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RESEARCH ARTICLE

Weighted Bonferroni Aggregation Operators on Complex q-Rung Orthopair 2-Tuple Linguistic Variables With Application to Green Supply Chain Management

ZEESHAN ALI[1], TAHIR MAHMOOD[2], AND MIIN-SHEN YANG[3]
[1]Department of Mathematics and Statistics, Riphah International University, Islamabad 46000, Pakistan
[2]Department of Mathematics and Statistics, International Islamic University Islamabad, Islamabad 44000, Pakistan
[3]Department of Applied Mathematics, Chung Yuan Christian University, Zhongli, Taoyuan 32023, Taiwan

Corresponding author: Miin-Shen Yang (msyang@math.cycu.edu.tw)

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ABSTRACT In this paper, we give advanced study in complex q-rung orthopair 2-tuple linguistic variables (CQRO2-TLVs). The major theme of the paper is to evaluate the novel concept of CQRO2-TLVs and their dominant operational laws so that it can be a competent procedure to assess ambiguous and erratic information in realistic decision problems. Furthermore, we derive the weighted Bonferroni aggregation operators with weighted Bonferroni mean (WBM) and weighted geometric Bonferroni mean (WGBM) based on the CQRO2-TLV information for exploring the complex q-rung orthopair 2-tuple linguistic WBM (CQRO2-TLWBM) and complex q-rung orthopair 2-tuple linguistic WGBM (CQRO2-TLWGBM) operators. Some flexible and reliable properties and theories for the CQRO2-TLWBM and CQRO2-TLWGBM operators are investigated. We then introduce two new techniques to manage the multi-attribute decision making (MADM) issues under the fuzzy environment based on these operators. We know that a green supply chain management integrates environmental, ethical, and social concerns that make more about environmental and social responsibility correlated with design, production and distribution. In this paper, we apply the proposed techniques to green supply chain management to express the efficacy and usefulness of the proposed techniques. We finally make the comparisons of the proposed operators with some existing operators that demonstrate the effectiveness of our proposed method.

INDEX TERMS Fuzzy sets, complex q-rung orthopair (CQRO) fuzzy sets, 2-tuple linguistic variables (2-TLVs), CQRO 2-TLVs, weighted Bonferroni aggregation operators, weighted Bonferroni mean (WBM), green supply chain.

I. INTRODUCTION

Multi-attribute decision making (MADM) is a procedure for choosing the most ideal alternative among all accessible choices. It is important in the decision sciences. For most parts, choices and decisions in our day-by-day life issues are very complicated and have a significant part as a result in which we cannot generally have crisp data. To manage such issues, Zadeh [1] in 1965 first gave the idea of fuzzy sets (FSs) in which FSs had been applied in various areas for handling uncertainty arising from vagueness and partial belongingness [2], [3]. However, the theory of FSs has only a belongingness grade. We notice that the non-belongingness grade is also presented in many real-life decisions because human beings have the freedom to take their decision in any direction. For this, intuitionistic FSs (IFSs) explored by Atanassov [4] are stretched out from FSs with membership and non-membership values in which they got various applications in the literature [5], [6], and [7]. Afterwards, Yager and Abbasov [8] proposed Pythagorean

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fuzzy sets (PFSs), an extended version of IFSs, as a compelling device for delineating the vulnerability of MADM issues. The PFSs are additionally described by the membership and non-membership degrees with a condition that the sum of the squares of both is not exceeded from a unit interval. PFSs are broader than IFSs in which they can take care of issues that IFSs cannot. For example, if a decision-making problem gives the membership and non-membership degrees as 0.7 and 0.6, respectively, at that point it is just substantial for PFSs. After the PFS was effectively introduced, numerous researchers started to consider it broadly and profoundly [9], [10].

Additionally, Yager [11] extended PFSs to be q-rung orthopair fuzzy sets (QROFSs) with a modified condition where the sum of q-powers of the membership and non-membership degrees is not exceeded from a unit interval. The QROFS is additionally described by the participation degree and the non-membership degree, whose aggregate of q-powers is not exactly or equivalent to 1. The QROFS is broader than PFSs and IFSs in which both IFSs and PFSs are special cases of QROFSs. The QROFS can take care of issues that the PFS and IFS cannot, for instance, if a DM problem gives the enrollment degree and the non-enrollment degree as 0.9 and 0.8, respectively, at that point it is just substantial for the QROFS. After the QROFS was effectively introduced, it got many advance studies [12], [13], [14]. Moreover, in our day-to-day life, vulnerability and ambiguity that are available in the information happen simultaneously with changes to the stage (periodicity) of the information. Along these lines, the current speculations are inadequate to think about this data, and thus, there is a data misfortune during the procedure. Thus, we may ask what will happen if we change the range of FSs into a unit disc in the complex plane. For coping with more adequate vulnerability and ambiguity information in our day-to-day life, Ramot et al. [15] proposed complex FSs (CFSs) which contain the complex-valued membership degrees in the form of complex numbers in a unit disc with the condition that the real part (also for imaginary part) of the membership degrees is between 0 and 1. As CFSs only consider the membership degree in the form of polar co-ordinates, Alkouri and Salleh [16] extended CFSs to complex IFSs (CIFSs) that contains the membership and non-membership degrees in the form of complex number in a unit disc with the condition that the sum of real part (also for imaginary part) of the membership and non-membership degrees is not exceeded from a unit interval. Additionally, Ullah et al. [17] and Akram and Naz [18] modified the CIFSs to explore the complex PFSs (CPFSs) with the condition of the sum of squares of real parts (also for imaginary parts) of the membership and non-membership degrees not exceeded a unit interval with applications in pattern recognition and decision making.

Afterwards, Liu et al. [19] proposed complex QROFSs (CQROFSs) and showed them as a compelling device for delineating the vulnerability of MADM issues. The CQROFS is additionally described by the membership and non-membership degrees, whose aggregate of q-powers of the real part (also for imaginary part) is not exactly or equivalent to 1. CQROFSs are broader than CIFSs and CPFSs. Now and again, CQROFSs can take care of issues that the CPFS and IFS cannot, for instance, if a DM problem gives the enrollment degree and the non- enrollment degree as $0.9e^{i2\pi(0.9)}$ and $0.8e^{i2\pi(0.8)}$, respectively, at that point it is just substantial for CQROFSs. Thus, CQROFSs demonstrate that they are more impressive for dealing with unsure issues. After the CQROFS was effectively introduced, numerous researchers started to consider it broadly and profoundly [20], [21]. In general, a linguistic variable is a variable whose qualities are words or sentences in the fake language. Zadeh [22] used linguistic variables in approximate reasoning. Different analysts had also investigated the philosophy of linguistic MADM issues [23], [24]. In some handy issues, the single linguistic term set cannot include those cases, which contains two terms like truth and lie grades. For managing such sorts of issues, Herrera and Martinez [25] established the 2-tuple linguistic computational model, and Herrera and Martinez [26] established the fuzzy 2-tuple linguistic representation model. Further, the intuitionistic fuzzy 2-tuple linguistic terms set was explored by Liu and Chen [27]. The Pythagorean fuzzy 2-tuple linguistic aggregation operators were explored by Wei et al. [28], and Ju et al. [29] considered 2-tuple linguistic variables based on QROFSs.

Addressing or collecting the information into a singleton set is a very complicated and challenging task for scholars in which the Bonferroni mean (BM) operator, investigated by Bonferroni [30], is very flexible and dominant for aggregating the collection of data. The BM operator became to be a compelling strategy to assess impeccably the interrelationship among the qualities. The BM operator had resuscitated broad consideration from specialists and different researchers that was also effectively used in various fields [31], [32], [33]. On the other hand, global warming and climate change had heavily affected environment and human life on the earth. Environmental issues have been increasing. To integrate environmental concerns into green supply chain management (GSCM) becomes more important. Ashley [34] considered the issue about designing for the environment. Jelinski et al. [35] gave the concepts and approaches in industrial ecology. Sarkis [36] considered a decision framework for GSCM. Srivastava [37] gave a state-of-the-art literature review for GSCM. More researchers made researches on GSCM broadly [38], [39]. Furthermore, there were some fuzzy techniques been applied in GSCM in the literature [40], [41], [42], and [43].

We observe that there is less study in the idea of 2-tuple linguistic variables based on CQROFSs and no any application of CQROFSs in GSCM in the literature. Keeping the advantages of the CQROFSs and 2-tuple linguistic variables, we make the advanced study in complex q-rung orthopair 2-tuple linguistic variables (CQRO2-TLVs). In this paper, we first give the novel concept of CQRO2-TLVs and their dominant operational laws, then develop the weighted Bonferroni aggregation operators (WBAOs) with CQRO2-TLVs, such as complex q-rung orthopair 2-tuple linguistic (CQRO2-TL) weighted Bonferroni mean (CQRO2-TLWBM) and CQRO2-TL weighted geometric Bonferroni mean (CQRO2-TLWGBM). We also apply these operators in GSCM and make the comparisons of the proposed operators with some existing operators. The main contributions of the paper are listed as follows:

1) To evaluate the novel concept of CQRO2-TLVs and their dominant operational laws.
2) To derive the WBM and WGBM operators based on CQRO2-TLVs, such as CQRO2-TLWBM and CQRO2-TLWGBM operators. Some flexible and reliable properties for both operators are also derived.
3) Based on the CQRO2-TLWBM and CQRO2-TLWGBM operators, we give two new techniques to manage the multi-attribute decision making (MADM) issues.
4) Since a green supply chain management (GSCM) is important in environment and human life, we apply the proposed techniques to GSCM to express the efficacy and usefulness of the proposed techniques.
5) The comparative analysis between the proposed and existing methods is made to demonstrate the effectiveness of the proposed method.

The rest of this article is organized as follows. In Section II, basic notions of CQROFSs, 2-TLVs and their WBM, WGBM operators are briefly reviewed. In Section III, we evaluate the novel concept of CQRO2-TLVs and their dominant operational laws. To examine the interrelationships among CQRO2-TLVs, the WBM and WGBM operators combined with CQRO2-TLVs are then proposed for exploring the CQRO2-TLWBM and CQRO2-TLWGBM operators. Furthermore, we also study their properties with more theorems. In Section IV, we introduce two new techniques to manage the MADM issues under the fuzzy environment and then apply it to GSCM to demonstrate the efficacy and usefulness of the proposed techniques. We also make comparative analysis on the proposed and existing operators to show the effectiveness of our proposed operators. Finally, we give conclusions in Section V.

II. PRELIMINARIES

Basic notions of CQROFSs, 2-TLVs and their WBM, WGBM operators are briefly reviewed in this section. Throughout, the symbols $\mathcal{X}_{UNI}$, $\mathcal{M}_{\mathcal{Q}_{CQRP}}, \mathcal{M}_{\mathcal{Q}_{CQIP}}, \mathcal{N}_{\mathcal{Q}_{CQRP}}$ and $\mathcal{N}_{\mathcal{Q}_{CQIP}}$ denote the universal set, real part and imaginary part of the complex-valued supporting grade, real part and imaginary part of the complex-valued supporting against grade, respectively.

Definition 1 (Liu et al. [19]): A CQROFS is an object of the following form:

$$\mathcal{Q}_{CQ}=\left\{\left(\tilde{x},\left(\mathcal{M}_{\mathcal{Q}_{CQ}}(\tilde{x}),\mathcal{N}_{\mathcal{Q}_{CQ}}(\tilde{x})\right)\right):\tilde{x}\in\mathcal{X}_{UNI}\right\} \tag{1}$$

where $\mathcal{M}_{\mathcal{Q}_{CQ}}(\tilde{x}) = \mathcal{M}_{\mathcal{Q}_{CQRP}}(\tilde{x})\, e^{i2\pi\left(\mathcal{M}_{\mathcal{Q}_{CQIP}}(\tilde{x})\right)}$ and $\mathcal{N}_{\mathcal{Q}_{CQ}}(\tilde{x})=\mathcal{N}_{\mathcal{Q}_{CQRP}}(\tilde{x})\, e^{i2\pi\left(\mathcal{N}_{\mathcal{Q}_{CQIP}}(\tilde{x})\right)}$ denote the complex grade of supporting and the complex grade of supporting against with the conditions: $0 \leq \mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQRP}} + \mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQRP}} \leq 1$ and $0 \leq \mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQIP}} + \mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQIP}} \leq 1$. Additionally, the complex grade of refusal is stated by: $\mathcal{R}_{CQR}(\tilde{x}) = \mathcal{R}_{CQRRP}(\tilde{x})\, e^{i2\pi(\mathcal{R}_{CQRIP}(\tilde{x}))} = \left(1-\left(\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQRP}} +\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQRP}}\right)\right)^{1/q_{SC}} e^{i2\pi\left(1-\left(\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQIP}}+\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQIP}}\right)\right)^{1/q_{SC}}}$. The complex q-rung orthopair fuzzy number (CQROFN) is stated by $\mathcal{Q}_{CQ}=\left(\mathcal{M}_{\mathcal{Q}_{CQRP}} e^{i2\pi\left(\mathcal{M}_{\mathcal{Q}_{CQIP}}\right)}, \mathcal{N}_{\mathcal{Q}_{CQRP}} e^{i2\pi\left(\mathcal{N}_{\mathcal{Q}_{CQIP}}\right)}\right)$.

Definition 2 (Liu et al. [19]): For any two CQROFNs

$$\mathcal{Q}_{CQ-1} = \left(\mathcal{M}_{\mathcal{Q}_{CQRP-1}} e^{i2\pi\left(\mathcal{M}_{\mathcal{Q}_{CQIP-1}}\right)}, \mathcal{N}_{\mathcal{Q}_{CQRP-1}} e^{i2\pi\left(\mathcal{N}_{\mathcal{Q}_{CQIP-1}}\right)}\right)$$

and

$$\mathcal{Q}_{CQ-2} = \left(\mathcal{M}_{\mathcal{Q}_{CQRP-2}} e^{i2\pi\left(\mathcal{M}_{\mathcal{Q}_{CQIP-2}}\right)}, \mathcal{N}_{\mathcal{Q}_{CQRP-2}} e^{i2\pi\left(\mathcal{N}_{\mathcal{Q}_{CQIP-2}}\right)}\right)$$

with $\varrho_{SC}, \sigma_{SC} \geq 1$, the followings are defined:

1. $$\mathcal{Q}_{CQ-1}\oplus\mathcal{Q}_{CQ-2} = \begin{pmatrix} \left(\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQRP-1}}+\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQRP-2}}-\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQRP-1}}\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQRP-2}}\right)^{\frac{1}{q_{SC}}}\times \\ e^{i2\pi\left(\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQIP-1}}+\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQIP-2}}-\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQIP-1}}\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQIP-2}}\right)^{\frac{1}{q_{SC}}}}, \\ \mathcal{N}_{\mathcal{Q}_{CQRP-1}}\mathcal{N}_{\mathcal{Q}_{CQRP-2}} e^{i2\pi\left(\mathcal{N}_{\mathcal{Q}_{CQIP-1}}\mathcal{N}_{\mathcal{Q}_{CQIP-2}}\right)} \end{pmatrix};$$
2. $$\mathcal{Q}_{CQ-1}\otimes\mathcal{Q}_{CQ-2} = \begin{pmatrix} \mathcal{M}_{\mathcal{Q}_{CQRP-1}}\mathcal{M}_{\mathcal{Q}_{CQRP-2}} e^{i2\pi\left(\mathcal{M}_{\mathcal{Q}_{CQIP-1}}\mathcal{M}_{\mathcal{Q}_{CQIP-2}}\right)}, \\ \left(\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQRP-1}}+\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQRP-2}}-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQRP-1}}\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQRP-2}}\right)^{\frac{1}{q_{SC}}}\times \\ e^{i2\pi\left(\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQIP-1}}+\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQIP-2}}-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQIP-1}}\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQIP-2}}\right)^{\frac{1}{q_{SC}}}} \end{pmatrix};$$
3. $$\varrho_{SC}\,\mathcal{Q}_{CQ-1} = \left(\left(1-\left(1-\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQRP-1}}\right)^{\varrho_{SC}}\right)^{\frac{1}{q_{SC}}} \times e^{i2\pi\left(1-\left(1-\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQIP-1}}\right)^{\varrho_{SC}}\right)^{\frac{1}{q_{SC}}}}, \mathcal{N}^{\varrho_{SC}}_{\mathcal{Q}_{CQRP-1}} e^{i2\pi\left(\mathcal{N}^{\varrho_{SC}}_{\mathcal{Q}_{CQIP-1}}\right)}\right);$$
4. $$\mathcal{Q}^{\varrho_{SC}}_{CQ-1} = \begin{pmatrix} \mathcal{M}^{\varrho_{SC}}_{\mathcal{Q}_{CQRP-1}} e^{i2\pi\left(\mathcal{M}^{\varrho_{SC}}_{\mathcal{Q}_{CQIP-1}}\right)}, \\ \left(1-\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQRP-1}}\right)^{\varrho_{SC}}\right)^{\frac{1}{q_{SC}}} e^{i2\pi\left(1-\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQIP-1}}\right)^{\varrho_{SC}}\right)^{\frac{1}{q_{SC}}}} \end{pmatrix}.$$

Definition 3 (Liu et al. [19]): For any CQROFN

$$\mathcal{Q}_{CQ-1} = \left(\mathcal{M}_{\mathcal{Q}_{CQRP-1}} e^{i2\pi\left(\mathcal{M}_{\mathcal{Q}_{CQIP-1}}\right)}, \mathcal{N}_{\mathcal{Q}_{CQRP-1}} e^{i2\pi\left(\mathcal{N}_{\mathcal{Q}_{CQIP-1}}\right)}\right),$$

the score function and accuracy function are defined as:

$$\mathcal{S}_{SF}\left(\mathcal{Q}_{CQ-1}\right) = \frac{1}{2}\left(\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQRP-1}} - \mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQRP-1}} + \mathcal{M}^{q_{SC}}_{\mathcal{Q}_{CQIP-1}} - \mathcal{N}^{q_{SC}}_{\mathcal{Q}_{CQIP-1}}\right) \tag{2}$$

$$\mathcal{H}_{AF}\left(\mathcal{Q}_{CQ-1}\right)=\frac{1}{2}\left(\mathcal{M}^{qsc}_{\mathcal{Q}_{CQRP-1}}+\mathcal{N}^{qsc}_{\mathcal{Q}_{CQRP-1}}+\mathcal{M}^{qsc}_{\mathcal{Q}_{CQIP-1}}+\mathcal{N}^{qsc}_{\mathcal{Q}_{CQIP-1}}\right) \tag{3}$$

where $\mathcal{S}_{SF}\left(\mathcal{Q}_{CQ-1}\right)\in[-1,1]$ and $\mathcal{H}_{AF}\left(\mathcal{Q}_{CQ-1}\right)\in[0,1]$. For examining the relationships between any two CQROFNs

$$\mathcal{Q}_{CQ-1}=\left(\mathcal{M}_{\mathcal{Q}_{CQRP-1}}e^{i2\pi\left(\mathcal{M}_{\mathcal{Q}_{CQIP-1}}\right)},\mathcal{N}_{\mathcal{Q}_{CQRP-1}}e^{i2\pi\left(\mathcal{N}_{\mathcal{Q}_{CQIP-1}}\right)}\right)$$

and

$$\mathcal{Q}_{CQ-2}=\left(\mathcal{M}_{\mathcal{Q}_{CQRP-2}}e^{i2\pi\left(\mathcal{M}_{\mathcal{Q}_{CQIP-2}}\right)},\mathcal{N}_{\mathcal{Q}_{CQRP-2}}e^{i2\pi\left(\mathcal{N}_{\mathcal{Q}_{CQIP-2}}\right)}\right),$$

the following properties are followed:

1. If $\mathcal{S}_{SF}\left(\mathcal{Q}_{CQ-1}\right)\geq\mathcal{S}_{SF}\left(\mathcal{Q}_{CQ-2}\right)\Longrightarrow\mathcal{Q}_{CQ-1}\geq\mathcal{Q}_{CQ-2}$;
2. If $\mathcal{S}_{SF}\left(\mathcal{Q}_{CQ-1}\right)=\mathcal{S}_{SF}\left(\mathcal{Q}_{CQ-2}\right)\Longrightarrow$
   1) If $\mathcal{H}_{AF}\left(\mathcal{Q}_{CQ-1}\right)\geq\mathcal{H}_{AF}\left(\mathcal{Q}_{CQ-2}\right)\Longrightarrow\mathcal{Q}_{CQ-1}\geq\mathcal{Q}_{CQ-2}$;
   2) If $\mathcal{H}_{AF}\left(\mathcal{Q}_{CQ-1}\right)=\mathcal{H}_{AF}\left(\mathcal{Q}_{CQ-2}\right)\Longrightarrow\mathcal{Q}_{CQ-1}=\mathcal{Q}_{CQ-2}$.

Definition 4 (Herrera and Martinez [25]): For a linguistic term set $S_{LT}=\left\{s_{S_{LT-0}},s_{S_{LT-1}},s_{S_{LT-2}},s_{S_{LT-3}},\ldots,s_{S_{LT-2g}}\right\}$ with $\beta_{SC}\in[0,1]$, the 2-tuple linguistic function Δ_{LT} is an object of the following form:

$$\Delta_{LT}:[0,1]\rightarrow S_{LT}\times\left[-\frac{1}{2g},\frac{1}{2g}\right\}$$
$$\Delta_{LT}\left(\beta_{SC}\right)=\left(s_{S_{LT-j}},\alpha_{SC}\right) \tag{4}$$
$$\text{with}\begin{cases}s_{S_{LT-j}} & j=round\left(\beta_{SC},g\right)\\ \alpha_{SC}=\beta_{SC}-\dfrac{j}{g} & \alpha_{SC}\in\left[-\dfrac{1}{2g},\dfrac{1}{2g}\right\}\end{cases} \tag{5}$$

The 2-tuple linguistic inverse function Δ^{-1}_{LT} is an object of the following form:

$$\Delta^{-1}_{LT}:S_{LT}\times\left[-\frac{1}{2g},\frac{1}{2g}\right\}\rightarrow[0,1] \tag{6}$$
$$\Delta^{-1}_{LT}\left(s_{S_{LT-j}},\alpha_{SC}\right)=\frac{j}{g}+\alpha_{SC}=\beta_{SC} \tag{7}$$

Definition 5: For any positive numbers $\mathcal{Q}_j, j=1,2,3,..,m$, the weighted Bonferroni mean (WBM) operator is an object of the following form:

$$WBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}_1,\mathcal{Q}_2,..,\mathcal{Q}_{\mathbf{m}}\right)=\left(\sum\nolimits_{\mathrm{j,k}=1}^{\mathbf{m}}\omega_{W-j}\omega_{W-k}\,\mathcal{Q}_{\mathrm{j}}^{s_{CQ}}\,\mathcal{Q}_{\mathrm{k}}^{t_{CQ}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}} \tag{8}$$

where $\omega_W=\left(\omega_{W-1},\omega_{W-2},\ldots,\omega_{W-m}\right)^T$ denotes the weight vector with the condition $\sum_{j=1}^{m}\omega_{W-j}=1$.

Definition 6: For any positive numbers $\mathcal{Q}_{\mathrm{j}}, \mathrm{j}=1,2,3,..,m$, the weighted geometric BM (WGBM) operator is an object of the following form:

$$WGBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}_1,\mathcal{Q}_2,..,\mathcal{Q}_{\mathbf{m}}\right)=\frac{1}{s_{CQ}+t_{CQ}}\prod\nolimits_{\mathrm{j,k}=1}^{\mathbf{m}}\left(s_{CQ}\mathcal{Q}_{\mathrm{j}}+t_{CQ}\mathcal{Q}_{\mathrm{k}}\right)^{\omega_{W-j}\omega_{W-k}} \tag{9}$$

where $\omega_W=\left(\omega_{W-1},\omega_{W-2},\ldots,\omega_{W-m}\right)^T$ denotes the weight vector with the condition $\sum_{j=1}^{m}\omega_{W-j}=1$.

III. THE PROPOSED CQRO2-TLV WITH CQRO2-TLWBM AND CQRO2-TLWGBM OPERATORS

In this section, we first propose the novel approach of CQRO2-TLV. We give the definitions of CQRO2-TLV with their operational laws, score function, accuracy function, and also examine the relationships among CQRO2-TLVs. We consider the weighted Bonferroni mean (WBM) and weighted geometric Bonferroni mean (WGBM) operators, and then propose the complex q-rung orthopair 2-tuple linguistic WBM (CQRO2-TLWBM) and complex q-rung orthopair 2-tuple linguistic WGBM (CQRO2-TLWGBM) operators.

A. COMPLEX Q-RUNG ORTHOPAIR 2-TUPLE LINGUISTIC VARIABLES (CQRO2-TLV)

Definition 7: A CQRO2-TLV is defined as an object of the following form:

$$\mathcal{Q}_{CQTL}=\left\{\left(\left(s_{S_{LT}(\tilde{x})},\alpha_{SC}\right),\left(\mathcal{M}_{\mathcal{Q}_{CQTL}}(\tilde{x}),\mathcal{N}_{\mathcal{Q}_{CQTL}}(\tilde{x})\right)\right):\tilde{x}\in\mathcal{X}_{UNI}\right\} \tag{10}$$

where $\mathcal{M}_{\mathcal{Q}_{CQTL}}=\mathcal{M}_{\mathcal{Q}_{RPTL}}e^{i2\pi\mathcal{M}_{\mathcal{Q}_{IPTL}}}$ and $\mathcal{N}_{\mathcal{Q}_{CQTL}}=\mathcal{N}_{\mathcal{Q}_{RPTL}}e^{i2\pi\mathcal{N}_{\mathcal{Q}_{IPTL}}}$, with the conditions: $0\leq\mathcal{M}^{qsc}_{\mathcal{Q}_{RPTL}}(\tilde{x})+\mathcal{N}^{qsc}_{\mathcal{Q}_{RPTL}}(\tilde{x})\leq1$, $0\leq\mathcal{M}^{qsc}_{\mathcal{Q}_{IPTL}}(\tilde{x})+\mathcal{N}^{qsc}_{\mathcal{Q}_{IPTL}}(\tilde{x})\leq1$ where the pair $\left(s_{S_{LT}(\tilde{x})},\alpha_{SC}\right)$ is called 2-tulpe linguistic variable with $\alpha_{SC}\in\left[-\frac{1}{2g},\frac{1}{2g}\right\}$ and $s_{S_{LT}(\tilde{x})}\in S_{LT}$. Moreover,

$$\zeta_{\mathcal{Q}_{CQTL}}(\tilde{x})=\zeta_{\mathcal{Q}_{RPTL}}e^{i2\pi\zeta_{\mathcal{Q}_{IPTL}}}=\left(1-\left(\mathcal{M}^{qsc}_{\mathcal{Q}_{RPTL}}(\tilde{x})+\mathcal{N}^{qsc}_{\mathcal{Q}_{RPTL}}(\tilde{x})\right)^{\frac{1}{qsc}}\right)e^{i2\pi\left(1-\left(\mathcal{M}^{qsc}_{\mathcal{Q}_{IPTL}}(\tilde{x})+\mathcal{N}^{qsc}_{\mathcal{Q}_{IPTL}}(\tilde{x})\right)^{\frac{1}{qsc}}\right)}$$

is called refusal grade, and the complex q-rung orthopair 2-tuple linguistic number (CQRO2-TLN) is represented by

$$\mathcal{Q}_{CQTL}=\left(\left(s_{S_{LT}},\alpha_{SC}\right),\left(\mathcal{M}_{\mathcal{Q}_{CQTL}},\mathcal{N}_{\mathcal{Q}_{CQTL}}\right)\right)=\left(\left(s_{S_{LT}},\alpha_{SC}\right),\left(\mathcal{M}_{\mathcal{Q}_{RPTL}}e^{i2\pi\mathcal{M}_{\mathcal{Q}_{IPTL}}},\mathcal{N}_{\mathcal{Q}_{RPTL}}e^{i2\pi\mathcal{N}_{\mathcal{Q}_{IPTL}}}\right)\right).$$

Definition 8: For any two CQRO2-TLVs

$$\mathcal{Q}_{CQTL-1}=\left(\left(s_{S_{LT-1}},\alpha_{SC-1}\right),\left(\mathcal{M}_{\mathcal{Q}_{RPTL-1}}e^{i2\pi\mathcal{M}_{\mathcal{Q}_{IPTL-1}}},\times\,\mathcal{N}_{\mathcal{Q}_{RPTL-1}}e^{i2\pi\mathcal{N}_{\mathcal{Q}_{IPTL-1}}}\right)\right)$$

and

$$\mathcal{Q}_{CQTL-2}=\left(\left(s_{S_{LT-2}},\alpha_{SC-2}\right),\left(\mathcal{M}_{\mathcal{Q}_{RPTL-2}}e^{i2\pi\mathcal{M}_{\mathcal{Q}_{IPTL-2}}},\times\,\mathcal{N}_{\mathcal{Q}_{RPTL-2}}e^{i2\pi\mathcal{N}_{\mathcal{Q}_{IPTL-2}}}\right)\right),$$

the following operators are defined:

1. $\mathcal{Q}_{CQTL-1} \oplus_{CQTL} \mathcal{Q}_{CQTL-2}$
$$= \left(\begin{array}{c} \Delta_{LT} \left(\Delta_{LT}^{-1} \left(s_{S_{LT-1}}, \alpha_{SC-1} \right) + \Delta_{LT}^{-1} \left(s_{S_{LT-2}}, \alpha_{SC-2} \right) \right), \\ \left(\begin{array}{c} \left(\begin{array}{c} \mathcal{M}_{\mathcal{Q}_{RPTL-1}}^{q_{CQ}} + \mathcal{M}_{\mathcal{Q}_{RPTL-2}}^{q_{CQ}} - \\ \mathcal{M}_{\mathcal{Q}_{RPTL-1}}^{q_{CQ}} \mathcal{M}_{\mathcal{Q}_{RPTL-2}}^{q_{CQ}} \end{array} \right)^{\frac{1}{q_{CQ}}} e^{i2\pi \left(\begin{array}{c} \mathcal{M}_{\mathcal{Q}_{IPTL-1}}^{q_{CQ}} + \mathcal{M}_{\mathcal{Q}_{IPTL-2}}^{q_{CQ}} - \\ \mathcal{M}_{\mathcal{Q}_{IPTL-1}}^{q_{CQ}} \mathcal{M}_{\mathcal{Q}_{IPTL-2}}^{q_{CQ}} \end{array} \right)^{\frac{1}{q_{CQ}}}}, \\ \left(\mathcal{N}_{\mathcal{Q}_{RPTL-1}} \mathcal{N}_{\mathcal{Q}_{RPTL-2}} \right) e^{i2\pi \left(\mathcal{N}_{\mathcal{Q}_{IPTL-1}} \mathcal{N}_{\mathcal{Q}_{IPTL-2}} \right)} \end{array} \right) \end{array} \right);$$
2. $\mathcal{Q}_{CQTL-1} \otimes_{CQTL} \mathcal{Q}_{CQTL-2}$
$$= \left(\begin{array}{c} \Delta_{LT} \left(\Delta_{LT}^{-1} \left(s_{S_{LT-1}}, \alpha_{SC-1} \right) \times \Delta_{LT}^{-1} \left(s_{S_{LT-2}}, \alpha_{SC-2} \right) \right), \\ \left(\begin{array}{c} \left(\mathcal{M}_{\mathcal{Q}_{RPTL-1}} \mathcal{M}_{\mathcal{Q}_{RPTL-2}} \right) e^{i2\pi \left(\mathcal{M}_{\mathcal{Q}_{IPTL-1}} \mathcal{M}_{\mathcal{Q}_{IPTL-2}} \right)}, \\ \left(\begin{array}{c} \mathcal{N}_{\mathcal{Q}_{RPTL-1}}^{q_{CQ}} + \mathcal{N}_{\mathcal{Q}_{RPTL-2}}^{q_{CQ}} - \\ \mathcal{N}_{\mathcal{Q}_{RPTL-1}}^{q_{CQ}} \mathcal{N}_{\mathcal{Q}_{RPTL-2}}^{q_{CQ}} \end{array} \right)^{\frac{1}{q_{CQ}}} e^{i2\pi \left(\begin{array}{c} \mathcal{N}_{\mathcal{Q}_{IPTL-1}}^{q_{CQ}} + \mathcal{N}_{\mathcal{Q}_{IPTL-2}}^{q_{CQ}} - \\ \mathcal{N}_{\mathcal{Q}_{IPTL-1}}^{q_{CQ}} \mathcal{N}_{\mathcal{Q}_{IPTL-2}}^{q_{CQ}} \end{array} \right)^{\frac{1}{q_{CQ}}}} \end{array} \right) \end{array} \right);$$
3. $\mathcal{Q}_{CQTL-1}^{\delta_{SC}}$
$$= \left(\begin{array}{c} \Delta_{LT} \left(\Delta_{LT}^{-1} \left(s_{S_{LT-1}}, \alpha_{SC-1} \right)^{\delta_{SC}} \right), \\ \left(\begin{array}{c} \mathcal{M}_{\mathcal{Q}_{RPTL-1}}^{\delta_{SC}} e^{i2\pi \mathcal{M}_{\mathcal{Q}_{IPTL-1}}^{\delta_{SC}}}, \\ \left(1 - \left(1 - \mathcal{N}_{\mathcal{Q}_{RPTL-1}}^{q_{CQ}} \right)^{\delta_{SC}} \right)^{\frac{1}{q_{CQ}}} e^{i2\pi \left(1 - \left(1 - \mathcal{N}_{\mathcal{Q}_{IPTL-1}}^{q_{CQ}} \right)^{\delta_{SC}} \right)^{\frac{1}{q_{CQ}}}} \end{array} \right) \end{array} \right);$$
4. $\delta_{SC} \mathcal{Q}_{CQTL-1}$
$$= \left(\begin{array}{c} \Delta_{LT} \left(\delta_{SC} \times \Delta_{LT}^{-1} \left(s_{S_{LT-1}}, \alpha_{SC-1} \right) \right), \\ \left(\begin{array}{c} \left(1 - \left(1 - \mathcal{M}_{\mathcal{Q}_{RPTL-1}}^{q_{CQ}} \right)^{\delta_{SC}} \right)^{\frac{1}{q_{CQ}}} e^{i2\pi \left(1 - \left(1 - \mathcal{M}_{\mathcal{Q}_{IPTL-1}}^{q_{CQ}} \right)^{\delta_{SC}} \right)^{\frac{1}{q_{CQ}}}}, \\ \mathcal{N}_{\mathcal{Q}_{RPTL-1}}^{\delta_{SC}} e^{i2\pi \mathcal{N}_{\mathcal{Q}_{IPTL-1}}^{\delta_{SC}}} \end{array} \right) \end{array} \right).$$

Definition 9: For any two CQRO2-TLVs

$$\mathcal{Q}_{CQTL-1} = \left(\left(s_{S_{LT-1}}, \alpha_{SC-1} \right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-1}} e^{i2\pi \mathcal{M}_{\mathcal{Q}_{IPTL-1}}}, \times \mathcal{N}_{\mathcal{Q}_{RPTL-1}} e^{i2\pi \mathcal{N}_{\mathcal{Q}_{IPTL-1}}} \right) \right)$$

and

$$\mathcal{Q}_{CQTL-2} = \left(\left(s_{S_{LT-2}}, \alpha_{SC-2} \right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-2}} e^{i2\pi \mathcal{M}_{\mathcal{Q}_{IPTL-2}}}, \times \mathcal{N}_{\mathcal{Q}_{RPTL-2}} e^{i2\pi \mathcal{N}_{\mathcal{Q}_{IPTL-2}}} \right) \right),$$

the score and accuracy functions are defined as (11), shown at the bottom of the next page.

$$\check{H} \left(\mathcal{Q}_{CQTL-1} \right) = \Delta_{LT}^{-1} \left(s_{S_{LT-1}}, \alpha_{SC-1} \right) \times \left(\mathcal{M}_{\mathcal{Q}_{RPTL-1}}^{q_{SC}} + \mathcal{M}_{\mathcal{Q}_{IPTL-1}}^{q_{SC}} + \mathcal{N}_{\mathcal{Q}_{RPTL-1}}^{q_{SC}} + \mathcal{N}_{\mathcal{Q}_{IPTL-1}}^{q_{SC}} \right) \tag{12}$$

Based on the above two notions, the compassion between two CQROF2-TLNs is given by:

1. If $\text{Ş} \left(\mathcal{Q}_{CQTL-1} \right) > \text{Ş} \left(\mathcal{Q}_{CQTL-2} \right)$, then $\mathcal{Q}_{CQTL-1} > \mathcal{Q}_{CQTL-2}$;
2. If $\text{Ş} \left(\mathcal{Q}_{CQTL-1} \right) = \text{Ş} \left(\mathcal{Q}_{CQTL-2} \right)$, then:
   1) If $\check{H} \left(\mathcal{Q}_{CQTL-1} \right) > \check{H} \left(\mathcal{Q}_{CQTL-2} \right)$, then $\mathcal{Q}_{CQTL-1} > \mathcal{Q}_{CQTL-2}$;
   2) If $\check{H} \left(\mathcal{Q}_{CQTL-1} \right) = \check{H} \left(\mathcal{Q}_{CQTL-2} \right)$, then $\mathcal{Q}_{CQTL-1} = \mathcal{Q}_{CQTL-2}$.

Example 1: For any two CQROF2-TLNs

$$\mathcal{Q}_{CQTL-1} = \left(\left(s_{S_{LT-2}}, 0.01 \right), \left(0.8 e^{i2\pi (0.8)}, 0.6 e^{i2\pi (0.6)} \right) \right)$$

and

$$\mathcal{Q}_{CQTL-2} = \left(\left(s_{S_{LT-4}}, -0.02 \right), \left(0.9 e^{i2\pi (0.9)}, 0.4 e^{i2\pi (0.4)} \right) \right)$$

for $q_{SC} = \delta_{SC} = 2$.

Then

1. $\mathcal{Q}_{CQTL-1} \oplus_{CQTL} \mathcal{Q}_{CQTL-2}$
$$= \left(\begin{array}{c} \Delta_{LT} \left(\Delta_{LT}^{-1} \left(s_{S_{LT-2}}, 0.01 \right) + \Delta_{LT}^{-1} \left(s_{S_{LT-4}}, -0.02 \right) \right), \\ \left(\begin{array}{c} \left(0.8^2 + 0.9^2 - 0.8^2 \times 0.9^2 \right)^{\frac{1}{2}} e^{i2\pi \left(0.8^2 + 0.9^2 - 0.8^2 \times 0.9^2 \right)^{\frac{1}{2}}}, \\ (0.6 \times 0.4)\, e^{i2\pi (0.6 \times 0.4)} \end{array} \right) \end{array} \right)$$
$$= \left(\begin{array}{c} \Delta_{LT} \left(\left(\frac{2}{4} + 0.01 \right) + \left(\frac{4}{4} - 0.02 \right) \right), \\ \left(\begin{array}{c} (0.9652) e^{i2\pi (0.9652)}, \\ (0.24) e^{i2\pi (0.24)} \end{array} \right) \end{array} \right) = \left(\begin{array}{c} \Delta_{LT} (1.49), \\ \left(\begin{array}{c} (0.9652) e^{i2\pi (0.9652)}, \\ (0.24) e^{i2\pi (0.24)} \end{array} \right) \end{array} \right) = \left(\begin{array}{c} \left(s_{S_{LT-4}}, 0.49 \right), \\ \left(\begin{array}{c} (0.9652) e^{i2\pi (0.9652)}, \\ (0.24) e^{i2\pi (0.24)} \end{array} \right) \end{array} \right).$$
2. $\mathcal{Q}_{CQTL-1} \otimes_{CQTL} \mathcal{Q}_{CQTL-2}$
$$= \left(\begin{array}{c} \Delta_{LT} \left(\Delta_{LT}^{-1} \left(s_{S_{LT-2}}, 0.01 \right) \times \Delta_{LT}^{-1} \left(s_{S_{LT-4}}, -0.02 \right) \right), \\ \left(\begin{array}{c} (0.8 \times 0.9)\, e^{i2\pi (0.8 \times 0.9)}, \\ \left(0.6^2 + 0.4^2 - 0.6^2 \times 0.4^2 \right)^{\frac{1}{2}} e^{i2\pi \left(0.6^2 + 0.4^2 - 0.6^2 \times 0.4^2 \right)^{\frac{1}{2}}} \end{array} \right) \end{array} \right) = \left(\begin{array}{c} \Delta_{LT} \left(\left(\frac{2}{4} + 0.01 \right) \times \left(\frac{4}{4} - 0.02 \right) \right), \\ \left(\begin{array}{c} (0.72) e^{i2\pi (0.72)}, \\ (0.68) e^{i2\pi (0.68)} \end{array} \right) \end{array} \right)$$
$$= \left(\begin{array}{c} \Delta_{LT} (0.4998), \\ \left(\begin{array}{c} (0.72) e^{i2\pi (0.72)}, \\ (0.68) e^{i2\pi (0.68)} \end{array} \right) \end{array} \right) = \left(\begin{array}{c} \left(s_{S_{LT-3}}, 0.1665 \right), \\ \left(\begin{array}{c} (0.9652) e^{i2\pi (0.9652)}, \\ (0.24) e^{i2\pi (0.24)} \end{array} \right) \end{array} \right).$$
3. $\mathcal{Q}_{CQTL-1}^{2}$
$$= \left(\begin{array}{c} \Delta_{LT} \left(\Delta_{LT}^{-1} \left(s_{S_{LT-2}}, 0.01 \right)^2 \right), \\ \left(\begin{array}{c} 0.8^2 e^{i2\pi 0.8^2}, \\ \left(1 - \left(1 - 0.6^2 \right)^2 \right)^{\frac{1}{2}} e^{i2\pi \left(1 - \left(1 - 0.6^2 \right)^2 \right)^{\frac{1}{2}}} \end{array} \right) \end{array} \right) = \left(\begin{array}{c} \Delta_{LT} \left((0.51)^2 \right), \\ \left(\begin{array}{c} 0.64 e^{i2\pi 0.64}, \\ \left(1 - (1 - 0.36)^2 \right)^{\frac{1}{2}} e^{i2\pi \left(1 - (1 - 0.36)^2 \right)^{\frac{1}{2}}} \end{array} \right) \end{array} \right)$$
$$= \left(\begin{array}{c} \Delta_{LT} (0.2601), \\ \left(\begin{array}{c} 0.64 e^{i2\pi 0.64}, \\ 0.7684 e^{i2\pi 0.7684} \end{array} \right) \end{array} \right) = \left(\begin{array}{c} \left(s_{S_{LT-1}}, 0.0101 \right), \\ \left(\begin{array}{c} 0.64 e^{i2\pi 0.64}, \\ 0.7684 e^{i2\pi 0.7684} \end{array} \right) \end{array} \right).$$
4. $2 \times \mathcal{Q}_{CQTL-1}$
$$= \left(\begin{array}{c} \Delta_{LT} \left(2 \times \Delta_{LT}^{-1} \left(s_{S_{LT-2}}, 0.01 \right) \right), \\ \left(\begin{array}{c} \left(1 - \left(1 - 0.8^2 \right)^2 \right)^{\frac{1}{2}} e^{i2\pi \left(1 - \left(1 - 0.8^2 \right)^2 \right)^{\frac{1}{2}}}, \\ 0.6^2 e^{i2\pi 0.6^2} \end{array} \right) \end{array} \right) = \left(\begin{array}{c} \Delta_{LT} (2 \times 0.51), \\ \left(\begin{array}{c} \left(1 - (1 - 0.64)^2 \right)^{\frac{1}{2}} e^{i2\pi \left(1 - (1 - 0.64)^2 \right)^{\frac{1}{2}}}, \\ 0.6^2 e^{i2\pi 0.6^2} \end{array} \right) \end{array} \right)$$
$$= \left(\begin{array}{c} \Delta_{LT} (1.02), \\ \left(\begin{array}{c} 0.9330 e^{i2\pi 0.9330}, \\ 0.36 e^{i2\pi 0.36} \end{array} \right) \end{array} \right) = \left(\begin{array}{c} \left(s_{S_{LT-4}}, 0.02 \right), \\ \left(\begin{array}{c} 0.9330 e^{i2\pi 0.9330}, \\ 0.36 e^{i2\pi 0.36} \end{array} \right) \end{array} \right).$$

Furthermore, we examine the interrelationship between two CQRO2-TLVs with the help of score functions with

$$\text{Ş} \left(\mathcal{Q}_{CQTL-1} \right) = \frac{\Delta_{LT}^{-1} \left(s_{S_{LT-2}}, 0.01 \right) \times \left(1 + 0.8^2 + 0.8^2 - 0.6^2 - 0.6^2 \right)}{4} = \frac{(0.51) \times (1.56)}{4} = 0.1989$$

and

$$\text{Ş} \left(\mathcal{Q}_{CQTL-2} \right) = \frac{\Delta_{LT}^{-1} \left(s_{S_{LT-4}}, -0.02 \right) \times \left(1 + 0.9^2 + 0.9^2 - 0.4^2 - 0.4^2 \right)}{4} = \frac{(0.98) \times (2.3)}{4} = 0.5635.$$

Thus, we have $\text{Ş} \left(\mathcal{Q}_{CQTL-2} \right) > \text{Ş} \left(\mathcal{Q}_{CQTL-1} \right)$. If $\text{Ş} \left(\mathcal{Q}_{CQTL-2} \right) = \text{Ş} \left(\mathcal{Q}_{CQTL-1} \right)$, then we use the accuracy function which will be discussed next in Eq. (15) of Theorem 2.

B. THE CQRO2-TLWBM AND CQRO2-TLWGBM OPERATORS

We present the novel approach of the CQRO2-TLWBM and CQRO2-TLWGBM operators with their properties and theorems.

Definition 10: For any family of CQRO2-TLVs with

$$\mathcal{Q}_{CQTL-j} = \left(\left(s_{S_{LT-j}}, \alpha_{SC-j} \right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \mathcal{M}_{\mathcal{Q}_{IPTL-j}}}, \times \mathcal{N}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \mathcal{N}_{\mathcal{Q}_{IPTL-j}}} \right) \right), \ j = 1, 2, 3, .., m,$$

the CQRO2-TLWBM operator is defined as the following form:

$$CQRO2-TLWBM^{s_{CQ}, t_{CQ}} \left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, .., \mathcal{Q}_{CQTL-\mathbf{m}} \right)$$
$$= \left(\oplus_{CQTL}{}_{j,k=1}^{m} \left(\omega_{W-j} \omega_{W-k} \left(\mathcal{Q}_{CQTL-j}^{s_{CQ}} \right.\right.\right.$$

$$\otimes_{CQTL}\mathcal{Q}^{t_{CQ}}_{CQTL-k}\Big)\Big)\Big)^{\frac{1}{s_{CQ}+t_{CQ}}} \tag{13}$$

where $\omega_W = (\omega_{W-1}, \omega_{W-2}, \ldots, \omega_{W-m})^T$ denotes the weight vector with the condition $\sum_{j=1}^{m} \omega_{W-j} = 1$.

Theorem 1: For any family of CQRO2-TLVs with

$$\mathcal{Q}_{CQTL-j} = \left(\left(s_{S_{LT-j}}, \alpha_{SC-j}\right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi\mathcal{M}_{\mathcal{Q}_{IPTL-j}}}, \times \mathcal{N}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi\mathcal{N}_{\mathcal{Q}_{IPTL-j}}}\right)\right), \; j = 1, 2, 3, .., m,$$

by using Definition 8 we can obtain the following aggregated value (14), as shown at the bottom of the next page.

Proof: We prove Eq. (14) with as shown in the equation at the bottom of the next page.

Then

$$\mathcal{Q}^{s_{SC}}_{CQTL-j} \otimes_{CQTL} \mathcal{Q}^{t_{SC}}_{CQTL-k} = \left(\begin{array}{l}\Delta_{LT}\left(\Delta^{-1}_{LT}\left(s_{S_{LT-j}}, \alpha_{SC-j}\right)^{s_{SC}} \Delta^{-1}_{LT}\left(s_{S_{LT-k}}, \alpha_{SC-k}\right)^{t_{SC}}\right), \\ \left(\begin{array}{l}\mathcal{M}^{s_{SC}}_{\mathcal{Q}_{RPTL-j}}\mathcal{M}^{t_{SC}}_{\mathcal{Q}_{RPTL-k}} e^{i2\pi\mathcal{M}^{s_{SC}}_{\mathcal{Q}_{IPTL-j}}\mathcal{M}^{t_{SC}}_{\mathcal{Q}_{IPTL-k}}}, \\ \left(1-\left(1-\mathcal{N}^{q_{CQ}}_{\mathcal{Q}_{RPTL-j}}\right)^{s_{SC}}\left(1-\mathcal{N}^{q_{CQ}}_{\mathcal{Q}_{RPTL-k}}\right)^{t_{SC}}\right)^{\frac{1}{q_{CQ}}} \times \\ e^{i2\pi\left(1-\left(1-\mathcal{N}^{q_{CQ}}_{\mathcal{Q}_{IPTL-j}}\right)^{s_{SC}}\left(1-\mathcal{N}^{q_{CQ}}_{\mathcal{Q}_{IPTL-k}}\right)^{t_{SC}}\right)^{\frac{1}{q_{CQ}}}}\end{array}\right)\end{array}\right)$$

further, as shown in the equation at the bottom of page 8.

Thus, Eq. (14) is proved. ■

Additionally, we can prove the properties of Idempotency, Monotonicity, and Boundedness.

Theorem 2: For any family of CQRO2-TLVs

$$\mathcal{Q}_{CQTL-j} = \left(\left(s_{S_{LT-j}}, \alpha_{SC-j}\right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi\mathcal{M}_{\mathcal{Q}_{IPTL-j}}}, \times \mathcal{N}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi\mathcal{N}_{\mathcal{Q}_{IPTL-j}}}\right)\right), \; j = 1, 2, 3, .., m,$$

if $\mathcal{Q}_{CQTL} = \mathcal{Q}_{CQTL-j}$, we have

$$CQRO2-TLWBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, .., \mathcal{Q}_{CQTL-\mathbf{m}}\right) = \mathcal{Q}_{CQTL} \tag{15}$$

Proof: We prove Eq. (15) by

$$CQRO2-TLWBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, .., \mathcal{Q}_{CQTL-\mathbf{m}}\right) = \left(\oplus_{CQTL}{}^{m}_{j,k=1}\left(\omega_{W-j}\omega_{W-k}\left(\mathcal{Q}^{s_{CQ}}_{CQTL}\otimes_{CQTL}\mathcal{Q}^{t_{CQ}}_{CQTL}\right)\right)\right)^{\frac{1}{s_{CQ}+t_{CQ}}}$$
$$= \mathcal{Q}_{CQTL}\left(\oplus_{CQTL}{}^{m}_{j,k=1}\omega_{W-j}\omega_{W-k}\right)^{\frac{1}{s_{CQ}+t_{CQ}}} = \mathcal{Q}_{CQTL}.$$

■

Theorem 3: For any two families of CQRO2-TLVs

$$\mathcal{Q}_{CQTL-j} = \left(\left(s_{S_{LT-j}}, \alpha_{SC-j}\right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi\mathcal{M}_{\mathcal{Q}_{IPTL-j}}}, \times\mathcal{N}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi\mathcal{N}_{\mathcal{Q}_{IPTL-j}}}\right)\right)$$

and

$$\mathcal{Q}_{CQTL-*j} = \left(\left(s_{S_{LT-*j}}, \alpha_{SC-*j}\right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-*j}} e^{i2\pi\mathcal{M}_{\mathcal{Q}_{IPTL-*j}}}, \times\mathcal{N}_{\mathcal{Q}_{RPTL-*j}} e^{i2\pi\mathcal{N}_{\mathcal{Q}_{IPTL-*j}}}\right)\right), \; j = 1, 2, 3, .., m,$$

if $\mathcal{Q}_{CQTL-j} \leq \mathcal{Q}_{CQTL-*j}$, then we have

$$CQRO2-TLWBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, .., \mathcal{Q}_{CQTL-\mathbf{m}}\right) \leq CQRO2-\;TLWBM^{s_{CQ},t_{CQ}} \times \left(\mathcal{Q}_{CQTL-*1}, \mathcal{Q}_{CQTL-*2}, .., \mathcal{Q}_{CQTL-*\mathbf{m}}\right) \tag{16}$$

Proof: By hypothesis, it is given that

$$\mathcal{Q}_{CQTL-j} \leq \mathcal{Q}_{CQTL-*j} \Longrightarrow \left(s_{S_{LT-j}}, \alpha_{SC-j}\right) \leq \left(s_{S_{LT-*j}}, \alpha_{SC-*j}\right), \mathcal{M}_{\mathcal{Q}_{RPTL-j}} \leq \mathcal{M}_{\mathcal{Q}_{RPTL-j}}, \mathcal{M}_{\mathcal{Q}_{IPTL-j}} \leq \mathcal{M}_{\mathcal{Q}_{IPTL-*j}}, \mathcal{N}_{\mathcal{Q}_{RPTL-j}} \leq \mathcal{N}_{\mathcal{Q}_{RPTL-*j}}, \mathcal{N}_{\mathcal{Q}_{IPTL-j}} \leq \mathcal{N}_{\mathcal{Q}_{IPTL-*j}}$$

Then

$$\Delta_{LT}\left(\sum_{j,k=1}^{m}\omega_{W-j}\omega_{W-k}\left(\Delta^{-1}_{LT}\left(s_{S_{LT-j}}, \alpha_{SC-j}\right)\right)^{s_{CQ}} \times \left(\Delta^{-1}_{LT}\left(s_{S_{LT-k}}, \alpha_{SC-k}\right)\right)^{t_{CQ}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}$$
$$\leq \Delta_{LT}\left(\sum_{j,k=1}^{m}\omega_{W-*j}\omega_{W-*k}\left(\Delta^{-1}_{LT}\left(s_{S_{LT-*j}}, \alpha_{SC-*j}\right)\right)^{s_{CQ}} \times \left(\Delta^{-1}_{LT}\left(s_{S_{LT-*k}}, \alpha_{SC-*k}\right)\right)^{t_{CQ}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}$$

Further, we check it for the real part of supporting grade, such that as shown in the equation at the bottom of page 9.

Similarly, we can find the imaginary part of supporting grade with

$$\left(\left(\prod_{j,k=1}^{m}\left(\begin{array}{c}1- \\ 1- \\ \mathcal{M}^{q_{SC}s_{SC}}_{\mathcal{Q}_{IPTL-j}}\mathcal{M}^{q_{SC}t_{SC}}_{\mathcal{Q}_{IPTL-k}}\end{array}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}$$
$$\leq \left(\left(\prod_{j,k=1}^{m}\left(\begin{array}{c}1- \\ 1- \\ \mathcal{M}^{q_{SC}s_{SC}}_{\mathcal{Q}_{IPTL-*j}}\mathcal{M}^{q_{SC}t_{SC}}_{\mathcal{Q}_{IPTL-*k}}\end{array}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}$$

Similarly, we can find the real and imaginary part of supporting against grade with as shown in the equation at the bottom of page 9. From the above analysis, we get the result that

$$CQRO2-TLWBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, .., \mathcal{Q}_{CQTL-\mathbf{m}}\right) \leq CQRO2-TLWBM^{s_{CQ},t_{CQ}}$$

$$\c{S}\left(\mathcal{Q}_{CQTL-1}\right) = \frac{\Delta^{-1}_{LT}\left(s_{S_{LT-1}}, \alpha_{SC-1}\right) \times \left(1 + \mathcal{M}^{q_{SC}}_{\mathcal{Q}_{RPTL-1}} + \mathcal{M}^{q_{SC}}_{\mathcal{Q}_{IPTL-1}} - \mathcal{N}^{q_{SC}}_{\mathcal{Q}_{RPTL-1}} - \mathcal{N}^{q_{SC}}_{\mathcal{Q}_{IPTL-1}}\right)}{4} \tag{11}$$

$$\times\left(\mathcal{Q}_{CQTL-*1},\mathcal{Q}_{CQTL-*2},..,\mathcal{Q}_{CQTL-*\mathbf{m}}\right)$$

■

Theorem 4: For any family of CQRO2-TLVs

$$\mathcal{Q}_{CQTL-j}=\left(\left(s_{S_{LT-j}},\alpha_{SC-j}\right),\left(\mathcal{M}_{\mathcal{Q}_{RPTL-j}}e^{i2\pi\mathcal{M}_{\mathcal{Q}_{IPTL-j}}},\times\ \mathcal{N}_{\mathcal{Q}_{RPTL-j}}e^{i2\pi\mathcal{N}_{\mathcal{Q}_{IPTL-j}}}\right)\right),\ j=1,2,3,..,m,$$

if

$$\mathcal{Q}^{+}_{CQTL}=\left(\max_j\left(s_{S_{LT-j}},\alpha_{SC-j}\right),\left(\begin{matrix}\max\limits_j\mathcal{M}_{\mathcal{Q}_{RPTL-j}}e^{i2\pi\max\limits_j\mathcal{M}_{\mathcal{Q}_{IPTL-j}}},\\ \min\limits_j\mathcal{N}_{\mathcal{Q}_{RPTL-j}}e^{i2\pi\min\limits_j\mathcal{N}_{\mathcal{Q}_{IPTL-j}}}\end{matrix}\right)\right)$$

and

$$\mathcal{Q}^{-}_{CQTL}=\left(\min_j\left(s_{S_{LT-j}},\alpha_{SC-j}\right),\left(\begin{matrix}\min\limits_j\mathcal{M}_{\mathcal{Q}_{RPTL-j}}e^{i2\pi\min\limits_j\mathcal{M}_{\mathcal{Q}_{IPTL-j}}},\\ \max\limits_j\mathcal{N}_{\mathcal{Q}_{RPTL-j}}e^{i2\pi\max\limits_j\mathcal{N}_{\mathcal{Q}_{IPTL-j}}}\end{matrix}\right)\right),$$

then we have that

$$\mathcal{Q}^{-}_{CQTL}\leq CQRO2-TLWBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}_{CQTL\text{–}1},\mathcal{Q}_{CQTL-2},..,\mathcal{Q}_{CQTL-\mathbf{m}}\right)\leq\mathcal{Q}^{+}_{CQTL}\tag{17}$$

Proof: By using Theorems 2 and 3, we have

$$CQRO2-TLWBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}^{+}_{CQTL},\mathcal{Q}^{+}_{CQTL},..,\mathcal{Q}^{+}_{CQTL}\right)=\mathcal{Q}^{+}_{CQTL}$$

$$CQRO2-TLWBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}_{CQTL-1},\mathcal{Q}_{CQTL-2},..,\mathcal{Q}_{CQTL-\mathbf{m}}\right)$$
$$=\left(\begin{matrix}\Delta_{LT}\left(\left(\sum_{j,k=1}^{m}\omega_{W-j}\omega_{W-k}\left(\Delta^{-1}_{LT}\left(s_{S_{LT-j}},\alpha_{SC-j}\right)\right)^{s_{CQ}}\left(\Delta^{-1}_{LT}\left(s_{S_{LT-k}},\alpha_{SC-k}\right)\right)^{t_{CQ}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\right),\\ \left(\begin{matrix}\left(\left(1-\prod_{j,k=1}^{m}\left(1-\mathcal{M}^{q_{SC}s_{SC}}_{\mathcal{Q}_{RPTL-j}}\mathcal{M}^{q_{SC}t_{SC}}_{\mathcal{Q}_{RPTL-k}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\times\\ e^{i2\pi\left(\left(1-\prod_{j,k=1}^{m}\left(1-\mathcal{M}^{q_{SC}s_{SC}}_{\mathcal{Q}_{IPTL-j}}\mathcal{M}^{q_{SC}t_{SC}}_{\mathcal{Q}_{IPTL-k}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}},\\ \left(1-\left(1-\prod_{j,k=1}^{m}\left(1-\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{RPTL-j}}\right)^{s_{SC}}\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{RPTL-k}}\right)^{t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\right)^{\frac{1}{q_{SC}}}\times\\ e^{i2\pi\left(1-\left(1-\prod_{j,k=1}^{m}\left(1-\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{IPTL-j}}\right)^{s_{SC}}\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{IPTL-k}}\right)^{t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\right)^{\frac{1}{q_{SC}}}}\end{matrix}\right)\end{matrix}\right)\tag{14}$$

$$\mathcal{Q}^{s_{SC}}_{CQTL-j}=\left(\begin{matrix}\Delta_{LT}\left(\left(\Delta^{-1}_{LT}\left(s_{S_{LT-j}},\alpha_{SC-j}\right)^{s_{SC}}\right)\right),\\ \left(\begin{matrix}\mathcal{M}^{s_{SC}}_{\mathcal{Q}_{RPTL-j}}e^{i2\pi\mathcal{M}^{s_{SC}}_{\mathcal{Q}_{IPTL-j}}},\\ \left(1-\left(1-\mathcal{N}^{q_{CQ}}_{\mathcal{Q}_{RPTL-j}}\right)^{s_{SC}}\right)^{\frac{1}{q_{CQ}}}e^{i2\pi\left(1-\left(1-\mathcal{N}^{q_{CQ}}_{\mathcal{Q}_{IPTL-j}}\right)^{s_{SC}}\right)^{\frac{1}{q_{CQ}}}}\end{matrix}\right)\end{matrix}\right)$$

$$\mathcal{Q}^{t_{SC}}_{CQTL-k}=\left(\begin{matrix}\Delta_{LT}\left(\left(\Delta^{-1}_{LT}\left(s_{S_{LT-k}},\alpha_{SC-k}\right)^{t_{SC}}\right)\right),\\ \left(\begin{matrix}\mathcal{M}^{t_{SC}}_{\mathcal{Q}_{RPTL-k}}e^{i2\pi\mathcal{M}^{t_{SC}}_{\mathcal{Q}_{IPTL-k}}},\\ \left(1-\left(1-\mathcal{N}^{q_{CQ}}_{\mathcal{Q}_{RPTL-k}}\right)^{t_{SC}}\right)^{\frac{1}{q_{CQ}}}e^{i2\pi\left(1-\left(1-\mathcal{N}^{q_{CQ}}_{\mathcal{Q}_{IPTL-k}}\right)^{t_{SC}}\right)^{\frac{1}{q_{CQ}}}}\end{matrix}\right)\end{matrix}\right)$$

$$
\begin{aligned}
&\oplus_{CQTL}{}_{j,k=1}^{m}\left(\omega_{W-j}\omega_{W-k}\left(\mathcal{Q}_{\mathrm{j}}^{s_{CQ}}\otimes_{CQTL}\mathcal{Q}_{\mathrm{k}}^{t_{CQ}}\right)\right)\\
&=\left(\begin{array}{l}
\Delta_{LT}\left(\sum_{j,k=1}^{m}\omega_{W-j}\omega_{W-k}\left(\Delta_{LT}^{-1}\left(s_{S_{LT-j}},\alpha_{SC-j}\right)\right)^{s_{CQ}}\left(\Delta_{LT}^{-1}\left(s_{S_{LT-k}},\alpha_{SC-k}\right)\right)^{t_{CQ}}\right),\\
\left(\begin{array}{l}
\left(\left(1-\prod_{j,k=1}^{m}\left(1-\mathcal{M}_{\mathcal{Q}_{RPTL-j}}^{q_{SC}s_{SC}}\mathcal{M}_{\mathcal{Q}_{RPTL-k}}^{q_{SC}t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{q_{SC}}}\right)\times\\
e^{i2\pi\left(\left(1-\prod_{j,k=1}^{m}\left(1-\mathcal{M}_{\mathcal{Q}_{IPTL-j}}^{q_{SC}s_{SC}}\mathcal{M}_{\mathcal{Q}_{IPTL-k}}^{q_{SC}t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{q_{SC}}}\right)},\\
\left(\left(1-\prod_{j,k=1}^{m}\left(1-\left(1-\mathcal{N}_{\mathcal{Q}_{RPTL-j}}^{q_{SC}}\right)^{s_{SC}}\left(1-\mathcal{N}_{\mathcal{Q}_{RPTL-k}}^{q_{SC}}\right)^{t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)\right)^{\frac{1}{q_{SC}}}\times\\
e^{i2\pi\left(\left(1-\prod_{j,k=1}^{m}\left(1-\left(1-\mathcal{N}_{\mathcal{Q}_{IPTL-j}}^{q_{SC}}\right)^{s_{SC}}\left(1-\mathcal{N}_{\mathcal{Q}_{IPTL-k}}^{q_{SC}}\right)^{t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)\right)^{\frac{1}{q_{SC}}}}
\end{array}\right)
\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
&CQRO2-TLWBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}_{CQTL-1},\mathcal{Q}_{CQTL-2},..,\mathcal{Q}_{CQTL-\mathbf{m}}\right)\\
&=\left(\oplus_{CQTL}{}_{j,k=1}^{m}\left(\omega_{W-j}\omega_{W-k}\left(\mathcal{Q}_{CQTL-\mathrm{j}}^{s_{CQ}}\otimes_{CQTL}\mathcal{Q}_{CQTL-\mathrm{k}}^{t_{CQ}}\right)\right)\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\\
&=\left(\begin{array}{l}
\Delta_{LT}\left(\sum_{j,k=1}^{m}\omega_{W-j}\omega_{W-k}\left(\Delta_{LT}^{-1}\left(s_{S_{LT-j}},\alpha_{SC-j}\right)\right)^{s_{CQ}}\left(\Delta_{LT}^{-1}\left(s_{S_{LT-k}},\alpha_{SC-k}\right)\right)^{t_{CQ}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}},\\
\left(\begin{array}{l}
\left(\left(1-\prod_{j,k=1}^{m}\left(1-\mathcal{M}_{\mathcal{Q}_{RPTL-j}}^{q_{SC}s_{SC}}\mathcal{M}_{\mathcal{Q}_{RPTL-k}}^{q_{SC}t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\times\\
e^{i2\pi\left(\left(1-\prod_{j,k=1}^{m}\left(1-\mathcal{M}_{\mathcal{Q}_{IPTL-j}}^{q_{SC}s_{SC}}\mathcal{M}_{\mathcal{Q}_{IPTL-k}}^{q_{SC}t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}},\\
\left(\left(1-\prod_{j,k=1}^{m}\left(1-\left(1-\mathcal{N}_{\mathcal{Q}_{RPTL-j}}^{q_{SC}}\right)^{s_{SC}}\left(1-\mathcal{N}_{\mathcal{Q}_{RPTL-k}}^{q_{SC}}\right)^{t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\right)\times^{\frac{1}{q_{SC}}}\\
e^{i2\pi\left(\left(1-\prod_{j,k=1}^{m}\left(1-\left(1-\mathcal{N}_{\mathcal{Q}_{IPTL-j}}^{q_{SC}}\right)^{s_{SC}}\left(1-\mathcal{N}_{\mathcal{Q}_{IPTL-k}}^{q_{SC}}\right)^{t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\right)^{\frac{1}{q_{SC}}}}
\end{array}\right)
\end{array}\right)
\end{aligned}
$$

and

$$CQRO2 - TLWBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}^{-}_{CQTL}, \mathcal{Q}^{-}_{CQTL}, .., \mathcal{Q}^{-}_{CQTL}\right) = \mathcal{Q}^{-}_{CQTL}.$$

Then

$$\begin{aligned}\mathcal{Q}^{-}_{CQTL} &= CQRO2 - TLWBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}^{-}_{CQTL}, \mathcal{Q}^{-}_{CQTL}, .., \mathcal{Q}^{-}_{CQTL}\right)\\ &\leq CQRO2 - TLWBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, .., \mathcal{Q}_{CQTL-\mathbf{m}}\right)\\ &\leq CQRO2 - TLWBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}^{+}_{CQTL}, \mathcal{Q}^{+}_{CQTL}, .., \mathcal{Q}^{+}_{CQTL}\right)\\ &= \mathcal{Q}^{+}_{CQTL}\end{aligned}$$

Hence

$$\mathcal{Q}^{-}_{CQTL} \leq CQRO2 - TLWBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, .., \mathcal{Q}_{CQTL-\mathbf{m}}\right) \leq \mathcal{Q}^{+}_{CQTL}$$

■

Definition 11: For any family of CQRO2-TLVs

$$\mathcal{Q}_{CQTL-j} = \left(\left(s_{S_{LT-j}}, \alpha_{SC-j}\right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \mathcal{M}_{\mathcal{Q}_{IPTL-j}}}, \times \mathcal{N}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \mathcal{N}_{\mathcal{Q}_{IPTL-j}}}\right)\right), j = 1, 2, 3, .., m,$$

$$\begin{aligned}\mathcal{M}^{q_{SC}s_{SC}}_{\mathcal{Q}_{RPTL-j}}\mathcal{M}^{q_{SC}t_{SC}}_{\mathcal{Q}_{RPTL-k}} \leq \mathcal{M}^{q_{SC}s_{SC}}_{\mathcal{Q}_{RPTL-*j}}\mathcal{M}^{q_{SC}t_{SC}}_{\mathcal{Q}_{RPTL-*k}} &\Longrightarrow \prod_{j,k=1}^{m}\left(1-\mathcal{M}^{q_{SC}s_{SC}}_{\mathcal{Q}_{RPTL-j}}\mathcal{M}^{q_{SC}t_{SC}}_{\mathcal{Q}_{RPTL-k}}\right)^{\omega_{W-j}\omega_{W-k}}\\ &\geq \prod_{j,k=1}^{m}\left(1-\mathcal{M}^{q_{SC}s_{SC}}_{\mathcal{Q}_{RPTL-*j}}\mathcal{M}^{q_{SC}t_{SC}}_{\mathcal{Q}_{RPTL-*k}}\right)^{\omega_{W-j}\omega_{W-k}}\\ &\Longrightarrow \left(\left(1-\prod_{j,k=1}^{m}\left(1-\mathcal{M}^{q_{SC}s_{SC}}_{\mathcal{Q}_{RPTL-j}}\mathcal{M}^{q_{SC}t_{SC}}_{\mathcal{Q}_{RPTL-k}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\\ &\leq \left(\left(1-\prod_{j,k=1}^{m}\left(1-\mathcal{M}^{q_{SC}s_{SC}}_{\mathcal{Q}_{RPTL-*j}}\mathcal{M}^{q_{SC}t_{SC}}_{\mathcal{Q}_{RPTL-*k}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\end{aligned}$$

$$\begin{aligned}&\left(1-\left(1-\prod_{j,k=1}^{m}\left(1-\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{RPTL-j}}\right)^{s_{SC}}\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{RPTL-k}}\right)^{t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\right)^{\frac{1}{q_{SC}}}\\ &\geq \left(1-\left(1-\prod_{j,k=1}^{m}\left(1-\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{RPTL-*j}}\right)^{s_{SC}}\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{RPTL-*k}}\right)^{t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\right)^{\frac{1}{q_{SC}}}\\ &\Longrightarrow \left(1-\left(1-\prod_{j,k=1}^{m}\left(1-\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{IPTL-j}}\right)^{s_{SC}}\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{IPTL-k}}\right)^{t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\right)^{\frac{1}{q_{SC}}}\\ &\geq \left(1-\left(1-\prod_{j,k=1}^{m}\left(1-\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{IPTL-*j}}\right)^{s_{SC}}\left(1-\mathcal{N}^{q_{SC}}_{\mathcal{Q}_{IPTL-*k}}\right)^{t_{SC}}\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\right)^{\frac{1}{q_{SC}}}\end{aligned}$$

the CQRO2-TLWGBM operator is defined as the following form:

$$CQRO2-TLWGBM^{\mathrm{s}_{CQ},\mathrm{t}_{CQ}} \times \left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, .., \mathcal{Q}_{CQTL-\mathbf{m}}\right) = \frac{1}{\mathrm{s}_{CQ}+\mathrm{t}_{CQ}}\left(\otimes_{CQTL}{}_{j,k=1}^{m}\left(\mathrm{s}_{CQ}\mathcal{Q}_{CQTL-j} \oplus_{CQTL}\mathrm{t}_{CQ}\mathcal{Q}_{CQTL-k}\right)\right)^{\omega_{W-j}\omega_{W-k}} \tag{18}$$

where $\omega_W = (\omega_{W-1}, \omega_{W-2}, \ldots, \omega_{W-m})^T$ denotes the weight vector with the condition $\sum_{j=1}^{m} \omega_{W-j} = 1$.

Theorem 5: For any family of CQRO2-TLV

$$\mathcal{Q}_{CQTL-j} = \left(\left(s_{S_{LT-j}}, \alpha_{SC-j}\right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \mathcal{M}_{\mathcal{Q}_{IPTL-j}}}, \times \mathcal{N}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \mathcal{N}_{\mathcal{Q}_{IPTL-j}}}\right)\right), j = 1, 2, 3, .., m,$$

the aggregated values by using Definition 8 are as follows (19), shown at the bottom of the next page.

Proof: Proof of this theorem is similar to the proof of Theorem 1. ■

Theorem 6: For any family of CQRO2-TLVs

$$\mathcal{Q}_{CQTL-j} = \left(\left(s_{S_{LT-j}}, \alpha_{SC-j}\right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \mathcal{M}_{\mathcal{Q}_{IPTL-j}}}, \times \mathcal{N}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \mathcal{N}_{\mathcal{Q}_{IPTL-j}}}\right)\right), j = 1, 2, 3, .., m,$$

if $\mathcal{Q}_{CQTL} = \mathcal{Q}_{CQTL-j}$, then

$$CQRO2-TLWGBM^{\mathrm{s}_{CQ},\mathrm{t}_{CQ}} \times \left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, .., \mathcal{Q}_{CQTL-\mathbf{m}}\right) = \mathcal{Q}_{CQTL} \tag{20}$$

Proof: Proof of this theorem is similar to the proof of Theorem 2. ■

Theorem 7: For any two families of CQRO2-TLVs

$$\mathcal{Q}_{CQTL-j} = \left(\left(s_{S_{LT-j}}, \alpha_{SC-j}\right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \mathcal{M}_{\mathcal{Q}_{IPTL-j}}}, \times \mathcal{N}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \mathcal{N}_{\mathcal{Q}_{IPTL-j}}}\right)\right)$$

and

$$\mathcal{Q}_{CQTL-*j} = \left(\left(s_{S_{LT-*j}}, \alpha_{SC-*j}\right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-*j}} e^{i2\pi \mathcal{M}_{\mathcal{Q}_{IPTL-*j}}}, \times \mathcal{N}_{\mathcal{Q}_{RPTL-*j}} e^{i2\pi \mathcal{N}_{\mathcal{Q}_{IPTL-*j}}}\right)\right), j = 1, 2, 3, .., m,$$

if $\mathcal{Q}_{CQTL-j} \leq \mathcal{Q}_{CQTL-*j}$, then we have

$$CQRO2-TLWGBM^{\mathrm{s}_{CQ},\mathrm{t}_{CQ}} \times \left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, .., \mathcal{Q}_{CQTL-\mathbf{m}}\right) \leq CQRO2-TLWGBM^{\mathrm{s}_{CQ},\mathrm{t}_{CQ}}\left(\mathcal{Q}_{CQTL-*1}, \mathcal{Q}_{CQTL-*2}, .., \mathcal{Q}_{CQTL-*\mathbf{m}}\right) \tag{21}$$

Proof: Proof of this theorem is similar to the proof of Theorem 3. ■

Theorem 8: For any family of CQRO2-TLVs

$$\mathcal{Q}_{CQTL-j} = \left(\left(s_{S_{LT-j}}, \alpha_{SC-j}\right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \mathcal{M}_{\mathcal{Q}_{IPTL-j}}}, \times \mathcal{N}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \mathcal{N}_{\mathcal{Q}_{IPTL-j}}}\right)\right), j = 1, 2, 3, .., m,$$

if

$$\mathcal{Q}_{CQTL}^{+} = \left(\max_j\left(s_{S_{LT-j}}, \alpha_{SC-}\right), \left(\begin{matrix}\max_j \mathcal{M}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \max_j \mathcal{M}_{\mathcal{Q}_{IPTL-j}}}, \\ \min_j \mathcal{N}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \min_j \mathcal{N}_{\mathcal{Q}_{IPTL-j}}}\end{matrix}\right)\right)$$

and

$$\mathcal{Q}_{CQTL}^{-} = \left(\min_j\left(s_{S_{LT-j}}, \alpha_{SC-}\right), \left(\begin{matrix}\min_j \mathcal{M}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \min_j \mathcal{M}_{\mathcal{Q}_{IPTL-j}}}, \\ \max_j \mathcal{N}_{\mathcal{Q}_{RPTL-j}} e^{i2\pi \max_j \mathcal{N}_{\mathcal{Q}_{IPTL-j}}}\end{matrix}\right)\right),$$

then we have

$$\mathcal{Q}_{CQTL}^{-} \leq CQRO2-TLWGBM^{\mathrm{s}_{CQ},\mathrm{t}_{CQ}}\left(\mathcal{Q}_{CQTL-1}, \mathcal{Q}_{CQTL-2}, .., \mathcal{Q}_{CQTL-\mathbf{m}}\right) \leq \mathcal{Q}_{CQTL}^{+} \tag{22}$$

Proof: Proof of this theorem is similar to the proof of Theorem 4

IV. APPLICATION TO GREEN SUPPLY CHAIN MANAGEMENT WITH COMPARATIVE ANALYSIS

In this section, we first construct the MADM technique by using the proposed CQRO2-TLWBM and CQRO2-TLWGBM operators, and then apply then to green supply chain management (GSCM) with comparative analysis. By using the explored operators of CQRO2-TLWBM and CQRO2-TLWGBM, we can construct the MADM technique to resolve problems for estimating ambiguous and impulsive information of realistic decision theory. To resolve such kind of issues, we choose the family of alternatives and their attributes based on weight vectors, whose information is stated as follows: $\mathbb{A}_{AL} = \{\mathbb{A}_{AL-1}, \mathbb{A}_{AL-2}, \ldots, \mathbb{A}_{AL-m}\}, \mathbb{G}_{AT} = \{\mathbb{G}_{AT-1}, \mathbb{G}_{AT-2}, \ldots, \mathbb{G}_{AT-n}\}, \omega_W = \{\omega_{W-1}, \omega_{W-2}, \ldots, \omega_{W-n}\}$ with the condition $\sum_{j=1}^{n} \omega_{W-j} = 1$. For addressing these problems effectively, we choose the complex q-rung orthopair 2-tuple linguistic information, which is in the form of $\mathcal{Q}_{CQTL-jk} = \left(\left(s_{S_{LT-jk}}, \alpha_{SC-jk}\right), \left(\mathcal{M}_{\mathcal{Q}_{RPTL-jk}} e^{i2\pi \mathcal{M}_{\mathcal{Q}_{IPTL-jk}}}, \mathcal{N}_{\mathcal{Q}_{RPTL-jk}} e^{i2\pi \mathcal{N}_{\mathcal{Q}_{IPTL-jk}}}\right)\right), j = 1, 2, 3, .., m, k = 1, 2, 3, .., n$. The procedures of the MADM technique is stated as follows:

*Step 1:*Based on CQRO2-TLVs, we construct a decision matrix, which is in the form of

$$\mathbb{R}_{DM} = \left(r_{jk}\right)_{m\times n}$$

*Step 2:*We normalize the decision matrix by using the following Eq. (23):

$$\mathbb{R}_{DM} = \left(r_{jk}\right)_{m\times n}$$

$$
=\begin{cases}\left(\left(s_{S_{LT-jk}},\alpha_{SC-jk}\right),\left(\mathcal{M}_{\mathcal{Q}_{RPTL-jk}}e^{i2\pi\mathcal{M}_{\mathcal{Q}_{IPTL-jk}}},\mathcal{N}_{\mathcal{Q}_{RPTL-jk}}e^{i2\pi\mathcal{N}_{\mathcal{Q}_{IPTL-jk}}}\right)\right) & \text{for benefit type}\\ \left(\left(s_{S_{LT-jk}},\alpha_{SC-jk}\right),\left(\mathcal{N}_{\mathcal{Q}_{RPTL-jk}}e^{i2\pi\mathcal{N}_{\mathcal{Q}_{IPTL-jk}}},\mathcal{M}_{\mathcal{Q}_{RPTL-jk}}e^{i2\pi\mathcal{M}_{\mathcal{Q}_{IPTL-jk}}}\right)\right) & \text{for cost type}\end{cases} \tag{23}
$$

Step 3: By using Eq. (14), we aggregate the normalized decision matrix.

Step 4: By using Eq. (11), we examine the score values of the aggregated values.

Step 5: Rank to all alternatives, and examine the best one.

Step 6: The end.

A. APPLICATION TO GREEN SUPPLY CHAIN MANAGEMENT

We know that global warming and climate change had heavily affected environment, and so integrating environmental concerns into green supply chain management (GSCM) is important. There were numerous researchers had applied fuzzy set and its extensions in GSCM, such as Wang et al. [40] and Krishankumar et al. [41], Riaz et al. [42] and Zulqarnain et al. [43]. However, there is no complex QROFSs (CQROFSs) applied in GSCM. We next apply these proposed operators of CQROFSs to GSCM. To choose the green suppliers (GSs) in GSCM based on CQRO2-TLVs, we choose the five GSs in the GSCM $\mathbb{O}_{CM-j}, j=1,2,3,4,5$. To resolve this problem, the expert chooses the four attributes, whose details are as follows:

1) $\mathcal{C}_{AT-1}$: Product quality factor;
2) $\mathcal{C}_{AT-2}$: Environmental factors;
3) $\mathcal{C}_{AT-3}$: Delivery factor;
4) $\mathcal{C}_{AT-4}$: Price factor.

To resolve the above issue, we choose a weight vector $\omega_W=(0.1,0.2,0.3,0.4)^T$. Thus, the procedures of the MADM technique is as follows:

Step 1: Based on the complex intuitionistic 2-tuple linguistic numbers (CI2-TLNs), we construct a decision matrix, which is in the form of Table 1.

Step 2: We normalize the decision matrix by using Eq. (23), which is given in step 1, if needed. But, the information of Table 1 is not needed to be normalized, so we have considered the data of Table 1 for continuing the processes of the MADM technique.

Step 3: By using Eq. (14), we aggregate the normalized decision matrix by using $q_{SC}=1$, which is as follows:

$$
\begin{aligned}
&\mathbb{O}_{\boldsymbol{CM-1}}\\
&=\left(\left(s_1,0.1144\right),\left(0.448e^{i2\pi(0.448)},0.609e^{i2\pi(0.609)}\right)\right)\\
&\mathbb{O}_{\boldsymbol{CM-2}}\\
&=\left(\left(s_1,0.1111\right),\left(0.4564e^{i2\pi(0.4564)},0.5971e^{i2\pi(0.5971)}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&CQRO2-TLWGBM^{s_{CQ},t_{CQ}}\left(\mathcal{Q}_{CQTL-1},\mathcal{Q}_{CQTL-2},..,\mathcal{Q}_{CQTL-\mathbf{m}}\right)\\
&=\left(\begin{array}{l}\Delta_{LT}\left(\frac{1}{s_{CQ}+t_{CQ}}\prod_{j,k=1}^{m}\left(\left(s_{CQ}\Delta_{LT}^{-1}\left(s_{S_{LT-j}},\alpha_{SC-j}\right)\right)+\left(t_{CQ}\Delta_{LT}^{-1}\left(s_{S_{LT-k}},\alpha_{SC-k}\right)\right)\right)^{\omega_{W-j}\omega_{W-k}}\right),\\
\left(\begin{array}{l}\left(\left(\prod_{j,k=1}^{m}\left(1-\left(1-\left(1-\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{RPTL-j}}\right)^{s_{SC}}\left(1-\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{RPTL-k}}\right)^{t_{SC}}\right)\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\right)^{\frac{1}{q_{SC}}}\times\\
e^{i2\pi\left(\left(\prod_{j,k=1}^{m}\left(1-\left(1-\left(1-\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{IPTL-j}}\right)^{s_{SC}}\left(1-\mathcal{M}^{q_{SC}}_{\mathcal{Q}_{IPTL-k}}\right)^{t_{SC}}\right)\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\right)^{\frac{1}{q_{SC}}}},\\
\left(\left(\prod_{j,k=1}^{m}\left(1-\left(1-\mathcal{N}^{q_{SC}s_{SC}}_{\mathcal{Q}_{RPTL-j}}\mathcal{N}^{q_{SC}t_{SC}}_{\mathcal{Q}_{RPTL-k}}\right)\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}\times\\
e^{i2\pi\left(\left(\prod_{j,k=1}^{m}\left(1-\left(1-\mathcal{N}^{q_{SC}s_{SC}}_{\mathcal{Q}_{IPTL-j}}\mathcal{N}^{q_{SC}t_{SC}}_{\mathcal{Q}_{IPTL-k}}\right)\right)^{\omega_{W-j}\omega_{W-k}}\right)^{\frac{1}{q_{SC}}}\right)^{\frac{1}{s_{CQ}+t_{CQ}}}}\end{array}\right)\end{array}\right)
\end{aligned} \tag{19}
$$

TABLE 1. Original decision matrix, whose values in the form of complex intuitionistic 2-tuple linguistic variables.

Symbols	$\mathcal{C}_{AT-1}$	$\mathcal{C}_{AT-2}$	$\mathcal{C}_{AT-3}$	$\mathcal{C}_{AT-4}$
$\mathbb{O}_{CM-1}$	$\left((s_4,0),\left(0.5e^{i2\pi(0.5)},0.3e^{i2\pi(0.3)}\right)\right)$	$\left((s_3,0.1),\left(0.6e^{i2\pi(0.6)},0.2e^{i2\pi(0.2)}\right)\right)$	$\left((s_1,0.2),\left(0.7e^{i2\pi(0.7)},0.1e^{i2\pi(0.1)}\right)\right)$	$\left((s_3,0.1),\left(0.8e^{i2\pi(0.8)},0.1e^{i2\pi(0.1)}\right)\right)$
$\mathbb{O}_{CM-2}$	$\left((s_1,0.2),\left(0.51e^{i2\pi(0.51)},0.31e^{i2\pi(0.31)}\right)\right)$	$\left((s_4,0.2),\left(0.61e^{i2\pi(0.61)},0.21e^{i2\pi(0.21)}\right)\right)$	$\left((s_2,0.3),\left(0.71e^{i2\pi(0.71)},0.11e^{i2\pi(0.11)}\right)\right)$	$\left((s_1,0.2),\left(0.81e^{i2\pi(0.81)},0.11e^{i2\pi(0.11)}\right)\right)$
$\mathbb{O}_{CM-3}$	$\left((s_2,0.1),\left(0.52e^{i2\pi(0.52)},0.32e^{i2\pi(0.32)}\right)\right)$	$\left((s_2,0.1),\left(0.62e^{i2\pi(0.62)},0.22e^{i2\pi(0.22)}\right)\right)$	$\left((s_3,0.1),\left(0.72e^{i2\pi(0.72)},0.12e^{i2\pi(0.12)}\right)\right)$	$\left((s_2,0.3),\left(0.82e^{i2\pi(0.82)},0.12e^{i2\pi(0.12)}\right)\right)$
$\mathbb{O}_{CM-4}$	$\left((s_3,0.2),\left(0.53e^{i2\pi(0.53)},0.33e^{i2\pi(0.33)}\right)\right)$	$\left((s_1,0.3),\left(0.63e^{i2\pi(0.63)},0.23e^{i2\pi(0.23)}\right)\right)$	$\left((s_1,0.3),\left(0.73e^{i2\pi(0.73)},0.13e^{i2\pi(0.13)}\right)\right)$	$\left((s_3,0.4),\left(0.83e^{i2\pi(0.83)},0.13e^{i2\pi(0.13)}\right)\right)$
$\mathbb{O}_{CM-5}$	$\left((s_4,0),\left(0.54e^{i2\pi(0.54)},0.34e^{i2\pi(0.34)}\right)\right)$	$\left((s_3,0.2),\left(0.64e^{i2\pi(0.64)},0.24e^{i2\pi(0.24)}\right)\right)$	$\left((s_4,0.4),\left(0.74e^{i2\pi(0.74)},0.14e^{i2\pi(0.14)}\right)\right)$	$\left((s_1,0.4),\left(0.84e^{i2\pi(0.84)},0.14e^{i2\pi(0.14)}\right)\right)$

TABLE 2. Original decision matrix, whose values in the form of complex Pythagorean 2-tuple linguistic variables.

Symbols	$\mathcal{C}_{AT-1}$	$\mathcal{C}_{AT-2}$	$\mathcal{C}_{AT-3}$	$\mathcal{C}_{AT-4}$
$\mathbb{O}_{CM-1}$	$\left((s_4,0),\left(0.8e^{i2\pi(0.8)},0.3e^{i2\pi(0.3)}\right)\right)$	$\left((s_3,0.1),\left(0.6e^{i2\pi(0.6)},0.5e^{i2\pi(0.5)}\right)\right)$	$\left((s_1,0.2),\left(0.7e^{i2\pi(0.7)},0.5e^{i2\pi(0.5)}\right)\right)$	$\left((s_3,0.1),\left(0.8e^{i2\pi(0.8)},0.3e^{i2\pi(0.3)}\right)\right)$
$\mathbb{O}_{CM-2}$	$\left((s_1,0.2),\left(0.81e^{i2\pi(0.81)},0.31e^{i2\pi(0.31)}\right)\right)$	$\left((s_4,0.2),\left(0.61e^{i2\pi(0.61)},0.51e^{i2\pi(0.51)}\right)\right)$	$\left((s_2,0.3),\left(0.71e^{i2\pi(0.71)},0.51e^{i2\pi(0.51)}\right)\right)$	$\left((s_1,0.2),\left(0.81e^{i2\pi(0.81)},0.31e^{i2\pi(0.31)}\right)\right)$
$\mathbb{O}_{CM-3}$	$\left((s_2,0.1),\left(0.82e^{i2\pi(0.82)},0.32e^{i2\pi(0.32)}\right)\right)$	$\left((s_2,0.1),\left(0.62e^{i2\pi(0.62)},0.52e^{i2\pi(0.52)}\right)\right)$	$\left((s_3,0.1),\left(0.72e^{i2\pi(0.72)},0.52e^{i2\pi(0.52)}\right)\right)$	$\left((s_2,0.3),\left(0.82e^{i2\pi(0.82)},0.32e^{i2\pi(0.32)}\right)\right)$
$\mathbb{O}_{CM-4}$	$\left((s_3,0.2),\left(0.83e^{i2\pi(0.83)},0.33e^{i2\pi(0.33)}\right)\right)$	$\left((s_1,0.3),\left(0.63e^{i2\pi(0.63)},0.53e^{i2\pi(0.53)}\right)\right)$	$\left((s_1,0.3),\left(0.73e^{i2\pi(0.73)},0.53e^{i2\pi(0.53)}\right)\right)$	$\left((s_3,0.4),\left(0.83e^{i2\pi(0.83)},0.33e^{i2\pi(0.33)}\right)\right)$
$\mathbb{O}_{CM-5}$	$\left((s_4,0),\left(0.84e^{i2\pi(0.84)},0.34e^{i2\pi(0.34)}\right)\right)$	$\left((s_3,0.2),\left(0.64e^{i2\pi(0.64)},0.54e^{i2\pi(0.54)}\right)\right)$	$\left((s_4,0.4),\left(0.74e^{i2\pi(0.74)},0.54e^{i2\pi(0.54)}\right)\right)$	$\left((s_1,0.4),\left(0.84e^{i2\pi(0.84)},0.34e^{i2\pi(0.34)}\right)\right)$

$$\mathbb{O}_{CM-3} = \left((s_1, 0.1205), \left(0.4649e^{i2\pi(0.4649)}, 0.5856e^{i2\pi(0.5856)}\right)\right)$$

$$\mathbb{O}_{CM-4} = \left((s_1, 0.1587), \left(0.4735e^{i2\pi(0.4735)}, 0.5745e^{i2\pi(0.4745)}\right)\right)$$

$$\mathbb{O}_{CM-5} = \left((s_1, 0.2432), \left(0.4823e^{i2\pi(0.4823)}, 0.5638e^{i2\pi(0.5638)}\right)\right)$$

*Step 4:*By using Eq. (11), we examine the score values of the aggregated values, which are as follows:

$$\begin{aligned}\mathcal{S}_{SF}\left(\mathbb{O}_{CM-1}\right) = 0.1235, \mathcal{S}_{SF}\left(\mathbb{O}_{CM-2}\right) &= 0.1297, \mathcal{S}_{SF}\left(\mathbb{O}_{CM-3}\right)\\ &= 0.1405, \mathcal{S}_{SF}\left(\mathbb{O}_{CM-4}\right)\\ &= 0.1631, \mathcal{S}_{SF}\left(\mathbb{O}_{CM-5}\right) = 0.2064\end{aligned}$$

*Step 5:*Rank to all alternatives, and examine the best one, which is discussed as follows:

$$\mathbb{O}_{CM-5} \geq \mathbb{O}_{CM-4} \geq \mathbb{O}_{CM-3} \geq \mathbb{O}_{CM-2} \geq \mathbb{O}_{CM-1}$$

The best one is $\mathbb{O}_{CM-5}$.

*Step 6:*The end.

To examine the proficiency and effectiveness of the proposed operators, we construct more numerical results, whose values are in the form of complex Pythagorean 2-tuple linguistic numbers (CP2-TLNs), complex q-rung orthopair 2-tuple linguistic numbers (CQRO2-TLNs), and intuitionistic 2-tuple linguistic numbers (I2-TLNs) to evaluate uncertain and awkward information in genuine decision issues. The decision matrix, whose information in the form of complex Pythagorean 2-tuple linguistic variables are stated in the form of Table 2. To resolve the above issue, we choose the weight vector $\omega_W = (0.1, 0.2, 0.3, 0.4)^T$, and then the procedures of the MADM technique are as follows:

*Step 1:*Based on CP2-TLVs, we construct the decision matrix, which is in the form of Table 2.

Step 2: We normalize the decision matrix by using Eq. (23), which is given in step 1, if needed. But the information of Table 2 is not needed to be normalized, and so we use the data of Table 2 for continuing the processes of the MADM technique.

Step 3: By using Eq. (14), we aggregate the normalized decision matrix by using $q_{SC} = 2$, which is as follows:

$$\mathbb{O}_{CM-1} = \left((s_1, 0.1144), \left(0.5745e^{i2\pi(0.5745)}, 0.7604e^{i2\pi(0.7604)}\right)\right)$$

$$\mathbb{O}_{CM-2} = \left((s_1, 0.1111), \left(0.5835e^{i2\pi(0.5835)}, 0.7545e^{i2\pi(0.7545)}\right)\right)$$

$$\mathbb{O}_{CM-3} = \left((s_1, 0.1205), \left(0.5926e^{i2\pi(0.5926)}, 0.7487e^{i2\pi(0.7487)}\right)\right)$$

TABLE 3. Original decision matrix, whose values in the form of CQRO2-TLVs.

Symbols	$\mathcal{C}_{AT-1}$	$\mathcal{C}_{AT-2}$	$\mathcal{C}_{AT-3}$	$\mathcal{C}_{AT-4}$
$\mathbb{O}_{CM-1}$	$\begin{pmatrix}(s_4,0),\\(0.8e^{i2\pi(0.8)},0.7e^{i2\pi(0.7)})\end{pmatrix}$	$\begin{pmatrix}(s_3,0.1),\\(0.9e^{i2\pi(0.9)},0.5e^{i2\pi(0.5)})\end{pmatrix}$	$\begin{pmatrix}(s_1,0.2),\\(0.9e^{i2\pi(0.9)},0.7e^{i2\pi(0.7)})\end{pmatrix}$	$\begin{pmatrix}(s_3,0.1),\\(0.8e^{i2\pi(0.8)},0.7e^{i2\pi(0.7)})\end{pmatrix}$
$\mathbb{O}_{CM-2}$	$\begin{pmatrix}(s_1,0.2),\\(0.81e^{i2\pi(0.81)},0.71e^{i2\pi(0.71)})\end{pmatrix}$	$\begin{pmatrix}(s_4,0.2),\\(0.91e^{i2\pi(0.91)},0.51e^{i2\pi(0.51)})\end{pmatrix}$	$\begin{pmatrix}(s_2,0.3),\\(0.91e^{i2\pi(0.91)},0.71e^{i2\pi(0.71)})\end{pmatrix}$	$\begin{pmatrix}(s_1,0.2),\\(0.81e^{i2\pi(0.81)},0.71e^{i2\pi(0.71)})\end{pmatrix}$
$\mathbb{O}_{CM-3}$	$\begin{pmatrix}(s_2,0.1),\\(0.82e^{i2\pi(0.82)},0.72e^{i2\pi(0.72)})\end{pmatrix}$	$\begin{pmatrix}(s_2,0.1),\\(0.92e^{i2\pi(0.92)},0.52e^{i2\pi(0.52)})\end{pmatrix}$	$\begin{pmatrix}(s_3,0.1),\\(0.92e^{i2\pi(0.92)},0.72e^{i2\pi(0.72)})\end{pmatrix}$	$\begin{pmatrix}(s_2,0.3),\\(0.82e^{i2\pi(0.82)},0.72e^{i2\pi(0.72)})\end{pmatrix}$
$\mathbb{O}_{CM-4}$	$\begin{pmatrix}(s_3,0.2),\\(0.83e^{i2\pi(0.83)},0.73e^{i2\pi(0.73)})\end{pmatrix}$	$\begin{pmatrix}(s_1,0.3),\\(0.93e^{i2\pi(0.93)},0.53e^{i2\pi(0.53)})\end{pmatrix}$	$\begin{pmatrix}(s_1,0.3),\\(0.93e^{i2\pi(0.93)},0.73e^{i2\pi(0.73)})\end{pmatrix}$	$\begin{pmatrix}(s_3,0.4),\\(0.83e^{i2\pi(0.83)},0.73e^{i2\pi(0.73)})\end{pmatrix}$
$\mathbb{O}_{CM-5}$	$\begin{pmatrix}(s_4,0),\\(0.84e^{i2\pi(0.84)},0.74e^{i2\pi(0.74)})\end{pmatrix}$	$\begin{pmatrix}(s_3,0.2),\\(0.94e^{i2\pi(0.94)},0.54e^{i2\pi(0.54)})\end{pmatrix}$	$\begin{pmatrix}(s_4,0.4),\\(0.94e^{i2\pi(0.94)},0.74e^{i2\pi(0.74)})\end{pmatrix}$	$\begin{pmatrix}(s_1,0.4),\\(0.84e^{i2\pi(0.84)},0.74e^{i2\pi(0.74)})\end{pmatrix}$

$$\mathbb{O}_{\boldsymbol{CM-4}} = \left((s_1, 0.1587), \left(0.6018e^{i2\pi(0.6018)}, 0.7427e^{i2\pi(0.7427)}\right)\right)$$

$$\mathbb{O}_{\boldsymbol{CM-5}} = \left((s_1, 0.2432), \left(0.6111e^{i2\pi(0.6111)}, 0.7367e^{i2\pi(0.7367)}\right)\right)$$

Step 4: By using Eq. (11), we examine the score values of the aggregated values, which are as follows:

$$S_{SF}\left(\mathbb{O}_{\boldsymbol{CM-1}}\right) = 0.0918, S_{SF}\left(\mathbb{O}_{\boldsymbol{CM-2}}\right) = 0.0979, S_{SF}\left(\mathbb{O}_{\boldsymbol{CM-3}}\right) = 0.1077, S_{SF}\left(\mathbb{O}_{\boldsymbol{CM-4}}\right) = 0.1269, S_{SF}\left(\mathbb{O}_{\boldsymbol{CM-5}}\right) = 0.1631$$

Step 5: Rank to all alternatives, and examine the best one, which is discussed below:

$$\mathbb{O}_{\boldsymbol{CM-5}} \geq \mathbb{O}_{\boldsymbol{CM-4}} \geq \mathbb{O}_{\boldsymbol{CM-3}} \geq \mathbb{O}_{\boldsymbol{CM-2}} \geq \mathbb{O}_{\boldsymbol{CM-1}}$$

The best one is $\mathbb{O}_{\boldsymbol{CM-5}}$.

Step 6: The end.

The decision matrix whose information in the form of complex q-rung orthopair 2-tuple linguistic variables is stated in the form of Table 3. To resolve the above issue, we choose the weight vector $\omega_W = (0.1, 0.2, 0.3, 0.4)^T$, and the procedures of the MADM technique are as follows:

Step 1: Based on CQRO2-TLVs, we construct the decision matrix with the form of Table 3.

Step 2: We normalize the decision matrix by using Eq. (23), which is given in step 1, if needed. But the information of Table 3 is not needed to be normalized, so we consider the data of Table 3 for continuing the processes of the MADM technique.

Step 3: By using Eq. (14), we aggregate the normalized decision matrix by using $q_{SC} = 5$, which is as follows:

$$\mathbb{O}_{\boldsymbol{CM-1}} = \left((s_1, 0.1144), \left(0.7774e^{i2\pi(0.7774)}, 0.909e^{i2\pi(0.909)}\right)\right)$$

$$\mathbb{O}_{\boldsymbol{CM-2}} = \left((s_1, 0.1111), \left(0.7874e^{i2\pi(0.7874)}, 0.9054e^{i2\pi(0.9054)}\right)\right)$$

$$\mathbb{O}_{\boldsymbol{CM-3}} = \left((s_1, 0.1205), \left(0.7977e^{i2\pi(0.7977)}, 0.9017e^{i2\pi(0.9017)}\right)\right)$$

$$\mathbb{O}_{\boldsymbol{CM-4}} = \left((s_1, 0.1587), \left(0.8082e^{i2\pi(0.8082)}, 0.8978e^{i2\pi(0.8978)}\right)\right)$$

$$\mathbb{O}_{\boldsymbol{CM-5}} = \left((s_1, 0.2432), \left(0.8189e^{i2\pi(0.8189)}, 0.8938e^{i2\pi(0.8938)}\right)\right)$$

Step 4: By using Eq. (11), we examine the score values of the aggregated values, which are as follows:

$$S_{SF}\left(\mathbb{O}_{\boldsymbol{CM-1}}\right) = 0.0595, S_{SF}\left(\mathbb{O}_{\boldsymbol{CM-2}}\right) = 0.0702, S_{SF}\left(\mathbb{O}_{\boldsymbol{CM-3}}\right) = 0.0841, S_{SF}\left(\mathbb{O}_{\boldsymbol{CM-4}}\right) = 0.1068, S_{SF}\left(\mathbb{O}_{\boldsymbol{CM-5}}\right) = 0.1469$$

Step 5: Rank to all alternatives, and examine the best one, which is discussed below:

$$\mathbb{O}_{\boldsymbol{CM-5}} \geq \mathbb{O}_{\boldsymbol{CM-4}} \geq \mathbb{O}_{\boldsymbol{CM-3}} \geq \mathbb{O}_{\boldsymbol{CM-2}} \geq \mathbb{O}_{\boldsymbol{CM-1}}$$

The best one is $\mathbb{O}_{\boldsymbol{CM-5}}$.

Step 6: The end.

B. COMPARATIVE ANALYSIS

In this subsection, we make comparisons between the proposed operators and some existing operators. These existing operators are as follows. Liu and Chen [27] discovered the theory of intuitionistic 2-tuple linguistic aggregation operators, Wei et al. [28] explored the Pythagorean 2-tuple linguistic aggregation operators, Tang et al. [32] presented the Pythagorean 2-tuple linguistic Bonferroni mean operators, and Ju et al. [29] pioneered the q-rung orthopair 2-tuple linguistic Maclaurin symmetric mean operators. The comparative analysis by using the information of Table 1 is shown in Table 4.

From the analysis, we get the results in which the explored operators of WBM and WGBM are given the same results, which are discussed in Table 4. The best alternative is $\mathbb{O}_{\boldsymbol{CM-5}}$. Both concepts provide that the first two terms are same, but the last three terms are different due to its

TABLE 4. Comparison between explored works with some existing works by using the information of Table 1.

Methods	Operators	Score values	Ranking Values
Liu & Chen [27]	WA	*not available*	–
	WG	*not available*	–
Wei et al. [28]	WA	*not available*	–
	WG	*not available*	–
Tang et al. [32]	WBM	*not available*	–
	WGBM	*not available*	–
Ju et al. [29]	WMSM	*not available*	–
	WGMSM	*not available*	–
Proposed work for q=1	WBM	$\mathcal{S}_{SF}(\mathbb{O}_{CM-1}) = 0.1235, \mathcal{S}_{SF}(\mathbb{O}_{CM-2}) = 0.1297, \mathcal{S}_{SF}(\mathbb{O}_{CM-3}) = 0.1405, \mathcal{S}_{SF}(\mathbb{O}_{CM-4}) = 0.1631, \mathcal{S}_{SF}(\mathbb{O}_{CM-5}) = 0.2064$	$\mathbb{O}_{CM-5} \geq \mathbb{O}_{CM-4} \geq \mathbb{O}_{CM-3} \geq \mathbb{O}_{CM-2} \geq \mathbb{O}_{CM-1}$
	WGBM	$\mathcal{S}_{SF}(\mathbb{O}_{CM-1}) = 0.2188, \mathcal{S}_{SF}(\mathbb{O}_{CM-2}) = 0.2123, \mathcal{S}_{SF}(\mathbb{O}_{CM-3}) = 0.2132, \mathcal{S}_{SF}(\mathbb{O}_{CM-4}) = 0.23, \mathcal{S}_{SF}(\mathbb{O}_{CM-5}) = 0.2715$	$\mathbb{O}_{CM-5} \geq \mathbb{O}_{CM-4} \geq \mathbb{O}_{CM-1} \geq \mathbb{O}_{CM-3} \geq \mathbb{O}_{CM-2}$
Proposed work for q=2	WBM	$\mathcal{S}_{SF}(\mathbb{O}_{CM-1}) = -0.008, \mathcal{S}_{SF}(\mathbb{O}_{CM-2}) = -0.0004, \mathcal{S}_{SF}(\mathbb{O}_{CM-3}) = 0.0064, \mathcal{S}_{SF}(\mathbb{O}_{CM-4}) = 0.015, \mathcal{S}_{SF}(\mathbb{O}_{CM-5}) = 0.0277$	$\mathbb{O}_{CM-5} \geq \mathbb{O}_{CM-4} \geq \mathbb{O}_{CM-3} \geq \mathbb{O}_{CM-2} \geq \mathbb{O}_{CM-1}$
	WGBM	$\mathcal{S}_{SF}(\mathbb{O}_{CM-1}) = 0.2942, \mathcal{S}_{SF}(\mathbb{O}_{CM-2}) = 0.2873, \mathcal{S}_{SF}(\mathbb{O}_{CM-3}) = 0.2905, \mathcal{S}_{SF}(\mathbb{O}_{CM-4}) = 0.3154, \mathcal{S}_{SF}(\mathbb{O}_{CM-5}) = 0.3746$	$\mathbb{O}_{CM-5} \geq \mathbb{O}_{CM-4} \geq \mathbb{O}_{CM-1} \geq \mathbb{O}_{CM-3} \geq \mathbb{O}_{CM-2}$
Proposed work for q=5	WBM	$\mathcal{S}_{SF}(\mathbb{O}_{CM-1}) = -0.074, \mathcal{S}_{SF}(\mathbb{O}_{CM-2}) = -0.068, \mathcal{S}_{SF}(\mathbb{O}_{CM-3}) = -0.064, \mathcal{S}_{SF}(\mathbb{O}_{CM-4}) = -0.063, \mathcal{S}_{SF}(\mathbb{O}_{CM-5}) = -0.062$	$\mathbb{O}_{CM-5} \geq \mathbb{O}_{CM-4} \geq \mathbb{O}_{CM-3} \geq \mathbb{O}_{CM-2} \geq \mathbb{O}_{CM-1}$
	WGBM	$\mathcal{S}_{SF}(\mathbb{O}_{CM-1}) = 0.3362, \mathcal{S}_{SF}(\mathbb{O}_{CM-2}) = 0.3287, \mathcal{S}_{SF}(\mathbb{O}_{CM-3}) = 0.3327, \mathcal{S}_{SF}(\mathbb{O}_{CM-4}) = 0.3618, \mathcal{S}_{SF}(\mathbb{O}_{CM-5}) = 0.4303$	$\mathbb{O}_{CM-5} \geq \mathbb{O}_{CM-4} \geq \mathbb{O}_{CM-1} \geq \mathbb{O}_{CM-3} \geq \mathbb{O}_{CM-2}$

TABLE 5. Comparison between explored works with some existing works by using the information of Table 2.

Methods	Operators	Score values	Ranking Values
Liu & Chen [27]	WA	*not available*	–
	WG	*not available*	–
Wei et al. [28]	WA	*not available*	–
	WG	*not available*	–
Tang et al. [32]	WBM	*not available*	–
	WGBM	*not available*	–
Ju et al. [29]	WMSM	*not available*	–
	WGMSM	*not available*	–
Proposed work for q=1	WBM	*not available*	–
	WGBM	*not available*	–
Proposed work for q=2	WBM	$\mathcal{S}_{SF}(\mathbb{O}_{CM-1}) = 0.0918, \mathcal{S}_{SF}(\mathbb{O}_{CM-2}) = 0.0979, \mathcal{S}_{SF}(\mathbb{O}_{CM-3}) = 0.1077, \mathcal{S}_{SF}(\mathbb{O}_{CM-4}) = 0.1269, \mathcal{S}_{SF}(\mathbb{O}_{CM-5}) = 0.1631$	$\mathbb{O}_{CM-5} \geq \mathbb{O}_{CM-4} \geq \mathbb{O}_{CM-3} \geq \mathbb{O}_{CM-2} \geq \mathbb{O}_{CM-1}$
	WGBM	$\mathcal{S}_{SF}(\mathbb{O}_{CM-1}) = 0.2506, \mathcal{S}_{SF}(\mathbb{O}_{CM-2}) = 0.243, \mathcal{S}_{SF}(\mathbb{O}_{CM-3}) = 0.2437, \mathcal{S}_{SF}(\mathbb{O}_{CM-4}) = 0.2625, \mathcal{S}_{SF}(\mathbb{O}_{CM-5}) = 0.3092$	$\mathbb{O}_{CM-5} \geq \mathbb{O}_{CM-4} \geq \mathbb{O}_{CM-1} \geq \mathbb{O}_{CM-2} \geq \mathbb{O}_{CM-3}$
Proposed work for q=3	WBM	$\mathcal{S}_{SF}(\mathbb{O}_{CM-1}) = 0.0122, \mathcal{S}_{SF}(\mathbb{O}_{CM-2}) = 0.019, \mathcal{S}_{SF}(\mathbb{O}_{CM-3}) = 0.0667, \mathcal{S}_{SF}(\mathbb{O}_{CM-4}) = 0.0377,$	$\mathbb{O}_{CM-5} \geq \mathbb{O}_{CM-3} \geq \mathbb{O}_{CM-4} \geq \mathbb{O}_{CM-2} \geq \mathbb{O}_{CM-1}$
		$\mathcal{S}_{SF}(\mathbb{O}_{CM-5}) = 0.0669$	
	WGBM	$\mathcal{S}_{SF}(\mathbb{O}_{CM-1}) = 0.307, \mathcal{S}_{SF}(\mathbb{O}_{CM-2}) = 0.2987, \mathcal{S}_{SF}(\mathbb{O}_{CM-3}) = 0.3007, \mathcal{S}_{SF}(\mathbb{O}_{CM-4}) = 0.3252, \mathcal{S}_{SF}(\mathbb{O}_{CM-5}) = 0.3844$	$\mathbb{O}_{CM-5} \geq \mathbb{O}_{CM-4} \geq \mathbb{O}_{CM-1} \geq \mathbb{O}_{CM-3} \geq \mathbb{O}_{CM-2}$

structure. Therefore, the best alternative is $\mathbb{O}_{CM-5}$ by using the explored WBM and WGBM operators.

From the above analysis, we get the results in which the proposed operators of WBM and WGBM are given the same

TABLE 6. Comparison between proposed works with some existing works by using the information of Table 3.

Methods	Operators	Score values	Ranking Values
Liu & Chen [27]	WA	*not available*	–
	WG	*not available*	–
Wei et al. [28]	WA	*not available*	–
	WG	*not available*	–
Tang et al. [32]	WBM	*not available*	–
	WGBM	*not available*	–
Ju et al. [29]	WMSM	*not available*	–
	WGMSM	*not available*	–
Proposed work for q=1	WBM	*not available*	–
	WGBM	*not available*	–
Proposed work for q=2	WBM	*not available*	–
	WGBM	*not available*	–
Proposed work for q=5	WBM	$\mathcal{S}_{SF}(\mathbb{O}_{CM-1}) = 0.0595, \mathcal{S}_{SF}(\mathbb{O}_{CM-2}) = 0.0702, \mathcal{S}_{SF}(\mathbb{O}_{CM-3}) = 0.0841, \mathcal{S}_{SF}(\mathbb{O}_{CM-4}) = 0.1068, \mathcal{S}_{SF}(\mathbb{O}_{CM-5}) = 0.1469$	$\mathbb{O}_{CM-5} \geq \mathbb{O}_{CM-4} \geq \mathbb{O}_{CM-3} \geq \mathbb{O}_{CM-2} \geq \mathbb{O}_{CM-1}$
	WGBM	$\mathcal{S}_{SF}(\mathbb{O}_{CM-1}) = 0.2738, \mathcal{S}_{SF}(\mathbb{O}_{CM-2}) = 0.2619, \mathcal{S}_{SF}(\mathbb{O}_{CM-3}) = 0.2588, \mathcal{S}_{SF}(\mathbb{O}_{CM-4}) = 0.274, \mathcal{S}_{SF}(\mathbb{O}_{CM-5}) = 0.3163$	$\mathbb{O}_{CM-5} \geq \mathbb{O}_{CM-4} \geq \mathbb{O}_{CM-1} \geq \mathbb{O}_{CM-2} \geq \mathbb{O}_{CM-3}$

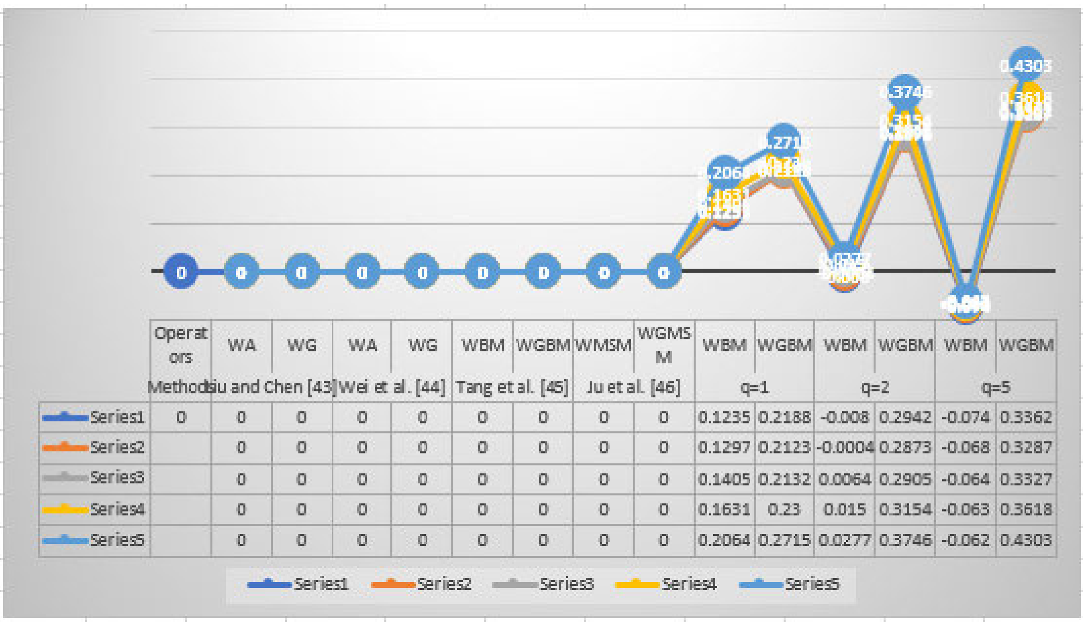

FIGURE 1. Geometrical representations of Table 4

results, which are discussed in Table 5. The best alternative is $\mathbb{O}_{CM-5}$. Both concepts provide that the first two terms are same, but the last three terms are different due to its structure. Therefore, the best alternative is $\mathbb{O}_{CM-5}$ by using the proposed WBM and WGBM operators. Furthermore, we get the results in which the proposed operators of WBM and WGBM are given the same results, which are shown in Table 6. The best alternative is $\mathbb{O}_{CM-5}$. Both concepts provide that the first two terms are same, but the last three terms are different due to its structure. Therefore, the best alternative is $\mathbb{O}_{CM-5}$ by using the proposed WBM and WGBM operators.

C. GRAPHICAL REPRESENTATIONS OF THE PROPOSED OPERATORS

The concept of CQRO2-TLV is extensive powerful and more proficient technique to cope with awkward information in genuine decision issues. We further demonstrate the compassions between the proposed operators and existing operators of Liu and Chen [27], Wei et al. [28], Tang et al. [32] and Ju et al. [29] with the help of figures. These figures are quite helpful for easily understanding the difference between values and their ranking. The geometrical representations of the information in Tables 4, 5, and 6 are discussed below,

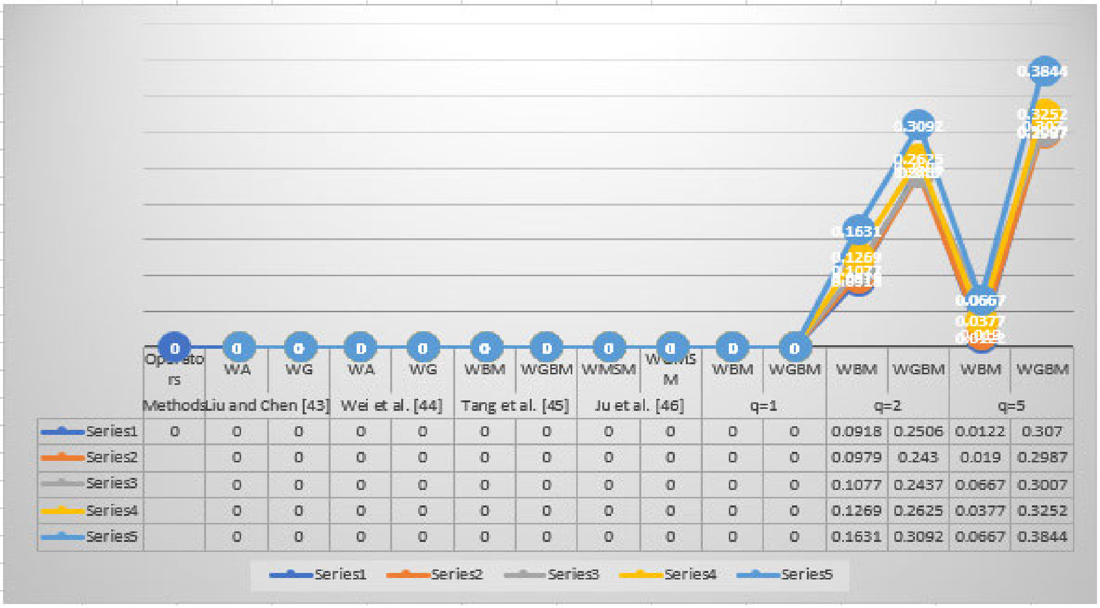

FIGURE 2. Geometrical representations of Table 5

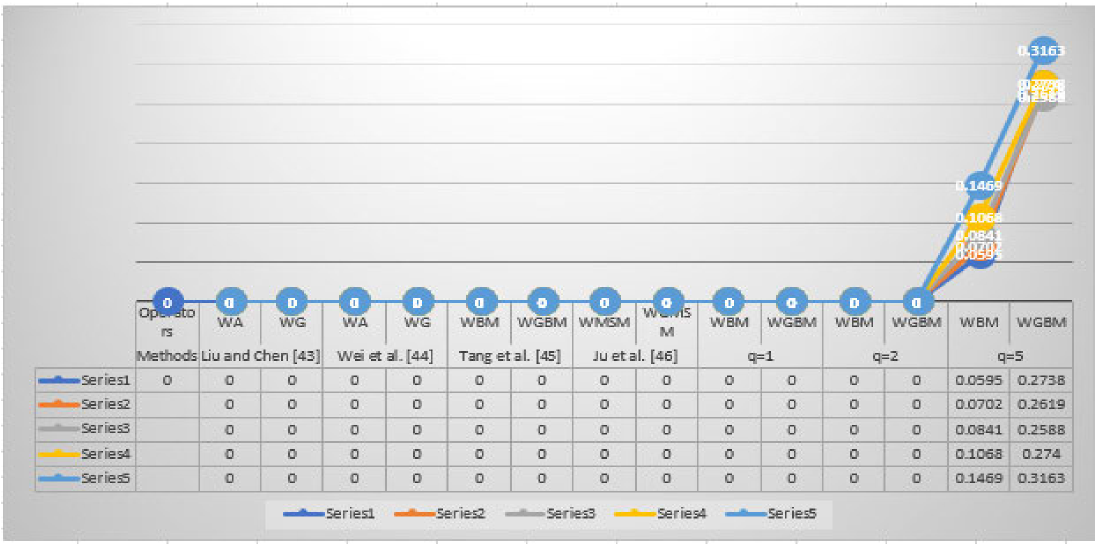

FIGURE 3. Geometrical representations of the Table 6

which express the reliability and superiority of the proposed operators based on CQRO2-TLVs.

The graphical interpretations of Table 4 are shown in Fig. 1. From Fig. 1, it is clear that, it contains five series which express the set of alternatives with different colors. These different colors denote the flow of the values, as shown in Fig. 1. Only the proposed works for q=1, q=2, and q=5, provide that the values and existing operators cannot solve it because it contains two-dimension information in a single set. From Fig. 1, we easily get the best alternative, which is $\mathbb{O}_{CM-5}$.

The graphical interpretations of Table 5 are shown in Fig. 2. From Fig. 2, it is clear that, it contains five series which express the set of alternatives with different colors. These different colors denote the flow of the values, as shown in Fig. 2. Only the proposed works for q=2, and q=5, provides that the values and existing operators cannot solve it because it contains two-dimension information in a single set. Based on Fig. 2, we easily get the value of the best alternative, which is $\mathbb{O}_{CM-5}$. Furthermore, the graphical interpretations of Table 6 are shown in Fig. 3. From Fig. 3, it is clear that, it contains five series which express the set of alternatives with different colors. These different colors denote the flow of the values as shown in Fig. 3. Only the proposed work for q=5, provides that the values and existing operators cannot solve it because it contains two-dimension information in a single set. According to Fig. 3, we easily get the value of the best alternative, which is $\mathbb{O}_{CM-5}$.

The reasons of these results are that, for Fig. 1, we choose the complex intuitionistic 2-tuple linguistic kind of information, for Fig. 2, we choose the complex Pythagorean 2-tuple linguistic kind of information, and for Fig. 3, we choose the complex q-rung orthopair 2-tuple linguistic kind of information. The parented work cannot solve it by using any one operator, but the existing work easily solve it by using explored work, due to its structure. The complex q-rung orthopair 2-tuple linguistic set contains 2-TLV, complex-valued supporting grade, and complex-valued supporting against grade with the conditions that is the sum of q-powers of the real part (also for imaginary part) of the supporting grade and supporting against grade cannot exceed from a unit interval. Therefore, the explored concept and operators are more proficient and extensive superior than these existing methods.

V. CONCLUSION

The proposed complex q-rung orthopair 2-tuple linguistic variable (CQRO2-TLV) is a mixture of the two various theories of CQROFS and 2-TLV. CQRO2-TLV comprises 2-TLV and the grades of supporting and supporting against in the complex plane. According to these properties and theories of CQRO2-TLV with its operational laws presented in the paper, it can be a competent procedure to assess ambiguous and erratic information in most decision problems. The two new techniques with the complex q-rung orthopair 2-tuple linguistic WBM (CQRO2-TLWBM) and complex q-rung orthopair 2-tuple linguistic WGBM (CQRO2-TLWGBM) operators are used to manage the MADM problems under fuzzy environment. Based on the CQRO2-TLWBM and CQRO2-TLWGBM operators, we constructed a MADM technique to cope ambiguous and impulsive information in real decision problems. We then applied it to a green supply chain management (GSCM) for choosing green suppliers (GSs). To examine the advantages of the presented MADM techniques, we used different methods, whose values are in the form of complex pythagorean 2-tuple linguistic numbers (CP2-TLNs), complex q-rung orthopair 2-tuple linguistic numbers (CQRO2-TLNs) and intuitionistic 2-tuple linguistic numbers (I2-TLNs), to make comparisons between the proposed method and some existing methods. Based on the obtained results, the proposed method by employing CQRO2-TLWBM and CQRO2-TLWGBM operators is well suited in handling different kinds of sets in the fuzzy environment with legitimacy and prevalence of the proposed technique by contrasting other existing operators. Although the proposed method is good in complex q-rung orthopair fuzzy environment. However, it can only handle the two-dimensional information with membership and non-membership. In the case of three-dimensional information, such as representing truth, falsity and indeterminacy, the proposed technique will fail because it can only represent two of them under two dimensions. For handling three-dimensional information, the spherical fuzzy sets [44] and T-spherical fuzzy sets [45], [46] need to be considered. In our future work, we will extend the proposed method to complex spherical fuzzy sets and complex T-spherical fuzzy sets so that three-dimensional information with truth, falsity and indeterminacy can be handled and applied in more extents of MADM problems.

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ZEESHAN ALI received the B.S. degree in mathematics from Abdul Wali Khan University Mardan, Pakistan, in 2016, and the M.S. degree in mathematics from International Islamic University Islamabad, Pakistan, in 2018, where he is currently pursuing the Ph.D. degree in mathematics. He has published more than 100 articles in reputed journals. His research interests include aggregation operators, fuzzy logic, and fuzzy decision making and their applications.

TAHIR MAHMOOD received the Ph.D. degree in mathematics from Quaid-i-Azam University, Islamabad, Pakistan, in 2012. He is currently an Assistant Professor in mathematics with the Department of Mathematics and Statistics, International Islamic University Islamabad, Pakistan. He has published more than 190 international publications and he has also produced more than 45 M.S. students and six Ph.D. students. His research interests include algebraic structures, fuzzy algebraic structures, and soft sets and their generalizations.

MIIN-SHEN YANG received the B.S. degree in mathematics from Chung Yuan Christian University, Taoyuan, Taiwan, in 1977, the M.S. degree in applied mathematics from National Chiao Tung University, Hsinchu, Taiwan, in 1980, and the Ph.D. degree in statistics from the University of South Carolina, Columbia, USA, in 1989. In 1989, he joined as a Faculty Member of the Department of Mathematics, Chung Yuan Christian University (CYCU), as an Associate Professor, where he has been a Professor, since 1994. From 1997 to 1998, he was a Visiting Professor with the Department of Industrial Engineering, University of Washington, Seattle, USA. From 2001 to 2005, he was the Chairperson of the Department of Applied Mathematics, CYCU. Since 2012, he has been a Distinguished Professor of the Department of Applied Mathematics. He was the Director of Chaplain's Office and the Dean of the College of Science, CYCU. His research interests include applications of statistics, fuzzy clustering, soft computing, pattern recognition, and machine learning. He was an Associate Editor of IEEE Transactions on Fuzzy Systems, from 2005 to 2011, and is an Editorial Board Member of the Computer Science and Engineering Section in the journal *Electronics* (MDPI).