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RESEARCH ARTICLE

Choquet-Integral Aggregation Operators Based on Hamacher t-Norm and t-Conorm for Complex Intuitionistic Fuzzy TOPSIS Technique to Deal With Socio-Economic Problems

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ABSTRACT Social and economic factors, such as education, employment, social support, income, and community safety, can directly affect how well and long we live. Under the presence of the above factors, we easily look at health decisions, afford medical care and housing, manage stress, and so on. In this analysis, we derive the Choquet-integral aggregation operators based on Hamacher t-norm and t-conorm for complex Atanassov intuitionistic fuzzy (CAIF) such as CAIF Choquet-integral Hamacher averaging (CAIFCIHA), CAIF Choquet-integral Hamacher ordered averaging (CAIFCIHOA), CAIF Choquet-integral Hamacher geometric (CAIFCIHG), CAIF Choquet-integral Hamacher ordered geometric (CAIFCIHOG) operators and exposed their properties. Additionally, we discover the Technique for Order Preference by Similarity to the Ideal Solution "TOPSIS" technique under the consideration of the above-derived theory. Moreover, it is very famous and interesting to note that various unpredicted factors such as road, diesel prices, weather, and traffic conditions affect the cost of transportation. Therefore, decision-makers deal with vague and unreliable information to estimate the cost of transportation. To evaluate the above problem, we select a transportation problem with CAIF parameters, and for its evaluation, a simple and well-known computational technique is derived and illustrated. Finally, we compared our derived results with some prevailing results to show the worth and reliability of the exposed approaches.

INDEX TERMS Complex intuitionistic fuzzy sets, choquet-integral aggregation operators, Hamacher t-norm and t-conorm, TOPSIS technique, socio-economic problems.

I. INTRODUCTION

To grow any society and civilization, we need to revise the social and economic problems because, without these two factors, any civilization cannot grow smoothly. The main factors are education, employment, food, facilities, support, income, and community safety. Without these factors, a good society's economy will fail or be finished very soon. From these, the transportation problem is also very important, especially for the Asian and European countries, because they failed to use them. Further, certain scholars have discussed the above problems under the presence of the different types of procedures, such as pattern recognition, artificial intelligence, machine learning, and decision-making, but during the decision-making process, they lost a lot of data due to classical data, because in classical data we have only two possibilities such as zero or one. To increase the range of the crips set to provide a wide space to experts for making a valuable decision, the theory of fuzzy analysis [1] is very famous and reliable because the range of the fuzzy set is very wide such as unit interval, which is the extended form of $\{0, 1\}$. Furthermore, because of ambiguity and complication,

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fuzzy set (FS) is not enough for evaluating awkward and unreliable information because the falsity of information is also a very important factor in real life, but fuzzy information only deals with truth grade, which is not enough. Therefore, intuitionistic FS (IFS) was exposed by Atanassov [2]. IFS is much more famous compared to FS because it is more modified than FS, and many people have derived different types of operators, methods, and measures based on FS and IFSs.

Further, Ramot et al. [3] discovered a new theory such as complex FS (CFS), where the grades in CFS is computed in the form of complex number whose real and unreal information form unit interval, where the main idea of FS is the special case of the CFS. The amplitude and phase terms are valuable and dominant in real life. For instance, when an enterprise wants to purchase new software for their employees, the owner of the shop is provided two types of data regarding each software such as the name and production date of the software, where the name and procedure date are stated in the amplitude and phase term of the complex values. But because of the same issues as FS, the CFS ignored the falsity of information, and because of these issues, the theory of CFS has received less attention from scholars. Therefore, the complex IFS (CIFS) [4] is very general and reliable for depicting awkward and unreliable data that cannot handle existing FSs, IFSs, and CFSs. The main structure of CIFS is solid and famous because it contains two different grades, truth and falsity, in the form of polar coordinates, where the major characteristics of CIFS are stated by $0 \leq \mathrm{u}_{s}^{rp}(\sigma) + \vartheta_{sa}^{rp}(\sigma) \leq 1$ and $0 \leq \mathrm{u}_{s}^{ip}(\sigma) + \vartheta_{sa}^{ip}(\sigma) \leq 1$. Many people have derived different types of operators, methods, and measures based on CFS and CIFSs.

TOPSIS technique is an efficient decision analysis method use to evaluate a suitable optimal option and make the decision in a decent way. TOPSIS technique or procedure is powerful, combining two valuable ideas, positive and negative ideal solutions, proposed by Hwang and Yoon [5] in 1981. Furthermore, in 1987, Yoon [6] also examined the compromise solution for discrete information, a simple TOPSIS technique. In 1993, Hwang et al. [7] separately exposed the theory of the TOPSIS technique with different measures. Furthermore, Bellman and Zadeh [8] exposed the decision-making procedure, and Huang [9] derived the Hamacher aggregation operators for IFSs. The idea of CI was invented by Choquet [10] in 1953. Furthermore, Akram et al. [11] exposed the Hamacher operators for CIFSs. Mahmood et al. [12] examined the Hamacher CI aggregation operators for IFSs. Jia and Wang [13] invented the CI operators for IFSs. Tan and Chen [14] proposed the CI operators for induced IFSs. The simple aggregation operators for IFS were derived by Xu [15]. Further, Xu and Yager [16] invented the geometric operators for IFSs. Zhao et al. [17] exposed the generalized or extended aggregation operators for IFSs. Recently, numerous research scientists worked on different fuzzy domains and derived several mathematical approaches to tackle awkward and vague type information during decision analysis [18], [19], [20], [21]. Garg and Rani [22], [23] exposed the novel aggregation operators and generalized geometric aggregation operators for CIFSs. Mahmood et al. [24] recently derived the Aczel-Alsina aggregation operators for CIFSs. We also studied a latest research work by utilizing the properties of Hamy mean aggregation models by incorporating the features of algebraic t-norms and t-conorms [25], [26]. A number of scholars utilized a novel approache of the TOPSIS technique in the decision-making process under different fuzzy domains [27], [28], [29], [30], [31]. It is very ambiguous and awkward how to develop a new operator or how to combine three different structures and expose a new one under the presence of the CIFS is a very challenging task for scholars. The above discussed mathematical strategies sometime unable to handel redundant or incomplete information of the human opinion. Motivated by the significance and efficient model of CIFS, they have a grat capability to expose the loss of information and acquire a smooth judgment during decision analysis. We see no one to use the well-known technique of the TOPSIS method by incorporating the theory of complex intuitionistic fuzzy information and the operational laws of the Hamacher t-norms. Although, there are many type of extensios of the triangular norm and use to aggregate complicated information about any real life application. But Hamcher t-norms attained a lot of attraction from numerous research scholars, have a capacity to provide a smooth and authentic aggregated information during decision-making process. Therefore, in this analysis, we aim to discuss how social and economic factors, such as education, employment, social support, income, and community safety, can directly affect how well and long we live. Under the presence of the above factors, we easily look at health decisions, afford medical care and housing, manage stress, and so on. In this analysis, we derive the CI aggregation operators based on Hamacher t-norm and t-conorm for CAIF values, which are listed below:

1) Expose the theory of CIFS with some reliable features to cope with awkward and uncertain situations of human opinion in a decent way.
2) Derive some robust operations of Hamacher t-norms in the light CIFS environments.
3) We derive new mathematical strategies under the system of complex intuitionistic fuzzy situationa and basic charateristics of the Hamacher t-norms namely CAIFCIHA, CAIFCIHOA, CAIFCIHG, and CAIFCIHOG operators with notable properties.
4) A novel study of the decision-making technique of the TOPSIS method under consideration the developed theory.
5) To show the credibility and effectiveness of derived research work, we studied a numerical example to demonstrate a reliable option based on the advanced decision-making approach of the TOPSIS method.

6) To compare our derived results with prevailing results to show the worth and reliability of the exposed approaches.

This analysis is summarized in the form: In Section II, we stated the old concept of CAIFS, Hamacher operational laws for CAIFS, and fuzzy measure (FS). In Section III, we invented the idea of CAIFCIHA, CAIFCIHOA, CAIFCIHG, and CAIFCIHOG operators and exposed their properties. In Section IV, we discovered the TOPSIS technique under the consideration of the above-derived theory. In Section V, we evaluated the above problem. We select a transportation problem with CAIF parameters, and for its evaluation, a simple and well-known computational technique is derived and illustrated. In Section VI, we compared our derived results with some prevailing results to show the worth and reliability of the exposed approaches. Concluding information is part of section VII.

II. PRELIMINARIES

In this scenario, we stated the old concept of CAIFS, Hamacher operational laws for CAIFS, and FS, which will help in evaluating or computing a new idea.

Definition 1 [4]: The term η_{if} used as a CA-IFS under the presence of the discourse X, such that:

$$\eta_{if}=\left\{\left(\left(\mathrm{u}_{s}^{rp}(\sigma),\mathrm{u}_{s}^{ip}(\sigma)\right),\left(\vartheta_{sa}^{rp}(\sigma),\vartheta_{sa}^{ip}(\sigma)\right)\right):\sigma\in X\right\} \tag{1}$$

Noticed that the pairs $(\mathrm{u}_s(\sigma),\mathrm{u}_s(\sigma))$ and $(\vartheta_{sa}(\sigma),\vartheta_{sa}(\sigma))$ used as truth and falsity information in the form of complex numbers with a valuable and dominant characteristic such as $0\le \mathrm{u}_{s}^{rp}(\sigma)+\vartheta_{sa}^{rp}(\sigma)\le 1$ and $0\le \mathrm{u}_{s}^{ip}(\sigma)+\vartheta_{sa}^{ip}(\sigma)\le 1$. Moreover, under the presence of the above information, we stated the theory of neutral grade such as: $^{\circ}\mathrm{F}_r(\sigma)=\left(^{\circ}\mathrm{F}_r^{rp}(\sigma),{}^{\circ}\mathrm{F}_r^{ip}(\sigma)\right)=\left(1-\left(\mathrm{u}_{s}^{rp}(\sigma)+\vartheta_{sa}^{rp}(\sigma)\right),1-\left(\mathrm{u}_{s}^{ip}(\sigma)+\vartheta_{sa}^{ip}(\sigma)\right)\right)$ and the simple form of CA-IF values (CA-IFVs) is stated by: $\eta_{if}^{\mu}=\left(\left(\mathrm{u}_{s_\mu}^{rp},\mathrm{u}_{s_\mu}^{ip}\right),\left(\vartheta_{sa_\mu}^{rp},\vartheta_{sa_\mu}^{ip}\right)\right),\mu=1,2,\ldots,\omega$.

Definition 2 [11]: Under two CA-IFVs $\eta_{if}^{\mu}=\left(\left(\mathrm{u}_{s_\mu}^{rp},\mathrm{u}_{s_\mu}^{ip}\right),\left(\vartheta_{sa_\mu}^{rp},\vartheta_{sa_\mu}^{ip}\right)\right),\mu=1,2$, we examined the score and accuracy information, such as

$$\eta_{S-if}^{\mu}=\frac{1}{2}\left(\mathrm{u}_{s_\mu}^{rp}-\vartheta_{sa_\mu}^{rp}+\mathrm{u}_{s_\mu}^{ip}-\vartheta_{sa_\mu}^{ip}\right)\in[-1,1] \tag{2}$$

$$\eta_{H-if}^{\mu}=\frac{1}{2}\left(\mathrm{u}_{s_\mu}^{rp}+\vartheta_{sa_\mu}^{rp}+\mathrm{u}_{s_\mu}^{ip}+\vartheta_{sa_\mu}^{ip}\right)\in[0,1] \tag{3}$$

For distinguishing any two CA-IFVs, we can help form the following properties such as when $\eta_{S-if}^{1}>\eta_{S-if}^{2}$, then we assume $\eta_{if}^{1}>\eta_{if}^{2}$, when $\eta_{S-if}^{1}<\eta_{S-if}^{2}$, then we assume $\eta_{if}^{1}<\eta_{if}^{2}$, when $\eta_{S-if}^{1}=\eta_{S-if}^{2}$, then we assume when $\eta_{H-if}^{1}>\eta_{H-if}^{2}$, then we assume $\eta_{if}^{1}>\eta_{if}^{2}$, when $\eta_{H-if}^{1}<\eta_{H-if}^{2}$, then we assume $\eta_{if}^{1}<\eta_{if}^{2}$.

Definition 3 [11]: Under two CA-IFVs $\eta_{if}^{\mu}=\left(\left(\mathrm{u}_{s_\mu}^{rp},\mathrm{u}_{s_\mu}^{ip}\right),\left(\vartheta_{sa_\mu}^{rp},\vartheta_{sa_\mu}^{ip}\right)\right),\mu=1,2$, we have (4)–(7), as shown at the bottom of the next page.

Definition 4 [8]: Under the presence of the collection of positive information, we concentrate on examining the idea of FM, such as:

$$\int\psi d\beth=\sum_{\mu=1}^{\omega}\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)\psi_{0(\mu)} \tag{8}$$

$$\beth\left(\overline{\overline{\eta}}\right)=\beth\left(\coprod_{\mu=1}^{\omega}\sigma_\mu\right)=\begin{cases}\frac{1}{\nabla}\left[\prod_{\mu=1}^{\omega}\left(1+\gamma\beth\left(\sigma_\mu\right)\right)-1\right] & \nabla\neq 0\\ \sum_{\sigma_\mu\in A}\beth\left(\sigma_\mu\right) & \nabla=0\end{cases} \tag{9}$$

Noticed that the permutation of $(1,2,\ldots,\omega)$ is stated by $0(\mu)$ with ordered $\psi_{0(1)}\ge\psi_{0(2)}\ge\ldots\ge\psi_{0(\omega)}$ and $(\overline{\overline{\eta}}=\theta$, $\overline{\overline{\eta}}_{0(\mu)}=\left\{\eta'_{0(1)},\eta'_{0(2)},\ldots,\eta'_{0(\mu)}\right\}$.

III. CAIF-CI HAMACHER OPERATIONS

In this section, we derived the CI aggregation operators and exposed their properties based on Hamacher t-norm and t-conorm for CAIF, such as CAIFCIHA, CAIFCIHOA, CAIFCIHG, and CAIFCIHOG operators.

Definition 5: Under n-family of CA-IFVs $\eta_{if}^{\mu}=\left(\left(\mathrm{u}_{s_\mu}^{rp},\mathrm{u}_{s_\mu}^{ip}\right),\left(\vartheta_{sa_\mu}^{rp},\vartheta_{sa_\mu}^{ip}\right)\right),\mu=1,2,\ldots,\omega$, the well-known theory of CAIFCIHA operator is established or proven by:

$$\begin{aligned}&\int\eta_{if}d\beth\\&=CAIFCIHA\left(\eta_{if}^{1},\eta_{if}^{2},\ldots,\eta_{if}^{\omega}\right)\\&=\left(\beth\left(\overline{\overline{\eta}}_{0(1)}\right)-\beth\left(\overline{\overline{\eta}}_{0(0)}\right)\right)\eta_{if}^{1}\oplus\left(\beth\left(\overline{\overline{\eta}}_{0(2)}\right)\right.\\&\quad\left.-\beth\left(\overline{\overline{\eta}}_{0(1)}\right)\right)\eta_{if}^{2}\oplus\ldots\oplus\left(\beth\left(\overline{\overline{\eta}}_{0(\omega)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\omega-1)}\right)\right)\eta_{if}^{\omega}\\&=\sum_{\mu=1}^{\omega}\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)\eta_{if}^{\mu}\\&=\oplus_{\mu=1}^{\omega}\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)\eta_{if}^{\mu}\end{aligned} \tag{10}$$

Theorem 1: Under n-family of CA-IFVs $\eta_{if}^{\mu}=\left(\left(\mathrm{u}_{s_\mu}^{rp},\mathrm{u}_{s_\mu}^{ip}\right),\left(\vartheta_{sa_\mu}^{rp},\vartheta_{sa_\mu}^{ip}\right)\right),\mu=1,2,\ldots,\omega$, we evaluate that the final aggregated value of Eq. (10) is again in the shape of CA-IFV such as shown in (11) at the bottom of page 5.

Proof: To derive the data in Eq. (11), the induction of mathematics is very famous and reliable. Therefore, for $\omega=2$, we get in $\left(\beth\left(\overline{\overline{\eta}}_{0(1)}\right)-\beth\left(\overline{\overline{\eta}}_{0(0)}\right)\right)\eta_{if}^{1}$ and $\left(\beth\left(\overline{\overline{\eta}}_{0(2)}\right)-\beth\left(\overline{\overline{\eta}}_{0(1)}\right)\right)\eta_{if}^{2}$, as shown at the bottom of page 5. Thus, $\left(\beth\left(\overline{\overline{\eta}}_{0(1)}\right)-\beth\left(\overline{\overline{\eta}}_{0(0)}\right)\right)\eta_{if}^{1}\oplus\left(\beth\left(\overline{\overline{\eta}}_{0(2)}\right)-\beth\left(\overline{\overline{\eta}}_{0(1)}\right)\right)\eta_{if}^{2}$, as shown at the bottom of page 6.

Noticed that the data in Eq. (11) is held for $\omega = 2$, further for $\omega = z$, we claim that the data in Eq. (11) is also correct, such as $CAIFCIHA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{z}\right)$, shown at the bottom of page 7.

$$\eta_{if}^{1} \oplus \eta_{if}^{2} \oplus \ldots \oplus \eta_{if}^{\omega} = \left(\begin{array}{c} \left(\frac{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\,\mathfrak{u}_{s_\mu}^{rp}\right)-\prod_{\mu=1}^{\omega}\left(1-\mathfrak{u}_{s_\mu}^{rp}\right)}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\,\mathfrak{u}_{s_\mu}^{rp}\right)+(\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\mathfrak{u}_{s_\mu}^{rp}\right)}, \frac{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\,\mathfrak{u}_{s_\mu}^{ip}\right)-\prod_{\mu=1}^{\omega}\left(1-\mathfrak{u}_{s_\mu}^{ip}\right)}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\,\mathfrak{u}_{s_\mu}^{ip}\right)+(\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\mathfrak{u}_{s_\mu}^{ip}\right)} \right), \\ \left(\frac{\Delta\prod_{\mu=1}^{\omega}\left(1-\mathfrak{u}_{s_\mu}^{rp}\right)-\Delta\prod_{\mu=1}^{\omega}\left(1-\mathfrak{u}_{s_\mu}^{rp}-\vartheta_{sa_\mu}^{rp}\right)}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\,\mathfrak{u}_{s_\mu}^{rp}\right)+(\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\mathfrak{u}_{s_\mu}^{rp}\right)}, \frac{\Delta\prod_{\mu=1}^{\omega}\left(1-\mathfrak{u}_{s_\mu}^{ip}\right)-\Delta\prod_{\mu=1}^{\omega}\left(1-\mathfrak{u}_{s_\mu}^{ip}-\vartheta_{sa_\mu}^{ip}\right)}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\,\mathfrak{u}_{s_\mu}^{ip}\right)+(\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\mathfrak{u}_{s_\mu}^{ip}\right)} \right) \end{array} \right) \tag{4}$$

$$\eta_{if}^{1} \otimes \eta_{if}^{2} \otimes \ldots \otimes \eta_{if}^{\omega} = \left(\begin{array}{c} \left(\frac{\Delta\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{rp}\right)-\Delta\prod_{\mu=1}^{\omega}\left(1-\mathfrak{u}_{s_\mu}^{rp}-\vartheta_{sa_\mu}^{rp}\right)}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\,\vartheta_{sa_\mu}^{rp}\right)+(\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{rp}\right)}, \frac{\Delta\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{ip}\right)-\Delta\prod_{\mu=1}^{\omega}\left(1-\mathfrak{u}_{s_\mu}^{ip}-\vartheta_{sa_\mu}^{ip}\right)}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\,\vartheta_{sa_\mu}^{ip}\right)+(\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{ip}\right)} \right), \\ \left(\frac{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\,\vartheta_{sa_\mu}^{rp}\right)-\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{rp}\right)}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\,\vartheta_{sa_\mu}^{rp}\right)+(\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{rp}\right)}, \frac{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\,\vartheta_{sa_\mu}^{ip}\right)-\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{ip}\right)}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\,\vartheta_{sa_\mu}^{ip}\right)+(\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{ip}\right)} \right) \end{array} \right) \tag{5}$$

$$\nabla\eta_{if}^{1} = \left(\begin{array}{c} \left(\begin{array}{c} \frac{\left(1+(\Delta-1)\,\mathfrak{u}_{s_1}^{rp}\right)^{\nabla}-\left(1-\mathfrak{u}_{s_1}^{rp}\right)^{\nabla}}{\left(1+(\Delta-1)\,\mathfrak{u}_{s_1}^{rp}\right)^{\nabla}-(\Delta-1)\left(1-\mathfrak{u}_{s_1}^{rp}\right)^{\nabla}}, \\ \frac{\left(1+(\Delta-1)\,\mathfrak{u}_{s_1}^{ip}\right)^{\nabla}-\left(1-\mathfrak{u}_{s_1}^{ip}\right)^{\nabla}}{\left(1+(\Delta-1)\,\mathfrak{u}_{s_1}^{ip}\right)^{\nabla}-(\Delta-1)\left(1-\mathfrak{u}_{s_1}^{ip}\right)^{\nabla}} \end{array} \right), \\ \left(\begin{array}{c} \frac{\Delta\left(1-\mathfrak{u}_{s_1}^{rp}\right)^{\nabla}-\Delta\left(1-\mathfrak{u}_{s_1}^{rp}-\vartheta_{sa_1}^{rp}\right)^{\nabla}}{\left(1+(\Delta-1)\,\mathfrak{u}_{s_1}^{rp}\right)^{\nabla}-(\Delta-1)\left(1-\mathfrak{u}_{s_1}^{rp}\right)^{\nabla}}, \\ \frac{\Delta\left(1-\mathfrak{u}_{s_1}^{ip}\right)^{\nabla}-\Delta\left(1-\mathfrak{u}_{s_1}^{ip}-\vartheta_{sa_1}^{ip}\right)^{\nabla}}{\left(1+(\Delta-1)\,\mathfrak{u}_{s_1}^{ip}\right)^{\nabla}-(\Delta-1)\left(1-\mathfrak{u}_{s_1}^{ip}\right)^{\nabla}} \end{array} \right) \end{array} \right) \tag{6}$$

$$\eta_{if}^{1\,\nabla} = \left(\begin{array}{c} \left(\begin{array}{c} \frac{\Delta\left(1-\vartheta_{sa_1}^{rp}\right)^{\nabla}-\Delta\left(1-\mathfrak{u}_{s_1}^{rp}-\vartheta_{sa_1}^{rp}\right)^{\nabla}}{\left(1+(\Delta-1)\,\vartheta_{sa_1}^{rp}\right)^{\nabla}-(\Delta-1)\left(1-\vartheta_{sa_1}^{rp}\right)^{\nabla}}, \\ \frac{\Delta\left(1-\vartheta_{sa_1}^{ip}\right)^{\nabla}-\Delta\left(1-\mathfrak{u}_{s_1}^{ip}-\vartheta_{sa_1}^{ip}\right)^{\nabla}}{\left(1+(\Delta-1)\,\vartheta_{sa_1}^{ip}\right)^{\nabla}-(\Delta-1)\left(1-\vartheta_{sa_1}^{ip}\right)^{\nabla}} \end{array} \right), \\ \left(\begin{array}{c} \frac{\left(1+(\Delta-1)\,\vartheta_{sa_1}^{rp}\right)^{\nabla}-\left(1-\vartheta_{sa_1}^{rp}\right)^{\nabla}}{\left(1+(\Delta-1)\,\vartheta_{sa_1}^{rp}\right)^{\nabla}-(\Delta-1)\left(1-\vartheta_{sa_1}^{rp}\right)^{\nabla}}, \\ \frac{\left(1+(\Delta-1)\,\vartheta_{sa_1}^{ip}\right)^{\nabla}-\left(1-\vartheta_{sa_1}^{ip}\right)^{\nabla}}{\left(1+(\Delta-1)\,\vartheta_{sa_1}^{ip}\right)^{\nabla}-(\Delta-1)\left(1-\vartheta_{sa_1}^{ip}\right)^{\nabla}} \end{array} \right) \end{array} \right) \tag{7}$$

For $\omega = z+1$, we discover that the data in Eq. (11) is also correct, such as $CAIFCIHA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right)$, shown at the bottom of page 8.
Therefore, we proved that the data in Eq. (11) is held for positive numbers.

Property 1 (Idempotency): Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(\mathrm{u}_{s_\mu}^{rp}, \mathrm{u}_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu = 1, 2, \ldots, \omega$, if $\eta_{if}^{\mu} = \eta_{if} = (\mathrm{u}_s, \vartheta_{sa}), \mu = 1, 2, \ldots, \omega$, then we get

$$CAIFCIHA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) = \eta_{if} \tag{12}$$

$$\begin{aligned}
&CAIFCIHA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right)\\
&= \left(\begin{array}{c}
\left(\begin{array}{c}
\frac{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} - \prod_{\mu=1}^{\omega}\left(1-\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} + (\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}},\\
\frac{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} - \prod_{\mu=1}^{\omega}\left(1-\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} + (\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}
\end{array}\right),\\
\left(\begin{array}{c}
\frac{\Delta\prod_{\mu=1}^{\omega}\left(1-\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} - \Delta\prod_{\mu=1}^{\omega}\left(1-\mathrm{u}_{s_\mu}^{rp}-\vartheta_{sa_\mu}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} + (\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}},\\
\frac{\Delta\prod_{\mu=1}^{\omega}\left(1-\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} - \Delta\prod_{\mu=1}^{\omega}\left(1-\mathrm{u}_{s_\mu}^{ip}-\vartheta_{sa_\mu}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} + (\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(\mu)}\right)-\daleth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}
\end{array}\right)
\end{array}\right)
\end{aligned} \tag{11}$$

$$\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(0)}\right)\right)\eta_{if}^{1} = \left(\begin{array}{c}
\left(\begin{array}{c}
\frac{\left(1+(\Delta-1)\mathrm{u}_{s_1}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)} - \left(1-\mathrm{u}_{s_1}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)}}{\left(1+(\Delta-1)\mathrm{u}_{s_1}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)} + (\Delta-1)\left(1-\mathrm{u}_{s_1}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)}},\\
\frac{\left(1+(\Delta-1)\mathrm{u}_{s_1}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)} - \left(1-\mathrm{u}_{s_1}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)}}{\left(1+(\Delta-1)\mathrm{u}_{s_1}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)} + (\Delta-1)\left(1-\mathrm{u}_{s_1}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)}}
\end{array}\right),\\
\left(\begin{array}{c}
\frac{\Delta\left(1-\mathrm{u}_{s_1}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)} - \Delta\left(1-\mathrm{u}_{s_1}^{rp}-\vartheta_{sa_1}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)}}{\left(1+(\Delta-1)\mathrm{u}_{s_1}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)} + (\Delta-1)\left(1-\mathrm{u}_{s_1}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)}},\\
\frac{\Delta\left(1-\mathrm{u}_{s_1}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)} - \Delta\left(1-\mathrm{u}_{s_1}^{ip}-\vartheta_{sa_1}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)}}{\left(1+(\Delta-1)\mathrm{u}_{s_1}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)} + (\Delta-1)\left(1-\mathrm{u}_{s_1}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(1)}\right)-\daleth\left(\overline{\eta}_{0(1-1)}\right)\right)}}
\end{array}\right)
\end{array}\right)$$

$$\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(1)}\right)\right)\eta_{if}^{2} = \left(\begin{array}{c}
\left(\begin{array}{c}
\frac{\left(1+(\Delta-1)\mathrm{u}_{s_2}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)} - \left(1-\mathrm{u}_{s_2}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)}}{\left(1+(\Delta-1)\mathrm{u}_{s_2}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)} + (\Delta-1)\left(1-\mathrm{u}_{s_2}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)}},\\
\frac{\left(1+(\Delta-1)\mathrm{u}_{s_2}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)} - \left(1-\mathrm{u}_{s_2}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)}}{\left(1+(\Delta-1)\mathrm{u}_{s_2}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)} + (\Delta-1)\left(1-\mathrm{u}_{s_2}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)}}
\end{array}\right),\\
\left(\begin{array}{c}
\frac{\Delta\left(1-\mathrm{u}_{s_2}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)} - \Delta\left(1-\mathrm{u}_{s_2}^{rp}-\vartheta_{sa_2}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)}}{\left(1+(\Delta-1)\mathrm{u}_{s_2}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)} + (\Delta-1)\left(1-\mathrm{u}_{s_2}^{rp}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)}},\\
\frac{\Delta\left(1-\mathrm{u}_{s_2}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)} - \Delta\left(1-\mathrm{u}_{s_2}^{ip}-\vartheta_{sa_2}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)}}{\left(1+(\Delta-1)\mathrm{u}_{s_2}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)} + (\Delta-1)\left(1-\mathrm{u}_{s_2}^{ip}\right)^{\left(\daleth\left(\overline{\eta}_{0(2)}\right)-\daleth\left(\overline{\eta}_{0(2-1)}\right)\right)}}
\end{array}\right)
\end{array}\right)$$

$$
\left(\sqsupset\left(\overline{\overline{\eta}}_{0(1)}\right)-\sqsupset\left(\overline{\overline{\eta}}_{0(0)}\right)\right)\eta_{if}^{1}\oplus\left(\sqsupset\left(\overline{\overline{\eta}}_{0(2)}\right)-\sqsupset\left(\overline{\overline{\eta}}_{0(1)}\right)\right)\eta_{if}^{2}
$$

$$
=\left(\begin{array}{c}\left(\begin{array}{c}\dfrac{\left(1+(\Delta-1)\,\mathrm{u}_{s_1}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}-\left(1-\mathrm{u}_{s_1}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}}{\left(1+(\Delta-1)\,\mathrm{u}_{s_1}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}+(\Delta-1)\left(1-\mathrm{u}_{s_1}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}},\\ \dfrac{\left(1+(\Delta-1)\,\mathrm{u}_{s_1}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}-\left(1-\mathrm{u}_{s_1}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}}{\left(1+(\Delta-1)\,\mathrm{u}_{s_1}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}+(\Delta-1)\left(1-\mathrm{u}_{s_1}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}}\end{array}\right),\\ \left(\begin{array}{c}\dfrac{\Delta\left(1-\mathrm{u}_{s_1}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}-\Delta\left(1-\mathrm{u}_{s_1}^{rp}-\vartheta_{sa_1}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}}{\left(1+(\Delta-1)\,\mathrm{u}_{s_1}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}+(\Delta-1)\left(1-\mathrm{u}_{s_1}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}},\\ \dfrac{\Delta\left(1-\mathrm{u}_{s_1}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}-\Delta\left(1-\mathrm{u}_{s_1}^{ip}-\vartheta_{sa_1}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}}{\left(1+(\Delta-1)\,\mathrm{u}_{s_1}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}+(\Delta-1)\left(1-\mathrm{u}_{s_1}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(1)})-\sqsupset(\overline{\overline{\eta}}_{0(1-1)})\right)}}\end{array}\right)\end{array}\right)
$$

$$
\oplus\left(\begin{array}{c}\left(\begin{array}{c}\dfrac{\left(1+(\Delta-1)\,\mathrm{u}_{s_2}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}-\left(1-\mathrm{u}_{s_2}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}}{\left(1+(\Delta-1)\,\mathrm{u}_{s_2}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}+(\Delta-1)\left(1-\mathrm{u}_{s_2}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}},\\ \dfrac{\left(1+(\Delta-1)\,\mathrm{u}_{s_2}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}-\left(1-\mathrm{u}_{s_2}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}}{\left(1+(\Delta-1)\,\mathrm{u}_{s_2}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}+(\Delta-1)\left(1-\mathrm{u}_{s_2}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}}\end{array}\right),\\ \left(\begin{array}{c}\dfrac{\Delta\left(1-\mathrm{u}_{s_2}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}-\Delta\left(1-\mathrm{u}_{s_2}^{rp}-\vartheta_{sa_2}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}}{\left(1+(\Delta-1)\,\mathrm{u}_{s_2}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}+(\Delta-1)\left(1-\mathrm{u}_{s_2}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}},\\ \dfrac{\Delta\left(1-\mathrm{u}_{s_2}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}-\Delta\left(1-\mathrm{u}_{s_2}^{ip}-\vartheta_{sa_2}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}}{\left(1+(\Delta-1)\,\mathrm{u}_{s_2}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}+(\Delta-1)\left(1-\mathrm{u}_{s_2}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(2)})-\sqsupset(\overline{\overline{\eta}}_{0(2-1)})\right)}}\end{array}\right)\end{array}\right)
$$

$$
=\left(\begin{array}{c}\left(\begin{array}{c}\dfrac{\prod_{\mu=1}^{2}\left(1+(\Delta-1)\,\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}-\prod_{\mu=1}^{2}\left(1-\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}}{\prod_{\mu=1}^{2}\left(1+(\Delta-1)\,\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}+(\Delta-1)\prod_{\mu=1}^{2}\left(1-\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}},\\ \dfrac{\prod_{\mu=1}^{2}\left(1+(\Delta-1)\,\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}-\prod_{\mu=1}^{2}\left(1-\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}}{\prod_{\mu=1}^{2}\left(1+(\Delta-1)\,\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}+(\Delta-1)\prod_{\mu=1}^{2}\left(1-\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}}\end{array}\right),\\ \left(\begin{array}{c}\dfrac{\Delta\prod_{\mu=1}^{2}\left(1-\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}-\Delta\prod_{\mu=1}^{2}\left(1-\mathrm{u}_{s_\mu}^{rp}-\vartheta_{sa_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}}{\prod_{\mu=1}^{2}\left(1+(\Delta-1)\,\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}+(\Delta-1)\prod_{\mu=1}^{2}\left(1-\mathrm{u}_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}},\\ \dfrac{\Delta\prod_{\mu=1}^{2}\left(1-\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}-\Delta\prod_{\mu=1}^{2}\left(1-\mathrm{u}_{s_\mu}^{ip}-\vartheta_{sa_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}}{\prod_{\mu=1}^{2}\left(1+(\Delta-1)\,\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}+(\Delta-1)\prod_{\mu=1}^{2}\left(1-\mathrm{u}_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\overline{\eta}}_{0(\mu)})-\sqsupset(\overline{\overline{\eta}}_{0(\mu-1)})\right)}}\end{array}\right)\end{array}\right)
$$

Property 2 (Monotonicity): Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu = 1, 2, \ldots, \omega$, if $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right) \leq \eta{*}_{if}^{\mu} = \left(\left(u{*}_{s_\mu}^{rp}, u{*}_{s_\mu}^{ip}\right), \left(\vartheta{*}_{sa_\mu}^{rp}, \vartheta{*}_{sa_\mu}^{ip}\right)\right)$, then we have

$$CAIFCIHA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) \leq CAIFCIHA\left(\eta{*}_{if}^{1}, \eta{*}_{if}^{2}, \ldots, \eta{*}_{if}^{\omega}\right) \tag{13}$$

Property 3 (Boundedness): Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu = 1, 2, \ldots, \omega$, if $\eta_{if}^{-} = \left(\left(\min_{\mu} u_{s_\mu}^{rp}, \min_{\mu} u_{s_\mu}^{ip}\right), \left(\max_{\mu} \vartheta_{sa_\mu}^{rp}, \max_{\mu} \vartheta_{sa_\mu}^{ip}\right)\right)$ and $\eta_{if}^{+} = \left(\left(\max_{\mu} u_{s_\mu}^{rp}, \max_{\mu} u_{s_\mu}^{ip}\right), \left(\min_{\mu} \vartheta_{sa_\mu}^{rp}, \min_{\mu} \vartheta_{sa_\mu}^{ip}\right)\right)$, then we have

$$\eta_{if}^{-} \leq CAIFCIHA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) \leq \eta_{if}^{+} \tag{14}$$

Definition 6: Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu = 1, 2, \ldots, \omega$, the well-known theory of CAIFCIHOA operator is established or proven by:

$$\begin{aligned}
&\int \eta_{if}\, d\beth \\
&= CAIFCIHOA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) \\
&= \left(\beth\left(\overline{\eta}_{0(1)}\right) - \beth\left(\overline{\eta}_{0(0)}\right)\right) \eta_{if}^{o(1)} \\
&\quad \oplus \left(\beth\left(\overline{\eta}_{0(2)}\right) - \beth\left(\overline{\eta}_{0(1)}\right)\right) \eta_{if}^{o(2)} \oplus \ldots \oplus \left(\beth\left(\overline{\eta}_{0(\omega)}\right)\right. \\
&\quad \left. - \beth\left(\overline{\eta}_{0(\omega-1)}\right)\right) \eta_{if}^{o(\omega)} = \sum_{\mu=1}^{\omega} \left(\beth\left(\overline{\eta}_{0(\mu)}\right) - \beth\left(\overline{\eta}_{0(\mu-1)}\right)\right) \eta_{if}^{o(\mu)} \\
&= \oplus_{\mu=1}^{\omega} \left(\beth\left(\overline{\eta}_{0(\mu)}\right) - \beth\left(\overline{\eta}_{0(\mu-1)}\right)\right) \eta_{if}^{o(\mu)}
\end{aligned} \tag{15}$$

where $o(\mu) \leq o(\mu - 1)$.

Theorem 2: Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu = 1, 2, \ldots, \omega$, we evaluate that the final aggregated value of Eq. (15) is again in the shape of CA-IFV such as (16), shown at the bottom of page 10.

Property 4 (Idempotency): Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu = 1, 2, \ldots, \omega$, if $\eta_{if}^{\mu} = \eta_{if} = \left(u_s, \vartheta_{sa}\right), \mu = 1, 2, \ldots, \omega$, then we get

$$CAIFCIHOA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) = \eta_{if} \tag{17}$$

Property 5 (Monotonicity): Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu = 1, 2, \ldots, \omega$, if $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right) \leq \eta{*}_{if}^{\mu} = \left(\left(u{*}_{s_\mu}^{rp}, u{*}_{s_\mu}^{ip}\right), \left(\vartheta{*}_{sa_\mu}^{rp}, \vartheta{*}_{sa_\mu}^{ip}\right)\right)$, then we have

$$CAIFCIHOA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) \leq CAIFCIHOA\left(\eta{*}_{if}^{1}, \eta{*}_{if}^{2}, \ldots, \eta{*}_{if}^{\omega}\right) \tag{18}$$

Property 6 (Boundedness): Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu = 1, 2, \ldots, \omega$, if $\eta_{if}^{-} = \left(\left(\min_{\mu} u_{s_\mu}^{rp}, \min_{\mu} u_{s_\mu}^{ip}\right), \left(\max_{\mu} \vartheta_{sa_\mu}^{rp}, \max_{\mu} \vartheta_{sa_\mu}^{ip}\right)\right)$ and $\eta_{if}^{+} = \left(\left(\max_{\mu} u_{s_\mu}^{rp}, \max_{\mu} u_{s_\mu}^{ip}\right), \left(\min_{\mu} \vartheta_{sa_\mu}^{rp}, \min_{\mu} \vartheta_{sa_\mu}^{ip}\right)\right)$, then

$$CAIFCIHA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{z}\right) = \left(\begin{array}{l}
\left(\dfrac{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{rp}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} - \prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{rp}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} + (\Delta-1)\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}, \right. \\
\left. \dfrac{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{ip}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} - \prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{ip}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} + (\Delta-1)\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}\right), \\
\left(\dfrac{\Delta\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} - \Delta\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}-\vartheta_{sa_\mu}^{rp}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{rp}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} + (\Delta-1)\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}, \right. \\
\left. \dfrac{\Delta\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} - \Delta\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}-\vartheta_{sa_\mu}^{ip}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{ip}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} + (\Delta-1)\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}\right)^{\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}\right)
\end{array}\right)$$

we have

$$\eta_{if}^{-} \leq CAIFCIHOA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) \leq \eta_{if}^{+} \tag{19}$$

Definition 7: Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right)$, $\mu = 1, 2, \ldots, \omega$, the

$$\begin{aligned}
&CAIFCIHA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right)\\
&= \left(\sqsupset\left(\overline{\eta}_{0(1)}\right) - \sqsupset\left(\overline{\eta}_{0(0)}\right)\right)\eta_{if}^{1}\\
&\quad \oplus \left(\sqsupset\left(\overline{\eta}_{0(2)}\right) - \sqsupset\left(\overline{\eta}_{0(1)}\right)\right)\eta_{if}^{2} \oplus \ldots \oplus \left(\sqsupset\left(\overline{\eta}_{0(z)}\right) - \sqsupset\left(\overline{\eta}_{0(z-1)}\right)\right)\eta_{if}^{z} \oplus \left(\sqsupset\left(\overline{\eta}_{0(z+1)}\right) - \sqsupset\left(\overline{\eta}_{0(z+1-1)}\right)\right)\eta_{if}^{z+1}\\
&= \sum_{\mu=1}^{z}\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right) - \sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)\eta_{if}^{\mu} \oplus \left(\sqsupset\left(\overline{\eta}_{0(z+1)}\right) - \sqsupset\left(\overline{\eta}_{0(z+1-1)}\right)\right)\eta_{if}^{z+1}\\
&= \oplus_{\mu=1}^{z}\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right) - \sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)\eta_{if}^{\mu} \oplus \left(\sqsupset\left(\overline{\eta}_{0(z+1)}\right) - \sqsupset\left(\overline{\eta}_{0(z+1-1)}\right)\right)\eta_{if}^{z+1}\\
&= \left(\begin{array}{c}
\left(\dfrac{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} - \prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} + (\Delta-1)\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}, \dfrac{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} - \prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} + (\Delta-1)\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}\right),\\
\left(\dfrac{\Delta\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} - \Delta\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}-\vartheta_{sa_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} + (\Delta-1)\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}, \dfrac{\Delta\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} - \Delta\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}-\vartheta_{sa_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} + (\Delta-1)\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}\right)
\end{array}\right)\\
&\quad \oplus \left(\sqsupset\left(\overline{\eta}_{0(z+1)}\right) - \sqsupset\left(\overline{\eta}_{0(z+1-1)}\right)\right)\eta_{if}^{z+1}\\
&= \left(\begin{array}{c}
\left(\dfrac{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} - \prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} + (\Delta-1)\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}, \dfrac{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} - \prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} + (\Delta-1)\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}\right),\\
\left(\dfrac{\Delta\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} - \Delta\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}-\vartheta_{sa_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} + (\Delta-1)\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{rp}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}, \dfrac{\Delta\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} - \Delta\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}-\vartheta_{sa_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}{\prod_{\mu=1}^{z}\left(1+(\Delta-1)u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)} + (\Delta-1)\prod_{\mu=1}^{z}\left(1-u_{s_\mu}^{ip}\right)^{\left(\sqsupset(\overline{\eta}_{0(\mu)})-\sqsupset(\overline{\eta}_{0(\mu-1)})\right)}}\right)
\end{array}\right)
\end{aligned}$$

well-known theory of CAIFCIHG operator is established or proven by:

$$\begin{aligned}
&\int \eta_{if}\, d\beth \\
&= CAIFCIHG\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) \\
&= \eta_{if}^{1\,(\beth(\overline{\eta}_{0(1)})-\beth(\overline{\eta}_{0(0)}))} \otimes \eta_{if}^{2\,(\beth(\overline{\eta}_{0(2)})-\beth(\overline{\eta}_{0(1)}))} \otimes \ldots \\
&\quad \otimes \eta_{if}^{\omega\,(\beth(\overline{\eta}_{0(\omega)})-\beth(\overline{\eta}_{0(\omega-1)}))} \\
&= \prod_{\mu=1}^{\omega} \eta_{if}^{\mu\,(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))} = \otimes_{\mu=1}^{\omega} \eta_{if}^{\mu\,(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))}
\end{aligned} \tag{20}$$

Theorem 3: Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(\mathrm{u}_{s_\mu}^{rp}, \mathrm{u}_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right)$, $\mu = 1, 2, \ldots, \omega$, we evaluate that the final aggregated value of Eq. (20) is again in the shape of CA-IFV such as in (21), as shown at the bottom of the next page.

Property 7 (Idempotency): Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(\mathrm{u}_{s_\mu}^{rp}, \mathrm{u}_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right)$, $\mu = 1, 2, \ldots, \omega$, if $\eta_{if}^{\mu} = \eta_{if} = (\mathrm{u}_s, \vartheta_{sa})$, $\mu = 1, 2, \ldots, \omega$, then we get

$$CAIFCIHG\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) = \eta_{if} \tag{22}$$

Property 8 (Monotonicity): Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(\mathrm{u}_{s_\mu}^{rp}, \mathrm{u}_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right)$, $\mu = 1, 2, \ldots, \omega$, if $\eta_{if}^{\mu} = \left(\left(\mathrm{u}_{s_\mu}^{rp}, \mathrm{u}_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right) \leq \eta^{*\mu}_{if} = \left(\left(\mathrm{u}^{*rp}_{s_\mu}, \mathrm{u}^{*ip}_{s_\mu}\right), \left(\vartheta^{*rp}_{sa_\mu}, \vartheta^{*ip}_{sa_\mu}\right)\right)$, then we have

$$\begin{aligned}
&CAIFCIHG\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) \\
&\quad \leq CAIFCIHG\left(\eta^{*1}_{if}, \eta^{*2}_{if}, \ldots, \eta^{*\omega}_{if}\right)
\end{aligned} \tag{23}$$

Property 9 (Boundedness): Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(\mathrm{u}_{s_\mu}^{rp}, \mathrm{u}_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right)$, $\mu = 1, 2, \ldots, \omega$,

$$\oplus \left(\begin{array}{c}
\left(\begin{array}{c}
\frac{\left(1+(\Delta-1)\,\mathrm{u}_{s_{z+1}}^{rp}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))} - \left(1-\mathrm{u}_{s_{z+1}}^{rp}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))}}{\left(1+(\Delta-1)\,\mathrm{u}_{s_{z+1}}^{rp}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))} + (\Delta-1)\left(1-\mathrm{u}_{s_{z+1}}^{rp}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))}}, \\
\frac{\left(1+(\Delta-1)\,\mathrm{u}_{s_{z+1}}^{ip}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))} - \left(1-\mathrm{u}_{s_{z+1}}^{ip}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))}}{\left(1+(\Delta-1)\,\mathrm{u}_{s_{z+1}}^{ip}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))} + (\Delta-1)\left(1-\mathrm{u}_{s_{z+1}}^{ip}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))}}
\end{array}\right), \\
\left(\begin{array}{c}
\frac{\Delta\left(1-\mathrm{u}_{s_{z+1}}^{rp}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))} - \Delta\left(1-\mathrm{u}_{s_{z+1}}^{rp}-\vartheta_{sa_{z+1}}^{rp}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))}}{\left(1+(\Delta-1)\,\mathrm{u}_{s_{z+1}}^{rp}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))} + (\Delta-1)\left(1-\mathrm{u}_{s_{z+1}}^{rp}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))}}, \\
\frac{\Delta\left(1-\mathrm{u}_{s_{z+1}}^{ip}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))} - \Delta\left(1-\mathrm{u}_{s_{z+1}}^{ip}-\vartheta_{sa_{z+1}}^{ip}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))}}{\left(1+(\Delta-1)\,\mathrm{u}_{s_{z+1}}^{ip}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))} + (\Delta-1)\left(1-\mathrm{u}_{s_{z+1}}^{ip}\right)^{(\beth(\overline{\eta}_{0(z+1)})-\beth(\overline{\eta}_{0(z+1-1)}))}}
\end{array}\right)
\end{array}\right)$$

$$= \left(\begin{array}{c}
\left(\begin{array}{c}
\frac{\prod_{\mu=1}^{z+1}\left(1+(\Delta-1)\,\mathrm{u}_{s_\mu}^{rp}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))} - \prod_{\mu=1}^{z+1}\left(1-\mathrm{u}_{s_\mu}^{rp}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))}}{\prod_{\mu=1}^{z+1}\left(1+(\Delta-1)\,\mathrm{u}_{s_\mu}^{rp}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))} + (\Delta-1)\prod_{\mu=1}^{z+1}\left(1-\mathrm{u}_{s_\mu}^{rp}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))}}, \\
\frac{\prod_{\mu=1}^{z+1}\left(1+(\Delta-1)\,\mathrm{u}_{s_\mu}^{ip}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))} - \prod_{\mu=1}^{z+1}\left(1-\mathrm{u}_{s_\mu}^{ip}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))}}{\prod_{\mu=1}^{z+1}\left(1+(\Delta-1)\,\mathrm{u}_{s_\mu}^{ip}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))} + (\Delta-1)\prod_{\mu=1}^{z+1}\left(1-\mathrm{u}_{s_\mu}^{ip}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))}}
\end{array}\right), \\
\left(\begin{array}{c}
\frac{\Delta\prod_{\mu=1}^{z+1}\left(1-\mathrm{u}_{s_\mu}^{rp}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))} - \Delta\prod_{\mu=1}^{z+1}\left(1-\mathrm{u}_{s_\mu}^{rp}-\vartheta_{sa_\mu}^{rp}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))}}{\prod_{\mu=1}^{z+1}\left(1+(\Delta-1)\,\mathrm{u}_{s_\mu}^{rp}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))} + (\Delta-1)\prod_{\mu=1}^{z+1}\left(1-\mathrm{u}_{s_\mu}^{rp}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))}}, \\
\frac{\Delta\prod_{\mu=1}^{z+1}\left(1-\mathrm{u}_{s_\mu}^{ip}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))} - \Delta\prod_{\mu=1}^{z+1}\left(1-\mathrm{u}_{s_\mu}^{ip}-\vartheta_{sa_\mu}^{ip}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))}}{\prod_{\mu=1}^{z+1}\left(1+(\Delta-1)\,\mathrm{u}_{s_\mu}^{ip}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))} + (\Delta-1)\prod_{\mu=1}^{z+1}\left(1-\mathrm{u}_{s_\mu}^{ip}\right)^{(\beth(\overline{\eta}_{0(\mu)})-\beth(\overline{\eta}_{0(\mu-1)}))}}
\end{array}\right)
\end{array}\right)$$

if $\eta_{if}^{-}=\left(\left(\min_{\mu} u_{s_\mu}^{rp}, \min_{\mu} u_{s_\mu}^{ip}\right),\left(\max_{\mu} \vartheta_{sa_\mu}^{rp}, \max_{\mu} \vartheta_{sa_\mu}^{ip}\right)\right)$ and $\eta_{if}^{+}=\left(\left(\max_{\mu} u_{s_\mu}^{rp}, \max_{\mu} u_{s_\mu}^{ip}\right),\left(\min_{\mu} \vartheta_{sa_\mu}^{rp}, \min_{\mu} \vartheta_{sa_\mu}^{ip}\right)\right)$, then we have

$$\eta_{if}^{-} \leq CAIFCIHG\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) \leq \eta_{if}^{+} \tag{24}$$

Definition 8: Under n-family of CA-IFVs $\eta_{if}^{\mu}=\left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right),\left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu=1,2,\ldots,\omega$, the well-known theory of CAIFCIHOG operator is established or proven by:

$$\begin{aligned}
&\int \eta_{if}\, d\beth \\
&= CAIFCIHOG\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) \\
&= \eta_{if}^{o(1)\left(\beth\left(\overline{\overline{\eta}}_{0(1)}\right)-\beth\left(\overline{\overline{\eta}}_{0(0)}\right)\right)} \otimes \eta_{if}^{o(2)\left(\beth\left(\overline{\overline{\eta}}_{0(2)}\right)-\beth\left(\overline{\overline{\eta}}_{0(1)}\right)\right)} \otimes \ldots \\
&\quad \otimes \eta_{if}^{o(\omega)\left(\beth\left(\overline{\overline{\eta}}_{0(\omega)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\omega-1)}\right)\right)} = \prod_{\mu=1}^{\omega} \eta_{if}^{o(\mu)\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}
\end{aligned}$$

$$\begin{aligned}
&CAIFCIHOA\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) \\
&= \left(\begin{array}{c}
\left(\begin{array}{c}
\dfrac{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1) u_{s_{o(\mu)}}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}-\prod_{\mu=1}^{\omega}\left(1-u_{s_{o(\mu)}}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1) u_{s_{o(\mu)}}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}+(\Delta-1) \prod_{\mu=1}^{\omega}\left(1-u_{s_{o(\mu)}}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}, \\
\dfrac{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1) u_{s_{o(\mu)}}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}-\prod_{\mu=1}^{\omega}\left(1-u_{s_{o(\mu)}}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1) u_{s_{o(\mu)}}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}+(\Delta-1) \prod_{\mu=1}^{\omega}\left(1-u_{s_{o(\mu)}}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}
\end{array}\right), \\
\left(\begin{array}{c}
\dfrac{\Delta \prod_{\mu=1}^{\omega}\left(1-u_{s_{o(\mu)}}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}-\Delta \prod_{\mu=1}^{\omega}\left(1-u_{s_{o(\mu)}}^{rp}-\vartheta_{sa_{o(\mu)}}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1) u_{s_{o(\mu)}}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}+(\Delta-1) \prod_{\mu=1}^{\omega}\left(1-u_{s_{o(\mu)}}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}, \\
\dfrac{\Delta \prod_{\mu=1}^{\omega}\left(1-u_{s_{o(\mu)}}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}-\Delta \prod_{\mu=1}^{\omega}\left(1-u_{s_{o(\mu)}}^{ip}-\vartheta_{sa_{o(\mu)}}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1) u_{s_{o(\mu)}}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}+(\Delta-1) \prod_{\mu=1}^{\omega}\left(1-u_{s_{o(\mu)}}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}
\end{array}\right)
\end{array}\right)
\end{aligned} \tag{16}$$

$$\begin{aligned}
&CAIFCIHG\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) \\
&= \left(\begin{array}{c}
\left(\begin{array}{c}
\dfrac{\Delta \prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}-\Delta \prod_{\mu=1}^{\omega}\left(1-u_{s_\mu}^{rp}-\vartheta_{sa_\mu}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1) \vartheta_{sa_\mu}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}+(\Delta-1) \prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}, \\
\dfrac{\Delta \prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}-\Delta \prod_{\mu=1}^{\omega}\left(1-u_{s_\mu}^{ip}-\vartheta_{sa_\mu}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1) \vartheta_{sa_\mu}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}+(\Delta-1) \prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}
\end{array}\right), \\
\left(\begin{array}{c}
\dfrac{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1) \vartheta_{sa_\mu}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}-\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1) \vartheta_{sa_\mu}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}+(\Delta-1) \prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{rp}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}, \\
\dfrac{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1) \vartheta_{sa_\mu}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}-\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1) \vartheta_{sa_\mu}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}+(\Delta-1) \prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_\mu}^{ip}\right)^{\left(\beth\left(\overline{\overline{\eta}}_{0(\mu)}\right)-\beth\left(\overline{\overline{\eta}}_{0(\mu-1)}\right)\right)}}
\end{array}\right)
\end{array}\right)
\end{aligned} \tag{21}$$

$$= \otimes_{\mu=1}^{\omega} \eta_{if}^{o(\mu)\left(\beth\left(\overline{\eta}_{0(\mu)}\right)-\beth\left(\overline{\eta}_{0(\mu-1)}\right)\right)} \tag{25}$$

where $o(\mu) \leq o(\mu-1)$.

Theorem 4: Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu = 1, 2, \ldots, \omega$, we evaluate that the final aggregated value of Eq. (25) is again in the shape of CA-IFV such as in (26), as shown at the bottom of the next page.

Property 10 (Idempotency): Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu = 1, 2, \ldots, \omega$, if $\eta_{if}^{\mu} = \eta_{if} = (u_s, \vartheta_{sa}), \mu = 1, 2, \ldots, \omega$, then we get

$$CAIFCIHOG\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) = \eta_{if} \tag{27}$$

Property 11 (Monotonicity): Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu = 1, 2, \ldots, \omega$, if $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right) \leq \eta^{*\mu}_{if} = \left(\left(u^{*rp}_{s_\mu}, u^{*ip}_{s_\mu}\right), \left(\vartheta^{*rp}_{sa_\mu}, \vartheta^{*ip}_{sa_\mu}\right)\right)$, then we have

$$CAIFCIHOG\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) \leq CAIFCIHOG\left(\eta^{*1}_{if}, \eta^{*2}_{if}, \ldots, \eta^{*\omega}_{if}\right) \tag{28}$$

Property 12 (Boundedness): Under n-family of CA-IFVs $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu = 1, 2, \ldots, \omega$, if $\eta_{if}^{-} = \left(\left(\min_{\mu} u_{s_\mu}^{rp}, \min_{\mu} u_{s_\mu}^{ip}\right), \left(\max_{\mu} \vartheta_{sa_\mu}^{rp}, \max_{\mu} \vartheta_{sa_\mu}^{ip}\right)\right)$ and $\eta_{if}^{+} = \left(\left(\max_{\mu} u_{s_\mu}^{rp}, \max_{\mu} u_{s_\mu}^{ip}\right), \left(\min_{\mu} \vartheta_{sa_\mu}^{rp}, \min_{\mu} \vartheta_{sa_\mu}^{ip}\right)\right)$, then we have

$$\eta_{if}^{-} \leq CAIFCIHOG\left(\eta_{if}^{1}, \eta_{if}^{2}, \ldots, \eta_{if}^{\omega}\right) \leq \eta_{if}^{+} \tag{29}$$

IV. TOPSIS METHOD BASED ON DECISION-MAKING PROCEDURE

The major influence of this procedure is to improve the worth of the decision-making procedure by utilizing the TOPSIS method in the presence of the derived operators for CA-IFVs. For this, we select the m-alternatives $\eta_{if}^{AL} = \left\{\eta_{if}^{a-1}, \eta_{if}^{a-2}, \ldots, \eta_{if}^{a-m}\right\}$ and n- attributes $\eta_{if}^{AT} = \left\{\eta_{if}^{at-1}, \eta_{if}^{at-2}, \ldots, \eta_{if}^{at-\omega}\right\}$. Further, to compute a matrix we needed to use the data of alternatives and their attributes, where each attribute in every alternative will be computed in the shape of CA-IFVs, such as the pairs $(u_s(\sigma), u_s(\sigma))$ and $(\vartheta_{sa}(\sigma), \vartheta_{sa}(\sigma))$ used as truth and falsity information in the form of complex numbers with a valuable and dominant characteristic such as $0 \leq u_s^{rp}(\sigma) + \vartheta_{sa}^{rp}(\sigma) \leq 1$ and $0 \leq u_s^{ip}(\sigma) + \vartheta_{sa}^{ip}(\sigma) \leq 1$. Moreover, under the presence of the above information, we stated the theory of neutral grade such as: $^{\circ}F_r(\sigma) = \left({}^{\circ}F_r^{rp}(\sigma), {}^{\circ}F_r^{ip}(\sigma)\right) = \left(1 - \left(u_s^{rp}(\sigma) + \vartheta_{sa}^{rp}(\sigma)\right), 1 - \left(u_s^{ip}(\sigma) + \vartheta_{sa}^{ip}(\sigma)\right)\right)$ and the simple form of CA-IFVs is stated: $\eta_{if}^{\mu} = \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right), \mu = 1, 2, \ldots, \omega$. Therefore, after this analysis, we aim to compute the main procedure of the TOPSIS technique, such as

Step 1: Assigning values of each attribute in every alternative will be possible in the form of cost or benefit types, therefore, if the values of attributes are arranged in the form of cost types, then with the use of the below idea, we require to normalize it, such as

$$F = \begin{cases} \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right) & \textit{for beωefit} \\ \left(\left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right), \left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right)\right) & \textit{for cost} \end{cases}$$

When the values of attributes are arranged in the form of benefit types, then not required to normalize it.

Step 2: The arranged values are required to be aggregated with the help of two basic theories such as *CAIFCIHA* operator or *CAIFCIHG* operator.

Step 3: After successfully evaluating the theory of aggregated information, we further find the positive and negative ideal solutions such as:

$$\eta_{if}^{+} = \left\{\begin{matrix} \left(\left(\max_{\mu} u_{s_{\mu1}}^{rp}, \max_{\mu} u_{s_{\mu1}}^{ip}\right), \left(\min_{\mu} \vartheta_{sa_{\mu1}}^{rp}, \min_{\mu} \vartheta_{sa_{\mu1}}^{ip}\right)\right), \\ \left(\left(\max_{\mu} u_{s_{\mu2}}^{rp}, \max_{\mu} u_{s_{\mu2}}^{ip}\right), \left(\min_{\mu} \vartheta_{sa_{\mu2}}^{rp}, \min_{\mu} \vartheta_{sa_{\mu2}}^{ip}\right)\right) \\ , \ldots, \left(\left(\max_{\mu} u_{s_{\mu\omega}}^{rp}, \max_{\mu} u_{s_{\mu\omega}}^{ip}\right), \left(\min_{\mu} \vartheta_{sa_{\mu\omega}}^{rp}, \min_{\mu} \vartheta_{sa_{\mu\omega}}^{ip}\right)\right) \end{matrix}\right\}$$

$$\eta_{if}^{-} = \left\{\begin{matrix} \left(\left(\min_{\mu} u_{s_{\mu1}}^{rp}, \min_{\mu} u_{s_{\mu1}}^{ip}\right), \left(\max_{\mu} \vartheta_{sa_{\mu1}}^{rp}, \max_{\mu} \vartheta_{sa_{\mu1}}^{ip}\right)\right), \\ \left(\left(\min_{\mu} u_{s_{\mu2}}^{rp}, \min_{\mu} u_{s_{\mu2}}^{ip}\right), \left(\max_{\mu} \vartheta_{sa_{\mu2}}^{rp}, \max_{\mu} \vartheta_{sa_{\mu2}}^{ip}\right)\right) \\ , \ldots, \left(\left(\min_{\mu} u_{s_{\mu\omega}}^{rp}, \min_{\mu} u_{s_{\mu\omega}}^{ip}\right), \left(\max_{\mu} \vartheta_{sa_{\mu\omega}}^{rp}, \max_{\mu} \vartheta_{sa_{\mu\omega}}^{ip}\right)\right) \end{matrix}\right\}$$

Step 4: With the help of the above information, we further find the discrimination measures, such as

$$D^{+}\left(\eta_{if}^{\mu}, \eta_{if}^{+}\right) = \frac{1}{\omega}\sum_{\mu=1}^{\omega}\left(\frac{\left(\begin{matrix} u_{s_\mu}^{rp} u_{s_\mu}^{+\,rp} + u_{s_\mu}^{rp} u_{s_\mu}^{+\,rp} + {}^{\circ}F_{r_\mu}^{rp}\,{}^{\circ}F_{r_\mu}^{+\,rp} + \\ u_{s_\mu}^{ip} u_{s_\mu}^{+\,ip} + u_{s_\mu}^{ip} u_{s_\mu}^{+\,ip} + {}^{\circ}F_{r_\mu}^{ip}\,{}^{\circ}F_{r_\mu}^{+\,ip} \end{matrix}\right)}{\sqrt{\begin{matrix} u_{s_\mu}^{rp\,2} + u_{s_\mu}^{rp\,2} + {}^{\circ}F_{r_\mu}^{rp\,2} + \\ u_{s_\mu}^{ip\,2} + u_{s_\mu}^{ip\,2} + {}^{\circ}F_{r_\mu}^{ip\,2} \end{matrix}} \times \sqrt{\begin{matrix} u^{rp+\,2}_{s_\mu} + u^{rp+\,2}_{s_\mu} + {}^{\circ}F^{rp+\,2}_{r_\mu} \\ + u^{ip+\,2}_{s_\mu} + u^{ip+\,2}_{s_\mu} + {}^{\circ}F^{ip+\,2}_{r_\mu} \end{matrix}}}\right)$$

$$D^{-}\left(\eta_{if}^{\mu}, \eta_{if}^{-}\right)$$

TABLE 1. Information matrix one.

	η_{if}^{at-1}	η_{if}^{at-2}	η_{if}^{at-3}	η_{if}^{at-4}
η_{if}^{a-1}	$((0.5,0.4),(0.3,0.2))$	$((0.51,0.41),(0.31,0.21))$	$((0.52,0.42),(0.32,0.22))$	$((0.53,0.43),(0.33,0.23))$
η_{if}^{a-2}	$((0.4,0.3),(0.4,0.3))$	$((0.41,0.31),(0.41,0.31))$	$((0.42,0.32),(0.42,0.32))$	$((0.43,0.33),(0.43,0.33))$
η_{if}^{a-3}	$((0.6,0.2),(0.1,0.3))$	$((0.61,0.21),(0.11,0.31))$	$((0.62,0.22),(0.12,0.32))$	$((0.63,0.23),(0.13,0.33))$
η_{if}^{a-4}	$((0.5,0.6),(0.2,0.1))$	$((0.51,0.61),(0.21,0.11))$	$((0.52,0.62),(0.22,0.12))$	$((0.53,0.63),(0.23,0.13))$

TABLE 2. Information matrix two.

	η_{if}^{at-1}	η_{if}^{at-2}	η_{if}^{at-3}	η_{if}^{at-4}
η_{if}^{a-1}	$((0.3,0.4),(0.2,0.1))$	$((0.31,0.41),(0.21,0.11))$	$((0.32,0.42),(0.22,0.12))$	$((0.33,0.43),(0.23,0.13))$
η_{if}^{a-2}	$((0.6,0.7),(0.1,0.1))$	$((0.61,0.71),(0.11,0.11))$	$((0.62,0.72),(0.12,0.12))$	$((0.63,0.73),(0.13,0.13))$
η_{if}^{a-3}	$((0.6,0.5),(0.2,0.3))$	$((0.61,0.51),(0.21,0.31))$	$((0.62,0.52),(0.22,0.32))$	$((0.63,0.53),(0.23,0.33))$
η_{if}^{a-4}	$((0.2,0.3),(0.2,0.3))$	$((0.21,0.31),(0.21,0.31))$	$((0.22,0.32),(0.22,0.32))$	$((0.23,0.33),(0.23,0.33))$

$$= \frac{1}{\omega}\sum_{\mu=1}^{\omega}\left(\frac{\begin{pmatrix}\mathrm{u}_{s_\mu}^{rp}\mathrm{u}_{s_\mu}^{-\ rp}+\mathrm{u}_{s_\mu}^{rp}\mathrm{u}_{s_\mu}^{-\ rp}+{}^{\circ}\mathrm{F}_{r_\mu}^{rp}\,{}^{\circ}\mathrm{F}_{r_\mu}^{-\ rp}+\\ \mathrm{u}_{s_\mu}^{ip}\mathrm{u}_{s_\mu}^{-\ ip}+\mathrm{u}_{s_\mu}^{ip}\mathrm{u}_{s_\mu}^{-\ ip}+{}^{\circ}\mathrm{F}_{r_\mu}^{ip}\,{}^{\circ}\mathrm{F}_{r_\mu}^{-\ ip}\end{pmatrix}}{\sqrt{\begin{matrix}\mathrm{u}_{s_\mu}^{rp\,2}+\mathrm{u}_{s_\mu}^{rp\,2}+{}^{\circ}\mathrm{F}_{r_\mu}^{rp\,2}+\\ \mathrm{u}_{s_\mu}^{ip\ 2}+\mathrm{u}_{s_\mu}^{ip\ 2}+{}^{\circ}\mathrm{F}_{r_\mu}^{ip\ 2}\end{matrix}}\times\sqrt{\begin{matrix}\mathrm{u}^{rp}{}_{s_\mu}^{-\ 2}+\mathrm{u}^{rp}{}_{s_\mu}^{-\ 2}+{}^{\circ}\mathrm{F}^{rp}{}_{r_\mu}^{-\ 2}\\ +\mathrm{u}^{ip}{}_{s_\mu}^{-\ 2}+\mathrm{u}^{ip}{}_{s_\mu}^{-\ 2}+{}^{\circ}\mathrm{F}^{ip}{}_{r_\mu}^{-\ 2}\end{matrix}}}\right)$$

Step 5: Under the above theories, we further find the closeness measures, such as

$$C^{\mu}=\frac{D^{-}\left(\eta_{if}^{\mu},\eta_{if}^{-}\right)}{D^{+}\left(\eta_{if}^{\mu},\eta_{if}^{+}\right)+D^{-}\left(\eta_{if}^{\mu},\eta_{if}^{-}\right)}$$

Noticed that $0\leq C^{\mu}\leq 1$.

Step 6: We aim to draw the ranking data for finding the best optimal form for the collection of preferences.

V. NUMERICAL EXAMPLE

In this section, the available or describing information is taken from [33] for evaluating or verifying the derived

$$CAIFCIHOG\left(\eta_{if}^{1},\eta_{if}^{2},\ldots,\eta_{if}^{\omega}\right)$$

$$=\left(\begin{array}{c}\left(\begin{array}{c}\dfrac{\Delta\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_{o(\mu)}}^{rp}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}-\Delta\prod_{\mu=1}^{\omega}\left(1-\mathrm{u}_{s_{o(\mu)}}^{rp}-\vartheta_{sa_{o(\mu)}}^{rp}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\vartheta_{sa_{o(\mu)}}^{rp}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}+(\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_{o(\mu)}}^{rp}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}},\\ \dfrac{\Delta\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_{o(\mu)}}^{ip}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}-\Delta\prod_{\mu=1}^{\omega}\left(1-\mathrm{u}_{s_{o(\mu)}}^{ip}-\vartheta_{sa_{o(\mu)}}^{ip}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\vartheta_{sa_{o(\mu)}}^{ip}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}+(\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_{o(\mu)}}^{ip}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}\end{array}\right),\\ \left(\begin{array}{c}\dfrac{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\vartheta_{sa_{o(\mu)}}^{rp}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}-\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_{o(\mu)}}^{rp}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\vartheta_{sa_{o(\mu)}}^{rp}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}+(\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_{o(\mu)}}^{rp}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}},\\ \dfrac{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\vartheta_{sa_{o(\mu)}}^{ip}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}-\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_{o(\mu)}}^{ip}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}{\prod_{\mu=1}^{\omega}\left(1+(\Delta-1)\vartheta_{sa_{o(\mu)}}^{ip}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}+(\Delta-1)\prod_{\mu=1}^{\omega}\left(1-\vartheta_{sa_{o(\mu)}}^{ip}\right)^{\left(\sqsupset\left(\overline{\eta}_{0(\mu)}\right)-\sqsupset\left(\overline{\eta}_{0(\mu-1)}\right)\right)}}\end{array}\right)\end{array}\right) \quad (26)$$

TABLE 3. Information matrix three.

	η_{if}^{at-1}	η_{if}^{at-2}	η_{if}^{at-3}	η_{if}^{at-4}
η_{if}^{a-1}	$((0.6,0.5),(0.2,0.3))$	$((0.61,0.51),(0.21,0.31))$	$((0.62,0.52),(0.22,0.32))$	$((0.63,0.53),(0.23,0.33))$
η_{if}^{a-2}	$((0.2,0.3),(0.2,0.3))$	$((0.21,0.31),(0.21,0.31))$	$((0.22,0.32),(0.22,0.32))$	$((0.23,0.33),(0.23,0.33))$
η_{if}^{a-3}	$((0.5,0.4),(0.3,0.2))$	$((0.51,0.41),(0.31,0.21))$	$((0.52,0.42),(0.32,0.22))$	$((0.53,0.43),(0.33,0.23))$
η_{if}^{a-4}	$((0.4,0.3),(0.4,0.3))$	$((0.41,0.31),(0.41,0.31))$	$((0.42,0.32),(0.42,0.32))$	$((0.43,0.33),(0.43,0.33))$

TABLE 4. Information matrix four.

	η_{if}^{at-1}	η_{if}^{at-2}	η_{if}^{at-3}	η_{if}^{at-4}
η_{if}^{a-1}	$((0.5,0.2),(0.1,0.3))$	$((0.51,0.21),(0.11,0.31))$	$((0.52,0.22),(0.12,0.32))$	$((0.53,0.23),(0.13,0.33))$
η_{if}^{a-2}	$((0.4,0.3),(0.2,0.4))$	$((0.41,0.31),(0.21,0.41))$	$((0.42,0.32),(0.22,0.42))$	$((0.43,0.33),(0.23,0.43))$
η_{if}^{a-3}	$((0.3,0.4),(0.3,0.2))$	$((0.31,0.41),(0.31,0.21))$	$((0.32,0.42),(0.32,0.22))$	$((0.33,0.43),(0.33,0.23))$
η_{if}^{a-4}	$((0.2,0.5),(0.4,0.1))$	$((0.21,0.51),(0.41,0.11))$	$((0.22,0.52),(0.42,0.12))$	$((0.22,0.52),(0.42,0.12))$

TABLE 5. CAIFCIHA and CAIFCIHG matrix.

	η_{if}^{at-1}	η_{if}^{at-2}
η_{if}^{a-1}	$((0.1979,0.1293),(0.1741,0.1101))$	$((0.2028,0.1337),(0.1874,0.1169))$
η_{if}^{a-2}	$((0.1916,0.1505),(0.2304,0.194))$	$((0.1964,0.1553),(0.2451,0.2053))$
η_{if}^{a-3}	$((0.2231,0.1199),(0.1047,0.1623))$	$((0.2285,0.1243),(0.1147,0.172))$
η_{if}^{a-4}	$((0.1554,0.2503),(0.1469,0.075))$	$((0.1601,0.2558),(0.1262,0.0848))$
	η_{if}^{at-3}	η_{if}^{at-4}
η_{if}^{a-1}	$((0.2078,0.1381),(0.2019,0.1238))$	$((0.2128,0.1426),(0.2177,0.1308))$
η_{if}^{a-2}	$((0.2014,0.1601),(0.2611,0.2173))$	$((0.2063,0.1649),(0.2786,0.2302))$
η_{if}^{a-3}	$((0.234,0.1288),(0.1253,0.1822))$	$((0.2396,0.1332),(0.1366,0.1932))$
η_{if}^{a-4}	$((0.1649,0.2615),(0.1659,0.0952))$	$((0.1697,0.2672),(0.176,0.1063))$

TABLE 6. CAIFCIHA and CAIFCIHG matrix.

	$CAIFCIHA$	$CAIFCIHG$
η_{if}^{a-1}	$((0.0821,0.0538),(0.0943,0.0534))$	$((0.1007,0.0607),(0.0763,0.047))$
η_{if}^{a-2}	$((0.0795,0.0626),(0.1259,0.0995))$	$((0.1027,0.0766),(0.1004,0.0835))$
η_{if}^{a-3}	$((0.0928,0.05),(0.0571,0.08))$	$((0.1076,0.0588),(0.0463,0.0696))$
η_{if}^{a-4}	$((0.0645,0.1043),(0.0743,0.0428))$	$((0.0758,0.119),(0.0631,0.0343))$

computational method of the CAIF transportation problem. Here, we considered all the benefit types of transportation which are stated in the form of CAIF values. It is very interesting to see that the benefit types of data obtained from derived approaches are the maximum that exists. For evaluating the above problem, we consider four main transportation problems as education, employment, electricity problems, and economic ratio which are stated as an alternative, and try to take our decision under the below criteria such as population ratio, uneducated people's ratio, poverty, and facilities. Therefore, after this analysis, we aim to compute the main procedure of the TOPSIS technique, such as

TABLE 7. Score information matrix.

	$CAIFCIHA$	$CAIFCIHG$
η_{if}^{a-1}	-0.0059	0.019
η_{if}^{a-2}	-0.0417	-0.0023
η_{if}^{a-3}	0.0028	0.0252
η_{if}^{a-4}	0.0259	0.0487

Step 1: Assigning values of each attribute in every alternative will be possible in the form of cost or benefit types, therefore, if the values of attributes are arranged in the form

TABLE 8. Comparison evaluation matrix.

Methods	Score Values	Ranking Values
Akram et al. [11]	$-0.0039, -0.0061, -0.0058, -0.0062$	$\eta_{if}^{a-1} \geq \eta_{if}^{a-3} \geq \eta_{if}^{a-2} \geq \eta_{if}^{a-4}$
Mahmood et al. [12]	********************************	********************************
Jia and Wang [13]	********************************	********************************
Tan and Chen [14]	********************************	********************************
Xu [15]	********************************	********************************
Xu and Yager [16]	********************************	********************************
Zhao et al. [17]	********************************	********************************
TOPSIS technique	0.502,0.5064,0.4999,0.4947	$C^2 \geq C^1 \geq C^3 \geq C^4$
CAIFCIHA	$-0.0059, -0.0417, 0.0028, 0.0259$	$\eta_{if}^{a-4} \geq \eta_{if}^{a-3} \geq \eta_{if}^{a-2} \geq \eta_{if}^{a-1}$
CAIFCIHG	$0.019, -0.0023, 0.0252, 0.0487$	$\eta_{if}^{a-4} \geq \eta_{if}^{a-3} \geq \eta_{if}^{a-2} \geq \eta_{if}^{a-1}$

of cost types, then with the use of the below idea, we require to normalize it, such as

$$F = \begin{cases} \left(\left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right), \left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right)\right) & \textit{for beΩefit} \\ \left(\left(\vartheta_{sa_\mu}^{rp}, \vartheta_{sa_\mu}^{ip}\right), \left(u_{s_\mu}^{rp}, u_{s_\mu}^{ip}\right)\right) & \textit{for cost} \end{cases} \tag{30}$$

When the values of attributes are arranged in the form of benefit types, then not required to normalize it. But the data in Table 1, Table 2, Table 3, and Table 4 are in the shape of benefit types.

Step 2: The arranged values are required to be aggregated with the help of two basic theories such as *CAIFCIHA* operator or *CAIFCIHG* operator, see Table 5.

Step 3: After successfully evaluating the theory of aggregated information, we further find the positive and negative ideal solutions such as:

$$\eta_{if}^{+} = \begin{Bmatrix} ((0.2231, 0.2503), (0.1047, 0.075)), \\ ((0.2285, 0.2558), (0.1147, 0.0848)), \\ ((0.234, 0.2615), (0.1253, 0.0952)), \\ ((0.2396, 0.2672), (0.1366, 0.1063)) \end{Bmatrix}$$

$$\eta_{if}^{-} = \begin{Bmatrix} ((0.1554, 0.1199), (0.2304, 0.1623)), \\ ((0.1601, 0.1243), (0.2451, 0.2053)), \\ ((0.1649, 0.1288), (0.2611, 0.2173)), \\ ((0.1697, 0.1332), (0.2786, 0.2302)) \end{Bmatrix}$$

Step 4: With the help of the above information, we further find the discrimination measures, such as

$$\begin{aligned} D^{+}\left(\eta_{if}^{1}, \eta_{if}^{+}\right) &= 0.9836, D^{+}\left(\eta_{if}^{2}, \eta_{if}^{+}\right) \\ &= 0.9725, D^{+}\left(\eta_{if}^{3}, \eta_{if}^{+}\right) \\ &= 0.9866, D^{+}\left(\eta_{if}^{4}, \eta_{if}^{+}\right) = 0.9965 \\ D^{-}\left(\eta_{if}^{1}, \eta_{if}^{-}\right) &= 0.9914, D^{-}\left(\eta_{if}^{2}, \eta_{if}^{-}\right) \\ &= 0.9977, D^{-}\left(\eta_{if}^{3}, \eta_{if}^{-}\right) \\ &= 0.9862, D^{-}\left(\eta_{if}^{4}, \eta_{if}^{-}\right) = 0.9757 \end{aligned}$$

Step 5: Under the above theories, we further find the closeness measures, such as

$$\begin{aligned} C^1 &= \frac{D^{-}\left(\eta_{if}^{1}, \eta_{if}^{-}\right)}{D^{+}\left(\eta_{if}^{1}, \eta_{if}^{+}\right) + D^{-}\left(\eta_{if}^{1}, \eta_{if}^{-}\right)} \\ &= \frac{0.9914}{0.9836 + 0.9914} = 0.502, C^2 = 0.5064, C^3 \\ &= 0.4999, C^4 = 0.4947 \end{aligned}$$

where $0 \leq C^{\mu} \leq 1$.

Step 6: We aim to draw the ranking data for finding the best optimal form for the collection of preferences, such as

$$C^2 \geq C^1 \geq C^3 \geq C^4$$

The valuable decision is C^2 according to the TOPSIS technique. Further, we separately evaluated the information in Table 5 by using the theory of CAIFCIHA and CAIFCIHG operators and try to derive the best one, therefore, the aggregated information is mentioned in Table 6.

Then, under the presence of the data in Table 6, the score values are stated in Table 7.

Finally, we aim to draw the ranking data for finding the best optimal form for the collection of preferences, such as

$$\eta_{if}^{a-4} \geq \eta_{if}^{a-3} \geq \eta_{if}^{a-2} \geq \eta_{if}^{a-1}$$
$$\eta_{if}^{a-4} \geq \eta_{if}^{a-3} \geq \eta_{if}^{a-2} \geq \eta_{if}^{a-1}$$

The valuable option is η_{if}^{a-4} according to the theory of CAIFCIHA and CAIFCIHG operators. Further, we evaluate the comparison between derived operators and prevailing operators in the presence of the information in Table 5.

VI. COMPARATIVE ANALYSIS

Comparison analysis is part of this section, where for comparison we have required some existing operators based on various prevailing ideas. Comparing the proposed work with some prevailing works are part of every valuable analysis. For this, we select the following theories, such as Akram et al. [11] exposed the Hamacher operators for CIFSs. Mahmood et al. [12] examined the Hamacher CI aggregation operators for IFSs. Jia and Wang [13] invented the CI operators for IFSs. Tan and Chen [14] proposed the CI operators for induced IFSs. The simple aggregation operators for IFS were derived by Xu [15]. Further, Xu and Yager [16] invented the geometric operators for IFSs. Zhao et al. [17] exposed the generalized or extended aggregation operators for IFSs. Therefore, the comparison information is listed in Table 8.

From the data in Table 8, we clear that the theory of Akram et al. [11] gives their best optimal as η_{if}^{a-1}, and the proposed theories are given C^2 and η_{if}^{a-4} According to the theory of the TOPSIS technique and CAIFCIHA and CAIFCIHG operators. The other existing laws failed because they proposed their ideas based on IFS which is the special case of the derived work.

VII. CONCLUSION

This research work presents some dominant mathematical strategies for solving the complicated real-life applications by incorporating the theory of complex intuitionistic fuzzy system. The key features of this research work are sated as follows:

1) We investigated some prominent operational laws of th Hamcher t-nom and t-conorm in the presence of compelx intuitionistic fuzzy information.
2) By utilizing the theory of complex intuitionistic fuzzy framework, we derived some mathematical approaches like CAIFCIHA, CAIFCIHOA, CAIFCIHG, and CAIFCIHOG operators.
3) To reveal the intensity and applicability of developed apporahces, some noteable characteristics are also studied.
4) We discussed an innovative approach of an advanced decision-making problem such as TOPSIS method under considering the diaganosed research work.
5) A numerical example is also established to evaluate a suitabl transportation system based on developed mathematical methodologies.
6) We compared our derived results with some prevailing results to show the worth and reliability of the exposed approaches.

Moreover, we examined our proposed research work having a lot of advantages, but some time decision-maker faces multichallenges to find out the solution of the real life application. To serve this such situaitons, we will explore our proposed research work in different fuzzy domains like bipolar complex fuzzy theory [34], Farmatean fuzzy sets [35], bipolar soft set [36] and t-shperical fuzzy hyper soft theory [37]. Furthermore, we will try to resolve some crucial real-life applications such as artificial intelligence, machine learning, and clustering analysis, because these applications are very valuable and feasible nowadays.

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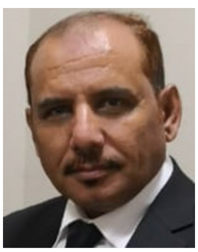

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