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RESEARCH ARTICLE

Rough Fuzzy Substructures of Quantale Module Under Soft Relations and Corresponding Decision-Making Methods

SAQIB MAZHER QURASHI[1], BANDER ALMUTAIRI[2], RANI SUMAIRA KANWAL[3], ANTHONY KARAGEORGOS[4], AND AYESHA SAEED[1]

[1]Department of Mathematics, Government College University Faisalabad, Faisalabad 38000, Pakistan
[2]Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia
[3]Department of Mathematics, Government College Women University Faisalabad, Faisalabad 38000, Pakistan
[4]School of Technology, University of Thessaly, 431 00 Karditsa Greece

Corresponding author: Anthony Karageorgos (karageorgos@uth.gr)

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ABSTRACT This research paper has developed a way of roughness of fuzzy substructures by using soft relations for developing rough fuzzy substructures in Quantale module. Thus, an innovative concept of fuzzy substructures of Quantale module under rough environment by soft relations, is presented. The lower and upper approximations of fuzzy subsets of quantale module are defined by aftersets and foresets. This relationship leads to various characterizations of the rough fuzzy substructures of quantale modules. Besides of more comprehensive results, soft compatible and soft complete relations are required with foresets and aftersets. Soft relations are further being used to determine upper (lower) approximation of fuzzy subsets of quantale module using foreset and afterset. Moreover, several characterizations of rough fuzzy substructures of quantale module are investigated. Furthermore, the algebraic relations of upper (lower) approximations of fuzzy quantale submodule and fuzzy quantale submodule ideals are studied with the help of soft relations under weak quantale module homomorphism. To illustrate that the suggested approach is superior to the given methods, examples are provided. At last, we describe decision-making methods by using rough fuzzy substructures of Quantale module under soft relations to deal with uncertainties in the real-world problems. To demonstrate the validity, applicability, and efficacy of the suggested method, a detailed example of the decision-making process is provided.

INDEX TERMS Quantale module and its substructures, rough sets, soft sets and fuzzy sets.

I. INTRODUCTION

In 1993, Abramsky and Vickers [1] proposed the Quantale module concept. Quantale modules drew the attention of many scientists and researchers. The idea of a module over a ring served as the idea for the quantale module [2]. It substitutes quantales for rings and complete latices for abelian groups. The quantale module was initially introduced by Abramsky and Vickers' for unified treatment of process semantics [1]. Rosenthal [3] shows that modules over a commutative unital quantaleas provides a class of full linear logic models.

The rough set is a formal approximation of a crisp set in terms of a pair of crisp sets that provide the lower and upper approximations of the original set, as first proposed by Pawlak [4] in 1982. In recent years, roughness has been applied to a variety of algebraic structures. Ali et al. [5] proposed roughness in hemirings in terms of the Pawlak approximation space and generalized approximation space. Quantales were considered universal sets by Yang and Xu [6] and the concepts of rough (prime, semi-prime, primary) ideals and prime radicals of upper rough ideals of quantales were introduced.

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Qurashi and Shabir [7] introduced the concept of roughness in the Q-module. Some extension of rough set model was presented by Zhang et al. [8]. The relation among topological spaces and hyperrings with rough sets was studied by Abughazalah et al. [9]. Yaqoob and Tang [10] rough set theory to investigate quasi and inner hyperfilters in ordered LA-semi hypergroups.

Molodtsov [11] presented soft set theory in 1999 as to generalize fuzzy set theory to deal with uncertainty in a parametric way. Soft set can be used as a parametrized family of subsets of a crisp universal set. Numerous authors used soft set theory to study various algebraic structures. Matrices in soft set theory and their applications in decision making problems were presented by Basu et al. [12].

Soft intersection semigroups, soft intersection ideals and bi-ideals of semigroup was presented by Sezer et al. [13]. The concepts of generalized finite soft equality, generalized finite soft union and generalized finite soft intersection of two soft sets were introduced by Abbas et al. [14]. Approximation of ideals in semigroups by soft relations were proposed by Kanwal and Shabir [15].

An application of soft vector spaces was discussed by Ali et al. [16]. The concept of generalized approximation of substructures in quantales by soft relations was introduced by Kanwal et al. [17] which is an extended notion of a rough quantale and a soft quantale. A Multi-attribute decision-making method in terms of complex q-rung orthopair via Einstein geometric aggregation operators were studied by Wu et.al. [18].

Zadeh [19] first proposed the fuzzy set notion in 1965. Since then researchers have applied this key set in a variety of fields. Qurashi and Shabir [20] presented generalized fuzzy substructure in quantale. Yaqoob et al. [21] presented generalized fuzzy hyperideals, generalized fuzzy bi-hyperideals and generalized fuzzy normal bi-hyperideals in ordered LA-semi hypergroups using the concept of generalized fuzzy sets. Soft binary relations were used by Bilal and Shabir [22] to approximate pythagorean fuzzy sets over dual universes. Recently, Fuzzy convexities were investigated via overlap functions by Pang [23]. Important Hamacher aggregation operators dependent on the interval-valued intuitionistic fuzzy numbers related to decision making was proposed by Liu [24].

Fuzzy formal contexts and fuzzy relations between objects of different types in the form of fuzzy relational context families were investigated by Boffa [25]. Qurashi and Shabir ([26], [27]) studied the roughness of fuzzy substructures in quantales w.r.t generalized approximation space in the form of $(\in, \in \bigvee q)$ and $(\in_\gamma, \in_\gamma \bigvee q_\delta)$. Hussain et al. [28] presented the notion of rough pythagorean fuzzy ideals in semigroups. Malik and Shabir [29] proposed the notion of rough fuzzy bipolar soft sets and its use in decision-making problems. Rough approximation of a fuzzy set in semigroups based on soft relations was presented by Kanwal and Shabir [30]. Different characterizations of important residual Implications in terms of Copulas were presented by Ji and Xie [31]. While Fuzzy Quasi-Normed spaces utilized to express open mapping and closed graph theorems were studied by Wu and Li [32]. Cubic Bipolar Fuzzy-VIKOR Method dependent on entropy measures was proposed by Riaz et al. [33]. The character and applications of aggregating intuitionistic uncertain linguistic variables to group decision making were proposed by Liu and Jin [34].

Zhan et al. [35] investigated the relationships among rough sets, soft sets and hemirings. He introduced the concept of soft rough hemirings, which is an extension of the rough hemiring notion. Several soft rough set results and topological structure of soft rough sets were presented by Riaz et al. [36]. Multigranulation roughness of intuitionistic fuzzy sets using soft relations and their applications in decision making were suggested by Anwar et al. [37]. Hussain et al. [38] studied pythagorean fuzzy soft rough sets and their applications in decision-making. Bera and Roy [39] developed a relation between rough soft set and fuzzy set and introduced fuzzy rough soft set. Shabir et al. [40] presented multigranulation roughness based on soft relations.

The term "soft relationship" was first used by Feng et al. [41]. Considering that soft relations were employed to more effectively that minimize some types of information was extended by Shabir et al. [42]. Additionally, decision-making methods including soft relations were connected to algebraic structures and gave rise to fascinating new study areas ([43], [44], [45], [46]).

Fuzzy logic was introduced by Zadehas a mechanism for computing with words, fuzzy logic has seen widespread use. This artificial intelligence method is perfect for effectively tackling the ambiguity, imprecision, and uncertainty present in a wide range of scientific and technological domains. Software systems employ it to make decisions, detect problems, provide recommendations, and more automatically. Computer network security system is important in different software houses and industries. Azam et al. [47] presented such type of selection dependent on complex intuitionistic fuzzy setting. Hybridization of fuzzy sets in terms of complex interval-valued intuitionistic fuzzy decision making problems including COVID-19 healthcare facilities was introduced by Khan et al. [48]. Different decision making techniques of physical and natural phenomena inclusive of orthopair fuzzy TOPSIS methods with incomplete weight and complex hesitant fuzzy sets with priority degrees and distance were explored by Khan et al. ([49], [50]). Aggregation operators fermatean fuzzy power Bonferroni in decision making were proposed by Ruan et al. [51]. Decision making process under linguistic q-rung orthopair fuzzy Einstein models and orthopair fuzzy membership grades were presented by Akram et al. [52] and Feng et al. [53]. Some algorithm with trapezoidal picture fuzzy numbers was studied by Akram et al. [54]. TOPSIS model with complex spherical fuzzy information was proposed by Akram et al. [55].

A. COMPARITIVE STUDY AND DISCUSSION

In this study, we have made a comparison between the suggested study and earlier research. First of all we will present a definition which was presented by Shabir and Kanwal [42]. The definition is as follows

Assume V is the subset of E (S.P) and $(\text{Л}, V)$ be a SBIR from M_1 *to* M_2 i.e., $\text{Л} : V \rightarrow P(M_1 \times M_2)$ .Thus, the LO_{ap} $(\underline{\text{Л}}^{\mathcal{M}}, V)$ and UP_{ap} $(\overline{\text{Л}}^{\mathcal{M}}, V)$ w.r.t the afterset of soft set $(\mathcal{M}, V)$ over M_2 are essentially two soft sets over M_1 are defined as

$$\underline{\text{Л}}^{\mathcal{M}}(v) = \{\gamma_1 \in M_1 : \emptyset \neq \gamma_1 \text{Л}(v) \subseteq \mathcal{M}(v)\} \text{ and}$$
$$\overline{\text{Л}}^{\mathcal{M}}(v) = \{\gamma_1 \in M_1 : \gamma_1 \text{Л}(v) \cap \mathcal{M}(v) \neq \emptyset\}.$$

For foresets the definition is as follows

$$^{\mathcal{N}}\underline{\text{Л}}(v) = \{\gamma_2 \in M_2 : \emptyset \neq \text{Л}(v)\gamma_2 \subseteq \mathcal{N}(v)\} \quad \text{and}$$
$$^{\mathcal{N}}\overline{\text{Л}}(v) = \{\gamma_2 \in M_2 : \text{Л}(v)\gamma_2 \cap \mathcal{N}(v) \neq \emptyset\} \; \forall \, v \in V.$$

In fact, in the above definition, equivalence class under equivalence relation (congruence) is no needed while in the following [5], [6], and [7], congruence relations are required. Fuzzification of the above definition was defined in [30] and equivalence (congruence) is not required in this definition. It is nothing but the generalization of the definition proposed by Dubois and Prade [57]. Now we will present comparative discussion in the form of a Table as follows.

TABLE 1. Comparison Table.

Sr. No	Previous Models	Proposed Models
1.	Approximations of fuzzy sets and their fuzzy algebraic structures by soft relations were proposed by Kanwal and Shabir in their core paper and further the proposed study was being applied to semigroups and quantales [30], [58].	Such type of approximations is not applied in fuzzy substructures in quantale module. This is first time that the authors have used to find out different characterizations of fuzzy soft substructures in quantale module by soft relations in this paper.
2.	Sometimes it seems difficult to find an equivalence relations and congruence relations. These problems has been seen while evaluating roughness in algebraic structure [5], [6], [7].	In the proposed model, neither we are need equivalence relation nor a congruence. We have much relax condition in this case. Moreover, such type of approximation is being applied first time in quantale module.
3.	Qurashi *et al.*,[57], Kanwal, and Shabir [30] conducted studies on rough approximations based on fuzzy substructures in quantale and semigroups along soft relations with respect to aftersets and foresets.	In this paper, authors used soft compatible relations by using congruence in quantale modules. Then these are applied to fuzzy subsets of them to find rough approximations. The idea is unique and applied for the first time.

B. DEFICIT AND RESEARCH GAP IN EXISTING LITERATURE

The literature mentioned above highlights the various contributions made by researchers in the fields of fuzzy sets, rough sets, and soft sets theories. The ideas of roughness of crisp sets, rough fuzzy sets, and rough soft sets have been applied to many algebraic structures, and they play a significant role in decision-making processes. Roughness dependent on set-valued homomorphism in quantale module was presented by Qurashi and Shabir [7]. However there are many open questions and problems which should be answered and addressed.

1. In the quantale module, approximation was performed using set-valued mapping and congruence relations. There was no mention of the decision-making process in these kinds of approximations. So it was necessary to make decision process in a new type of approximation in quantale module.

2. Numerous contributions to classical quantale module theory are known yet, its generalization is not discussed much. There is no discussion of fuzzy substructures or how to approximate them using set-valued homomorphism and congruence in quantale module.

3. Literature already in existence studies various crisp and soft substructures under rough environment of quantale modules via soft relations ([43], [44]). Moreover, such type of roughness via soft relations has also been discussed in different structures like semigroups and quantale ([15], [17]). Given that quantale modules' fuzzy substructures are a generalization of their substructures, it is necessary to comprehend how these fuzzy substructures are characterized with respect to soft relations.

4. Roughness of fuzzy sets in semigroups by soft relations was applied by Kanwal and Shabir [30]. Naturally, one would wonder how soft relations will handle the roughness of fuzzy substructures in quantale module. This is a reasonable concern to pose.

5. Some fundamental and important theorems of quantale module homomorphism and soft quantale module homomorphism were discussed in ([43], [44]). Consequently, a discussion of these important theorems in the context of quantale module homomorphism via fuzzy sets is imperative.

The final objective of this study is to close the knowledge gap in the body of current literature and tackle the aforementioned remaining issues.

C. MERITS AND LIMITATIONS OF THE PROPOSED MODEL

Intuitionistic fuzzy substructures in quantale modules can benefit from the results demonstrated in this study. Multi-granulation roughness of Intuitionistic fuzzy based on soft relations was presented by Anwar et al. [37], As intuitionistic fuzzy substructures in quantale module are easily defined from fuzzy substructures in quantale modules so present study can apply to find rough intuitionistic fuzzy substructures in quantale module by soft relations. It is noted that roughness in Pythagorean fuzzy sets defined by soft relations was suggested by Bilal and Shabir [22], so such kind of Pythagorean fuzzy substructures in quantale module can be defined in quantale module by soft relations. However, can we apply soft relations to define picture fuzzy substructures, bipolar fuzzy substructures and q-rung orthopair fuzzy substructures in quantale module under rough environment? For these generalized structures, therefore, independent research is advised. This represents our research's major limitation. Moreover, decision-making techniques are used to define rough fuzzy structures in quantale modules for the first time. These are the merits of the work we have proposed.

D. MOTIVATION AND GOAL OF THE PROPOSED WORK

Although there have been numerous contributions to the theory of quantale module, yet its generalization has not received the appropriate attention or input. That is in the quantale module, generalization including fuzzy and rough fuzzy substructures received less attention. There are, as far as we are aware, a few articles on the generalization of quantale module. Although roughness with congruence relations and set-valued homomorphism [7] is available in the literature yet there is no such attempt to find roughness which is without the above techniques. More generalized form of roughness of substructures and soft substructures were performed in [43] and [44]. But these generalized models in quantale modules did not contain roughness of fuzzy substructures. Thus, this is our main motivation to define roughness of fuzzy substructures in quantale module by soft relations without equivalence and congruence and we are motivated to take help from aftersets and foresets.

The paper's specifics are as follows. In introduction, there are sub headings which specify our targets. The introduction Section includes comparative study, research gaps, merits and limitations and finally motivation of the proposed study. Section II presents some important definitions relating to fuzzy sets and fuzzy quantale module substructures. Further, soft sets, soft binary relations, and rough sets are examined. Section III describes a few characterizations of fuzzy subsets of quantale modules with the use soft relations. Additionally, section IV will express several rough fuzzy substructures with respect to aftersets and foresets using soft relations. The upper (lower) approximations of fuzzy substructures and homomorphic images are described in Section V together with quantale module homomorphism. The section VI helps us to understand how rough fuzzy substructures are utilized to have better understanding of decision making problems. An example for better understanding is also added to understand. At the last section the whole paper is captured in the conclusion.

TABLE 2. List of acronyms/abbreviations.

Acronyms	Representation
Θ	Binary operation on quantale
$\mathbb{Q}$	Left action on Quantale module
UP_{AP}	Upper approximation
LO_{AP}	Lower approximation
$\mathbb{Q}_{SM}$	Quantale submodule
$\mathbb{Q}_{ID}$	Quantale submodule ideal
$STBR$	Soft binary relation
$STCR$	Soft compatible relation
$STCMR$	Soft complete relation
F_{sst}	Fuzzy subset
$FSTS$	Fuzzy soft set
$F\mathbb{Q}_{SM}$	Fuzzy quantale submodule
$FST\mathbb{Q}_{SM}$	Fuzzy soft quantale submodule
$F\mathbb{Q}_{ID}$	Fuzzy quantale submodule ideal
$FST\mathbb{Q}_{ID}$	Fuzzy soft quantale submodule ideal
$WM\!\!\!,H$	Weak quantale module homomorphism

II. PRELIMINARIES

In In this section, we go over some fundamental ideas about substructures and fuzzy substructures of quantale module, as well as the conclusions that go along with them. This will be beneficial for our next work.

Definition 1 [56]: Let $\mathbb{Q}$ be a complete lattice. Define an associative binary operation Θ on $\mathbb{Q}$ satisfying :

1) $g\Theta\left(\vee_{j\in J}h_j\right)=\vee_{j\in J}\left(g\Theta h_j\right)$;
2) $\left(\vee_{j\in J}g_j\right)\Theta h=\vee_{j\in J}\left(g_j\Theta h\right)$.
   $\forall\, g,\ h\in\mathbb{Q}$ and $\{g_j\},\ \{h_j\}\subseteq\mathbb{Q}\ (j\in J)$.
   Then $(\mathbb{Q},\ \Theta)$ is a quantale. Let $\mathcal{F}_i,\ \mathcal{F}_1,\ \mathcal{F}_2\subseteq\mathbb{Q}$. Then the following are defined;

$$F_1\Theta F_2=\{f_1\Theta f_2\ :\ f_1\in F_1\ ,\ f_2\in F_2\};$$
$$F_1\vee F_2=\{f_1\vee f_2\ :\ f_1\in F_1\ ,\ f_2\in F_2\};$$

and $\vee_{i\in I}F_i=\{\vee_{i\in I}f_i\ :\ f_i\in F_i\}$.

Throughout the paper, for quantales the symbol $\mathbb{Q}_1$ and $\mathbb{Q}_2$ will be used. The top element and bottom element will be expressed by $\mathcal{T}$ and $\mathcal{L}$ respectively.

Definition 2 [1]: Let M be a sup-lattice and let $\mathbb{Q}$ be a quantale. Define a left action $\mathbb{Q}:\ \mathbb{Q}\times M\longrightarrow M$. Then M is called left $\mathbb{Q}-$ module over the quantale $\mathbb{Q}$ if it satisfies the following conditions:

1) $(\vee_{i\in I}p_i)\,\mathbb{Q}\,q=\vee_{i\in I}\left(p_i\mathbb{Q}\,q\right)$;
2) $p\,\mathbb{Q}\left(\vee_{j\in J}q_j\right)=\vee_{j\in J}\left(p\,\mathbb{Q}\,q_j\right)$;
3) $(p\,\Theta\, r)\ \mathbb{Q}\,q=p\,\mathbb{Q}\ (r\,\mathbb{Q}\,q)$.
   for any $p, r\in\mathbb{Q}$, $\{p_i\}\subseteq\mathbb{Q}\,(i\in I), q\in M$, and $\{q_j\}\subseteq M\,(j\in J)$.
   In this paper, M for left $\mathbb{Q}-$ module over the quantale $\mathbb{Q}$ will be used. For a $\mathbb{Q}-$ module M, $A\subseteq\mathbb{Q}$ *and* $m\in M$ we have :

$$A\,\mathbb{Q}\,m=\ \{a\,\mathbb{Q}\,m\mid a\in A\};$$

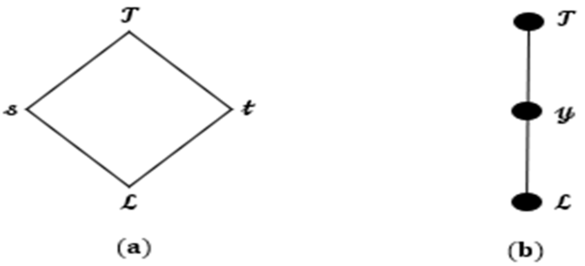

FIGURE 1. (a) Description of $\mathbb{Q}$ (b) Description of $\mathbb{M}$.

TABLE 3. Binary operation subject to $\circleddash$.

$\circleddash$	$\mathcal{L}$	$\mathscr{s}$	$\mathscr{t}$	$\mathcal{T}$
$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
$\mathscr{s}$	$\mathcal{L}$	$\mathscr{s}$	$\mathcal{L}$	$\mathscr{s}$
$\mathscr{t}$	$\mathcal{L}$	$\mathcal{L}$	$\mathscr{t}$	$\mathscr{t}$
$\mathcal{T}$	$\mathcal{L}$	$\mathscr{s}$	$\mathscr{t}$	$\mathcal{T}$

TABLE 4. Left action subject to $\circledcirc$.

$\circledcirc$	$\mathcal{L}$	$\mathscr{y}$	$\mathcal{T}$
$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
$\mathscr{y}$	$\mathcal{L}$	$\mathscr{y}$	$\mathscr{y}$
$\mathcal{T}$	$\mathcal{L}$	$\mathscr{y}$	$\mathcal{T}$

$$A \circledcirc B = \{a \circledcirc b \mid a\in A,\ b\in B\} \text{ where } B\subseteq \mathbb{M}.$$

For $\mathcal{A}, \mathcal{B}, \mathcal{A}_i \subseteq \mathbb{M}$ $(i \in I)$. We write

$$\mathcal{A}\vee\mathcal{B} = \{a\vee b \mid \in\mathcal{A},\ b\in\mathcal{B}\}$$

and $\vee_{i\in I}\mathcal{A}_i = \{\vee_{i\in I\, i} \mid {}_i\in\mathcal{A}_i\}$.

Example 1: Let $\mathbb{Q} = \{\mathcal{L}, \mathscr{s}, \mathscr{t}, \mathcal{T}\}$ be the complete lattice as shown in Fig. 1(a) and operation $\circleddash$ on $\mathbb{Q}$ is shown in Table 3. Then $(\mathbb{Q}, \circleddash)$ is a quantale. Let $\mathbb{M} = \{\mathcal{L}, \mathscr{y}, \mathcal{T}\}$ be a sup lattice. The lattice diagram of $\mathbb{M}$ is given in Fig. 1(b). Let $\circledcirc : \mathbb{Q} \times \mathbb{M} \rightarrow \mathbb{M}$ be the left action on $\mathbb{M}$ as shown in Table 4. Then it is easy to verify that $\mathbb{M}$ is $\mathbb{Q}-$ module

Definition 3 [1]: Let $\mathbb{M}$ be a $\mathbb{Q}-$ module. If a subset $\mathbb{M}_1\subseteq\mathbb{M}$ satisfies the following axioms for any $w\in\mathbb{M}_1$, $\{w_i\} \subseteq \mathbb{M}_1$ and $\lambda\in\mathbb{Q}$.

1) $\vee_{i\in I} w_i \in \mathbb{M}_1\ \forall\, w_i \in \mathbb{M}_1$;
2) $\lambda \circledcirc w\in \mathbb{M}_1\ \forall\, \lambda\in\mathbb{Q},\ \forall\, w\in\mathbb{M}_1$.

Then $\mathbb{M}_1$ is called $\mathbb{Q}-$ submodule $(\mathbb{Q}_{SM})$ of $\mathbb{M}$.

Definition 4 [1]: Let I be a subset of $\mathbb{Q}-$ module $\mathbb{M}$. Then I is called $\mathbb{Q}-$ ideal $(\mathbb{Q}_{\text{ID}})$ of $\mathbb{M}$ if the following hold :

1) $A \subseteq I$ *implies* $\vee A \subseteq I$;
2) $e\in I$ *and* $d\le e$ *implies* $d\in I$ *where* $d\in\mathbb{M}$;
3) $e\in I$ *implies* $\lambda\circledcirc e\in I$ *for all* $\lambda \in$

Definition 5: If $\mathbb{J}$ is a mapping given by $\mathbb{J} : \mathbb{G} \rightarrow P(\mathbb{M})$ where $\mathbb{G} \subseteq$ E (S.O.P), then the pair $(\mathbb{J}, \mathbb{G})$ is called a soft set over $\mathbb{M}$.

1) $(\mathbb{J}, \mathbb{G})$ is called soft $\mathbb{Q}_{SM}$ over $\mathbb{M}$ if $\mathbb{J}(u)$ is a $\mathbb{Q}_{SM}$ for all $u\in\mathbb{G}$.
2) $(\mathbb{J}, \mathbb{G})$ is called soft $\mathbb{Q}_{ID}$ over $\mathbb{M}$ if $\mathbb{J}(u)$ is a $\mathbb{Q}_{ID}$ for all $u\in\mathbb{G}$.

Definition 6 [11]: Let $\mathbb{J} : \mathbb{G} \rightarrow P(\mathbb{M}_1 \times \mathbb{M}_2)$ where $\mathbb{G} \subseteq$ E (S.O.P). Then $(\mathbb{J}, \mathbb{G})$ is called a STBR from a quantale module $\mathbb{M}_1$ to $\mathbb{M}_2$.

Definition 7 [19]: A function $\beta : \mathbb{M} \longrightarrow [0,\ 1]$ is known as fuzzy subset (F_{sst}) of $\mathbb{M}$. Let β and μ be two F_{sst} of $\mathbb{M}$. Then $\beta \subseteq \mu$ if and only if $\beta(m) \le \mu(m)$ for all $m\in\mathbb{M}$. Clearly $\beta = \mu$ if and only if $\beta \subseteq \mu$ and $\mu \subseteq \beta$. Let β and μ be two F_{sst} of $\mathbb{M}$. Then the union and intersection of β and μ are

$$(\beta\cup\mu)(m) = \text{Max}\ \{\beta(m)\ ,\ \mu(m)\}$$
$$(\beta\cap\mu)(m) = \text{Min}\ \{\beta(m)\ ,\ \mu(m)\}$$

for all $m\in\mathbb{M}$.

Definition 8: Let β be a F_{sst} of a quantale module $\mathbb{M}$ and $\alpha \in [0,\ 1]$. Then

$$\beta_\alpha = \{m\in\mathbb{M} \mid \beta(m) \ge \alpha\};\ \beta_{\alpha^+} = \{m\in\mathbb{M} \mid \beta(m) > \alpha\}$$

are called $\alpha-$cut and strong $\alpha-$cut of $F_{sst}\beta$, respectively.

Definition 9: Let $\mathbb{Q}$ be a quantale and $\mathbb{M}_1$ be $\mathbb{Q}-$ module and η be F_{sst} of $\mathbb{M}_1$.Then η is said to be fuzzy $\mathbb{Q}-$ submodule, if for any $e\in\mathbb{M}_1$ and $q\in\mathbb{Q}$, the following conditions hold:

1) $\eta(\vee_{i\in I} e_i) \ge \wedge_{i\in I}\eta(e_i)\quad \forall e_i\in\mathbb{M},\ \forall i \in I$;
2) $\eta(q\circledcirc_1 e) \ge \eta(e)$.

Definition 10: A pair $(\mathbb{J}, \mathbb{G})$ is called a FSTS over $\mathbb{U}$ if $\mathbb{J}$ is a mapping given by $\mathbb{J} : \mathbb{G}\rightarrow \mathfrak{F}(\mathbb{U})$ and $\mathbb{G}$ is a subset of E (the set of parameters) and $\mathfrak{F}(\mathbb{U})$ is the set of all fuzzy subsets of $\mathbb{U}$.

Let $\mathbb{M}$ be a quantale module and $(\mathbb{J}, \mathbb{G})$ be FSTS over $\mathbb{M}$. Then

$(\mathbb{J}, \mathbb{G})$ is called a *FST*$\mathbb{Q}_{SM}$ over $\mathbb{M}$ if $\mathbb{J}(u)$ is a $F\mathbb{Q}_{SM}$ of $\mathbb{M}$ for all $u\in\mathbb{G}$ with $\mathbb{J}(u)\neq\Phi$.

$(\mathbb{J}, \mathbb{G})$ is called a *FST*$\mathbb{Q}_{ID}$ over $\mathbb{M}$ if $\mathbb{J}(u)$ is a $F\mathbb{Q}_{ID}$ of $\mathbb{M}$ for all $u\in\mathbb{G}$ with $\mathbb{J}(u)\neq\Phi$.

Definition 11 [11]: Let $\mathbb{M}$ be a finite set ϑ be an equivalence relation on it. So $(\mathbb{M}, \vartheta)$ is referred as an approximation space. Let $\mathcal{U}$ represent a subset of $\mathbb{M}$. The union of the equivalence classes of $\mathbb{M}$ then may or may not be written as $\mathcal{U}$. If $\mathcal{U}$ can be expressed as a union of certain equivalence classes of $\mathbb{M}$, then we say that $\mathcal{U}$ is defined. Otherwise, it is called not definable. The lower and higher approximations of $\mathcal{U}$ are two definable subsets that can be used to approximate $\mathcal{U}$ in the event that it cannot be definable. These approximations are defined as follows :

$$\underline{\vartheta}(\text{U}) = \{\text{m}\in\mathbb{M} : [\text{m}]_\vartheta\subseteq\text{U}\}$$
$$\text{and } \overline{\vartheta}(\text{U}) = \{\text{m}\in\mathbb{M} : [\text{m}]_\vartheta\cap\text{U}\neq\Phi\}.$$

A rough set is a pair $(\underline{\vartheta}(\mathcal{U}), \overline{\vartheta}(\mathcal{U}))$ if $\underline{\vartheta}(\mathcal{U}) \neq \overline{\vartheta}(\mathcal{U})$.

Definition 12 [57]: Dubois and Prade introduced rough fuzzy sets and fuzzy rough sets. Let $\mathcal{X}$ be a non-empty finite set called the universe set and ϑ be an equivalence relation on

$\mathcal{X}$. Then $(\mathcal{X},\ \vartheta)$ is called an approximation space. Let $\mathcal{C}$ be a fuzzy subset of x. If $x\in\mathcal{X}$, then

$$\underline{\vartheta}(\mathcal{C})(x) = \bigwedge_{z\in[x]_\vartheta}\mathcal{C}()\text{ and }\overline{\vartheta}(\mathcal{C})(x) = \vee_{z\in[x]_\vartheta}\mathcal{C}(z)$$

Then $\underline{\vartheta}(\mathcal{C})$ is called lower approximation and $\overline{\vartheta}(\mathcal{C})$ is called upper approximation of the fuzzy subset $\mathcal{C}$. *If* $\underline{\vartheta}(\mathcal{C})(x) \neq \overline{\vartheta}(\mathcal{C})(x)$, then $\beta(\mathcal{C}) = (\underline{\vartheta}(\mathcal{C})(x), \overline{\vartheta}(\mathcal{C})(x))$ is called a rough fuzzy set w.r.t ϑ.

III. ROUGHNESS OF FUZZY SET IN QUANTALE MODULE BY SOFT RELATIONS

This section serve as to initiate the study of the notion of approximation of fuzzy substructures by soft relation in quantale modules and establish many fundamental aspects of this phenomena.

Definition 13 [30]: Let $(\mathbb{J}, \mathbb{G})$ be a *STBR* from a quantale module M_1 to a quantale module M_2. That is $\mathbb{J} : \mathbb{G} \to P(M_1 \times M_2)$. For a $F_{sst}\,\beta$ of M_2, the $UP_{AP}(\overline{\mathbb{J}}^\beta, \mathbb{G})$ and the $L0_{AP}(\underline{\mathbb{J}}^\beta, \mathbb{G})$ of β w.r.t aftersets are the two fuzzy soft sets (FSTS) over M_1 defined as follows;

$$\overline{\mathbb{J}}^\beta(u)(\lambda_1) = \begin{cases} \bigvee_{c\,\in\lambda_1\mathbb{J}(u)}\beta(c) & if\ \lambda_1\mathbb{J}(u)\neq\Phi \\ 0 & if\ \lambda_1\mathbb{J}(u) = \Phi \end{cases}$$

and

$$\underline{\mathbb{J}}^\beta(u)(\lambda_1) = \begin{cases} \bigwedge_{c\,\in\lambda_1\mathbb{J}(u)}\beta(c) & if\ \lambda_1\mathbb{J}(u)\neq\Phi \\ 0 & if\ \lambda_1\mathbb{J}(u) = \Phi \end{cases}$$

For a $F_{sst}\,\mu$ of M_1, the $UP_{AP}({}^\mu\overline{\mathbb{J}}, G)$ and the $L0_{AP}({}^\mu\underline{\mathbb{J}}, G)$ of μ w.r.t the foresets, are the two *FSTS* over M_2 defined as follows

$${}^\mu\overline{\mathbb{J}}(u)(\lambda_2) = \begin{cases} \bigvee_{c\,\in\mathbb{J}(u)\lambda_2}\mu(c) & if\ \ \mathbb{J}(u)\lambda_2\neq\Phi \\ 0 & if\ \ \mathbb{J}(u)\lambda_2 = \Phi \end{cases}$$

and

$${}^\mu\underline{\mathbb{J}}(u)(\lambda_2) = \begin{cases} \bigwedge_{c\,\in\mathbb{J}(u)\lambda_2}\mu(c) if\ \ \mathbb{J}(u)\lambda_2\neq\Phi \\ 0 \qquad if\ \ \mathbb{J}(u)\lambda_2 = \Phi \end{cases}$$

for all $u\in\mathbb{G}$.

here $\lambda_1\mathbb{J}(u) = \{\lambda_2 \in M_2 : (\lambda_1, \lambda_2) \in\mathbb{J}(u)\}$ is called the afterset of λ_1 and $\mathbb{J}(u)\lambda_2 = \{\lambda_1 \in M_1 : (\lambda_1, \lambda_2) \in\mathbb{J}(u)\}$ is called the foreset of λ_2.

Moreover, for each $\beta\in\mathcal{F}(K_2)$ $\overline{\mathbb{J}}^\beta(u) : \mathbb{G}\to \mathcal{F}(K_1)$ and $\underline{\mathbb{J}}^\beta(u)$: $\mathbb{G}\to \mathcal{F}(K_1)$ *a* nd for each $\mu\in\mathcal{F}(K_1)$ ${}^\mu\overline{\mathbb{J}}(u)$: $\mathbb{G} \to F(K_2)$ and ${}^\mu\underline{\mathbb{J}}(u)$: $\mathbb{G}\to \mathcal{F}(K_2)$.

Definition 14: A *STBR* $(\mathbb{J}, \mathbb{G})$ from M_1 to M_2 i.e., $\mathbb{J} : \mathbb{G} \to P(M_1\times M_2)$ is called soft compatible relation (*STCR*) if for all $f \in M_1$ and $g \in M_2$, $\{f_j\} \subseteq M_1$, $\{g_j\} \subseteq M_2$ for j $\in J$ and for all $u \in G$, we have

1) $(f_j,\ g_j)\in\mathbb{J}(u)\Rightarrow(\bigvee_{j\in J}f_j, \bigvee_{j\in J} g_j)\in\mathbb{J}(u)$;
2) $(f\ ,\ g)\in\mathbb{J}(u)\Rightarrow (\lambda@_1 f\ ,\ \lambda@_2 g) \in\mathbb{J}(u) \forall \lambda\in\mathbb{Q}$.

Definition 15: A *STCR* $(\mathbb{J}, \mathbb{G})$ from M_1 to M_2 w.r.t afterset is called soft complete relation (*STCMR*) if for all $v, w\in M_1$, $\lambda\in\mathbb{Q}$ we have :

1) $v\mathbb{J}(u)\vee w\mathbb{J}(u) = (v\vee w)\,\mathbb{J}(u)$;
2) $\lambda\mathbb{J}(u)@_2 v\mathbb{J}(u) = (\lambda@_1 v)\,\mathbb{J}(u)$.
   A *STCR* if satisfies condition (i) w.r.t the aftersets only, then we say it is $\vee-$ complete.

A *STCR* if satisfies condition (ii) only w.r.t the aftersets, then we say it is $@-$ complete.

A *STCR* $(\mathbb{J}, \mathbb{G})$ from $M_1 to M_2$ w.r.t foreset is called soft complete relation (*STCMR*) if for all $v, w\in M_2$, $\lambda\in\mathbb{Q}$ we have:

1) $\mathbb{J}(u)v\vee\mathbb{J}(u)w = \mathbb{J}(u)\,(v\vee w)$;
2) $\mathbb{J}(u)\lambda@_1\ \mathbb{J}(u)v = \mathbb{J}(u)\,(\lambda@_2 v)$.

Theorem 1 [30]: Let $(\mathbb{J}, \mathbb{G})$ and $(\mathbb{R}, \mathbb{G})$ be two *STBR* from a non-empty set M_1 to M_2 and β_1, β_2 be non-empty F_{sst} of M_2. Then for all $u\in\mathbb{G}$, we have

1) $\beta_1\leq \beta_2 \Rightarrow \overline{\mathbb{J}}^{\beta_1}(u) \leq \overline{\mathbb{J}}^{\beta_2}(u)$;
2) $\beta_1\leq \beta_2 \Rightarrow \underline{\mathbb{J}}^{\beta_1}(u) \leq \underline{\mathbb{J}}^{\beta_2}(u)$;
3) $\left(\overline{\mathbb{J}}^{\beta_1}, \mathbb{G}\right) \cap \left(\overline{\mathbb{J}}^{\beta_2}, \mathbb{G}\right) \supseteq \left(\overline{\mathbb{J}}^{\beta_1\cap\,\beta_2}, \mathbb{G}\right)$;
4) $\left(\underline{\mathbb{J}}^{\beta_1}, \mathbb{G}\right) \cap \left(\underline{\mathbb{J}}^{\beta_2}, \mathbb{G}\right) = \left(\underline{\mathbb{J}}^{\beta_1\cap\,\beta_2}, \mathbb{G}\right)$;
5) $\left(\overline{\mathbb{J}}^{\beta_1}, \mathbb{G}\right) \cup \left(\overline{\mathbb{J}}^{\beta_2}, \mathbb{G}\right) = \left(\overline{\mathbb{J}}^{\beta_1\cup\beta_2}, \mathbb{G}\right)$;
6) $\left(\underline{\mathbb{J}}^{\beta_1}, \mathbb{G}\right) \cup \left(\underline{\mathbb{J}}^{\beta_2}, \mathbb{G}\right) \subseteq \left(\underline{\mathbb{J}}^{\beta_1\cup\,\beta_2}, \mathbb{G}\right)$;
7) $(\mathbb{J}, \mathbb{G})\subseteq(\mathbb{R}, \mathbb{G})$ implies $\left(\overline{\mathbb{J}}^{\beta_1}, \mathbb{G}\right) \subseteq \left(\overline{\mathbb{R}}^{\beta_1}, \mathbb{G}\right)$;
8) $(\mathbb{J}, \mathbb{G}) \subseteq (\mathbb{R}, \mathbb{G})$ implies $\left(\underline{\mathbb{J}}^{\beta_1}, \mathbb{G}\right) \supseteq \left(\underline{\mathbb{R}}^{\beta_1}, \mathbb{G}\right)$.

Theorem 2 [30]: Let $(\mathbb{J}, \mathbb{G})$ and $(\mathbb{R}, \mathbb{G})$ be two *STBR* from a non-empty set M_1 and M_2 and μ_1, μ_2 be non-empty F_{sst} of M_1. Then for all $u\in\mathbb{G}$, we have:

1) $\mu_1\leq \mu_2 \Rightarrow {}^{\mu_1}\overline{\mathbb{J}}(u)\leq{}^{\mu_2}\overline{\mathbb{J}}(u)$;
2) $\mu_1\leq \mu_2 \Rightarrow {}^{\mu_1}\underline{\mathbb{J}}(u) \leq {}^{\mu_2}\underline{\mathbb{J}}(u)$;
3) $\left({}^{\mu_1}\overline{\mathbb{J}},\mathbb{G}\right) \cap \left({}^{\mu_2}\overline{\mathbb{J}},\mathbb{G}\right) \supseteq \left({}^{\mu_1\cap\mu_2}\overline{\mathbb{J}},\mathbb{G}\right)$;
4) $\left({}^{\mu_1}\underline{\mathbb{J}},\mathbb{G}\right) \cap \left({}^{\mu_2}\underline{\mathbb{J}},\mathbb{G}\right) = \left({}^{\mu_1\cap\,\mu_2}\underline{\mathbb{J}},\mathbb{G}\right)$;
5) $\left({}^{\mu_1}\underline{\mathbb{J}},\mathbb{G}\right) \cup \left({}^{\mu_2}\underline{\mathbb{J}},\mathbb{G}\right) \supseteq \left({}^{\mu_1\cup\,\mu_2}\underline{\mathbb{J}},\mathbb{G}\right)$;
6) $\left({}^{\mu_1}\overline{\mathbb{J}},\mathbb{G}\right) \cup \left({}^{\mu_2}\overline{\mathbb{J}},\mathbb{G}\right) = \left({}^{\mu_1\cup\,\mu_2}\overline{\mathbb{J}},\mathbb{G}\right)$;
7) $(\mathbb{J}, \mathbb{G}) \subseteq (\mathbb{R}, \mathbb{G})$ implies $\left({}^{\mu_1}\overline{\mathbb{J}},\mathbb{G}\right) \subseteq \left({}^{\mu_1}\overline{\mathbb{R}},\mathbb{G}\right)$;
8) $(\mathbb{J}, \mathbb{G})\subseteq(\mathbb{R}, \mathbb{G})$implies $\left({}^{\mu_1}\underline{\mathbb{J}},\mathbb{G}\right) \supseteq \left({}^{\mu_1}\underline{\mathbb{R}},\mathbb{G}\right)$.

Theorem 3 [30]: Let $(\mathbb{J}, \mathbb{G})$ and $(\mathbb{R}, \mathbb{G})$ be *STBR* from M_1 to M_2. If β is a F_{sst} of M_2. Then

$$\left(\overline{(\mathbb{J}\cap\mathbb{R})}^\beta, \mathbb{G}\right) \subseteq \left(\overline{\mathbb{J}}^\beta, \mathbb{G}\right) \cap \left(\overline{\mathbb{R}}^\beta, \mathbb{G}\right);$$

$$(\underline{(\mathbb{J}\cap\mathbb{R})}^\beta, \mathbb{G}) \supseteq \left(\underline{\mathbb{J}}^\beta, \mathbb{G}\right) \cup ({}^\beta\underline{\mathbb{R}}, \mathbb{G}).$$

Proof: The proof is simple and obtained by parts 4 and 5 of Theorem 2.

The equality in the preceding Theorem 3 is disproved by the example that follows.

Example 2: Assume $\mathbb{Q}_1 = \{\mathcal{L}, d, \mathcal{T}\}$ and $\mathbb{Q}_2 = \{\mathcal{L}', j, k, \mathcal{T}'\}$ be two complete lattices shown in Fig. 2(a)

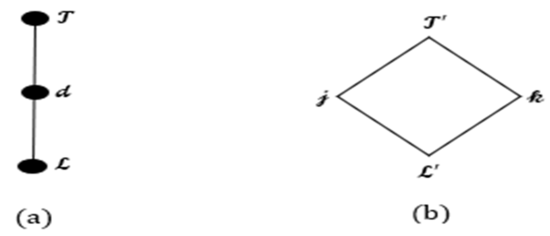

FIGURE 2. (a) Description of $\mathbb{Q}_1$ (b) Description of $\mathbb{Q}_2$

TABLE 5. Left action subject to $\mathfrak{Q}_1$.

$\mathfrak{Q}_1$	$\mathcal{L}$	d	$\mathcal{T}$
$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
d	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
$\mathcal{T}$	$\mathcal{L}$	d	$\mathcal{T}$

TABLE 6. Left action subject to $\mathfrak{Q}_2$.

$\mathfrak{Q}_2$	$\mathcal{L}'$	j	k	$\mathcal{T}'$
$\mathcal{L}'$	$\mathcal{L}'$	$\mathcal{L}'$	$\mathcal{L}'$	$\mathcal{L}'$
j	$\mathcal{L}'$	j	k	$\mathcal{T}'$
k	$\mathcal{L}'$	j	k	$\mathcal{T}'$
$\mathcal{T}'$	$\mathcal{L}'$	j	k	$\mathcal{T}'$

and Fig. 2(b) respectively where $\ominus_1$ and $\ominus_2$ on $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are associative binary operations and defined as :

1) $a \ominus_1 b = a$
2) $a \ominus_2 b = \mathcal{L}'$

By $\ominus_1$ and $\ominus_2$, we have $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are quantales, respectively and M_1 and M_2 are quantale modules by Table 5 and Table 6.

Consider $\mathbb{G} = \{u_1, u_2\}$ and $\mathfrak{J} : \mathbb{G} \longrightarrow P(\mathrm{M}_1 \times M_2)$, $\mathfrak{R} : G \longrightarrow P(\mathrm{M}_1 \times M_2)$ be defined by:

$$\mathfrak{J}(u_1) = \left\{ \begin{matrix} (\mathcal{L},\mathcal{L}'), (d, j), (\mathcal{T}, k), (d,\mathcal{T}') \\ (\mathcal{L},\mathcal{T}'), (\mathcal{L}, k), (d, k), (\mathcal{L}, j) \end{matrix} \right\}$$
$$\mathfrak{R}(u_1) = \left\{ \begin{matrix} (\mathcal{L},\mathcal{L}'), (d, k), (\mathcal{T}, j), (d,\mathcal{L}'), (\mathcal{T},\mathcal{T}') \\ (\mathcal{L}, k), (\mathcal{L}, j), (d, j), (d,\mathcal{T}') \end{matrix} \right\}$$
$$(\mathfrak{J}\cap\mathfrak{R})(u_1) = \left\{ \begin{matrix} (\mathcal{L},\mathcal{L}'), (d, k), (d, j) \\ (d,\mathcal{T}'), (\mathcal{L}, k), (\mathcal{L}, j) \end{matrix} \right\}$$

The aftersets w.r.t $\mathfrak{J}(u_1)$ and $\mathfrak{R}(u_1)$ are as follows:

$$\mathcal{L}\mathfrak{J}(u_1) = \{\mathcal{L}', j, k, \mathcal{T}'\}, d\mathfrak{J}(u_1) = \{j, k, \mathcal{T}'\}$$
$$\text{and} \quad \mathcal{T}\mathfrak{J}(u_1) = \{k\}.$$
$$\mathcal{L}\mathfrak{R}(u_1) = \{\mathcal{L}', j, k\}, d\mathfrak{R}(u_1) = \{\mathcal{L}', j, k, \mathcal{T}'\}$$
$$\text{and} \quad \mathcal{T}\mathfrak{R}(u_1) = \{j, \mathcal{T}'\}.$$

The aftersets w.r.t $(\mathfrak{J}\cap\mathfrak{R})(u_1)$ are as follows :

$$\mathcal{L}(\mathfrak{J}\cap\mathfrak{R})(u_1) = \{\mathcal{L}', j, k\}, d(\mathfrak{J}\cap\mathfrak{R})(u_1) = \{j, k, \mathcal{T}'\}$$

and $\mathcal{T}(\mathfrak{J}\cap\mathfrak{R})(u_1) =$ f.

Define $\beta_1 : \mathrm{M}_2 \rightarrow [0, 1]$ by,

$$\beta_1 = \frac{0.6}{\mathcal{L}'} + \frac{0.7}{j} + \frac{0.5}{k} + \frac{0.3}{\mathcal{T}'}$$

Then β_1 is a F_{sst} of M_2.

$$\overline{\mathfrak{J}}^{\beta_1}(u_1) = \frac{0.7}{\mathcal{L}} + \frac{0.7}{d} + \frac{0.5}{\mathcal{T}}$$
$$\overline{\mathfrak{R}}^{\beta_1}(u_1) = \frac{0.7}{\mathcal{L}} + \frac{0.7}{d} + \frac{0.7}{\mathcal{T}}$$
$$\overline{(\mathfrak{J}\cap\mathfrak{R})}^{\beta_1}(u_1) = \frac{0.7}{\mathcal{L}} + \frac{0.7}{d} + \frac{0}{\mathcal{T}}$$

This shows that $\overline{\mathfrak{J}}^{\beta_1}(u_1) \cap \overline{\mathfrak{R}}^{\beta_1}(u_1) \neq \overline{(\mathfrak{J}\cap\mathfrak{R})}^{\beta_1}(u_1)$.

Now, define $\beta_2 : \mathrm{M}_2 \rightarrow [0, 1]$ by,

$$\beta_2 = \frac{0.2}{\mathcal{L}'} + \frac{0.5}{j} + \frac{0.7}{k} + \frac{1}{\mathcal{T}'}$$

Then β_2 is a F_{sst} of M_2 .

$$\underline{\mathfrak{J}}^{\beta_2}(u_1) = \frac{0.2}{\mathcal{L}} + \frac{0.5}{d} + \frac{0.7}{\mathcal{T}}$$
$$\underline{\mathfrak{R}}^{\beta_2}(u_1) = \frac{0.2}{\mathcal{L}} + \frac{0.2}{d} + \frac{0.5}{\mathcal{T}}$$
$$\underline{(\mathfrak{J}\cap\mathfrak{R})}^{\beta_2}(u_1) = \frac{0.2}{\mathcal{L}} + \frac{0.5}{d} + \frac{0}{\mathcal{T}}$$

This shows that $\underline{\mathfrak{J}}^{\beta_2}(u_1) \cup \underline{\mathfrak{R}}^{\beta_2}(u_1) \neq \underline{(\mathfrak{J}\cap\mathfrak{R})}^{\beta_2}(u_1)$.

Theorem 4 [30]: Let $(\mathfrak{J}, \mathbb{G})$ and $(\mathfrak{R}, \mathbb{G})$ be a *STBR* from a non-empty set M_1 to M_2. If μ is a F_{sst} of M_1 then,

$$\left({}^{\mu}\overline{(\mathfrak{J}\cap\mathfrak{R})}, \mathbb{G}\right) \subseteq \left({}^{\mu}\overline{\mathfrak{J}}, \mathbb{G}\right) \cap \left({}^{\mu}\overline{\mathfrak{R}}, \mathbb{G}\right);$$
$$\left({}^{\mu}\underline{(\mathfrak{J}\cap\mathfrak{R})}, \mathbb{G}\right) \supseteq \left({}^{\mu}\underline{\mathfrak{J}}, \mathbb{G}\right) \cup \left({}^{\mu}\underline{\mathfrak{R}}, \mathbb{G}\right).$$

Proof: The proof is simple and obtained by parts 4 and 5 of Theorem 2.

The equality in the preceding Theorem 4 is disproved by the example that follows.

Example 3: Examine about the quantales from Example 2. Let $\mathbb{G} = \{u_1, u_2\}$ and $\mathfrak{J} : \mathbb{G} \rightarrow P(\mathrm{M}_1 \times \mathrm{M}_2)$, $\mathfrak{R} : \mathbb{G} \rightarrow P(\mathrm{M}_1 \times \mathrm{M}_2)$ be the *STBR* defined by :

$$\mathfrak{J}(u_1) = \left\{ \begin{matrix} (\mathcal{L},\mathcal{L}'), (d, j), (\mathcal{T}, k), (d,\mathcal{T}') \\ (\mathcal{L},\mathcal{T}'), (\mathcal{L}, k), (d, k), (\mathcal{L}, j) \end{matrix} \right\}$$
$$\mathfrak{R}(u_1) = \left\{ \begin{matrix} (\mathcal{L},\mathcal{L}'), (d, k), (\mathcal{T}, j), (d,\mathcal{L}'), (\mathcal{T},\mathcal{T}') \\ (\mathcal{L}, k), (\mathcal{L}, j), (d, j), (d,\mathcal{T}') \end{matrix} \right\}$$
$$(\mathfrak{J}\cap\mathfrak{R})(u_1) = \left\{ \begin{matrix} (\mathcal{L},\mathcal{L}'), (d, k), (d, j) \\ (d,\mathcal{T}'), (\mathcal{L}, k), (\mathcal{L}, j) \end{matrix} \right\}$$

The foresets w.r.t $\mathfrak{J}(u_1)$ and $\mathfrak{R}(u_1)$ are as follows:

$$\mathfrak{J}(u_1)\mathcal{L}' = \{\mathcal{L}\}, \ \mathfrak{J}(u_1)j = \{\mathcal{L}, d\},$$
$$\mathfrak{J}(u_1)k = \{\mathcal{L}, d, \mathcal{T}\}, \ \mathfrak{J}(u_1)\mathcal{T}' = \{\mathcal{L}, d\}.$$
$$\mathfrak{R}(u_1)\mathcal{L}' = \{\mathcal{L}, d\}, \ \mathfrak{R}(u_1)j = \{\mathcal{L}, d, \mathcal{T}\},$$
$$\mathfrak{R}(u_1)k = \{\mathcal{L}, d\}, \ \mathfrak{R}(u_1)\mathcal{T}' = \{d, \mathcal{T}\}.$$

The foresets w.r.t $(\mathfrak{J}\cap\mathfrak{R})(u_1)$ are as follows:

$$(\mathfrak{J}\cap\mathfrak{R})(u_1)\mathcal{L}' = \{\mathcal{L}\}, \ (\mathfrak{J}\cap\mathfrak{R})(u_1)j = \{\mathcal{L}, d\},$$

$$(\mathcal{J}\cap\mathcal{R})(u_1)\hbar = \{\mathcal{L}, d\}, (\mathcal{J}\cap\mathcal{R})(u_1)\mathcal{T}' = \{d\}.$$

Define $\mu_1 : M_1 \to [0, 1]$ by,

$$\mu_1 = \frac{0.4}{\mathcal{L}} + \frac{0.3}{d} + \frac{0.5}{\mathcal{T}}$$

Then μ_1 is a F_{sst} of M_1. But,

$$^{\mu_1}\overline{\mathcal{J}}(u_1) = \frac{0.4}{\mathcal{L}'} + \frac{0.4}{j} + \frac{0.5}{\hbar} + \frac{0.4}{\mathcal{T}'}$$

$$^{\mu_1}\overline{\mathcal{R}}(u_1) = \frac{0.4}{\mathcal{L}'} + \frac{0.5}{j} + \frac{0.4}{\hbar} + \frac{0.5}{\mathcal{T}'}$$

$$^{\mu_1}\overline{(\mathcal{J}\cap\mathcal{R})}(u_1) = \frac{0.4}{\mathcal{L}'} + \frac{0.4}{j} + \frac{0.4}{\hbar} + \frac{0.3}{\mathcal{T}'}$$

This shows that $^{\mu_1}\overline{\mathcal{J}}(u_1)\cap{}^{\mu_1}\overline{\mathcal{R}}(u_1)\neq{}^{\mu_1}\overline{(\mathcal{J}\cap\mathcal{R})}(u_1)$.

Define $\mu_2 : M_1 \to [0, 1]$ by, $\mu_2 = \frac{0.4}{\mathcal{L}} + \frac{0.6}{d} + \frac{0}{\mathcal{T}}$

Then μ_2 is a F_{sst} of M_1. But,

$$^{\mu_2}\underline{\mathcal{J}}(u_1) = \frac{0.4}{\mathcal{L}'} + \frac{0.4}{j} + \frac{0}{\hbar} + \frac{0.4}{\mathcal{T}'}$$

$$^{\mu_2}\underline{\mathcal{R}}(u_1) = \frac{0.4}{\mathcal{L}'} + \frac{0}{j} + \frac{0.4}{\hbar} + \frac{0}{\mathcal{T}'}$$

$$^{\mu_2}\underline{(\mathcal{J}\cap\mathcal{R})}(u_1) = \frac{0.4}{\mathcal{L}'} + \frac{0.4}{j} + \frac{0.4}{\hbar} + \frac{0.6}{\mathcal{T}'}$$

This shows that $^{\mu_2}\underline{\mathcal{J}}(u_1)\cup{}^{\mu_2}\underline{\mathcal{R}}(u_1)\neq{}^{\mu_2}\underline{(\mathcal{J}\cap\mathcal{R})}(u_1)$.

IV. ROUGH FUZZY SUBSTRUCTURES IN QUANTALE MODULE BY SOFT RELATIONS

By using two distinct quantale modules, we are considering STCR in the following section. The fuzzy substructures of quantale module M_2 are taken and approximated by aftersets to produce the fuzzy substructures of M_1. Furthermore, the lower and upper approximation of fuzzy substructures of M_1 by foresets gives fuzzy substructures of M_2.

Definition 16: Let $\mathbb{Q}$ be a quantale and M_1 be $\mathbb{Q}-$ module and η be F_{sst} of M_1. Then η is said to be fuzzy $\mathbb{Q}-$ submodule, if for any $e\in M_1$ and $q\in\mathbb{Q}$, the followin hold :

1) $\eta(\vee_{i\in I}e_i) \geq \wedge_{i\in I}\eta(e_i)$
2) $\eta(q \circledcirc_1 e) \geq \eta(e)$.

Definition 17: Let $\mathbb{Q}$ be a quantale and M_1 be $\mathbb{Q}-$ module and η be F_{sst} of M. η is said to be fuzzy $\mathbb{Q}-$ submodule ideal, if the following conditions hold:

1) $f\leq e \Rightarrow \eta(e) \leq \eta(f)$;
2) $\eta(e\bigvee f) \geq \eta(e) \bigwedge \eta(f)$;
3) $\eta(q\circledcirc_1 e) \geq \eta(e)\ \forall\, e,\ f\in M_1, q\in\mathbb{Q}$.

Definition 18: Let $(\mathcal{J}, \mathbb{G})$ be a *STBR* from M_1 *to* M_2 and β be a non-empty F_{sst} of M_2. Then β is termed as generalized upper rough (GUR) fuzzy quantale sub-module (URF $\mathbb{Q}_{SM}$) of M_1 w.r.t aftersets if $UP_{AP}(\overline{\mathcal{J}}^{\beta}, \mathbb{G})$ is a F $\mathbb{Q}_{SM}$ of M_1.

Definition 19: Let $(\mathcal{J}, \mathbb{G})$ be a *STBR* from M_1 *to* M_2 and β be a non-empty F_{sst} of M_2. Then β is termed as generalized upper rough (GUR) fuzzy left (right) $\mathbb{Q}_{ID}$ (quantale sub-module ideal) of M_1 w.r.t aftersets if $UP_{AP}(\overline{\mathcal{J}}^{\beta}, \mathbb{G})$ is fuzzy left (right) $\mathbb{Q}_{ID}$ of M_1.

Definition 20: Let $(\mathcal{J}, \mathbb{G})$ be a *STBR* from $M_1 to M_2$ and μ be a non-empty F_{sst} of M_1. Then μ is termed as generalized upper rough (GUR) fuzzy quantale sub-module (URF $\mathbb{Q}_{SM}$) of M_2 w.r.t foresets if $UP_{AP}(^{\mu}\overline{\mathcal{J}},\mathbb{G})$ is F $\mathbb{Q}_{SM}$ of M_2.

Definition 21: Let $(\mathcal{J}, \mathbb{G})$ be a *STBR* from M_1 *to* M_2 and μ be a non-empty of F_{sst} M_1. Then μ is termed as generalized upper rough (GUR) fuzzy left (right) $\mathbb{Q}_{ID}$ (quantale sub-module ideal) of M_2 w.r.t foresets if $UP_{AP}(^{\mu}\overline{\mathcal{J}},\mathbb{G})$ is fuzzy left (right) $\mathbb{Q}_{ID}$ of M_2.

Theorem 5: Let $(\mathcal{J}, \mathbb{G})$ be a *STCR* and β be a $F\mathbb{Q}_{SM}$ of M_2. Then β is a GUR $FST\mathbb{Q}_{SM}$ of M_1 w.r.t aftersets.

Proof: As β is a $F\mathbb{Q}_{SM}$ of M_2. So, we have $\beta\left(\bigvee_{i\in I}p_i\right) \geq \bigwedge_{i\in I}\beta(p_i)$ and $\beta(q \circledcirc_2 \hbar) \geq \beta(\hbar)\ \forall q\in\mathbb{Q}$ and $\hbar, p_i \in M_2$. Since $(\mathcal{J}, \mathbb{G})$ be a *STCR* so, we have $g\mathcal{J}(u)\vee\hbar\mathcal{J}(u) \subseteq (g\vee\hbar)\,\mathcal{J}(u)$ for all $u\in\mathbb{G}$ and $g, \hbar\in M_1$.

Let $\hbar_i \in M_1$ for some $i \in I$. Then

$$\begin{aligned}
&\bigwedge_{i\in I}\overline{\mathcal{J}}^{\beta}(u)(\hbar_i)\\
&= \overline{\mathcal{J}}^{\beta}(u)(\hbar_1) \wedge \overline{\mathcal{J}}^{\beta}(u)(\hbar_2), \ldots, \overline{\mathcal{J}}^{\beta}(u)(\hbar_i)\\
&= \left(\bigvee_{a_1\in\hbar_1\mathcal{J}(u)}\beta(a_1)\right) \wedge \left(\bigvee_{a_2\in\hbar_2\mathcal{J}(u)}\beta(a_2)\right)\\
&\quad \wedge, \ldots, \wedge \left(\bigvee_{a_i\in\hbar_i\mathcal{J}(u)}\beta(a_i)\right)\\
&= \bigvee_{a_1\in\hbar_1\mathcal{J}(u),\ldots a_i\in\hbar_i\mathcal{J}(u)}[\beta(a_1)\wedge\beta(a_2)\wedge,..,\wedge\beta(a_i)]\\
&= \bigvee_{a_1\vee a_2\vee,\ldots,\vee a_i\in\hbar_1\mathcal{J}(u)\vee\hbar_2\mathcal{J}(u)\vee,\ldots,\vee\hbar_i\mathcal{J}(u)}\left[\bigwedge_{i\in I}\beta(a_i)\right]\\
&= \bigvee_{\vee_{i\in I}a_i\in(\hbar_1\vee\hbar_2\vee,\ldots,\vee\hbar_i)\mathcal{J}(u)}[\bigwedge_{i\in I}\beta(a_i)]\\
&= \bigvee_{\vee_{i\in I}a_i\in\vee_{i\in I}\hbar_i\mathcal{J}(u)}[\bigwedge_{i\in I}\beta(a_i)]\\
&\leq \bigvee_{\vee_{i\in I}a_i\in\vee_{i\in I}\hbar_i\mathcal{J}(u)}\beta(\bigvee_{i\in I}(a_i))\\
&= \bigvee_{c\in\vee_{i\in I}\hbar_i\mathcal{J}(u)}\beta(c)\\
&= \overline{\mathcal{J}}^{\beta}(u)(\vee_{i\in I}\hbar_i)
\end{aligned}$$

Hence $\overline{\mathcal{J}}^{\beta}(u)(\vee_{i\in I}\hbar_i) \geq \bigwedge_{i\in I}\overline{\mathcal{J}}^{\beta}(u)(\hbar_i)\ \forall\ \hbar_i \in M_1$ and for all $u\in\mathbb{G}$.

As $(\mathcal{J}, \mathbb{G})$ be a *STCR*. So, we have $q\mathcal{J}(u)\circledcirc_2\hbar\mathcal{J}(u)\subseteq(q\circledcirc_1\hbar)\mathcal{J}(u)\ \forall q\in\mathbb{Q},\ \hbar\in M_1$ and for all $u\in\mathbb{G}$.

Consider $q\in\mathbb{Q}, \hbar\in M_1$ and

$$\begin{aligned}
\overline{\mathcal{J}}^{\beta}(u)(m) &= \bigvee_{\hbar\in m\mathcal{J}(u)}\beta(\hbar)\\
&\leq \bigvee_{\hbar\in m\mathcal{J}(u)}\beta(q\circledcirc_2\hbar)\\
&= \bigvee_{q\circledcirc_2\hbar\in q\circledcirc_2 m\mathcal{J}(u)}\beta(q\circledcirc_2\hbar)\\
&\leq \bigvee_{q\circledcirc_2\hbar\in(q\circledcirc_1 m)\mathcal{J}(u)}\beta(q\circledcirc_2\hbar)\\
&= \bigvee_{c\in(q\circledcirc_1 m)\mathcal{J}(u)}\beta(c)\\
&= \overline{\mathcal{J}}^{\beta}(u)(q\circledcirc_1 m)
\end{aligned}$$

Hence $\overline{\mathcal{J}}^{\beta}(u)(q\circledcirc_1 m) \geq \overline{\mathcal{J}}^{\beta}(u)(m)\ \forall\ q\in\mathbb{Q}, m\in M_1$. Thus, $\overline{\mathcal{J}}^{\beta}(u)$ is a $F\mathbb{Q}_{SM}$ of M_1. Consequently β is a GUR $FST\mathbb{Q}_{SM}$ of M_1 w.r.t aftersets.

Theorem 6: Let $(\mathcal{J}, \mathbb{G})$ be a *STCR* and μ be a $F\mathbb{Q}_{SM}$ of M_1. Then μ is a GUR $FST\mathbb{Q}_{SM}$ of M_2 w.r.t foresets.

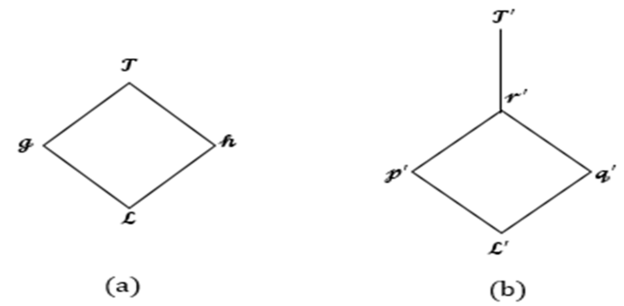

FIGURE 3. (a) Description of $\mathbb{Q}_1$ (b) Description of $\mathbb{Q}_2$.

TABLE 7. Left action subject to $\mathbb{Q}_1$.

$\mathbb{Q}_1$	$\mathcal{L}$	$\mathcal{g}$	$\mathcal{h}$	$\mathcal{T}$
$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
$\mathcal{g}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
$\mathcal{h}$	$\mathcal{L}$	$\mathcal{g}$	$\mathcal{h}$	$\mathcal{T}$
$\mathcal{T}$	$\mathcal{L}$	$\mathcal{g}$	$\mathcal{h}$	$\mathcal{T}$

TABLE 8. Left action subject to $\mathbb{Q}_2$.

$\mathbb{Q}_2$	$\mathcal{L}'$	$\mathcal{p}'$	$\mathcal{q}'$	$\mathcal{r}'$	$\mathcal{T}'$
$\mathcal{L}'$	$\mathcal{L}'$	$\mathcal{p}'$	$\mathcal{q}'$	$\mathcal{r}'$	$\mathcal{T}'$
$\mathcal{p}'$	$\mathcal{L}'$	$\mathcal{p}'$	$\mathcal{q}'$	$\mathcal{r}'$	$\mathcal{T}'$
$\mathcal{q}'$	$\mathcal{L}'$	$\mathcal{p}'$	$\mathcal{q}'$	$\mathcal{r}'$	$\mathcal{T}'$
$\mathcal{r}'$	$\mathcal{L}'$	$\mathcal{p}'$	$\mathcal{q}'$	$\mathcal{r}'$	$\mathcal{T}'$
$\mathcal{T}'$	$\mathcal{L}'$	$\mathcal{p}'$	$\mathcal{q}'$	$\mathcal{r}'$	$\mathcal{T}'$

Proof: The proof is obvious.

Now we consider an Example for our better understanding to show that converse of Theorem 5 and 6 is not true.

Example 4: Assume $\mathbb{Q}_1 = \{\mathcal{L}, \mathcal{g}, \mathcal{h}, \mathcal{T}\}$ and $\mathbb{Q}_2 = \{\mathcal{L}', \mathcal{p}', \mathcal{q}', \mathbb{O}', \mathcal{T}'\}$ be two complete lattices as shown in Fig. 3(a) and Fig. 3(b), respectively. Then $\odot_1$ and $\odot_2$ on $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are associative binary operations defined as :

1) $a \odot_1 b = a \wedge b$
2) $a \odot_2 b = \mathcal{L}'$

Then $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are quantales by $\odot_1$ and $\odot_2$, respectively and M_1 and M_2 are quantale modules by Table 5 and Table 6.

Consider $G = \{u_1, u_2\}$ and $\mathbb{J} : G \longrightarrow P(\mathrm{M}_1 \times M_2)$ be defined by:

$$\mathbb{J}(u_1) = \left\{ \begin{array}{c} (\mathcal{L},\mathcal{L}'), (\mathcal{g},\mathcal{p}'), (\mathcal{h},\mathcal{q}'), (\mathcal{L},\mathcal{p}'), (\mathcal{L},\mathcal{q}') \\ (\mathcal{L},\mathcal{r}'), (\mathcal{T},\mathcal{T}'), (\mathcal{L},\mathcal{T}'), (\mathcal{h},\mathcal{T}') \\ (\mathcal{g},\mathcal{q}'), (\mathcal{T},\mathcal{L}'), (\mathcal{h},\mathcal{L}'), (\mathcal{g},\mathcal{L}') \end{array} \right\}$$

$$\mathbb{J}(u_2) = \left\{ \begin{array}{c} (\mathcal{T},\mathcal{L}'), (\mathcal{g},\mathcal{p}'), (\mathcal{h},\mathcal{q}'), (\mathcal{T},\mathcal{r}'), (\mathcal{h},\mathcal{r}') \\ (\mathcal{g},\mathcal{r}'), (\mathcal{L},\mathcal{p}'), (\mathcal{L},\mathcal{q}'), (\mathcal{L},\mathcal{r}') \end{array} \right\}$$

Then $(\mathbb{J}, \mathbb{G})$ is a *STCR*. Aftersets w.r.t $\mathbb{J}(u_1)$ and $\mathbb{J}u_2)$ are as follows :

$$\mathcal{L}\mathbb{J}(u_1) = \{\mathcal{L}', \mathcal{p}', \mathcal{q}', \mathcal{r}', \mathcal{T}'\}, \ \mathcal{g}\mathbb{J}(u_1) = \{\mathcal{L}', \mathcal{p}', \mathcal{q}'\},$$
$$\mathcal{h}\mathbb{J}(u_1) = \{\mathcal{L}', \mathcal{q}', \mathcal{T}'\}, \ \mathcal{T}\mathbb{J}(u_1) = \{\mathcal{L}', \mathcal{T}'\}$$
$$\mathcal{L}\mathbb{J}(u_2) = \{\mathcal{p}', \mathcal{q}', \mathcal{r}'\}, \ \mathcal{g}\mathbb{J}(u_2) = \{\mathcal{p}', \mathcal{r}'\}$$
$$\mathcal{h}\mathbb{J}(u_2) = \{\mathcal{q}', \mathcal{r}'\}, \mathcal{T}\mathbb{J}(u_2) = \{\mathcal{L}', \mathcal{r}'\}.$$

Let $\beta : \mathrm{M}_2 \to [0, 1]$ be defined by,

$$\beta = \frac{1}{\mathcal{L}'} + \frac{0.5}{\mathcal{p}'} + \frac{0.4}{\mathcal{q}'} + \frac{0.3}{\mathcal{r}'} + \frac{0.2}{\mathcal{T}'}$$

Then β is not a $F\mathbb{Q}_{SM}$ of M_2 . However,

$$\overline{\mathbb{J}}^{\beta}(u_1) = \frac{1}{\mathcal{L}} + \frac{1}{\mathcal{g}} + \frac{1}{\mathcal{h}} + \frac{1}{\mathcal{T}}$$
$$\overline{\mathbb{J}}^{\beta}(u_2) = \frac{0.5}{\mathcal{L}} + \frac{0.5}{\mathcal{g}} + \frac{0.4}{\mathcal{h}} + \frac{1}{\mathcal{T}}$$

This shows that $\overline{\mathbb{J}}^{\beta}(u_1)$ and $\overline{\mathbb{J}}^{\beta}(u_2)$ are $F\mathbb{Q}_{SM}$ of M_1 but β is not a $F\mathbb{Q}_{SM}$ of M_2.

Hence β isgeneralized upper rough $FST\mathbb{Q}_{SM}$ of M_1 w.r.t aftersets.

Foresets w.r.t $\mathbb{J}(u_1)$ and $\mathbb{J}(u_2)$ are as follows:

$\mathbb{J}(u_1)\mathcal{L}' = \{\mathcal{L}, \mathcal{g}, \mathcal{h}, \mathcal{T}\}$, $\mathbb{J}(u_1)\mathcal{p}' = \{\mathcal{L}, \mathcal{g}\}$, $\mathbb{J}(u_1)\mathcal{q}' = \{\mathcal{L}, \mathcal{g}, \mathcal{h}\}$, $\mathbb{J}(u_1)\mathbb{O}' = \{\mathcal{L}\}$, $\mathbb{J}(u_1)\mathcal{T}' = \{\mathcal{L}, \mathcal{h}, \mathcal{T}\}$

$\mathbb{J}(u_2)\mathcal{L}' = \{\mathcal{T}\}$, $\mathbb{J}(u_2)\mathcal{p}' = \{\mathcal{L}, \mathcal{g}\}$, $\mathbb{J}(u_2)\mathcal{q}' = \{\mathcal{L}, \mathcal{h}\}$, $\mathbb{J}(u_2)\mathbb{O}' = \{\mathcal{L}, \mathcal{g}, \mathcal{h}, \mathcal{T}\}$, $\mathbb{J}(u_2)\mathcal{T}' = \text{f}$.

Let $\mu : \mathrm{M}_1 \to [0, 1]$ be defined by,

$$\mu = \frac{1}{\mathcal{L}} + \frac{0.5}{\mathcal{g}} + \frac{0.7}{\mathcal{h}} + \frac{0.3}{\mathcal{T}}$$

Then μ is not a $F\mathbb{Q}_{SM}$ of M_1. However,

$${}^{\mu}\overline{\mathbb{J}}(u_1) = \frac{1}{\mathcal{L}'} + \frac{1}{\mathcal{p}'} + \frac{1}{\mathcal{q}'} + \frac{1}{\mathcal{r}'} + \frac{1}{\mathcal{T}'}$$
$${}^{\mu}\overline{\mathbb{J}}(u_2) = \frac{0.3}{\mathcal{L}'} + \frac{1}{\mathcal{p}'} + \frac{1}{\mathcal{q}'} + \frac{1}{\mathcal{r}'} + \frac{0}{\mathcal{T}'}$$

This shows that ${}^{\mu}\overline{\mathbb{J}}(u_1)$ and ${}^{\mu}\overline{\mathbb{J}}(u_2)$ are $F\mathbb{Q}_{SM}$ of M_2 but μ is not a $F\mathbb{Q}_{SM}$ of M_1.

Hence μ is GUR, $FST\mathbb{Q}_{SM}$ of M_2 w.r.t foresets.

Definition 22: Let $(\mathbb{J}, \mathbb{G})$ be a *STBR* from M_1 *to* M_2 and β be a non-empty F_{sst} of M_2. Then β is termed as generalized lower rough fuzzy quantale sub-module (LRF $\mathbb{Q}_{SM}$) of M_1 w.r.t aftersets if $LO_{AP}(\underline{\mathbb{J}}^{\beta}, \mathbb{G})$ is a F $\mathbb{Q}_{SM}$ of M_1.

Definition 23: Let $(\mathbb{J}, \mathbb{G})$ be a *STBR* from M_1 *to* M_2 and β be a non-empty F_{sst} of M_2. Then β is termed as generalized lower rough (GLR) fuzzy left (right) $\mathbb{Q}_{ID}$ (quantale sub-module ideal) of M_1 w.r.t aftersets if $LO_{AP}(\underline{\mathbb{J}}^{\beta}, \mathbb{G})$ is fuzzy left (right) $\mathbb{Q}_{ID}$ of M_1.

Definition 24: Let $(\mathbb{J}, \mathbb{G})$ be a *STBR* from M_1 *to* M_2 and μ be a non-empty F_{sst} of M_1. Then μ is termed as generalized lower rough (GLR) fuzzy quantale sub-module (LRF $\mathbb{Q}_{SM}$) of M_2 w.r.t foresets if $LO_{AP}({}^{\mu}\underline{\mathbb{J}}, \mathbb{G})$ is F $\mathbb{Q}_{SM}$ of M_2.

Definition 25: Let $(\mathbb{J}, \mathbb{G})$ be a *STBR* from M_1 *to* M_2 and μ be a non-empty of F_{sst} M_1. Then μ is termed as generalized lower rough (GLR) fuzzy left (right) $\mathbb{Q}_{ID}$ (quantale sub-module ideal) of M_2 w.r.t foresets if $LO_{AP}({}^{\mu}\underline{\mathbb{J}}, \mathbb{G})$ is fuzzy left (right) $\mathbb{Q}_{ID}$ of M_2.

Theorem 7: Let $(\mathbb{J}, \mathbb{G})$ be a *STCMR* and β be a $F\mathbb{Q}_{SM}$ of M_2. Then β is a GLR $FST\mathbb{Q}_{SM}$ of M_1 w.r.t aftersets.

Proof: As β is a $F\mathbb{Q}_{SM}$ of M_2. So, we have $\beta\left(\bigvee_{i\in I}\mathscr{p}_i\right) \geq \bigwedge_{i\in I}\beta(\mathscr{p}_i)$ and $\beta(\mathscr{q}\odot_2\mathscr{h}) \geq \beta(\mathscr{h})\ \forall \mathscr{q}\in\mathbb{Q}$ *and* $\mathscr{h}, \mathscr{p}_i \in M_2$. Since $(\mathcal{J}, \mathbb{G})$ be a *STCMR*. So, we have $\mathscr{g}\mathcal{J}(u)\vee\mathscr{h}\mathcal{J}(u) = (\mathscr{g}\vee\mathscr{h})\mathcal{J}(u)$ for all $u\in\mathbb{G}$ and $g, \mathscr{h}\in M_1$. Let $\mathscr{w}_i \in M_1$ for some, $i \in I$. Consider

$$\underline{\mathcal{J}}^{\beta}(u)\left(\bigvee_{i\in I}\mathscr{w}_i\right) = \bigwedge_{e\in(\vee_{i\in I}\mathscr{w}_i)\mathcal{J}(u)}\beta(e) = \bigwedge_{e\in\mathscr{w}_1\mathcal{J}(u)\vee\mathscr{w}_2\mathcal{J}(u)\vee,\ldots,\vee\mathscr{w}_i\mathcal{J}(u)}\beta(e)$$

Since $e\in\mathscr{w}_1\mathcal{J}(u)\vee\mathscr{w}_2\mathcal{J}(u)\vee,\ldots,\vee\mathscr{w}_i\mathcal{J}(u)$ so there have $e_1 \in \mathscr{w}_1\mathcal{J}(u), e_2 \in \mathscr{w}_2\mathcal{J}(u), \ldots, e_i \in \mathscr{w}_i\mathcal{J}(u)$ such that $e = \bigvee_{i\in I} e_i$. Hence,

$$\begin{aligned}
&\underline{\mathcal{J}}^{\beta}(u)\left(\bigvee_{i\in I}\mathscr{w}_i\right)\\
&= \bigvee_{i\in I} e_i \in \mathscr{w}_1\mathcal{J}(u)\vee,\ldots,\vee\mathscr{w}_i\mathcal{J}(u)^{\wedge}\beta\left(\bigvee_{i\in I}e_i\right)\\
&\geq \bigvee_{i\in I} e_i \in \mathscr{w}_1\mathcal{J}(u)\vee\mathscr{w}_2\mathcal{J}(u)\vee,\ldots,\vee\mathscr{w}_i\mathcal{J}(u)^{\wedge}\bigwedge_{i\in I}\beta(e_i)\\
&= \bigwedge_{e_1\in\mathscr{w}_1\mathcal{J}(u),\ldots,e_i\in\mathscr{w}_i\mathcal{J}(u)}[\beta(e_1)\wedge\beta(e_2)\wedge,\ldots,\wedge\beta(e_i)]\\
&= \left(\bigwedge_{e_1\in\mathscr{w}_1\mathcal{J}(u)}\beta(e_1)\right)\wedge\left(\bigwedge_{e_2\in\mathscr{w}_2\mathcal{J}(u)}\beta(e_2)\right)\\
&\quad\wedge,\ldots,\wedge\left(\bigwedge_{e_i\in\mathscr{w}_i\mathcal{J}(u)}\beta(e_i)\right)\\
&= \underline{\mathcal{J}}^{\beta}(u)(\mathscr{w}_1)\wedge\underline{\mathcal{J}}^{\beta}(u)(\mathscr{w}_2)\wedge,\ldots,\wedge\underline{\mathcal{J}}^{\beta}(u)(\mathscr{w}_i)\\
&= \bigwedge_{i\in I}\underline{\mathcal{J}}^{\beta}(u)(\mathscr{w}_i)
\end{aligned}$$

Hence $\underline{\mathcal{J}}^{\beta}(u)(\vee_{i\in I}\mathscr{w}_i) \geq \bigwedge_{i\in I}\underline{\mathcal{J}}^{\beta}(u)(\mathscr{w}_i)\ \forall\ \mathscr{w}_i \in M_1$ and for all $u\in\mathbb{G}$. As $(\mathcal{J}, \mathbb{G})$ be a *STCMR*. So, we have $\mathscr{q}\odot_2\mathscr{w}\mathcal{J}(u) = (\mathscr{q}\odot_1\mathscr{w})\mathcal{J}(u)\ \forall\mathscr{q}\in\mathbb{Q},\ \mathscr{w}\in M_1$ and for all $u\in\mathbb{G}$.

Consider $\mathscr{q}\in\mathbb{Q}$, $\mathscr{n}\in M_1$ and

$$\underline{\mathcal{J}}^{\beta}(u)(\mathscr{q}\odot_1\mathscr{n}) = \bigwedge_{\mathscr{w}\in(\mathscr{q}\odot_1\mathscr{n})\mathcal{J}(u)}\beta(\mathscr{w}) = \bigwedge_{\mathscr{w}\in\mathscr{q}\odot_2\mathscr{n}\mathcal{J}(u)}\beta(\mathscr{w})$$

As $w\in\mathscr{q}\odot_2\mathscr{n}\mathcal{J}(u)$, so there have $\mathscr{k}\in\mathscr{n}\mathcal{J}(u)$ such that $\mathscr{w} = \mathscr{q}\odot_2\mathscr{k}$.

$$\begin{aligned}
\underline{\mathcal{J}}^{\beta}(u)(\mathscr{q}\odot_1\mathscr{n}) &= \bigwedge_{\mathscr{q}\odot_2\mathscr{k}\in\mathscr{q}\odot_2\mathscr{n}\mathcal{J}(u)}\beta(\mathscr{q}\odot_2\mathscr{k})\\
&\geq \mathscr{q}\odot_2\mathscr{k}\in\mathscr{q}\odot_2\mathscr{n}\mathcal{J}(u)^{\wedge}\beta(\mathscr{k})\\
&= \bigwedge_{\mathscr{k}\in\mathscr{n}\mathcal{J}(u)}\beta(\mathscr{k}) = \underline{\mathcal{J}}^{\beta}(u)(\mathscr{n})
\end{aligned}$$

Hence, $\underline{\mathcal{J}}^{\beta}(u)(\mathscr{q}\odot_1\mathscr{n}) \geq \underline{\mathcal{J}}^{\beta}(u)(\mathscr{n})\ \forall\mathscr{q}\in\mathbb{Q},\mathscr{n}\in M_1$. Thus, $\underline{\mathcal{J}}^{\beta}(u)$ is a $F\mathbb{Q}_{SM}$ of M_1. Consequently, β is a GLR $FST\mathbb{Q}_{SM}$ of M_1 w.r.t aftersets.

Theorem 8: Let $(\mathcal{J}, \mathbb{G})$ be a *STCMR* and μ be a $F\mathbb{Q}_{SM}$ of M_1. Then μ is a GLR, $FST\mathbb{Q}_{SM}$ of M_2 w.r.t foresets.

Proof: The proof is obvious.

Now we consider an Example for our better understanding to show that converse of Theorem 7 and 8 are not true.

Example 5: Consider the quantale modules in Example IV.9. Let $\mathbb{G} = \{u_1, u_2\}$ and $\mathcal{J} : \mathbb{G} \to P(M_1 \times M_2)$ be defined by,

$$\mathcal{J}(u_1) = \left\{\begin{array}{c}(\mathcal{L},\mathcal{L}'),(\mathcal{L},\mathscr{p}'),(\mathcal{L},\mathscr{q}'),(\mathcal{L},\mathscr{r}'),(\mathscr{g},\mathcal{L}'),\\(\mathscr{g},\mathscr{p}'),(\mathscr{g},\mathscr{q}'),(\mathscr{g},\mathscr{r}')(\mathscr{h},\mathcal{L}'),(\mathscr{h},\mathscr{p}'),\\(\mathscr{h},\mathscr{q}'),(\mathscr{h},\mathscr{r}'),(\mathcal{T},\mathcal{L}'),(\mathcal{T},\mathscr{p}'),(\mathcal{T},\mathscr{q}'),\\(\mathcal{T},\mathscr{r}')\end{array}\right\}$$

$$\mathcal{J}(u_2) = \left\{\begin{array}{c}(\mathcal{L},\mathcal{L}'),(\mathcal{L},\mathscr{q}'),(\mathcal{L},\mathscr{r}'),(\mathscr{g},\mathcal{L}'),\\(\mathscr{g},\mathscr{q}'),(\mathscr{g},\mathscr{r}'),(\mathscr{h},\mathcal{L}'),(\mathscr{h},\mathscr{q}'),\\(\mathscr{h},\mathscr{r}'),(\mathcal{T},\mathcal{L}'),(\mathcal{T},\mathscr{q}'),(\mathcal{T},\mathscr{r}')\end{array}\right\}$$

Now, aftersets w.r.t $\mathcal{J}(u_1)$ and $\mathcal{J}(u_2)$ are given below;

$$\mathcal{L}\mathcal{J}(u_1) = \{\mathcal{L}', \mathscr{p}', \mathscr{q}', \mathscr{r}'\},\ \mathscr{g}\mathcal{J}(u_1) = \{\mathcal{L}', \mathscr{p}', \mathscr{q}', \mathscr{r}'\}$$
$$\mathscr{h}\mathcal{J}(u_1) = \{\mathcal{L}', \mathscr{p}', \mathscr{q}', \mathscr{r}'\},\ \mathcal{T}\mathcal{J}(u_1) = \{\mathcal{L}', \mathscr{p}', \mathscr{q}', \mathscr{r}'\}$$
$$\mathcal{L}\mathcal{J}(u_2) = \{\mathcal{L}', \mathscr{q}', \mathscr{r}'\},\ \mathscr{g}\mathcal{J}(u_2) = \{\mathcal{L}', \mathscr{q}', \mathscr{r}'\}$$
$$\mathscr{h}\mathcal{J}(u_2) = \{\mathcal{L}', \mathscr{q}', \mathscr{r}'\},\ \mathcal{T}\mathcal{J}(u_2) = \{\mathcal{L}', \mathscr{q}', \mathscr{r}'\}.$$

Then $(\mathcal{J}, \mathbb{G})$ is a *STCMR* w.r.t aftersets.

Define $\beta : M_2 \to [0, 1]$ by,

$$\beta = \frac{1}{\mathcal{L}'} + \frac{0.7}{\mathscr{p}'} + \frac{0.5}{\mathscr{q}'} + \frac{0.3}{\mathscr{r}'} + \frac{0.8}{\mathcal{T}'}$$

Then β is not a $F\mathbb{Q}_{SM}$ of M_2. But

$$\underline{\mathcal{J}}^{\beta}(u_1) = \frac{0.3}{\mathcal{L}} + \frac{0.3}{\mathscr{g}} + \frac{0.3}{\mathscr{h}} + \frac{0.3}{\mathcal{T}}$$
$$\underline{\mathcal{J}}^{\beta}(u_2) = \frac{0.3}{\mathcal{L}} + \frac{0.3}{\mathscr{g}} + \frac{0.3}{\mathscr{h}} + \frac{0.3}{\mathcal{T}}$$

This shows that $\underline{\mathcal{J}}^{\beta}(u_1)$ and $\underline{\mathcal{J}}^{\beta}(u_2)$ are $F\mathbb{Q}_{SM}$ of M_1.

Hence β is generalized lower rough $FST\mathbb{Q}_{SM}$ of M_1 w.r.t aftersets.

Now define $\mathcal{J} : \mathbb{G} \to P(M_1 \times M_2)$. Then

$$\mathcal{J}(u_1) = \left\{\begin{array}{c}(\mathcal{L},\mathcal{L}'),(\mathcal{L},\mathscr{p}'),(\mathcal{L},\mathscr{q}'),(\mathcal{L},\mathscr{r}'),(\mathcal{L},\mathcal{T}')\\(\mathscr{g},\mathcal{L}'),(\mathscr{g},\mathscr{p}'),(\mathscr{g},\mathscr{q}'),(\mathscr{g},\mathscr{r}'),(\mathscr{g},\mathcal{T}')\end{array}\right\}$$
$$\mathcal{J}(u_2) = \left\{\begin{array}{c}(\mathcal{L},\mathcal{L}'),(\mathcal{L},\mathscr{p}'),(\mathcal{L},\mathscr{q}'),(\mathcal{L},\mathscr{r}'),(\mathcal{L},\mathcal{T}')\\(\mathscr{h},\mathcal{L}'),(\mathscr{h},\mathscr{p}'),(\mathscr{h},\mathscr{q}'),(\mathscr{h},\mathscr{r}'),(\mathscr{h},\mathcal{T}')\end{array}\right\}$$

Now, foresets w.r.t $\Gamma(u_1)$ and $\Gamma(u_2)$ are as follows : $\mathcal{J}(u_1)\mathcal{L}' = \{\mathcal{L},\mathscr{g}\}$, $\mathcal{J}(u_1)\mathscr{p}' = \{\mathcal{L},\mathscr{g}\}$, $\mathcal{J}(u_1)\mathscr{q}' = \{\mathcal{L},\mathscr{g}\}$, $\mathcal{J}(u_1)\mathscr{r}' = \{\mathcal{L},\mathscr{g}\}$, $\mathcal{J}(u_1)\mathcal{T}' = \{\mathcal{L},\mathscr{g}\}$

$\mathcal{J}(u_2)\mathcal{L}' = \{\mathcal{L},\mathscr{h}\}$, $\mathcal{J}(u_2)\mathscr{p}' = \{\mathcal{L},\mathscr{h}\}$, $\mathcal{J}(u_2)\mathscr{q}' = \{\mathcal{L},\mathscr{h}\}$, $\mathcal{J}(u_2)\mathscr{r}' = \{\mathcal{L},\mathscr{h}\}$, $\mathcal{J}(u_2)\mathcal{T}' = \{\mathcal{L},\mathscr{h}\}$.

Then $(\mathcal{J}, \mathbb{G})$ is a *STCMR* wr.t foresets.

Define $\mu : M_1 \to [0, 1]$ by,

$$\mu = \frac{1}{\mathcal{L}} + \frac{0.4}{\mathscr{g}} + \frac{0.5}{\mathscr{h}} + \frac{0.3}{\mathcal{T}}$$

Then μ is not a $F\mathbb{Q}_{SM}$ of M_1. But

$${}^{\mu}\underline{\mathcal{J}}(u_1) = \frac{0.4}{\mathcal{L}'} + \frac{0.4}{\mathscr{p}'} + \frac{0.4}{\mathscr{q}'} + \frac{0.4}{\mathscr{r}'} + \frac{0.4}{\mathcal{T}'}$$
$${}^{\mu}\underline{\mathcal{J}}(u_2) = \frac{0.5}{\mathcal{L}'} + \frac{0.5}{\mathscr{p}'} + \frac{0.5}{\mathscr{q}'} + \frac{0.5}{\mathscr{r}'} + \frac{0.5}{\mathcal{T}'}$$

Thus ${}^{\mu}\underline{\mathcal{J}}(u_1)$ and ${}^{\mu}\underline{\mathcal{J}}(u_2)$ are $F\mathbb{Q}_{SM}$ of M_2. Hence, μ is GLR, $FST\mathbb{Q}_{SM}$ of M_2 w.r.t foresets.

Proposition 1: Let $(J, \mathbb{G})$ be a *STCR*. Let β be a F_{sst} of M_2. Then for each $\alpha \in [0, 1]$, the following hold:

$$(\overline{J}^{\beta}(u))_{\alpha} = \overline{J}^{\beta_{\alpha}}(u);$$
$$(\underline{J}^{\beta}(u))_{\alpha} = \underline{J}^{\beta_{\alpha}}(u);$$

Proof: 1. Let $m \in (\overline{J}^{\beta}(u))_{\alpha} \iff \overline{J}^{\beta}(u)(m) \geq \alpha \iff \bigvee_{e\,\in mJ(u)} \beta(e) \geq \alpha \iff \beta(e) \geq \alpha$ for some $e \in mJ(u) \iff mJ(u) \cap \beta_{\alpha} \neq \emptyset \iff m \in \overline{J}^{\beta_{\alpha}}(u)$.

2. Let $m \in (\underline{J}^{\beta}(u))_{\alpha} \iff \underline{J}^{\beta}(u)(m) \geq \alpha \iff \bigwedge_{e\,\in mJ(u)} \beta(a) \geq \alpha \iff \beta(e) \geq \alpha$ for all $e \in mJ(u) \iff mJ(u) \subseteq \beta_{\alpha} \iff m \in \underline{J}^{\beta_{\alpha}}(u)$.

Remark 1: The Proposition 1 also holds w.r.t foreset.

Theorem 9: Let β be a $F\mathbb{Q}_{SM}$ of M_2 and $(J, \mathbb{G})$ be a *STCMR*. Then $\underline{J}^{\beta}(u)$, $[\overline{J}^{\beta}(u)]$ is a $FST\mathbb{Q}_{SM}$ of M_1 w.r.t aftersets if and only if for each $\alpha \in [0, 1]$, $\underline{J}^{\beta_{\alpha}}(u)$, $[\overline{J}^{\beta_{\alpha}}(u)$ where $\beta_{\alpha} \neq \phi]$, is a $F\mathbb{Q}_{SM}$ of M_1 for all $u \in \mathbb{G}$.

Proof: 1. Let $\underline{J}^{\beta}(u)$ is a $FST\mathbb{Q}_{SM}$ of M_1 and $\rho_i \in \underline{J}^{\beta_{\alpha}}(u)$ for some $i \in I$. Then $\underline{J}^{\beta}(u)(\rho_i) \geq \alpha\ \forall i \in I$. But $\underline{J}^{\beta}(u)$ is a $FST\mathbb{Q}_{SM}$. So, we have $\underline{J}^{\beta}(u)\left(\bigvee_{i\in I} \rho_i\right) \geq \bigwedge_{i\in I} \underline{J}^{\beta}(u)(\rho_i) \geq \alpha$ this implies that $\underline{J}^{\beta}(u)\left(\bigvee_{i\in I} \rho_i\right) \geq \alpha$. Consequently, $\bigvee_{i\in I} \rho_i \in \underline{J}^{\beta_{\alpha}}(u)$.

2. Let $\rho \in \underline{J}^{\beta_{\alpha}}(u)$ and $q \in \mathbb{Q}$, then $\underline{J}^{\beta}(u)(\rho) \geq \alpha$. Since $\underline{J}^{\beta}(u)$ is a $FST\mathbb{Q}_{SM}$. So, we have $\underline{J}^{\beta}(u)(q\circledcirc_1 \rho) \geq \underline{J}^{\beta}(u)(\rho) \geq \alpha \implies \underline{J}^{\beta}(u)(q\circledcirc_1 \rho) \geq \alpha$. Consequently, $q\circledcirc_1 \rho \in \underline{J}^{\beta_{\alpha}}(u)$. Hence, $\underline{J}^{\beta_{\alpha}}(u)$ is a $F\mathbb{Q}_{SM}$ of M_1 for all $u \in \mathbb{G}$.

Converse part is obvious.

Theorem 10: Let $(J, \mathbb{G})$ be a *STCMR* and β be a $F\mathbb{Q}_{ID}$ of M_2. Then β is a GUR, $FST\mathbb{Q}_{ID}$ of M_1 w.r.t aftersets.

Proof: 1. As β is a $F\mathbb{Q}_{ID}$ of M_2. So, we have $\beta\left(e \vee f\right) = \beta(e) \wedge \beta\left(f\right)$ and $\beta(q \circledcirc_2 e) \geq \beta(e)\ \forall q \in \mathbb{Q}, e, f \in M_2$. Since $(J, \mathbb{G})$ be a *STCMR*. So, we have $mJ(u) \vee nJ(u) = (m \vee n) J(u)$ for all $u \in \mathbb{G}$ and $m, n \in M_1$. Consider,

$$\overline{J}^{\beta}(u)(m \vee n) = \bigvee_{d\,\in (m\vee n)J(u)} \beta(d)$$
$$= \bigvee_{d\,\in mJ(u)\vee nJ(u)} \beta(d)$$

Since $d \in mJ(u) \vee nJ(u)$, so there is $e \in mJ(u)$ *and* $f \in nJ(u)$ such that $d = e \vee f$. Consequently,

$$\overline{J}^{\beta}(u)(m \vee n) = \bigvee_{e\vee f \in mJ(u)\vee nJ(u)} \beta(e \vee f)$$
$$= \bigvee_{e\vee f \in mJ(u)\vee nJ(u)} \left[\beta(e) \wedge \beta(f)\right]$$
$$= \left[\bigvee_{e\in mJ(u)} \beta(e)\right] \wedge \left[\bigvee_{f\in nJ(u)} \beta(f)\right]$$
$$= \overline{J}^{\beta}(u)(m) \wedge \overline{J}^{\beta}(u)(n)$$

Hence $\overline{J}^{\beta}(u)(m \vee n) = \overline{J}^{\beta}(u)(m) \wedge \overline{J}^{\beta}(u)(n)\ \forall\, m, n \in M_1$ *and* $\forall\, u \in \mathbb{G}$.

2. As $(J, \mathbb{G})$ be a *STCMR*. So, we have $J(u) \circledcirc_2 mJ(u) = (q \circledcirc_1 m) J(u)\ \forall q \in \mathbb{Q},\ m \in M_1$ and $\forall\, u \in \mathbb{G}$.

Consider $q \in \mathbb{Q}, m \in M_1$ and

$$\overline{J}^{\beta}(u)(m) = \bigvee_{d\in mJ(u)} \beta(d)$$
$$\leq \bigvee_{d\in mJ(u)} \beta(q \circledcirc_2 d)$$
$$= \bigvee_{q\circledcirc_2 d\in q\circledcirc_2 mJ(u)} \beta(q\circledcirc_2 d)$$
$$= \bigvee_{q\circledcirc_2 d\in (q\circledcirc_1 m)J(u)} \beta(q\circledcirc_2 d)$$
$$= \bigvee_{c\in (q\circledcirc_1 m)J(u)} \beta(c) = \overline{J}^{\beta}(u)(q\circledcirc_1 m)$$

Hence $\overline{J}^{\beta}(u)(q \circledcirc_1 m) \geq \overline{J}^{\beta}(u)(m)\ \forall q \in \mathbb{Q}, m \in M_1$. Thus, $\overline{J}^{\beta}(u)$ is a $F\mathbb{Q}_{ID}$ of M_1. Consequently, β is a GUR, $FST\mathbb{Q}_{ID}$ of M_1 w.r.t aftersets.

Theorem 11: Let $(J, \mathbb{G})$ be a *STCMR* and μ be a $F\mathbb{Q}_{ID}$ of M_1. Then μ is a GUR, $FST\mathbb{Q}_{ID}$ of M_2 w.r.t foresets.

Proof: The proof is obvious.

Now we consider an Example for our better understanding to show that converse of Theorem 10 and 11 is not true.

Example 6: Consider the quantale modules in Example III.7 Let $\mathbb{G} = \{u_1, u_2\}$ and $J: \mathbb{G} \to P(M_1 \times M_2)$ be defined by :

$$J(u_1) = \left\{\begin{array}{c} (\mathcal{L}, \mathcal{L}'), (\mathcal{L}, j), (\mathcal{L}, k), (\mathcal{L}, \mathcal{T}'), \\ (d, \mathcal{L}'), (d, j), (d, k), (d, \mathcal{T}'), \\ (\mathcal{T}, \mathcal{L}'), (\mathcal{T}, j), (\mathcal{T}, k)(\mathcal{T}, \mathcal{T}') \end{array}\right\}$$
$$J(u_2) = \left\{\begin{array}{c} (\mathcal{L}, \mathcal{L}'), (\mathcal{L}, k), (\mathcal{L}, \mathcal{T}'), (d, \mathcal{L}'), (d, k) \\ (d, \mathcal{T}'), (\mathcal{T}, \mathcal{L}'), (\mathcal{T}, k), (\mathcal{T}, \mathcal{T}') \end{array}\right\}$$

Now, the aftersets in terms of $J(u_1)$ and $J(u_2)$ are as follows;

$$\mathcal{L}J(u_1) = \{\mathcal{L}', j, k, \mathcal{T}'\},\ J(u_1) = \{\mathcal{L}', j, k, \mathcal{T}'\}$$
$$\mathcal{T}J(u_1) = \{\mathcal{L}', j, k, \mathcal{T}'\}$$
$$\mathcal{L}J(u_2) = \{\mathcal{L}', k, \mathcal{T}'\},\ J(u_2) = \{\mathcal{L}', k, \mathcal{T}'\}$$
$$\mathcal{T}J(u_2) = \{\mathcal{L}', k, \mathcal{T}'\}.$$

Then $(J, \mathbb{G})$ is a *STCMR* w.r.t aftersets.

Let $\beta : M_2 \to [0, 1]$ be defined by,

$$\beta = \frac{0.4}{\mathcal{L}'} + \frac{0.7}{j} + \frac{0.6}{k} + \frac{0.2}{\mathcal{T}'}$$

Then β is not a $F\mathbb{Q}_{ID}$ of M_2 .But

$$\overline{J}^{\beta}(u_1) = \frac{0.7}{\mathcal{L}} + \frac{0.7}{d} + \frac{0.7}{\mathcal{T}}$$
$$\overline{J}^{\beta}(u_2) = \frac{0.6}{\mathcal{L}} + \frac{0.6}{d} + \frac{0.6}{\mathcal{T}}$$

This shows that $\overline{J}^{\beta}(u_1)$ and $\overline{J}^{\beta}(u_2)$ are $F\mathbb{Q}_{ID}$ of M_1 but β is not a $F\mathbb{Q}_{ID}$ of M_2.

Hence β is GUR, $FST\mathbb{Q}_{ID}$ of M_1 w.r.t aftersets.

Now define $J: \mathbb{G} \to P(M_1 \times M_2)$ by :

$$J(u_1) = \left\{\begin{array}{c} (\mathcal{L}, \mathcal{L}'), (d, \mathcal{L}'), (\mathcal{L}, j), (d, j) \\ (\mathcal{L}, k), (d, k), (\mathcal{L}, \mathcal{T}'), (d, \mathcal{T}') \end{array}\right\}$$
$$J(u_2) = \left\{\begin{array}{c} (d, \mathcal{L}'), (\mathcal{T}, \mathcal{L}'), (d, j), (\mathcal{T}, j) \\ (d, k), (\mathcal{T}, k), (d, \mathcal{T}'), (\mathcal{T}, \mathcal{T}') \end{array}\right\}$$

Now, foresets w.r.t $J(u_1)$ and $J(u_2)$ are as follows :

$$J(u_1)\mathcal{L}' = \{\mathcal{L}, d\},\ J(u_1) j = \{\mathcal{L}, d\},\ J(u_1) k = \{\mathcal{L}, d\}$$

and $\mathcal{J}(u_1)\mathcal{T}' = \{\mathcal{L}, d\}$

$$\mathcal{J}(u_2)\mathcal{L}' = \{d, \mathcal{T}\},\ \mathcal{J}(u_2)j' = \{d, \mathcal{T}\},\ \mathcal{J}(u_2)k = \{d, T\} \text{ and } \mathcal{J}(u_2)\mathcal{T}' = \{d, \mathcal{T}\}.$$

Then $(\mathcal{J}, \mathbb{G})$ is a *STCMR* w.r.t foresets.

Let $\mu : M_1 \to [0, 1]$ be defined by,

$$\mu = \frac{0.6}{\mathcal{L}} + \frac{0.4}{d} + \frac{1}{\mathcal{T}}$$

Then μ is not a $F\mathbb{Q}_{ID}$ of M_1. But

$$^{\mu}\overline{\mathcal{J}}(u_1) = \frac{0.6}{\mathcal{L}'} + \frac{0.6}{j} + \frac{0.6}{k} + \frac{0.6}{\mathcal{T}'}$$
$$^{\mu}\overline{\mathcal{J}}(u_1) = \frac{1}{\mathcal{L}'} + \frac{1}{j} + \frac{1}{k} + \frac{1}{\mathcal{T}'}$$

This shows that $^{\mu}\overline{\mathcal{J}}(u_1)$ and $^{\mu}\overline{\mathcal{J}}(u_2)$ are $F\mathbb{Q}_{ID}$ of M_2 but μ is not a $F\mathbb{Q}_{ID}$ of M_1.

Hence, μ is GUR, $FST\mathbb{Q}_{ID}$ of M_2 w.r.t foresets.

Theorem 12: Let $(\mathcal{J}, \mathbb{G})$ be a *STCMR* and β be a $F\mathbb{Q}_{ID}$ of M_2. Then β is a GLR, $FST\mathbb{Q}_{ID}$ of M_1 w.r.t aftersets.

Proof: 1. As β is a $F\mathbb{Q}_{ID}$ of M_2. So, we have $\beta(e \vee f) \geq \beta(e) \wedge \beta(f)$ and $\beta(q \odot_2 f) \geq \beta(f)\ \forall q \in \mathbb{Q}, e, f \in M_2$. Since $(\mathcal{J}, \mathbb{G})$ be a *STCMR*. So, we have $s\mathcal{J}(u) \vee t\mathcal{J}(u) = (s \vee t)\mathcal{J}(u)$ for all $u \in \mathbb{G}$ and $s, t \in M_1$. Consider

$$\underline{\mathcal{J}}^{\beta}(u)(s \vee t) = \bigwedge_{d \in (s \vee t)\mathcal{J}(u)} \beta(d) = \bigwedge_{d \in s\mathcal{J}(u) \vee t\mathcal{J}(u)} \beta(d)$$

Since, $d \in s\mathcal{J}(u) \vee t\mathcal{J}(u)$, so there is $e \in s\mathcal{J}(u)$ *and* $f \in t\mathcal{J}(u)$ such that $d = e \vee f$. Consequently

$$\begin{aligned}\underline{\mathcal{J}}^{\beta}(u)(s \vee t) &= \bigwedge_{e \vee f \in s\mathcal{J}(u) \vee t\mathcal{J}(u)} \beta(e \vee f) \\ &= \bigwedge_{e \vee f \in s\mathcal{J}(u) \vee t\mathcal{J}(u)} [\beta(e) \wedge \beta(f)] \\ &= \left[\bigwedge_{e \in s\mathcal{J}(u)} \beta(e)\right] \wedge \left[\bigwedge_{f \in t\mathcal{J}(u)} \beta(f)\right] \\ &= \underline{\mathcal{J}}^{\beta}(u)(s) \wedge \underline{\mathcal{J}}^{\beta}(u)(t)\end{aligned}$$

Hence, $\underline{\mathcal{J}}^{\beta}(u)(s \vee t) = \underline{\mathcal{J}}^{\beta}(u)(s) \wedge \underline{\mathcal{J}}^{\beta}(u)(t)\ \forall s, t \in M_1$ *and* $\forall u \in \mathbb{G}$.

2. As $(\mathcal{J}, \mathbb{G})$ be a *STCMR*. So, we have $q \odot_2 s\mathcal{J}(u) = (q \odot_1 s)\mathcal{J}(u)\ \forall q \in \mathbb{Q},\ s \in M_1$ and $\forall u \in \mathbb{G}$.

Consider $q \in \mathbb{Q}, s \in M_1$ and

$$\underline{\mathcal{J}}^{\beta}(u)(q \odot_1 s) = \bigwedge_{d \in (q \odot_1 s)\mathcal{J}(u)} \beta(d) = \bigwedge_{d \in q \odot_2 s\mathcal{J}(u)} \beta(d)$$

As $d \in q \odot_2 s\mathcal{J}(u)$, so there have $f \in s\mathcal{J}(u)$ such that $d = q \odot_2 f$.

$$\begin{aligned}\underline{\mathcal{J}}^{\beta}(u)(q \odot_1 s) &= \bigwedge_{q \odot_2 f \in q \odot_2 s\mathcal{J}(u)} \beta(q \odot_2 f) \\ &\geq \bigwedge_{q \odot_2 f \in q \odot_2 s\mathcal{J}(u)} \beta(f) \\ &= \bigwedge_{f \in s\mathcal{J}(u)} \beta(f) = \underline{\mathcal{J}}^{\beta}(u)(s)\end{aligned}$$

Hence, $\underline{\mathcal{J}}^{\beta}(u)(q \odot_1 s) \geq \underline{\mathcal{J}}^{\beta}(u)(s)\ \forall q \in \mathbb{Q}, s \in M_1$. Thus, $\underline{\mathcal{J}}^{\beta}(u)$ is a $F\mathbb{Q}_{ID}$ of M_1. Consequently, β is a GLR, $FST\mathbb{Q}_{ID}$ of M_1 w.r.t aftersets.

Theorem 13: Let $(\mathcal{J}, \mathbb{G})$ be a *STCMR* and μ be a $F\mathbb{Q}_{ID}$ of M_1. Then μ is a GLR, $FST\mathbb{Q}_{ID}$ of M_2 w.r.t foresets.

Proof: The proof is clear.

Theorem 14: Let β be a $F\mathbb{Q}_{ID}$ of M_2 and $(\mathcal{J}, \mathbb{G})$ be a *STCMR*. Then $\underline{\mathcal{J}}^{\beta}(u)$ $[\overline{\mathcal{J}}^{\beta}(u)]$ is a $FST\mathbb{Q}_{ID}$ of M_1 w.r.t aftersets if and only if for each $\alpha \in [0, 1]$, $\underline{\mathcal{J}}^{\beta_\alpha}(u)$, $[\overline{\mathcal{J}}^{\beta_\alpha}(u)$ where $\beta_\alpha \neq \phi]$ is an $F\mathbb{Q}_{ID}$ of M_1 for all $u \in \mathbb{G}$.

Proof: Let $\underline{\mathcal{J}}^{\beta}(u)$ is a $FST\mathbb{Q}_{ID}$ of M_1.

1. Let $p, q \in \underline{\mathcal{J}}^{\beta_\alpha}(u)$. Then $\underline{\mathcal{J}}^{\beta}(u)(p) \geq \alpha$ and $\underline{\mathcal{J}}^{\beta}(u)(q) \geq \alpha$. Since $\underline{\mathcal{J}}^{\beta}(u)$ is a $FST\mathbb{Q}_{ID}$ of M_1. So, we have $\underline{\mathcal{J}}^{\beta}(u)(p \vee q) = \underline{\mathcal{J}}^{\beta}(u)(p) \wedge \underline{\mathcal{J}}^{\beta}(u)(q) \geq \alpha \implies \underline{\mathcal{J}}^{\beta}(u)(p \vee q) \geq \alpha$. Consequently, $p \vee q \in \underline{\mathcal{J}}^{\beta_\alpha}(u)$.

2. Let $\rho \in \underline{\mathcal{J}}^{\beta_\alpha}(u)$ and $q \in \mathbb{Q}$, Then $\underline{\mathcal{J}}^{\beta}(u)(\rho) \geq \alpha$. Since $\underline{\mathcal{J}}^{\beta}(u)$ is a $FST\mathbb{Q}_{ID}$, So, we have $\underline{\mathcal{J}}^{\beta}(u)(q \odot_1 \rho) \geq \underline{\mathcal{J}}^{\beta}(u)(\rho) \geq \alpha \implies \underline{\mathcal{J}}^{\beta}(u)(q \odot_1 \rho) \geq \alpha$. Consequently, $q \odot_1 \rho \in \underline{\mathcal{J}}^{\beta_\alpha}(u)$. With similar arguments, we have $\rho \odot_1 q \in \underline{\mathcal{J}}^{\beta_\alpha}(u)$. Hence, $\underline{\mathcal{J}}^{\beta_\alpha}(u)$ is a $F\mathbb{Q}_{ID}$ of M_1 for all $u \in \mathbb{G}$.

Conversely, let $\underline{\mathcal{J}}^{\beta_\alpha}(u)$ is a $F\mathbb{Q}_{ID}$ of M_1.

1. Let $s, t \in M_1$.

Consider

$$\underline{\mathcal{J}}^{\beta}(u)(s \vee t) = \bigwedge_{d \in (s \vee t)\mathcal{J}(u)} \beta(d) = \bigwedge_{d \in s\mathcal{J}(u) \vee t\mathcal{J}(u)} \beta(d).$$

Since, $d \in s\mathcal{J}(u) \vee t\mathcal{J}(u)$, so there have $e \in s\mathcal{J}(u)$ *and* $f \in t\mathcal{J}(u)$ such that $d = e \vee f$.

Consequently,

$$\begin{aligned}\underline{\mathcal{J}}^{\beta}(u)(s \vee t) &= \bigwedge_{e \vee f \in s\mathcal{J}(u) \vee t\mathcal{J}(u)} \beta(e \vee f) \\ &= \bigwedge_{e \vee f \in s\mathcal{J}(u) \vee t\mathcal{J}(u)} [\beta(e) \wedge \beta(f)] \\ &= \left[\bigwedge_{e \in s\mathcal{J}(u)} \beta(e)\right] \wedge \left[\bigwedge_{f \in t\mathcal{J}(u)} \beta(f)\right] \\ &= \underline{\mathcal{J}}^{\beta}(u)(s) \wedge \underline{\mathcal{J}}^{\beta}(u)(t)\end{aligned}$$

Hence, $\underline{\mathcal{J}}^{\beta}(u)(s \vee t) = \underline{\mathcal{J}}^{\beta}(u)(s) \wedge \underline{\mathcal{J}}^{\beta}(u)(t)\ \forall s, t \in M_1$ and $\forall u \in \mathbb{G}$. 2. As $(\mathcal{J}, \mathbb{G})$ be a *STCMR*. So, we have $q \odot_2 s\mathcal{J}(u) = (q \odot_1 s)\mathcal{J}(u)\ \forall q \in \mathbb{Q},\ s \in M_1$ and $\forall u \in \mathbb{G}$.

Consider $q \in \mathbb{Q}, s \in M_1$ and

$$\underline{\mathcal{J}}^{\beta}(u)(q \odot_1 s) = \bigwedge_{d \in (q \odot_1 s)\mathcal{J}(u)} \beta(d) = \bigwedge_{d \in q \odot_2 s\mathcal{J}(u)} \beta(d)$$

As $d \in q \odot_2 s\mathcal{J}(u)$, so there is $f \in s\mathcal{J}(u)$ such that $d = q \odot_2 f$.

$$\begin{aligned}\underline{\mathcal{J}}^{\beta}(u)(q \odot_1 s) &= \bigwedge_{q \odot_2 f \in q \odot_2 s\mathcal{J}(u)} \beta(q \odot_2 f) \\ &\geq \bigwedge_{q \odot_2 f \in q \odot_2 s\mathcal{J}(u)} \beta(f) \\ &= \bigwedge_{f \in s\mathcal{J}(u)} \beta(f) = \underline{\mathcal{J}}^{\beta}(u)(s)\end{aligned}$$

Hence $\underline{\mathcal{J}}^{\beta}(u)(q \odot_1 s) \geq \underline{\mathcal{J}}^{\beta}(u)(s)\ \forall q \in \mathbb{Q}, s \in M_1$. Thus, $\underline{\mathcal{J}}^{\beta}(u)$ is a $FST\mathbb{Q}_{ID}$ of M_1.

V. HOMOMORPHISM PROBLEMS ON GENERALIZED ROUGH FUZZY SUBSTRUCTURES

Some interesting problems on homomorphism of quantale module are introduced here by using rough fuzzy soft substructures of quantale modules.

Definition 26 [1]: Consider $(\mathbb{M}_1, \odot_1)$ and $(\mathbb{M}_2, \odot_2)$ be two $\mathbb{Q}-$ modules. A map $\mathcal{H} : \mathbb{M}_1 \longrightarrow \mathbb{M}_2$ is called a weak $\mathbb{Q}-$ module homomorphism (WMH) if

1) $\mathcal{H}(\mathscr{g} \vee \hbar) = \mathcal{H}(\mathscr{g}) \vee \mathcal{H}(\hbar)$;
2) $\mathcal{H}(\lambda \odot_1 \mathscr{g}) = \lambda \odot_2 \mathcal{H}(\mathscr{g})$.
   for all $\lambda \in \mathbb{Q}$ *and* $\mathscr{g}, \in \mathbb{M}_1$.

We say a weak quantale homomorphism $\mathcal{H} : \mathbb{M}_1 \longrightarrow \mathbb{M}_2$ is said to be an epimorphism if $\mathcal{H}$ is on to $\mathbb{M}_2$ if $\mathcal{H}$ is one-one then it is called called a monomorphism. If $\mathcal{H}$ is bijective, then it is called an isomorphism.

Lemma 1: Let $\mathcal{H} : \mathbb{M}_1 \longrightarrow \mathbb{M}_2$ be a surjective WMH and $(\mathbb{J}_2, \mathbb{G})$ be a $STBR$ on $\mathbb{M}_2$. Set $\mathbb{J}_1(u) = \{(\mathscr{y}, z) \in \mathbb{M}_1 \times \mathbb{M}_1 : (\mathcal{H}(\mathscr{y}), \mathcal{H}(z)) \in \mathbb{J}_2(u)\}$ for all $u \in \mathbb{G}$. Then for all $u \in \mathbb{G}$:

1) $(\mathbb{J}_1, \mathbb{G})$ is $STCR$ if $(\mathbb{J}_2, \mathbb{G})$ is $STCR$.
2) $(\mathbb{J}_1, \mathbb{G})$ is $STCMR$ w.r.t aftersets (w.r.t foresets)if $(\mathbb{J}_2, \mathbb{G})$ is $STCMR$ w.r.t aftersets (w.r.t foresets)and $\mathcal{H}$ is one-one.
3) $\mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u)) = \overline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$for $\mathcal{R} \subseteq \mathbb{M}_1$.
4) $\mathcal{H}(\underline{\mathbb{J}}_1^{\mathcal{R}}(u)) \subseteq \underline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$ for all $u \in \mathbb{G}$ and if $\mathcal{H}$ is one $-$ one, then $\mathcal{H}(\underline{\mathbb{J}}_1^{\mathcal{R}}(u)) = \underline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$.
5) Let $\mathcal{H} : \mathbb{M}_1 \longrightarrow \mathbb{M}_2$ be one-one. Then $\mathcal{H}(x) \in (\overline{\mathbb{J}}_1^{\mathcal{R}}(u)) \Longleftrightarrow x \in (\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$.

Proof: 1 and 2 are obvious.

3) Let $\gamma \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$ for some $\gamma \in \mathbb{M}_2$. Then there exists $d \in \mathbb{M}_1$ such that $d \in \overline{\mathbb{J}}_1^{\mathcal{R}}(u)$ and $\mathcal{H}(d) = \gamma$. Since, $d \in \overline{\mathbb{J}}_1^{\mathcal{R}}(u) \Longrightarrow d\mathbb{J}_1(u) \cap \mathcal{R} \neq \phi$. Thus, there exists $\in d\mathbb{J}_1(u) \cap \mathcal{R}$ such that $(d, \ell) \in \mathbb{J}_1(u)$ and $\in \mathcal{R}$. This shows that $(\mathcal{H}(d), \mathcal{H}(\ell)) \in \mathbb{J}_2(u) \Longrightarrow \mathcal{H}(\ell) \in \mathcal{H}(d)\mathbb{J}_2(u)$. Moreover, $\mathcal{H}(\ell) \in \mathcal{H}(\mathcal{R})$. Thus, $\mathcal{H}(\ell) \in \mathcal{H}(d)\mathbb{J}_2(u) \cap \mathcal{H}(\mathcal{R}) \Rightarrow \mathcal{H}(d)\mathbb{J}_2(u) \cap \mathcal{H}(\mathcal{R}) \neq \phi$. Hence, $\gamma = \mathcal{H}(d) \in \overline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$. Consequently, $\mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u)) \subseteq \overline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$.

Conversely, let $w \in \overline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$. Then $w\mathbb{J}_2(u) \cap \mathcal{H}(\mathcal{R}) \neq \phi$, so, there exists $t \in w\mathbb{J}_2(u) \cap \mathcal{H}(\mathcal{R})$ such that $(w, t) \in \mathbb{J}_2(u)$ and $t \in \mathcal{H}(\mathcal{R})$. Since, $\mathcal{H}$ is onto so there is $m \in \mathcal{R}$ and $n \in \mathbb{M}_1$ such that $t = \mathcal{H}(m)$ and $w = \mathcal{H}(n)$. Thus, $(\mathcal{H}(n), \mathcal{H}(m)) = (w, t) \in \mathbb{J}_2(u) \Longrightarrow (n, m) \in \mathbb{J}_1(u)$. This implies that $m \in n\mathbb{J}_1(u) \bigcap \mathcal{R}$.

So, we have $n \in \overline{\mathbb{J}}_1^{\mathcal{R}}(u)$. i.e., $w = \mathcal{H}(n) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Cconsequently, $\overline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u) \subseteq \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Hence, $\mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u)) = \overline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$ for all $u \in \mathbb{G}$.

4. Let $v \in \mathcal{H}(\underline{\mathbb{J}}_1^{\mathcal{R}}(u))$ for all $u \in \mathbb{G}$. Then there exists $d \in \underline{\mathbb{J}}_1^{\mathcal{R}}(u)$ such that $\mathbb{J}_1(u) \subseteq \mathcal{R}$ and $\mathcal{H}(d) = v$. Let $p \in v\mathbb{J}_2(u)$. Then there is $q \in \mathbb{M}_1$ such that $\mathcal{H}(q) = p$, and $\mathcal{H}(q) \in \mathcal{H}(d)\mathbb{J}_2(u)$, i.e., $(\mathcal{H}(d), \mathcal{H}(q)) \in \mathbb{J}_2(u)$. Hence, $(d, q) \in \mathbb{J}_1(u)$. i.e., $q \in \mathbb{J}_1(u) \subseteq \mathcal{R}$ and so $\mathcal{H}(q) \in \mathcal{H}(\mathcal{R})$ *and* $v\mathbb{J}_2(u) \subseteq \mathcal{H}(\mathcal{R})$. This shows that $v \in \underline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$. Hence, $\mathcal{H}(\underline{\mathbb{J}}_1^{\mathcal{R}}(u)) \subseteq \underline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$ for all $u \in \mathbb{G}$.

Now $v \in \underline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}$. Then there exists $\ell \in \mathbb{M}_1$ such that $\mathcal{H}(\ell) = v$ and $\mathcal{H}(\ell)\mathbb{J}_2(u) \subseteq \mathcal{H}(\mathcal{R})$. Let $q \in \ell\mathbb{J}_1(u)$, i.e., $(\ell, q) \in \mathbb{J}_1(u)$. Then $(\mathcal{H}(\ell), \mathcal{H}(q)) \in \mathbb{J}_2(u)$. i.e., $\mathcal{H}(q) \in \mathcal{H}(\ell)\mathbb{J}_2(u) \subseteq \mathcal{H}(\mathcal{R})$, and so $q \in \mathcal{R}$ Thus, $\ell\mathbb{J}_1(u) \subseteq \mathcal{R}$, which gives $\ell \in \underline{\mathbb{J}}_1^{\mathcal{R}}(u)$. Hence $v = \mathcal{H}(\ell) \in \mathcal{H}(\underline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Consequently, $\underline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})} \subseteq \mathcal{H}(\underline{\mathbb{J}}_1^{\mathcal{R}}(u))$ for all $u \in \mathbb{G}$.

5. Let $x \in \overline{\mathbb{J}}_1^{\mathcal{R}}(u)$ for all $u \in \mathbb{G}$. Then $\mathcal{H}(x) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$ for all $u \in \mathbb{G}$. Conversely, suppose that $\mathcal{H}(x) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Then there is $x' \in \overline{\mathbb{J}}_1^{\mathcal{R}}(u)$ such that $\mathcal{H}(x) = \mathcal{H}(x')$. Since $\mathcal{H}$ is one-one, we get $x = x' \in \overline{\mathbb{J}}_1^{\mathcal{R}}(u)$.

Theorem 15: Let $\mathcal{H}$ be a surjective WMH and $(\mathbb{J}_2, \mathbb{G})$ be a $STCR$ w.r.t aftersets on $\mathbb{M}_2$. Set

$$\mathbb{J}_1(\mathrm{u}) = \{(\mathscr{y}, z) \in \mathbb{M}_1 \times \mathbb{M}_1 : (\mathcal{H}(\mathscr{y}), \mathcal{H}(z)) \in \mathbb{J}_2(\mathrm{u})\}$$

for all $\mathrm{u} \in \mathbb{G}$. Then for all $\phi \not{f} \neq \mathcal{R} \subseteq \mathbb{M}_1$ and $\mathrm{u} \in \mathbb{G}$, the following hold;

1) $\overline{\mathbb{J}}_1^{\mathcal{R}}(u)$is $\mathbb{Q}_{SM}$of $\mathbb{M}_1$ if and only if $\overline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$is $\mathbb{Q}_{SM}$ of $\mathbb{M}_2$for all $u \in \mathbb{G}$.
2) $\overline{\mathbb{J}}_1^{\mathcal{R}}(u)$is $\mathbb{Q}_{ID}$of $\mathbb{M}_1$ if and only if $\overline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$ is $\mathbb{Q}_{ID}$ of $\mathbb{M}_2$ for all $u \in \mathbb{G}$.

Proof: 1. Let $\overline{\mathbb{J}}_1^{\mathcal{R}}(u)$ is $\mathbb{Q}_{SM}$ of $\mathbb{M}_1$ for all $u \in \mathbb{G}$. Then we have to show that $\overline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$ is $\mathbb{Q}_{SM}$ of $\mathbb{M}_2$ for all $u \in \mathbb{G}$. By Lemma 1(5), we have $\mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u)) = \overline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$ for $\mathcal{R} \subseteq \mathbb{M}_1$ and for all $u \in \mathbb{G}$.

(i) Let $\gamma_j \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$ and for all $u \in \mathbb{G}$ and for all $j \in J$. Then $\mu_j \in \overline{\mathbb{J}}_1^{\mathcal{R}}(u)$ be such that $\mathcal{H}(\mu_j) = \gamma_j$. Since $\overline{\mathbb{J}}_1^{\mathcal{R}}(u)$ is $\mathbb{Q}_{SM}$. So, we have $\bigvee_{j \in J}\mu_j \in \overline{\mathbb{J}}_1^{\mathcal{R}}(u)$. Then by Lemma 1(5), we have $\mathcal{H}(\bigvee_{j \in J}\mu_j) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Since $\mathcal{H}$ is WMH. So, we have $\bigvee_{j \in J}\mathcal{H}(\mu_j) = \mathcal{H}(\bigvee_{j \in J}\mu_j) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Hence, $\bigvee_{j \in J}\mathcal{H}(\mu_j) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Consequently, $\bigvee_{j \in J}\gamma_j \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$.

(ii) Let $e \in \mathbb{Q}$ and $\not{f} \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Then there is $w \in \overline{\mathbb{J}}_1^{\mathcal{R}}(u)$ such that $\mathcal{H}(w) = \not{f}$. Since $\overline{\mathbb{J}}_1^{\mathcal{R}}(u)$ is an $\mathbb{Q}_{SM}$ of $\mathbb{M}_1$. So, we have $e \odot_1 w \in \overline{\mathbb{J}}_1^{\mathcal{R}}(u)$. By Lemma 1(5), we have $\mathcal{H}(e \odot_1 w) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Since $\mathcal{H}$ is WMH. So, we have $e \odot_2 \mathcal{H}(w) = \mathcal{H}(e \odot_1 w) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. This implies that $e \odot_2 \not{f} = e \odot_2 \mathcal{H}(w) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Hence $e \odot_2 \not{f} \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. With similar arguments, we have $\not{f} \odot_2 e \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Thus, $\mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u)) = \overline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$ is $\mathbb{Q}_{SM}$ of $\mathbb{M}_2$ from (i)-(ii) for all $u \in \mathbb{G}$.

Conversely, let $\mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u)) = \overline{\mathbb{J}}_2^{\mathcal{H}(\mathcal{R})}(u)$ is an $\mathbb{Q}_{SM}$ of $\mathbb{M}_2$ for all $u \in \mathbb{G}$.

(i) Let $\omega_j \in \overline{\mathbb{J}}_1^{\mathcal{R}}(u)$ for all $u \in \mathbb{G}$. Then $\mathcal{H}(\omega_j) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Since $\mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$ is $\mathbb{Q}_{SM}$. So, we have $\bigvee_{j \in J}\mathcal{H}(w_j) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Since, $\mathcal{H}$ is WMH. So, we have $\mathcal{H}(\bigvee_{j \in J}\omega_j) = \bigvee_{j \in J}\mathcal{H}(w_j) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Hence, $\mathcal{H}(\bigvee_{j \in J}\omega_j) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$. Then by Lemma 1(5), we have $\bigvee_{j \in J}\omega_j \in \overline{\mathbb{J}}_1^{\mathcal{R}}(u)$.

(ii) Let $e \in \mathbb{Q}$ and $\not{f} \in \overline{\mathbb{J}}_1^{\mathcal{R}}(u)$. Then by Lemma 1(5), we have $\mathcal{H}(\not{f}) \in \mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$.Since $\mathcal{H}(\overline{\mathbb{J}}_1^{\mathcal{R}}(u))$ is $\mathbb{Q}_{SM}$ of

M_2. So, we have $e \circledcirc_2 \mathcal{H}(f) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Since $\mathcal{H}$ is WMH. So, we have $\mathcal{H}(e \circledcirc_1 f) = e \circledcirc_2 \mathcal{H}(f) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Hence, $\mathcal{H}(e \circledcirc_1 f) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Then by Lemma 1(5), we have $e \circledcirc_1 f \in \overline{J}_1^{\mathcal{R}}(u)$. With similar arguments, we have $f \circledcirc_1 e \in \overline{J}_1^{\mathcal{R}}(u)$. Thus, $\overline{J}_1^{\mathcal{R}}(u)$ is $\mathbb{Q}_{SM}$ of M_1 from (i)-(ii) for all $u \in \mathbb{G}$.

2. Let $\overline{J}_1^{\mathcal{R}}(u)$ is $\mathbb{Q}_{ID}$ of M_1 for all $u \in \mathbb{G}$. We have to show that $\overline{J}_2^{\mathcal{H}(\mathcal{R})}(u)$ is $\mathbb{Q}_{ID}$ of M_2 for all $u \in \mathbb{G}$. By Lemma 1(5), we have $\mathcal{H}(\overline{J}_1^{\mathcal{R}}(u)) = \overline{J}_2^{\mathcal{H}(\mathcal{R})}(u)$ for $\mathcal{R} \subseteq M_1$ and $\forall\, u \in \mathbb{G}$.

(i) Let $p, q \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$ for all $u \in \mathbb{G}$. Then $e, f \in \overline{J}_1^{\mathcal{R}}(u)$ be such that $\mathcal{H}(e) = p$, $\mathcal{H}(f) = q$. Since, $\overline{J}_1^{\mathcal{R}}(u)$ is $\mathbb{Q}_{ID}$ and $\mathcal{H}$ is WMH. So, we have $p \vee q = \mathcal{H}(e) \vee \mathcal{H}(f) = \mathcal{H}(e \vee f) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Hence, $p \vee q \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$.

(ii) Let $e \leq f \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Then there exist $\omega_1 \in M_1$ and $\omega_2 \in \overline{J}_1^{\mathcal{R}}(u)$ such that $e = \mathcal{H}(\omega_1)$ and $f = \mathcal{H}(\omega_2)$. Since $\mathcal{H}(\omega_1) \leq \mathcal{H}(\omega_2)$. So, we have $\mathcal{H}(\omega_1 \vee \omega_2) = \mathcal{H}(\omega_1) \vee \mathcal{H}(\omega_2) = \mathcal{H}(\omega_2) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. By Lemma 1(5), we have $\omega_1 \vee \omega_2 \in \overline{J}_1^{\mathcal{R}}(u)$. Since $\overline{J}_1^{\mathcal{R}}(u)$ is $\mathbb{Q}_{ID}$ and $\omega_1 \leq \omega_1 \vee \omega_2$, we have $\omega_1 \in \overline{J}_1^{\mathcal{R}}(u)$ and $e = \mathcal{H}(\omega_1) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Hence, $e \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$.

(iii) Let $e \in \mathbb{Q}$ and $f \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Then there is $w \in \overline{J}_1^{\mathcal{R}}(u)$ such that $\mathcal{H}(w) = f$. Since $\overline{J}_1^{\mathcal{R}}(u)$ is $\mathbb{Q}_{ID}$ of M_1. So, we have $e \circledcirc_1 w \in \overline{J}_1^{\mathcal{R}}(u)$. By Lemma 1(5), we have $\mathcal{H}(e \circledcirc_1 w) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Since $\mathcal{H}$ is WMH. So, we have $e \circledcirc_2 \mathcal{H}(w) = \mathcal{H}(e \circledcirc_1 w) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. This implies that $e \circledcirc_2 f = e \circledcirc_2 \mathcal{H}(w) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Hence, $e \circledcirc_2 f \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. With similar arguments, we have $f \circledcirc_2 e \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Thus, $\mathcal{H}(\overline{J}_1^{\mathcal{R}}(u)) = \overline{J}_2^{\mathcal{H}(\mathcal{R})}(u)$ is $\mathbb{Q}_{ID}$ of M_2 from (i)-(iii) for all $u \in \mathbb{G}$.

Conversely, let $\mathcal{H}(\overline{J}_1^{\mathcal{R}}(u)) = \overline{J}_2^{\mathcal{H}(\mathcal{R})}(u)$ is $\mathbb{Q}_{ID}$ of M_2 for all $u \in \mathbb{G}$.

(i) Let , $w_2 \in \overline{J}_1^{\mathcal{R}}(u)$ for all $u \in \mathbb{G}$. Then $\mathcal{H}(w_1), \mathcal{H}(w_2) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Since $\mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$ is an $\mathbb{Q}_{ID}$. So, we have $\mathcal{H}(w_1 \vee w_2) = \mathcal{H}(w_1) \vee \mathcal{H}(w_2) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Then by Lemma 1(5), we have $\vee w_2 \in \overline{J}_1^{\mathcal{R}}(u)$.

(ii) Let $m_1 \leq m_2 \in \overline{J}_1^{\mathcal{R}}(u)$, then $\mathcal{H}(m_1) \leq \mathcal{H}(m_2) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Since $\mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$ is $\mathbb{Q}_{ID}$. So, we have $\mathcal{H}(m_1) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Thus, by Lemma V. 2(5), we have $m_1 \in \overline{J}_1^{\mathcal{R}}(u)$. (iii) Let $e \in \mathbb{Q}$ and $f \in \overline{J}_1^{\mathcal{R}}(u)$. Then by Lemma V. 2(5), we have $\mathcal{H}(f) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Since $\mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$ is $\mathbb{Q}_{ID}$ of M_2. So, we have $e \circledcirc_2 \mathcal{H}(f) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Since $\mathcal{H}$ is WMH. So, we have $\mathcal{H}(e \circledcirc_1 f) = e \circledcirc_2 \mathcal{H}(f) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Hence, $\mathcal{H}(e \circledcirc_1 f) \in \mathcal{H}(\overline{J}_1^{\mathcal{R}}(u))$. Then by Lemma 1(5), we have $e \circledcirc_1 f \in \overline{J}_1^{\mathcal{R}}(u)$. With similar arguments, we have $f \circledcirc_1 e \in \overline{J}_1^{\mathcal{R}}(u)$.

Thus, $\overline{J}_1^{\mathcal{R}}(u)$ is $\mathbb{Q}_{ID}$ of M_1 from (i)-(iii) for all $u \in \mathbb{G}$.

Proposition 2: Let $\mathcal{H}$ be a surjective WMH and $(J_2, \mathbb{G})$ be a $STCR$ on M_2. Set $J_1(u) = \{(y, z) \in M_1 \times M_1 : (\mathcal{H}(y), \mathcal{H}(z)) \in J_2(u)\}$

For all $u \in \mathbb{G}$. Then for all $\phi \neq \mathcal{R} \subseteq M_1$ and for all $u \in \mathbb{G}$, we have

1) $\underline{J}_1^{\mathcal{R}}(u)$ is $\mathbb{Q}_{SM}$ of M_1 if and only if $\underline{J}_2^{\mathcal{H}(\mathcal{R})}(u)$ is $\mathbb{Q}_{SM}$ of M_2.
2) $\underline{J}_1^{\mathcal{R}}(u)$is $\mathbb{Q}_{ID}$ of M_1 if and only if $\underline{J}_2^{\mathcal{H}(\mathcal{R})}(u)$ is $\mathbb{Q}_{ID}$ of M_2.

Proof: The proof is simple and like Theorem 15.

Theorem 16: Let $\mathcal{F} : M_1 \longrightarrow M_2$ be a surjective WMH and $(J_2, \mathbb{G})$ be a $STCR$ on M_2 and μ be a F_{sst} of M_1. Set

$$J_1(u) = \{(s, t) \in M_1 \times M_1 : (\mathcal{F}(s), \mathcal{F}(t)) \in J_2(u)\}$$

Then following hold for all $u \in \mathbb{G}$:

1) $\overline{J}_1^{\beta}(u)$ is $F\mathbb{Q}_{ID}$ of M_1 if and only if $\overline{J}_2^{\mathcal{F}(\beta)}(u)$ is $F\mathbb{Q}_{ID}$ of M_2;
2) $\overline{J}_1^{\beta}(u)$ is $F\mathbb{Q}_{SM}$ of M_1 if and only if $\overline{J}_2^{\mathcal{F}(\beta)}(u)$ is $F\mathbb{Q}_{SM}$ of M_2. where

$$\mathcal{F}(\mu)(y) = \begin{cases} \bigvee\limits_{x \in \mathcal{F}^{-1}(y)} \mu(x) & if \;\; \mathcal{F}(y) \neq \phi \;\; \forall y \in M_2 \\ 0 & otherwise \end{cases}$$

Proof: 1. Note that $(\mathcal{F}(\beta))_{\alpha^+} = \mathcal{F}(\beta_{\alpha^+})$ for each $\alpha \in [0, 1]$, also $\left(\overline{J}_1^{\beta}(u)\right)_{\alpha^+} \neq \phi$ if and only if $\overline{J}_2^{(\mathcal{F}(\beta))_{\alpha^+}}(u) \neq \phi$. Let $\overline{J}_1^{\beta}(u)$ is a $F\mathbb{Q}_{ID}$ of M_1, then $\overline{J}_2^{(\mathcal{F}(\beta))_{\alpha^+}}(u) \neq f$ if $\overline{J}_1^{\beta_{\alpha^+}}(u) \neq \Phi$ for all $\alpha \in [0, 1]$. By Theorem IV.26, we have $\overline{J}_1^{\beta_{\alpha^+}}(u)$ is a $F\mathbb{Q}_{ID}$ of M_1 . Also, by using Proposition 1, we obtain $\left(\overline{J}_1^{\beta}(u)\right)_{\alpha^+}$ is a $F\mathbb{Q}_{ID}$ of M_1 . Now by Theorem 15 and Proposition 1, we have $\overline{J}_2^{(\mathcal{F}(\beta))_{\alpha^+}}(u) = \left(\overline{J}_2^{\mathcal{F}(\beta)}(u)\right)_{\alpha^+} = \overline{J}_2^{\mathcal{F}(\beta_{\alpha^+})}(u)$ is $F\mathbb{Q}_{ID}$ of M_2. Thus, by Theorem IV.26, we have $\overline{J}_2^{\mathcal{F}(\beta)}(u)$ is a $F\mathbb{Q}_{ID}$ of M_2.

Conversely, suppose $\overline{J}_2^{\mathcal{F}(\beta)}(u)$ is a $F\mathbb{Q}_{ID}$ of M_2. By Theorem IV.26 and Proposition 1, we have $\overline{J}_2^{(\mathcal{F}(\beta))_{\alpha^+}}(u) = \left(\overline{J}_2^{\mathcal{F}(\beta)}(u)\right)_{\alpha^+} = \overline{J}_2^{\mathcal{F}(\beta_{\alpha^+})}(u)$ is an $F\mathbb{Q}_{ID}$ of M_2. Thus, from Theorem V. 3, we have $\overline{J}_1^{\beta_{\alpha^+}}(u)$ is a $\mathbb{Q}_{ID}$ of M_1. Hence by Theorem 14, we have $\overline{J}_1^{\beta}(u)$ is a $F\mathbb{Q}_{ID}$ of M_1.

Theorem 14 provides a proof for 2 that is similar.

Theorem 15 provides similar proof for the following Proposition.

Proposition 3: Let $(J_2, \mathbb{G})$ be a $STCR$ w.r.t aftersets on M_2 and β be a F_{sst} of M_2. Let $\mathcal{F} : M_1 \longrightarrow M_2$ be a surjective WMH. Set

$$J_1(u) = \{(s, t) \in M_1 \times M_1 : (\mathcal{F}(s), \mathcal{F}(t)) \in J_2(u)\}$$

Then following hold for all $u \in \mathbb{G}$;

1) $\underline{J}_1^{\beta}(u)$ is an $F\mathbb{Q}_{ID}$ of M_1 if and only if $\underline{J}_2^{\mathcal{F}(\beta)}(u)$ is a $F\mathbb{Q}_{ID}$ of M_2;
2) $\underline{J}_1^{\beta}(u)$ is an $F\mathbb{Q}_{SM}$ of M_1 if and only if $\underline{J}_2^{\mathcal{F}(\beta)}(u)$ is a $F\mathbb{Q}_{SM}$ of M_2.

VI. APPLICATION IN DECISION-MAKING PROBLEM

In this section, soft binary relations-based decision-making techniques are suggested. They are based on fuzzy soft rough set theory. With this method, decision-makers' data can be used and no more information is needed. As a result, the outcomes should avoid the paradoxical outcomes.

Algorithm 1:

Here, with respect to aftersets, is a description of an algorithm for the solution of a decision-making problem. Following is the decision-making algorithm:

1) Determine the lower fuzzy soft set approximation $\underline{\mathfrak{J}}^{\beta}$ and upper fuzzy soft set approximation $\overline{\mathfrak{J}}^{\beta}$ of the fuzzy set β w.r.t aftersets ;
2) Determine the sum. That is sum of lower approximation $\sum_{i=1}^{3}\underline{\mathfrak{J}}^{\beta}(u_i)(y_j)$ and the sum of upper approximation $\sum_{i=1}^{3}\overline{\mathfrak{J}}^{\beta}(u_i)(y_j)$ corresponding to each i w.r.t aftersets ;
3) Determine the value that is choice value $\alpha_j = \sum_{i=1}^{3}\underline{\mathfrak{J}}^{\beta}(u_i)(y_j)+\sum_{i=1}^{3}\overline{\mathfrak{J}}^{\beta}(u_i)(y_j)$, $y_j \in U$ w.r.t aftersets;
4) The best decision is $y_k = max_j(\alpha_j)$;
5) The worst decision is $y_k = min_j(\alpha_j)$;
6) If k has more than one value, then select any one of y_{k_1} and y_{k_2}.

Algorithm 2:

Here, with respect to foresets, is a description of an algorithm for the solution of a decision-making problem. Following is the decision-making algorithm:

1) Find the approximation that is the lower fuzzy soft set approximation ${}^{\mu}\underline{\mathfrak{J}}$ and upper fuzzy soft set approximation ${}^{\mu}\overline{\mathfrak{J}}$ of the fuzzy set μ w.r.t foresets ;
2) Find the sum. That is sum of lower approximation $\sum_{i=1}^{3}{}^{\mu}\underline{\mathfrak{J}}(u_i)(y_j)$ and the sum of upper approximation $\sum_{i=1}^{3}{}^{\mu}\overline{\mathfrak{J}}(u_i)(y_j)$ corresponding to each i w.r.t foresets ;
3) Determine the value that is choice value $\alpha'_j = \sum_{i=1}^{3}{}^{\mu}\underline{\mathfrak{J}}(u_i)(y_j)+\sum_{i=1}^{3}{}^{\mu}\overline{\mathfrak{J}}(u_i)(y_j)$, $y_j \in U$ w.r.t foresets ;
4) The best decision is $y_k = max_j(\alpha'_j)$;
5) The worst decision is $y_k = min_j(\alpha'_j)$;
6) If k has more than one value, then select any one of y_{k_1} and y_{k_2}. by:

Example 7: Suppose Mr. Z wants to buy a refrigerator for his house. Let $\mathrm{M}_1 = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ = the sets of all available colors and $\mathrm{M}_2 = \{r_1, r_2, r_3, r_4, r_5, r_6\}$ = the models of all colors available and the set of attributes $G = \{u_1, u_2, u_3\}$ = the set of brands = $u_1 = haier$, $u_2 = dawlance$, $u_3 = pel\}$. Define $\mathfrak{J} : G \longrightarrow P(\mathrm{M}_1 \times M_2)$

$$\mathfrak{J}(u_1) = \left\{ \begin{array}{c} (\ell_1, r_1), (\ell_2, r_3), (\ell_3, r_2), (\ell_3, r_6), (\ell_2, r_5), \\ (\ell_4, r_5), (\ell_4, r_4), (\ell_1, r_3), (\ell_1, r_2), (\ell_1, r_6), \\ (\ell_1, r_5), (\ell_3, r_5), (\ell_1, r_4), (\ell_2, r_4), (\ell_3, r_4) \end{array} \right\}$$

$$\mathfrak{J}(u_2) = \left\{ \begin{array}{c} (\ell_4, r_6), (\ell_4, r_4), (\ell_2, r_4), (\ell_3, r_5), (\ell_3, r_2), \\ (\ell_1, r_3), (\ell_1, r_6), (\ell_4, r_1), (\ell_2, r_1), (\ell_1, r_5), \\ (\ell_1, r_2), (\ell_3, r_1), (\ell_1, r_1) \end{array} \right\}$$

$$\mathfrak{J}(u_3) = \left\{ \begin{array}{c} (\ell_1, r_5), (\ell_1, r_1), (\ell_1, r_3), (\ell_1, r_6), (\ell_1, r_2), \\ (\ell_2, r_5), (\ell_1, r_4), (\ell_2, r_6), (\ell_2, r_2), (\ell_2, r_4) \end{array} \right\}$$

which represents the relation between colors and models available in brand u_i for $1 \le i \le 3$. Then

$$\ell_1\mathfrak{J}(u_1) = \{r_1, r_2, r_3, r_4, r_5, r_6\},$$
$$\ell_2\mathfrak{J}(u_1) = \{r_3, r_4, r_5\},\ \ell_3\mathfrak{J}(u_1) = \{r_2, r_4, r_5, r_6\},$$
$$\ell_4\mathfrak{J}(u_1) = \{r_4, r_5\},$$
$$\ell_1\mathfrak{J}(u_2) = \{r_1, r_2, r_3, r_5, r_6\},\ \ell_2\mathfrak{J}(u_2) = \{r_1, r_4\},$$
$$\ell_3\mathfrak{J}(u_2) = \{r_1, r_2, r_5\},\ \ell_4\mathfrak{J}(u_2) = \{r_1, r_4, r_6\},$$
$$\ell_1\mathfrak{J}(u_3) = \{r_1, r_2, r_3, r_5, r_6\},$$
$$\ell_2\mathfrak{J}(u_3) = \{r_1, r_4, r_5, r_6\},\ \ell_3\mathfrak{J}(u_3) = \Phi \text{ and } \ell_4\mathfrak{J}(u_3) = \Phi,$$

where $\ell_i\mathfrak{J}(u_j)$ represents the models of the colors ℓ_i available in brand u_j.

Also,

$$\mathfrak{J}(u_1)r_1 = \{\ell_1\},\ \mathfrak{J}(u_1)r_2 = \{\ell_1, \ell_3\},$$
$$\mathfrak{J}(u_1)r_3 = \{\ell_1, \ell_2\}, \mathfrak{J}(u_1)r_4 = \{\ell_1, \ell_2, \ell_3, \ell_4\},$$
$$\mathfrak{J}(u_1)r_5 = \{\ell_1, \ell_2, \ell_3, \ell_4\},\ \mathfrak{J}(u_1)r_6 = \{\ell_1, \ell_3\},$$
$$\mathfrak{J}(u_2)r_1 = \{\ell_1, \ell_2, \ell_3, \ell_4\},\ \mathfrak{J}(u_2)r_2 = \{\ell_1, \ell_3\},$$
$$\mathfrak{J}(u_2)r_3 = \{\ell_1\},\ \mathfrak{J}(u_2)r_4 = \{\ell_2, \ell_4\},$$
$$\mathfrak{J}(u_2)\ell r_5' = \{\ell_1, \ell_3\},\ \mathfrak{J}(u_2)r_6 = \{\ell_1, \ell_4\},$$
$$\mathfrak{J}(u_1)r_1 = \{\ell_1\},\ \mathfrak{J}(u_1)r_2 = \{\ell_1, \ell_2\},$$
$$\mathfrak{J}(u_1)r_3 = \{\ell_1\},\ \mathfrak{J}(u_1)r_4 = \{\ell_1, \ell_2\},$$
$$\mathfrak{J}(u_1)r_5 = \{\ell_1, \ell_2\},\ \mathfrak{J}(u_1)r_6 = \{\ell_1, \ell_2\},$$

where $\mathfrak{J}(u_j)r_i$ represents the colors of the models r_i available in brand u_j.

Define $\beta : \mathrm{M}_2 \rightarrow [0, 1]$ which represents the preference of the models given by Mr. Z such that

$$\beta(r_1) = 0.5,\ \beta(r_2) = 0.8,\ \beta(r_3) = 1,$$
$$\beta(r_4) = 0.2,\ \beta(r_5) = 0.9,\ \beta(r_6) = 0.$$

Define $\mu : \mathrm{M}_1 \rightarrow [0, 1]$ which represents the preference of the colors given by Mr. Z such that

$$\mu(\ell_1) = 0.6,\ \mu(\ell_2) = 0.7,\ \mu(\ell_3) = 0.5,\ \mu(\ell_4) = 0.3.$$

After using the algorithm, take a look at the following table.

Here the choice value $\alpha_j = \sum_{i=1}^{3}\underline{\mathfrak{J}}^{\beta}(u_i)(y_j) + \sum_{i=1}^{3}\overline{\mathfrak{J}}^{\beta}(u_i)(y_j)$ is calculated w.r.t aftersets and the choice

TABLE 9. The decision algorithm's outputs with regard to aftersets.

	$\underline{\mathbb{J}}^{\beta}(u_1)$	$\underline{\mathbb{J}}^{\beta}(u_2)$	$\underline{\mathbb{J}}^{\beta}(u_3)$	$\overline{\mathbb{J}}^{\beta}(u_1)$	$\overline{\mathbb{J}}^{\beta}(u_2)$	$\overline{\mathbb{J}}^{\beta}(u_3)$	Choice value α_j
ℓ_1	0	0	0	1	1	1	3
ℓ_2	0.2	0.2	0	1	0.5	0.9	2.8
ℓ_3	0	0.5	0	0.9	0.9	0	2.3
ℓ_4	0.2	0	0	0.9	0.5	0	1.6

TABLE 10. The decision algorithm's outputs with regard to foresets.

	$^{\mu}\underline{\mathbb{J}}(u_1)$	$^{\mu}\underline{\mathbb{J}}(u_2)$	$^{\mu}\underline{\mathbb{J}}(u_3)$	$^{\mu}\overline{\mathbb{J}}(u_1)$	$^{\mu}\overline{\mathbb{J}}(u_2)$	$^{\mu}\overline{\mathbb{J}}(u_3)$	Choice value α'_j
r_1	0.6	0.3	0.6	0.6	0.7	0.6	3.4
r_2	0.5	0.5	0.6	0.6	0.6	0.7	3.5
r_3	0.6	0.6	0.6	0.7	0.6	0.6	3,7
r_4	0.3	0.3	0.6	0.7	0.7	0.7	3.3
r_5	0.3	0.5	0.6	0.7	0.6	0.7	3.4
r_6	0.5	0.3	0.6	0.6	0.6	0.7	3.3

value $\alpha'_j = \sum_{i=1}^{3} {}^{\mu}\underline{\mathbb{J}}(u_i)(y_j) + \sum_{i=1}^{3} {}^{\mu}\overline{\mathbb{J}}(u_i)(y_j)$ is calculated w.r.t foresets.

Since the maximum choice value is $\alpha_{k} = 3 = \alpha_1$ so, the decision is in favour of choosing the color ℓ_1, Furthermore, the color ℓ_4 is totally ignored. Hence, Mr. Z will select the refrigerator of color ℓ_1 for his house and he won't select the color ℓ_4 w.r.t the aftersets. Similarly, the maximum choice value is $\alpha'_j = 3.7 = \alpha_3$, so the decision is in favor of choosing model ω_3. Furthermore, the model ω_4 and ω_6 are totally ignored. Hence Mr. Z will select the refrigerator of model ω_3 for his house.

VII. CONCLUSION WITH BENEFITS AND DISADVANTAGES

In this paper, a novel concept of rough fuzzy subsets (substructures) are proposed which is based on soft relations with the aid of aftersets and foresets. The new planned study has numerous fuzzy algebraic properties which are also thoroughly affirmed. A new approximation technique in quantale module is employed and is based on soft compatible relation and further soft complete. According to Section V, connections under homomorphism problems are totally affirmed, sufficient conditions of rough fuzzy sub-modules and rough fuzzy sub-module ideals are obtained, and they are all thoroughly proved. Hence, the innovative generalization's approximation processing structure may be applied to different algebraic data fields. The main advantages are

(a) A new definition of roughness of fuzzy subsets in quantale module is proposed. This type of approach is not applied before in quantale module.
(b) The rough fuzzy substructures are introduced and many related examples are given to make the sense more clear.
(c) These rough fuzzy structures are presented under quantale module homomorphism and discussed their relations.
(d) The proposed model are then subjected to decision making problems to solve real word problems.

During the study of the proposed model, we have taken the left action. If we consider the same model but with right action then it will be difficult to proceed. This is the main disadvantage of our work.

However, the following topics could be taken into consideration for future scope as an extension of current work:

(1) Constructing rough fuzzy sets under soft relations to other algebras, including groups, rings, hyperrings, and so forth, that are connected to aftersets and foresets;

(2) Examining soft relations to study soft rough fuzzy submodules. In other words, we can explore various characterizations of soft rough fuzzy quantale module under soft relations and swap out quantale module substructures for fuzzy substructures in quantale module;

(3) Looking at several approaches to decision-making based on fuzzy rough sets that have soft relationships to other algebraic structures;

AUTHOR CONTRIBUTIONS

A. S: Conceptualization, Methodology, writing—original draft, Writing—review & editing; RSK: Methodology, Writing—original draft, Writing—review & editing;: SMQ: Methodology, Funding acquisition, Writing—original draft, Writing—review & editing; B. A: Writing—review & editing; A. K: Writing—review & editing;.

DATA AVAILABILITY

The paper includes the information used to verify the study's findings.

DECLARATIONS

Conflict of Interest:

It is demonstrated that we have no conflict of interest.

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SAQIB MAZHER QURASHI received the M.Sc., M.Phil., and Ph.D. degrees in mathematics from Quaid-i-Azam University, Islamabad, Pakistan. He is currently an Assistant Professor with the Department of Mathematics, Government College University Faisalabad. He has published many articles in international journals. He has also produced ten M.Phil. student. His current research interests include fuzzy algebraic structures, soft algebraic structures and their applications in quantale, and quantale modules.

BANDER ALMUTAIRI received the Ph.D. degree from the University of East Angila, Norwich, U.K., in 2013. He is currently an Assistant Professor with the Department of Mathematics, King Saud University, Riyadh, Saudi Arabia. His research interests include number theory and discrete mathematics.

RANI SUMAIRA KANWAL received the M.Sc., M.Phil., and Ph.D. degrees in mathematics from Quaid-i-Azam University, Islamabad, Pakistan. She is currently an Assistant Professor with the Department of Mathematics, Government College Women University Faisalabad. She has published many articles in international journals. Her current research interests include fuzzy algebraic structures and soft algebraic structures and their applications in decision making problems.

ANTHONY KARAGEORGOS received the degree in applied mathematics and in computer science and the Ph.D. degree in agent-based software engineering. He is currently a Professor in intelligent information systems with the School of Technology, University of Thessaly, Greece, and the Head of the Applied Informatics and Digital Technologies (AIDigiLab) Laboratory. Among his main research involvements in AIDigiLab are the development of intelligent systems for personalized e-commerce and mass manufacturing. He has received funding for research projects in Greece and EU, which resulted in patents and several international journals and conference publications. He has extensive experience in self-organizing systems and decentralized intelligence. His current research interests include combining immersive reality and decentralized intelligence, self-organizing and intelligent systems, the IoT, and data analytics.

AYESHA SAEED received the M.Sc. and M.Phil. degrees from Government College University Faisalabad, Pakistan, in 2020 and 2022, respectively. Her current research interests include substructures and fuzzy substructures of quantale module and their applications in decision making.