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RESEARCH ARTICLE

Generalized *m*-Polar Fuzzy Planar Graph and Its Application

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ABSTRACT Planarity of crisp graphs is a well-established field, whereas planarity within a fuzzy framework has seen recent development and extensive exploration. In an *m*-polar fuzzy graph (*m*PFG), each node and edge is associated with *m*-components, connected through minimal relationships. However, if one desires to incorporate maximum, average, or other intermediate relationships between nodes and edges, the *m*PFG concept becomes inadequate as in the *m*-polar fuzzy model, only minimum relation is considered. To address this limitation, a generalized model of *m*PFG is introduced in this article, allowing for a broader range of relationships to be considered simultaneously. This paper also discusses the properties of generalized *m*-polar fuzzy environments and generalized *m*-polar fuzzy graphs (G*m*PFGs), highlighting their isomorphism. Several significant findings and insights are presented in this paper. The article delves into the properties and characteristics of generalized *m*-polar fuzzy planar graphs (G*m*PFPGs) and explores various intriguing aspects related to them. Additionally, a novel concept of a generalized *m*-polar fuzzy dual graph (G*m*PFDG) is introduced, derived from G*m*PFPGs. The paper establishes a relationship between the dual of a G*m*PFG and G*m*PFG, examining their properties in the context of dual G*m*PFPGs. Lastly, the article discusses an illustrative example of a social group network problem assessing the group's activity based on attributes such as cooperation, team spirit, awareness, controlling power, good behaviour, and creativity.

INDEX TERMS Fuzzy graph, *m*-polar fuzzy graph, planarity of generalized *m*-polar fuzzy graph, dual of generalized *m*-polar fuzzy planar graph, fuzzy face.

I. INTRODUCTION

A. RESEARCH BACKGROUND AND RELATED WORKS

A fuzzy graph (FG) is a mathematical representation that extends traditional graphs by allowing edges and vertices to have degrees of membership rather than binary values, thereby accommodating uncertainty and imprecision in modeling relationships between entities. In a fuzzy graph, edges signify varying degrees of connection or strength of relationships, and vertices possess degrees of membership reflecting their affiliation with the graph. FGs find applications in fields such as pattern recognition, decision-making under uncertainty, and social network analysis, offering a versatile framework for capturing and analyzing complex relationships in scenarios where precise, binary relationships are inadequate. FG theory has significantly contributed to technical advancements, particularly in the development of rule-based expert systems for engineers. Moreover, graph theory plays a vital role in establishing connectivity principles across various fields, including algebra, geometry, topology, computer science, number theory, optimization, and operations research. Rosenfeld's groundbreaking work in 1975 [1] marked the inception of fuzzy relations and the development of FGs. Subsequent

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research, as demonstrated by Mathew and Sunitha [2], Anjali and Mathew [3], Mordeson and Nair [4], and Sunitha and Mathew [5], expanded upon this foundation, focusing primarily on operations involving FGs, fuzzy paths, the complement of FGs, fuzzy sub-graphs, and fuzzy trees. Samanta and Pal [6] pioneered the concept of fuzzy planar graphs (FPGs), and Samanta et al. further contributed to the field by introducing fuzzy coloring for FGs in their work [7]. Bhattacharya and Pal [8] conducted research on the fifth sustainable development goal, which focuses on gender equality in India. Their work involved the application of Mathematics of uncertainty and the analysis of fuzzy graphs. Additionally, they [9] explored the fuzzy tree covering number for fuzzy graphs and its practical implications in the context of the electricity distribution system. Furthermore, they [10] applied fuzzy graph theory to address facility location problems, with a specific case study in the Indian banking system.

Many real-world challenges have been resolved using data from various sources in our world. This method of data collection exemplifies multi-polarity. The concept of FG or bipolar FG cannot structure this kind of polarity well. The introduction of m-polar fuzzy sets (mPFS) to graph theory has been instrumental in addressing various aspects of graph structures. Initially, Ghorai and Pal [11] pioneered the field by establishing the concept of mPFG. Subsequently, they expanded their research to include the notion of mPFPGs [12], as well as investigating the dual and faces of mPFPGs [13] and exploring some isomorphic properties of mPFGs [14]. The genus value of mPFGs was introduced by Mandal et al. [15]. Additionally, Mandal et al. conducted a comprehensive study on various types of arcs within mPFGs [16]. Akram and Adeel examined the concept of m-polarity within FGs and line graphs [17], while Akram et al. discussed specific edge features within mPFGs [18]. Further contributions to the field include Mahapatra and Pal's introduction of fuzzy coloring for mPFGs [19] and, more recently, Mahapatra et al.'s initiation of fractional coloring on FGs [20]. Sasikala and Divya [21] studied an innovative perspective on Fermatean neutrosophic Dombi fuzzy graphs, and Uma and Nandhitha [22] explored the application of fuzzy and neutrosophic Poisson distribution in a rapid switching system. Abdel-Basset et al. [23] conducted a study on sustainability assessment for the optimal location of electric vehicle charging stations, presenting a conceptual framework for integrating green energy into smart cities. They also [24] performed an effective analysis of risk assessment and mitigation strategies for photovoltaic power plants based on real data, addressing strategies, challenges, perspectives, and sustainability. Li et al. [25] examined the MAGDM model with Aczel-Alsina aggregation operators applied to neutrosophic entropy elements within the context of neutrosophic multi-valued sets. In contrast, Bu et al. [26] delved into the topic of neutrosophic Pseudo-t-Norm and its associated neutrosophic residual implication.

Another characteristic of FGs is that their edge membership value (MV) does not exceed the MVs of their adjacent nodes. However, a feature of generalized FGs is that their edge MVs are equal to any relation of their end node MVs. That is, edge MV can be determined with different suitable forms or relations like maximum, minimum, average, etc., between two nodes of that edge for different corresponding problems. The same case for mPFGs has one restriction for determining edge MVs, but in GmPFGs, the edge MVs can be determined by using different relations or restrictions with respect to different problems. For example, we have taken a social group which is represented by G3PFG corresponding to the attribute awareness, cooperation and good behaviors of each person in the social group. Every person is treated as a node, and the edges are attached by adjacent nodes. Here, the MVs of the node are considered with respect to the attributes of awareness, cooperation and good behavior of the person. Every edge represents the relation between two persons. MVs can be derived from the 'relationship' parameter between two persons in a group. Now, the question is, "how will a relationship be constructed?" The relationship is defined as the maximum between two node MVs. This relation will measure the maximum awareness between two persons, cooperation between two persons and good behavior between two persons.

The inequality restriction of edges is the base of FGs. Samanta and Sarkar [27] presented the idea of generalized graphs to eliminate such a requirement. In Samanta and Sarkar [28], [29], [30], several generalized FG properties were covered. Isomorphism in FGs was initially explored by Bhutani in 1989, as documented in [31]. Nagoorgani and Malarvizhi [32], [33], Nagorgani and Latha [34], [35] added several intriguing aspects to it. The condition of homomorphism and isomorphism in generalized FGs was generalized by Samanta and Sarkar [36]. For fundamental terminologies and definitions, see [37].

B. MOTIVATION OF THE WORK

In our real world, the resolution of numerous issues relies on data gathered from diverse sources, illustrating the concept of multi-polarity in data collection. Traditional graph models like FGs or bipolar FGs may not adequately capture such polarity. An mPFG is characterized by a single "minimum" relation between vertices and edges component-wise, while a GmPFG can encompass multiple relations between vertices and edges for determining edge MVs. This means that edge MVs can be calculated using various suitable forms or relationships, such as maximum, minimum, average, and more, between the two nodes of that edge to address different corresponding problems. Consider a social group graph model aimed at assessing group activity based on attributes like cooperation, team spirit, awareness, controlling power, good behaviour, and creativeness. These attributes inherently entail uncertainty, making the 6-polar fuzzy model a suitable choice, while standard fuzzy, intuitionistic fuzzy, or bipolar fuzzy systems are less effective in addressing this challenge.

The m-polar fuzzy model is the preferred approach, yet it too falls short when users require relationships beyond the minimum between nodes and edges. Consequently, GmPFG systems prove more effective in handling the intricacies of fuzziness than any other fuzzy system. The incorporation of illustrations and relevant theorems in creating and interpreting such mPFGs enhances the existing GmPFG concepts and bolsters their suitability for addressing complex challenges.

C. NOVELTY OF THE WORK

GmPFGs are a relatively new concept in the field of fuzzy set theory and graph theory, and their novelty lies in their ability to represent and analyze complex relationships and uncertainty in a wide range of applications. Here are some aspects of their novelty:

(i) GmPFGs combine the principles of m-polar fuzzy set theory, which deals with uncertainty and vagueness, with graph theory, which focuses on representing relationships between objects. This integration allows for a more comprehensive representation of real-world systems where relationships are not always crisp but involve degrees of uncertainty.

(ii) An mPFG is characterized by a single "minimum" relation between vertices and edges component-wise, while a GmPFG can encompass multiple relations between vertices and edges for determining edge MVs. This means that edge MVs can be calculated using various suitable forms or relationships, such as maximum, minimum, average, and more, between the two nodes of that edge to address different corresponding problems.

(iii) GmPFGs have found applications in various fields, including social network analysis, image processing, pattern recognition, transportation systems, and more. Their ability to model uncertainty and imprecision in these domains makes them a valuable tool for solving real-world problems.

(iv) Researchers have developed a robust mathematical framework for GmPFGs, including definitions, operations, and algorithms for various graph-related tasks. This framework provides a solid basis for conducting research and solving problems involving GmPFGs.

In summary, the novelty of GmPFGs lies in their ability to bridge the gap of mPFG, providing a powerful framework for modeling and analyzing uncertain and complex relationships in various applications. As researchers continue to explore and develop this concept, its potential to address real-world problems with a high degree of uncertainty becomes increasingly apparent.

D. FRAMEWORK OF THIS STUDY

This article is structured as follows: Section II discusses a few important features that are necessary for this study. We have mentioned the concept of generalized mPFG and presented some theory on its aspect in section III. We have initiated a brand-new conception called GmPFPGs in section IV. We also investigated a detailed description of it through proper examples and studied different types of properties. Based on the above concept, we also investigated some features. In section V, we have mentioned faces on GmPFPG and discussed several features along with proper justification. In section VI, we initiate a brand-new notion of dual of GmPFPGs along with its different features. In section VII, a practical implementation based on social group to explain whether the group is active or not with respect to attributes cooperation, team spirit, awareness, controlling power, good behavior, creativeness, etc. has been given. In section VIII, we have discussed the theoretical implications, managerial insights, and policy implications of the study. In section IX, we have outlined several advantages, constraints and drawbacks inherent to the proposed study. In section X, some concluding remarks of our study have been made.

E. CONTRIBUTION OF THE WORK

The article makes a significant contribution by introducing planarity within a generalized m-polar fuzzy environment and exploring its properties under isomorphism. It also delves into the intricate details of GmPFPGs, establishes a relationship between dual GmPFGs and GmPFGs, and offers a practical application in analyzing group dynamics based on various attributes, thus advancing the understanding of fuzzy graph theory and its real-world applications. The article's primary contributions encompass the following points.

- Introduction of planarity in a generalized m-polar fuzzy environment.
- Discussion of properties under isomorphism.
- Exploration of faces of GmPFPGs.
- Presentation of intriguing details about GmPFPGs.
- Initiation of a GmPFDG derived from GmPFPGs.
- Establishment of a relationship between dual GmPFGs and GmPFGs.
- Discussion of properties in the context of dual GmPFPGs.
- Inclusion of a real-world application in social groups.
- Determination of group activity based on attributes such as cooperation, team spirit, awareness, controlling power, good behavior, and creativity.

F. NOTATIONS AND SYMBOLS

In Table 1, we have given some notations and abbreviation forms which are applied in the entire work for the construction of the article.

II. PRELIMINARIES

In this context, let's briefly revisit several definitions associated with mPFG, including concepts such as strong mPFG, complete mPFG, and paths within mPFG.

In this article, the notation $p_s : [0, 1]^m \rightarrow [0, 1]$ represents the s^{th} material projection mapping, with s ranging from 1 to m, i.e., $s = 1(1)m$.

TABLE 1. Full name and their abbreviation.

Full name	Abbreviation
Fuzzy graph	FG
m-polar fuzzy graph	mPFG
Generalized m-polar fuzzy graph	GmPFG
Underlying crisp graph	UCG
Membership value	MV
m-polar fuzzy set	mPFS
Generalized m-polar fuzzy multi-graph	GmPFMG
Generalized m-polar fuzzy planar graph	GmPFPG
Fuzzy planar graph	FPG
Generalized m-polar fuzzy dual graph	GmPFDG
Degree of planarity	DOP
Strong edge	SE

Definition 1 [14]: Consider $H = (\tilde{V}, \sigma, \gamma)$ as an mPFG derived from the underlying crisp graph (UCG) $H^* = (\tilde{V}, \tilde{E})$, where both σ and γ are mappings from $\tilde{V}$ to $[0, 1]^m$ and from $\tilde{V} \times \tilde{V}$ to $[0, 1]^m$, respectively. Here, σ and γ represent mPFS for $\tilde{V}$ and $\tilde{V} \times \tilde{V}$, respectively, satisfying the relationship $p_s \circ \gamma(x, w) \leq \{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)\}$ for every $s = 1(1)m$ and $(x, w) \in \tilde{V} \times \tilde{V}$, while also ensuring $\gamma(x, w) = 0$ for every $(x, w) \in (\tilde{V} \times \tilde{V} - \tilde{E})$.

Definition 2 [11]: $H = (\tilde{V}, \sigma, \gamma)$ is considered a complete mPFG when $p_s \circ \gamma(x, w) = \{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)\}$ holds for all x and w in $\tilde{V}$, with s ranging from 1 to m.

Definition 3 [14]: $H = (\tilde{V}, \sigma, \gamma)$ is classified as an mPF strong graph when the equation $p_s \circ \gamma(x, w) = \{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)\}$ holds for every $(x, w) \in \tilde{E}$, with s ranging from 1 to m.

Definition 4 [18]: Lets $H = (\tilde{V}, \sigma, \gamma)$ to be a mPFG as well as $P : x_1, x_2, \ldots, x_k$ to be a path in H. $S(P)$ denotes the strength of P, which is defined as $S(P) = (\min_{1\leq s<j\leq k} p_1 \circ \gamma(x_s, x_j), \min_{1\leq s<j\leq k} p_2 \circ \gamma(x_s, x_j), \ldots, \min_{1\leq s<j\leq k} p_m \circ \gamma(x_s, x_j)) = (\gamma_1^n(x_s, x_j), \gamma_2^n(x_s, x_j), \ldots, \gamma_m^n(x_s, x_j))$.

The strength of connectedness of the path between x_1 and x_k is given as follows:

$CONN_G(x_1, x_k) = (p_1 \circ \gamma(x_s, x_j)^\infty, p_2 \circ \gamma(x_s, x_j)^\infty, \ldots, p_m \circ \gamma(x_s, x_j)^\infty)$, where $(p_s \circ \gamma(x_s, x_j)^\infty) = \max_{n\in N}(\gamma_s^n(x_s, x_j))$.

Definition 5 [19]: For a mPFG, an edge $(x, w), x, w \in \tilde{V}$ is considered independently strong in $H = (\tilde{V}, \sigma, \gamma)$ if $\frac{1}{2}\{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)\} \leq p_s \circ \gamma(x, w), s = 1(1)m$. If not, it is perceived as weak on its own. To measure the strength of an edge (x, w) is

$$p_s \circ I(x, w) = \frac{p_s \circ \gamma(x, w)}{p_s \circ \sigma(x) \wedge p_s \circ \sigma(w)}, \quad s = 1(1)m.$$

Definition 6 [11]: Consider two mPFGs, denoted as $H = (\tilde{V}, \sigma, \gamma)$ and $H' = (\tilde{V}', \sigma', \gamma')$, associated with the UCGs $H^* = (\tilde{V}, \tilde{E})$ and $H'^* = (\tilde{V}', \tilde{E}')$, respectively. An isomorphism between these graphs is defined as a bijection denoted by $g : \tilde{V} \rightarrow \tilde{V}'$, which satisfies the conditions:

$$p_s \circ \sigma(x) = p_s \circ \sigma'(g(x)) \text{ and } p_s \circ \gamma(x, w) = p_s \circ \gamma'(g(x), g(w))$$

for all x and w in $\tilde{V}$ and for every $s = 1(1)m$. In such cases, H is said to be isomorphic to H'.

Definition 7 [11]: Between $H = (\tilde{V}, \sigma, \gamma)$ and $H' = (\tilde{V}', \sigma', \gamma')$, a co-weak isomorphism is a bijective homomorphism $g : \tilde{V} \rightarrow \tilde{V}'$ that satisfies

$$p_s \circ \gamma(x, w) = p_s \circ \gamma'(g(x), g(w))$$

for all $x, w \in \tilde{V}$ and every $s = 1(1)m$.

III. GENERALIZED M-POLAR FUZZY GRAPHS

Here, we will initiate a new concept of mPFG through which we can handle a lot of relationships between nodes and edges called generalized mPFG, where nodes and edges are interlinked through many relationships like max, average, difference, etc.

Definition 8: Consider $\acute{V}$ a set that is not void. Two mappings are taken as below: $C : \acute{V} \rightarrow [0, 1]^m$ and $D : \acute{V} \times \acute{V} \rightarrow [0, 1]^m$. Also, let $\tilde{A} = \{(p_s \circ C(t), p_s \circ C(w)) | p_s \circ D(t, w) > 0\}$, for every $s = 1(1)m$. The triplet $(\acute{V}, C, D)$ is said to be GmPFG in the event that one exists $\phi : \tilde{A} \rightarrow [0, 1]$ such that for all $s = 1(1)m$,

$$p_s \circ D(t, w) = \phi((p_s \circ C(t), p_s \circ C(w))),$$

where $t, w \in \acute{V}$. Here, $p_s \circ C(t), s = 1(1)m, t \in \acute{V}$ is the s^{th} MV of the node t and $p_s \circ D(t, w), s = 1(1)m, t, w \in \acute{V}$ is the s^{th} MV of the edge (t, w). Here, in case all the components of any edge are zero, i.e., $(0, 0, \ldots, 0)$, then for this case, the edge does not exist.

Note 1: For GmPFG, the relation between node and edge MVs are connected by $\phi : \tilde{A} \rightarrow [0, 1]$ such that

$$p_s \circ D(t, w) = \phi((p_s \circ C(t), p_s \circ C(w))),$$

every $s = 1(1)m$ and $t, w \in \acute{V}$. In this case, the domain of ϕ is the set $\tilde{A}$ of a pair of elements of C that are adjacent. An element of $\tilde{A}$ is now associated with an edge MV. Suppose we have taken a social group which is represented by G3PFG corresponding to the attributes of awareness, cooperation and good behavior of each person in the social group. Every person is taken as a node, and adjacent nodes are connected by edges. Here, the MVs of the node are considered with respect to the attributes of awareness, cooperation and good behavior of the person. Every edge represents the relation between two persons. The MVs can be collected from the 'relationship' between two persons in a group. Now, the question is, "how will relationship, i.e., ϕ be constructed?" For example, ϕ can be assumed as

$$\phi(p_s \circ C(t), p_s \circ C(w)) = p_s \circ C(t) \vee p_s \circ C(w),$$

for every $s = 1, 2, 3$. This function will measure the maximum awareness between two persons, cooperation between two persons and good behavior between two persons.

Similarly, depending on the various situations, we can choose the function $\phi : \tilde{A} \to [0, 1]$ as follows

$$(a)\ \phi(p_s \circ C(t), p_s \circ C(w)) = p_s \circ C(t) \wedge p_s \circ C(w),$$
$$(b)\ \phi(p_s \circ C(t), p_s \circ C(w)) = p_s \circ C(t) \vee p_s \circ C(w),$$
$$(c)\ \phi(p_s \circ C(t), p_s \circ C(w)) = \frac{1}{2}(p_s \circ C(t) + p_s \circ C(w)),$$
$$(d)\ \phi(p_s \circ C(t), p_s \circ C(w)) = p_s \circ C(t) \times p_s \circ C(w)$$

for all $s = 1(1)m$ *and* $t, w \in \acute{V}$.

***Note** 2:* In Definition 8 of GmPFG, if we consider the function $\phi : \tilde{A} \to [0, 1]$ as $\phi(p_s \circ C(t), p_s \circ C(w)) = p_s \circ C(t) \wedge p_s \circ C(w)$ for every $s = 1(1)m$ and for all $(t, w) \in E$, then for this case a GmPFG will be called strong mPFG. So, we can easily form a strong mPFG from a GmPFG.

Example 1: Consider $\Gamma = (\acute{V}, C, D)$ be a G3PFG. Let $\acute{V} = \{t_1, t_2, t_3, t_4\}$ be the nodes set and $\{(t_1, t_3), (t_1, t_4), (t_2, t_3)\}$ be the edges set with $C(t_1) = (0.2, 0.3, 0.5)$, $C(t_2) = (0.5, 0.3, 0.4)$, $C(t_3) = (0.6, 0.7, 0.2)$, $C(t_4) = (0.1, 0.4, 0.3)$. Now, we have considered the function $\phi : \tilde{A} \to [0, 1]$ such that $\phi(p_s \circ C(x), p_s \circ C(w)) = p_s \circ C(x) \vee p_s \circ C(w)$, for every $s = 1(1)m$ and $C(x), C(w)$ indicate the MVs of nodes x and w respectively. Here, $\tilde{A} = \{(C(t_1), C(t_3)), (C(t_1), C(t_4)), (C(t_2), C(t_3))\} = \{((0.2, 0.3, 0.5), (0.6, 0.7, 0.2)), ((0.2, 0.3, 0.5), (0.1, 0.4, 0.3)), ((0.5, 0.3, 0.4), (0.6, 0.7, 0.2))\}$. Then $D(t_1, t_3) = (0.2 \vee 0.6, 0.3 \vee 0.7, 0.5 \vee 0.2) = (0.6, 0.7, 0.5)$, $D(t_1, t_4) = (0.2, 0.4, 0.5)$, $D(t_2, t_3) = (0.6, 0.7, 0.4)$. The pictorial representation of G3PFG is shown in Fig. 1.

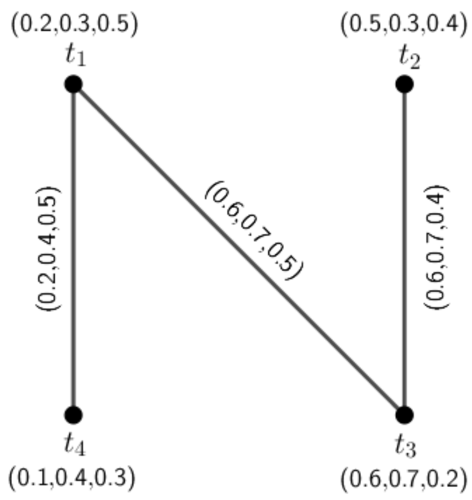

FIGURE 1. **An example of G3PFG.**

Definition 9: Take $\Gamma = (\acute{V}, C, D)$ be a GmPFG. A edge (t, u) is defined to be effective edge if $p_s \circ D(t, u) \geq \frac{1}{2} max\{p_s \circ C(t), p_s \circ C(u)\}$, for every $s = 1(1)m$ and $t, u \in \acute{V}$. A GmPFG is said to be effective if it satisfies the relation $p_s \circ D(t, u) \geq \frac{1}{2} max\{p_s \circ C(t), p_s \circ C(u)\}$, for every $s = 1(1)m$ and for all $t, u \in \acute{V}$.

Example 2: From Example 1, we have seen that the edge (t_1, t_4) is effective because the condition of effectiveness, i.e., $p_s \circ D(t_1, t_4) \geq \frac{1}{2} max\{p_s \circ C(t_1), p_s \circ C(t_4)\}$ has satisfied for every $s = 1, 2, 3$. That is $0.2 > \frac{1}{2} max\{0.2, 0.1\}$, $0.4 > \frac{1}{2} max\{0.3, 0.4\}$ and $0.5 > \frac{1}{2} max\{0.5, 0.3\}$. Similarly, the edges (t_1, t_3) and (t_2, t_3) are effective. Therefore, the graph Γ is effective.

Definition 10: Let Γ be a GmPFG. If each node pair are attached by effective edges, then Γ is said to be complete; otherwise, it is called incomplete GmPFG.

Example 3: Let $\Gamma = (\acute{V}, C, D)$ be a G3PFG with node set $\{(a_1, (0.4, 0.3, 0.6)), (a_2, (0.3, 0.2, 0.5)), (a_3, (0.2, 0.4, 0.7)), (a_4, (0.6, 0.5, 0.4))\}$ and edge set $\{(a_1, a_2), (a_1, a_3), (a_1, a_4), (a_2, a_3), (a_2, a_4), (a_3, a_4)\}$. We define the function $\phi : \tilde{A} \to [0, 1]$, where $\tilde{A} = \{(p_s \circ C(a), p_s \circ C(b)) | p_s \circ D(a, b) > 0\}$, such as $\phi(p_s \circ C(a), p_s \circ C(b)) = \frac{1}{2}(p_s \circ C(a) + p_s \circ C(b))$, for every $s = 1, 2, 3$ and $a, b \in \acute{V}$. With the help of the above function, we have evaluated the MVs of all edges of Γ, which are shown in Fig. 2. Now, we have seen that the edge

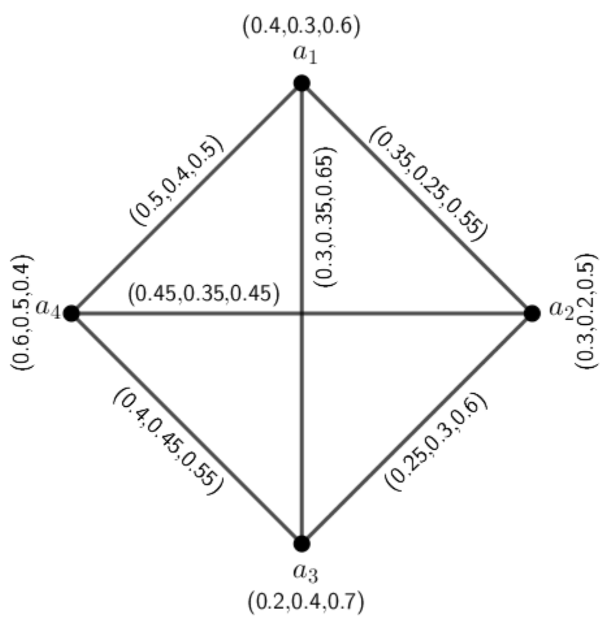

FIGURE 2. **An example of complete G3PFG.**

(a_1, a_2) is effective because the condition of effectiveness, i.e., $p_s \circ D(a_1, a_2) \geq \frac{1}{2} max\{p_s \circ C(a_1), p_s \circ C(a_2)\}$ has satisfied for every $s = 1, 2, 3$. That is $0.35 > \frac{1}{2} max\{0.4, 0.3\}$, $0.25 > \frac{1}{2} max\{0.3, 0.2\}$ and $0.55 > \frac{1}{2} max\{0.6, 0.5\}$. Similarly, the other edges (a_1, a_3), (a_1, a_4), (a_2, a_3), (a_2, a_4) and (a_3, a_4) are effective. As a result, effective edges connect each pair of nodes. Therefore, the graph Γ is a complete G3PFG.

Definition 11: Let $\Gamma_1 = (\acute{V}_1, C_1, D_1)$ and $\Gamma_2 = (\acute{V}_2, C_2, D_2)$ be two GmPFGs. Also, let $\tilde{h} : \acute{V}_1 \to \acute{V}_2$, $\phi_1 : \tilde{A}_1 \to [0, 1]$ and $\phi_2 : \tilde{A}_2 \to [0, 1]$ be the functions where $\tilde{A}_1 = \{p_s \circ C_1(t) | t \in \acute{V}\}$, for every $s = 1(1)m$ and $\tilde{A}_2 = \{(p_s \circ C_2(t), p_s \circ C_2(u)) | t, u \in \acute{V}_2, (t, u) \textit{ is an edge in } \Gamma_2\}$, for every $s = 1(1)m$ such that $p_s \circ C_2(t) = \phi_1(p_s \circ C_1(\tilde{h}(t)))$ and

$p_s \circ D_2(t,u) = \phi_2(p_s \circ D_1(\tilde{h}(t), \tilde{h}(u)))$, for every $s = 1(1)m$. After that, it is said that the function $\tilde{h}$ is a homomorphism between two GmPFGs.

Definition 12: Let's say that $\Gamma_1 = (\acute{V}_1, C_1, D_1)$ and $\Gamma_2 = (\acute{V}_2, C_2, D_2)$ are two homomorphic GmPFGs. Now, consider ϕ_1, an identity mapping and $\tilde{h}$, a bijective homomorphism from $\acute{V}_1$ to $\acute{V}_2$. After that, the isomorphism $\tilde{h}$ is called to be weak between two GmPFGs.

Definition 13: Let's say that $\Gamma_1 = (\acute{V}_1, C_1, D_1)$ and $\Gamma_2 = (\acute{V}_2, C_2, D_2)$ are two homomorphic GmPFGs. Now, consider ϕ_2, an identity mapping and $\tilde{h}$, a bijective homomorphism from $\acute{V}_1$ to $\acute{V}_2$. After that, the isomorphism $\tilde{h}$ is called to be co-weak between two GmPFGs.

Definition 14: Suppose $\Gamma_1 = (\acute{V}_1, C_1, D_1)$ and $\Gamma_2 = (\acute{V}_2, C_2, D_2)$ be two homomorphic GmPFGs. Now, consider ϕ_1 and ϕ_2, both identity mapping and $\tilde{h}$, a bijective homomorphism from $\acute{V}_1$ to $\acute{V}_2$. After that, $\tilde{h}$ is called to be isomorphism between two GmPFGs.

Definition 15: Consider $\acute{V}$ a set that is not void and C be an mPFS on $\acute{V}$. Assume $D = \{((x,w), D^j(x,w)), j = 1(1)p_{xw} : (x,w) \in \acute{V} \times \acute{V}\}$, where $p_{xw} = max\{j | D^j(x,w) \neq 0\}$ be an m-polar fuzzy multi-set of $\acute{V} \times \acute{V}$. Consider $\tilde{A} = \{(p_s \circ C(x), p_s \circ C(w)) | p_s \circ D^j(x,w) > 0, j = 1(1)p_{xw}\}$, for every $s = 1(1)m$. Then $\Gamma = (\acute{V}, C, D)$ is called to be generalized m-polar fuzzy multi-graph (GmPFMG) if there exists a function $\phi : \tilde{A} \rightarrow [0,1]$ such that for every $s = 1(1)m$,

$$p_s \circ D^j(x,w) = \phi((p_s \circ C(x), p_s \circ C(w))),$$

$s = 1(1)m, j = 1(1)p_{xw}$ and for all $x, w \in \acute{V}$. Here, $C(x)$ and $D(x,w)$ stand in for the MV of the node x and the edge (x,w) in Γ, respectively. It should be noted that there could be more than one edge connecting nodes x and w. It can be noted that there may be multiple edges between the nodes x and w. The number of edges between the nodes x and w is represented by p_{xw}, and the MV of the j^{th} edge connecting them is denoted by $D^j(x,w)$.

IV. GENERALIZED *M*-POLAR FUZZY PLANAR GRAPHS

According to crisp graph theory, a graph is said to be planar if it is drawn so that no two edges overlap other than at their endpoints. It is referred to as non-planar if it is not. Again, we are aware that any graph that contains either K_5 or $K_{3,3}$ is also non-planar in the crisp meaning. This identical idea will be examined in a GmPFPGs.

Let $\Gamma = (\acute{V}, C, D)$ be a GmPFMG and the graph contains just one crossover between the edges $((t,v), D(t,v))$ and $((w,y), D(w,y))$ for a particular geometric shapes. The graph is called to not cross if $D(t,v) = (1,1,\ldots,1)$ and $D(w,y) = (0,0,\ldots,0)$. Similarly, if $p_s \circ D(t,v) = \phi((p_s \circ C(t), p_s \circ C(v)))$, $s = 1(1)m$ has value near to 1 and $p_s \circ D(w,y) = \phi((p_s \circ C(w), p_s \circ C(y)))$, $s = 1(1)m$ has value near to 0, the intersection won't matter for planarity. If $p_s \circ D(t,v) = \phi((p_s \circ C(t), p_s \circ C(v)))$ and $p_s \circ D(w,y) = \phi((p_s \circ C(w), p_s \circ C(y)))$, $s = 1(1)m$ are close to 1, after that the crossover becomes vital for planarity. Therefore, a value corresponding to the point is assigned, known as the intersecting value, if there is a point where two edges cross.

Definition 16: An edge's $((t,v), D^j(t,v))$ strength is determined by $I_{(t,v)} = (I^1_{(t,v)}, I^2_{(t,v)}, \ldots, I^m_{(t,v)})$, where $I^s_{(t,v)} = \frac{p_s \circ D^j(t,v)}{min\{p_s \circ C(t), p_s \circ C(v)\}}$, $s = 1(1)m, j = 1(1)p_{tv}$ and $t, v \in \acute{V}$. Here $p_s \circ D^j(t,v) = \phi((p_s \circ C(t), p_s \circ C(v)))$, for every $s = 1(1)m$, $j = 1(1)p_{tv}$ and $t, v \in \acute{V}$.

Definition 17: Consider $\Gamma = (\acute{V}, C, D)$ be a GmPFMG. An edge (u,v) in Γ is called to be generalized m-polar fuzzy strong if for every $s = 1(1)m$, $I^s_{(u,v)} \geq 0.5$. Instead, it is referred to as generalized m-polar fuzzy weak.

When two edges cross at a point in a GmPFMG, that point is specified as follows. Let in a GmPFMG $\Gamma = (\acute{V}, C, D)$, two edges $((u_1,v_1), D^j(u_1,v_1))$ and $((u_2,v_2), D^k(u_2,v_2))$ cross at a point P, where k and j are fixed positive integers. $\mathcal{I}_{\mathcal{P}} = (\mathcal{I}^1_P, \mathcal{I}^2_P, \ldots, \mathcal{I}^m_P)$, where $\mathcal{I}^s_P = \frac{I^s_{(u_1,v_1)} + I^s_{(u_2,v_2)}}{2}$, $s = 1(1)m$ represents the intersecting value at the point P. The 'planarity' of a GmPFMG diminishes as the number of intersecting points rises. The idea of a GmPFPG is described below using these ideas.

Definition 18: Let $\Gamma = (\acute{V}, C, D)$ be a GmPFMG, and for a particular geometric shape, the points of crossing between the edges are symbolized by $Q_1, Q_2, \ldots, Q_k$. Then Γ is called to be GmPFPG with degree of planarity (DOP) $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m)$, where $\mathcal{Q}_s = \frac{1}{1+\{\mathcal{I}^s_{Q_1} + \mathcal{I}^s_{Q_2} + \cdots + \mathcal{I}^s_{Q_k}\}}$, $s = 1(1)m$.

***Note** 3:* $\mathcal{Q}$ is bounded, since for every $s = 1(1)m$, $0 < \mathcal{Q}_s \leq 1$.

Example 4: Consider $\Gamma = (\acute{V}, C, D)$ be a GmPFMG which is shown in Fig. 3. Let $\acute{V} = \{u_1, u_2, u_3, u_4\}$ be the node set with $C(u_1) = (0.3, 0.2, 0.4)$, $C(u_2) = (0.5, 0.4, 0.6)$, $C(u_3) = (0.3, 0.4, 0.6)$ and $C(u_4) = (0.2, 0.6, 0.7)$. Now, we have considered the function $\phi : \tilde{A} \rightarrow [0,1]$ such that

$$p_s \circ D^j(u,w) = \phi((p_s \circ C(u), p_s \circ C(w))) \quad (1)$$
$$= min\{p_s \circ C(u), p_s \circ C(w)\} \quad (2)$$

for every $s = 1(1)m$, $j = 1(1)p$ and $C(u), C(w)$ indicate the MVs of nodes u and w respectively. Using (2), we have determined all the MVs of edges, which is shown in Fig. 3.

Q_1 is a point between the edges $((u_1,u_3), (0.3,0.2,0.4))$ and $((u_2,u_4), (0.2,0.4,0.6))$, Q_2 is also between the edges $((u_1,u_3), (0.3,0.2,0.4))$ and $((u_2,u_4), (0.2,0.4,0.6))$. Now, the strength of an edge (a,b) is $I^s_{(a,b)} = p_s \circ I_{(a,b)} = \frac{p_s \circ D(a,b)}{p_s \circ C(a) \wedge p_s \circ C(b)}$, for every $s = 1,2,3$ and $D(a,b)$ indicates the MV of the edge (a,b) and $C(a), C(b)$ indicate the MVs of the vertices a and b respectively. Then, for the edge $((u_1,u_3), (0.3,0.2,0.4))$, the strength is $I_{(u_1,u_3)} = (1,1,1)$, and the edge $((u_2,u_4), (0.2,0.4,0.6))$, is $I_{(u_2,u_4)} = (1,1,1)$. Now, at the point P, we calculate the value $\mathcal{I}_P = (\mathcal{I}^1_P, \mathcal{I}^2_P, \mathcal{I}^3_P)$ which represents intersecting value at that point, where $\mathcal{I}^s_P = \frac{I^s_{(u_1,v_1)} + I^s_{(u_2,v_2)}}{2}$ for $s = 1,2,3$. The values that intersect are $\mathcal{I}_{Q_1} = (\mathcal{I}^1_{Q_1}, \mathcal{I}^2_{Q_1}, \mathcal{I}^3_{Q_1}) = (1,1,1)$ and $\mathcal{I}_{Q_2} =$

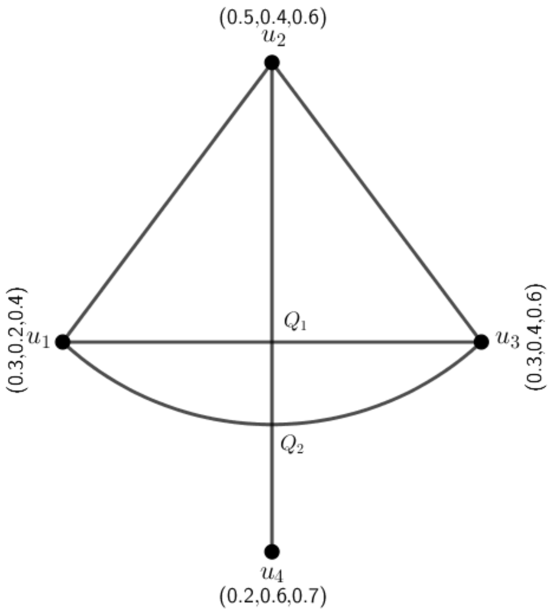

FIGURE 3. A G3PFMG.

$(\mathcal{I}^1_{Q_2}, \mathcal{I}^2_{Q_2}, \mathcal{I}^3_{Q_2}) = (1, 1, 1)$. Now, the DOP is defined by $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3)$, where $\mathcal{Q}_s = \frac{1}{1+(\mathcal{I}^s_{Q_1}+\mathcal{I}^s_{Q_2})}$, $s = 1, 2, 3$. As a result, the DOP of the G3PFMG is $(\frac{1}{1+1+1}, \frac{1}{1+1+1}, \frac{1}{1+1+1})$, i.e., $(0.33, 0.33, 0.33)$.

Now we will check the DOP of the graph for other forms of ϕ function in the following ways:

(i) If we consider the function ϕ in the form

$$\phi((p_s \circ C(u), p_s \circ C(w))) = max\{p_s \circ C(u), p_s \circ C(w)\}, \tag{3}$$

for every $s = 1(1)m$ and $C(u), C(w)$ indicate the MVs of nodes u and w respectively. Using the function (3), we can determine all MVs of the edges which are $((u_1, u_2), (0.5, 0.4, 0.6))$, $((u_1, u_3), (0.3, 0.4, 0.6))$, $((u_1, u_3), (0.3, 0.4, 0.6))$, $((u_2, u_3), (0.5, 0.4, 0.6))$, $((u_2, u_4), (0.5, 0.6, 0.7))$. Then in the similar way we can determine the strength values of all the edges which are $I_{(u_1,u_2)} = (1.67, 2, 1.5)$, $I_{(u_1,u_3)} = (1, 2, 1.5)$, $I_{(u_1,u_3)} = (1, 2, 1.5)$, $I_{(u_2,u_3)} = (1.67, 1, 1)$, $I_{(u_2,u_4)} = (2.5, 1.5, 1.67)$. In this case, the intersecting values of the intersecting points Q_1 and Q_2 respectively are $\mathcal{I}_{Q_1} = (1.75, 1.75, 1.58)$ and $\mathcal{I}_{Q_2} = (1.75, 1.75, 1.58)$. So, the DOP of the G3PFMG is $(0.22, 0.22, 0.24)$.

(ii) If we consider the function ϕ in the form

$$\phi((p_s \circ C(u), p_s \circ C(w))) = \frac{1}{2}\{p_s \circ C(u) + p_s \circ C(w)\}, \tag{4}$$

for every $s = 1(1)m$ and $C(u), C(w)$ indicate the MVs of nodes u and w respectively. Then this case the DOP of the G3PFMG is $(0.26, 0.26, 0.3)$.

(iii) If we consider the function ϕ in the form

$$\phi((p_s \circ C(u), p_s \circ C(w))) = p_s \circ C(u) \times p_s \circ C(w) \tag{5}$$

for every $s = 1(1)m$. Then, in this case, the DOP of the G3PFMG is $(0.56, 0.5, 0.43)$.

Theorem 1: Let $\Gamma = (\acute{V}, C, D)$ be a GmPFMG with $\phi((p_s \circ C(u), p_s \circ C(v))) = min\{p_s \circ C(u), p_s \circ C(u)\}$, for every $s = 1(1)m$ and for all $u, v \in \acute{V}$. Then the DOP $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \ldots,$
$\mathcal{Q}_m)$ is given by $\mathcal{Q}_s = \frac{1}{1+r}$, $s = 1(1)m$, here, r is the number of locations where the edges of Γ crosses.

Proof: Since Γ is a GmPFMG, we get

$$p_s \circ D^j(u, v) = \phi((p_s \circ C(u), p_s \circ C(v))) \tag{6}$$

for each $s = 1(1)m, j = 1(1)p_{uv}$ and $u, v \in \acute{V}$. Given,

$$\phi((p_s \circ C(u), p_s \circ C(v))) = min\{p_s \circ C(u), p_s \circ C(v)\} \tag{7}$$

for every $s = 1(1)m$ and for each $u, v \in \acute{V}$. Let $Q_1, Q_2, \ldots, Q_r$ be the intersection locations of the edges in Γ. In Γ, for the edge (u, v), from Definition 16, we have $I^s_{(u,v)} = \frac{p_s \circ D^j(u,v)}{min\{p_s \circ C(u), p_s \circ C(v)\}} = \frac{min\{p_s \circ C(u), p_s \circ C(v)\}}{min\{p_s \circ C(u), p_s \circ C(v)\}} = 1, s = 1(1)m$, using 6 and 7. Therefore, for the location Q_1, which is a location of crossing between edges (u, v) and (x, y), the crossing value is $\mathcal{I}_{Q_1} = (\mathcal{I}^1_{Q_1}, \mathcal{I}^2_{Q_1}, \ldots \mathcal{I}^m_{Q_1}) = (\frac{I^1_{(u,v)}+I^1_{(x,y)}}{2}, \frac{I^2_{(u,v)}+I^2_{(x,y)}}{2}, \ldots, \frac{I^m_{(u,v)}+I^m_{(x,y)}}{2}) = (\frac{1+1}{2}, \frac{1+1}{2}, \ldots, \frac{1+1}{2}) = (1, 1, \ldots, 1)$. Therefore $\mathcal{I}_{Q_r} = (1, 1, \ldots, 1)$, where r is the number of intersection locations. Now, for $s = 1(1)m$, $\mathcal{Q}_s = \frac{1}{1+(\mathcal{I}^s_{Q_1}+\mathcal{I}^s_{Q_2}+\cdots+\mathcal{I}^s_{Q_r})} = \frac{1}{1+r}$. Therefore, the planarity $\mathcal{Q}$ is given by $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m)$, where $\mathcal{Q}_s = \frac{1}{1+r}$, $s = 1(1)m$.

Theorem 2: Let $\Gamma = (\acute{V}, C, D)$ be a GmPFPG with DOP $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m)$ such that $\mathcal{Q}_s > 0.5$ for $s = 1(1)m$. Hence, there can only be one crossing of generalized m-polar fuzzy strong edges (SEs) in Γ.

Proof: Take Q_1 and Q_2 be the locations of crossing between two generalized m-polar fuzzy SEs in Γ and this the least possibility for Γ in the said sense. For any generalized m-polar fuzzy SE $((u, v), D^j(u, v))$, $I^s_{(u,v)} \geq 0.5$, $s = 1(1)m$ and $j = 1(1)p_{uv}$. Thus, the crossing of SEs $((u, v), D^j(u, v))$ and $((x, y), D^k(x, y))$, where k and j are fixed positive integers and for $s = 1(1)m$, $\frac{I^s_{(u,v)}+I^s_{(x,y)}}{2} \geq 0.5$, that is $\mathcal{I}^s_{Q_1} \geq 0.5$. Likewise, $\mathcal{I}^s_{Q_1} \geq 0.5$. So, $1 + \mathcal{I}^s_{Q_1} + \mathcal{I}^s_{Q_2} \geq 2$, i.e., $\mathcal{Q}_s = \frac{1}{1+\mathcal{I}^s_{Q_1}+\mathcal{I}^s_{Q_2}} \leq 0.5$. Here, for $s = 1(1)m$, $\mathcal{Q}_s > 0.5$, which is a contradiction. As a result, there can be no more than one point awarded.

Theorem 3: Let $\Gamma = (\acute{V}, C, D)$ be a GmPFPG with DOP $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m)$. If $\mathcal{Q}_s \geq 0.67$, $s = 1(1)m$, so there is no location of crossing between two generalized m-polar fuzzy SEs for Γ.

Proof: Let us assume that between two generalized m-polar fuzzy SEs $((u, v), D^j(u, v))$ and $((x, y), D^k(x, y))$, where k and j are fixed positive integers, the point of intersection be P. For any generalized m-polar fuzzy SE $((u, v), D^j(u, v))$, we get $I^s_{(u,v)} \geq 0.5$, $s = 1(1)m$. In case of finding minimum score of $I^s_{(u,v)}, I^s_{(x,y)}, \mathcal{I}^s_P = 0.5$, $s = 1(1)m$. Then, $\mathcal{Q}_s = \frac{1}{1+0.5} < 0.67$ for $s = 1(1)m$, a contradiction. As a result, Γ does not have any locations where two generalized m-polar fuzzy SEs cross.

Definition 19: A GmPFPG Γ with DOP $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m)$ is said to be strong GmPFPG if $\mathcal{Q}_s \geq 0.67$, $s = 1(1)m$.

Edges with sufficient strength play a crucial role in modeling certain projects, while those with low strength can be disregarded. These significant edges are referred to as "considerable edges," as defined below.

Definition 20: Let $\Gamma = (\acute{V}, C, D)$ be a GmPFPG. Let f be a given number, such that $0 < f < 0.5$. An edge $((u, v), D(u, v))$ is said to be considerable edge if for $s = 1(1)m$, $\frac{p_s \circ D(u,v)}{min\{p_s \circ C(u), p_s \circ C(v)\}} \geq f$. Otherwise, it is called a non-considerable edge. For a GmPFMG Γ, a multi-edge $((u, v), D^j(u, v))$ is said to be considerable edge if for $s = 1(1)m$, $\frac{p_s \circ D^j(u,v)}{min\{p_s \circ C(u), p_s \circ C(v)\}} \geq f, j = 1(1)p_{uv}$.

Theorem 4: Let Γ be a GmPFPG with DOP $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m)$ be such that $\mathcal{Q}_s > 0.5$, $s = 1(1)m$ and considerable number f. Then the number of points of intersection between considerable edges in Γ is at most $[\frac{1}{f}]$ or $\frac{1}{f} - 1$ according to $\frac{1}{f}$ is not an integer or an integer respectively.

Proof: Let $\Gamma = (\acute{V}, C, D)$ be a GmPFPG where $D = \{((u, v), D^j(u, v)), j = 1(1)p_{uv} : (u, v) \in \acute{V} \times \acute{V}\}$. Let $0 < f < 0.5$ be the considerable number. For any considerable edge $((u, v), D^j(u, v))$, we have $p_s \circ D^j(u, v) \geq f \times min\{p_s \circ C(u), p_s \circ C(v)\}$, $s = 1(1)m$, where $p_s \circ D^j(u, v) = \phi(p_s \circ C(u), p_s \circ C(v))$, for every $s = 1(1)m$. This implies that $I^s_{(u,v)} \geq f$ for $s = 1(1)m$. Let $Q_1, Q_2, \ldots, Q_q$ be the q numbers intersecting points between the considerable edges. Also let, between the considerable edges $((u, v), D^j(u, v))$ and $((x, y), D^k(x, y))$, where k and j are fixed positive integers, the location of crossing be Q_1. We have $I^s_{(u,v)} \geq f$ and $I^s_{(x,y)} \geq f$ for $s = 1(1)m$. This implies that $I^s_{(u,v)} + I^s_{(x,y)} \geq 2f$, i.e., $\frac{I^s_{(u,v)} + I^s_{(x,y)}}{2} \geq f$ for $s = 1(1)m$. Then $\mathcal{I}^s_{Q_1} = \frac{I^s_{(u,v)} + I^s_{(x,y)}}{2} \geq f$, i.e., $\mathcal{I}^s_{Q_1} \geq f$ for $s = 1(1)m$. Now, we have taken the sum of crossing values on q locations, i.e., $\sum_{n=1}^{q} \mathcal{I}^s_{Q_n} \geq fq$ for $s = 1(1)m$,

$$\text{or, } 1 + \sum_{n=1}^{q} \mathcal{I}^s_{Q_n} \geq 1 + fq$$

$$\text{or, } \frac{1}{1 + \sum_{n=1}^{q} \mathcal{I}^s_{Q_n}} \leq \frac{1}{1 + fq}$$

$$\text{or, } \mathcal{Q}_s \leq \frac{1}{1 + fq}, \text{ by the Definition 18.}$$

Hence $\mathcal{Q}_s \leq \frac{1}{1+fq}$, for $s = 1(1)m$ and given $\mathcal{Q}_s > 0.5$, for $s = 1(1)m$. This implies that $0.5 < \mathcal{Q}_s \leq \frac{1}{1+fq}$, for $s = 1(1)m$.

$$\text{That is, } 0.5 < \frac{1}{1 + fq}$$

$$\text{or, } \frac{1}{2} < \frac{1}{1 + fq}$$

$$\text{or, } 2 > 1 + fq$$

$$\text{or, } 1 > fq.$$

So, $q < \frac{1}{f}$. Hence the values of q are given by $q = \begin{cases} [\frac{1}{f}], \text{ if } \frac{1}{f} \text{ is not an integer,} \\ \frac{1}{f} - 1, \text{ if } \frac{1}{f} \text{ is an integer.} \end{cases}$

Theorem 5: Complete GmPFGs K_5 and $K_{3,3}$ have a DOP of $(0.5, 0.5, \ldots, 0.5)$.

Proof: The DOP for a complete GmPFG is $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m)$, where $\mathcal{Q}_s = \frac{1}{1+r}$ and r represents number of locations of crossings of edges in Γ. We know in crisp that one crossover between two edges cannot be avoided for a particular geometric shape of a complete graph K_5 and $K_{3,3}$. Therefore, $r = 1$ for the complete K_5 and $K_{3,3}$ GmPFG. Therefore, $\mathcal{Q}_s = \frac{1}{1+r} = \frac{1}{1+1} = 0.5$. Hence, the complete G$m$PFG K_5 and $K_{3,3}$ have with DOP $(0.5, 0.5, \ldots, 0.5)$.

Theorem 6: The complete GmPFGs K_5 and $K_{3,3}$ can not be strong GmPFGs.

Proof: The entire GmPFG K_5 and $K_{3,3}$ are GmPFGs with DOP $(0.5, 0.5, \ldots, 0.5)$, respectively, are obtained from Theorem 5. Again, according to Definition 19, a GmPFPG is considered to be strong if $\mathcal{Q}_s \geq 0.67$, for $s = 1(1)m$. In this instance, the requirement for a strong GmPFG is not fulfilled. Hence, the complete GmPFG K_5 and $K_{3,3}$ can not be strong GmPFG.

V. FACES OF GENERALIZED *M*-POLAR FUZZY PLANAR GRAPH

One of the most crucial components of a GmPFPG is its face. A GmPFG's faces are each region bordered by a few edges. There must be at least two edges for a GmPFG to have two faces. There must be one face on each and every GmPFPG that is referred to as the outer face. One face of the GmPFPG may be decreased from the supplied GmPFPG if one edge of the GmPFG is eliminated. We'll define the face of the GmPFG next.

Definition 21: Let $\Gamma = (\acute{V}, C, D)$ be a GmPFPG and $D = \{((t, v), D^j(t, v)) : (t, v) \in \acute{V} \times \acute{V} \text{ and } j = 1(1)p\}$. Hence, the face of G$m$PFG of Γ is a region bounded by edges. $E \subseteq \acute{V} \times \acute{V}$ of Γ.

Definition 22: In a GmPFPG, a face's strength is indicated by the notation $(I^1_F, I^2_F, \ldots, I^m_F)$ and is defined by $I^s_F = \wedge\{I^s_{(t,v)} : \forall(t, v) \in E\}$, for $s = 1(1)m$.

Definition 23: For every $s = 1(1)m$, a face of the GmPFPG is considered to be strong if $I^s_F \geq 0.5$. It is regarded to be weak otherwise.

Example 5: To explain the strength of faces of a G3PFPG and the powerful faces we see as G3PFPG $\Gamma = (\acute{V}, C, D)$ having node set $\acute{V} = \{u_1, u_2, u_3, u_4, u_5\}$ with $C(u_1) = (0.2, 0.3, 0.2)$, $C(u_2) = (0.6, 0.4, 0.2)$, $C(u_3) = (0.2, 0.4, 0.6)$, $C(u_4) = (0.3, 0.4, 0.5)$ and $C(u_5) = (0.4, 0.6, 0.5)$. Now, we have considered the function $\phi : \tilde{A} \rightarrow [0, 1]$ such that

$$p_s \circ D^j(t, v) = \phi((p_s \circ C(t), p_s \circ C(v))) \quad (8)$$

$$= \frac{1}{2}(p_s \circ C(t) + p_s \circ C(v)), \quad (9)$$

for every $s = 1(1)m$, $j = 1(1)p$ and $C(t), C(v)$ indicate the MVs of nodes t and v respectively. Using (9), we have determined all the MVs of edges, which is shown in Fig. 4.

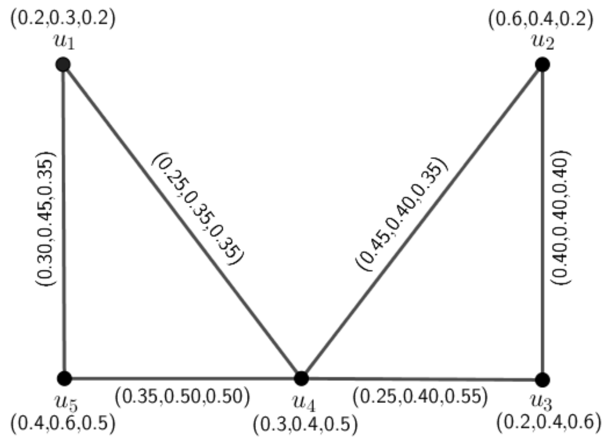

FIGURE 4. An example of G3PFPG.

There are three faces with the symbols F_1, F_2, and F_3, and face F_1 is bounded by the edges (u_1, u_4), (u_4, u_5), (u_5, u_1) and face F_2 is bounded by the edges (u_2, u_3), (u_3, u_4), (u_4, u_2) and F_3 is an outer face of the given G3PFG. In Table 2, the strength values of edges are listed.

TABLE 2. Strength values of edges of G3PFPG, shown in Fig. 4.

Edges	Strength values
(u_1, u_4)	$(1.25, 1.16, 1.75)$
(u_1, u_5)	$(1.5, 1.5, 1.75)$
(u_4, u_2)	$(1.5, 1, 1.75)$
(u_3, u_4)	$(1.25, 1, 1.1)$
(u_4, u_5)	$(1.16, 1.25, 1)$
(u_2, u_3)	$(2, 1, 2)$

Now, the strength of the faces F_1, F_2 and F_3 are $(1.16, 1.16, 1)$, $(1.25, 1, 1.1)$ and $(1.16, 1, 1)$ respectively. Therefore, F_1, F_2 *and* F_3 are strong faces of Γ.

VI. GENERALIZED *M*-POLAR FUZZY DUAL GRAPH

This section introduces the dual idea of the GmPFPG, the GmPFDG, and a new GmPFG. In an GmPFDG, the vertices are the strong faces of the GmPFPG, and if two vertices' corresponding faces share an edge, then there is an edge between them.

Definition 24: Assuming that $\Gamma = (\acute{V}, C, D)$ be a GmPFPG and $F_1, F_2, \ldots, F_p$ are its strong faces of Γ. Then $\Gamma_1 = (\acute{V}_1, C_1, D_1)$ be a GmPFDG of Γ, where the node will be u_s corresponding to the strong face F_s, for $s = 1, 2, \ldots, p$ and there will be a edge between two nodes u_s and u_j if their corresponding faces have a common edge, that is, for each common face between F_s and F_j, there exists a edge in between u_s and u_j of Γ_1 and the MV of the node u_k will be determined by the relation $p_s \circ C_1(u_k) = max\{p_s \circ D^j(t, v) : j = 1(1)p\}$, $s = 1(1)m$, where (t, v) be a boundary edge of the F_k.

The following method provides the MV of edges: $j = 1(1)p$, $p_s \circ D_1^j(t, v) = p_s \circ D^j(w, y)$, $s = 1(1)m$ and $((w, y), D^j(w, y))$ be one of the edges shared by two faces. Be the number of common edges between two faces F_k, F_l, and $p_s \circ ((t, v), D^j(t, v))$ be one of the edges of Γ_1.

If the equivalent GmPFPG contains a pendant edge and the MV of the self loop is the same as the pendent edge, there will be a self-loop in the GmPFDG.

Due to the fact that the GmPFDG is derived from the GmPFPG, the GmPFDG's planarity value will be $(\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m)$, where $\mathcal{Q}_s = 1$, $s = 1(1)m$.

Example 6: To explain G3PFDG concept, we consider a G3PFPG $\Gamma = (\acute{V}, C, D)$ having node set $\acute{V} = \{u_1, u_2, u_3, u_4, u_5\}$ with $C(u_1) = (0.4, 0.5, 0.3)$, $C(u_2) = (0.5, 0.4, 0.5)$, $C(u_3) = (0.6, 0.5, 0.8)$, $C(u_4) = (0.3, 0.7, 0.6)$ and $C(u_5) = (0.5, 0.6, 0.4)$. Now, we have considered the function $\phi : \tilde{A} \to [0, 1]$ such that

$$p_s \circ D^j(t, w) = \phi((p_s \circ C(t), p_s \circ C(w))) \tag{10}$$
$$= \max\{p_s \circ C(t)), p_s \circ C(w)\} \tag{11}$$

for every $s = 1(1)m$, $j = 1(1)p$ and $C(t), C(w)$ indicate the MVs of nodes t and w respectively. Using (11), we have determined all the MVs of edges, which is shown in Fig. 5.

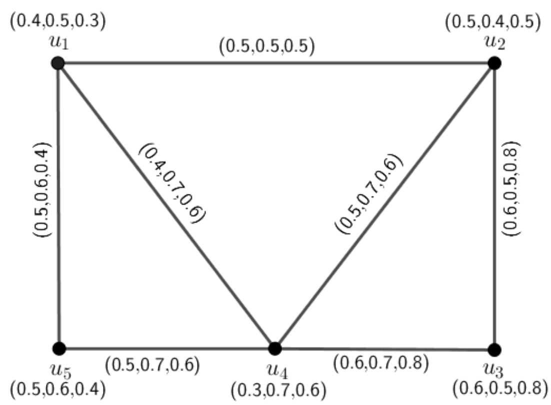

FIGURE 5. A generalized 3PFPG.

We now determine the strength of the edges, and Table 3 has the strength values.

TABLE 3. Strength values of edges of G3PFPG, shown in Fig. 5.

Edges	Strength values
(u_1, u_2)	$(1.25, 1.25, 1.67)$
(u_2, u_4)	$(1.67, 1.75, 1.2)$
(u_4, u_1)	$(1.33, 1.4, 2)$
(u_4, u_5)	$(1.67, 1.17, 1.5)$
(u_5, u_1)	$(1.25, 1.2, 1.33)$
(u_2, u_3)	$(1.2, 1.25, 1.6)$
(u_3, u_4)	$(2, 1.4, 1.33)$

Here, we can see that every edge has a strength value that is more than or equal to 0.5. Therefore, in the given G3PFPG, all edges are SEs. There are four faces, designated F_1, F_2, F_3, F_4, in the given G3PFPG depicted in Fig. 5. The following criteria define the faces' boundaries:

(a) The SEs $(u_1, u_2), (u_2, u_4), (u_4, u_1)$ surround the face F_1.

(b) The SEs $(u_2, u_3), (u_3, u_4), (u_4, u_2)$ surround the face F_2.

(c) The SEs $(u_1, u_4), (u_4, u_5), (u_5, u_1)$ surround face F_3.

(d) The face F_4 on the supplied G3PFPG in Fig. 5 is an outer face.

The faces' strengths are then calculated and are listed in Table 4.

TABLE 4. Strength values of faces of G3PFPG, shown in Fig. 5.

Faces	Strength values
F_1	$(1.25, 1.25, 1.2)$
F_2	$(1.2, 1.25, 1.2)$
F_3	$(1.25, 1.17, 1.33)$
F_4	$(1.2, 1.17, 1.2)$

Here, we can see that every face has a strength value that is more than or equal to 0.5. Thus, every face is a strong face. All of the faces are strong; thus we take the nodes of each strong face into consideration. For $s = 1, 2, 3, 4$, let v_s be the corresponding nodes of the faces F_s. The dual G3PFG's node set is then $\acute{V}_1 = \{v_1, v_2, v_3, v_4\}$, and if the matching faces F_s and F_j share a boundary edge, a edge will exist between v_s and v_j. The border between F_s and F_j's faces has the same number of edges as the distance between v_s and v_j. Fig. 6 presents the equivalent dual graph Γ_1 of the supplied G3PFPG Γ of Fig. 5.

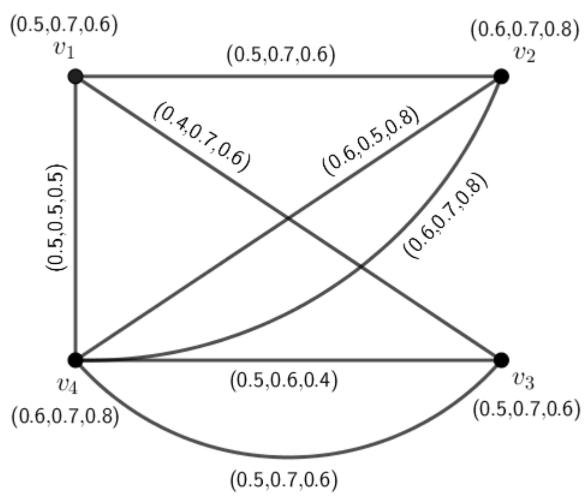

FIGURE 6. Dual G3PFPG of Fig. 5.

Note 4: Assume that Γ is a strong GmPFPG whose number of nodes, edges, and strong faces are v, e and f, respectively. If Γ_1 is the dual graph of Γ in the generalized m-polar fuzzy, then

(a) f is the same as the number of nodes in Γ_1,

(b) Γ_1 has the same number of edges as e,

(c) Γ_1 has the same number of faces as v.

Note 5: The number of strong faces in a GmPFDG of a GmPFPG is fewer than or equal to the number of nodes in the GmPFPG, assuming that the GmPFDG does not contain all of the strong generalized m-polar fuzzy faces.

Theorem 7: Let Γ_1 be a GmPFDG of a GmPFPG $\Gamma = (\acute{V}, C, D)$ having all SEs when $\phi(p_s \circ C(u), p_s \circ C(z)) = \wedge\{p_s \circ C(u), p_s \circ C(z)\}$, for every $s = 1(1)m$ and $u, z \in \acute{V}$. Then, the MV of edges of GmPFPG and GmPFDG are the same.

Proof: Since Γ is a GmPFPG having all SEs then we have $\phi(p_s \circ C(u), p_s \circ C(z)) = min\{p_s \circ C(u), p_s \circ C(z)\}$, for each $s = 1(1)m$ and $u, z \in \acute{V}$ and let $\Gamma_1 = (\acute{V}_1, C_1, D_1)$ be the GmPFDG of Γ. Since Γ is planar, Γ_1 has no points where any two edges intersect. Let Γ's strong faces be $\{F_1, F_2, \ldots, F_k\}$. From the definition of GmPFDG we know that for $j = 1(1)p$, $p_s \circ D_1^j(t, v) = p_s \circ D^j(w, y)$, $s = 1(1)m$ and $((w, y), D^j(w, y))$ be one of the edges shared by two faces. Be the number of common edges between two faces F_k, F_l, and $p_s \circ ((t, v), D^j(t, v))$ be one of the edges of Γ_1. Make (w, y) a boundary edge on the Γ. Therefore, for any $j = 1(1)p$, the dual graphs $((w, y), D^j(w, y))$ and $((t, v), D_1^j(t, v))$ will be identical. Given that there are no weak edges, both Γ and Γ_1 have the same number of edges by Note 4. Therefore, the proof is completed.

Theorem 8: Give Γ_1 and Γ_2 two GmPFGs, where Γ_1 is a planar. If Γ_1 and Γ_2 are isomorphic, then Γ_2 is also a GmPFPG with the same DOP as Γ_1.

Proof: Let $\tilde{h} : \Gamma_1 \rightarrow \Gamma_2$ be a generalized m-polar fuzzy isomorphism between Γ_1 and Γ_2. The nodes and edges MV between Γ_1 and Γ_2 are preserved because $\tilde{h}$ is an isomorphism between two GmPFGs, that is, $p_s \circ C_1(x) = p_s \circ C_2(\tilde{h}(x))$ and $p_s \circ D_1(x, w) = p_s \circ D_2(\tilde{h}(x), \tilde{h}(w))$ for all x and w in V_1 and for every $s = 1(1)m$ and Γ_1 is also planar. As a result, the MVs of the nodes and edges will be the same in both Γ_1 and Γ_2, and the number of points at which two edges cross will likewise be the same. So, Γ_2 is planer. Since Γ_2 has the same number of vertices, edges, and is planar, it is indeed a GmPFG with the same DOP as Γ_1. The DOP is, therefore, the same in Γ_1 and Γ_2. This demonstrates that Γ_2 is a GmPFPG with the same DOP as Γ_1.

Theorem 9: For any $s = 1(1)m$ and $u, z \in \acute{V}$, there is a co-weak isomorphism between the GmPFPG $\Gamma = (\acute{V}, C, D)$ with $\phi(p_s \circ C(u), p_s \circ C(z)) = \wedge\{p_s \circ C(u), p_s \circ C(z)\}$ and the dual of the graph Γ.

Proof: Given Γ be a GmPFPG with $\phi(p_s \circ C(u), p_s \circ C(z)) = min\{p_s \circ C(u), p_s \circ C(z)\}$, for every $s = 1(1)m$ and $u, z \in \acute{V}$. Let Γ_1 and Γ_2 be the duals of Γ and Γ_1, respectively. We need to demonstrate that Γ and Γ_2 are co-weak isomorphic. Since they are mirror images of one another, the number of strong faces in Γ_1 is equal to the number of nodes in Γ. Again, because Γ_2 is a dual of Γ_1, the two are equivalent in terms of the number of nodes and the number of strong faces in Γ_1. Therefore, there are exactly the same number of nodes in Γ and Γ_2. Additionally, the number of edges in a GmPFPG and those in its dual GmPFG are equal, by Note 4. Again, we know from the definition of dual that the MV of edges in the GmPFDG and GmPFPG are equal. A co-weak isomorphism between Γ and Γ_2 can thus be defined. Hence completes the proof.

VII. APPLICATION

In this section, we discuss an application of GmPFG. We are aware that generalized FGs have a variety of applications in decision-making challenges and are utilized to manage uncertainty resulting from issues in our daily lives. However, in this case, we have applied the polarity and planarity concepts to generalized FGs. Here, we take into account a social group to explain the value of G3PFPG. We have used the G3PFPG concept.

People have a natural tendency to form social groups, whether they are friendships, families, or entire communities. This inclination is fundamental to the human experience. A social group refers to any gathering of two or more individuals who recognize their connection as a unique social entity. Social groups play a crucial role in daily life and help individuals navigate their environment.

A. MODEL CONSTRUCTION

Suppose we have taken a social group of 10 persons in Figure 7, which is represented by G3PFPG $\widetilde{\Gamma} = (\acute{V}, C, D)$ corresponding to the attributes cooperation, controlling power and team spirit of each person of the social group. These attributes are taken as the first, second and third components of the graph. Every person is treated as a node, and the connection between any two persons is represented by an edge.

Now, we show that the connectivity of each person to another person in the social group is strong or not, which indicates whether the group is active or not depending on the attributes of the person's cooperation, controlling power, team spirit, creativeness, etc.

To find out whether the social group is strong or not i.e., active or not, we use the DOP of the G3PFPG model, which represents the social group with respect to the attributes of persons cooperation, controlling power and team spirit.

B. ILLUSTRATION OF MEMBERSHIP VALUES

Here, MVs of the node are considered with respect to the attributes of cooperation, controlling power and team spirit of the person. The node MVs of the G3PFPG $\widetilde{\Gamma}$ shown in Figure 7 are considered as in Table 5.

Every edge represents the relation or connection between two persons. The MVs can be derived from the parameter 'relationship' between two persons in a group. The relationship is defined as the average between two node MVs.

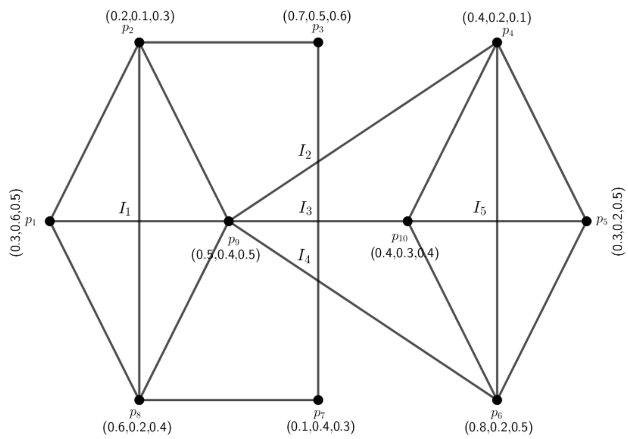

FIGURE 7. Graphical representation of social group in G3PFPG.

TABLE 5. Vertex membership values of $\widetilde{\Gamma}$.

Vertex	Membership values
p_1	$(0.3, 0.6, 0.5)$
p_2	$(0.2, 0.1, 0.3)$
p_3	$(0.7, 0.5, 0.6)$
p_4	$(0.4, 0.2, 0.1)$
p_5	$(0.3, 0.2, 0.5)$
p_6	$(0.8, 0.2, 0.5)$
p_7	$(0.1, 0.4, 0.3)$
p_8	$(0.6, 0.2, 0.4)$
p_9	$(0.5, 0.4, 0.5)$
p_{10}	$(0.4, 0.3, 0.4)$

That is

$$\phi(p_s \circ C(x), p_s \circ C(w)) = \frac{1}{2}\{p_s \circ C(x) + p_s \circ C(w)\},$$

for every $s = 1, 2, 3$. This relation will measure the average MVs of cooperation between two persons, controlling power between two persons and team spirit between two persons. Now we have to show whether the social group is active or inactive, determining the DOP of the model of G3PFPG with respect to the attributes cooperation, controlling power and team spirit of the person. For illustration, the MVs of edges are listed in Table 6.

For determining the DOP of G3PFPG, firstly, we find out the strength of all the edges in $\widetilde{\Gamma}$. Now, the the strength of an edge (a, b) is $I^s_{(a,b)} = p_s \circ I_{(a,b)} = \frac{p_s \circ D(a,b)}{p_s \circ C(a) \wedge p_s \circ C(b)}$, for every $s = 1, 2, 3$ and $D(a, b)$ indicates the MV of the edge (a, b) and $C(a)$, $C(b)$ indicate the MVs of the vertices a and b respectively. So the strength of the edge (p_1, p_2) is $I_{(p_1,p_2)} = (\frac{p_1 \circ D(a,b)}{p_1 \circ C(a) \wedge p_1 \circ C(b)}, \frac{p_2 \circ D(a,b)}{p_2 \circ C(a) \wedge p_2 \circ C(b)}, \frac{p_3 \circ D(a,b)}{p_3 \circ C(a) \wedge p_3 \circ C(b)}) = (\frac{0.25}{0.3 \wedge 0.2}, \frac{0.35}{0.6 \wedge 0.1}, \frac{0.4}{0.5 \wedge 0.3}) = (1.25, 3.5, 1.33)$. Similarly, we have determined the strength of all the edges which are listed in the Table 7.

Clearly, all edges are strong, which we have seen in Table 7. In the G3PFPG model, there are five intersecting points

TABLE 6. Membership values of each edge of $\widetilde{\Gamma}$.

Edge	Membership values
(p_1, p_2)	$(0.25, 0.35, 0.4)$
(p_2, p_3)	$(0.45, 0.3, 0.45)$
(p_1, p_9)	$(0.4, 0.5, 0.5)$
(p_2, p_9)	$(0.35, 0.25, 0.4)$
(p_1, p_8)	$(0.45, 0.4, 0.45)$
(p_2, p_8)	$(0.4, 0.15, 0.35)$
(p_3, p_7)	$(0.4, 0.0.45, 0.45)$
(p_4, p_5)	$(0.35, 0.2, 0.3)$
(p_4, p_6)	$(0.6, 0.2, 0.3)$
(p_4, p_{10})	$(0.4, 0.25, 0.25)$
(p_4, p_9)	$(0.45, 0.3, 0.3)$
(p_5, p_6)	$(0.55, 0.2, 0.5)$
(p_5, p_{10})	$(0.35, 0.25, 0.45)$
(p_6, p_9)	$(0.65, 0.3, 0.5)$
(p_6, p_{10})	$(0.6, 0.25, 0.45)$
(p_7, p_8)	$(0.35, 0.3, 0.35)$
(p_8, p_9)	$(0.55, 0.3, 0.45)$
(p_9, p_{10})	$(0.45, 0.35, 0.45)$

TABLE 7. Strength values of each edge of $\widetilde{\Gamma}$.

Edge	Strength values
(p_1, p_2)	$(1.25, 3.5, 1.33)$
(p_2, p_3)	$(2.25, 3, 1.5)$
(p_1, p_9)	$(1.33, 1.25, 1)$
(p_2, p_9)	$(1.75, 2.5, 1.33)$
(p_1, p_8)	$(1.5, 2, 1.12)$
(p_2, p_8)	$(2, 1.5, 1.16)$
(p_3, p_7)	$(4, 1.12, 1.5)$
(p_4, p_5)	$(1.16, 1, 3)$
(p_4, p_6)	$(1.5, 1, 3)$
(p_4, p_{10})	$(1, 1.25, 2.5)$
(p_4, p_9)	$(1.12, 1.5, 3)$
(p_5, p_6)	$(1.83, 1, 1)$
(p_5, p_{10})	$(1.16, 1.25, 1.12)$
(p_6, p_9)	$(1.3, 1.5, 1)$
(p_6, p_{10})	$(1.5, 1.25, 1.12)$
(p_7, p_8)	$(3.5, 1.5, 1.16)$
(p_8, p_9)	$(1.1, 1.5, 1.12)$
(p_9, p_{10})	$(1.12, 1.16, 1.12)$

denoted as I_1, I_2, I_3, I_4, I_5. At point P, we compute the value $\mathcal{I}_P$, which represents the intersecting value at that location. $\mathcal{I}_P$ is calculated as $\left(\mathcal{I}_P^1, \mathcal{I}_P^2, \mathcal{I}_P^3\right)$, where each component is given by $\mathcal{I}_P^s = \frac{I^s_{(u_1,v_1)}+I^s_{(u_2,v_2)}}{2}$ for $s = 1, 2, 3$. So one intersecting point is I_1 whose value is $\mathcal{I}_{I_1} = (\mathcal{I}^1_{I_1}, \mathcal{I}^2_{I_1}, \mathcal{I}^3_{I_1}) = (\frac{I^1_{(p_2,p_8)}+I^1_{(p_1,p_9)}}{2}, \frac{I^2_{(p_2,p_8)}+I^2_{(p_1,p_9)}}{2}, \frac{I^3_{(p_2,p_8)}+I^3_{(p_1,p_9)}}{2}) = (\frac{2+1.33}{2}, \frac{1.5+1.25}{2}, \frac{1.16+1}{2}) = (1.66, 1.37, 1.08)$. Similarly, the intersecting values of the points I_2, I_3, I_4 and I_5 are respectively given by $(2.56, 1.31, 2.25), (2.56, 1.14, 1.31), (2.65, 1.31, 1.25)$ and $(1.33, 1.12, 2.06)$. Now, the DOP is defined by $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3)$, where $\mathcal{Q}_s = \frac{1}{1+(\mathcal{I}^s_{I_1}+\mathcal{I}^s_{I_2}+\cdots+\mathcal{I}^s_{I_5})}$, $s = 1, 2, 3$. For $s = 1$, $\mathcal{Q}_1 = \frac{1}{1+(1.66+2.56+2.56+2.65+1.33)} = 0.08$. Similarly, for $s = 2$, $\mathcal{Q}_2 = 0.13$ and for $s = 3$, $\mathcal{Q}_3 = 0.11$. Therefore the DOP of $\widetilde{\Gamma}$ is $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3) = (0.08, 0.13, 0.11)$.

C. DECISION MAKING

The DOP of the G3PFPG $\widetilde{\Gamma}$ is $(0.08, 0.13, 0.11)$. We know that if the DOP of the graph is greater than or equal to 0.67, then the planar graph is strong. But in this model, each component of the DOP of G3PFPG $\widetilde{\Gamma}$ is not more than 0.67 by Definition 19. This means that the corresponding planar graph is not strong. This implies that the connectivity of each person to another person in the social group is not strong, which indicates the corresponding social group is not active depending on the attributes of the person's cooperation, controlling power and team spirit. So the bonding of the group is not better. Hence, this social group is an inactive group.

VIII. THEORETICAL IMPLICATIONS, MANAGERIAL INSIGHTS, AND POLICY IMPLICATIONS OF THE STUDY

The theoretical implications of GmPFGs represent a significant advancement in the field of FG theory. The introduction of GmPFGs extends the traditional notions of mPFGs by incorporating multiple levels of uncertainty and polarization. An mPFG is characterized by a single "minimum" relation between vertices and edges component-wise, while a GmPFG can encompass multiple relations between vertices and edges for determining edge MVs. This means that edge MVs can be calculated using various suitable forms or relationships, such as maximum, minimum, average, and more, between the two nodes of that edge to address different corresponding problems. This allows for a more nuanced representation of relationships in complex systems where the degree of membership may vary across different dimensions. Theoretical developments in GmPFGs provide a richer framework for modeling real-world phenomena, fostering a deeper understanding of uncertainty and variability in graph structures.

From a managerial perspective, insights derived from GmPFGs can be invaluable in decision-making processes. Managers often deal with uncertain and dynamic environments, and the ability to model and analyze relationships with multiple dimensions of uncertainty provides a more realistic representation of the complexities they face. GmPFGs offer a versatile tool for managers to assess and adapt their strategies in situations where traditional crisp graph models fall short. This can lead to more informed decision-making, improved risk management, and a better grasp of the intricate relationships within the systems they oversee.

Policy implications arise from the potential application of GmPFGs in diverse domains such as transportation, communication networks, and social systems. Policymakers can leverage the insights provided by these graphs to design more flexible and adaptive policies that account for the inherent uncertainties in real-world systems. The incorporation of GmPFG theory into policy frameworks enables a more nuanced understanding of complex interdependencies, fostering policies that are better aligned with the dynamic nature of modern societies. This theoretical framework can thus contribute to the development of policies that are more resilient and responsive to the challenges posed by uncertainty and variability in various domains.

IX. ADVANTAGES AND LIMITATIONS OF THE PROPOSED WORK

In our real world, the resolution of numerous issues relies on data gathered from diverse sources, illustrating the concept of multi-polarity in data collection. Traditional graph models like FGs or bipolar FGs may not adequately capture such polarity. An mPFG is characterized by a single ''minimum'' relation between vertices and edges component-wise, while a GmPFG can encompass multiple relations between vertices and edges for determining edge MVs. This means that edge MVs can be calculated using various suitable forms or relationships, such as maximum, minimum, average, and more, between the two nodes of that edge to address different corresponding problems. For example, we have taken a social group which is represented by G3PFG corresponding to the attribute awareness, cooperation and good behaviors of each person in the social group. Every person is treated as a node, and the edges are attached by adjacent nodes. Here, the MVs of the node are considered with respect to the attributes of awareness, cooperation and good behavior of the person. Every edge represents the relation between two persons. MVs can be derived from the 'relationship' parameter between two persons in a group. Now, the question is, ''how will a relationship be constructed?'' The relationship is defined as the maximum between two node MVs. This relation will measure the maximum awareness between two persons, cooperation between two persons and good behavior between two persons.

Some additional benefits of the proposed model include:

(i) In this work, individuals can analyze MVs within a multi-polar fuzzy environment in a specific manner.
(ii) A GmPFG can encompass multiple relations between vertices and edges for determining edge MVs. This means that edge MVs can be calculated using various suitable forms or relationships.
(iii) GmPFGs provide a more expressive modeling framework than traditional mPFGs. They can capture a wider range of relationships and associations among elements in the graph.
(vi) In decision-making processes, GmPFGs can provide more informative and accurate results by considering membership degrees. This can lead to more robust and well-informed decisions.
(v) GmPFGs can model complex systems, such as biological networks, social networks, and ecological systems, where the relationships between elements are inherently uncertain and dynamic.

GmPFGs are mathematical structures that extend the concept of traditional mPFGs to incorporate fuzzy sets and fuzzy logic. While they are a useful tool for modeling uncertainty and imprecision in various applications, they also have some limitations. Here are some limitations of GmPFGs:

(i) In this environment, it is not permissible to utilize the negative MVs of the characters.
(ii) Non-heterogeneous types of data are not suitable for use in this context.
(iii) When the MVs of the characters are provided within various interval-valued m-polar fuzzy environments, the application of GmPFG is not feasible.
(vi) GmPFGs can become quite complex, especially when dealing with a large number of nodes and edges.
(v) Acquiring and modeling fuzzy data to construct GmPFGs can be challenging. Fuzzy data may not always be readily available or easy to represent accurately.

Despite these limitations, GmPFGs can be valuable in scenarios where uncertainty and imprecision play a crucial role, such as in decision support systems and expert systems. However, their use should be carefully considered in light of these limitations and the specific characteristics of the problem being addressed.

X. CONCLUSION

In this article, we explore the concept of planarity within a generalized m-polar fuzzy environment and discuss their properties under isomorphism. Several important findings are presented, including a comprehensive examination of the faces of GmPFPGs and various intriguing facts about them. We also introduce a GmPFDG derived from GmPFPGs and establish a relationship between the dual of GmPFGs and GmPFGs, delving into their properties within the context of dual GmPFPGs. Additionally, we present a real-life application that assesses a social group's activity based on attributes such as cooperation, team spirit, awareness, controlling power, good behaviour, and creativity. The future scope of GmPFGs holds great potential for further development and application in various fields. Some areas of future research and application include: GmPFGs can be applied to network analysis, such as social networks, transportation networks, and communication networks, to model and analyze uncertain or imprecise connections and relationships. Additionally, our ongoing research endeavors to extend the concept of GmPFGs into diverse domains, including intuitionistic FGs, bipolar FGs, picture FGs, and fuzzy soft graphs, among others.

DECLARATIONS

Conflict of interest The authors affirm that they do not possess any conflicts of interest.

Ethics approval The authors of this article did not conduct any studies involving human participants or animals.

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DATA AVAILABILITY STATEMENT

No data are used in the study.

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