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RESEARCH ARTICLE

On Distance-Based Attribute Reduction With α, β-Level Intuitionistic Fuzzy Sets

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ABSTRACT Attribute reduction, often referred to as feature selection, is a vital step in data preprocessing aimed at eliminating unnecessary attributes and enhancing the efficiency of classification models. Intuitionistic fuzzy sets are widely acknowledged for their highly effective approach to the attribute reduction problem in decision tables, especially in the context of noisy decision tables with low classification accuracy. Nonetheless, the computational complexity of this approach increases significantly due to the additional incorporation of a non-membership component when calculating the significance measures of each attribute. Besides, some intuitionistic fuzzy equivalence classes may include objects with relatively low similarity or high diversity degrees generated from noisy data. Performing calculations on these objects can be time-consuming and lead to inefficiencies in the attribute reduction process. To address the aforementioned issue, we first define an α, β-level set. This allows us to eliminate the mentioned noisy objects from the intuitionistic fuzzy equivalence classes and transform them into α, β-level intuitionistic fuzzy equivalence classes. Subsequently, we construct a distance measure between two α, β-level intuitionistic fuzzy partitions and define a new reduct to preserve the distance between two α, β-level intuitionistic fuzzy partitions generated by the condition attribute set and the decision attribute set. Finally, we propose a novel and efficient heuristic attribute reduction algorithm to find the new reduct, in which we also use a significance measure based on the α, β-level intuitionistic fuzzy partition distance to determine the vital attribute at each step of the algorithm. Obviously, our algorithm is also applied to various benchmark datasets for comparative analysis against existing algorithms. The experimental results demonstrate that the proposed algorithm not only improves the accuracy of the classification model but also significantly reduces execution time when compared to intuitionistic fuzzy rough set-based algorithms applied to noisy datasets with high dimensionality.

INDEX TERMS Attribute reduction, noisy data, intuitionistic fuzzy sets, α, β-level intuitionistic fuzzy sets, distance measures.

I. INTRODUCTION

Attribute reduction plays a crucial role in data preprocessing as it focuses on eliminating redundant and unnecessary attributes. This process is essential for improving the performance of machine learning and data mining algorithms.

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The traditional rough set theory introduced by Pawlak [1] offers an effective solution for the attribute reduction problem in complete decision tables. Usually, attribute reduction approaches have two main directions: methods based on indiscernibility matrix, such as [2] and [4], and significance metric functions, such as [5], [6], [7] and [8]. These approaches are efficient for the decision tables only containing categorical attributes [9], [10]. However, when the decision tables have the numerical value domain, these methods have to divide the value domain of each numerical attribute into intervals corresponding to discrete values. To solve the attribute reduction problem on the original decision table, Dubois et al. [11], [12] proposed a fuzzy rough set (FRS) model. Based on the FRS model, the researchers constructed attribute reduction methods using different measures on the decision tables that contain numeric attributes. Some typical methods were proposed based on fuzzy membership functions [13], fuzzy positive regions [14], [47], fuzzy information entropy [15], [39], and fuzzy distance [16]. Jensen and Sen [17] developed the theory of the dependency function in the traditional rough set model into the fuzzy occurrence. Then, they designed attribute reduction algorithms based on the concept of FRS in [5], [9], [10], [15], [18], [19], [20], [21], [22], [23], [24], [25], [26], and [27]. One development direction from fuzzy rough set theory is fuzzy neighborhood rough sets. According to this approach, Sun [44] built uncertainty measures and designed a feature selection algorithm based on fuzzy neighborhood multigranulation rough sets. Zhang [45] defined fuzzy neighborhood relative decision entropy and proposed an attribute reduction algorithm. Sun [46] introduced a new method using multilabel fuzzy neighborhood rough sets and maximum relevance minimum redundancy. This approach is considered suitable for multilabel datasets that have missing labels. Additionally, several studies have expanded incremental algorithms to solve issues with incomplete dynamic decision tables. Specifically, Giang et al. proposed hybrid incremental algorithms for adding and deleting object sets based on the tolerance rough set [3]. Later, Thang et al. also formulated incremental algorithms for two cases of supplementing and removing attribute sets [43]. In general, their experimental results show that the reduction attribute algorithms following the fuzzy rough set approach have better results than traditional algorithms for the decision tables with continuous and numerical value domains. Unfortunately, Hung et al. [28] have shown that attribute reduction by the FRS approach is less effective in noisy or low classification accuracy data sets.

Recently, some researchers have proposed using the intuitionistic fuzzy rough set (IFRS) model to solve the attribute reduction problem. This model has minimized the noisy information because the added non-membership function component can adjust the noise objects to the correct classifier [29]. In the case of noisy or low classification accuracy data sets, the attribute reduction algorithms based on the IFRS model have better processing abilities than the FRS model-based algorithms. Following this approach, the authors in the work [30] constructed an IFRS model based on granular structures. Then they proposed a filtering algorithm (IFPR) to select attributes on the decision table. Experimental results show that the IFPR algorithm performs superior to the algorithms [31], [32], [33], [34] according to the FRS approach. Giang et al. in work [35] recently constructed a distance measure based on the intuitionistic fuzzy set (IFS) model and proposed the IFDBAR algorithm to find the reduct on the decision table. The experimental results of the algorithm are compared with the algorithm in [16] and have proven effective on noisy data. However, the disadvantage of the attribute reduction algorithms based on the IFS according to the filter approach is the slow computational processing speed. The addition of non-membership in the model explains this. The attribute selection process of the algorithms goes through each loop and uses the significance measure of each attribute. The significance measure is determined based on the cardinality of the intuitionistic fuzzy information granulations, including the similarity and diversity degrees. Therefore, the algorithms following this approach must compute more than those following the FRS approach. In addition, the IFS model still has some limitations that need to be improved, as follows: Firstly, some elements in the intuitionistic fuzzy information granulations with small similarity degrees will contribute little to the cardinality calculation process. Therefore, the algorithm will survive many redundant and unnecessary computations. Secondly, the information granulations on the noisy data set will contain much false information that affects the selection process of necessary attributes. As a result, the attribute subset generated by the algorithm has low classification accuracy. In this paper, we propose an attribute reduction algorithm based on the IFS approach and supplement α, β levels to reduce the computational redundancy and noise of the intuitionistic fuzzy information granulations. The main content of the paper is as follows:

1. In Part II, we summarize some basic concepts of IFS theory and intuitionistic fuzzy relation.
2. In Part III, we present the α, β-level concept and the definition of an IFS according to α, β-level.
3. In Part IV, we propose an attribute reduction algorithm based on the intuitionistic fuzzy partition distance using the filter approach.
4. Parts V and VI show the results and discussion of the proposed algorithm. Finally, there are summaries of the paper.

II. PRELIMINARIES

This part will present basic concepts related to the IFS model. These concepts will be an essential foundation for proposing an attribute reduction algorithm presented in the third part of the paper. Some basic concepts can be cited in [3], [31], [35], [36], [37], and [38].

A. DECISION TABLES AND INTUITIONISTIC FUZZY SETS

Firstly, a decision table is a pair of $DT = (U, C \cup D)$, where U is a finite nonempty set of objects, also known as the universe, C and D are finite nonempty sets of attributes such that each $a \in C \cup D$ determines a map $a : U \rightarrow V_a$, where V_a is the value set of a. Then, for $u \in U$ and $a \in C \cup D$, the value of a for u is written as $a(u)$. Here, we shall call C as condition attributes and D as decision attributes.

TABLE 1. An example decision table.

U	c_1	c_2	c_3	c_4	d
u_1	0.8	0.6	1	0	0
u_2	0.8	0	0.2	0.8	1
u_3	0.6	0.8	0.6	0.4	0
u_4	0	0.6	0	1	1

Not losing the comprehensive characteristics, hypothesis D only has one decision attribute d (if D has many attributes, a transformation that can be reduced to an attribute [37]). From this, we consider the decision table $DT = (U, C \cup \{d\})$.

Example 1: A decision table $DT = (U, C \cup \{d\})$ as Table 1 where $U = \{u_1, u_2, u_3, u_4\}$ and $C = \{c_1, c_2, c_3, c_4\}$.

Definition 1: Given a decision table $DT = (U, C \cup \{d\})$, an intuitionistic fuzzy set (IFS) P on U has the form $P = \{\langle u, \mu_P(u), \vartheta_P(u)\rangle \,|u \in U\}$, with $\mu_P(u) : U \rightarrow [0,1]$ and $\vartheta_P(u) : U \rightarrow [0,1]$ are respectively the membership and non-membership degrees of u in P such that $0 \leq \mu_P(u) + \vartheta_P(u) \leq 1$, $\forall u \in U$.

The hesitant degree of u in P is determined by $\pi_P(u) = 1 - \mu_P(u) - \vartheta_P(u)$. When $\pi_P(u) = 0, \forall u \in U$, IFS becomes a traditional fuzzy set. The cardinality of P is denoted as $|P|$ determined by the formula [40]:

$$|P| = \sum_{u\in U} \frac{1 + \mu_P(u) - \vartheta_P(u)}{2} \tag{1}$$

Consider two IFSs P and Q on U, we will define several set operations to compare them as follow [38]:

1) $P \subseteq Q$ iff $\mu_P(u) \leq \mu_Q(u)$ and $\vartheta_P(u) \geq \vartheta_Q(u)$ for any $u \in U$.
2) $P = Q$ iff $P \subseteq Q$ and $Q \subseteq P$.
3) $P \cap Q = \left\{\left(u, \min\left(\mu_P(u), \mu_Q(u)\right), \max\left(\vartheta_P(u), \vartheta_Q(u)\right)\right)\right\}$.
4) $P \cup Q = \left\{\left(u, \max\left(\mu_P(u), \mu_Q(u)\right), \min\left(\vartheta_P(u), \vartheta_Q(u)\right)\right)\right\}$.

To facilitate the deployment of definitions and calculation formulas later, an IFS P will sometimes be briefly represented by two components, membership and non-membership. Specifically, $P = \{\langle \mu_P(u), \vartheta_P(u)\rangle \,|u \in U\}$.

B. INTUITIONISTIC FUZZY EQUIVALENCE CLASSES

Definition 2: Let U be a finite nonempty set of objects. An intuitionistic fuzzy binary relation R on $U \times U$ is defined as follows:

$$R = \{((u, v), \mu_R(u, v), \vartheta_R(u, v)) \,|(u, v) \in U \times U\} \tag{2}$$

where $\mu_R(u, v) \in [0, 1]$ and $\vartheta_R(u, v) \in [0, 1]$ are the similarity and diversity degrees, respectively. The pair $(\mu_R(u, v), \vartheta_R(u, v))$ is called an intuitionistic fuzzy number between two objects u and v, which satisfies $0 \leq \mu_R(u, v) + \vartheta_R(u, v) \leq 1$. Then, R is called an intuitionistic fuzzy equivalence relation (IFER) if R satisfies:

1) Reflexive: $\mu_R(u, u) = 1$ and $\vartheta_R(u, u) = 0$, $\forall u \in U$.
2) Symmetric: $\mu_R(u, v) = \mu_R(v, u)$ and $\vartheta_R(u, v) = \vartheta_R(v, u)$, $\forall u, v \in U$.
3) Transitive:
$\mu_R(u, v) \geq \max_{t\in U} \{\min(\mu_R(u, t), \mu_R(t, v))\}$;
$\vartheta_R(u, v) \leq \min_{t\in U} \{\max(\vartheta_R(u, t), \vartheta_R(t, v))\}$, $\forall u, v \in U$.

Given a decision table $DT = (U, C \cup \{d\})$, each attribute subset $A \subseteq C$ determines an IFER, denoted as R_A. The IFER R_A generates an intuitionistic fuzzy partition (IFP) on U, $\mathscr{P}_A = \{R_A[u] \,|u \in U\}$, in which $R_A[u] = \left\{\left(v, \mu_{R_A[u]}(v), \vartheta_{R_A[u]}(v)\right) |v \in U\right\}$ is an intuitionistic fuzzy equivalence class (IFEC) of u according to R_A and can be called an information granulation. It is easy to see that each IFEC $R_A[u]$ is an IFS on U. To simplify the denotation, for each object v, we denote $R_A[u](v) = (\mu_A[u](v), \vartheta_A[u](v))$.

For $A, B \subseteq C$, we have $R_A[u] = \cap_{a\in A} R_a[u]$ and $R_{A\cup B}[u] = R_A[u] \cap R_B[u]$. This means that $R_{A\cup B}[u](v) = (\min\{\mu_A[u](v), \mu_B[u](v)\}, \max\{\vartheta_A[u](v), \vartheta_B[u](v)\})$ and $\mathscr{P}_{A\cup B} = \mathscr{P}_A \cap \mathscr{P}_B$.

III. α, β-LEVEL INTUITIONISTIC FUZZY SETS

As mentioned, the fuzzy rough set theory is ineffective in dealing with decision tables with low initial classification accuracy, while the IFS theory still has limitations in terms of processing time as it involves calculations on both membership and non-membership functions. For addressing above issues, we shall introduce in this section a new extension of IFSs, called α, β-level IFSs, to handle the noise of misclassification and perturbation.

We continue to consider an intuitionistic fuzzy equivalence class $R_A[u]$. Formally, let α and β be two real numbers in the range [0,1] with $\alpha + \beta \leq 1$. Then, the ordinary set based on level α, β of the IFS $R_A[u]$ is a crisp set and called as an α, β-level set [32]:

$$R_A^{\{\alpha,\beta\}}[u] = \{v \in U \,|\mu_A[u](v) \geq \alpha \wedge \vartheta_A[u](v) \leq \beta\} \tag{3}$$

Next, we construct an IFS $R_A^{\alpha,\beta}[u]$ by combining each element $R_A^{\{\alpha,\beta\}}[u]$ with the similarity and diversity degrees. More accurately, $R_A^{\alpha,\beta}[u]$ is an IFS in U with the similarity and diversity degrees of each object $v \in U$ as

$$R_A^{\alpha,\beta}[u](v) = \left(\mu_A^{\alpha,\beta}[u](v), \vartheta_A^{\alpha,\beta}[u](v)\right) = \begin{cases} R_A[u](v) & \text{if } v \in R_A^{\{\alpha,\beta\}}[u] \\ (0, 1) & \text{otherwise} \end{cases} \tag{4}$$

It is easy to see that $R_A^{\alpha,\beta}[u]$ will be formed based on adjusting some intuitionistic fuzzy numbers from the intuitionistic fuzzy equivalence class $R_A[u]$. These intuitionistic fuzzy numbers have a similarity degree less than α or a diversity degree higher than β. Accordingly, if an object is removed, its similarity and diversity degrees will be represented in $R_A^{\alpha,\beta}[u]$ as $(0, 1)$.

In this paper, we shall call $R_A^{\alpha,\beta}[u]$ as an α, β-level intuitionistic fuzzy equivalence class of u. Therefore, a family $\left\{R_A^{\alpha,\beta}[u] : u \in U\right\}$ will generate an intuitionistic fuzzy partition on U. To ensure clarity, this family will be denoted $\mathscr{P}_A^{\alpha,\beta}$ and more specifically called an α, β-level intuitionistic fuzzy partition.

Suppose we are given two α, β-level intuitionistic fuzzy partitions $\mathscr{P}_A^{\alpha,\beta}$ and $\mathscr{P}_B^{\alpha,\beta}$. We say that $\mathscr{P}_A^{\alpha,\beta}$ is finer than

$\mathscr{P}_B^{\alpha,\beta}$, denoted $\mathscr{P}_A^{\alpha,\beta} \preceq \mathscr{P}_B^{\alpha,\beta}$, if for all $u \in U$, $R_A^{\alpha,\beta}[u] \subseteq R_B^{\alpha,\beta}[u]$. Then, we will present some significant properties of the α,β-level intuitionistic fuzzy equivalence class and the α,β-level intuitionistic fuzzy partition.

Proposition 1: Let $DT = (U, C \cup \{d\})$ and $A, B \subseteq C$.

1) If $A \subseteq B$, then $R_B^{\{\alpha,\beta\}}[u] \subseteq R_A^{\{\alpha,\beta\}}[u] \subseteq U, \forall u \in U$.
2) $\mathscr{P}_A^{\alpha,\beta} \preceq \mathscr{P}_A$.
3) $R_{A\cup B}^{\alpha,\beta}[u] = R_A^{\alpha,\beta}[u] \cap R_B^{\alpha,\beta}[u], \forall u \in U$.

Proof: We will skip the simple proofs of properties 1 and 2. To prove the property 3, we consider the following two cases:

For any $v \in R_{A\cup B}^{\alpha,\beta}[u]$, we have $\mu_{A\cup B}[u](v) \geq \alpha$ and $\vartheta_{A\cup B}[u](v) \leq \beta$.

$$\Leftrightarrow \begin{cases} \min\{\mu_A[u](v), \mu_B[u](v)\} \geq \alpha \\ \max\{\vartheta_A[u](v), \vartheta_B[u](v)\} \leq \beta \end{cases}$$

Hence, $v \in R_A^{\alpha,\beta}[u] \cap R_B^{\alpha,\beta}[u]$.

Proved similar to $v \in R_A^{\alpha,\beta}[u] \cap R_B^{\alpha,\beta}[u]$, we have

$$\Leftrightarrow \begin{cases} \min\{\mu_A[u](v), \mu_B[u](v)\} \geq \alpha \\ \max\{\vartheta_A[u](v), \vartheta_B[u](v)\} \leq \beta \end{cases}$$

$$\Leftrightarrow \begin{cases} \mu_{A\cup B}[u](v) \geq \alpha \\ \vartheta_{A\cup B}[u](v) \leq \beta \end{cases} \text{Hence, } v \in R_{A\cup B}^{\alpha,\beta}[u].$$

From the above two cases, we have $R_{A\cup B}^{\alpha,\beta}[u] = R_A^{\alpha,\beta}[u] \cap R_B^{\alpha,\beta}[u]$. The Proposition is proved.

Because the α,β-level intuitionistic equivalence class is also an intuitionistic fuzzy set, its fundamental operations are similar to an intuitionistic fuzzy set. Consider $R_A^{\alpha,\beta}[u]$ and $R_B^{\alpha,\beta}[u]$ are two α,β-level intuitionistic fuzzy equivalence classes with $A, B \subseteq C$. Then, we have some fundamental operations as follows:

1) $R_A^{\alpha,\beta}[u] \subseteq R_B^{\alpha,\beta}[u]$ iff $\mu_A^{\alpha,\beta}[u](v) \leq \mu_B^{\alpha,\beta}[u](v)$ and $\vartheta_A^{\alpha,\beta}[u](v) \geq \vartheta_B^{\alpha,\beta}[u](v)$ for any $v \in U$.
2) $R_A^{\alpha,\beta}[u] = R_B^{\alpha,\beta}[u]$ iff $R_A^{\alpha,\beta}[u] \subseteq R_B^{\alpha,\beta}[u]$ and $R_B^{\alpha,\beta}[u] \subseteq R_A^{\alpha,\beta}[u]$.
3) $R_A^{\alpha,\beta}[u] \cap R_B^{\alpha,\beta}[u] = \left\{v, \min\left(\mu_A^{\alpha,\beta}[u](v), \mu_B^{\alpha,\beta}[u](v)\right), \max\left(\vartheta_A^{\alpha,\beta}[u](v), \vartheta_B^{\alpha,\beta}[u](v)\right)\right\}$ for any $v \in U$.
4) $R_A^{\alpha,\beta}[u] \cup R_B^{\alpha,\beta}[u] = \left\{v, \max\left(\mu_A^{\alpha,\beta}[u](v), \mu_B^{\alpha,\beta}[u](v)\right), \min\left(\vartheta_A^{\alpha,\beta}[u](v), \vartheta_B^{\alpha,\beta}[u](v)\right)\right\}$ for any $v \in U$.

Obviously, $\mathscr{P}_A^{0,1} = \mathscr{P}_A$. Therefore, from Property 2 we can see that $\mathscr{P}_A^{\alpha,\beta}$ is an extension of $\mathscr{P}_A$. Information granulations of $\mathscr{P}_A^{\alpha,\beta}$ will only be characterized by the intuitionistic fuzzy numbers that satisfy two conditions of α,β-level sets. The remaining elements will not contribute too much to the cardinality calculation process.

Proposition 2: Let $DT = (U, C \cup \{d\})$ be a decision table. If $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$ then $\mathscr{P}_A^{\alpha_2,\beta_2} \preceq \mathscr{P}_A^{\alpha_1,\beta_1}$.

Proof: We found that $\forall u \in U, \alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$ we always have $R_A^{\{\alpha_2,\beta_2\}}[u] \subseteq R_A^{\{\alpha_1,\beta_1\}}[u]$. We consider the following three cases:

In the case $v \in R_A^{\{\alpha_2,\beta_2\}}[u] \Rightarrow v \in R_A^{\{\alpha_1,\beta_1\}}[u]$, then $\mu_A^{\alpha_1,\beta_1}[u](v) = \mu_A^{\alpha_2,\beta_2}[u](v) = \mu_A[u](v)$ and $\vartheta_A^{\alpha_1,\beta_1}[u](v) = \vartheta_A^{\alpha_2,\beta_2}[u](v) = \vartheta_A[u](v)$.

In the case $v \in U \backslash R_A^{\{\alpha_1,\beta_1\}}[u] \Rightarrow v \notin U \backslash R_A^{\{\alpha_1,\beta_1\}}[u]$ and $v \notin U \backslash R_A^{\{\alpha_2,\beta_2\}}[u]$, then $\mu_A^{\alpha_1,\beta_1}[u](v) = \mu_A^{\alpha_2,\beta_2}[u](v) = 0$ and $\vartheta_A^{\alpha_1,\beta_1}[u](v) = \vartheta_A^{\alpha_2,\beta_2}[u](v) = 1$.

In the case $v \in R_A^{\{\alpha_1,\beta_1\}}[u] \backslash R_A^{\{\alpha_2,\beta_2\}}[u]$, it is easy to see that $R_A^{\{\alpha_2,\beta_2\}}[u](v) = (0, 1)$ and $R_A^{\{\alpha_1,\beta_1\}}[u](v) = R_A[u](v) = (\mu_A[u](v), \vartheta_A[u](v))$. In which, $\mu_A[u](v) \geq 0$ and $\vartheta_A[u](v) \leq 1$.

Thus, for all $v \in U$, $\mu_A^{\alpha_2,\beta_2}[u](v) \leq \mu_A^{\alpha_1,\beta_1}[u](v)$ and $\vartheta_A^{\alpha_2,\beta_2}[u](v) \geq \vartheta_A^{\alpha_1,\beta_1}[u](v)$.

Imply, $R_A^{\alpha_2,\beta_2}[u] \subseteq R_A^{\alpha_1,\beta_1}[u]$ for any $u \in U$ which means $\mathscr{P}_A^{\alpha_2,\beta_2} \preceq \mathscr{P}_A^{\alpha_1,\beta_1}$. This completes of proof.

We next propose the distance between two α,β-level intuitionistic fuzzy equivalence classes.

Consider $R_A^{\alpha,\beta}[u]$ and $R_B^{\alpha,\beta}[u]$ with $A, B \subseteq C$, the distance between $R_A^{\alpha,\beta}[u]$ and $R_B^{\alpha,\beta}[u]$ denoted by $\mathscr{D}\left(R_A^{\alpha,\beta}[u], R_B^{\alpha,\beta}[u]\right)$:

$$\mathscr{D}\left(R_A^{\alpha,\beta}[u], R_B^{\alpha,\beta}[u]\right) = \frac{1}{2} * \sup_{v\in U}\left\{\left|\mu_A^{\alpha,\beta}[u](v) - \mu_B^{\alpha,\beta}[u](v)\right| + \left|\vartheta_A^{\alpha,\beta}[u](v) - \vartheta_B^{\alpha,\beta}[u](v)\right|\right\} \quad (5)$$

Our proposed distance measure characterizes the degree of similarity between two α,β-level intuitionistic fuzzy equivalence classes through the difference of the similarity and diversity degrees. The following is some important properties of this distance.

Proposition 3: Given a decision table $DT = (U, C \cup \{d\})$, $A, B \subseteq C$ and two α,β-level intuitionistic fuzzy equivalence classes, $R_A^{\alpha,\beta}[u]$, $R_B^{\alpha,\beta}[u]$, then we have

1) $\mathscr{D}\left(R_A^{\alpha,\beta}[u], R_B^{\alpha,\beta}[u]\right) \geq 0$; $\mathscr{D}\left(R_A^{\alpha,\beta}[u], R_B^{\alpha,\beta}[u]\right) = 0$ iff $R_A^{\alpha,\beta}[u] = R_B^{\alpha,\beta}[u]$.
2) $\mathscr{D}\left(R_A^{\alpha,\beta}[u], R_B^{\alpha,\beta}[u]\right) = \mathscr{D}\left(R_B^{\alpha,\beta}[u], R_A^{\alpha,\beta}[u]\right)$
3) $\mathscr{D}\left(R_A^{\alpha,\beta}[u], R_E^{\alpha,\beta}[u]\right) + \mathscr{D}\left(R_E^{\alpha,\beta}[u], R_B^{\alpha,\beta}[u]\right) \geq \mathscr{D}\left(R_A^{\alpha,\beta}[u], R_B^{\alpha,\beta}[u]\right)$

Proof:

1. It is easily provable that $\mathscr{D}\left(R_A^{\alpha,\beta}[u], R_B^{\alpha,\beta}[u]\right) \geq 0$ and $\mathscr{D}\left(R_A^{\alpha,\beta}[u], R_B^{\alpha,\beta}[u]\right) = 0$ iff $R_A^{\alpha,\beta}[u] = R_B^{\alpha,\beta}[u]$.

2. We have

$$\begin{aligned}\mathscr{D}\left(R_A^{\alpha,\beta}[u], R_B^{\alpha,\beta}[u]\right) &= \frac{1}{2}\sup_{v\in U}\left\{\left|\mu_A^{\alpha,\beta}[u](v) - \mu_B^{\alpha,\beta}[u](v)\right| + \left|\vartheta_A^{\alpha,\beta}[u](v) - \vartheta_B^{\alpha,\beta}[u](v)\right|\right\} \\ &= \frac{1}{2}\sup_{v\in U}\left\{\left|\mu_B^{\alpha,\beta}[u](v) - \mu_A^{\alpha,\beta}[u](v)\right| + \left|\vartheta_B^{\alpha,\beta}[u](v) - \vartheta_A^{\alpha,\beta}[u](v)\right|\right\} \\ &= \mathscr{D}\left(R_B^{\alpha,\beta}[u], R_A^{\alpha,\beta}[u]\right)\end{aligned}$$

3. Because, we have $\mu_A^{\alpha,\beta}[u](v)$ is inverse ratio of $\vartheta_A^{\alpha,\beta}[u](v)$. Thus, consider $v \in U$, if the maximum

value of $\left|\mu_A^{\alpha,\beta}[u](v)-\mu_E^{\alpha,\beta}[u](v)\right|$ occurs at v then $\left|\vartheta_A^{\alpha,\beta}[u](v)-\vartheta_E^{\alpha,\beta}[u](v)\right|$ also reaches the maximum value at v. Hence, we have

$$\mathscr{D}\left(R_A^{\alpha,\beta}[u],R_E^{\alpha,\beta}[u]\right)=\frac{1}{2}\sup_{v\in U}\left|\mu_A^{\alpha,\beta}[u](v)-\mu_E^{\alpha,\beta}[u](v)\right| +\frac{1}{2}\sup_{v\in U}\left|\vartheta_A^{\alpha,\beta}[u](v)-\vartheta_E^{\alpha,\beta}[u](v)\right|,$$

$$\mathscr{D}\left(R_E^{\alpha,\beta}[u],R_B^{\alpha,\beta}[u]\right)=\frac{1}{2}\sup_{v\in U}\left|\mu_E^{\alpha,\beta}[u](v)-\mu_B^{\alpha,\beta}[u](v)\right| +\frac{1}{2}\sup_{v\in U}\left|\vartheta_E^{\alpha,\beta}[u](v)-\vartheta_B^{\alpha,\beta}[u](v)\right|.$$

It is not difficult to see that $\frac{1}{2}\sup_{v\in U}\left|\mu_A^{\alpha,\beta}[u](v)-\mu_E^{\alpha,\beta}[u](v)\right| + \frac{1}{2}\sup_{v\in U}\left|\mu_E^{\alpha,\beta}[u](v)-\mu_B^{\alpha,\beta}[u](v)\right| \geq \frac{1}{2}\sup_{v\in U}\left\{\left|\mu_A^{\alpha,\beta}[u](v)-\mu_E^{\alpha,\beta}[u](v)\right| + \left|\mu_E^{\alpha,\beta}[u](v)-\mu_B^{\alpha,\beta}[u](v)\right|\right\} \geq \frac{1}{2}\sup_{v\in U}\left|\mu_A^{\alpha,\beta}[u](v)-\mu_E^{\alpha,\beta}[u](v)\right|$, we similarly have $\frac{1}{2}\sup_{v\in U}\left|\vartheta_A^{\alpha,\beta}[u](v)-\vartheta_E^{\alpha,\beta}[u](v)\right| + \frac{1}{2}\sup_{v\in U}\left|\vartheta_E^{\alpha,\beta}[u](v)-\vartheta_B^{\alpha,\beta}[u](v)\right| \geq \frac{1}{2}\sup_{v\in U}\left|\vartheta_A^{\alpha,\beta}[u](v)-\vartheta_B^{\alpha,\beta}[u](v)\right|$.

Therefore $\mathscr{D}\left(R_A^{\alpha,\beta}[u],R_E^{\alpha,\beta}[u]\right)+\mathscr{D}\left(R_E^{\alpha,\beta}[u],R_B^{\alpha,\beta}[u]\right)\geq \mathscr{D}\left(R_A^{\alpha,\beta}[u],R_B^{\alpha,\beta}[u]\right)$.

From the distance measure built on two α,β-level intuitionistic fuzzy equivalence classes, we expand on two α,β-level intuitionistic fuzzy partitions.

$$\mathscr{D}\left(\mathscr{P}_A^{\alpha,\beta},\mathscr{P}_B^{\alpha,\beta}\right)=\frac{1}{|U|}\sum_{u\in U}\mathscr{D}\left(R_A^{\alpha,\beta}[u],R_B^{\alpha,\beta}[u]\right) \tag{6}$$

It can be easily seen that the distance between two α,β-level intuitionistic fuzzy partitions is the average distance measures between α,β-level intuitionistic fuzzy equivalence classes of all objects in the universe. This measure will serve as a fundamental foundation for developing a new criteria to effectively select important attributes in the next section of the paper.

IV. ATTRIBUTE REDUCTION BASED ON α,β-LEVEL INTUITIONISTIC FUZZY PARTITION DISTANCE

This part provides an efficient algorithm to find a subset of critical attributes in decision tables. In particular, main steps of the algorithm is constructed through two parts. Part I defines a novel reduct and the significance of attributes based on the distance measure. Then, Part II presents an algorithm to extract the new reduct. A specific example of the proposed algorithm through each calculation step is also shown.

It is well known that a reduct of a decision table must preserve both information and consistency. This implies that the attribute's evaluative criteria should encompass both the condition and decision attributes. Hence, to meet this requirement, the distance measure must incorporate intuitionistic fuzzy partitions of the decision attributes as well. We continue to construct Proposition 4 as follows.

Proposition 4: Let $DT=(U,C\cup D)$ and $A\subseteq C$. We have: $\mathscr{D}\left(\mathscr{P}_A^{\alpha,\beta},\mathscr{P}_{A\cup\{d\}}^{\alpha,\beta}\right)=$

$$\frac{1}{2|U|}\sum_{u\in U}\sup_{v\in R_A^{\{\alpha,\beta\}}[u]}\left\{1+\mu_A^{\alpha,\beta}[u](v)-\vartheta_A^{\alpha,\beta}[u](v)\right\} \tag{7}$$

According to the formula (5):

$$\begin{aligned}&\mathscr{D}\left(\mathscr{P}_A^{\alpha,\beta},\mathscr{P}_{A\cup\{d\}}^{\alpha,\beta}\right)\\&=\frac{1}{2|U|}\sum_{u\in U}\sup_{v\in U}\left\{\left|\mu_A^{\alpha,\beta}[u](v)-\mu_{A\cup\{d\}}^{\alpha,\beta}[u](v)\right| + \left|\vartheta_A^{\alpha,\beta}[u](v)-\vartheta_{A\cup\{d\}}^{\alpha,\beta}[u](v)\right|\right\}\\&=\frac{1}{2|U|}\sum_{u\in U}\sup_{v\in U}\left\{\left|\mu_A^{\alpha,\beta}[u](v) -\min\left\{\mu_A^{\alpha,\beta}[u](v),\mu_{\{d\}}^{\alpha,\beta}[u](v)\right\}\right| + \left|\vartheta_A^{\alpha,\beta}[u](v)-\max\left\{\vartheta_A^{\alpha,\beta}[u](v),\vartheta_{\{d\}}^{\alpha,\beta}[u](v)\right\}\right|\right\}\end{aligned}$$

With $v\in U\backslash R_A^{\{\alpha,\beta\}}[u]$, we have $\mu_A^{\alpha,\beta}[u](v)=0$ and $\vartheta_A^{\alpha,\beta}[u](v)=1$, therefore: $\left|\mu_A^{\alpha,\beta}[u](v)-\min\left\{\mu_A^{\alpha,\beta}[u](v),\mu_{\{d\}}^{\alpha,\beta}[u](v)\right\}\right|+\left|\vartheta_A^{\alpha,\beta}[u](v)-\max\left\{\vartheta_A^{\alpha,\beta}[u](v),\vartheta_{\{d\}}^{\alpha,\beta}[u](v)\right\}\right|=0$

With $v\in R_A^{\{\alpha,\beta\}}[u]$, we consider the following two cases: $R_{\{d\}}[u](v)=(1,0)$

$$\Rightarrow\begin{cases}\min\left\{\mu_{R_A[u]}^{\alpha,\beta}(v),\mu_{R_{\{d\}}[u]}(v)\right\}=\mu_{R_A[u]}(v)\\ \max\left\{\vartheta_{R_A[u]}^{\beta,\beta}(v),\vartheta_{R_{\{d\}}[u]}(v)\right\}=\vartheta_{R_A[u]}(v)\end{cases}$$

$$\Rightarrow\left|\mu_A^{\alpha,\beta}[u](v)-\min\left\{\mu_A^{\alpha,\beta}[u](v),\mu_{\{d\}}^{\alpha,\beta}[u](v)\right\}\right| +\left|\vartheta_A^{\alpha,\beta}[u](v)-\max\left\{\vartheta_A^{\alpha,\beta}[u](v),\ \vartheta_{\{d\}}^{\alpha,\beta}[u](v)\right\}\right|=0$$

$R_{\{d\}}[u](v)=(0,1)$

$$\Rightarrow\begin{cases}\min\left\{\mu_{R_A[u]}^{\alpha,\beta}(v),\mu_{R_{\{d\}}[u]}(v)\right\}=\mu_{R_{\{d\}}[u]}(v)\\ \max\left\{\vartheta_{R_A[u]}^{\alpha,\beta}(v),\vartheta_{R_{\{d\}}[u]}(v)\right\}=\vartheta_{R_{\{d\}}[u]}(v)\end{cases}$$

$$\begin{aligned}&\Rightarrow\left|\mu_A^{\alpha,\beta}[u](v)-\min\left\{\mu_A^{\alpha,\beta}[u](v),\mu_{\{d\}}^{\alpha,\beta}[u](v)\right\}\right| +\left|\vartheta_A^{\alpha,\beta}[u](v)-\max\left\{\vartheta_A^{\alpha,\beta}[u](v),\ \vartheta_{\{d\}}^{\alpha,\beta}[u](v)\right\}\right|\\&=\left|\mu_A^{\alpha,\beta}[u](v)\right|+\left|\vartheta_A^{\alpha,\beta}[u](v)-1\right|\Rightarrow\mathscr{D}\left(\mathscr{P}_A^{\alpha,\beta},\mathscr{P}_{A\cup\{d\}}^{\alpha,\beta}\right)\\&=\frac{1}{2|U|}\sum_{u\in U}\sup_{v\in R_A^{\{\alpha,\beta\}}[u]}\left\{\mu_A^{\alpha,\beta}[u](v)+\left|\vartheta_A^{\alpha,\beta}[u](v)-1\right|\right\}\end{aligned}$$

Because $\vartheta_A^{\alpha,\beta}[u](v)\leq 1\Rightarrow\left|\vartheta_A^{\alpha,\beta}[u](v)-1\right|=1-\vartheta_A^{\alpha,\beta}[u](v)\Rightarrow\mathscr{D}\left(\mathscr{P}_A^{\alpha,\beta},\mathscr{P}_{A\cup\{d\}}^{\alpha,\beta}\right)$

$$=\frac{1}{2|U|}\sum_{u\in U}\sup_{v\in R_A^{\{\alpha,\beta\}}[u]}\left\{1+\mu_A^{\alpha,\beta}[u](v)-\vartheta_A^{\alpha,\beta}[u](v)\right\}$$

Clearly, formula (7) only calculates with the intuitionistic fuzzy numbers of objects in the α,β-level set. Therefore, it leads to a significant reduction in the algorithm's computational time.

Proposition 5: Let $DT = (U, C \cup \{d\})$ and $A, B \subseteq C$. If $A \subseteq B$ then $\mathscr{D}\left(\mathscr{P}_A^{\alpha,\beta}, \mathscr{P}_{A\cup\{d\}}^{\alpha,\beta}\right) \geq \mathscr{D}\left(\mathscr{P}_B^{\alpha,\beta}, \mathscr{P}_{B\cup\{d\}}^{\alpha,\beta}\right)$.

Proof: Because $A \subseteq B$, we have: $\mu_A^{\alpha,\beta}[u](v) \geq \mu_B^{\alpha,\beta}[u](v)$ and $\vartheta_A^{\alpha,\beta}[u](v) \leq \vartheta_B^{\alpha,\beta}[u](v)$ $\forall u, v \in U$, then $1+\mu_A^{\alpha,\beta}[u](v) - \vartheta_A^{\beta,\beta}[u](v) \geq 1+\mu_B^{\alpha,\beta}[u](v) - \vartheta_B^{\alpha,\beta}[u](v)$. Thus, $\mathscr{D}\left(\mathscr{P}_A^{\alpha,\beta}, \mathscr{P}_{A\cup\{d\}}^{\alpha,\beta}\right) \geq \mathscr{D}\left(\mathscr{P}_B^{\alpha,\beta}, \mathscr{P}_{B\cup\{d\}}^{\alpha,\beta}\right)$. The proposition is proved.

Proposition 5 provides the anti-monotone property of the size of the conditional attribute subset. This means that for any $A \subseteq C$, the smaller A is, the larger $\mathscr{D}\left(\mathscr{P}_A^{\alpha,\beta}, \mathscr{P}_{A\cup\{d\}}^{\alpha,\beta}\right)$ is. This is considered as a criterion for selecting atribute in the attribute reduction algorithm.

Definition 3: Let $DT = (U, C \cup \{d\})$. A subset $B \subseteq C$ is called a reduct of C if

1) $\mathscr{D}\left(\mathscr{P}_B^{\alpha,\beta}, \mathscr{P}_{B\cup\{d\}}^{\alpha,\beta}\right) = \mathscr{D}\left(\mathscr{P}_C^{\alpha,\beta}, \mathscr{P}_{C\cup\{d\}}^{\alpha,\beta}\right)$
2) $\forall B' \subset B$, $\mathscr{D}\left(\mathscr{P}_{B'}^{\alpha,\beta}, \mathscr{P}_{B'\cup\{d\}}^{\alpha,\beta}\right) > \mathscr{D}\left(\mathscr{P}_B^{\alpha,\beta}, \mathscr{P}_{B\cup\{d\}}^{\alpha,\beta}\right)$

This implies that for any $a \in B$, if $\mathscr{D}\left(\mathscr{P}_{B\setminus\{a\}}^{\alpha,\beta}, \mathscr{P}_{B\setminus\{a\}\cup\{d\}}^{\alpha,\beta}\right) \neq \mathscr{D}\left(\mathscr{P}_B^{\alpha,\beta}, \mathscr{P}_{B\cup\{d\}}^{\alpha,\beta}\right)$, then we say that a is indispensable in B. In contrast, a will be called a redundant attribute in B. The redundant attribute may not provide more classification information. Moreover, it can confuse the learning algorithm during training. Therefore, it should be removed from the condition attribute set before classification learning.

Definition 4: Let $DT = (U, C \cup \{d\})$, $B \subseteq C$ and $a \in C/B$. Then the significance of the attribute a with respect to B, denoted $SIG_B(a)$, is determined by the formula:

$$SIG_B(a) = \mathscr{D}\left(\mathscr{P}_B^{\alpha,\beta}, \mathscr{P}_{B\cup\{d\}}^{\alpha,\beta}\right) - \mathscr{D}\left(\mathscr{P}_{B\cup\{a\}}^{\alpha,\beta}, \mathscr{P}_{B\cup\{a\}\cup\{d\}}^{\alpha,\beta}\right) \quad (8)$$

It can be easily seen that $SIG_B(a) \geq 0$. Consider any attribute $a \in C$, its significance for an attribute subset characterizes the classification quality of b. We can see the alteration of the certainty degree. If the value of $SIG_B(a)$ is higher, then the attribute b will be more essential. This measure can be considered as a criterion for selecting the necessary attributes. Based on this definition, we design an effective algorithm to extract an optimal attribute subset from a given decision table.

In the following, we will propose a heuristic algorithm for selecting a subset of necessary attributes on the decision table to enhance the efficiency of classification models. This algorithm comprises two main stages, initialize and filter. In the first stage of the algorithm, α, β-level intuitionistic fuzzy partitions are calculated for each attribute in the decision table. The algorithm then initializes B as the set containing the crucial attributes of the table and starts incorporating attribute a_0 into B, ensuring that $\mathscr{D}\left(\mathscr{P}_{\{a_0\}}^{\alpha,\beta}, \mathscr{P}_{\{a_0\}\cup\{d\}}^{\alpha,\beta}\right)$ holds the minimum value. In the next stage, the algorithm continues to add each attribute into the selected attribute subset B with the maximum significance in each iteration until the condition stops happening. The stop condition has to satisfy a given threshold δ. The algorithm is designed as Algorithm 3.

Algorithm 1 Attribute Reduction Based on the α, β-Level Intuitionistic Fuzzy Partition Distance (IFPD)

Input: A decision table $DT = (U, C \cup \{d\})$, levels α, β and a threshold δ.
Output: One reduct B

```
    // Initialize stage
1:  Compute 𝒫_{d}
2:  for a ∈ C do
3:      if a is a continous numeric value domain attribute then
4:          Compute 𝒫^{α,β}_{a}
5:      else
6:          𝒫^{α,β}_{a} = 𝒫_{a}
7:      end if
8:  end for
9:  Compute 𝒫^{α,β}_C := ⋂_{a∈C} 𝒫^{α,β}_{a}
10: B:={a_0} which satisfies:
        𝒟(𝒫^{α,β}_{a_0}, 𝒫^{α,β}_{a_0}∪{d}) = Min_{a∈C} {𝒟(𝒫^{α,β}_{a}, 𝒫^{α,β}_{a}∪{d})}
    // Filter stage
11: while 𝒟(𝒫^{α,β}_B, 𝒫^{α,β}_{B∪{d}}) − 𝒟(𝒫^{α,β}_C, 𝒫^{α,β}_{C∪{d}}) > δ do
12:     for a ∈ C\B do
13:         Compute 𝒟(𝒫^{α,β}_{B∪{a}}, 𝒫^{α,β}_{B∪{a}∪{d}})
14:         Compute SIG_B(a)
15:     end for
16:     Select a_0 which satisfies:
            SIG_B(a_0) = Max_{a∈C\B} {SIG_B(a)}
17:     B := B ∪ {a_0}
18: end while
19: Return B
```

We now analyze the computational complexity of the IFPD. Suppose that $|C|$, $|U|$ are the number of condition attributes and the number of objects in the decision table respectively. The complexity of computing the intuitionistic fuzzy partition $\mathscr{P}_C^{\alpha,\beta}$ is $O\left(|C| * |U|^2\right)$. Hence, the complexity of computing $\mathscr{D}\left(\mathscr{P}_C^{\alpha,\beta}, \mathscr{P}_{C\cup\{d\}}^{\alpha,\beta}\right)$ in line 11 is $O\left(|C| * |U|^2\right)$. In the loop While (from line 11 to line 18), the complexity of computing $\mathscr{D}\left(\mathscr{P}_{B\cup\{a\}}^{\alpha,\beta}, \mathscr{P}_{B\cup\{a\}\cup\{d\}}^{\alpha,\beta}\right)$ is $O(|U|^2)$, the complexity of $SIG_B(a)$ in line 14 is $O\left(|U|^2\right)$. Besides, the complexity of the loop For (from line 12 to line 15) is $O\left(|C| * |U|^2\right)$. Thus, the complexity of the loop While is $O\left(|C|^2 * |U|^2\right)$. The overall time complexity of the algorithm is $O\left(|C|^2 * |U|^2\right)$. Let us employ an example to show the idea of the new algorithm.

Example 2: A decision table $DT = (U, C \cup \{d\})$ as Table 1 with $\alpha = 0.45$, $\beta = 0.4$ and $\delta = 0.0$.

Implement the steps of the algorithm IFPD:

Step 1: Initialize:

For each attribute $c \in C$, the similarity degree of u and v according to the relation R_c is determined by the formula: $\mu_{R_c}(u,v) = 1 - |c(u) - c(v)|$. The diversity degree according to the relation R_c is determined by the formula: $\vartheta_{R_c}(u,v) = \left(1-\mu_{R_c}(u,v)\right)/\left(1-\lambda*\mu_{R_c}(u,v)\right)$ with $\lambda = 1$. Using $\alpha = 0.45, \beta = 0.4$ to compute the partitions $\mathscr{P}_{\{c\}}^{\alpha,\beta}, \forall c \in C$ and $\mathscr{P}_{\{d\}}$, thereby we have:

$$\mathscr{P}_C^{\alpha,\beta} = \begin{bmatrix} (1.00, 0.00) & (0.00, 1.00) & (0.60, 0.25) & (0.00, 1.00) \\ (0.00, 1.00) & (1.00, 0.00) & (0.00, 1.00) & (0.00, 1.00) \\ (0.60, 0.25) & (0.00, 1.00) & (1.00, 0.00) & (0.00, 1.00) \\ (0.00, 1.00) & (0.00, 1.00) & (0.00, 1.00) & (1.00, 0.00) \end{bmatrix}$$

According to Proposition 4, we have $\mathscr{D}\left(\mathscr{P}_C^{\alpha,\beta}, \mathscr{P}_{C\cup\{d\}}^{\alpha,\beta}\right)=0$, $\mathscr{D}\left(\mathscr{P}_{\{c_1\}}^{\alpha,\beta}, \mathscr{P}_{\{c_1\}\cup\{d\}}^{\alpha,\beta}\right) = 0.71$, $\mathscr{D}\left(\mathscr{P}_{\{c_2\}}^{\alpha,\beta}, \mathscr{P}_{\{c_2\}\cup\{d\}}^{\alpha,\beta}\right) = 0.71$, $\mathscr{D}\left(\mathscr{P}_{\{c_3\}}^{\alpha,\beta}, \mathscr{P}_{\{c_3\}\cup\{d\}}^{\alpha,\beta}\right) = 0.34$, $\mathscr{D}\left(\mathscr{P}_{\{c_4\}}^{\alpha,\beta}, \mathscr{P}_{\{c_4\}\cup\{d\}}^{\alpha,\beta}\right) = 0.34$. Because $\mathscr{D}\left(\mathscr{P}_{\{c_3\}}^{\alpha,\beta}, \mathscr{P}_{\{c_3\}\cup\{d\}}^{\alpha,\beta}\right)$ is the smallest, so add c_3 into the set B. Then $B = \{c_3\}$.

Step 2: Compute in the filter stage

Consider remaining attributes:

$\mathscr{D}\left(\mathscr{P}_{B\cup\{c_1\}}^{\alpha,\beta}, \mathscr{P}_{B\cup\{c_1\}\cup\{d\}}^{\alpha,\beta}\right) = 0.34$

$\mathscr{D}\left(\mathscr{P}_{B\cup\{c_2\}}^{\alpha,\beta}, \mathscr{P}_{B\cup\{c_2\}\cup\{d\}}^{\alpha,\beta}\right) = 0$

$\mathscr{D}\left(\mathscr{P}_{B\cup\{c_4\}}^{\alpha,\beta}, \mathscr{P}_{B\cup\{c_4\}\cup\{d\}}^{\alpha,\beta}\right) = 0.34$.

Therefore c_2 is selected because $SIG_B(B\cup\{c_2\}) = 0.34$ and holds the maximum value. Then $B = B\cup\{c_2\} = \{c_3, c_2\}$. Because $\mathscr{D}\left(\mathscr{P}_{B\cup\{c_2\}}^{\alpha,\beta}, \mathscr{P}_{B\cup\{c_2\}\cup\{d\}}^{\alpha,\beta}\right) = \mathscr{D}\left(\mathscr{P}_C^{\alpha,\beta}, \mathscr{P}_{C\cup\{d\}}^{\alpha,\beta}\right) = 0$ the algorithm stops. Thus, the reduct of the given decision table is $B = \{c_3, c_2\}$.

V. NUMERICAL EXPERIMENT

In the previous sections, we presented a promising attribute reduction method based on the α,β-level intuitionistic fuzzy sets approach. In this section, we demonstrate some experiments to prove the efficiency of the proposed method (IFPD) based on the comparison process with some state-of-the-art methods. Algorithms include the distance between two intuitionistic fuzzy partitions-based algorithm (IFDAR) [35], the intuitionistic fuzzy positive region-based algorithm (IFPR) [30], the intuitionistic fuzzy information entropy-based algorithm (IFIE) [33] and the algorithm of fitting model based on IFS feature selection (FMIF) [34].

A. EXPERIMENT GOALS AND PLANS

As mentioned above, we will present the results of five algorithms based on attribute subset search strategies of the decision table. The evaluation process will focus on the criteria of classification accuracy, size of reduct and computational time.

Experimental environment: All algorithms are coded by Python programming languages and implemented in a computer with a Windows 10 operating system, Xeon processor, and 16GB of memory. The main goal of the experiment is to illustrate the classification efficiency of algorithms on reduced data sets. We use two classification models: K-nearest-neighbor (K-NN, K=10) and support vector machine (SVM). The classification results are given out by a tenfold cross-validation, where each part will be used for the algorithms to find the reduct and check the classification accuracy.

TABLE 2. Decription of data sets.

No.s	Dataset	Attributes	Samples	Classes
1	Vehicle	18	846	4
2	Satimage	36	6430	6
3	Ozone	72	2534	2
4	Qsar	41	1055	2
5	Robot	90	164	5
6	Triazines	60	186	2
7	Movement	90	360	15
8	Sona	60	208	2
9	Agnostic	48	4562	2
10	Tecator	124	240	2
11	LSVT	310	126	2
12	PD	754	756	2
13	warpAR10P	2400	130	10
14	Tumors	7129	60	2
15	Leukemia	7129	72	2
	AVERAGE	1224	1182	4

Experimental data: The data sets during the experiment are presented in Table 2, downloaded from the UCI machine learning repository [41]. To demonstrate the efficiency of the algorithm, we will show datasets with diverse characteristics in attribute domains, the number of decision classes and the number of data dimensions. Five illustrated datasets with low initial classification accuracy ($< 85\%$) are Vehicle, Robot, Sona, Movement, Triazines, four datasets with a large number of objects are Qsar, Ozone, Satimage, Agnostic and six datasets with a large number of attributes are Tecator, LSVT, PD, warpAR10P, Tumors, Leukemia.

Before reduction, we construct intuitionistic fuzzy equivalence relations including the similarity and diversity degrees between two objects u and v of attribute a.

If the value of attribute a is a continuous value type, then:

$$\mu_{\{a\}}[u](v) = 1 - \frac{a(u) - a(v)}{\max(a) - \min(a)} \tag{9}$$

The above formula determines the similarity degree of object u with object v, in which max(a) and min(a) are maximal and minimal values corresponding to attribute a. In essence, the denominator component of the formula above is the process of min-max data normalizing to ensure that the values in the decision table are in the range [0,1]. Finally, we base on the formula of Sugeno and Terano [42] to calculate the diversity degree.

$$\vartheta_{\{a\}}[u](v) = \frac{1 - \mu_{\{a\}}[u](v)}{1 + \lambda_a * \mu_{\{a\}}[u](v)} \ with\ \lambda_a > 0 \tag{10}$$

Clearly, if value $\lambda_a = 0$, will becomes a traditional fuzzy set. With $\lambda_a > 0$, we can see that similarity and diversity degrees are inversely proportional to each other and satisfy the properties of a intuitionistic fuzzy number in IFS: $0 \leq \mu_{\{a\}}[u](v) + \vartheta_{\{a\}}[u](v) \leq 1$. We recommend the value λ_a of attribute a according to the following formula:

$$\lambda_a = \begin{cases} 1 \text{ if } \sigma_a = 0 \\ \beta_a/\sigma_a \text{ if } \sigma_a > 0 \end{cases} \tag{11}$$

in which, $\sigma_a = \sqrt{\frac{1}{|U|-1}\sum_{u\in U}(a(u)-\bar{a})^2}$ is the standard deviation of the value domain of the attribute a, $\beta_a = \left|\mathscr{P}^F_{\{a\}\cup\{d\}}\right| / \left|\mathscr{P}^F_{\{d\}}\right|$ is the consistency of attribute a in the decision table, $\mathscr{P}^F_{\{a\}\cup\{d\}}$ is a fuzzy partition of $\{a\}\cup\{d\}$ and $\mathscr{P}^F_{\{d\}}$ is the fuzzy partition of the decision attribute d. Clearly, when $\mu_{\{a\}}[u](v)$ has a small value which will lead to a small consistency of the attribute a and $\vartheta_{\{a\}}[u](v)$ has a big value. It is emphasized that the elements with relatively low similarity degrees will be removed easily in the intuitionistic fuzzy equivalence classes based on the condition of α, β-level IFSs. In contrast, the elements have a low ability to remove when $\mu_{\{a\}}[u](v)$ is large. Therefore, these elements will be kept and used in the next computation.

If the value of a is a categorical value, then:

$$\mu_{\{a\}}[u](v) = \begin{cases} 1, a(u) = a(v) \\ 0, a(u) \neq a(v) \end{cases} \tag{12}$$

$$\vartheta_{\{a\}}[u](v) = 1 - \mu_{\{a\}}[u](v) \tag{13}$$

B. EVALUATION OF THE REDUCT

Table 3 records the average size of reducts obtained from five algorithms when running ten times for each dataset. Obviously, on the datasets, the reduct size of the five algorithms is all smaller than the original set of attributes. For example, on the Sona dataset, the number of attributes is reduced by 64.83% from the IFBR algorithm and the number of attributes is reduced by 90.33% from the IFPD algorithm compared to the original set of attributes. Regarding data Ozone, the size of attributes obtained by the algorithm FMIF and the algorithm IFPD decreased to 92.22% and 94.02%, respectively.

The methods show very effective processing ability with high dimensional data sets when removing hundreds or even thousands of attributes on the original datasets is possible. It can be easily seen that the reducts obtained by algorithms IFPD and IFIE often have the smallest size for almost all data sets, while the reducts of algorithm IFBDAR is higher than that of other algorithms. It should be emphasized that algorithm IFDBAR deals ineffectively with high-dimensional datasets, such as LSVT, PD, and warpAR10P. The average size of reducts obtained by the IFPD algorithm is the best when reducing to 98.85% the number of attributes. Thus, concerning the sizes of reducts, the proposed method reduced very effectively, especially for high-dimensional data sets.

Next, we analyze the execution time of five algorithms IFPD, IFDBAR, IFBR, IFIE and FMIF. For small and medium-dimensional datasets, all five algorithms are high-speed. For example, on the Robot dataset, the execution time of five algorithms is 0.019, 0.013, 0.579, 0.023, 0.074 (s). On the Qsar dataset, the running times are 0.058, 0.151, 0.548, 0.217, and 0.226 (s) respectively. However, the IFPR algorithm takes longer than other algorithms to process on large dimensional datasets. For example, on the warpAR10P dataset, the processing time of the IFPR algorithm is up to 61.72s

TABLE 3. Number of selected attributes of diffirent algorithms.

No.s	Dataset	IFPD	IFDBAR	IFBR	IFIE	FMIF
1	Vehicle	13.7	13.2	12	**6.6**	12.7
2	Satimage	**17.1**	17.3	17.2	18.6	29.1
3	Ozone	**4.3**	4.8	7.1	12.5	5.6
4	Qsar	18	32.5	20.6	**14.4**	23
5	Robot	6.6	3.2	4.7	**2.8**	5.4
6	Triazines	12.2	41	13.5	**11.8**	12.8
7	Movement	**14.9**	21.4	17.6	39.5	15.9
8	Sona	**5.8**	47.8	21.1	15.2	15.6
9	Agnostic	**22.5**	45.5	33.3	31	30.2
10	Tecator	11.3	22.7	11.7	**3.7**	7.4
11	LSVT	12.3	118.4	15.1	11.3	**9.2**
12	PD	45.1	360.5	44.7	52.1	**42.7**
13	warpAR10P	20.3	238.8	**14.2**	37.5	18.8
14	Tumors	2.4	2.3	7	5.2	**1.3**
15	Leukemia	3.7	6	7.1	11.7	**3.4**
	AVERAGE	**14.0**	65.03	16.46	18.26	15.54

TABLE 4. Computational time of difference algorithms.

No.s	Dataset	IFPD	IFDBAR	IFBR	IFIE	FMIF
1	Vehicle	**0.017**	0.026	0.240	0.027	0.091
2	Satimage	**2.692**	7.437	5.300	12.44	22.70
3	Ozone	**0.107**	0.305	0.792	0.935	0.392
4	Qsar	**0.058**	0.151	0.548	0.217	0.226
5	Robot	0.019	**0.013**	0.579	0.023	0.074
6	Triazines	**0.021**	0.079	0.665	0.060	0.066
7	Movement	**0.071**	0.138	7.703	0.500	1.064
8	Sona	**0.011**	0.079	0.772	0.087	0.225
9	Agnostic	**1.885**	3.776	4.719	5.012	3.346
10	Tecator	0.069	0.119	1.087	**0.042**	0.121
11	LSVT	**0.113**	0.911	2.744	0.263	0.223
12	PD	**1.966**	27.58	26.07	7.752	7.312
13	warpAR10P	**1.469**	16.68	61.72	7.448	15.53
14	Tumors	**0.464**	0.587	29.58	2.868	0.753
15	Leukemia	**0.726**	1.562	29.22	6.695	2.311
	AVERAGE	**0.646**	3.963	11.45	2.958	3.629

Table 4 shows that the execution time of the proposed IFPD algorithm is the fastest on 13 datasets. It reveals our method based on the α, β-level IFSs approach improved the computational time. The proposed algorithm is capable of working with datasets that have different features. It is easy to see that the larger the value of α, the shorter the computational time. The IFPD algorithm only computes intuitionistic fuzzy information granulations containing elements that contribute mainly to the calculation. In Table 4, the proposed method has superior time performance over IFSs-based methods for multidimensional data sets, such as warpAR10P, Leukemia,

TABLE 5. Comparison on classification accuracies of reduced data with SVM.

No.s	Dataset	IFPD	IFDBAR	IFBR	IFIE	FMIF	Raw
1	Vehicle	**0.763±0.05**	0.759±0.05	0.759±0.05	0.678±0.05	**0.763±0.04**	0.752±0.04
2	Satimage	0.896±0.01	0.891±0.01	0.894±0.01	0.888±0.01	**0.899±0.01**	0.900±0.01
3	Ozone	**0.937±0.01**	**0.937±0.01**	**0.937±0.01**	**0.937±0.01**	**0.937±0.01**	0.937±0.01
4	Qsar	**0.864±0.03**	0.860±0.04	0.862±0.03	0.847±0.03	**0.864±0.03**	0.854±0.03
5	Robot	0.452±0.11	0.423±0.08	0.434±0.12	**0.493±0.10**	0.463±0.12	0.397±0.08
6	Triazines	**0.791±0.08**	0.769±0.11	0.785±0.08	0.769±0.10	0.769±0.08	0.764±0.01
7	Movement	**0.792±0.07**	0.783±0.06	0.783±0.08	0.756±0.09	0.781±0.09	0.775±0.08
8	Sona	**0.852±0.09**	0.793±0.11	0.822±0.08	0.822±0.02	0.822±0.09	0.793±0.10
9	Agnostic	**0.827±0.02**	0.821±0.02	0.824±0.02	0.824±0.02	0.825±0.02	0.821±0.02
10	Tecator	**0.942±0.05**	0.921±0.08	0.933±0.05	0.912±0.05	0.925±0.06	0.933±0.05
11	LSVT	0.896±0.12	0.825±0.13	0.847±0.14	0.872±0.14	**0.913±0.08**	0.809±0.12
12	PD	**0.866±0.03**	0.859±0.04	0.849±0.04	0.865±0.04	0.857±0.04	0.857±0.03
13	warpAR10P	0.723±0.15	0.615±0.17	0.654±0.20	**0.738±0.22**	0.654±0.16	0.677±0.15
14	Tumors	**0.667±0.16**	0.650±0.15	0.617±0.11	0.650±0.12	**0.667±0.16**	0.650±0.17
15	Leukemia	**0.930±0.12**	0.862±0.15	0.871±0.13	**0.930±0.10**	0.914±0.15	0.875±0.10
	AVERAGE	**0.813±0.07**	0.785±0.08	0.791±0.08	0.795±0.08	0.804±0.08	0.786±0.07

TABLE 6. Comparison on classification accuracies of reduced data with KNN.

No.s	Dataset	IFPD	IFDBAR	IFBR	IFIE	FMIF	Raw
1	Vehicle	**0.695±0.06**	0.686±0.06	0.691±0.07	0.688±0.04	0.688±0.04	0.694±0.06
2	Satimage	0.896±0.01	0.897±0.01	0.900±0.01	0.895±0.01	**0.902±0.01**	0.904±0.01
3	Ozone	0.940±0.01	0.938±0.02	0.939±0.01	**0.941±0.02**	0.940±0.01	0.940±0.01
4	Qsar	0.862±0.05	0.855±0.04	0.848±0.04	**0.873±0.03**	0.850±0.04	0.854±0.04
5	Robot	**0.677±0.08**	0.567±0.11	0.622±0.13	0.635±0.11	0.616±0.14	0.488±0.16
6	Triazines	0.736±0.09	0.710±0.08	0.747±0.08	0.731±0.08	**0.764±0.08**	0.710±0.09
7	Movement	0.647±0.09	**0.658±0.10**	0.647±0.12	0.636±0.12	0.642±0.12	0.644±0.08
8	Sona	0.760±0.12	0.707±0.10	0.706±0.08	**0.779±0.11**	0.754±0.12	0.707±0.10
9	Agnostic	**0.823±0.02**	0.820±0.01	0.819±0.01	0.818±0.01	0.819±0.01	0.819±0.01
10	Tecator	0.904±0.06	0.900±0.06	0.912±0.06	0.896±0.06	**0.917±0.04**	0.842±0.07
11	LSVT	**0.881±0.13**	0.770±0.12	0.826±0.13	0.849±0.13	0.866±0.11	0.770±0.09
12	PD	**0.849±0.04**	0.824±0.01	0.827±0.04	0.848±0.03	0.831±0.03	0.847±0.03
13	warpAR10P	**0.600±0.17**	0.508±0.18	0.554±0.19	0.569±0.19	0.569±0.14	0.408±0.15
14	Tumors	0.683±0.18	0.650±0.20	0.600±0.21	0.617±0.22	**0.733±0.21**	0.666±0.19
15	Leukemia	**0.957±0.07**	0.889±0.13	0.900±0.10	0.932±0.09	0.930±0.12	0.750±0.14
	AVERAGE	**0.794±0.08**	0.759±0.08	0.769±0.07	0.780±0.07	0.787±0.06	0.736±0.08

PD, and Satimage. In addition, the proposed distance has an improvement compared to the IFDBAR algorithm when eliminating the process of calculating the union of the partitions of the conditional attribute set with the decision attribute d.

Finally, we will analyze the classification ability of reducts returned from the methods through two classifiers: SVM (kernel = "RBF") and KNN (K=10). Tables 5 and 6 present the classification results from the reducts of five algorithms, where "Raw" is the classification accuracy class of the original data. It can be confirmed that the proposed IFPD algorithm selected important attributes very efficiently for all data sets. In particular, when comparing the raw data in 30 cases of 15 data sets for two classifiers (Tab. 5, 6), there are only 2 cases that our reduct accuracy is slightly lower than the raw data.

There are 26 cases, the IFPD algorithm has higher classification accuracy than the original data set and 2 cases have the same accuracy as the original data. The average classification accuracy of the IFPD algorithm is the highest among five algorithms. The improvement is most shown in noise datasets with low initial classification accuracy. For example, on the Leukemia dataset, the SVM classification accuracy of the reduct obtained from the FMIF and IFPD algorithms increased by 4.4% and 6.28% respectively. The accuracy of the LSVT dataset with KNN classifier increased significantly from 77% to 88.1%, 82.6%, 84.9%, and 86.6% on four algorithms IFPD, IFPR, IFIE and FMIF.

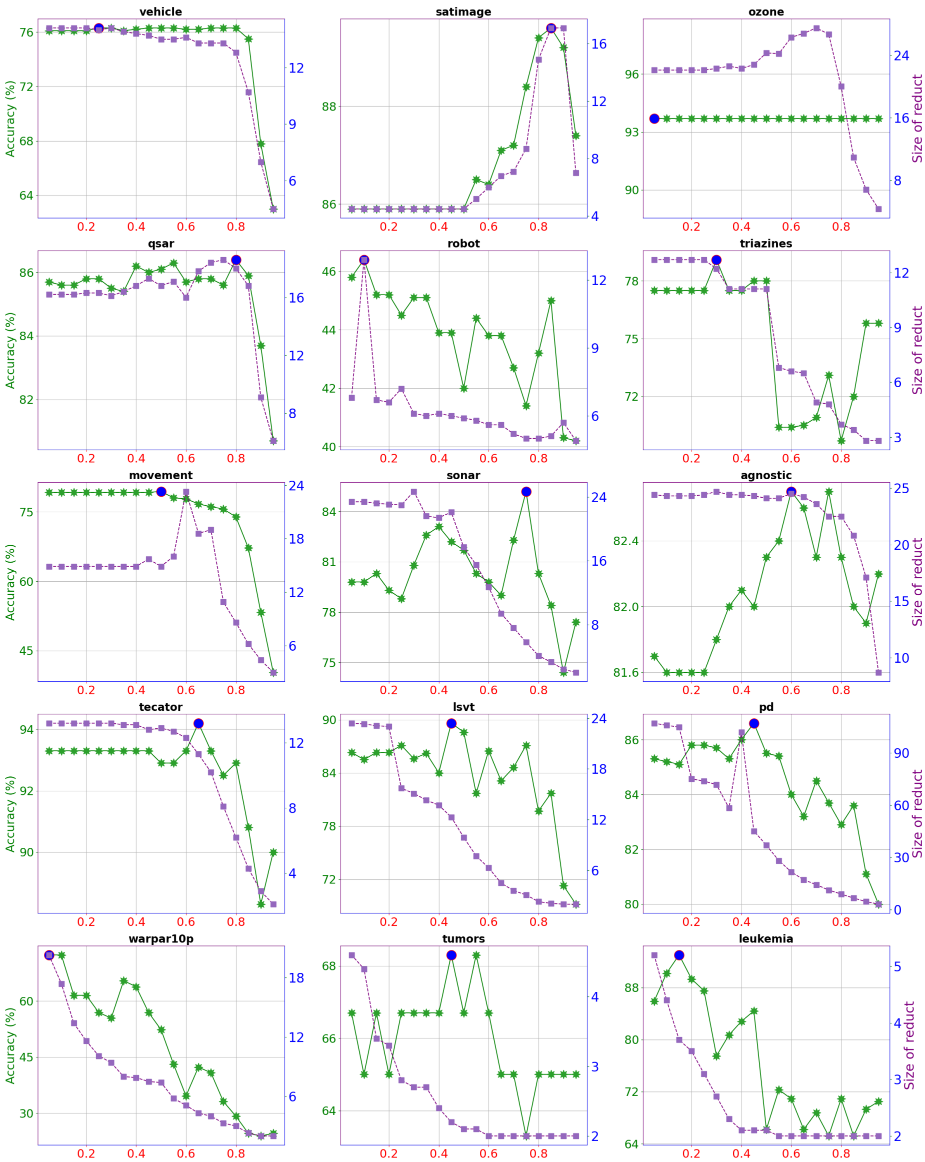

FIGURE 1. Classification accuracy and size of reduct of the proposed algorithm varying in the level α with the SVM classifier.

The average classification accuracy of the IFPD algorithm is the highest of the two classifiers. According to Table 5 and Table 6, mostly, IFPD algorithms have better classification results than the remaining four algorithms. Compared with the original data, the average classification accuracy of IFPD is improved by 2.7% and 5.8% on SVM and KNN, respectively. In addition, the average standard deviation of the IFPD algorithm is also approximately the same as other algorithms. It is easy to see that the α, β-level set restricts the influence of objects in the information granulation that have low similarity or high diversity degrees. Intuitively, these objects might be generated by noise and could be the reason for the diminished effectiveness of the classification models. Furthermore, our proposed measure is especially well-suited for α, β-level intuitionistic fuzzy partitions, emphasizing calculations exclusively on objects that play a significant role in the attribute evaluation process. Therefore, this indicates the stability of the algorithm's efficiency compared to the rest, particularly on datasets with low initial classification rates, such as Vehicle, Robot, Movement and Tumors. This stability comes from the fact that the information granulations created from the proposed algorithm contain beneficial information-carrying objects that increase the efficiency of the attribute evaluation measure and contribute to improving the classification accuracy as well as the computational time.

C. THE EFFECT OF THE PARAMETERS ON THE EFFICIENCY OF THE ALGORITHM

For the proposed method, there are three parameters for the model. The first and second parameters are the value α, β

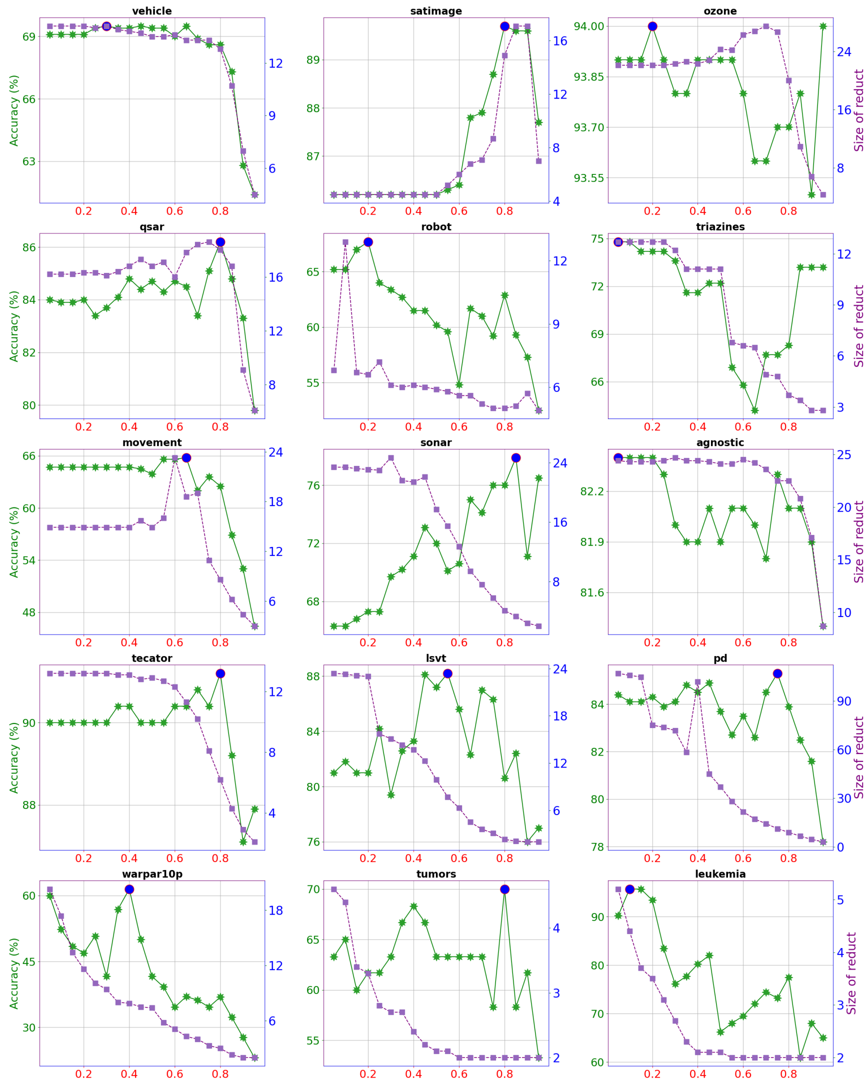

FIGURE 2. Classification accuracy and size of reduct of the proposed algorithm varying in the level α with the KNN classifier.

TABLE 7. Model parameter value.

No.s	Datasets	IFPD		FMIF
		α	β	ε
1	Vehicle	0.45	0.38	0.80
2	Satimage	0.85	0.08	0.85
3	Ozone	0.95	0.03	0.95
4	Qsar	0.80	0.11	0.80
5	Robot	0.20	0.67	0.55
6	Triazines	0.30	0.54	0.10
7	Movement	0.05	0.90	0.55
8	Sona	0.75	0.14	0.55
9	Agnostic	0.75	0.14	0.60
10	Tecator	0.65	0.21	0.20
11	LSVT	0.45	0.38	0.55
12	PD	0.45	0.38	0.40
13	warpAR10P	0.05	0.90	0.10
14	Tumors	0.40	0.43	0.65
15	Leukemia	0.15	0.74	0.20

to adjust the α, β-level when construct α, β-sets. Different values of α, β can lead to the selection of different subsets of attributes. If the value α is higher, the minor β, the more elements in the intuitionistic information granulation are removed. This makes the distance measure value smaller and gradually converges to the stopping condition of the algorithm. When the algorithm converges fast, the number of attributes decreases, the accuracy efficiency can be affected. However, we just traverse the parameter α in computing α, β-level intuitionistic fuzzy partitions. For the compute of β, we also base on the formula in [42] with $(1-\alpha)/(1+\alpha)$. For each data set, we traverse the values α and select a set of parameters to obtain the reduct with high efficiency. We gradually adjust the value of α from 0 to 1 with each step of 0.05 and are shown in Figure 1, 2.

The third parameter is the value δ is the stopping threshold of the proposed algorithm. Value δ also affects the size of the reduct; more particularly, value δ with the larger the number, the smaller the number of selected attributes, and vice versa. However, this parameter does not affect the selection of attributes. For datasets with large dimensionality, we set the threshold δ as 0.05. We set threshold δ on the small and medium-sized datasets as 0.01 and 0, respectively. Table 7 gives the model's parameter values on 15 datasets and compares them with the FMIF algorithm.

VI. CONCLUSION

Attribute reduction is a pivotal problem in the data preprocessing step to reduce the number of redundant attributes and improve the classification ability. According to the intuitionistic fuzzy sets approach, this paper first builds an α, β-level to eliminate objects with small similarity or high diversity degrees in the intuitionistic fuzzy equivalence class to transform them into α, β-level intuitionistic fuzzy equivalence classes. These objects cause computational redundancy and can be created by noisy data. In other words, the α, β-level intuitionistic fuzzy set has removed the impact of noisy objects in the decision table. This not only helps determine crucial attributes but also accelerates the computation process. After that, this paper constructed the suitable distance between two α, β-level intuitionistic fuzzy partitions and proposed a novel algorithm by using intuitionistic fuzzy partition distance to find an optimal attribute subset in the decision table. The experimental results indicate that our method can reduce computational time and enhance classification accuracy, particularly when dealing with datasets containing numerical values and a large number of objects. Additionally, our algorithm demonstrates superior performance when compared to algorithms based on the intuitionistic fuzzy set approach. In the future, we will continue to expand the properties of α, β-level intuitionistic fuzzy sets to apply for discrete, categorical value datasets. In particular, we will build some calculation techniques α, β level on each attribute to get better results for the method.

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