarily treated. Numerous plants and herbs are grown in the swamp. Microorganisms grow and degrade the organic industrial waste that is presented in the industrial water on the roots, leaves, and stems of these plants. Wetland plants breathe out fresh air after absorbing carbon dioxide [39], [40].

4) ANAEROBIC LAGOON

It is a substantial and profound type of basin on the earth. It is employed in a number of processes, such as the decomposition of sludge, the breakdown of water-soluble organic compounds, and the settling of suspended materials. Industrial wastewater and microbes both enter the basin. While smells are controlled by surface aeration, the process is kept going in an anaerobic environment. Surface aeration is essential because, without it, heat and an unpleasant scent are produced. Methane gas produced by this method can be collected and used in other operations [41].

5) MEMBRANE FILTRATION

This effective method to filter pollutants from industrial wastewater uses membranes with different porosity diameters. The membranes must be pushed through with the industrial effluent. Even micron-sized particles are removed from the wastewater by the membrane, outperforming conventional wastewater treatment solutions in terms of effluent quality [42], [43].

6) METHODOLOGY

Consider that we have a set of five alternatives WWTP's $A = \{A_1, A_2, A_3, A_4, A_5\}$ which are Activated Sludge (AS-A_1), Sequential Batch Reactor(SBR $-A_2$), Constructed Wetland (CW-A_3), Anaerobic Lagoon (AL-A_4), and Membrane Filtration (MF-A_5). There are many criteria relating to the above discussed alternatives, but we will suppose some criteria $C = \{C_1, C_2, C_3, C_4, C_5\}$ which are Chemical Consumption (CC)-(C_1), Sludge Production (SP)-(C_2), Environmental Impact(EI)-(C_3), Energy Consumption(EC)-(C_4), Efficency(E)-(C_5). These all five criteria further sub-divided into many sub-criteria as shown in Fig. 4.

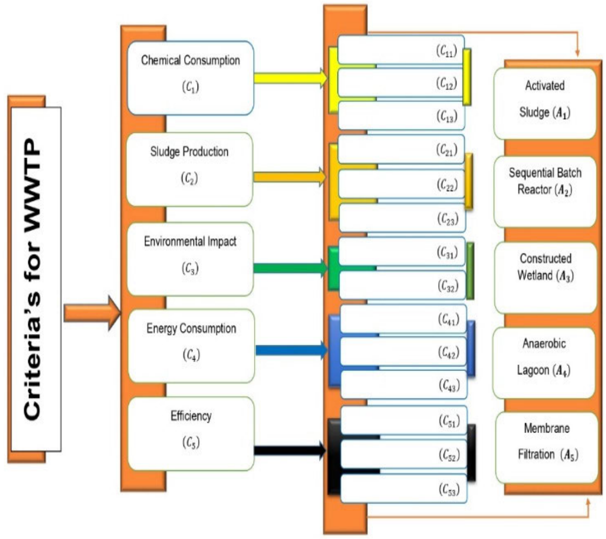

FIGURE 4. Structure of parameters and alternatives.

TABLE 2. Representation of PFHSM-P.

Criteria / Alternatives	C_1	C_2	C_3	C_4	C_5
A_1	(0.5,0.3)	(0.6,0.2)	(0.4,0.7)	(0.5,0.6)	(0.9,0.3)
A_2	(0.6,0.2)	(0.5,0.4)	(0.5,0.6)	(0.7,0.4)	(0.7,0.2)
A_3	(0.3,0.6)	(0.4,0.5)	(0.6,0.7)	(0.8,0.4)	(0.5,0.5)
A_4	(0.6,0.6)	(0.3,0.6)	(0.8,0.5)	(0.5,0.7)	(0.6,0.3)

TABLE 3. Representation of PFHSM-Q.

Criteria / Alternatives	C_1	C_2	C_3	C_4	C_5
A_1	(0.4,0.6)	(0.8,0.4)	(0.5,0.6)	(0.6,0.7)	(0.7,0.4)
A_2	(0.3,0.5)	(0.6,0.3)	(0.8,0.6)	(0.7,0.5)	(0.7,0.4)
A_3	(0.5,0.4)	(0.4,0.6)	(0.8,0.3)	(0.7,0.5)	(0.3,0.5)
A_4	(0.5,0.6)	(0.5,0.3)	(0.7,0.5)	(0.5,0.6)	(0.6,0.6)

Now, we are going to construct the PHSS under the described criteria. Let $C = \{C_1, C_2, C_3, C_4, C_5\}$ be the set

TABLE 4. Comparisons of the proposed study with existing studies.

Researcher	Truthiness	Falseness	Attributes	Sub-attributes	Matrix Theory	Aggregation Operators	Matrix Algorithms
Guleria and Bajaj [17]	✓	✓	✓	✗	✓	✗	✗
Zulqarnain et al. [28]	✓	✓	✓	✓	✗	✗	✗
Proposed methods	✓	✓	✓	✓	✓	✓	✓

of criteria and $A = \{A_1, A_2, A_3, A_4, A_5\}$ be the set of alternatives (WWTP's) which are Activated Sludge, Sequential Batch Reactor, Constructed Wetland, Anaerobic Lagoon and Membrane Filtration. Also, we know that the all criteria are further divided into sub-criteria. So, we consider some criteria according to our discussion, as shown in Fig. 4. Due to the heavy weightage sub-criteria, we just suppose the linguistic terms of sub-criteria. Let $\acute{K} : C_1 \times C_2 \times C_3 \times C_4 \times C_5 \rightarrow \mathcal{P}(A)$ be defined by $\acute{K}(C_1 \times C_2 \times C_3 \times C_4 \times C_5) = \{A_1, A_2, A_3, A_4\}$ that is a hypersoft set and its PFHSM variantsare defined in Tables 2 and 3.

The matrix form of the P and Q are

$$P = \begin{pmatrix} (0.5, 0.3) & (0.6, 0.2) & (0.4, 0.7) & (0.5, 0.6) & (0.9, 0.3) \\ (0.6, 0.2) & (0.5, 0.4) & (0.5, 0.6) & (0.7, 0.4) & (0.7, 0.2) \\ (0.3, 0.6) & (0.4, 0.5) & (0.6, 0.7) & (0.8, 0.4) & (0.5, 0.5) \\ (0.6, 0.6) & (0.3, 0.6) & (0.8, 0.5) & (0.5, 0.7) & (0.6, 0.3) \end{pmatrix}$$

$$Q = \begin{pmatrix} (0.4, 0.6) & (0.8, 0.4) & (0.5, 0.6) & (0.6, 0.7) & (0.7, 0.4) \\ (0.3, 0.5) & (0.6, 0.3) & (0.8, 0.6) & (0.7, 0.5) & (0.7, 0.4) \\ (0.5, 0.4) & (0.4, 0.6) & (0.8, 0.3) & (0.7, 0.5) & (0.3, 0.5) \\ (0.5, 0.6) & (0.5, 0.3) & (0.7, 0.5) & (0.5, 0.6) & (0.6, 0.6) \end{pmatrix}$$

Now calculate the value matrix by using Definition 12:

$$v(P) = \begin{pmatrix} 0.16 & 0.32 & -0.33 & -0.11 & 0.72 \\ 0.32 & 0.09 & -0.11 & 0.33 & 0.45 \\ -0.27 & -0.09 & 0.55 & 0.48 & 0 \\ 0 & -0.27 & 0.39 & -0.24 & 0.27 \end{pmatrix}$$

$$v(Q) = \begin{pmatrix} -0.2 & 0.48 & -0.11 & -0.13 & 0.33 \\ -0.16 & 0.27 & 0.28 & 0.24 & 0.33 \\ 0.09 & -0.2 & 0.55 & 0.24 & -0.16 \\ -0.11 & 0.16 & 0.24 & -0.11 & 0 \end{pmatrix}$$

Calculate the utility matrix by using Definition 13:

$$u(P, Q) = \begin{pmatrix} -0.04 & -0.16 & -0.22 & 0.02 & 0.39 \\ 0.48 & -0.18 & -0.39 & 0.09 & 0.12 \\ -0.36 & 0.11 & -0.68 & 0.24 & 0.16 \\ 0.11 & -0.43 & 0.15 & -0.13 & 0.27 \end{pmatrix}$$

Calculate the score matrix by using Definition 14:

$$s(P) = \begin{bmatrix} 0.01 \\ 0.12 \\ 0.53 \\ 0.03 \end{bmatrix}$$

We now select the optimal value that is the highest in the column, and so they can make preferences ranking for the alternatives. Here, the optimal solution is the third alternative in the set. From the ranking point of view, CW is first, and then SBR, AL and AS are at second, third, and fourth, respectively.

The proposed work is the merger concept of the hypersoft set and PFSs. As PFSs are the refined form of IFSs which is more reliable for the trueness and falseness concept. Hypersoft set structure is a better structure to deal with the further bifurcations under discussion of criteria. So, the merger of these two theories is PFHSS. We enhance the concept of PFHSSs to PFHSMs which is the gap in the literature. We fill the gap by proposing PFHSMs with their aggregations, properties, theorems, and propositions with their proofs. We propose the two new algorithms by using PFHSMs with the choice matrix, weighted choice matrix, score matrix, and utility matrix, respectively. Totally, the proposed work has its novelty, especially applied in the selection of wastewater treatment plants.

To further demonstrate the effectiveness of the proposed methodsusing PFHSMs, we make the comparisons of the proposed method with the methods of Guleria and Bajaj [17] and Zulqarnain et al. [28], as shown in Table 4. On the other hand, we need to mention that the constructed structure in PFHSSs should be more complex than the structure in PFSSs. However, the proposed PFHSM Algorithms can be well used in this complex structure of PFHSSs.

V. CONCLUSION

The Pythagorean Fuzzy Hypersoft Set (PFHSS) combine both PFSs and HSSs to enhance the representation of uncertainty and ambiguity. The PFHSS becomes a model that aims to get the usage of a multi-argument domain for estimating the relevant parameters so that it offers a more flexible and reliable framework for handling membership degrees that may be stated in both language and numerical terms. Since the

PFHSS takes the extra classification of parameters into their appropriate parametric valued sets, we first explored more avenues in which several axiomatic findings, operational outcomes, and aggregation strategies were first presented under the PFHSS environment. Further, there is less to consider Pythagorean fuzzy hypersoft matrices (PFHSMs) in the literature, and we then proposed these PFHSMs in the paper. The basic properties and aggregation processes of PFHSMs are also presented. Thirdly, utilizing the suggested choice matrix, the PFHSM matrix theory is applied to the decision-making scenarios of real mobile selection. At the end of the paper, we discussed a very real-life problem that is affecting human health. Wastewater treatment is crucial for preserving the environment and the public health. It comprises purifying wastewater of contaminants and pollutants so that it may be utilized for other things or discharged safely into the environment. It is essential to protect the environment and public health by removing toxins from domestic, industrial, and commercial sewage. We applied our proposed algorithm in the selection of wastewater treatment plants. The proposed work is a novel technique to solve multi-attributive decision-making (MDAM) problems. There are many MADM techniques, such as TOPSIS, SAW, AHP, VIKOR, etc. In future works, we will reconstruct these algorithms and apply them to these MADM techniques of TOPSIS, SAW, AHP, VIKOR, etc. under the PFHSSs and PFHSMs. We should also work on more similarity and distance measures on PFHSSs and PFHSMs. Furthermore, Peng and Selvachandran [44] had given two algorithms for solving MADM problems under Pythagorean fuzzyenvironment. We will extend these algorithms by using the proposed PFHSMs, and apply these to financial risk evaluation of new energy vehicle industry [45] and optimal cache placement policy [46] under the PFHSSenvironment.

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