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RESEARCH ARTICLE

A New Algebraic Structure of Complex Pythagorean Fuzzy Subfield

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ABSTRACT The concept of complex Pythagorean fuzzy set (**CPFS**) is recent development in the field of fuzzy set (**FS**) theory. The significance of this concept lies in the fact that this theory assigned membership grades ψ and non-membership grades $\hat{\psi}$ from unit circle in plane, i.e., in the form of a complex number with the condition $(\psi)^2 + (\hat{\psi})^2 \leq 1$ instead from [0, 1] interval. This is an expressive technique for dealing with uncertain circumstances. The aim of this study is to proceed the classification of the unique framework of **CPFS** in algebraic structure that is field theory and examine its numerous algebraic features. Also, we initiate the important examples and results of certain field. Furthermore, we illustrate that every complex Pythagorean fuzzy subfield (**CPFSF**) generates two Pythagorean fuzzy subfields (**PFSFs**). We also prove many useful algebraic aspects of this notion for a **CPFSF**. Moreover, we demonstrate that intersection of two complex Pythagorean fuzzy subfields (**CPFSFs**) is also **CPFSF**. Additionally, we discuss the novel idea of level subsets of **CPFSFs** and demonstrate that level subset of **CPFSF** form subfield. Additionally, we improve the application of this theory to show the concept of the direct product of two **CPFSFs** is also a **CPFSF** and produce several novel results on direct product of **CPFSFs**. Finally, we explore the homomorphic images and inverse images of **CPFSFs**.

INDEX TERMS Complex Pythagorean fuzzy set, complex Pythagorean fuzzy subfield, level subset of complex Pythagorean fuzzy subfield, product of complex Pythagorean fuzzy sets, product of complex Pythagorean fuzzy subfields.
AMS (MOS) Subject Classifications: 11E57,08A72,03E72.

I. INTRODUCTION

A field is an algebraic structure which play a significant role in number theory, algebra and many other areas of mathematics. Fields serve as development notions in various mathematical domains. Field theory is an extremely helpful mathematical area that is used an extensively in the study of electronic circuits, coding theory, cryptography, cyber security and combinatorial mathematics. McEliece [14] introduced finite fields to both the engineering and computer sciences in 2012. In addition, algebraic coding theory defined by the concept of finite fields was studied. Additionally, covered finite fields in-depth in his treatment of coding theory courses in Volkswage. Zadeh [1] pioneered the concept of fuzzy sets (**FSs**) in 1965. The fuzzy set (**FS**) theory provides an approach for scientifically describing the uncertainty associated with human mental processes like intelligent and thinking. In many real-world sectors, this innovative idea is used to effectively simulate uncertainty. A membership value (**MV**) with the range [0,1] defines a **FS** such that **FS** $\mathbb{Q}$ of a classical set $\mathbb{R}$ is define as $\mathbb{Q} = \{(s, \eta_{\mathbb{Q}}(s)) : s \in \mathbb{R}\}$, thus $\eta_{\mathbb{Q}} : \mathbb{R} :\longrightarrow [0, 1]$ is known as **MV**. This idea also help us to find the better way to solve challenges in ordinary situations through effective decision-making. The **MV** of a variable in

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a **FS** is a single integer between 0 and 1. In 1986, Atanassov established intuitionistic fuzzy sets (**IFSs**). The intuitionistic fuzzy set (**IFS**) is defined by the membership value **MV** and non-membership value (**NMV**) with scale [0,1]. The most broad description of a **FS** is an **IFS**. Numerous areas, including decision-making, modelling theories, pattern detection and medical diagnostics, can benefit from the implementation of **IFS**. **IFS** is used in numerous fields, such as neural network modelling, gas pipeline network vulnerability analysis and medical diagnostics. Rosenfeld [2] introduced the notion of fuzzy subgroups and created a link between the concept of fuzzy theory and group theory. In 1989, Biswas [5] initiated intuitionistic fuzzy subgroups (**IFSGs**) and studied the fundamental results. Osman [6] defined the t-fuzzy subfield in 1989 and different types of its properties was prevalent. Additionally, the idea of t-fuzzy subalgebra is presented, along with the idea of t-fuzzy vector subspaces of a t-fuzzy subfield of a field. In 1990, Malik and Mordeson [7] established some fundamental characteristics of fuzzy subfields (**FSFs**) with certain field. Field extensions characteristics were described in terms of their **FSFs** and vice versa. As way of illustration, they demonstrated that a field extension is finite dimensional KF iff the image of each **FSFs** A of K such that $\{x \in K \ldots A(1) \geq A(1)\} \supseteq F$ is finite. In 1992, Malik and Mordeson [8] presented necessary and sufficient of extensions of fuzzy subrings and fuzzy ideals. The idea of fuzzy algebraic field extensions was introduced by Mordeson [9] in 1992. Also, identified the conditions under an extension of a fuzzy algebraic field has a distinct maximum fuzzy intermediate field that is both fuzzy separable algebraically and fuzzy completely inseparable. In 1992, Mordeson [10] demonstrated that each fuzzy subfield of the additive group F is the direct sum of its fuzzy subgroups. The sup property is present in the fuzzy intermediate field (**FIF**) of such an expansion, which is a property shared by all **FIF** with the sup property, as explained by Volf [11] using intermediate fields with a chain. A linguistic interval-valued intuitionistic fuzzy set (**IFS**) with **MV**Â and **NMV** expressed by the interval-valued linguistic variables was proposed by Garg and Kumar [26] in 2019 to negotiate with ambiguity and unclear details during in the decision-making process. Before defining the operational rules, score and accuracy functions of linguistic interval-valued **IFS**, it begins with a brief discussion of part of the analysis. Then, a number of aggregating operators are suggested to aggregate the linguistic interval-valued **IFS** intelligence on the basis of these operational principles. To demonstrated the efficacy and validity of suggested operators, some properties and inequalities are established. Another method for solving multi-attribute group decision making issues in the linguistic interval-valued **IFS** environment has been described and it is based on the proposed operators. Using the basic arithmetic operations in 1989, Buckley [4] defined the fuzzy complex numbers (**FCNs**) and illustrated the closure of **FCNs**. In 2011, Sharma [13] discussed the (α, β)-cut of **IFS** and defined the mapping of two **IFSs**.

In 2013, Yager [19], [20] was the pioneered of a Pythagorean fuzzy set (**PFS**), where the sum of the squares of **MV** and **NMV** belongs to [0,1]. Therefore, **PFS** has made a substantial contribution to our understanding of problem-solving in decision-making. The ideas of Pythagorean fuzzy isomorphisms and Pythagorean fuzzy normal subgroups were suggested in 2022 by Razaq et al. [43]. Also, investigated the basic properties of Pythagorean fuzzy normal subgroups and showed the vital results of Pythagorean fuzzy isomorphism. In 2023, Razaq et al. [38] discussed the Pythagorean fuzzy sets on ring structure. Moreover, defined the Pythagorean fuzzy ideals of a classical ring and investigated some fundamental operations of Pythagorean fuzzy ideals.

A powerful framework for defining novel machine learning techniques is a complex fuzzy set. In 2002, Ramot [12] explored the novel idea of complex fuzzy sets (**CFSs**). The membership function (**MF**) is the richness of the complex fuzzy set (**CFS**). Instead of being restricted to the range [0, 1], as is the case with a typical fuzzy **MF**, this area covers to the complex unit plane's circle. As a result, the **CFSs** offers a framework for mathematically expressing membership in a set in terms of a complex number (**CN**). In 2013, Alkouri and Salleh [17] introduced the complex intuitionistic fuzzy set (**CIFS**), where the **MF** and non-membership function (**NMF**) of **CIFS** have values in the complex plane's unit circle. A number of properties were studied together with the introduction of two main procedures. Additionally, by establishing a connection between the ideas of **CIFS** and **CFSs**, they drawn a conclusion about the benefit of applying these two procedures to the world of complex fuzzy numbers (**CFNs**). In 2013, Akram [18] introduced the notion of bipolar fuzzy soft Lie sub-algebras and investigated some of their properties. Also initiated the concept of an $(\in, \in \vee q)$-bipolar fuzzy Lie sub-algebra and presented some of its properties.

In 2013, Thirunavukarasu et al. [16] used the idea of complex fuzzy relation to illustrate a feasible application that includes complex fuzzy representation of solar activity, forecasting issues, time series, signal processing application and compare the two national economies. A fuzzy subgroup whose **MF** accepts values in the complex plane's unit circle is referred to as a complex fuzzy subgroup. In 2017, Alsarahead and Ahmad [23] examined some of the traits of the complex fuzzy subgroup and described it. Using the idea of complex intuitionistic fuzzy subspace as a base in 2016, Husban [21] introduced a novel concept of complex intuitionistic fuzzy subrings (**CIFSRs**) and expanded the definition of intuitionistic fuzzy space from the real range of **MF** and **NMF**, [0, 1], to the complex range of **MF** and **NMF** unit disc in the complex plane. Moreover, introduced **CIFSRs** and studied their methodology as a result of this generalization. In 2018, [25] explored certain structures of fuzzy Lie algebras, fuzzy Lie super algebras and fuzzy n-Lie algebras.

In 2019, Akram and Naz [27] introduced a novel idea of complex Pythagorean fuzzy graphs (**CPFGs**). Also, initiated the regular and edge regular graphs in a complex Pythagorean fuzzy situation. Furthermore, described scenarios when the graphic structure of characteristics is unclear. Also create a powerful multi-attribute decision-making method based on Pythagorean fuzzy graphs. In 2021, Ma et al. [36] presented the complex Pythagorean fuzzy multi-criteria optimization and compromise solution method, a novel approach to multi-criteria group decision-making challenges. It is made to deal with a lot of ambiguity and reluctance, which are common in judgments made by people. Expert judgments about the merits of each alternative and the relative weights of the criteria can be expressed in language using the **CPF − VIKOR** approach. In 2022, Xiao and Pedrycz [39] studied quantum decisions from the negation perspective. Specifically, complex evidence theory (CET) is considered to be effective to express and handle uncertain information in a complex plane.In 2022, Akram et al. [40] presented a new study on the elimination and choice translating reality (**ELECTRE**) family of approaches, which advances and evolves outranking decision-making methodologies. Its main goal is to explained the components and used of the **ELECTRE** II approach for group decision making in a complex Pythagorean fuzzy framework. In 2023, Xiao [45] presented a new quantum model of **GQET**, which provides a new perspective to express and handle the generalized quantum mass function with more explicit physical meanings. In 2021, Alolaiyan et al. [35] established the existence of (α, β)-complex fuzzy normal subgroups of a given group and show that any complex fuzzy subgroup is a (α, β)-complex fuzzy subgroup. In 2022, Alharbi and Alghazawi [37] introduced (ρ, η)-**CFS** and demonstrated the basics examples of the group under (ρ, η)-**CFS**. Gulzar et al. [29] introduced the innovative idea of complex intuitionistic fuzzy subgroups (**CIFSGs**) and established that each **CIFSGs** creates two intuitionistic fuzzy subgroups. With the help of this philosophy, they developed the idea of level subsets of a **CIFS** and go through all of its essential algebraic properties. Furthermore, they demonstrated the existence of a subgroup in the level subset of the **CIFSGs**. The homomorphic image and pre-image of the **CIFSGs** are also investigated under group homomorphism. The motivation and contribution of the current work is given as follows

1) Ramot [12] pioneered the innovative idea of **CFSs**. The **MF** is an extension of the **CFS**. As a result of this area being expanded to the circle of the complex unit plane rather than being constrained to the interval [0, 1], it provides a foundation for mathematically describing set membership in terms of a complex number.
2) Alkouri and Salleh [17] presented the **CIFS** in which the **MF** and **NMF** of the **CIFS** have values in the complex plane's unit circle. Gulzar et al. [29] introduced the innovative idea of (**CIFSGs**) and discussed vital results of group theory. Gulzar et al. [32] was introduced the concept of direct product of two **CIFSRs** and examined its different algebraic properties. The level subsets of the direct product of two **CIFSs** can also be developed. Furthermore, the homomorphic image of the direct product of subrings was explored.
3) Ullah et al. [30] purposed the idea of complex Pythagorean fuzzy set (**CPFS**) and generalized some distance measures. In comparison to **FSs**, **IFSs**, bipolar fuzzy sets (**BFSs**), **PFS**,**CFS** and **CIFS**, the **CPFS** is more appropriate and adaptable.
4) The idea of complex Pythagorean fuzzy set is not yet apply on subfield. In this work, we apply the **CPFS** on field theory, also discuss fundamental results, level subset, homomorphic images, inverse homomorphic images and direct product under the framework of complex Pythagorean fuzzy subfield.

To achieve this the remaining article is structure as follows: The essential definitions of **CPFSs** and **PFSFs** are the foundation of section II, and important conclusions from this innovative theory are crucial to our later study. In section III we present the new concept of **CPFSFs** and describe their fundamental characteristics. Every **CPFSF** generates two **PFSFs** also prove in this section. We also establish the level subset of **CPFSs** and argue that **CPFSF** level subsets define subfields of fields. In section IV the homomorphic images and homomorphic inverse images of **CPFSF** of certain field is initiate. The direct product of **CPFSFs** is discuss in section V along with an investigation of the algebraic structures underlying this theory.

The following Figure 1 shows our proposed model complex Pythagorean fuzzy subfield, a new algebraic structure at the junction of fuzzy subfields (established since 1971) and complex Pythagorean fuzzy sets. Various novel results enhance the new theory.

II. PRELIMINARIES

Some definitions and notions that are essential for the conception of later sections are examined in this section.

Definition 1 ([3]): An **IFS** $\mathbb{S}$ of universal set $\mathbb{T}$ form $\mathbb{S} = \{< \top, \varrho_{\mathbb{S}}(\top), \hat{\varrho}_{\mathbb{S}}(\top) >: \top \in \mathbb{T}\}$, where $\varrho_{\mathbb{S}}$ **MV** and $\hat{\varrho}_{\mathbb{S}}$ **NMV** of $\top$ from [0, 1], such that $0 \leq \varrho_{\mathbb{S}}(\top) + \hat{\varrho}_{\mathbb{S}}(\top) \leq 1, \forall \top \in \mathbb{T}$.

Definition 2 ([5]): Suppose that $\mathbb{S}$ is an **IFS** of $\dot{\mathbb{W}}$, where $\dot{\mathbb{W}}$ is any group. Then, **IFS** is said to be an **IFSG** of $\dot{\mathbb{W}}$, if it hold these properties:

1) $\varrho_{\mathbb{S}}(\top\hbar) \geq \min\{\varrho_{\mathbb{S}}(\top), \varrho_{\mathbb{S}}(\hbar)\}$,
2) $\varrho_{\mathbb{S}}(\top^{-1}) \geq \varrho_{\mathbb{S}}(\top)$,
3) $\hat{\varrho}_{\mathbb{S}}(\top\hbar) \leq \max\{\hat{\varrho}_{\mathbb{S}}(\top), \hat{\varrho}_{\mathbb{S}}(\hbar)\}$,
4) $\hat{\varrho}_{\mathbb{S}}(\top^{-1}) \leq \hat{\varrho}_{\mathbb{S}}(\top)$, for all $\top, \hbar \in \dot{\mathbb{W}}$.

Definition 3 ([15]): Every **CIFS** $\mathbb{S}$ of set $\mathbb{T}$ form $\mathbb{S} = \{< \top, \theta_{\mathbb{S}}(\top), \hat{\theta}_{\mathbb{S}}(\top) >: \top \in \mathbb{T}\}$, where the **MF** $\theta_{\mathbb{S}}(\top) = \varrho_{\mathbb{S}}(\top)e^{i\varphi_{\mathbb{S}}(\top)}$ and it is described as $\theta_{\mathbb{S}} : \mathbb{T} \to \{z \in \mathbb{C} : |z| \leq 1\}$ and **NMF** $\hat{\theta}_{\mathbb{S}}(\top) = \hat{\varrho}_{\mathbb{S}}(\top)e^{i\hat{\varphi}_{\mathbb{S}}(\top)}$ and it is described as $\hat{\theta}_{\mathbb{S}} : \mathbb{T} \to \{z \in \mathbb{C} : |z| \leq 1\}$, where $\mathbb{C} \in$ **CNs**.

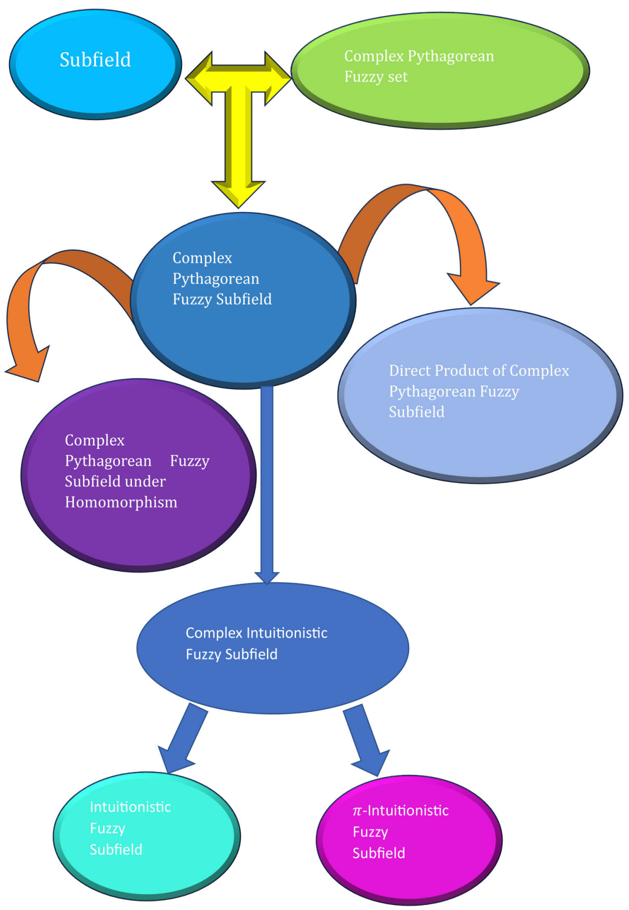

FIGURE 1. **Flowchart of complex Pythagorean fuzzy subfield.**

In order for the total of the **MV** and **NMV** to being restricted within the complex plane's unit disc, these **MF** and **NMF** must acquire all of the **MV** and **NMV**, respectively, where $i = \sqrt{-1}$ $\varrho_{\mathbb{S}}(\intercal)$, $\hat{\varrho}_{\mathbb{S}}(\intercal)$, $\varphi_{\mathbb{S}}(\intercal)$, and $\hat{\varphi}_{\mathbb{S}}(\intercal)$ are real valued such that $0 \leq \varrho_{\mathbb{S}}(\intercal) + \hat{\varrho}_{\mathbb{S}}(\intercal) \leq 1$ and $0 \leq \varphi_{\mathbb{S}}(\intercal) + \hat{\varphi}_{\mathbb{S}}(\intercal) \leq 2\pi$. In the interest of remaining simple, we'll use throughout this article $\theta_{\mathbb{S}}(\intercal) = \varrho_{\mathbb{S}}(\intercal)e^{i\varphi_{\mathbb{S}}(\intercal)}$, $\theta_{\mathbb{N}}(\intercal) = \varrho_{\mathbb{N}}(\intercal)e^{i\varphi_{\mathbb{N}}(\intercal)}$ as **MF** and $\hat{\theta}_{\mathbb{S}}(\intercal) = \hat{\varrho}_{\mathbb{S}}(\intercal)e^{i\hat{\varphi}_{\mathbb{S}}(\intercal)}$, $\hat{\theta}_{\mathbb{N}}(\intercal) = \hat{\varrho}_{\mathbb{N}}(\intercal)e^{i\hat{\varphi}_{\mathbb{N}}(\intercal)}$ as **NMF** of **CIFSs** $\mathbb{S}$ and $\mathbb{N}$, respectively.

Theorem 1 ([5]): An **IFSG** is the intersection of two **IFSGs** from the group $\dot{\mathbb{G}}$.

Definition 4 ([17]): Suppose that $\mathbb{S}$ and $\mathbb{N}$ are **CIFSs** of $\mathbb{T}$ and $\mathbb{L}$, respectively. Then, intersection of **CIFSs** $\mathbb{S}$ and $\mathbb{N}$ is described as:

$$\mathbb{S} \cap \mathbb{N} = \{< (\intercal), \theta_{\mathbb{S}\cap\mathbb{N}}(\intercal), \hat{\theta}_{\mathbb{S}\cap\mathbb{N}}(\intercal) >\}.$$

Where

$$\begin{aligned}\theta_{\mathbb{S}\cap\mathbb{N}}(\intercal) &= \varrho_{\mathbb{S}\cap\mathbb{N}}(\intercal)e^{i\varphi_{\mathbb{S}\cap\mathbb{N}}(\intercal)}\\ &= \min\{\varrho_{\mathbb{S}}(\intercal), \varrho_{\mathbb{N}}(\intercal)\}e^{i\min\{\varphi_{\mathbb{S}}(\intercal),\varphi_{\mathbb{N}}(\intercal)\}},\\ \hat{\theta}_{\mathbb{S}\cap\mathbb{N}}(\intercal) &= \hat{\varrho}_{\mathbb{S}\cap\mathbb{N}}(\intercal)e^{i\hat{\varphi}_{\mathbb{S}\cap\mathbb{N}}(\intercal)}\\ &= \max\{\hat{\varrho}_{\mathbb{S}}(\intercal), \hat{\varrho}_{\mathbb{N}}(\intercal)\}e^{i\max\{\hat{\varphi}_{\mathbb{S}}(\intercal),\hat{\varphi}_{\mathbb{N}}(\intercal)\}}.\end{aligned}$$

Definition 5 ([17]): Suppose that $\mathbb{S}$ and $\mathbb{N}$ are **CIFSs** of sets $\mathbb{T}$ and $\mathbb{L}$, respectively. Then, union of **CIFSs** $\mathbb{S}$ and $\mathbb{N}$ is defined as: $\mathbb{S} \cup \mathbb{N} = \{< (\intercal), \theta_{\mathbb{S}\cup\mathbb{N}}(\intercal), \hat{\theta}_{\mathbb{S}\cup\mathbb{N}}(\intercal) >\}$.
Where $\theta_{\mathbb{S}\cup\mathbb{N}}(\intercal) = \varrho_{\mathbb{S}\cup\mathbb{N}}(\intercal)e^{i\varphi_{\mathbb{S}\cup\mathbb{N}}(\intercal)}$
$= \max\{\varrho_{\mathbb{S}}(\intercal), \varrho_{\mathbb{N}}(\intercal)\}e^{i\max\{\varphi_{\mathbb{S}}(\intercal),\varphi_{\mathbb{N}}(\intercal)\}}$,
$\hat{\theta}_{\mathbb{S}\cup\mathbb{N}}(\intercal) = \hat{\varrho}_{\mathbb{S}\cup\mathbb{N}}(\intercal)e^{i\hat{\varphi}_{\mathbb{S}\cup\mathbb{N}}(\intercal)}$
$= \min\{\hat{\varrho}_{\mathbb{S}}(\intercal), \hat{\varrho}_{\mathbb{N}}(\intercal)\}e^{i\min\{\hat{\varphi}_{\mathbb{S}}(\intercal),\hat{\varphi}_{\mathbb{N}}(\intercal)\}}$.

III. CHARACTERISTICS OF COMPLEX PYTHAGOREAN FUZZY SUBFIELDS

In this section, we present **CPFSFs** and level subsets of **CPFSFs**. Furthermore, we demonstrate that **CPFSFs** generates two **PFSGs**. Also, initiate the **CPFSF** as a level subset and illustrate that **CPFSF** level subset form subgroup and examine some of this phenomenon's algebraic features.

Definition 6: Suppose that $\mathbb{S} = \{< \intercal, \psi_{\mathbb{S}}(\intercal), \hat{\psi}_{\mathbb{S}}(\intercal) >: (\psi_{\mathbb{S}}(\intercal))^2 + (\hat{\psi}_{\mathbb{S}}(\intercal))^2 \leq 1, \intercal \in \mathbb{K}\}$ is a **PFS**. Then, π-Pythagorean fuzzy set(π-**PFS**) $\mathbb{S}_\pi$ described as $\mathbb{S}_\pi = \{< \intercal, \psi_{\mathbb{S}_\pi}(\intercal), \hat{\psi}_{\mathbb{S}_\pi}(\intercal) >: \intercal \in \mathbb{K}\}$, where the function $(\psi_{\mathbb{S}_\pi}(\intercal))^2 = 2\pi(\psi_{\mathbb{S}}(\intercal))^2$ and $(\hat{\psi}_{\mathbb{S}_\pi}(\intercal))^2 = 2\pi(\hat{\psi}_{\mathbb{S}}(\intercal))^2$ denote the **MV** and **NMV** of an element $\intercal$ of $\mathbb{K}$, respectively and fulfill the following properties $0 < (\psi_{\mathbb{S}_\pi}(\intercal))^2 + (\hat{\psi}_{\mathbb{S}_\pi}(\intercal))^2 \leq 2\pi$.

Definition 7: Every π-**PFS** $\mathbb{S}_\pi$ of field $\mathbb{K}$ is called π-Pythagorean fuzzy subfield (π-**PFSF**) of $\mathbb{K}$, $\forall\, \intercal$, Ꝁ $\in \mathbb{K}$ if:

1) $(\psi_{\mathbb{S}_\pi}(\intercal - \ddot{\mathfrak{z}}))^2 \geq \min\{(\psi_{\mathbb{S}_\pi}(\intercal))^2, (\psi_{\mathbb{S}_\pi}(\hbar))^2\}$,
2) $(\psi_{\mathbb{S}_\pi}(\intercal\hbar))^2 \geq \min\{(\psi_{\mathbb{S}_\pi}(\intercal))^2, (\psi_{\mathbb{S}_\pi}(\hbar))^2\}$,
3) $(\psi_{\mathbb{S}_\pi}(\intercal^{-1}))^2 \geq (\psi_{\mathbb{S}_\pi}(\intercal))^2$,
4) $(\hat{\psi}_{\mathbb{S}_\pi}(\intercal - \hbar))^2 \leq \max\{(\hat{\psi}_{\mathbb{S}_\pi}(\intercal))^2, (\hat{\psi}_{\mathbb{S}_\pi}(\hbar))^2\}$,
5) $(\hat{\psi}_{\mathbb{S}_\pi}(\intercal\hbar))^2 \leq \max\{(\hat{\psi}_{\mathbb{S}_\pi}(\intercal))^2, (\hat{\psi}_{\mathbb{S}_\pi}(\hbar))^2\}$,
6) $(\hat{\psi}_{\mathbb{S}_\pi}(\intercal^{-1}))^2 \leq (\hat{\psi}_{\mathbb{S}_\pi}(\intercal))^2$.

Theorem 2: [13] Every π-**PFS** $\mathbb{S}_\pi$ of field $\mathbb{H}$ is a π-**PFSF** of $\mathbb{H}$ iff $\mathbb{S}$ is **PFSF** of $\mathbb{K}$.

Definition 8: Suppose that $\mathbb{S}$ and $\mathbb{N}$ are **CPFSs** of $\mathbb{K}$. Then,

1) Homogeneous **CPFS** is defined as a **CPFS** with $\mathbb{S}$, if $\forall\, \ddot{\mathrm{j}}, \ddot{\mathrm{t}} \in \mathbb{K}$, we have
   a $(\mu_{\mathbb{S}}(\intercal))^2 \leq (\mu_{\mathbb{S}}(\hbar))^2$ iff $(\zeta_{\mathbb{S}}(\intercal))^2 \leq (\zeta_{\mathbb{S}}(\hbar))^2$,
   b $(\hat{\mu}_{\mathbb{S}}(\intercal))^2 \geq (\hat{\mu}_{\mathbb{S}}(\hbar))^2$ iff $(\hat{\omega}_{\mathbb{S}}(\intercal))^2 \geq (\hat{\omega}_{\mathbb{S}}(\hbar))^2$.
2) Every **CPFS** $\mathbb{S}$ is said to be homogeneous **CPFS** by $\mathbb{N}$, if $\forall\, \dot{\mathrm{j}} \in \mathbb{K}$, we have
   a $(\mu_{\mathbb{S}}(\intercal))^2 \leq (\mu_{\mathbb{N}}(\intercal))^2$ iff $(\zeta_{\mathbb{S}}(\intercal))^2 \leq (\zeta_{\mathbb{N}}(\intercal))^2$,
   b $(\hat{\mu}_{\mathbb{S}}(\intercal))^2 \geq (\hat{\mu}_{\mathbb{N}}(\intercal))^2$ iff $(\hat{\omega}_{\mathbb{S}}(\intercal))^2 \geq (\hat{\omega}_{\mathbb{N}}(\intercal))^2$.

We will refer to **CPFS** as homogeneous **CPFS** throughout this article.

Definition 9: A **CPFS** $\mathbb{S} = \{< (\intercal, \chi_{\mathbb{S}}\intercal, \hat{\chi}_{\mathbb{S}}\hbar) >: \intercal \in \mathbb{K}\}$ of field $\mathbb{K}$ is called a **CPFSF**, $\forall\, \intercal, \hbar \in \mathbb{K}$ if

1) $(\Upsilon_{\mathbb{S}}(\intercal - \hbar))^2 \geq \min\{(\Upsilon_{\mathbb{S}}(\intercal))^2, (\Upsilon_{\mathbb{S}}(\hbar))^2\}$,
2) $(\Upsilon_{\mathbb{S}}(\ddot{\mathrm{j}}\hbar))^2 \geq \min\{(\Upsilon_{\mathbb{S}}(\intercal))^2, (\Upsilon_{\mathbb{S}}(\hbar))^2\}$,
3) $(\Upsilon_{\mathbb{S}}(\intercal^{-1}))^2 \geq (\Upsilon_{\mathbb{S}}(\intercal))^2$,
4) $(\hat{\Upsilon}_{\mathbb{S}}(\dot{\mathrm{j}} - \ddot{\mathrm{t}}))^2 \leq \max\{(\hat{\Upsilon}_{\mathbb{S}}(\intercal))^2, (\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\}$,
5) $(\hat{\Upsilon}_{\mathbb{S}}(\intercal\hbar))^2 \leq \max\{(\hat{\Upsilon}_{\mathbb{S}}(\intercal))^2, (\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\}$,
6) $(\hat{\Upsilon}_{\mathbb{S}}(\intercal^{-1}))^2 \leq (\hat{\Upsilon}_{\mathbb{S}}(\intercal))^2$.

In addition, we illustrate the definition of **CPFSF** as follow:

1) $(\mu_{\mathbb{S}}(\top-\check{\mathfrak{r}}))^2 e^{i(\zeta_{\mathbb{S}}(\top-\hbar))^2} \geq \min\{(\mu_{\mathbb{S}}(\top))^2, (\mu_{\mathbb{S}}(\hbar))^2\} e^{i\min\{(\zeta_{\mathbb{S}}(\top))^2,(\omega_{\mathbb{S}}(\hbar))^2\}}$,
2) $(\mu_{\mathbb{S}}(\top\hbar))^2 e^{i(\zeta_{\mathbb{S}}(\top\hbar))^2} \geq \min\{(\mu_{\mathbb{S}}(\top))^2, (\mu_{\mathbb{S}}(\hbar))^2\} e^{i\min\{(\zeta_{\mathbb{S}}(\top))^2,(\omega_{\mathbb{S}}(\hbar))^2\}}$,
3) $(\mu_{\mathbb{S}}(\top^{-1}))^2 e^{i(\zeta_{\mathbb{S}}(\top^{-1}))^2} \geq (\mu_{\mathbb{S}}(\top))^2 e^{i(\zeta_{\mathbb{S}}\hbar)^2}$,
4) $(\hat{\mu}_{\mathbb{S}}(\top-\hbar))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top-\hbar))^2} \leq \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2, (\hat{\mu}_{\mathbb{S}}(\hbar))^2\} e^{i\max\{(\hat{\zeta}_{\mathbb{S}}(\top))^2, (\hat{\zeta}_{\mathbb{S}}(\hbar))^2\}}$
5) $(\hat{\mu}_{\mathbb{S}}(\top\hbar))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top\hbar))^2} \leq \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2, (\hat{\mu}_{\mathbb{S}}(\hbar))^2\} e^{i\max\{(\hat{\zeta}_{\mathbb{S}}(\top))^2, (\hat{\zeta}_{\mathbb{S}}(\hbar))^2\}}$
6) $(\hat{\mu}_{\mathbb{S}}(\top^{-1}))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top^{-1}))^2} \leq (\hat{\mu}_{\mathbb{S}}(\top))^2 e^{i(\hat{\omega}_{\mathbb{S}}(\top))^2}$, for all $\top, \hbar \in \mathbb{K}$.

Theorem 3: Assume that $\mathbb{S}$ is a **CPFS** of field $\mathbb{K}$. Then, $\mathbb{S}$ is said to be a **CPFSF** of $\mathbb{K}$ if and only if

1) The **FS** $\overline{\mathbb{S}} = \{<\top,\ \mu_{\mathbb{S}}(\hbar),\ \hat{\mu}_{\mathbb{S}}(\hbar)>: 0 < (\mu_{\mathbb{S}}(\top))^2 + (\hat{\mu}_{\mathbb{S}}(\top))^2 \leq 1,\ \top \in \mathbb{K}\}$ is a **PFSF**.
2) The π-**FS** $\underline{\mathbb{S}} = \{<\top, \omega_{\mathbb{S}}(\hbar),\ \hat{\omega}_{\mathbb{S}}(\hbar)>: 0 < (\omega_{\mathbb{S}}(\top))^2 + (\hat{\omega}_{\top}(\top))^2 \leq 2\pi,\ \top \in \mathbb{K}\}$ is called π-**PFSG**.

Proof: Suppose that $\mathbb{S}$ is **CPFSF** and $\top, \hbar \in \mathbb{H}$.
Then, $(\mu_{\mathbb{S}}(\top-\hbar))^2 e^{i(\zeta_{\mathbb{S}}(\top-\hbar))^2} = (\Upsilon_{\mathbb{S}}(\top-\hbar))^2$
$\geq \min\{(\Upsilon_{\mathbb{S}}(\top))^2,\ (\Upsilon_{\mathbb{S}}(\hbar))^2\}$
$= \min\{(\mu_{\mathbb{S}}(\top))^2 e^{i(\zeta_{\mathbb{S}}(\top))^2},\ (\mu_{\mathbb{S}}(\hbar))^2 e^{i(\zeta_{\mathbb{S}}(\hbar))^2}\}$
$= \min\{(\mu_{\mathbb{S}}(\top))^2,\ (\mu_{\mathbb{S}}(\hbar))^2\} e^{i\min\{(\zeta_{\mathbb{S}}(\top))^2,\ (\zeta_{\mathbb{S}}(\hbar))^2\}}$. Since, $\mathbb{S}$ is homogeneous, so
$(\mu_{\mathbb{S}}(\top-\hbar))^2 \geq \min\{(\mu_{\mathbb{S}}(\top))^2, (\mu_{\mathbb{S}}(\hbar))^2\}$
and $(\zeta_{\mathbb{S}}(\top-\hbar))^2 \geq \min\{(\zeta_{\mathbb{S}}(\top))^2, (\zeta_{\mathbb{S}}(\hbar))^2\}$.

$$\begin{aligned}
&(\mu_{\mathbb{S}}(\top\hbar))^2 e^{i(\zeta_{\mathbb{S}}(\top\hbar))^2}\\
&= (\Upsilon_{\mathbb{S}}(\top\hbar))^2\\
&\geq \min\{(\Upsilon_{\mathbb{S}}(\top))^2, (\Upsilon_{\mathbb{S}}(\hbar))^2\}\\
&= \min\{(\mu_{\mathbb{S}}(\top))^2 e^{i(\zeta_{\mathbb{S}}(\top))^2},\ (\mu_{\mathbb{S}}(\hbar))^2 e^{i(\zeta_{\mathbb{S}}(\hbar))^2}\}\\
&= \min\{(\mu_{\mathbb{S}}(\top))^2,\ (\mu_{\mathbb{S}}(\hbar))^2\} e^{i\min\{(\zeta_{\mathbb{S}}(\top))^2,\ (\omega_{\mathbb{S}}(\hbar))^2\}}.
\end{aligned}$$

$\because$ $\mathbb{S}$ is homogeneous, so $(\mu_{\mathbb{S}}(\top\hbar))^2 \geq \min\{(\mu_{\mathbb{S}}(\top))^2, (\mu_{\mathbb{S}}(\hbar))^2\}$ and $(\zeta_{\mathbb{S}}(\top\hbar))^2 \geq \min\{(\zeta_{\mathbb{S}}(\top))^2, (\zeta_{\mathbb{S}}(\hbar))^2\}$.

Moreover, $(\mu_{\mathbb{S}}(\top^{-1}))^2 e^{i(\zeta_{\mathbb{S}}(\top^{-1}))^2} = (\Upsilon_{\mathbb{S}}(\top^{-1}))^2 \geq (\Upsilon_{\mathbb{S}}(\top))^2 = (\mu_{\mathbb{S}}(\top))^2 e^{i(\zeta_{\mathbb{S}}(\top))^2}$
$\Rightarrow (\mu_{\mathbb{S}}(\top^{-1}))^2 \geq (\mu_{\mathbb{S}}(\top))^2$ and, $(\zeta_{\mathbb{S}}(\top^{-1}))^2 \geq (\omega_{\mathbb{S}}(\top))^2$.

$$\begin{aligned}
&(\hat{\mu}_{\mathbb{S}}(\top-\hbar))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top-\hbar))^2}\\
&= (\hat{\Upsilon}_{\mathbb{S}}(\top-\hbar))^2\\
&\leq \max\{(\hat{\Upsilon}_{\mathbb{S}}(\top))^2,\ (\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\}\\
&= \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2 e^{i(\hat{\omega}_{\mathbb{S}}(\top))^2},\ (\hat{\mu}_{\mathbb{S}}(\hbar))^2 e^{i(\hat{\omega}_{\mathbb{S}}(\hbar))^2}\}\\
&= \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2,\ (\hat{\mu}_{\mathbb{S}}(\hbar))^2\} e^{i\max\{(\hat{\omega}_{\mathbb{S}}\hbar)^2,\ (\hat{\zeta}_{\mathbb{S}}(\hbar))^2\}}.
\end{aligned}$$

Therefore $\mathbb{S}$ is homogeneous, so
$(\hat{\mu}_{\mathbb{S}}(\top-\hbar))^2 \leq \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2, (\hat{\mu}_{\mathbb{S}}(\hbar))^2\}$ and $(\hat{\omega}_{\mathbb{S}}(\top-\hbar))^2 \leq \max\{(\hat{\omega}_{\mathbb{S}}(\top))^2, (\hat{\omega}_{\mathbb{S}}(\hbar))^2\}$.

$$\begin{aligned}
&(\hat{\mu}_{\mathbb{S}}(\top\hbar))^2 e^{i(\hat{\omega}_{\mathbb{S}}(\top\hbar))^2}\\
&= (\hat{\Upsilon}_{\mathbb{S}}(\top\hbar))^2\\
&\leq \max\{(\hat{\Upsilon}_{\mathbb{S}}(\top))^2, (\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\}\\
&= \max\{(\hat{\mu}_{\mathbb{S}\top} e^{i(\hat{\omega}_{\mathbb{S}}(\top))^2},\ (\hat{\mu}_{\mathbb{S}}(\hbar))^2 e^{i(\hat{\omega}_{\mathbb{S}}(\hbar))^2}\}\\
&= \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2,\ (\hat{\mu}_{\mathbb{S}}(\hbar))^2\} e^{i\max\{(\hat{\omega}_{\mathbb{S}}(\top))^2,\ (\hat{\omega}_{\mathbb{S}}(\hbar))^2\}}.
\end{aligned}$$

By homogeneity of $\mathbb{S}$
$(\hat{\mu}_{\mathbb{S}}(\top\hbar))^2 \leq \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2, (\hat{\mu}_{\mathbb{S}}(\hbar))^2\}$ and $(\hat{\omega}_{\mathbb{S}}(\top\hbar))^2 \leq \max\{(\hat{\omega}_{\mathbb{S}}(\top))^2, (\hat{\omega}_{\mathbb{S}}(\hbar))^2\}$.
Furthermore,

$$\begin{aligned}
(\hat{\mu}_{\mathbb{S}}(\top^{-1}))^2 e^{i(\hat{\omega}_{\mathbb{S}}(\top^{-1}))^2} &= (\hat{\Upsilon}_{\mathbb{S}}(\top^{-1}))^2 \leq (\hat{\Upsilon}_{\mathbb{S}}(\top))^2\\
&= (\hat{\mu}_{\mathbb{S}}(\top))^2 e^{i(\hat{\omega}_{\mathbb{S}}(\top))^2}\\
(\hat{\mu}_{\mathbb{S}}(\top^{-1}))^2 \leq (\hat{\mu}_{\mathbb{S}}(\top))^2 &\text{ and, } (\hat{\omega}_{\mathbb{S}}(\top^{-1}))^2\\
&\leq (\hat{\omega}_{\mathbb{S}}(\top))^2.
\end{aligned}$$

Accordingly, $\overline{\mathbb{S}}$ is a **PFSF** and $\underline{\mathbb{S}}$ is π-**PFSF**.

Alternatively, assume that $\overline{\mathbb{S}}$ is a **PFSF** and $\underline{\mathbb{S}}$ is a π-**PFSF**. Then,

$$\begin{aligned}
(\mu_{\mathbb{S}}(\top-\hbar))^2 &\geq \min\{(\mu_{\mathbb{S}}(\top))^2, (\mu_{\mathbb{S}}(\hbar))^2\}\\
(\mu_{\mathbb{S}}(\top\hbar))^2 &\geq \min\{(\mu_{\mathbb{S}}(\top))^2, (\mu_{\mathbb{S}}(\hbar))^2\}\\
(\mu_{\mathbb{S}}(\top^{-1}))^2 &\geq (\mu_{\mathbb{S}}(\top))^2\\
(\hat{\mu}_{\mathbb{S}}(\top-\hbar))^2 &\leq \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2, (\hat{\mu}_{\mathbb{S}}(\hbar))^2\}\\
(\hat{\mu}_{\mathbb{S}}(\top\hbar))^2 &\leq \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2, (\hat{\mu}_{\mathbb{S}}(\hbar))^2\}\\
(\hat{\mu}_{\mathbb{S}}(\top^{-1}))^2 &\leq (\hat{\mu}_{\mathbb{S}}(\top))^2\\
(\zeta_{\mathbb{S}}(\top-\hbar))^2 &\geq \min\{(\zeta_{\mathbb{S}}(\top))^2, (\zeta_{\mathbb{S}}(\hbar))^2\}\\
(\zeta_{\mathbb{S}}(\top\hbar))^2 &\geq \min\{(\zeta_{\mathbb{S}}(\top))^2, (\zeta_{\mathbb{S}}(\hbar))^2\}\\
(\zeta_{\mathbb{S}}(\top^{-1}))^2 &\geq (\zeta_{\mathbb{S}}(\top))^2\\
(\hat{\zeta}_{\mathbb{S}}(\top-\hbar))^2 &\leq \max\{(\hat{\zeta}_{\mathbb{S}}(\top))^2, (\hat{\zeta}_{\mathbb{S}}(\hbar))^2\}\\
(\hat{\zeta}_{\mathbb{S}}(\top\hbar))^2 &\leq \max\{(\hat{\zeta}_{\mathbb{S}}(\top))^2, (\hat{\zeta}_{\mathbb{S}}(\hbar))^2\}\\
(\hat{\zeta}_{\mathbb{S}}(\top^{-1}))^2 &\leq (\hat{\zeta}_{\mathbb{S}}(\top))^2.
\end{aligned}$$

For this, we assume that

$$\begin{aligned}
&(\Upsilon_{\mathbb{S}}(\top-\hbar))^2\\
&= (\mu_{\mathbb{S}}(\top-\hbar))^2 e^{i(\zeta_{\mathbb{S}}(\top-\hbar))^2}\\
&\geq \min\{(\mu_{\mathbb{S}}(\top))^2, (\mu_{\mathbb{S}}(\hbar))^2\} e^{i\min\{(\zeta_{\mathbb{S}}(\top))^2,\ (\zeta_{\mathbb{S}}(\hbar))^2\}}\\
&= \min\{(\mu_{\mathbb{S}}(\top))^2 e^{i(\zeta_{\mathbb{S}}(\top))^2},\ (\mu_{\mathbb{S}}(\hbar))^2 e^{i(\zeta_{\mathbb{S}}(\hbar))^2}\}\\
&= \min\{(\Upsilon_{\mathbb{S}}(\top))^2,\ (\Upsilon_{\mathbb{S}}(\hbar))^2\}.
\end{aligned}$$

For this, we consider

$$\begin{aligned}
(\Upsilon_{\mathbb{S}}(\top\hbar))^2 &= (\mu_{\mathbb{S}}(\top\hbar))^2 e^{i(\zeta_{\mathbb{S}}(\top\hbar))^2}\\
&\geq \min\{(\mu_{\mathbb{S}}(\top))^2,\ (\mu_{\mathbb{S}}(\hbar))^2\} e^{i\min\{(\zeta_{\mathbb{S}}(\top))^2,\ (\zeta_{\mathbb{S}}(\hbar))^2\}}\\
&= \min\{(\mu_{\mathbb{S}}(\top))^2 e^{i(\zeta_{\mathbb{S}}(\top))^2},\ (\mu_{\mathbb{S}}(\hbar))^2 e^{i(\zeta_{\mathbb{S}}(\hbar))^2}\}\\
&= \min\{(\Upsilon_{\mathbb{S}}(\top))^2, (\Upsilon_{\mathbb{S}}(\hbar))^2\}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
(\Upsilon_{\mathbb{S}}(\top^{-1}))^2 &= (\mu_{\mathbb{S}}(\top^{-1}))^2 e^{i(\zeta_{\mathbb{S}}(\top^{-1}))^2}\\
&\geq (\mu_{\mathbb{S}}(\top))^2 e^{i(\zeta_{\mathbb{S}}(\hbar)^2)=(\Upsilon_{\mathbb{S}}(\top))^2}.
\end{aligned}$$

Consider,

$$
\begin{aligned}
&(\hat{\Upsilon}_{\mathbb{S}}(\top-\hbar))^2\\
&=(\hat{\mu}_{\mathbb{S}}(\top-\hbar))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top-\hbar))^2}\\
&\leq \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2,\ (\hat{\mu}_{\mathbb{S}}(\hbar))^2\} e^{i\ \max\{(\hat{\zeta}_{\mathbb{S}}(\top))^2,\ (\hat{\zeta}_{\mathbb{S}}(\hbar))^2\}}\\
&=\max\{(\hat{\mu}_{\mathbb{S}}(\top))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top))^2},\ (\hat{\mu}_{\mathbb{S}}(\hbar))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\hbar))^2}\}\\
&=\max\{(\hat{\Upsilon}_{\mathbb{S}}(\top))^2,\ (\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\}\\
&(\hat{\Upsilon}_{\mathbb{S}}(\top\hbar))^2\\
&=(\hat{\mu}_{\mathbb{S}}(\top\hbar))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top\hbar))^2}\\
&\leq \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2, (\hat{\mu}_{\mathbb{S}}(\hbar))^2\} e^{i\ \max\{(\hat{\zeta}_{\mathbb{S}}(\top))^2,\ (\hat{\zeta}_{\mathbb{S}}(\hbar))^2\}}\\
&=\max\{(\hat{\mu}_{\mathbb{S}}(\top))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top))^2},\ (\hat{\mu}_{\mathbb{S}}(\hbar))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\hbar))^2}\}\\
&=\max\{(\hat{\Upsilon}_{\mathbb{S}}(\top))^2, (\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\}.
\end{aligned}
$$

Further,

$$
\begin{aligned}
(\hat{\Upsilon}_{\mathbb{S}}(\top^{-1}))^2 &= (\hat{\mu}_{\mathbb{S}}(\top^{-1}))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top^{-1}))^2}\\
&\leq (\hat{\mu}_{\mathbb{S}}(\top))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top))^2} = (\hat{\Upsilon}_{\mathbb{S}}(\top))^2.
\end{aligned}
$$

Thus, $\mathbb{S}$ is **CPFSF**.

The intersection of two **CPFSFs** is shown to be a **CPFSF** in the following result.

Theorem 4: Intersection of two **CPFSFs** of field $\mathbb{K}$ is **CPFSF**.

Proof: Let $\mathbb{S}$ and $\mathbb{N}$ be two **CPFSFs** of field $\mathbb{K}$, for all $\eta, \hbar \in \mathbb{K}$. By Theorem 3, we get

The **PFS** $\{(\eta, \mu_{\mathbb{S}}(\eta), \hat{\mu}_{\mathbb{S}}(\top)),\ 0 < (\mu_{\mathbb{S}}(\top))^2 + (\hat{\mu}_{\mathbb{S}}(\top))^2 \leq 1, \forall \top \in \mathbb{H}\}$ and $\{(\top, \mu_{\mathbb{N}}(\top), \hat{\mu}_{\mathbb{N}}(\top)), 0 < (\mu_{\mathbb{N}}(\top))^2 + (\hat{\mu}_{\mathbb{N}}(\top))^2 \leq 1, \forall \top \in \mathbb{H}\}$ are **PFSGs** of a field $\mathbb{H}$. From Theorem 1, we obtain $\{(\top, \mu_{\mathbb{S}\cap\mathbb{N}}(\top), \hat{\mu}_{\mathbb{S}\cap\mathbb{N}}(\top)), 0 < (\mu_{\mathbb{S}\cap\mathbb{N}}(\top))^2 + (\hat{\mu}_{\mathbb{S}\cap\mathbb{N}}(\top))^2 \leq 1, \top \in \mathbb{H}\}$ is **PFSG** of a field $\mathbb{H}$.

The π-**PFS** $\{(\top, \zeta_{\mathbb{S}}(\top), \hat{\zeta}_{\mathbb{S}}(\top)), 0 < (\zeta_{\mathbb{S}}(\top))^2 + (\hat{\zeta}_{\mathbb{S}}(\top))^2 \leq 2\pi,\ \top \in \mathbb{H}\}$ and $\{(\top, \zeta_{\mathbb{N}}(\top), \hat{\zeta}_{\mathbb{N}}(\top)0 < (\zeta_{\mathbb{N}}(\top))^2 + (\hat{\zeta}_{\mathbb{N}}(\top))^2 \leq 2\pi, \top \in \mathbb{H}\}$ are π-**PFSGs** of a field $\mathbb{H}$. By Theorem 1 and 2, we obtain $\{(\top, \zeta_{\mathbb{S}\cap\mathbb{N}}(\top), \hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}(\top), 0 < (\zeta_{\mathbb{S}\cap\mathbb{N}}(\top))^2 + (\hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}(\top))^2 \leq 2\pi,\ \top \in \mathbb{H}\}$ is π-**PFSG** of $\mathbb{H}$.

Consider,

$$
\begin{aligned}
&(\Upsilon_{\mathbb{S}\cap\mathbb{N}}(\top-\nu))^2\\
&=(\mu_{\mathbb{S}\cap\mathbb{N}}(\top-\nu))^2 e^{i(\zeta_{\mathbb{S}\cap\mathbb{N}}(\top-\nu))^2}\\
&\geq \min\{(\mu_{\mathbb{S}\cap\mathbb{N}}(\top))^2, (\mu_{\mathbb{S}\cap\mathbb{N}}(\nu))^2\} e^{\min\{i(\zeta_{\mathbb{S}\cap\mathbb{N}}(\top))^2,\ (\zeta_{\mathbb{S}\cap\mathbb{N}}(\nu))^2\}}\\
&\quad \text{[By homogeneity of}\mathbf{CPFS}]\\
&=\min\{(\mu_{\mathbb{S}\cap\mathbb{N}}(\top))^2 e^{i(\zeta_{\mathbb{S}\cap\mathbb{N}}(\top))^2},\ (\zeta_{\mathbb{S}\cap\mathbb{N}}(\nu))^2 e^{i(\zeta_{\mathbb{S}\cap\mathbb{N}}(a))^2}\}\\
&=\min\{(\Upsilon_{\mathbb{S}\cap\mathbb{N}}(\top))^2,\ (\Upsilon_{\mathbb{S}\cap\mathbb{N}}(\nu))^2\}\\
&(\Upsilon_{\mathbb{S}\cap\mathbb{N}}(\top))^2\\
&=(\mu_{\mathbb{S}\cap\mathbb{N}}(\eta(\top))))^2 e^{i(\zeta_{\mathbb{S}\cap\mathbb{N}}(\eta(\top))))^2}\\
&\geq \min\{(\mu_{\mathbb{S}\cap\mathbb{N}}(\top))^2, (\mu_{\mathbb{S}\cap\mathbb{N}}((\top))))^2\} e^{\min\{i(\zeta_{\mathbb{S}\cap\mathbb{N}}(\top))^2,\ (\zeta_{\mathbb{S}\cap\mathbb{N}}((\top))))^2\}}\\
&\quad \text{[By homogeneity of}\mathbf{CPFS}]\\
&=\min\{(\mu_{\mathbb{S}\cap\mathbb{N}}(\top))^2 e^{i(\zeta_{\mathbb{S}\cap\mathbb{N}}(\top))^2},\ (\zeta_{\mathbb{S}\cap\mathbb{N}}((\top))))^2 e^{i(\zeta_{\mathbb{S}\cap\mathbb{N}}((\top))))^2}\}\\
&=\min\{(\Upsilon_{\mathbb{S}\cap\mathbb{N}}(\top))^2,\ (\Upsilon_{\mathbb{S}\cap\mathbb{N}}((\top))))^2\}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
(\Upsilon_{\mathbb{S}\cap\mathbb{N}}(\top^{-1}))^2 &= \mu_{\mathbb{S}\cap\mathbb{N}}(\top^{-1}))^2 e^{i(\zeta_{\mathbb{S}\cap\mathbb{N}}(\top^{-1}))^2}\\
&\geq \{\mu_{\mathbb{S}\cap\mathbb{N}}(\top))^2\} e^{\{i(\zeta_{\mathbb{S}\cap\mathbb{N}}(\top))^2\}}\\
(\Upsilon_{\mathbb{S}\cap\mathbb{N}}(\top^{-1}))^2 &\geq (\Upsilon_{\mathbb{S}\cap\mathbb{N}}(\top))^2.
\end{aligned}
$$

Assume that,

$$
\begin{aligned}
&(\hat{\Upsilon}_{\mathbb{S}\cap\mathbb{N}}(\top-(\top)))^2\\
&=\hat{\mu}_{\mathbb{S}\cap\mathbb{N}}(\top-(\top))^2 e^{i(\hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}(\top-\hbar)^2}\\
&\leq \max\{(\hat{\mu}_{\mathbb{S}\cap\mathbb{N}}(\top))^2, (\hat{\mu}_{\mathbb{S}\cap\mathbb{N}}(\top))^2\} e^{\max\{i(\hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}(\top))^2,\ \zeta_{\mathbb{S}\cap\mathbb{N}}(\top))^2\}}\\
&=\max\{(\hat{\mu}_{\mathbb{S}\cap\mathbb{N}}(\top))^2 e^{i(\hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}(\top))^2},\ (\hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}(\top))^2 e^{i(\hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}(\top))^2}\}\\
&=\max\{(\hat{\Upsilon}_{\mathbb{S}\cap\mathbb{N}}(\top))^2,\ (\hat{\Upsilon}_{\mathbb{S}\cap\mathbb{N}}(\top))^2\}\\
&(\hat{\Upsilon}_{\mathbb{S}\cap\mathbb{N}}(\top(\top))^2\\
&=(\hat{\mu}_{\mathbb{S}\cap\mathbb{N}}(\top(\top))^2 e^{i(\hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}(\top(\top))^2}\\
&\leq \max\{(\hat{\mu}_{\mathbb{S}\cap\mathbb{N}}(\top)^2, (\hat{\mu}_{\mathbb{S}\cap\mathbb{N}}((\top))^2\} e^{\max\{i(\hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}(\top))^2,\ \zeta_{\mathbb{S}\cap\mathbb{N}}((\top))^2\}}\\
&=\max\{(\hat{\mu}_{\mathbb{S}\cap\mathbb{N}}(\top))^2 e^{i(\hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}(\top))^2},\ (\hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}((\top))^2 e^{i(\hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}((\top))^2}\}\\
&=\max\{(\hat{\Upsilon}_{\mathbb{S}\cap\mathbb{N}}(\top))^2,\ (\hat{\Upsilon}_{\mathbb{S}\cap\mathbb{N}}((\top))^2\}
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
(\hat{\Upsilon}_{\mathbb{S}\cap\mathbb{N}}(\top^{-1}))^2 &= (\hat{\mu}_{\mathbb{S}\cap\mathbb{N}}(\top^{-1}))^2 e^{i(\hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}(\top^{-1}))^2}\\
&\leq \{(\hat{\mu}_{\mathbb{S}\cap\mathbb{N}}(\top))^2\} e^{\{i(\hat{\zeta}_{\mathbb{S}\cap\mathbb{N}}(\top))^2\}}\\
(\hat{\Upsilon}_{\mathbb{S}\cap\mathbb{N}}(\top^{-1}))^2 &\leq (\hat{\Upsilon}_{\mathbb{S}\cap\mathbb{N}}(\top))^2.
\end{aligned}
$$

Consequently, the proof is complete.

Remark 1: The union of two **CPFSFs** of field $\mathbb{H}$ may not be **CPFSF** of field $\mathbb{H}$.

The union of two **CPFSFs** may not be aÂ Â **CPFSF**, as shown by the example below.

Example 1: Let $\mathbb{Z}_{13}$ be a field, where $\mathbb{Z}_{13} = \{0, 1, 2, 3, \ldots, 12\}$. Assume that $\mathcal{S}$ and $\mathcal{N}$ are two **CPFSF** of field $\mathbb{Z}_{13}$ and defined as

$$
\Upsilon_{\mathcal{S}}(\mathfrak{q}) = \begin{cases} 0.3e^{\frac{i\pi}{2}} & \text{If } \mathfrak{q}\ \in 3\mathcal{Z}\\ 0 & \text{otherwise.}\end{cases}
$$

$$
\hat{\Upsilon}_{\mathcal{S}}(\mathfrak{q}) = \begin{cases} 0.2e^{\frac{i\pi}{9}} & \text{If } s\ \in 3\mathcal{Z}\\ 0.5e^{\frac{i\pi}{3}} & \text{otherwise.}\end{cases}
$$

$$
\Upsilon_{\mathcal{N}}(\mathfrak{q}) = \begin{cases} 0.2e^{\frac{i\pi}{3}} & \text{If } \mathfrak{n}\ \in 2\mathcal{Z}\\ 0.02e^{\frac{i\pi}{8}} & \text{otherwise.}\end{cases}
$$

$$
\hat{\Upsilon}_{\mathcal{N}}(\mathfrak{q}) = \begin{cases} 0.4e^{\frac{i\pi}{8}} & \text{If } \mathfrak{n}\ \in 2\mathcal{Z}\\ 0.6e^{\frac{i\pi}{2}} & \text{otherwise.}\end{cases}.
$$

It is simple to prove that $\mathcal{S}$ and $\mathcal{N}$ are two **CPFSFs** of field $\mathbb{Z}_{13}$. By Definition 5 $\mathcal{S} \cup \mathcal{N} = \{(\mathfrak{q}, \Upsilon_{\mathcal{S}\cup\mathcal{N}}, \hat{\Upsilon}_{\mathcal{S}\cup\mathcal{N}})\}$.

Therefore,

$$\Upsilon_{\mathcal{S}\cup\mathcal{N}}(\mathfrak{q}) = \begin{cases} 0.3\mathfrak{e}^{\frac{i\pi}{2}} & \text{If } \mathfrak{q} \in 3\mathcal{Z} \\ 0.2\mathfrak{e}^{\frac{i\pi}{3}} & \text{If } \mathfrak{q} \in 2\mathcal{Z}-3\mathcal{Z} \\ 0.02\mathfrak{e}^{\frac{i\pi}{8}} & \text{otherwise.} \end{cases}$$

$$\hat{\Upsilon}_{\mathcal{S}\cup\mathcal{N}}(\mathfrak{q}) = \begin{cases} 0.2\mathfrak{e}^{\frac{i\pi}{4}} & \text{If } \mathfrak{q} \in 3\mathcal{Z} \\ 0.4\mathfrak{e}^{\frac{i\pi}{8}} & \text{If } \mathfrak{q} \in 2\mathcal{Z}-3\mathcal{Z} \\ 0.5\mathfrak{e}^{\frac{i\pi}{3}} & \text{otherwise.} \end{cases}.$$

Take $\mathfrak{n} = 9$ and $\tilde{\mathfrak{a}} = 4$. Then $\Upsilon_{\mathcal{S}\cup\mathcal{N}}(9) = 0.3\mathfrak{e}^{\frac{i\pi}{2}}$ and $\Upsilon_{\mathcal{S}\cup\mathcal{N}}(4) = 0.2\mathfrak{e}^{\frac{i\pi}{3}}$, then $\Upsilon_{\mathcal{S}\cup\mathcal{N}}(9-4) = \Upsilon_{\mathcal{S}\cup\mathcal{N}}(5) = 0.02\mathfrak{e}^{\frac{i\pi}{8}}$ and $\min\{\Upsilon_{\mathcal{S}}(9), \hat{\Upsilon}_{\mathcal{N}}(4)\} = \min\{0.3\mathfrak{e}^{\frac{i\pi}{2}}, 0.2\mathfrak{e}^{\frac{i\pi}{3}}\} = 0.2\mathfrak{e}^{\frac{i\pi}{3}}$. clearly, $\Upsilon_{\mathcal{S}\cup\mathcal{N}}(9-4) < \min\{\Upsilon_{\mathcal{S}}(9), \hat{\Upsilon}_{\mathcal{N}}(4)\}$. This condition does not holds. Accordingly, $\mathcal{S}\cup\mathcal{N}$ is not **CPFSF** of $\mathcal{H}$.

Definition 10: Suppose that $\mathbb{S} = \{<\mathfrak{n}, \Upsilon_{\mathbb{S}}(\mathfrak{n}), \hat{\Upsilon}_{\mathbb{S}}(\mathfrak{n})>: \mathfrak{n} \in \mathbb{H}\}$ is a **CPFS** of $\mathbb{H}$, for all $\mathrm{c}, \hat{\mathrm{c}} \in [0,1]$ and $\mathrm{u}, \hat{\mathrm{u}} \in [0, 2\pi]$. The level subset of **CPFS** is described as
$\mathbb{S}^{(\hat{c},\hat{u})}_{(c,u)} = \{\mathfrak{n} \in \mathbb{H} : (\mu_{\mathbb{S}}(\mathfrak{n}))^2 \geq \mathrm{c}, (\zeta_{\mathbb{S}}(\mathfrak{n}))^2 \geq \mathrm{u}, (\hat{\mu}_{\mathbb{S}}(\mathfrak{n}))^2 \leq \hat{\mathrm{c}}, (\hat{\zeta}_{\mathbb{S}}(\mathfrak{n}))^2 \leq \hat{\mathrm{u}}\}$.
For $\hat{\mathrm{u}} = 0 = \mathrm{u}$, we get, $\mathbb{S}^{\hat{c}}_{c} = \{\mathfrak{n} \in \mathbb{H} : (\mu_{\mathbb{S}}(\mathfrak{n}))^2 \geq \mathrm{c}, (\hat{\mu}_{\mathbb{S}}(\mathfrak{n}))^2 \leq \hat{\mathrm{c}}\}$ and for $\hat{\mathrm{u}} = 0 = \mathrm{u}$, we get, $\mathbb{S}^{\hat{u}}_{u} = \{\mathfrak{n} \in \mathbb{H} : (\zeta_{\mathbb{S}}(\mathfrak{n}))^2 \geq \mathrm{u}, (\hat{\zeta}_{\mathbb{S}}(\mathfrak{n}))^2 \leq \hat{\mathrm{u}}\}$.

Theorem 5: Suppose that $\mathbb{S}$ be **CPFSF** of field $\mathbb{H}$. Then $\mathbb{S}^{(\hat{c},\hat{u})}_{(c,u)}$ is a subfield of field $\mathbb{H}$, $\forall\, \mathrm{c}, \hat{\mathrm{c}} \in [0,1]$ and $\mathrm{u}, \hat{\mathrm{u}} \in [0, 2\pi]$, where $(\mu_{\mathbb{S}}(\mathfrak{e}))^2 \geq \mathrm{c}$, $(\zeta_{\mathbb{S}}(\mathfrak{e}))^2 \geq \mathrm{u}$, $(\hat{\mu}_{\mathbb{S}}(\mathfrak{e}))^2 \leq \hat{\mathrm{c}}$, $(\hat{\zeta}_{\mathbb{S}}(\mathfrak{e}))^2 \leq \hat{\mathrm{u}}$, where $\mathfrak{e}$ is the unit element and $\mathfrak{e} \in \mathbb{H}$.

Proof: Consider $\mathbb{S}^{(\hat{c},\hat{u})}_{(c,u)}$, since $\mathfrak{e} \in \mathbb{S}^{(\hat{c},\hat{u})}_{(c,u)}$. Assume that any two elements $\top, \hbar \in \mathbb{S}^{(\hat{c},\hat{u})}_{(c,u)}$. Then

$$(\mu_{\mathbb{S}}(\top))^2 \geq \mathrm{c},\ (\zeta_{\mathbb{S}}(\top))^2 \geq \mathrm{u},\ (\hat{\mu}_{\mathbb{S}}(\top))^2 \leq \hat{\mathrm{c}},\ (\hat{\zeta}_{\mathbb{S}}(\top))^2 \leq \mathrm{u},$$

and $(\mu_{\mathbb{S}}(\hbar))^2 \geq \mathrm{c}, (\zeta_{\mathbb{S}}(\hbar))^2 \geq \mathrm{u}, (\hat{\mu}_{\mathbb{S}}(\hbar))^2 \leq \hat{\mathrm{c}}, (\hat{\zeta}_{\mathbb{S}}(\hbar))^2 \leq \hat{\mathrm{u}}$.

Now we suppose that,

$$\begin{aligned} &(\mu_{\mathbb{S}}(\top-\hbar))^2\mathfrak{e}^{i(\zeta_{\mathbb{S}}(\top-\hbar))^2} \\ &= (\Upsilon_{\mathbb{S}}(\top-\hbar))^2 \\ &\geq \min\{(\Upsilon_{\mathbb{S}}(\top))^2,\ (\Upsilon_{\mathbb{S}}(\hbar))^2\} \\ &= \min\{(\mu_{\mathbb{S}}\top)^2\mathfrak{e}^{i(\zeta_{\mathbb{S}}(\top))^2},\ (\mu_{\mathbb{S}}(\hbar))^2\mathfrak{e}^{i(\zeta_{\mathbb{S}}(\hbar))^2}\} \\ &= \min\{(\mu_{\mathbb{S}}(\top))^2,\ (\mu_{\mathbb{S}}(\hbar))^2\}\ \mathfrak{e}^{i\ \min\{(\zeta_{\mathbb{S}}(\top))^2,\ (\zeta_{\mathbb{S}}(\hbar))^2\}}. \end{aligned}$$

$\because \mathbb{S}$ is homogeneous, so

$$\begin{aligned} &(\mu_{\mathbb{S}}(\top-\hbar))^2 \\ &\geq \min\{(\mu_{\mathbb{S}}(\top))^2, b(\mu_{\mathbb{S}}(\hbar))^2\} = \min\{\mathrm{c},\mathrm{c}\} = \mathrm{c}, \\ &(\zeta_{\mathbb{S}}(\top-\hbar))^2 \\ &\geq \min\{(\zeta_{\mathbb{S}}(\top))^2, (\zeta_{\mathbb{S}}(\hbar))^2\} = \min\{\mathrm{u},\mathrm{u}\} = \mathrm{u}. \\ &(\mu_{\mathbb{S}}(\top(\top))))^2\mathfrak{e}^{i(\zeta_{\mathbb{S}}(\top(\top))))^2} \\ &= \Upsilon_{\mathbb{S}}(\top(\top))))^2 \\ &\geq \min\{(\Upsilon_{\mathbb{S}}(\top))^2, (\Upsilon_{\mathbb{S}}(\hbar))^2\} \\ &= \min\{(\mu_{\mathbb{S}}(\top))^2\mathfrak{e}^{i(\zeta_{\mathbb{S}}(\top))^2},\ (\mu_{\mathbb{S}}(\hbar))^2\mathfrak{e}^{i(\zeta_{\mathbb{S}}(\hbar))^2}\} \\ &= \min\{(\mu_{\mathbb{S}}(\top))^2,\ (\mu_{\mathbb{S}}(\hbar))^2\}\ \mathfrak{e}^{i\ \min\{(\zeta_{\mathbb{S}}(\top))^2,\ (\zeta_{\mathbb{S}}(\hbar))^2\}}. \end{aligned}$$

Therefore $\mathbb{S}$ is homogenous,

$$(\mu_{\mathbb{S}}(\top(\top))))^2 \geq \min\{(\mu_{\mathbb{S}}(\top))^2, (\mu_{\mathbb{S}}(\hbar))^2\} = \min\{\mathrm{c},\mathrm{c}\} = \mathrm{c},$$
$$(\zeta_{\mathbb{S}}(\top(\top))))^2 \geq \min\{(\zeta_{\mathbb{S}}(\top))^2, (\zeta_{\mathbb{S}}(\hbar))^2\} = \min\{\mathrm{u},\mathrm{u}\} = \mathrm{u}.$$

Further,

$$\begin{aligned} &(\hat{\mu}_{\mathbb{S}}(\top-\hbar))^2\mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}}(\top-\hbar))^2} \\ &= (\hat{\Upsilon}_{\mathbb{S}}(\top-\hbar))^2 \\ &\leq \max\{(\hat{\Upsilon}_{\mathbb{S}}(\top))^2, (\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\} \\ &= \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2\mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}}(\top))^2}, (\hat{\mu}_{\mathbb{S}}(\hbar))^2\mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}}(\hbar))^2}\} \\ &= \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2,\ (\hat{\mu}_{\mathbb{S}}(\hbar))^2\}\ \mathfrak{e}^{i\ \max\{(\hat{\zeta}_{\mathbb{S}}(\top))^2,\ (\hat{\zeta}_{\mathbb{S}}(\hbar))^2\}}. \end{aligned}$$

By homogeneity, so

$$\begin{aligned} &(\hat{\mu}_{\mathbb{S}}(\top-\hbar))^2 \\ &\leq \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2, (\hat{\mu}_{\mathbb{S}}(\hbar))^2\} = \max\{\hat{\mathrm{c}},\hat{\mathrm{c}}\} = \hat{\mathrm{c}}, \\ &(\hat{\zeta}_{\mathbb{S}}(\top-\hbar))^2 \\ &\leq \max\{(\hat{\zeta}_{\mathbb{S}}(\top))^2, (\hat{\zeta}_{\mathbb{S}}(\hbar))^2\} = \max\{\hat{\mathrm{u}},\hat{\mathrm{u}}\} = \hat{\mathrm{u}}. \\ &(\hat{\mu}_{\mathbb{S}}(\top\hbar))^2\mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}}(\top\hbar))^2} \\ &= (\hat{\Upsilon}_{\mathbb{S}}(\top\hbar))^2 \\ &\leq \max\{(\hat{\Upsilon}_{\mathbb{S}}(\top))^2, (\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\} \\ &= \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2\mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}}(\top))^2}, (\hat{\mu}_{\mathbb{S}}(\hbar))^2\mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}}(\hbar))^2}\} \\ &= \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2,\ (\hat{\mu}_{\mathbb{S}}(\hbar))^2\}\ \mathfrak{e}^{i\ \max\{(\hat{\zeta}_{\mathbb{S}}(\top))^2,\ (\hat{\zeta}_{\mathbb{S}}(\hbar))^2\}}. \end{aligned}$$

By homogeneity, so

$$(\hat{\mu}_{\mathbb{S}}(\top\hbar))^2 \leq \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2, (\hat{\mu}_{\mathbb{S}}(\hbar))^2\} = \max\{\hat{\mathrm{c}},\hat{\mathrm{c}}\} = \hat{\mathrm{c}},$$
$$(\hat{\zeta}_{\mathbb{S}}(\top\hbar))^2 \leq \max\{(\hat{\zeta}_{\mathbb{S}}(\top))^2, (\hat{\zeta}_{\mathbb{S}}(\hbar))^2\} = \max\{\hat{\mathrm{u}},\hat{\mathrm{u}}\} = \hat{\mathrm{u}}.$$

This implies that $\top\check{\mathfrak{r}} \in \mathbb{S}^{(\hat{c},\hat{u})}_{(c,u)}$.
Also, $(\mu_{\mathbb{S}}(\top^{-1}))^2\mathfrak{e}^{i(\zeta_{\mathbb{S}}(\top^{-1}))^2} = (\Upsilon_{\mathbb{S}}(\top^{-1}))^2 \geq (\Upsilon_{\mathbb{S}}(\top))^2 = (\mu_{\mathbb{S}}(\top))^2\mathfrak{e}^{i(\zeta_{\mathbb{S}}(\top))^2}$
$(\mu_{\mathbb{S}}(\top^{-1}))^2 \geq (\mu_{\mathbb{S}}(\top))^2 \geq r$ and $(\zeta_{\mathbb{S}}(\top^{-1}))^2 \geq (\zeta_{\mathbb{S}}(\top))^2 \geq t$ (by homogeneity).
Moreover, we have
$(\hat{\mu}_{\mathbb{S}}(\top^{-1}))^2\mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}}(\top^{-1}))^2} = (\hat{\Upsilon}_{\mathbb{S}}(\top^{-1}))^2 \leq (\hat{\Upsilon}_{\mathbb{S}}(\top))^2 = (\hat{\mu}_{\mathbb{S}}(\top))^2\mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}}(\top))^2}$
$(\hat{\mu}_{\mathbb{S}}(\top^{-1}))^2 \leq (\hat{\mu}_{\mathbb{S}}(\top))^2 \leq \hat{\mathrm{c}}$ and, $(\hat{\zeta}_{\mathbb{S}}(\top^{-1}))^2 \leq (\hat{\zeta}_{\mathbb{S}}(\top))^2 \leq \hat{t}$
$\Rightarrow \top^{-1} \in \mathbb{S}^{(\hat{c},\hat{u})}_{(c,u)}$. Hence, $\mathbb{S}^{(\hat{c},\hat{u})}_{(c,u)}$ is subfield.

Theorem 6: Suppose that $\mathbb{S}^{(\hat{c},\hat{u})}_{(c,u)}$ is a subfield of $\mathbb{H}$, then $\mathbb{S}$ is **CPFSF** of $\mathbb{H}\forall \mathrm{c} \in [0,1]$ and $\mathrm{u} \in [0, 2\pi]$, where $(\mu_{\mathbb{S}}(\mathfrak{e}))^2 \geq \mathrm{c}$, $(\zeta_{\mathbb{S}}(\mathfrak{e}))^2 \geq \mathrm{u}$, $(\hat{\mu}_{\mathbb{S}}\mathfrak{e}))^2 \leq \hat{\mathrm{c}}$, $(\hat{\zeta}_{\mathbb{S}}(\mathfrak{e}))^2 \leq \hat{\mathrm{u}}$, where $\mathfrak{e}$ is the unit element of $\mathbb{H}$.

Proof: Let $\min\{\mu_{\mathbb{S}}((\top))^2, \mu(S(((\top))))^2\} = \mathrm{c}$,
$\min\{\zeta_{\mathbb{S}}((\top))^2, \zeta_{\mathbb{S}}((\hbar))^2\} = \mathrm{u}$, and
$\max\{\hat{\mu}_{\mathbb{S}}(\top)^2, \hat{\mu}_{\mathbb{S}}(((\top)))^2)\} = \hat{\mathrm{c}}$, $\max\{\hat{\zeta}_{\mathbb{S}}((\top)^2), \hat{\zeta}_{\mathbb{S}}(((\top))))^2\} = \hat{\mathrm{u}}$. Then we have $\mu_{\mathbb{S}}((\top)^2) \geq \mathrm{c}$, $\hat{\mu}_{\mathbb{S}}(\top)^2 \leq \hat{\mathrm{c}}$, $\zeta_{\mathbb{S}}((\top)^2) \geq \mathrm{u}$, $\hat{\zeta}_{\mathbb{S}}((\top)^2) \leq \hat{\mathrm{u}}and\mu_{\mathbb{S}}(((\top))^2) \geq \mathrm{c}$, $\hat{\mu}_{\mathbb{S}}(((\top))^2) \leq$

$\hat{c}$, $\zeta_{\mathbb{S}}(((\mathsf{T})))^2) \geq u$, $\hat{\zeta}_{\mathbb{S}}(((\mathsf{T})))^2) \leq \hat{u}$. This implies that $\mathsf{T} \in \mathbb{S}_{(c,u)}^{(\hat{c},\hat{u})}$ and $\breve{\mathfrak{r}} \in \mathbb{S}_{(c,u)}^{(\hat{c},\hat{u})}$.

$\because$ $\mathbb{S}^{(\hat{c},\hat{u})}(c, u)$ is subfield, so $\mathsf{T}\breve{\mathfrak{r}} \in_{\mathbb{S}}^{(\hat{c},\hat{u})} (c, u)$. Then we have

$$(\mu_{\mathbb{S}}(\mathsf{T}-\hbar))^2 \geq c \text{ and } (\zeta_{\mathbb{S}}(\mathsf{T}-\hbar))^2 \geq u$$
$$(\hat{\mu}_{\mathbb{S}}(\mathsf{T}-\hbar))^2 \leq \hat{c} \text{ and } (\hat{\zeta}_{\mathbb{S}}(\mathsf{T}-\hbar))^2 \leq \hat{u}.$$

Implies that $(\mu_{\mathbb{S}}(\mathsf{T}-\hbar))^2 \geq \min\{(\mu_{\mathbb{S}}(\mathsf{T}))^2, (\mu_{\mathbb{S}}(\hbar))^2\}$ and $(\zeta_{\mathbb{S}}(\mathsf{T}-\hbar))^2 \geq \min\{(\zeta_{\mathbb{S}}(\mathsf{T}))^2, (\zeta_{\mathbb{S}}(\hbar))^2\}$

$$(\hat{\mu}_{\mathbb{S}}(\mathsf{T}-\hbar))^2 \leq \max\{(\hat{\mu}_{\mathbb{S}}(\mathsf{T}))^2, (\hat{\mu}_{\mathbb{S}}(\hbar))^2\}$$

and

$$(\hat{\zeta}_{\mathbb{S}}(\mathsf{T}-\hbar))^2 \leq \max\{(\hat{\zeta}_{\mathbb{S}}(\mathsf{T}))^2, (\hat{\zeta}_{\mathbb{S}}(\hbar))^2\}.$$
$$\mu_{\mathbb{S}}(\mathsf{T}\hbar))^2 \geq c \text{ and } (\zeta_{\mathbb{S}}(\mathsf{T}\hbar))^2 \geq u$$
$$(\hat{\mu}_{\mathbb{S}}(\mathsf{T}\hbar))^2 \leq \hat{c} \text{ and } (\hat{\zeta}_{\mathbb{S}}(\mathsf{T}\hbar))^2 \leq \hat{u}.$$

Implies that

$$(\mu_{\mathbb{S}}(\mathsf{T}\hbar))^2 \geq \min \mu_{\mathbb{S}}(\mathsf{T})^2, (\mu_{\mathbb{S}}(\hbar))^2$$

and

$$(\zeta_{\mathbb{S}}(\mathsf{T}\hbar))^2 \geq \min\{\zeta_{\mathbb{S}}\mathsf{T}, (\zeta_{\mathbb{S}}(\hbar))^2\}$$
$$(\hat{\mu}_{\mathbb{S}}(\mathsf{T}\hbar))^2 \leq \max\{(\hat{\mu}_{\mathbb{S}}(\mathsf{T}))^2, (\hat{\mu}_{\mathbb{S}}(\hbar))^2\}$$

and

$$(\hat{\zeta}_{\mathbb{S}}(\mathsf{T}\hbar))^2 \leq \max\{(\hat{\zeta}_{\mathbb{S}}(\mathsf{T}))^2, (\hat{\zeta}_{\mathbb{S}}(\hbar))^2\}.$$

Thus,

$$\begin{aligned}
&(\Upsilon_{\mathbb{S}}(\mathsf{T}-\hbar))^2 \\
&= (\mu_{\mathbb{S}}(\mathsf{T}-\hbar))^2 \mathfrak{e}^{\mathrm{i}(\mu_{\mathbb{S}}(\mathsf{T}-\hbar))^2} \\
&\geq \min\{(\mu_{\mathbb{S}}(\mathsf{T}))^2, (\mu_{\mathbb{S}}(\hbar))^2\}\ \mathfrak{e}^{\mathrm{i}\min\{(\zeta_{\mathbb{S}}(\mathsf{T}))^2, (\zeta_{\mathbb{S}}(\hbar))^2\}} \\
&= \min\{(\mu_{\mathbb{S}}(\mathsf{T}))^2 \mathfrak{e}^{\mathrm{i}(\zeta_{\mathbb{S}}(\mathsf{T}))^2}, (\mu_{\mathbb{S}}(\hbar))^2 \mathfrak{e}^{\mathrm{i}(\zeta_{\mathbb{S}}(\hbar))^2}\} \\
&(\Upsilon_{\mathbb{S}}(\mathsf{T}-\hbar))^2 \\
&\geq \min\{(\Upsilon_{\mathbb{S}}(\mathsf{T}))^2, (\Upsilon_{\mathbb{S}}(\hbar))^2\}. \\
&(\hat{\Upsilon}_{\mathbb{S}}(\mathsf{T}-\ddot{\mathfrak{z}}))^2 \\
&= (\hat{\mu}_{\mathbb{S}}(\mathsf{T}-\ddot{\mathfrak{z}}))^2 \mathfrak{e}^{\mathrm{i}(\hat{\mu}_{\mathbb{S}}(\mathsf{T}-\ddot{\mathfrak{z}}))^2} \\
&\leq \max\{(\hat{\mu}_{\mathbb{S}}(\mathsf{T}))^2, (\hat{\mu}_{\mathbb{S}}(\ddot{\mathfrak{z}}))^2\} \mathfrak{e}^{\mathrm{i}\max\{(\hat{\zeta}_{\mathbb{S}}(\mathsf{T}))^2, (\hat{\zeta}_{\mathbb{S}}(\ddot{\mathfrak{z}}))^2\}} \\
&= \max\{(\hat{\mu}_{\mathbb{S}}(\mathsf{T}))^2 \mathfrak{e}^{\mathrm{i}(\hat{\zeta}_{\mathbb{S}}(\mathsf{T}))^2}, (\hat{\mu}_{\mathbb{S}}(\ddot{\mathfrak{z}}))^2 \mathfrak{e}^{\mathrm{i}(\hat{\zeta}_{\mathbb{S}}(\ddot{\mathfrak{z}}))^2}\} \\
&((\hat{\Upsilon}_{\mathbb{S}}(\mathsf{T}-\ddot{\mathfrak{z}}))^2 \\
&\leq \max\{(\hat{\Upsilon}_{\mathbb{S}}(\mathsf{T}))^2, (\hat{\Upsilon}_{\mathbb{S}}(\ddot{\mathfrak{z}}))^2\} \\
&(\Upsilon_{\mathbb{S}}(\mathsf{T}\hbar))^2 \\
&= (\mu_{\mathbb{S}}(\mathsf{T}\hbar))^2 \mathfrak{e}^{\mathrm{i}(\mu_{\mathbb{S}}(\mathsf{T}\hbar))^2} \\
&\geq \min\{(\mu_{\mathbb{S}}(\mathsf{T}))^2, (\mu_{\mathbb{S}}(\hbar))^2\}\ \mathfrak{e}^{\mathrm{i}\min\{(\zeta_{\mathbb{S}}(\mathsf{T}))^2, (\zeta_{\mathbb{S}}(\hbar))^2\}} \\
&= \min\{(\mu_{\mathbb{S}}(\mathsf{T}))^2 \mathfrak{e}^{\mathrm{i}(\zeta_{\mathbb{S}}(\mathsf{T}))^2}, (\mu_{\mathbb{S}}(\hbar))^2 \mathfrak{e}^{\mathrm{i}(\zeta_{\mathbb{S}}(\hbar))^2}\} \\
&(\Upsilon_{\mathbb{S}}(\mathsf{T}\hbar))^2 \\
&\geq \min\{(\Upsilon_{\mathbb{S}}(\mathsf{T}))^2, (\Upsilon_{\mathbb{S}}(\hbar))^2\}. \\
&\hat{\Upsilon}_{\mathbb{S}}(\mathsf{T}\hbar) \\
&= \hat{\mu}_{\mathbb{S}}(\mathsf{T}\hbar) \mathfrak{e}^{\mathrm{i}\hat{\mu}_{\mathbb{S}}(\mathsf{T}\hbar)} \\
&\leq \max\{\hat{\mu}_{\mathbb{S}}(\mathsf{T}), \hat{\mu}_{\mathbb{S}}(\hbar)\}\ \mathfrak{e}^{\mathrm{i}\max\{\hat{\zeta}_{\mathbb{S}}(\mathsf{T}), \hat{\zeta}_{\mathbb{S}}(\hbar)\}} \\
&= \max\{\hat{\mu}_{\mathbb{S}}(\mathsf{T}) \mathfrak{e}^{\mathrm{i}\hat{\zeta}_{\mathbb{S}}(\mathsf{T})}, \hat{\mu}_{\mathbb{S}}(\hbar) \mathfrak{e}^{\mathrm{i}\hat{\zeta}_{\mathbb{S}}(\hbar)}\} \\
&\hat{\Upsilon}_{\mathbb{S}}(\mathsf{T}\hbar) \\
&\leq \max\{\hat{\Upsilon}_{\mathbb{S}}(\mathsf{T}), \hat{\Upsilon}_{\mathbb{S}}(\hbar)\}.
\end{aligned}$$

Further, consider $\mathsf{T} \in \mathbb{H}$ be any element. Let $(\mu_{\mathbb{S}}(\mathsf{T}))^2 = c$, $(\zeta_{\mathbb{S}}(\mathsf{T}))^2 = u$, $((\hat{\mu}_{\mathbb{S}}(\mathsf{T}))^2 = \hat{c}$ and $((\hat{\zeta}_{\mathbb{S}}(\mathsf{T}))^2 = \hat{u}$.

Then, $(\mu_{\mathbb{S}}(\mathsf{T}))^2 \geq c$, and $(\zeta_{\mathbb{S}}(\mathsf{T}))^2 \geq u$, and $((\hat{\mu}_{\mathbb{S}}(\mathsf{T}))^2 \leq \hat{c}$, $((\hat{\zeta}_{\mathbb{S}}(\mathsf{T}))^2 \leq \hat{u}$ is true. Implies that $\mathsf{T} \in \mathbb{S}_{(c,u)}^{(\hat{c},\hat{u})}$.

$\because \mathbb{S}_{(c,u)}^{(\hat{c},\hat{u})}$ is subfield. So, $\mathsf{T}^{-1} \in \mathbb{S}_{(c,u)}^{(\hat{c},\hat{u})} \Rightarrow (\mu_{\mathbb{S}}(\mathsf{T}^{-1}))^2 \geq c$, $(\zeta_{\mathbb{S}}(\mathsf{T}^{-1}))^2 \geq u$ and $(\hat{\zeta}_{\mathbb{S}}(\mathsf{T}^{-1}))^2 \leq \hat{c}$, $(\hat{\zeta}_{\mathbb{S}}(\mathsf{T}^{-1})^2 \leq \hat{u}, \Rightarrow$ $(\mu_{\mathbb{S}}(\mathsf{T}^{-1}))^2 \geq (\mu_{\mathbb{S}}(\mathsf{T}))^2$, $(\zeta_{\mathbb{S}}(\mathsf{T}^{-1}))^2 \geq (\zeta_{\mathbb{S}}(\mathsf{T}))^2$, and $(\hat{\mu}_{\mathbb{S}}(\mathsf{T}^{-1}))^2 \leq (\hat{\mu}_{\mathbb{S}}(\mathsf{T}))^2$, $(\hat{\zeta}_{\mathbb{S}}(\mathsf{T}^{-1}))^2 \leq (\hat{\zeta}_{\mathbb{S}}(\mathsf{T}))^2$. Consider that,

$$\begin{aligned}
(\Upsilon_{\mathbb{S}}(\mathsf{T}^{-1}))^2 &= (\mu_{\mathbb{S}}(\mathsf{T}^{-1}))^2 \mathfrak{e}^{\mathrm{i}(\zeta_{\mathbb{S}}(\mathsf{T}^{-1}))^2} \\
&\geq (\mu_{\mathbb{S}}(\mathsf{T}))^2 \mathfrak{e}^{\mathrm{i}(\zeta_{\mathbb{S}}(\mathsf{T}))^2} \\
&= (\Upsilon_{\mathbb{S}}(\mathsf{T}))^2. \\
(\hat{\Upsilon}_{\mathbb{S}}(\mathsf{T}^{-1}))^2 &= (\hat{\mu}_{\mathbb{S}}(\mathsf{T}^{-1}))^2 \mathfrak{e}^{\mathrm{i}}(\hat{\zeta}_{\mathbb{S}}(\mathsf{T}^{-1}))^2 \\
&\leq (\hat{\mu}_{\mathbb{S}}(\mathsf{T}))^2 \mathfrak{e}^{\mathrm{i}}(\hat{\zeta}_{\mathbb{S}}(\mathsf{T}))^2 \\
&= (\hat{\Upsilon}_{\mathbb{S}}(\mathsf{T}))^2.
\end{aligned}$$

Corollary 1: Suppose that $\mathbb{S}$ is a **CPFSF** of any field $\mathbb{H}$. Then, $\mathbb{S}_{c}^{\hat{c}}$ and $\mathbb{S}_{u}^{\hat{u}}$ level subsets is a subfields of field $\mathbb{H}$, $\forall$ $(\hat{c})$, $(c) \in [0, 1]$ and $\hat{u}$, $u \in [0, 2\pi]$, where $(\mu_{\mathbb{S}}(\mathfrak{e}))^2 \geq c$, $(\zeta_{\mathbb{S}}(\mathfrak{e}))^2 \geq u$ and $(\hat{\mu}_{\mathbb{S}}(\mathfrak{e}))^2 \leq \hat{c}$, $(\hat{\zeta}_{\mathbb{S}}(\mathfrak{e}))^2 \leq \hat{u}$, where $\mathfrak{e}$ is the unit element and $\mathfrak{e} \in \mathbb{H}$.

IV. HOMOMORPHISM OF COMPLEX INTUITIONISTIC FUZZY SUBFIELDS

The homomorphic image and pre-image of **CPFSF** are defined in this section. We exceed few more **CPFSF** results that are related to certain field homomorphism.

Definition 11: Suppose that $\mathbb{K}$ and $\mathbb{F}$ are two fields and $\mho : \mathbb{K} \to \mathbb{F}$ is a homomorphism from $\mathbb{K}$ to $\mathbb{F}$. Assume that $\mathbb{S}$ and $\mathbb{N}$ are two **CPFSF** of fields $\mathbb{K}$ and $\mathbb{F}$, respectively, $\forall$ $\mathsf{T} \in \mathbb{K}$, $\breve{\mathfrak{r}} \in \mathbb{F}$. The set $\mho(\mathbb{S})(\mathsf{T}) = \{\mathsf{T}, \mho(\Upsilon_{\mathbb{S}}(\mathsf{T}), \mho(\hat{\Upsilon}_{\mathbb{S}})(\mathsf{T})\}$ is image of $\mathbb{S}$, where

$$\mho(\Upsilon_{\mathbb{S}})(\mathsf{T}) = \begin{cases} sup\{\Upsilon_{\mathbb{S}}(\mathsf{T}), & \text{PF } \mho(\mathsf{T}) = \mathsf{T}\}, \quad \mho^{-1}(\mathsf{T}) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$
$$\mho(\hat{\Upsilon}_{\mathbb{S}})(\mathsf{T}) = \begin{cases} inf\{\hat{\Upsilon}_{\mathbb{S}}(\mathsf{T}), & \text{PF } \mho(\mathsf{T}) = \mathsf{T}\}, \quad \mho^{-1}(\mathsf{T}) \neq \emptyset \\ 1, & \text{otherwise.} \end{cases}$$

. The set $\mho^{-1}(\mathbb{N})(\mathsf{T}) = \{\mathsf{T}, \mho^{-1}(\Upsilon_{\mathbb{N}})(\mathsf{T}), \mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\mathsf{T})\}$ is called pre image of $\mathbb{N}$, where

$$\mho^{-1}(\Upsilon_{\mathbb{N}})(\mathsf{T}) = (\Upsilon_{\mathbb{N}})(\mho(\mathsf{T}))$$
$$\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\mathsf{T}) = (\hat{\Upsilon}_{\mathbb{N}})(\mho(\mathsf{T})), \forall\ \mathsf{T} \in \mathbb{K}$$

Theorem 7: [13] Suppose that $\mathbb{K}$ and $\mathbb{F}$ are two fields and $\mho : \mathbb{K} \to \mathbb{F}$ is a homomorphism from $\mathbb{K}$ to $\mathbb{F}$. Assume that $\mathbb{S}$ and $\mathbb{N}$ are **PFSF** of fields $\mathbb{K}$ and $\mathbb{F}$, respectively. Then $\mho(\mathbb{S})$ is **PFSF** of $\mathbb{F}$ and $\mho^{-1}(\mathbb{N})$ is **PFSF** of $\mathbb{K}$.

Lemma 1: Suppose that $\mathbb{K}$ and $\mathbb{F}$ are two fields and $\mho : \mathbb{K} \to \mathbb{F}$ is a homomorphism from $\mathbb{K}$ to $\mathbb{F}$. CPFS $\mathbb{S}$ and $\mathbb{N}$ are two **CPFSF** of given fields. Then,

1) $(\mho(\Upsilon_{\mathbb{S}})(\top))^2 = (\mho(\mu_{\mathbb{S}})(\top))^2 e^{i(\mho(\varphi_{\mathbb{S}})(\top))^2}$, for all $\top \in \mathbb{F}$,
2) $(\mho(\hat{\Upsilon}_{\mathbb{S}})(\top))^2 = (\mho(\hat{\mu}_{\mathbb{S}})(\top))^2 e^{i(\mho(\hat{\zeta}_{\mathbb{S}})(\top))^2}$, for all $\top \in \mathbb{F}$,
3) $(\mho^{-1}(\Upsilon_{\mathbb{N}})(\top))^2 = (\mho^{-1}(\mu_{\mathbb{N}})(\top))^2 e^{(i\mho^{-1}(\zeta_{\mathbb{N}})(\top))^2}$, for all $\top \in \mathbb{K}$,
4) $(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\top))^2 = (\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\top))^2 e^{i(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\top))^2}$, for all $\top \in \mathbb{K}$.

Proof: Suppose that

$$\begin{aligned}
&(\mho(\Upsilon_{\mathbb{S}})\top)^2\\
&= \max\{(\Upsilon_{\mathbb{S}}(\top))^2,\ \text{PF } \mho(\top) = \top\}\\
&= \max\{(\mu_{\mathbb{S}}(\top))^2 e^{i(\mho(\zeta_{\mathbb{S}})(\top))^2},\ \text{PF } \mho(\top) = \top\}\\
&= \max\{(\mu_{\mathbb{S}}(\top))^2, \text{PF } \mho(\top) = \top\}\ e^{i\ \max\{((\zeta_{\mathbb{S}})(\top))^2,\ \text{PF } \mho(\top)=\top\}}\\
&= (\mho(\mu_{\mathbb{S}})\top)^2 e^{i(\mho(\zeta_{\mathbb{S}})(\top))^2}.
\end{aligned}$$

Hence,

$$(\mho(\Upsilon_{\mathbb{S}})(\top))^2 = (\mho(\mu_{\mathbb{S}})(\top))^2 e^{i(\mho(\zeta_{\mathbb{S}})(\top))^2}$$

Assume that

$$\begin{aligned}
&(\mho(\hat{\Upsilon}_{\mathbb{S}})(\top))^2\\
&= \min\{(\hat{\Upsilon}_{\mathbb{S}}(\top))^2,\ \text{PF } \mho(\top) = \top\}\\
&= \min\{(\hat{\mu}_{\mathbb{S}}(\top))^2 e^{(i\mho(\hat{\zeta}_{\mathbb{S}})(\top))^2},\ \text{PF } \mho(\top) = \top\}\\
&= \min\{(\hat{\mu}_{\mathbb{S}}(\top))^2, \text{PF } \mho(\top) = \top\}\ e^{i\min\{((\hat{\zeta}_{\mathbb{S}})(\top))^2,\ \text{PF } \mho(\top)=\top\}}\\
&= (\mho(\hat{\mu}_{\mathbb{S}})(\top))^2 e^{i(\mho(\hat{\zeta}_{\mathbb{S}})(\top))^2}
\end{aligned}$$

Hence,

$$(\mho(\hat{\Upsilon}_{\mathbb{S}})(\top))^2 = (\mho(\hat{\mu}_{\mathbb{S}})(\top))^2 e^{i(\mho(\hat{\zeta}_{\mathbb{S}})(\top))^2}.$$

Consider,

$$\begin{aligned}
(\mho^{-1}(\Upsilon_{\mathbb{N}})(\top))^2 &= (\Upsilon_{\mathbb{N}}(\mho(\top)))^2\\
&= (\mu_{\mathbb{N}}(\mho(\top)))^2 e^{i(\zeta_{\mathbb{N}}(\mho(\top)))^2}\\
&= (\mho^{-1}(\mu_{\mathbb{N}})(\top))^2 e^{i(\mho^{-1}(\zeta_{\mathbb{N}})(\top))^2}.
\end{aligned}$$

Consequently,

$$(\mho^{-1}(\Upsilon_{\mathbb{N}})(\top))^2 = (\mho^{-1}(\mu_{\mathbb{N}})(\top))^2 e^{i(\mho^{-1}(\zeta_{\mathbb{N}})(\top))^2}.$$

Consider,

$$\begin{aligned}
(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\top))^2 &= (\hat{\Upsilon}_{\mathbb{N}}(\mho(\top)))^2\\
&= (\hat{\mu}_{\mathbb{N}}(\mho(\top)))^2 e^{i(\hat{\zeta}_{\mathbb{N}}(\mho(\top)))^2}\\
&= (\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\top))^2 e^{i(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\top))^2}.
\end{aligned}$$

Consequently,

$$(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\top))^2 = (\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\top))^2 e^{i(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\top))^2}.$$

The homomorphic image of **CPFSF**, according to the result, is always **CPFSF**.

Theorem 8: Suppose that $\mathbb{K}$ and $\mathbb{F}$ are two fields and $\mho : \mathbb{K} \to \mathbb{F}$ is a homomorphism from $\mathbb{K}$ to $\mathbb{F}$. PF $\mathbb{S}$ is a **CPFSF** of field $\mathbb{K}$. Then, $\mho(\mathbb{S})$ is **CPFSF** of $\mathbb{F}$.

Proof: Clearly, $\overline{\mathbb{S}} = \{< \hbar, \mu_{\mathbb{S}}(\hbar), \hat{\mu}_{\mathbb{S}}(\hbar) >: \hbar \in \mathbb{K}, \mu_{\mathbb{S}}(\hbar), \hat{\mu}_{\mathbb{S}}(\hbar) \in [0\ 1]\}$ and $\underline{\mathbb{S}} = \{< \hbar, \zeta_{\mathbb{S}}(\hslash), \hat{\zeta}_{\mathbb{S}}(\hbar) >: \hbar \in \mathbb{K}, \zeta_{\mathbb{S}}(\hbar), \hat{\zeta}_{\mathbb{S}}(\hbar) \in [0\ 2\pi]\}$ are **PFSF** and π-**PFSF**, respectively. In the view of Theorem 3 and in the view of Theorem 7 the homomorphic image of $\overline{\mathbb{S}} = \{< \hbar, \mu_{\mathbb{S}}(\hbar), \hat{\mu}_{\mathbb{S}}(\hbar) >: \hbar \in \mathbb{K}, 0 < (\zeta_{\mathbb{S}}(\hbar))^2 + (\hat{\zeta}_{\mathbb{S}}(\hbar))^2 \leq 1\}$ is **PFSF** and $\underline{\mathbb{S}} = \{< \hbar, \zeta_{\mathbb{S}}(\hslash), \hat{\zeta}_{\mathbb{S}}(\hbar) >: \hbar \in \mathbb{K}, 0 < (\zeta_{\mathbb{S}}(\hslash))^2 + \hat{\zeta}_{\mathbb{S}}(\hslash)^2 \leq 2\pi\}$ is π-**PFSF** $\forall\ \hbar, \hslash \in \mathbb{F}$. Then,

$$\begin{aligned}
(\mho(\mu_{\mathbb{S}})(\hbar - \hslash))^2 &\geq \min\{(\mho(\mu_{\mathbb{S}})(\hbar))^2, (\mho(\mu_{\mathbb{S}})(\hslash))^2\}\\
(\mho(\mu_{\mathbb{S}})((\hbar\hslash))^2 &\geq \min\{(\mho(\mu_{\mathbb{S}})(\hbar))^2, (\mho(\mu_{\mathbb{S}})(\hslash))^2\}\\
(\mho(\mu_{\mathbb{S}})(\hbar^{-1}))^2 &\geq (\mho(\mu_{\mathbb{S}})(\hbar))^2\\
(\mho(\hat{\mu}_{\mathbb{S}})((\hbar - \hslash))^2 &\leq \max\{(\mho(\hat{\mu}_{\mathbb{S}})(\hbar))^2, (\mho(\hat{\mu}_{\mathbb{S}})(\hslash))^2\}\\
(\mho(\hat{\mu}_{\mathbb{S}})((\hbar\hslash))^2 &\leq \max\{(\mho(\hat{\mu}_{\mathbb{S}})(\hbar))^2, (\mho(\hat{\mu}_{\mathbb{S}})(\hslash))^2\}\\
(\mho(\hat{\mu}_{\mathbb{S}})(\hbar^{-1}))^2 &\leq (\mho(\hat{\mu}_{\mathbb{S}})(\hbar))^2\\
(\mho(\zeta_{\mathbb{S}})(\hbar - \hslash)) &\geq \min\{(\mho(\zeta_{\mathbb{S}})(\hbar))^2, (\mho(\zeta_{\mathbb{S}})(\hslash))^2\}\\
(\mho(\zeta_{\mathbb{S}})(\hbar\hslash))^2 &\geq \min\{(\mho(\zeta_{\mathbb{S}})(\hbar))^2, (\mho(\zeta_{\mathbb{S}})(\hslash))^2\}\\
(\mho(\zeta_{\mathbb{S}})(\hbar^{-1}))^2 &\geq (\mho(\zeta_{\mathbb{S}})(\hbar))^2\\
(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar - \hslash))^2 &\leq \max\{(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar))^2, (\mho(\hat{\zeta}_{\mathbb{S}})(\hslash))^2\}\\
(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar\hslash))^2 &\leq \max\{(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar))^2, (\mho(\hat{\zeta}_{\mathbb{S}})(\hslash))^2\}\\
(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar^{-1}))^2 &\leq (\mho(\hat{\zeta}_{\mathbb{S}})(\hbar))^2.
\end{aligned}$$

By Lemma 1 [1], Since

$$\begin{aligned}
&(\mho(\Upsilon_{\mathbb{S}})(\hbar - \hslash))^2\\
&= (\mho(\mu_{\mathbb{S}})(\hbar - \hslash))^2)e^{i(\mho(\zeta_{\mathbb{S}})(\hbar-\hslash))^2},\quad \forall\ \hbar, \hslash \in \mathbb{F}\\
&\geq \min\{(\mho(\mu_{\mathbb{S}})(\hbar))^2, (\mho(\mu_{\mathbb{S}})(\hslash))^2\}\ e^{i\ \min\{(\mho(\zeta_{\mathbb{S}})(\hbar))^2,(\mho(\zeta_{\mathbb{S}})(\hslash))^2\}}\\
&\geq \min\{(\mho(\mu_{\mathbb{S}})(\hbar))^2 e^{i\mho(\zeta_{\mathbb{S}})(\hbar))^2}, (\mho(\mu_{\mathbb{S}})(\hslash))^2 e^{i(\mho(\zeta_{\mathbb{S}})(\hslash))^2}\}\\
&= \min\{(\mho(\Upsilon_{\mathbb{S}})(\hbar))^2, (\mho(\Upsilon_{\mathbb{S}})(\hslash))^2\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&(\mho(\Upsilon_{\mathbb{S}})(\hbar - \hslash))^2\\
&\geq \min\{(\mho(\Upsilon_{\mathbb{S}})(\hbar))^2, (\mho(\Upsilon_{\mathbb{S}}(\hslash))^2\}\\
&(\mho(\Upsilon_{\mathbb{S}})(\hbar\hslash))^2\\
&= (\mho(\mu_{\mathbb{S}})(\hbar\hslash))^2 e^{i(\mho(\zeta_{\mathbb{S}})(\hbar\hslash))^2},\quad \forall\ \hbar, \hslash \in \mathbb{F}\\
&\geq \min\{(\mho(\mu_{\mathbb{S}})(\hbar))^2, (\mho(\mu_{\mathbb{S}})(\hslash))^2\}\ e^{i\ \min\{(\mho(\zeta_{\mathbb{S}})(\hbar))^2,(\mho(\zeta_{\mathbb{S}})(\hslash))^2\}}\\
&\geq \min\{(\mho(\mu_{\mathbb{S}})(\hbar))^2 e^{i(\mho(\zeta_{\mathbb{S}})(\hbar))^2}, (\mho(\mu_{\mathbb{S}})(\hslash))^2 e^{i(\mho(\zeta_{\mathbb{S}})(\hslash))^2}\}\\
&= \min\{(\mho(\Upsilon_{\mathbb{S}})(\hbar))^2, (\mho(\Upsilon_{\mathbb{S}})(\hslash))^2\}.
\end{aligned}$$

Accordingly, $(\mho(\Upsilon_{\mathbb{S}})(\hbar\hslash))^2 \geq \min\{(\mho(\Upsilon_{\mathbb{S}})(\hbar))^2, (\mho(\Upsilon_{\mathbb{S}}(\hslash))^2\}$. Moreover,

$$(\mho(\Upsilon_{\mathbb{S}})(\hbar^{-1}))^2 = (\mho(\mu_{\mathbb{S}})(\hbar^{-1}))^2 e^{i(\mho(\zeta_{\mathbb{S}})(\hbar^{-1}))^2},\ \forall\ \hbar \in \mathbb{F}$$

$$\geq (\mho(\mu_{\mathbb{S}})(\hbar))^2 e^{i(\mho(\zeta_{\mathbb{S}})(\hbar))^2}$$
$$= (\mho(\Upsilon_{\mathbb{S}})(\hbar))^2.$$

Thus, $(\mho(\Upsilon_{\mathbb{S}})(\hbar^{-1}))^2 \geq (\mho(\Upsilon_{\mathbb{S}})(\hbar))^2$.

From Lemma 1[2], we know that,

$$\begin{aligned}
&(\mho(\hat{\Upsilon}_{\mathbb{S}})(\hbar - \hbar))^2\\
&= (\mho(\hat{\mu}_{\mathbb{S}})(\hbar - \hbar))^2 e^{i(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar-\hbar))^2}, \quad \forall\, \hbar, \hbar \in \mathbb{F}\\
&\leq \max\{(\mho(\hat{\mu}_{\mathbb{S}})(\hbar))^2, (\mho(\hat{\mu}_{\mathbb{S}})(\hbar))^2\}\ e^{i\ \max\{(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar))^2,(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar))^2\}}\\
&\leq \max\{(\mho(\hat{\mu}_{\mathbb{S}})(\hbar))^2 e^{i(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar))^2}, (\mho(\hat{\mu}_{\mathbb{S}})(\hbar))^2 e^{i(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar))^2}\}\\
&= \max\{(\mho(\hat{\Upsilon}_{\mathbb{S}})(\hbar))^2, (\mho(\hat{\Upsilon}_{\mathbb{S}})(\hbar))^2\}.
\end{aligned}$$

In consequence,

$$\begin{aligned}
&(\mho(\hat{\Upsilon}_{\mathbb{S}})(\hbar - \hbar))^2\\
&\leq \max\{(\mho(\hat{\Upsilon}_{\mathbb{S}})\hbar)^2, (\mho(\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\}\\
&(\mho(\hat{\Upsilon}_{\mathbb{S}})(\hbar\hbar))^2\\
&= (\mho(\hat{\mu}_{\mathbb{S}})(\hbar\hbar))^2 e^{i(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar\hbar))^2}, \quad \forall\, \hbar, \hbar \in \mathbb{F}\\
&\leq \max\{(\mho(\hat{\mu}_{\mathbb{S}})(\hbar))^2, (\mho(\hat{\mu}_{\mathbb{S}})(\hbar))^2\}\ e^{i\ \max\{(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar))^2,(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar))^2\}}\\
&\leq \max\{(\mho(\hat{\mu}_{\mathbb{S}})(\hbar))^2 e^{i(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar))^2}, (\mho(\hat{\mu}_{\mathbb{S}})(\hbar))^2 e^{i(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar))^2}\}\\
&= \max\{(\mho(\hat{\Upsilon}_{\mathbb{S}})(\hbar))^2, (\mho(\hat{\Upsilon}_{\mathbb{S}})(\hbar))^2\}.
\end{aligned}$$

Consequently,
$(\mho(\hat{\Upsilon}_{\mathbb{S}})(\hbar\hbar))^2 \leq \max\{(\mho(\hat{\Upsilon}_{\mathbb{S}})(\hbar))^2, (\mho(\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\}$. Moreover,

$$\begin{aligned}
(\mho(\hat{\Upsilon}_{\mathbb{S}})(\hbar^{-1}))^2 &= (\mho(\hat{\mu}_{\mathbb{S}})(\hbar^{-1}))^2 e^{i(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar^{-1}))^2}, \ \forall\, \hbar \in \mathbb{F}\\
&\leq (\mho(\hat{\mu}_{\mathbb{S}})(\hbar))^2 e^{i(\mho(\hat{\zeta}_{\mathbb{S}})(\hbar))^2}\\
&= (\mho(\hat{\Upsilon}_{\mathbb{S}})(\hbar))^2.
\end{aligned}$$

. Thus,

$$(\mho(\hat{\Upsilon}_{\mathbb{S}})(\hbar^{-1}))^2 \leq (\mho(\hat{\Upsilon}_{\mathbb{S}})(\hbar))^2.$$

Hence, the proof is demonstrated.

According to the following conclusion, **CPFSF** is the inverse homomorphic image of **CPFSF**.

Theorem 9: Assume that $\mathbb{K}$ and $\mathbb{F}$ are two fields and $\mho : \mathbb{K} \rightarrow \mathbb{F}$ is a homomorphism from $\mathbb{K}$ to $\mathbb{F}$. A CPFS $\mathbb{N}$ is a **CFSF** of $\mathbb{F}$. Then $\mho^{-1}(\mathbb{N})$ is **CPFSF** of $\mathbb{K}$.

Proof: Consider that, $\overline{\mathbb{N}} = \{<\hbar, \mu_{\mathbb{N}}(\hbar), \hat{\mu}_{\mathbb{N}}(\hbar) >: \hbar \in \mathbb{K}, 0 < (\mu_{\mathbb{N}}(\hbar))^2 + (\hat{\mu}_{\mathbb{N}}(\hbar))^2 \leq 1\}$ is a **PFSF** and $\underline{\mathbb{N}} = \{<\hbar, \zeta_{\mathbb{N}}(\hbar), \hat{\zeta}_{\mathbb{N}}(\hbar) >: \hbar \in \mathbb{K}, 0 < (\zeta_{\mathbb{N}}(\hbar))^2 + (\hat{\zeta}_{\mathbb{N}}(\hbar))^2 \leq 2\pi\}$ is π-**PFSF**. Then, in the view of Theorem 3 and in the view of Theorem 7 the inverse image of $\overline{\mathbb{N}} = \{<\hbar, \mu_{\mathbb{N}}(\hbar), \hat{\mu}_{\mathbb{N}}(\hbar) >: \hbar \in \mathbb{K}, \mu_{\mathbb{N}}(\hbar)$ and $\hat{\mu}_{\mathbb{N}}(\hbar) \in [0, 1]\}$ is a **PFSF** and $\underline{\mathbb{N}} = \{<\hbar, \zeta_{\mathbb{N}}(\hbar), \hat{\zeta}_{\mathbb{N}}(\hbar) >: \hbar \in \mathbb{K}, \zeta_{\mathbb{N}}(\hbar)$ and $\hat{\zeta}_{\mathbb{N}}(\hbar) \in [0, 2\pi]\}$ is π-**PFSF**, $\forall\, \hbar, \hbar \in \mathbb{K}$.

Then,

$$\begin{aligned}
(\mho^{-1}(\mu_{\mathbb{N}})(\hbar - \hbar))^2 &\geq \min\{(\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2\}\\
(\mho^{-1}(\mu_{\mathbb{N}})(\hbar\hbar))^2 &\geq \min\{(\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2\}\\
(\mho^{-1}(\mu_{\mathbb{N}})(\hbar^{-1}))^2 &\geq (\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2\\
(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar - \hbar))^2 &\leq \max\{(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar))^2\}\\
(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar\hbar))^2 &\leq \max\{(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar))^2\}\\
(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar^{-1}))^2 &\leq (\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar))^2\\
(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar - \hbar))^2 &\geq \min\{(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\zeta_{\mathbb{N}})(\hbar))^2\}\\
(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar\hbar))^2 &\geq \min\{(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\zeta_{\mathbb{N}})(\hbar))^2\}\\
(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar^{-1}))^2 &\geq (\mho^{-1}(\zeta_{\mathbb{N}})(\hbar))^2\\
(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar - \tilde{\mathfrak{b}}))^2 &\leq \max\{(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar))^2\}\\
(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar\hbar))^2 &\leq \max\{(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar))^2\}\\
(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar^{-1}))^2 &\leq (\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar))^2
\end{aligned}$$

From Lemma 1[3], since

$$\begin{aligned}
&(\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar - \hbar))^2\\
&= (\mho^{-1}(\mu_{\mathbb{N}})(\hbar - \hbar))^2 e^{i(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar-\hbar))^2},\\
&\qquad \forall\, \hbar, \hbar \in \mathbb{K}\\
&\geq \min\{(\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2\}\\
&e^{i\min\{(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar))^2,(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar))^2\}}\\
&\geq \min\{(\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2 e^{i(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar)},\\
&(\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2 e^{i(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar))^2}\}\\
&= \min\{(\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar))^2\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar - \hbar))^2\\
&\geq \min\{(\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar))^2\}\\
&(\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar\hbar))^2\\
&= (\mho^{-1}(\mu_{\mathbb{N}})(\hbar\hbar))^2 e^{i(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar\hbar))^2},\\
&\qquad \forall\, \hbar, \hbar \in \mathbb{K}\\
&\geq \min\{(\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2\}\\
&e^{i\min\{(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar))^2,(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar))^2\}}\\
&\geq \min\{(\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2 e^{i(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar))^2},\\
&(\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2 e^{i(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar))^2}\}\\
&= \min\{(\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar))^2\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&(\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar\hbar))^2\\
&\geq \min\{(\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar))^2, (\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar))^2\}.
\end{aligned}$$

Further,

$$\begin{aligned}
&(\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar^{-1}))^2\\
&= (\mho^{-1}(\mu_{\mathbb{N}})(\hbar^{-1}))^2 e^{i(\mho^{-1}(\zeta_{\mathbb{N}})(\hbar^{-1}))^2},\\
&\quad \forall\ \hbar \in \mathbb{K}\\
&\geq (\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2\ e^{i(\mho^{-1}(\mu_{\mathbb{N}})(\hbar))^2}\\
&\geq\ (\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar))^2.
\end{aligned}$$

As a result, we get

$$(\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar^{-1}))^2 \geq (\mho^{-1}(\Upsilon_{\mathbb{N}})(\hbar))^2.\$$$

By using the Lemma 1[4], we have,

$$(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar-\hbar))^2$$
$$=(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar-\hbar))^2 e^{i(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar-\hbar))^2}, \quad \forall\, \hbar,\hbar\in\mathbb{K}$$
$$\leq \max\{(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar))^2,(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar))^2\} e^{i\max\{(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar))^2,(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar))^2\}}$$
$$\leq \max\{(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar))^2 e^{i(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar))^2}, (\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar))^2 e^{i(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar)^2}\}$$
$$=\max\{(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar))^2,(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar))^2\}.$$

Consequently,

$$(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar-\hbar))^2$$
$$\leq \max\{(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar))^2,(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar))^2\}$$
$$(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar\hbar)$$
$$=(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar\hbar)e^{i(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar\hbar)}, \forall\, \hbar,\hbar\in\mathbb{K}$$
$$\leq \max\{(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar),(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar)\} e^{i\max\{(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar),(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar)\}}$$
$$\leq \max\{(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar)e^{i(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar)},(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar)e^{i(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar)}\}$$
$$=\max\{(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar),(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar)\}.$$

Therefore, $(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar\hbar)\leq \max\{(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar),(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar)\}$. Further,

$$(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar^{-1}))^2=(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar^{-1}))^2 e^{i(\mho^{-1}(\hat{\zeta}_{\mathbb{N}})(\hbar^{-1}))^2}, \quad \forall\ \hbar\in\mathbb{K}$$
$$\leq (\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar))^2\ e^{i(\mho^{-1}(\hat{\mu}_{\mathbb{N}})(\hbar))^2}$$
$$\leq\ (\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar))^2$$
$$(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar^{-1}))^2\leq(\mho^{-1}(\hat{\Upsilon}_{\mathbb{N}})(\hbar))^2.$$

The proof ends at this instance.

V. CHARACTERISTICS OF THE DIRECT PRODUCT OF COMPLEX PYTHAGOREAN FUZZY SUBFIELDS

The innovative **CPFSFs** direct product framework is presented in this section. To define the direct product of **CPFS**, we utilize the idea of **CPFS**. We demonstrate that the direct product of two **CPFSFs** is a **CPFSF** and examine its characteristics.

Definition 12: Suppose that $\mathbb{S}$ and $\mathbb{N}$ are two π-**CPFSs** of sets $\mathbb{K}_1$ and $\mathbb{K}_2$, respectively. The Cartesian product of π-**CPFS** $\mathbb{S}$ and $\mathbb{N}$ is defined as $(\mathbb{S}_\pi\times\mathbb{N}_\pi)(\mathrm{j},\hbar)=\{<(\mathrm{j},\hbar),\phi_{\mathbb{S}_\pi\times\mathbb{N}_\pi}(\mathrm{j},\hbar),\hat{\phi}_{\mathbb{S}_\pi\times\mathbb{N}_\pi}(\mathrm{j},\hbar)>0<(\phi_{\mathbb{S}_\pi\times\mathbb{N}_\pi}(\mathrm{j},\hbar))^2+(\hat{\phi}_{\mathbb{S}_\pi\times\mathbb{N}_\pi}(\mathrm{j},\hbar))^2\leq 1\},\forall\,\mathrm{j}\in\mathbb{K}_1,\ \hbar\in\mathbb{K}_2$.

Remark 2: Suppose that $\mathbb{S}$ and $\mathbb{N}$ are two π-**PFSRs** of $\mathbb{K}_1$ and $\mathbb{K}_2$, respectively. Then $\mathbb{S}_\pi\times\mathbb{N}_\pi$ is π-**PFSR** of $\mathbb{K}_1\times\mathbb{K}_2$.

Remark 3: A π-**PFSR** $\mathbb{S}_\pi\times\mathbb{N}_\pi$ of field $\mathbb{K}_1\times\mathbb{K}_2$ is a π-**FSR** of $\mathbb{K}_1\times\mathbb{K}_2$ if and only if $\mathbb{S}\times\mathbb{N}$ is **PFSG** of $\mathbb{K}_1\times\mathbb{K}_2$

Definition 13: Suppose that $\mathbb{S}$ and $\mathbb{N}$ are two **CPFSs**. The Cartesian product of given two sets defined by a function

$$\mathbb{S}\times\mathbb{N}$$
$$=\{<\mathrm{j},\hbar,\Upsilon_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar),\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar)>\},$$
$$(\Upsilon_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2$$
$$=(\varrho_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2 e^{i(\varphi_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2}$$
$$=\min\{(\varrho_{\mathbb{S}}(\intercal))^2,(\mu_{\mathbb{N}}(\hbar))^2\}e^{i\min\{(\zeta_{\mathbb{S}}(\intercal))^2,(\zeta_{\mathbb{N}}(\hbar))^2\}},$$
$$(\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2$$
$$=(\hat{\mu}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2 e^{i(\hat{\zeta}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2}$$
$$=\max\{(\hat{\mu}_{\mathbb{S}}(\intercal))^2,(\hat{\mu}_{\mathbb{N}}(\hbar))^2\}e^{i\max\{(\hat{\zeta}_{\mathbb{S}}(\intercal))^2,(\hat{\zeta}_{\mathbb{N}}(\hbar))^2\}}.$$

For the purpose of simplicity, we will use citations throughout this article.
$(\Upsilon_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2=(\mu_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2 e^{i(\zeta_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar)^2}$ and $(\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2=(\hat{\mu}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2 e^{i(\hat{\zeta}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2}$ for the **MF** and **NMF**, respectively, of cartesian product of **CPFS** $\mathbb{S}\times\mathbb{N}$.

According to the following theorem, two **CPFSFs** have a cartesian product of **CPFSF**.

Theorem 10: Let $\mathbb{S}$ and $\mathbb{N}$ be two **CPFSFs** of $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Then $\mathbb{S}\times\mathbb{N}$ is **CPFSF** of $\mathbb{H}_1\times\mathbb{H}_2$.

Proof: Let $\intercal,\hbar\in\mathbb{H}_1$ and $\hbar,\tilde{\mathrm{b}}\in\mathbb{H}_2$ be an elements. Then $(\intercal,\hbar),\ (\tilde{\mathrm{a}},\tilde{\mathrm{b}})\in\mathbb{H}_1\times\mathbb{H}_2$.
Consider

$$(\Upsilon_{\mathbb{S}\times\mathbb{N}}((\intercal,\hbar)-(\tilde{\mathrm{a}},\tilde{\mathrm{b}}))^2$$
$$=(\Upsilon_{\mathbb{S}\times\mathbb{N}}(\intercal-\tilde{\mathrm{a}},\hbar-\tilde{\mathrm{b}}))^2$$
$$=(\mu_{\mathbb{S}\times\mathbb{N}}(\intercal-\tilde{\mathrm{a}},\hbar-\tilde{\mathrm{b}}))^2 e^{i(\zeta_{\mathbb{S}\times\mathbb{N}}(\intercal-\tilde{\mathrm{a}},\hbar-\tilde{\mathrm{b}}))^2}$$
$$=\min\{(\mu_{\mathbb{S}}(\intercal-\hbar))^2,(\mu_{\mathbb{N}}(\hbar-\tilde{\mathrm{b}}))^2\} e^{i(\min\{(\zeta_{\mathbb{S}}(\intercal-\hbar))^2,(\zeta_{\mathbb{N}}(\hbar-\tilde{\mathrm{b}}))^2\}}$$
$$=\min\{(\mu_{\mathbb{S}}(\intercal-\hbar))^2 e^{i(\zeta_{\mathbb{S}}(\intercal-\hbar))^2},(\mu_{\mathbb{N}}(\hbar-\tilde{\mathrm{b}}))^2 e^{i(\zeta_{\mathbb{S}}(\hbar-\tilde{\mathrm{b}}))^2})\}$$
$$=\min\{(\Upsilon_{\mathbb{S}}(\intercal-\hbar))^2,(\Upsilon_{\mathbb{N}}(\intercal-\hbar))^2\}$$
$$\geq\min\{\min\{(\Upsilon_{\mathbb{S}}(\intercal))^2,(\Upsilon_{\mathbb{S}}(\intercal))^2\},\ \min\{(\Upsilon_{\mathbb{N}}(\hbar))^2,\Upsilon_{\mathbb{N}}(\tilde{\mathrm{b}}))^2\}\}$$
$$=\min\{\min\{((\Upsilon_{\mathbb{S}}(\intercal))^2,(\Upsilon_{\mathbb{N}}(\hbar))^2\},\ \min\{(\Upsilon_{\mathbb{S}}(\intercal))^2,(\Upsilon_{\mathbb{N}}(\tilde{\mathrm{b}}))^2\}\}$$
$$\geq\min\{(\Upsilon_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2,(\Upsilon_{\mathbb{S}\times\mathbb{N}}(\tilde{\mathrm{a}},\tilde{\mathrm{b}}))^2\}$$
$$(\Upsilon_{\mathbb{S}\times\mathbb{N}}((\intercal,\hbar)-(\tilde{\mathrm{a}},\tilde{\mathrm{b}})))^2$$
$$\geq\min\{(\Upsilon_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2,(\Upsilon_{\mathbb{S}\times\mathbb{N}}(\tilde{\mathrm{a}},\tilde{\mathrm{b}}))^2\}.$$
$$(\Upsilon_{\mathbb{S}\times\mathbb{N}}((\intercal,\hbar)(\tilde{\mathrm{a}},\tilde{\mathrm{b}})))^2$$
$$=(\Upsilon_{A\times\mathbb{N}}(\mathrm{j}\tilde{a},(\intercal))\tilde{\mathrm{b}}))^2$$
$$=(\mu_{\mathbb{S}\times\mathbb{N}}(\mathrm{j}\tilde{a},(\intercal))\tilde{b}))^2 e^{i(\zeta_{\mathbb{S}\times\mathbb{N}}(\mathrm{j}\tilde{a},(\intercal))\tilde{b}))^2}$$
$$=\min\{\mu_{\mathbb{S}}(\mathrm{j}\hbar)^2,\mu_{\mathbb{N}}(a\tilde{b})^2\}e^{i\ \min\{(\zeta_{\mathbb{S}}(\mathrm{j}\hbar))^2,(\zeta_{\mathbb{N}}a\tilde{b})^2\}}$$
$$=\min\{(\mu_{\mathbb{S}}(\mathrm{j},\hbar))^2 e^{i(\zeta_{\mathbb{S}}(\intercal\hbar))^2},(\mu_{\mathbb{N}}(a\tilde{b}))^2 e^{i(\zeta_{\mathbb{S}}(\tilde{a}\tilde{b}))^2}\}$$
$$=\min\{(\Upsilon_{\mathbb{S}}(\intercal\hbar))^2,(\Upsilon_{\mathbb{N}}(a\tilde{b}))^2\}$$

$$\geq \min\{\min\{(\Upsilon_{\mathbb{S}}(\intercal))^2, (\Upsilon_{\mathbb{S}}(\intercal))^2\}, \min\{(\Upsilon_{\mathbb{N}}(\tilde{a})^2, (\Upsilon_{\mathbb{N}}(\tilde{b}))^2\}\}$$
$$= \min\{\min\{(\Upsilon_{\mathbb{S}}(\intercal))^2, (\Upsilon_{\mathbb{N}}(\hbar))^2\}, \min\{(\Upsilon_{\mathbb{S}}(\tilde{a}))^2, (\Upsilon_{\mathbb{N}}(\tilde{b}))^2\}\}$$
$$\geq \min\{(\Upsilon_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2, (\Upsilon_{\mathbb{S}\times\mathbb{N}}(\tilde{a},\tilde{b}))^2\}$$
$$(\Upsilon_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar)(\tilde{a},\tilde{b}))^2$$
$$\geq \min\{(\Upsilon_{\mathbb{S}\times\mathbb{N}}(\intercal),\hbar)^2, (\Upsilon_{\mathbb{S}\times\mathbb{N}}(\tilde{a},\tilde{b}))^2\}.$$

Further,

$$(\Upsilon_{\mathbb{S}\times\mathbb{N}}(\intercal^{-1},\hbar^{-1}))^2$$
$$= (\mu_{\mathbb{S}\times\mathbb{N}}(\intercal^{-1},\hbar^{-1}))^2 \mathfrak{e}^{i(\zeta_{\mathbb{S}\times\mathbb{N}}(\intercal^{-1},\hbar^{-1}))^2}$$
$$= \min\{(\mu_{\mathbb{S}}(\intercal^{-1}))^2, (\mu_{\mathbb{N}}(\hbar^{-1}))^2\}\, \mathfrak{e}^{i\min\{(\zeta_{\mathbb{S}}(\intercal^{-1}))^2,(\zeta_{\mathbb{N}}(\hbar^{-1})^2)\}}$$
$$= \min\{(\mu_{\mathbb{S}}(\intercal^{-1}))^2 \mathfrak{e}^{i(\zeta_{\mathbb{S}}(\intercal^{-1}))^2}, (\mu_{\mathbb{N}}(\hbar^{-1}))^2 \mathfrak{e}^{i(\zeta_{\mathbb{S}}(y^{-1}))^2}\}$$
$$= \min\{(\Upsilon_{\mathbb{S}}(\intercal^{-1}))^2, (\Upsilon_{\mathbb{N}}(\hbar^{-1}))^2\}$$
$$\geq \min\{(\Upsilon_{\mathbb{S}}(\intercal)^2), (\Upsilon_{\mathbb{N}}(\hbar))^2\}.$$

Consequently,

$$(\Upsilon_{\mathbb{S}\times\mathbb{N}}(\intercal^{-1},\hbar^{-1}))^2 \geq (\Upsilon_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2.$$

Assume that,

$$(\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}((\intercal,\hbar)-(\intercal,\tilde{b})))^2$$
$$= (\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal-\intercal,\hbar-\tilde{b}))^2$$
$$= (\hat{\mu}_{\mathbb{S}\times\mathbb{N}}(\intercal-\intercal,\hbar-\tilde{b}))^2 \mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}\times\mathbb{N}}(\intercal-\intercal,\hbar-\tilde{b}))^2}$$
$$= \max\{(\hat{\mu}_{\mathbb{S}}(\intercal-\hbar))^2, (\hat{\mu}_{\mathbb{N}}(\hbar-\tilde{b}))^2\}\, \mathfrak{e}^{i\max\{(\hat{\zeta}_{\mathbb{S}}(\intercal-\hbar))^2,(\hat{\zeta}_{\mathbb{N}}(y-b))^2\}}$$
$$= \max\{(\hat{\mu}_{\mathbb{S}}(\intercal-\hbar))^2 \mathfrak{e}^{(i\hat{\zeta}_{\mathbb{S}}(\intercal-\hbar))^2}, (\hat{\mu}_{\mathbb{N}}(\hbar-\tilde{b}))^2 \mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}}(\hbar-\tilde{b}))^2}\}$$
$$= \max\{(\hat{\Upsilon}_{\mathbb{S}}(\intercal-\hbar))^2, (\Upsilon_{\mathbb{N}}(\hbar-\tilde{b}))^2\}$$
$$\leq \max\{\max\{(\hat{\Upsilon}_{\mathbb{S}}(\intercal))^2, (\hat{\Upsilon}_{\mathbb{S}}(\intercal))^2\}, \max\{(\hat{\Upsilon}_{\mathbb{N}}(\hbar))^2, (\hat{\Upsilon}_{\mathbb{N}}(\tilde{b}))^2\}\}$$
$$= \max\{\max\{(\hat{\Upsilon}_{\mathbb{S}}(\intercal))^2, (\hat{\Upsilon}_{\mathbb{N}}(\hbar))^2\}, \max\{(\hat{\Upsilon}_{\mathbb{S}}(\intercal)^2), (\Upsilon_{\mathbb{N}}(\tilde{b}))^2\}\}$$
$$\leq \max\{(\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2, (\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal,\tilde{b}))^2\}$$
$$(\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}((\intercal,\hbar)-(\intercal,\tilde{b})))^2$$
$$\leq \max\{(\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2, (\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal,\tilde{b}))^2\}.$$

Assume that,

$$(\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}((\intercal,\hbar)(\tilde{a},\tilde{b})))^2$$
$$= (\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal\tilde{a},(\intercal))\tilde{b}))^2$$
$$= (\hat{\mu}_{\mathbb{S}\times\mathbb{N}}(\intercal\tilde{a},(\intercal))\tilde{b}))^2 \mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}\times\mathbb{N}}(\intercal\tilde{a},(\intercal))\tilde{b}))^2}$$
$$= \max\{(\hat{\mu}_{\mathbb{S}}(\intercal\hbar))^2, (\hat{\mu}_{\mathbb{N}}(a\tilde{b}))^2\}\, \mathfrak{e}^{i\max\{(\hat{\zeta}_{\mathbb{S}}(\intercal\hbar))^2,(\hat{\zeta}_{\mathbb{N}}(\tilde{a}\tilde{b}))^2\}}$$
$$= \max\{(\hat{\mu}_{\mathbb{S}}(\intercal\hbar))^2 \mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}}(\intercal\hbar))^2}, (\hat{\mu}_{\mathbb{N}}(\tilde{a}\tilde{b}))^2 \mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}}(\tilde{a}\tilde{b}))^2}\}$$
$$= \max\{(\hat{\Upsilon}_{\mathbb{S}}(\intercal\hbar))^2, (\Upsilon_{\mathbb{N}}(a\tilde{b}))^2\}$$
$$\leq \max\{\max\{(\hat{\Upsilon}_{\mathbb{S}}(\intercal))^2, (\hat{\Upsilon}_{\mathbb{S}}((\intercal))))^2\}, \max\{(\hat{\Upsilon}_{\mathbb{N}}(\tilde{a}))^2, (\hat{\Upsilon}_{\mathbb{N}}(\tilde{b}))^2\}\}$$
$$= \max\{\max\{(\hat{\Upsilon}_{\mathbb{S}}(\intercal))^2, (\hat{\Upsilon}_{\mathbb{N}}(\hbar))^2\}, \max\{(\hat{\Upsilon}_{\mathbb{S}}(\tilde{a}))^2, (\Upsilon_{\mathbb{N}}(\tilde{b}))^2\}\}$$
$$\leq \max\{(\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2, (\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\tilde{a},\tilde{b}))^2\}$$
$$(\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}((\intercal,\hbar)(\tilde{a},\tilde{b})))^2$$
$$\leq \max\{(\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2, (\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\tilde{a},\tilde{b}))^2\}.$$

Further,

$$(\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal^{-1},\hbar^{-1}))^2$$
$$= (\hat{\mu}_{\mathbb{S}\times\mathbb{N}}(\intercal^{-1},\hbar^{-1}))^2 \mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}\times\mathbb{N}}(\intercal^{-1},\hbar^{-1}))^2}$$
$$= \max\{(\mu_{\mathbb{S}}(\intercal^{-1}))^2, (\hat{\mu}_{\mathbb{N}}(\hbar^{-1}))^2\}\, \mathfrak{e}^{i\max\{(\hat{\zeta}_{\mathbb{S}}(\intercal^{-1}))^2,(\hat{\zeta}_{\mathbb{N}}(\hbar^{-1}))^2\}}$$
$$= \max\{(\hat{\mu}_{\mathbb{S}}(\intercal^{-1}))^2 \mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}}(\intercal^{-1}))^2}, (\mu_{\mathbb{N}}(\hbar^{-1}))^2 \mathfrak{e}^{i(\hat{\zeta}_{\mathbb{S}}(\hbar^{-1}))^2}\}$$
$$= \max\{(\hat{\Upsilon}_{\mathbb{S}}(\intercal^{-1}))^2, (\hat{\Upsilon}_{\mathbb{N}}(\hbar^{-1}))^2\}$$
$$\leq \max\{(\hat{\Upsilon}_{\mathbb{S}}(\intercal))^2, (\hat{\Upsilon}_{\mathbb{N}}(\hbar))^2\}.$$

Consequently,

$$(\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal^{-1},\hbar^{-1}))^2 \leq (\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2.$$

Consequently, the demonstration is complete.

Corollary 2: . Let $\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_{\hbar}$ be **CPFSFs** of $\mathbb{K}_1, \mathbb{K}_2, \ldots, \mathbb{K}_{\hbar}$, respectively. Then, $\mathbb{S}_1 \times \mathbb{S}_2 \times, \ldots, \times \mathbb{S}_{\hbar}$ is **CPFSF** of $\mathbb{K}_1 \times \mathbb{K}_2 \times \ldots \times \mathbb{K}_{\hbar}$.

Remark 4: Suppose that $\mathbb{S}$ and $\mathbb{N}$ are two **CPFSs** of $\mathbb{K}_1$ and $\mathbb{K}_2$, respectively and $\mathbb{S}_1 \times \mathbb{S}_2$ are **CPFSF** of $\mathbb{K}_1 \times \mathbb{K}_2$. Then it is not compulsory both $\mathbb{S}_1$ and $\mathbb{S}_2$ should be **CPFSF** of $\mathbb{K}_1$ and $\mathbb{K}_2$, respectively.

Remark 5: Suppose that $\mathbb{S} \times \mathbb{N}$ are two **CPFSF** of field $\mathbb{K}_1 \times \mathbb{K}_2$. Let $\mathfrak{e} \in \mathbb{K}_1$ and $\mathfrak{e}' \in \mathbb{K}_2$. Then,

$$(\mu_{\mathbb{S}\times\mathbb{N}}(\mathfrak{e},\mathfrak{e}'))^2 \geq (\mu_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2,$$
$$(\zeta_{\mathbb{S}\times\mathbb{N}}(\mathfrak{e},\mathfrak{e}'))^2 \geq (\zeta_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2,$$
$$(\hat{\mu}_{\mathbb{S}\times\mathbb{N}}(\mathfrak{e},\mathfrak{e}'))^2 \leq (\hat{\mu}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2$$
$$(\hat{\zeta}_{\mathbb{S}\times\mathbb{N}}(\mathfrak{e},\mathfrak{e}'))^2 \leq (\hat{\zeta}_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2,\ \forall\, \intercal \in \mathbb{K}_1,\ \hbar \in \mathbb{K}_2.$$

Theorem 11: Suppose that $\mathbb{S}$ and $\mathbb{N}$ are two **CFS** of fields $\mathbb{K}_1$ and $\mathbb{K}_2$. If $\mathbb{S} \times \mathbb{N}$ is a **CPFSF**of $\mathbb{K}_1 \times \mathbb{K}_2$, the following statements must all be true, at least one of them and let $\mathfrak{e} \in \mathbb{K}_1$ and $\mathfrak{e}' \in \mathbb{K}_2$. Then,

1) $(\mu_{\mathbb{S}}(\mathfrak{e}))^2 \leq (\mu_{\mathbb{N}}(\hbar))^2$, $(\zeta_{\mathbb{S}}(\mathfrak{e}))^2 \leq (\zeta_{\mathbb{N}}(\hbar))^2$ and $(\hat{\mu}_{\mathbb{S}}(\mathfrak{e}))^2 \geq (\hat{\mu}_{\mathbb{N}}(\hbar))^2$, $(\hat{\zeta}_{\mathbb{S}}(\mathfrak{e}))^2 \geq (\hat{\zeta}_{\mathbb{N}}(\hbar))^2$, $\forall\, \hbar \in \mathbb{K}_2$
2) $(\mu_{\mathbb{N}}(\mathfrak{e}'))^2 \leq (\mu_{\mathbb{S}}(\intercal))^2$, $(\zeta_{\mathbb{N}}(\mathfrak{e}'))^2 \leq (\zeta_{\mathbb{S}}(\intercal))^2$, $(\hat{\mu}_{\mathbb{N}}(\mathfrak{e}'))^2 \geq (\mu_{\mathbb{S}}(\intercal))^2$, $(\hat{\zeta}_{\mathbb{N}}(\mathfrak{e}'))^2 \geq (\hat{\zeta}_{\mathbb{S}}(\intercal))^2$, $\forall\, \intercal \in \mathbb{K}_1$

Proof: Let $\mathbb{S} \times \mathbb{N}$ be a **CPFSF** of $\mathbb{K}_1 \times \mathbb{K}_2$. On the other hand, suppose that the propositions [1] and [2] are wrong. Then, $\forall\, \intercal \in \mathbb{K}_1$ and $\forall\, \hbar \in \mathbb{K}_2$ such that

1) $(\mu_{\mathbb{S}}(\mathfrak{e}))^2 \leq (\mu_{\mathbb{N}}(\hbar))^2$, $(\zeta_{\mathbb{S}}(\mathfrak{e}))^2 \leq (\zeta_{\mathbb{N}}(\hbar))^2$ and $(\hat{\mu}_{\mathbb{S}}(\mathfrak{e}))^2 \geq (\hat{\mu}_{\mathbb{N}}(\hbar)^2)$,$(\hat{\zeta}_{\mathbb{S}}(\mathfrak{e}))^2 \geq (\hat{\zeta}_{\mathbb{N}}\hbar)^2\ \forall\, \hbar \in \mathbb{K}_2$
2) $(\mu_{\mathbb{N}}(\mathfrak{e}'))^2 \leq (\mu_{\mathbb{S}}(\intercal), \zeta_{\mathbb{N}}(\mathfrak{e}'))^2 \leq (\zeta_{\mathbb{S}}(\intercal))^2$ and $(\hat{\mu}_{\mathbb{N}}(\mathfrak{e}'))^2 \geq (\mu_{\mathbb{S}}(\intercal))^2$,$(\hat{\zeta}_{\mathbb{N}}(\mathfrak{e}'))^2 \geq (\hat{\zeta}_{\mathbb{S}}(\intercal))^2\ \forall\, \intercal \in \mathbb{K}_1$.

Consider,

$$(\Upsilon_{\mathbb{S}\times\mathbb{N}}(\intercal,\hbar))^2$$

$$= \min\{(\mu_{\mathbb{S}}(\top))^2, (\mu_{\mathbb{N}}(\hbar))^2\} e^{i\min\{(\zeta_{\mathbb{S}}(\top))^2,(\zeta_{\mathbb{N}}\hbar)^2\}}$$
$$\geq \min\{(\mu_{\mathbb{S}}(\mathfrak{e}))^2, (\mu_{\mathbb{N}}(\mathfrak{e}'))^2\} e^{i\min\{(\zeta_{\mathbb{S}}(\mathfrak{e}))^2,(\zeta_{\mathbb{N}}(\mathfrak{e}'))^2\}}$$
$$= (\Upsilon_{\mathbb{S}\times\mathbb{N}}(\mathfrak{e}, \mathfrak{e}'))^2.$$

and

$$(\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\top, \hbar))^2$$
$$= \max\{(\hat{\mu}_{\mathbb{S}}(\top))^2, (\hat{\mu}_{\mathbb{N}}(\hbar))^2\} e^{i\max\{(\hat{\zeta}_{\mathbb{S}}(\top))^2,(\hat{\zeta}_{\mathbb{N}}(\hbar))^2\}}$$
$$\leq \max\{(\hat{\mu}_{\mathbb{S}}(\mathfrak{e}))^2, (\hat{\mu}_{\mathbb{N}}(\mathfrak{e}'))^2\} e^{i\max\{(\hat{\zeta}_{\mathbb{S}}(\mathfrak{e}))^2,(\hat{\zeta}_{\mathbb{N}}(\mathfrak{e}'))^2\}}$$
$$= (\hat{\Upsilon}_{\mathbb{S}\times\mathbb{N}}(\mathfrak{e}, \mathfrak{e}'))^2.$$

$\because \mathbb{S} \times \mathbb{N}$ is **CPFSF**.

As a result, it stands to reason that at most one of the below assertions must be true.

1) $(\mu_{\mathbb{S}}(\mathfrak{e}))^2 \leq (\mu_{\mathbb{N}}(\hbar))^2$, $(\zeta_{\mathbb{S}}(\mathfrak{e}))^2 \leq (\zeta_{\mathbb{N}}(\hbar))^2$ and $(\hat{\mu}_{\mathbb{S}}(\mathfrak{e}))^2 \geq (\hat{\mu}_{\mathbb{N}}(\hbar))^2$, $(\hat{\zeta}_{\mathbb{S}}(\mathfrak{e}))^2 \geq (\hat{\zeta}_{\mathbb{N}}(\hbar))^2$, $\forall\, \hbar \in \mathbb{K}_2$
2) $(\mu_{\mathbb{N}}(\mathfrak{e}'))^2 \leq (\mu_{\mathbb{S}}(\top))^2$, $(\zeta_{\mathbb{N}}(\mathfrak{e}'))^2 \leq (\zeta_{\mathbb{S}}(\top)^2$ and $(\hat{\mu}_{\mathbb{N}}(\mathfrak{e}'))^2 \geq (\mu_{\mathbb{S}}(\top))^2$, $(\hat{\zeta}_{\mathbb{N}}(\mathfrak{e}'))^2 \geq (\hat{\zeta}_{\mathbb{S}}(\top))^2$, $\forall\, \top \in \mathbb{K}_1$

Theorem 12: Suppose that $\mathbb{S}$ and $\mathbb{N}$ **CPFSs** of $\mathbb{K}_1$ and $\mathbb{K}_2$ and $(\mu_{\mathbb{N}}(\mathfrak{e}'))^2 \geq (\mu_{\mathbb{S}}(\top))^2$, $(\zeta_{\mathbb{N}}(\mathfrak{e}'))^2 \geq (\zeta_{\mathbb{S}}(\top))^2$, $(\hat{\mu}_{\mathbb{N}}(\mathfrak{e}'))^2 \leq (\hat{\mu}_{\mathbb{S}}(\top)$, $\hat{\zeta}_{\mathbb{N}}(\mathfrak{e}'))^2 \leq (\hat{\zeta}_{\mathbb{S}}(\top))^2$ $\forall\, \top \in \mathbb{K}_1$, $\mathfrak{e}'$ is identity of $\mathbb{K}_2$. If $\mathbb{S} \times \mathbb{N}$ is **CPFSF** of $\mathbb{K}_1 \times \mathbb{K}_2$, then $\mathbb{S}$ is **CPFSF** of $\mathbb{K}_1$.

Proof: Suppose that $(\top, \mathfrak{e}'), (\hbar, \mathfrak{e}')$ are elements of $\mathbb{K}_1 \times \mathbb{K}_2$. By given condition $\mu_{\mathbb{N}}(\mathfrak{e}') \geq \mu_{\mathbb{S}}(\top)$ and $\zeta_{\mathbb{N}}(\mathfrak{e}') \geq \zeta_{\mathbb{S}}(\top)$, for all $\top, \hbar \in \mathbb{K}_1$ and $\mathfrak{e}' \in \mathbb{K}_2$.

Consider,

$$(\Upsilon_{\mathbb{S}}(\top - \hbar))^2$$
$$= (\mu_{\mathbb{S}}(\top - \hbar))^2 e^{i(\zeta_{\mathbb{S}}(\top-\hbar))^2}$$
$$= \min\{(\mu_{\mathbb{S}}(\top - \hbar))^2 e^{i(\zeta_{\mathbb{S}}(\top-\hbar))^2}, (\mu_{\mathbb{N}}(\mathfrak{e}' - \mathfrak{e}'))^2 e^{i(\zeta_{\mathbb{N}}(\mathfrak{e}'-\mathfrak{e}'))^2}\}$$
$$= (\mu_{\mathbb{S}\times\mathbb{N}}((\top, \mathfrak{e}')(\top, \mathfrak{e}')))^2 e^{i(\zeta_{\mathbb{S}\times\mathbb{N}}((\top, \mathfrak{e}')(\top,\mathfrak{e}')))^2}$$
$$\geq \min\{(\mu_{\mathbb{S}\times\mathbb{N}}(\top, \mathfrak{e}'), \mu_{\mathbb{S}\times\mathbb{N}}(\top, \mathfrak{e}'))^2\} e^{i\min\{(\zeta_{\mathbb{S}\times\mathbb{N}}(\top, \mathfrak{e}'))^2, (\zeta_{\mathbb{S}\times\mathbb{N}}(\top, \mathfrak{e}'))^2\}}$$
$$= \min\{\min\{(\mu_{\mathbb{S}}(\top), \mu_{\mathbb{N}}(\mathfrak{e}'))^2\}, \min\{(\mu_{\mathbb{S}}(\hbar), \mu_{\mathbb{N}}(\mathfrak{e}'))^2\}\} e^{i\min\{\min\{(\zeta_{\mathbb{N}}(\mathfrak{e}'))^2\}, \min\{\zeta, \min\{(\zeta_{\mathbb{S}}(\hbar), \zeta_{\mathbb{N}}(\mathfrak{e}'))^2\}\}}$$
$$= \min\{(\Upsilon_{\mathbb{S}}(\top), \Upsilon_{\mathbb{S}}(\hbar))^2\}$$

Thus, $(\Upsilon_{\mathbb{S}}(\top - \hbar))^2 \geq \min\{(\Upsilon_{\mathbb{S}}(\top), \Upsilon_{\mathbb{S}}(\hbar))^2\}$

$$(\Upsilon_{\mathbb{S}}(\top\hbar))^2$$
$$= (\mu_{\mathbb{S}}(\top\hbar))^2 e^{i(\zeta_{\mathbb{S}}(\top\hbar))^2}$$
$$= \min\{(\mu_{\mathbb{S}}(\top\hbar))^2 e^{i(\zeta_{\mathbb{S}}(\top\hbar))^2}, (\mu_{\mathbb{N}}(\mathfrak{e}'\mathfrak{e}'))^2 e^{i(\zeta_{\mathbb{N}}(\mathfrak{e}'\mathfrak{e}'))^2}\}$$
$$= (\mu_{\mathbb{S}\times\mathbb{N}}((\top, \mathfrak{e}')(\top, \mathfrak{e}')))^2 e^{i(\zeta_{\mathbb{S}\times\mathbb{N}}((\top,\mathfrak{e}')(\top,\mathfrak{e}')))^2}$$
$$\geq \min\{(\mu_{\mathbb{S}\times\mathbb{N}}(\top, \mathfrak{e}'))^2, (\mu_{\mathbb{S}\times\mathbb{N}}(\top, \mathfrak{e}'))^2\} e^{i\min\{(\zeta_{\mathbb{S}\times\mathbb{N}}(\top,\mathfrak{e}'))^2,(\zeta_{\mathbb{S}\times\mathbb{N}}(\top,\mathfrak{e}'))^2\}}$$
$$= \min\{\min\{(\mu_{\mathbb{S}}(\top), \mu_{\mathbb{N}}(\mathfrak{e}'))^2\}, \min\{(\mu_{\mathbb{S}}(\hbar), \mu_{\mathbb{N}}(\mathfrak{e}'))^2\}\} e^{i\min\{\min\{(\zeta_{\mathbb{N}}(\mathfrak{e}'))^2\}, \min\{\zeta, \min\{(\zeta_{\mathbb{S}}(\hbar))^2, (\zeta_{\mathbb{N}}(\mathfrak{e}'))^2\}\}}$$
$$= \min\{(\Upsilon_{\mathbb{S}}(\top))^2, (\Upsilon_{\mathbb{S}}(\hbar))^2\}.$$

Thus, $(\Upsilon_{\mathbb{S}}(\top\hbar))^2 \geq \min\{(\Upsilon_{\mathbb{S}}(\top), \Upsilon_{\mathbb{S}}(\hbar))^2\}$.
Further,

$$(\hat{\Upsilon}_{\mathbb{S}}(\top - \hbar))^2$$
$$= (\hat{\mu}_{\mathbb{S}}(\top - \hbar))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top-\hbar))^2}$$
$$= \{\max\{(\hat{\mu}_{\mathbb{S}}(\top - \hbar))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top-\hbar))^2}, (\hat{\mu}_{\mathbb{N}}(\mathfrak{e}' - \mathfrak{e}'))^2 e^{i(\hat{\zeta}_{\mathbb{N}}(\mathfrak{e}'-\mathfrak{e}'))^2}\}\}$$
$$= \{(\hat{\mu}_{\mathbb{S}\times\mathbb{N}}((\top, \mathfrak{e}')(\hbar, \mathfrak{e}')))^2\} e^{i\{(\hat{\zeta}_{\mathbb{S}\times\mathbb{N}}((\top,\mathfrak{e}')(\hbar,\mathfrak{e}')))^2\}}$$
$$\leq \max\{(\hat{\mu}_{\mathbb{S}\times\mathbb{N}}(\top, \mathfrak{e}'))^2, (\hat{\mu}_{\mathbb{S}\times\mathbb{N}}(\hbar, \mathfrak{e}'))^2\} e^{i\max\{(\hat{\zeta}_{\mathbb{S}\times\mathbb{N}}(\top,\mathfrak{e}'))^2,(\zeta_{\mathbb{S}\times\mathbb{N}}(\hbar,\mathfrak{e}'))^2\}}$$
$$= \max\{\max\{(\hat{\mu}_{\mathbb{S}}(\top))^2, (\hat{\mu}_{\mathbb{N}}(\mathfrak{e}'))^2\}, \max\{(\hat{\mu}_{\mathbb{S}}(\hbar))^2, (\hat{\mu}_{\mathbb{N}}(\mathfrak{e}'))^2\}\} e^{i\max\{\max\{(\hat{\zeta}_{\mathbb{N}}(\mathfrak{e}'))^2\}, \max\{\zeta, \max\{(\hat{\zeta}_{\mathbb{S}}(\hbar))^2, (\zeta_{\mathbb{N}}(\mathfrak{e}'))^2\}\}\}}$$
$$= \max\{(\hat{\Upsilon}_{\mathbb{S}}(\top))^2, (\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\}.$$

Thus, $(\hat{\Upsilon}_{\mathbb{S}}(\top - \hbar))^2 \leq \max\{(\hat{\Upsilon}_{\mathbb{S}}(\top))^2, (\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\}$.
Further,

$$(\hat{\Upsilon}_{\mathbb{S}}(\top\hbar))^2$$
$$= (\hat{\mu}_{\mathbb{S}}(\top\hbar))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top\hbar))^2}$$
$$= \{\max\{(\hat{\mu}_{\mathbb{S}}(\top\hbar))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\top\hbar))^2}, (\hat{\mu}_{\mathbb{N}}(\mathfrak{e}'\mathfrak{e}'))^2 e^{i(\hat{\zeta}_{\mathbb{N}}(\mathfrak{e}'\mathfrak{e}'))^2}\}\}$$
$$= \{(\hat{\mu}_{\mathbb{S}\times\mathbb{N}}((\top, \mathfrak{e}')(\hbar, \mathfrak{e}')))^2\} e^{i\{(\hat{\zeta}_{\mathbb{S}\times\mathbb{N}}((\top,\mathfrak{e}')(\hbar,\mathfrak{e}')))^2\}}$$
$$\leq \max\{(\hat{\mu}_{\mathbb{S}\times\mathbb{N}}(\top, \mathfrak{e}'))^2, (\hat{\mu}_{\mathbb{S}\times\mathbb{N}}(\hbar, \mathfrak{e}'))^2\} e^{i\max\{(\hat{\zeta}_{\mathbb{S}\times\mathbb{N}}(\top, \mathfrak{e}'))^2, (\zeta_{\mathbb{S}\times\mathbb{N}}(\hbar, \mathfrak{e}'))^2\}}$$
$$= \max\{\max\{(\hat{\mu}_{\mathbb{S}}(\top))^2, (\hat{\mu}_{\mathbb{N}}(\mathfrak{e}'))^2\}, \max\{(\hat{\mu}_{\mathbb{S}}(\hbar))^2, (\hat{\mu}_{\mathbb{N}}(\mathfrak{e}'))^2\}\} e^{i\max\{\max\{(\hat{\zeta}_{\mathbb{N}}(\mathfrak{e}'))^2\}, \max\{\zeta, \max\{(\hat{\zeta}_{\mathbb{S}}(\hbar))^2, (\zeta_{\mathbb{N}}(\mathfrak{e}'))^2\}\}\}}$$
$$= \max\{(\hat{\Upsilon}_{\mathbb{S}}(\top))^2, (\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\}$$

Thus,
$(\hat{\Upsilon}_{\mathbb{S}}(\top\hbar))^2 \leq \max\{(\hat{\Upsilon}_{\mathbb{S}}(\top))^2, (\hat{\Upsilon}_{\mathbb{S}}(\hbar))^2\}$.
Also, let

$$(\Upsilon_{\mathbb{S}}(\top^{-1}))^2$$
$$= (\mu_{\mathbb{S}}(\top^{-1}))^2 e^{i(\zeta_{\mathbb{S}}(\top^{-1}))^2}$$
$$= \min\{(\mu_{\mathbb{S}}(\top^{-1}))^2 e^{i(\zeta_{\mathbb{S}}(\top^{-1}))^2}, (\mu_{\mathbb{N}}((\mathfrak{e}')^{-1}))^2 e^{i(\zeta_{\mathbb{N}}((\mathfrak{e}')^{-1}))^2}\}$$
$$= \min\{(\mu_{\mathbb{S}}(\top^{-1}))^2, (\mu_{\mathbb{N}}((\mathfrak{e}')^{-1}))^2\} e^{i\min\{(\zeta_{\mathbb{S}}(\top^{-1}))^2,(\zeta_{\mathbb{N}}((\mathfrak{e}')^{-1}))^2\}}$$
$$= (\mu_{\mathbb{S}\times\mathbb{N}}(\top^{-1}), (\mathfrak{e}')^{-1})^2 e^{i(\zeta_{\mathbb{S}\times\mathbb{N}}(\top^{-1},(\mathfrak{e}')^{-1}))^2}$$
$$\geq (\mu_{\mathbb{S}\times\mathbb{N}}(\top, \mathfrak{e}'))^2 e^{i(\zeta_{\mathbb{S}\times\mathbb{N}}(\top,\mathfrak{e}'))^2}$$
$$= \min\{(\mu_{\mathbb{S}}(\top))^2, (\mu_{\mathbb{N}}(\mathfrak{e}'))^2\} e^{i\min\{(\zeta_{\mathbb{N}}\top)^2,(\zeta_{\mathbb{N}}(\mathfrak{e}'))^2\}}$$
$$= \min\{(\mu_{\mathbb{S}}(\top))^2, (\mu_{\mathbb{S}}(\top))^2\} e^{i\min\{(\zeta_{\mathbb{S}}(\top))^2,(\zeta_{\mathbb{S}}(\top))^2\}}$$
$$= (\mu_{\mathbb{S}}(\top))^2 e^{i(\zeta_{\mathbb{S}}(\top))^2} = (\Upsilon_{\mathbb{S}}(\top))^2.$$

Consequently, we have
$(\Upsilon_{\mathbb{S}}(\tau^{-1}))^2 \geq (\Upsilon_{\mathbb{S}}(\tau)^2$. Moreover,

$$
\begin{aligned}
&(\hat{\Upsilon}_{\mathbb{S}}(\tau^{-1}))^2 \\
&= (\hat{\mu}_{\mathbb{S}}(\tau^{-1}))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\tau^{-1}))^2} \\
&= \max\{(\hat{\mu}_{\mathbb{S}}(\tau^{-1}))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\tau^{-1}))^2}, (\hat{\mu}_{\mathbb{N}}((\mathfrak{e}')^{-1}))^2 e^{i(\hat{\zeta}_{\mathbb{N}}((\mathfrak{e}')^{-1}))^2}\} \\
&= \max\{(\hat{\mu}_{\mathbb{S}}(\tau^{-1}))^2, (\hat{\mu}_{\mathbb{N}}((\mathfrak{e}')^{-1}))^2\} \\
&\quad e^{i\max\{(\hat{\zeta}_{\mathbb{S}}(\tau^{-1}))^2, (\hat{\zeta}_{\mathbb{N}}((\mathfrak{e}')^{-1}))^2\}} \\
&= (\hat{\hat{\mu}}_{\mathbb{S}\times\mathbb{N}}(\tau^{-1}), (\mathfrak{e}')^{-1})^2 e^{i(\hat{\zeta}_{\mathbb{S}\times\mathbb{N}}(\tau^{-1}, (\mathfrak{e}')^{-1}))^2} \\
&\leq (\hat{\hat{\mu}}_{\mathbb{S}\times\mathbb{N}}(\tau), \mathfrak{e}'))^2 e^{i(\hat{\zeta}_{\mathbb{S}\times\mathbb{N}}(\tau, \mathfrak{e}'))^2} \\
&= \max\{(\hat{\mu}_{\mathbb{S}}(\tau))^2, (\hat{\mu}_{\mathbb{N}}(\mathfrak{e}'))^2\}\, e^{i\max\{(\hat{\zeta}_{\mathbb{S}}(\tau))^2, (\hat{\zeta}_{\mathbb{N}}(\mathfrak{e}'))^2\}} \\
&= \max\{(\hat{\mu}_{\mathbb{S}}(\tau))^2, (\hat{\mu}_{\mathbb{S}}(\tau))^2\}\, e^{i\max\{(\hat{\zeta}_{\mathbb{S}}(\tau))^2, (\hat{\zeta}_{\mathbb{S}}(\tau))^2\}} \\
&= (\hat{\mu}_{\mathbb{S}}(\tau))^2 e^{i(\hat{\zeta}_{\mathbb{S}}(\tau))^2} = (\hat{\Upsilon}_{\mathbb{S}}(\tau))^2.
\end{aligned}
$$

As a result, we have $(\hat{\Upsilon}_{\mathbb{S}}(\tau^{-1}))^2 \leq (\hat{\Upsilon}_{\mathbb{S}}(\tau))^2$.

Thus, this represented our assertion.

Theorem 13: Let $\mathbb{S}$ and $\mathbb{N}$ two **CPFSs** of $\mathbb{K}_1$ and $\mathbb{K}_2$ such that $\mu_{\mathbb{S}}(\mathfrak{e}) \geq \mu_{\mathbb{N}}(\hbar)$ and $\zeta_{\mathbb{S}}(\mathfrak{e}) \geq \zeta_{\mathbb{N}}(\hbar)$, $\forall\, \hbar \in \mathbb{K}_2$ and $\mathfrak{e}$ is a unit element of $\mathbb{K}_1$. If $\mathbb{S} \times \mathbb{N}$ is **CPFSF** of $\mathbb{K}_1 \times \mathbb{K}_2$, then $\mathbb{N}$ is a **CPFSF** of $\mathbb{K}_2$.

Proof: Theorem 11 is used in the proof of this theorem.

Corollary 3: Suppose that $\mathbb{S}$ and $\mathbb{N}$ are two **CPFSs** of $\mathbb{K}_1$ and $\mathbb{K}_2$, respectively. If $\mathbb{S} \times \mathbb{N}$ is **CPFSF** of $\mathbb{K}_1 \times \mathbb{K}_2$, then $\mathbb{S}$ is a **CPFSF** of $\mathbb{K}_1$ or $\mathbb{N}$ is a **CPFSF** of $\mathbb{K}_2$.

VI. CONCLUSION

In this manuscript, we presented the unique framework, **CPFSF**, level subset and intersection of **CPFS**. Every **CPFSF** generates two **PFSFs**, which we have demonstrated and examined in detail. In addition to proving that the level subset of the **CPFSF** forms a subfield of the certain field and also examined some of the level subset's algebraic features. Furthermore, we demonstrated that intersection of two **CPFSFs** is also **CPFSFs**. We also showed the level subset of the **CPFSF** form subfield and described the new idea behind level subsets of **CPFSFs**. Moreover, we expended this theory to show the concept of the direct product of two **CPFSFs** is also a **CPFSFs** and developed fundamental results about direct product of **CPFSFs**. We initiated the homomorphic image and homomorphic inverse image of **CPFSFs**. Future work will involve extending the initial strategy to various algebraic structures, which will then be applied to various fields like ring theory field theory and module theory.

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