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RESEARCH ARTICLE

Block Algebra-Based Consistency Checking With Cardinal Direction Relations in 3D Space

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ABSTRACT Consistency checking, as a key and challenging problem in the research field of qualitative spatial reasoning with direction relations in 3D space, has received a lot of attention. It is widely used in 3D spatial configuration and anomaly detection in urban planning. To enrich and enhance the ability of reasoning and predict with 3D cardinal direction relations, a new approach for spatial projection on each axis is proposed on the basis of the 3DR27 model for cardinal direction relations in 3D space presented in our previous work. This paper divides the consistency checking of spatial direction relation networks into two processes. Firstly, the projection method is employed to determine whether a network with three-dimensional cardinal directional relations is a convex relation network. Then, by means of the good calculation properties of interval algebra and the mapping between 3D rectangular cardinal direction and 3-block algebra, an algorithm for consistency checking is proposed, which can be used to determine whether one or more than one solution can be found to satisfy the given network constraints. The results of theoretical analysis and verification show that our method is correct and complete. This method effectively improves the ability of intelligent analysis and processing for complex 3D spatial direction relations.

INDEX TERMS Consistency checking, qualitative spatial reasoning, 3D cardinal direction relations, block algebra, convex relation.

I. INTRODUCTION

As the premise of spatial cognition and reasoning, spatial direction relation describes the order relation between spatial objects, which is widely used in the field of spatial analysis and processing [1], spatial-temporal database [2], [3], [4], computer graphics [5], [6], [7] and natural language processing [8], [9], [10]. With the application of spatial direction relation, we are not satisfied with the simple description and storage of spatial direction relation, and it also requires that the spatial database system has the ability to intelligent prediction, analysis, and qualitative reasoning with directions [11], [12], [13]. The qualitative spatial reasoning problem has recently been formulated as a constraint satisfaction problem and addressed using traditional algorithms, such as path consistency [14]. One of the most important challenges in this field is the identification of useful and tractable classes of spatial constraints, as well as the research on effective consistency checking algorithms, minimum network computation, and so on. Several kinds of useful spatial constraints have been studied so far, e.g., Topological constraints remain connectivity and separateness constant when the spatial entities are under the condition of topological relation transformation such as rotating and scaling [15], [16]. Distance constraints describe the distance size relations between two spatial object targets [9], [17]. The cardinal direction constraints describe how regions of 3D space are placed relative to one another utilizing a coordinate system (e.g., region a is northeast of region b) [18], [19], [20], [21].

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In this paper, we focus on the problem of consistency checking with cardinal direction constraints in 3D space [22], [23], [24]. Currently, The models of Hao et al. [25] and Li et al. [26] is the most expressive model for subdivision within 3D space. These models are able to divide the spatial objects themselves so that they can have more precise expression. However, they have problems such as high computational complexity and limited reasoning capabilities. In this paper, we introduce the 3DR27 model for cardinal direction relations in 3D space presented in our recent work, which solves these problems by providing a computationally efficient and formalized reasoning approach [27].

What's more, interval algebra was proposed by Allen [28] and further extended to n-dimensional space by Balbiani [29]. It has been widely utilized in the field of spatiotemporal reasoning. Currently, interval algebra is predominantly used in the research of two-dimensional spatial relations and has not been extensively explored in three-dimensional space [22], [30], [31]. Due to the good computational properties of the interval algebra, so we combine it as a constraint satisfaction problem (CSPs) and establish the mapping between three-dimensional block algebra and three-dimensional rectangular cardinal relations. This enables the consistency checking of 3D basic rectangular relations.

We will study the problem of checking the consistency of a given set of cardinal direction constraints in the 3DR27 model. Checking the consistency of a set of constraints in a mode of spatial information is a fundamental problem and has received a lot of attention in the literature [3], [7], [14], [32], [33], [34]. Algorithms for consistency checking are of immediate use in various situations as following.

- Propagating relations and detecting inconsistencies in a given set of 3D spatial relations [3], [7].
- Preprocessing spatial queries so that inconsistent queries are detected or the search space is pruned [14], [34].
- Study of 3D spatial configuration with consistency checking [32], [33].

The major contributions of this paper can be summarized as follows.

- The research gives several important definitions to introduce the 3DR27 model. The model is currently one of the most expressive models for qualitative reasoning with 3D cardinal relations.
- Then, introducing Allen's interval algebra theory and extending the interval algebra to three dimensions.
- By the theoretical framework, the paper builds the mapping between three-dimensional block algebra and three-dimensional rectangular cardinal relations. Then, we study the problem of checking the consistency of a given set of cardinal direction constraints and propose the algorithm for the problem.
- The algorithm includes two processes. Determine whether it is a convex relation and conduct the path consistency checking based on the convex relation. Then, the paper gives examples to prove the algorithm's correctness and improve the powers of reasoning and analysis for the complex 3D direction relations, and then can better meet the complicated applications of 3D spatial data.

This article is structured as follows. Section II introduces the related work about the consistency checking of cardinal direction relations. Section III presents the 3D cardinal direction relation model, the interval algebra, and builds the connection between them with a unified formula. Section IV, we propose an algorithm for convex relational network. Section V and Section VI discusses the implementation of the proposed consistency checking algorithms and presents the results of analysis and verification. Finally, Section VII concludes the paper.

II. RELATED WORKS

Consistency checking is an important component of qualitative spatial direction relation inference [35], [36], [37], [38]. It ensures the logical coherence and integrity of the inferred relations. When reasoning qualitatively about spatial directions, it is crucial to maintain consistency to avoid contradictions and ensure reliable results [14]. Consistency checking involves verifying that the inferred qualitative spatial relations adhere to a set of predefined rules or constraints. These rules define the logical consistency between different directional relations and help maintain the coherence of the overall spatial representation. By performing consistency checks, we can identify and rectify any inconsistencies or contradictions that may arise during the inference process [3], [7].

The consistency checking in the two-dimensional space has been relatively mature. Initially, Ligozat [18] proposed the binary constraint network (BCN) algorithm for consistency checking of basic spatial directional relations between points, based on the block algebra theory, and it was proven to be NP-complete. Subsequently, The genetic algorithm proposed by Liu et al. [19] has been widely used for consistency checking of basic directional relations based on the projection model. He represented the basic directional relations in two-dimensional space as points and applied them to qualitative reasoning such as inversion, composition, and intersection operations. However, Considering the practical applications, spatial objects cannot always be treated as point objects. For example, Spain is approximately northeast of Portugal. But most people would agree that "northeast" does not accurately describe the relation between Spain and Portugal. (see Fig. 1).

To reduce the influence of the size and shape of space objects and to improve the accuracy of expression and reasoning with direction relations. Spiros Skiadopoulos et al. [20] proposed an ad-hoc algorithm (which will be called SK-CON) for consistency checking of a network of basic cardinal constraints with variables ranging over REG* of disconnected regions. The algorithm has a relatively high complexity of $O(n^5)$, and it is not guaranteed to work when

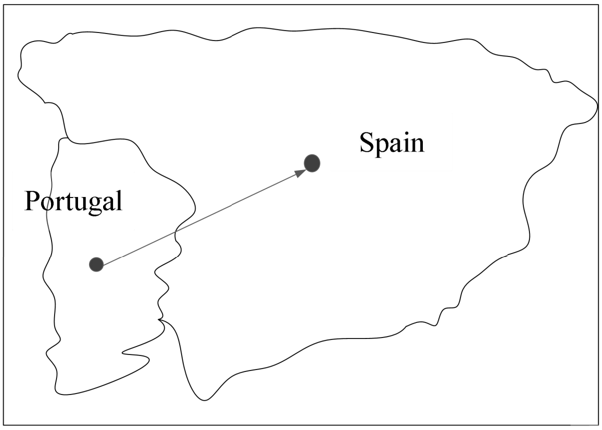

FIGURE 1. Problem with point approximations.

the domain is restricted to the set *REG* of connected regions. Thereby, Navarrete et al. [21] devised a new REG-BCON algorithm between connected regions based on the algorithm of Skiadopoulos, which has the complexity of $O(n^4)$. As a matter of fact, the consistency checking of a set of unrestricted cardinal directions is a NP hard problem [39].

The algorithm keeps the first 2 steps of the algorithm SK-CON, and adds to the Helly's topoliogical theorem [40], which gives the key to decide if a solution can be obtained for a set of basic cardinal constraints.

The aforementioned work focuses solely on the consistency checking of basic directional relations between points or regions in 2D space. The algorithms involved have relatively high computational complexity and mostly rely on path consistency under the assumption of a convex relation network, which has certain limitations. Considering that Euclidean space is three-dimensional in our world. Chen et al. [22] proposed a TCD model for cardinal direction relation in three dimensional space. Based on the smallest cubic TCD relations and original relations, explain the correlations between basic TCD relations and block algebra. And an $O(n^4)$ algorithm to check the consistency of a set of basic TCD constraints over simple blocks is given. But through the analysis and comparison found that the algorithm although solved with a non-constructive method, the time complexity of this algorithm is relatively high. Liu et al. [23] devised an algorithm for consistency checking with 3D rectangular cardinal direction relation network by extending the 2D convex relation network to 3D space, which laid the foundation for further research on finding consistency consistent scenes in the network of 3D cardinal direction relations.

In recent years, some researchers have also explored the application of graph theory to consistency checking. Liu et al. [24] researched on the consistency checking with the network of cardinal direction relations between points and regions by means of visual detection of the spatial graph in vertical which is obtained by merging and condensing the partition graph of the spatial constraints according to certain rules. But this method determines the consistency by checking whether there is a ring in the spatial graph in vertical. For this approach, the spatial object considered is a closed region, it can not deal with the object that has holes or that is disconnected. Kong [41] present a graph model to visually represent direction specifications, and perform a consistency checking on the graph model rather than through a constraint solver used by Skiadopoulos and Koubarakis [17]. As a matter of fact, in the field of graph-based consistency checking, the consideration of spatial objects is not comprehensive enough, and the study of three-dimensional space is largely blank, with a focus mainly on two-dimensional spatial relations.

To sum up, the researches on the consistency checking with the network of direction relations in 3D space are rare at present. Most of the existing methods make use of the composition operation to implement consistency checking on the basis of the determination of convex relation. And the consistency checking problem has been posed as a constraint satisfaction problem and solved using traditional algorithms like path consistency. But this kind of method has many problems, such as complicated calculation and low efficiency which to a certain extent hinders the deep application of 3Dspatial direction relations. In this paper, we will focus on the work by incorporating interval algebra theory and establishing a one-to-one mapping between three-dimensional rectangular directional relations and block algebra [342] based on the previous work [23]. The aim is to generalize the good computational properties of interval algebra theory to 3D spaces. It facilitates analysis for consistency checking of directional relations in three-dimensional space. Then, The study examines the convexity of relations along each axis and proposes an algorithm to search for path consistency under the assumption of a convex relation network which can be can be performed in $O(n^3)$. Theoretical analysis and experiments prove that the algorithm is correct and complete, and these detailed processes are described in later sections.

III. THREE-DIMENSIONAL CARDINAL DIRECTION RELATION AND INTERVAL ALGEBRA

In this section, we provided a detailed overview of the methodology used in the article. Firstly, we introduced the 3DR27 model for three-dimensional spatial direction relations, building upon our previous work. This model was employed due to its computational feasibility and excellent formal reasoning capabilities. Moreover, we introduced interval algebra and extended it to three dimensions space, establishing a corresponding connection with three-dimensional rectangular directional relations. This laid a solid foundation for the subsequent proposal of a consistency checking algorithm. The overall research technical route of our article is depicted in Fig. 2.

A. THE MODEL OF THREE-DIMENSIONAL CARDINAL DIRECTION RELATION

In recent years, many models for representing and reasoning with direction relations in 3D space have been proposed

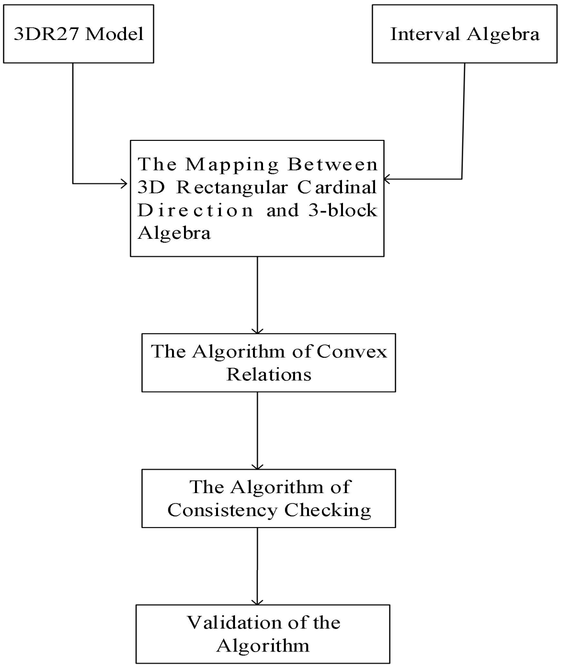

FIGURE 2. The overall technical route.

in [25], [26], [43], and [44]. But these models are complicated in calculation and poor in reasoning, which cannot meet the demand of real applications. Our study starts with 3DR27 model which is easy to calculate and carry out formal reasoning [27]. Let us show this model formally through the following definitions. Objects concerned in this study are homogeneous to the closed box ($\{(x,y,z): x^2 + y^2 + z^2 \leq 1\}$). The set of these objects will be denoted by T_{box}. Let $a \in T_{box}$. The greatest lower bound of the projection of a on the x-axis (respectively y-axis, z-axis) is denoted by $inf_x(a)$ (respectively $inf_y(a)$, $inf_z(a)$). The least upper bound or the supremum of the projection of a on the x-axis (respectively y-axis, z-axis) is denoted by $sup_x(a)$ (respectively $sup_y(a)$ $sup_z(a)$). The minimum bounding box of a object a, denoted by $mbb(a)$, is the cube formed by the straight lines $x = inf_x(a)$, $x = sup_x(a)$, $y = inf_y(a)$, $y = sup_y(a)$ $z = inf_z(a)$ and $z = sup_z(a)$.

Definition 1: Let $a \in T_{box}$ and a be reference object. The straight lines $x = inf_x(a)$, $x = sup_x(a)$, $y = inf_y(a)$, $y = sup_y(a)$ $z=inf_z(a)$ and $z = sup_z(a)$ forming $mbb(a)$ divide the space into 27 areas (see Fig. 3) which we call tiles of a. These tiles will be denoted by $UNW(a)$, $UN(a)$, $UNE(a)$, $UW(a)$, $UB(a)$, $UE(a)$, $USW(a)$, $US(a)$, $USE(a)$, $RNW(a)$, $RN(a)$, $RNE(a)$, $RW(a)$, $RB(a)$, $RE(a)$, $RSW(a)$, $RS(a)$, $RSE(a)$, $DNW(a)$, $DN(a)$, $DNE(a)$, $DW(a)$, $DB(a)$, $DE(a)$, $DSW(a)$, $DS(a)$ and $DSE(a)$, respectively.

Let us now consider two arbitrary objects a and b in T_{box}. Let object b be related to object a through a cardinal direction relation. Object b will be called the primary object while object a will be called the reference object. If b is included in tile $UNW(a)$ of a then we say that b is up-northwest of a and we write b UNW a. Similarly, we can define up-north(UN), up-south(US), up-northeast(UNE), up-west(UW),

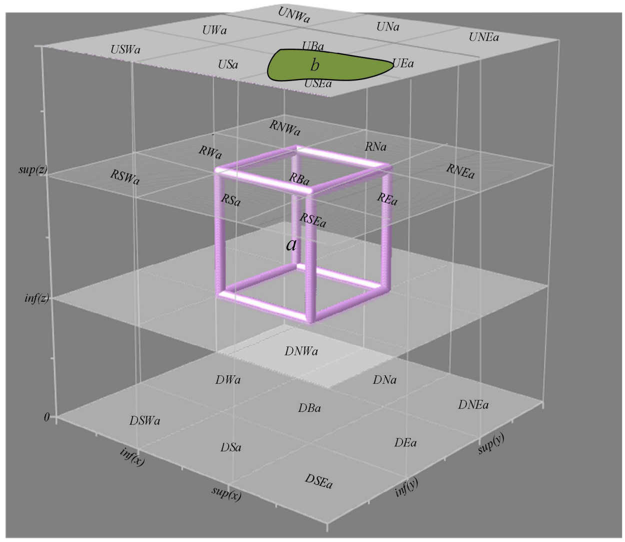

FIGURE 3. Reference object direction partition.

up-bounding box(UB), up-southwest(USW), up-east(UE), up-southeast (USE), north (RN), south (RS),northeast (RNE), west (RW), bounding box (RB), southwest (RSW), east (RE), southeast (RSE), northwest (RNW), down-north(DN), down-south(DS), down-northeast(DNE), down-west(DW), down-bounding box (DB), down-southwest (DSW), down-east (DE), down-southeast (DSE) and down- northwest (DNW) relations.

Definition 2: A basic 3D cardinal direction relation is an expression $R_1 : \ldots : R_k$ where $R_1, \ldots, R_k \in \{UNW, UN, UNE, UW, UB, UE, USW, US, USE, RNW, RN, RNE, RW, RB, RE, RSW, RS, RSE, DNW, DN, DNE, DW, DB, DE, DSW, DS, DSE\}$, $1 \leq k \leq 27$, and $R_i \neq R_j$ for every i, j such that $1 \leq i, j \leq k$ and $i \neq j$, and there exist objects $a_1, \ldots, a_k \in T_{box}$ such that $a_1 \in R_1(b), \ldots, a_k \in R_k(b)$ and $a_1 \cup \cdots \cup a_k \in T_{box}$. for any reference object $b \in T_{box}$. The set of basic 3D cardinal direction relations in this model is denoted by TD.

For example, in Fig. 3, b lies partly in the tile UB (a), partly in the tile US (a), partly in the tile USE (a), and partly in the tile UE (a), then b UB:US:USE:UE a and we say that b is partly up-bounding, partly up-south, partly up-southeast, and partly up-east of a.

Definition 3: Let $R \in TD$, R is a rectangle iff there exist two cubes a and b (the sides are both parallel to the x,y,z axes) such that a R b, otherwise R is non-rectangular. There are 216 sets of 3D rectangular cardinal direction relations is denoted by TD_{rec}.

B. INTERVAL ALGEBRA

In this study, we will employ block algebra to study the problem of consistency checking with cardinal direction relations defined by 3DR27 model. Let us shortly introduce the block algebra. Allen [28] firstly proposed the interval algebra which defined 13 basic interval relations as:

$$B_{int} = \{p, m, o, s, d, f, pi, mi, oi, si, di, fi, eq\} \quad (1)$$

TABLE 1. Composing table of six interval relations.

$R_1 \circ R_2$	{p,m}	{fi,o}	{di}	{eq,s,d,f}	{si,oi}	{pi,mi}
{p,m}	{p}	{p}	{p}	{p,m,o,d,s}	{p,m,o,d,s}	all
{fi,o}	{p,m}	{p,m,o,fi}	{p,m,o,fi,di}	{o,fi,eq,s,d,f}	{o,fi,di,eq,s,d,f,si ,oi}	{di,si,oi,pi,mi}
{di}	{p,m,o,fi,di}	{o,fi,di}	{di}	{o,fi,di,eq,s,d,f, si,oi}	{di,oi,si}	{di,si,oi,pi,mi}
{eq,s,d,f}	{p,m}	{p,m,o,fi,eq,s,d,f}	all	{eq,s,d,f}	{eq,s,d,f,oi,si,pi, mi}	{pi,mi}
{si,oi}	{p,m,o,fi,di}	{o,fi,di,eq,s,d,f,si,oi}	{di,oi,si,pi,mi}	{eq,s,d,f,oi,si}	{oi,si,pi,mi}	{pi,mi}
{pi,mi}	all	{d,f,oi,pi,mi}	{pi}	{d,f,oi,pi,mi}	{pi}	{pi}

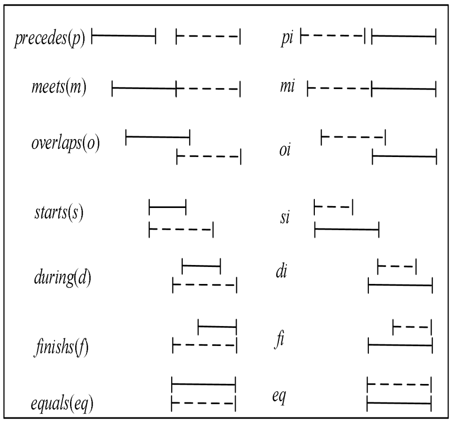

FIGURE 4. 13 basic interval algebra relations.

to describe the relations between two finite intervals as shown in Fig. 4. Allen's interval algebra facilitates formal reasoning which plays an important role in temporal relation reasoning and spatial calculus.

Definition 4: Let $R \in 2^{Bint}$, the inverse relation of *R*, denoted by R^{-1}, is another interval algebra relation which satisfies the following. For arbitrary intervals *a* and *b*, if *a* R^{-1} b holds, iff *b R a* also holds.

Definition 5: Let R_1,R_2 $\in 2^{Bint}$, the composition of R_1 and R_2, denoted by $R_1 \circ R_2$, is another interval algebra relation from 2^{Bint} which satisfies the following. $R_1 \circ R_2$ contains all relations $Q \in 2^{Bint}$ such that there exists interval *a, b, c* such that *a* R_1 *b*, *b* R_2 *c* and *a Q c* holds.

Thirteen relations in interval algebra can be divided into 6 groups of basic relations $k_1 = \{d, s, f, e\}$, $k_2 = \{m, b\}$, $k_3 = \{mi, bi\}$, $k_4 = \{fi, o\}$, $k_5 = \{si, oi\}$ and $k_6 = \{di\}$, which is defined formally shown as Fig. 5. by means of the endpoints of primary object *b* and reference object *a*.

According to the definition of interval algebra and Definition 5, an inference table for composing the six interval algebra relations $k_1 = \{d, s, f, e\}$, $k_2 = \{m, b\}$, $k_3 = \{mi, bi\}$, $k_4 = \{fi, o\}$, $k_5 = \{si, oi\}$ and $k_6 = \{di\}$ is presented as Table 1.

number	combined interval relation	expression of points
1	{p,m}	sup(a)⩽inf(b)
2	{o,fi}	inf(a)<inf(b),inf(b)<sup(a)⩽sup(b)
3	{di}	inf(a)<inf(b),sup(a)>sup(b)
4	{eq,s,d,f}	inf(a)⩾inf(b),sup(a)⩽sup(b)
5	{si,oi}	inf(b) ⩽inf(a) ⩽inf(b),sup(a) >sup(b)
6	{pi,mi}	inf(a)⩽sup(b)

FIGURE 5. 6 interval algebra relations.

C. ALGEBRA AND 3D RECTANGULAR CARDINAL DIRECTION RELATIONS

Balbiani [29] introduced *n*-dimensional block algebra which is the *n*-dimensional extension of the interval algebra. For every integer $n \geq 1$, a basic *n*-dimensional block algebra relation is an *n*-tuple like $(p_1, p_2, \ldots\ldots, p_n)$ where $p_i \in \{p, m, o, s, d, f, \text{pi}, mi, oi, si, di, fi, eq\}$, for every $i \in \{1,2,\ldots, n\}$, and for arbitrary *n* –blocks *a* and *b*, *a* and *b* satisfy the algebra relation $(p_1, p_2, \ldots\ldots, p_n)$ if and only if the two intervals a_i and b_i satisfy the interval algebra relation p_i where a_i and b_i are the projection of *a* and *b* onto the *i*-th axis respectively, for every $i \in \{1,2,\ldots, n\}$.

In this paper, we only focus on three dimensional block algebra. The three dimensional block algebra will be used to study the problem of consistency checking with the network of the rectangular cardinal direction relations defined by 3DR27 model. Therefore, it is necessary to establish the equivalent relationship between three dimensional block algebra relations and the rectangular cardinal direction relations defined by 3DR27 model. As a matter of fact, there exist a mapping between basic 3D rectangular cardinal direction relations and 3-block algebra relations which presented in our recent work [42] is shown as Fig. 6.

The Cartesian set of BT_{int} is used in Fig. 5. to represent a 3D algebra relation, and the operator $\oplus$ as defined in equation (2) is used to express a basic 3D rectangular cardinal direction.

$$m_1 : \ldots : m_j \oplus n_1 : \ldots : n_k = m_1 n_1 : \ldots m_1\ n_k : m_j n_1 : \ldots : m_j n_k \qquad (2)$$

where m_1:...: $m_j \in \{B,S,SE,E,NE,N,NW,W,SW\}$, n_1:...: $n_k \in \{U,R,D\}$, $1 \leq j \leq 9$, $1 \leq k \leq 3$. For instance, *NW*: $N \oplus U$:*R*=*UNW*:*UN*:*RNW*:*RN*.

Theorem 1: Let $R \in TD_{rec}$, x_i, y_i, $z_i \in BT_{int}$. If $f(R) = xi \times yi \times zi$ holds, then we say that the mapping between 3D rectangular cardinal direction and 3-block algebra. We can use the Cartesian product of block algebra to represent the three dimensional rectangular direction relation.

Proof: According to the Fig. 6, we know that the 3D rectangular cardinal direction is transformed into an equivalent three dimensional block algebra network. $R \in TD_{rec}$, the set of three-dimensional block algebra relation is represented by the Cartesian product on each axis as shown in Fig. 6, the three-dimensional block algebra is projected to the x, y and z axes, where x_i, y_i, $z_i \in BT_{int}$, then $f(R) = x_i \times y_i \times z_i$ indicates that the three-dimensional block algebra relation and thus the theorem holds.

For instance, we know that the relation *UN*:*UNE* are 3D rectangular direction relation. Next, we project it onto each axis and represent it using interval algebra. This establishes a one-to-one correspondence between three-dimensional direction relation and interval algebra. As shown in Fig. 6, we have:

$$f(UN : UNE) = K_5 \times K_3 \times K_3. \tag{3}$$

Theorem 2: Let $R \in TD_{rec}$, x_i, y_i, $z_i \in BT_{int}$. If $f(xi \times yi \times zi) = R$ holds, then we say that the mapping between 3-block algebra and 3D rectangular cardinal direction.

Proof: According to the Fig. 6, we know that the three dimensional block algebra network is transformed into an equivalent 3D rectangular cardinal direction. The three-dimensional block algebra is projected to the x, y, and z axes, where x_i, y_i, $z_i \in BT_{int}$, $R \in TD_{rec}$, and the set of three-dimensional block algebra relation is represented by the $f(xi \times yi \times zi)$, as shown in Fig. 6, then $f(xi \times yi \times zi) = R$ indicates that the three-dimensional rectangular cardinal direction relation and thus the theorem holds.

Given that the interval algebra projection on each axis results in 6 algebra relations, there are a total of 216 possible relations when projected onto all three axes, as shown in Fig. 6. For example, if the interval algebra relation on the x-axis is K_2, the relation on the y-axis is K_5, and the relation on the z-axis is K_2, we can determine that the 3D cardinal rectangular direction relation is *DW*:*DNW*. We have:

$$f(K_2 \times K_5 \times K_2) = DW : DNW \tag{4}$$

IV. THE ALGORITHM OF CONVEX RELATIONS

The determination of convex relations is the basis and prerequisite of the consistency checking with direction relations. Although different composition algorithms are used, the final decision principle is mostly based on the theory of path consistency [45], [46]. Therefore, the determination of convex relations is a key problem in the consistency checking with direction relations.

$\oplus$	*U*	*R*	*D*	*U:R*	*R:D*	*U:R:D*
B	$k_1\times k_1\times k_3$	$k_1\times k_1\times k_1$	$k_1\times k_1\times k_2$	$k_1\times k_1\times k_5$	$k_1\times k_1\times k_4$	$k_1\times k_1\times k_6$
S	$k_1\times k_2\times k_3$	$k_1\times k_2\times k_1$	$k_1\times k_2\times k_2$	$k_1\times k_2\times k_5$	$k_1\times k_2\times k_4$	$k_1\times k_2\times k_6$
N	$k_1\times k_3\times k_3$	$k_1\times k_3\times k_1$	$k_1\times k_3\times k_2$	$k_1\times k_3\times k_5$	$k_1\times k_3\times k_4$	$k_1\times k_3\times k_6$
E	$k_3\times k_1\times k_3$	$k_3\times k_1\times k_1$	$k_3\times k_1\times k_2$	$k_3\times k_1\times k_5$	$k_3\times k_1\times k_4$	$k_3\times k_1\times k_6$
W	$k_2\times k_1\times k_3$	$k_2\times k_1\times k_1$	$k_2\times k_1\times k_2$	$k_2\times k_1\times k_5$	$k_2\times k_1\times k_4$	$k_2\times k_1\times k_6$
NE	$k_3\times k_3\times k_3$	$k_3\times k_3\times k_1$	$k_3\times k_3\times k_2$	$k_3\times k_3\times k_5$	$k_3\times k_3\times k_4$	$k_3\times k_3\times k_6$
NW	$k_2\times k_3\times k_3$	$k_2\times k_3\times k_1$	$k_2\times k_3\times k_2$	$k_2\times k_3\times k_5$	$k_2\times k_3\times k_4$	$k_2\times k_3\times k_6$
SE	$k_3\times k_2\times k_3$	$k_3\times k_2\times k_1$	$k_3\times k_2\times k_2$	$k_3\times k_2\times k_5$	$k_3\times k_2\times k_4$	$k_3\times k_2\times k_6$
SW	$k_2\times k_2\times k_3$	$k_2\times k_2\times k_1$	$k_2\times k_2\times k_2$	$k_2\times k_2\times k_5$	$k_2\times k_2\times k_4$	$k_2\times k_2\times k_6$
S:SW	$k_4\times k_2\times k_3$	$k_4\times k_2\times k_1$	$k_4\times k_2\times k_2$	$k_4\times k_2\times k_5$	$k_4\times k_2\times k_4$	$k_4\times k_2\times k_6$
S:SE	$k_5\times k_2\times k_3$	$k_5\times k_2\times k_1$	$k_5\times k_2\times k_2$	$k_5\times k_2\times k_5$	$k_5\times k_2\times k_4$	$k_5\times k_2\times k_6$
N:NW	$k_4\times k_3\times k_3$	$k_4\times k_3\times k_1$	$k_4\times k_3\times k_2$	$k_4\times k_3\times k_5$	$k_4\times k_3\times k_4$	$k_4\times k_3\times k_6$
N:NE	$k_5\times k_3\times k_3$	$k_5\times k_3\times k_1$	$k_5\times k_3\times k_2$	$k_5\times k_3\times k_5$	$k_5\times k_3\times k_4$	$k_5\times k_3\times k_6$
B:W	$k_4\times k_1\times k_3$	$k_4\times k_1\times k_1$	$k_4\times k_1\times k_2$	$k_4\times k_1\times k_5$	$k_4\times k_1\times k_4$	$k_4\times k_1\times k_6$
B:E	$k_5\times k_1\times k_3$	$k_5\times k_1\times k_1$	$k_5\times k_1\times k_2$	$k_5\times k_1\times k_5$	$k_5\times k_1\times k_4$	$k_5\times k_1\times k_6$
B:S	$k_1\times k_4\times k_3$	$k_1\times k_4\times k_1$	$k_1\times k_4\times k_2$	$k_1\times k_4\times k_5$	$k_1\times k_4\times k_4$	$k_1\times k_4\times k_6$
B:N	$k_1\times k_5\times k_3$	$k_1\times k_5\times k_1$	$k_1\times k_5\times k_2$	$k_1\times k_5\times k_5$	$k_1\times k_5\times k_4$	$k_1\times k_5\times k_6$
W:SW	$k_2\times k_4\times k_3$	$k_2\times k_4\times k_1$	$k_2\times k_4\times k_2$	$k_2\times k_4\times k_5$	$k_2\times k_4\times k_4$	$k_2\times k_4\times k_6$
W:NW	$k_2\times k_5\times k_3$	$k_2\times k_5\times k_1$	$k_2\times k_5\times k_2$	$k_2\times k_5\times k_5$	$k_2\times k_5\times k_4$	$k_2\times k_5\times k_6$
E:SE	$k_3\times k_4\times k_3$	$k_3\times k_4\times k_1$	$k_3\times k_4\times k_2$	$k_3\times k_4\times k_5$	$k_3\times k_4\times k_4$	$k_3\times k_4\times k_6$
E:NE	$k_3\times k_5\times k_3$	$k_3\times k_5\times k_1$	$k_3\times k_5\times k_2$	$k_3\times k_5\times k_5$	$k_3\times k_5\times k_4$	$k_3\times k_5\times k_6$
S:SW:SE	$k_6\times k_2\times k_3$	$k_6\times k_2\times k_1$	$k_6\times k_2\times k_2$	$k_6\times k_2\times k_5$	$k_6\times k_2\times k_4$	$k_6\times k_2\times k_6$
N:NW:NE	$k_6\times k_3\times k_3$	$k_6\times k_3\times k_1$	$k_6\times k_3\times k_2$	$k_6\times k_3\times k_5$	$k_6\times k_3\times k_4$	$k_6\times k_3\times k_6$
B:W:E	$k_6\times k_1\times k_3$	$k_6\times k_1\times k_1$	$k_6\times k_1\times k_2$	$k_6\times k_1\times k_5$	$k_6\times k_1\times k_4$	$k_6\times k_1\times k_6$
B:N:S	$k_1\times k_6\times k_3$	$k_1\times k_6\times k_1$	$k_1\times k_6\times k_2$	$k_1\times k_6\times k_5$	$k_1\times k_6\times k_4$	$k_1\times k_6\times k_6$
W:NW:SW	$k_2\times k_6\times k_3$	$k_2\times k_6\times k_1$	$k_2\times k_6\times k_2$	$k_2\times k_6\times k_5$	$k_2\times k_6\times k_4$	$k_2\times k_6\times k_6$
E:NE:SE	$k_3\times k_6\times k_3$	$k_3\times k_6\times k_1$	$k_3\times k_6\times k_2$	$k_3\times k_6\times k_5$	$k_3\times k_6\times k_4$	$k_3\times k_6\times k_6$
B:S:SW:W	$k_4\times k_4\times k_3$	$k_4\times k_4\times k_1$	$k_4\times k_4\times k_2$	$k_4\times k_4\times k_5$	$k_4\times k_4\times k_4$	$k_4\times k_4\times k_6$
B:W:NW:N	$k_4\times k_5\times k_3$	$k_4\times k_5\times k_1$	$k_4\times k_5\times k_2$	$k_4\times k_5\times k_5$	$k_4\times k_5\times k_4$	$k_4\times k_5\times k_6$
B:S:E:SE	$k_5\times k_4\times k_3$	$k_5\times k_4\times k_1$	$k_5\times k_4\times k_2$	$k_5\times k_4\times k_5$	$k_5\times k_4\times k_4$	$k_5\times k_4\times k_6$
B:N:NE:E	$k_5\times k_5\times k_3$	$k_5\times k_5\times k_1$	$k_5\times k_5\times k_2$	$k_5\times k_5\times k_5$	$k_5\times k_5\times k_4$	$k_5\times k_5\times k_6$
B:S:SW:W:NW:N	$k_4\times k_6\times k_3$	$k_4\times k_6\times k_1$	$k_4\times k_6\times k_2$	$k_4\times k_6\times k_5$	$k_4\times k_6\times k_4$	$k_4\times k_6\times k_6$
B:S:SE:E:NE:N	$k_5\times k_6\times k_3$	$k_5\times k_6\times k_1$	$k_5\times k_6\times k_2$	$k_5\times k_6\times k_5$	$k_5\times k_6\times k_4$	$k_5\times k_6\times k_6$
B:S:SW:W:E:SE	$k_6\times k_4\times k_3$	$k_6\times k_4\times k_1$	$k_6\times k_4\times k_2$	$k_6\times k_4\times k_5$	$k_6\times k_4\times k_4$	$k_6\times k_4\times k_6$
B:W:NW:N:NE:E	$k_6\times k_5\times k_3$	$k_6\times k_5\times k_1$	$k_6\times k_5\times k_2$	$k_6\times k_5\times k_5$	$k_6\times k_5\times k_4$	$k_6\times k_5\times k_6$
B:S:SW:W:NW:N:NE:E:SE	$k_6\times k_6\times k_3$	$k_6\times k_6\times k_1$	$k_6\times k_6\times k_2$	$k_6\times k_6\times k_5$	$k_6\times k_6\times k_4$	$k_6\times k_6\times k_6$

FIGURE 6. The mapping between 3D rectangular cardinal direction and 3-block algebra.

From Fig. 6, we can see that each rectangular cardinal direction relation defined by 3DR27 model corresponds to a 3-block Algebra relation which can be expressed as the Cartesian product of the following six interval algebra relations $P_c = \{k_1 = \{d, s, f, e\}, k_2 = \{m, b\}, k_3 = \{mi, bi\}, k_4 = \{fi, o\}, k_5 = \{si, oi\}, k_6 = \{di\}\}$. It is obvious that the six interval algebra relations are all convex relations. A random combination of these six relations labeled {1,2,3,4,5,6} yields the following convex relations network of interval algebra {1}, {1,2,3},{1,2},{1,2,3,4,5},{1,2,4}, {1,2,3,4,5,6},{2},{2,3},{2,4} {2,3,4}, {2,3,4,5}, {2,3,4,5,6}, {3},{3,5},{3,5,6},{4},{4,5},{4,5,6},{5},{5,6},{6}. All the convex rectangular algebra relations can be represented by Fig. 7. Similarly, in three dimension space, we can get all the convex three dimensional block algebra relations by using the lattice presented in Fig. 8.

Definition 6: An interval relation p and q of the form $p \leq q$, $[p, q]$ satisfying the set of elements between all intervals from the closed interval p to the closed interval q is a convex interval relation [46].

For example, $\{p, m, o, fi, di\}$ is a convex relation due to that it corresponds to interval $[p, di]$, but $\{o,d,s,f\}$ is not a convex relation because the interval $[o, f]$ is missing $\{fi, eq\}$ between them.

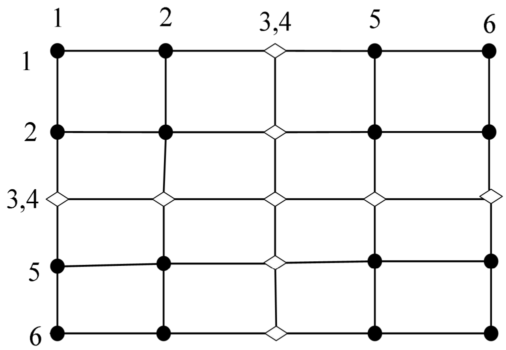

FIGURE 7. 2D convex relation network.

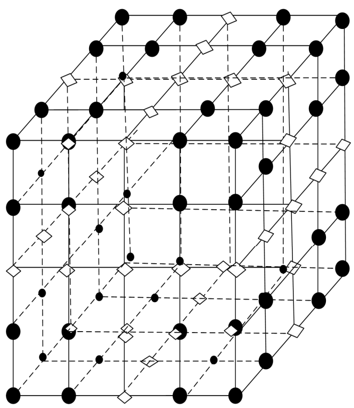

FIGURE 8. 3D convex relation network.

Definition 7: Let $(A,B,C)\in TD_{cube}$, $(D,E,F)\in TD_{cube}$. $(A,B,C) \leq (D,E,F)$ forms a convex three dimensional block algebra relation network iff $A \leq D$, $B\leq E$, $C \leq F$. $(TD_{cube}, \leq)$ defines an ordered grid of coordinate intervals.

Theorem 3: A convex region R of three dimensional block algebra relations is a Cartesian product of its projections, and the projection of its decomposition onto each axis is a convex interval relation [47].

Theorem 4: The result of the composition operation and the inverse operation on convex rectangular relations is also a convex rectangular relation [47].

A convex interval is an interval with convex relations. The three dimensional block algebra relation meet the definition of convex relation which corresponds to a region in Fig. 8. Let $TD_{rec} = \{d_1,\ldots d_n\}(1\leq n\leq 216)$ be a set of 3D rectangular cardinal direction relation which equivalently corresponds to three dimensional block algebra relations $TD_{cube} = \{d_1',\ldots d_n'\}$ $(1\leq n \leq 216)$ where $d_i' \in TD_{cube}$ corresponds to a point in Fig. 7. In the following discussion, the decomposition of d_i' in TD_{cube} on the x-axis, y-axis and z-axis are denoted by d_{ix}', d_{iy}' and d_{iz}' respectively. A convex relation in the Fig. 8 includes all the points from the minimum coordinate to the maximum coordinate.

For example, we can see that the region $(1.1.1) \sim (2.2.2)$ contains (1.1.1), (1.1.2), (2.1.1), (2.1.2), (2.2.1) and (2.2.2) from Fig. 8 which forms a convex region of three dimensional block algebra relations. It is easy to see that a convex region of three dimensional block algebra relations can be represented by it projections on three axes.

Theorem 5: The projection of a convex region R in three dimensional block algebra relations on x-axis (y-axis, and z-axis respectively) is a convex interval relation.

Proof: According to the definition of convex region, we have that a convex region R corresponds to a three dimensional block algebra relation. By Theorem 3, we know that the projection of R on x-axis (y-axis and z-axis respectively) is a convex interval relation and the boundary relation of the projection on each axis is also convex relation and thus the theorem holds.

Theorem 6: Assume that L_c is the convex interval corresponding to the TD_{cube}. The sets of boundary relations corresponding to L_c are D_{lx}, D_{ly}, D_{lz} on each axis, similarly, The sets of boundary relations corresponding to TD_{cube} are D_x, D_y, D_z on each axis, If $D_{lx} = D_x$, $D_{ly} = D_y$ and $D_{lz} = D_z$, TD_{cube} is a convex interval algebra relation. Equivalently, TD_{rec} is also convex relation network of the cardinal direction relation.

Proof: TD_{cube} is transformed from TD_{rec}. According to the definition of (1), the 3D rectangular algebra relations transformed from the 3DR27 model are also convex relations. According to Theorem 3, we can know that if the interval relations on each axis is continuous interval, the TD_{cube} also is convex relation. If the boundary relations of the set of rectangular relations TD_{cube} are equal to the boundary relations of the convex relations it corresponds to, it is known that the set of rectangular algebra relations TD_{cube} is also continuous interval in every dimension. Similarly, according to Theorem 5, it is known that TD_{cube} is convex. According to the definition of equation (2), the mapping between 3D rectangular cardinal direction and 3-block algebra, TD_{cube} is convex direction relation, so TD_{rec} is also convex relation.

A cardinal direction relation R defined by 3DR27 model can be transformed to an equivalent three dimensional block algebra relations TD_{rec} by means of Fig. 6. TD_{rec} is in the form of the Cartesian product of its projection on x-axis, y-axis and z-axis. We firstly determine whether its projection on x-axis (y-axis and z-axis respectively) is a convex relation. If the three interval algebra relations are all convex relations, then we need to further determine whether a cardinal direction relation corresponding to the Cartesian product of the three interval algebra relations is rectangular. If it is rectangular, then R is convex relation, otherwise it is not. Based on the above discussion, using Theorem 5, Theorem 6 and the mapping between three dimensional block algebra and 3D rectangular cardinal direction relations presented in Fig. 5 as our basis, an algorithm for determining whether a 3D basic cardinal direction relation is convex relation is presented as follows.

This algorithm is used to determine whether the 3D basic directional relation is convex or not. Firstly, the 3D spatial cardinal direction relation R is transformed into a 3D rectangular cardinal direction relation TD_{rec} by Definition 3, and

Algorithm CheckConvex( )

input: Direction relation R

output: true for convex relations, false for non-convex relations

begin

While $(R \neq \varnothing)$

$TD_{rec} \leftarrow R$; /* transform of the 3D cardinal direction relations into a rectangular cardinal direction relation by definition 3 */

$x_i \times y_i \times z_i \leftarrow f(TDrec)$/* transform the 3D rectangular cardinal direction relations into a Cartesian product of algebra relation on the three axes using Theorem 1 */

if $(x_i \cap P_c = \varnothing)$

  return false;

if $(y_i \cap P_c = \varnothing)$

  return false;

if $(z_i \cap P_c = \varnothing)$ /* by definition 6, determine whether the relation is convex on the x, y, z-axis */

  return false;

else if $(f(xi \times yi \times zi) = TDrec)$ /* use Theorem 2 to perform a Cartesian product operation on the interval algebra relations on the three axes to see whether they can be transformed into 3D rectangular cardinal direction relations */

  return true;

else

  return false;

end

then TD_{rec} is transformed into a Cartesian product of interval algebra relations on the three axes (x, y, z) by using Definition 6 to determine whether the interval algebra relations on each axis are the convex relation. Finally, we use Theorem 2 to perform a Cartesian product operation on the interval algebra relations on the three axes to check whether they can be transformed into 3D rectangular cardinal direction relations.

V. NETWORK CONSISTENCY CHECKING FOR 3D SPATIAL CARDINAL RELATIONS

The consistency checking with cardinal directions means that given a series of spatial objects and cardinal direction relations, determine whether the spatial network formed by these cardinal direction relations are consistent, that is to say, whether one or more than one solution can be found to satisfy the network constraints. Liu et al. [19] researched the problem of consistency checking with cardinal directions defined by direction relation matrix model in 2D space the by means of rectangular algebra. In this paper, we will employ the three dimensional block algebra to solve the problem of consistency checking with cardinal directions direction 3DR27 model in 3D space. Now, we give some definitions and theorems for further research.

Definition 8: Let (n,R) be a three dimensional block algebra. If R_{ij} is convex relation for each $i, j \in \{1,\ldots,n\}$, then we say that (n, R) is a convex relation. If $R_{ik} \neq \varnothing \wedge R_{ik} \subseteq R_{ij} \circ R_{jk}$ holds for each $i, j, k \in \{1,\ldots,n\}$, then we say that (n, R) is a path consistency network.

Theorem 7: A three dimensional block algebra constraint network (n, R) is path consistent iff it is a convex relation network and it is path consistent.

Proof: The 3D rectangular algebra constraint network is known to be a convex relation network, indicating that the projection of interval coordinates onto each axis satisfies the convex relation that R_{ij} is 2^{TDcube}, and the convex relation network is path consistent, *i*.e., there exists $i, j, k \subseteq (n)$, and satisfies $R_{ik} \neq \varnothing, R_{ik} \subseteq R_{ij} \circ R_{jk}$, Therefore, the network is a path consistent network.

From the above definitions and theorems, the path consistency checking algorithm based on 3DR27 model for 3D rectangular cardinal direction relation is presented as following.

Algorithm Consistency(n, R)

input: direction relation network (n, R)

output: true for the direction relation is consistency network, false for non-consistency network

begin

Boolean *flag*=true //performing the determination of convex relational networks

begin

for i 1 **to** n

{**for** j 1 **to** n

 {if($i \neq j$ and $R_{ij} \neq \varnothing$)

  { *flag*=CheckConvex(R);

/*call the CheckConvex() function to perform the determination of the convex relational network */

if(*flag*=false)

  return false;

end}}}

else push(stack) /*judging path consistency while stack is not empty do */

 {**for** i1 **to** n-2

  {**for** j(i+1) **to** n-1

   {**for** $k(j+1)$ **to** n

    $R_{ij} = \text{pop(stack)}$;

$f(Q) = f(R_{ij}) \cap f(R_{ik}) \circ f(R_{kj})$

/*The composing operations of the algebra of the x-axis, y-axis, z-axis interval are shown in Table 1 */

$Q = f(xQ \times yQ \times zQ)$ /*Transformation of 3D interval algebra into 3D rectangular cardinal direction relations */

if $(Q \neq \varnothing)$

 {**if** $(Q \neq R_{ij})$

  {$R_{ij} = Q$;

  push(R_{ij})}}

  else return false;}

return true;

end

This algorithm performs the convex relation checking firstly, and the time complexity is $o(n^2)$ for a direction relation constraint network (n, R). Then the consistency checking is performed, and the time cost is $o(n^3)$ due that there are $n(n\text{-}1)(n\text{-}2)/6$ triangles in the network (n, R). Therefore, the total time complexity of our algorithm is $o(n^3)$.

VI. EXAMPLE ANALYSIS

In section V, we have presented an algorithm Consistency (n, R) to solve the problem of the consistency checking with the constraint network of cardinal direction relations based on the 3DR27 model. The algorithm PathConsistency(n, R) has been implemented in Java language programming which runs on Intellij IDEA 2022.1.1 platform. In this section, we will verify the correctness of the proposed algorithm PathConsistency(n, R) by comparing the execution result of this algorithm with the result of manual checking.

Let us consider the following problem. Given a direction relation constraint network *a RE b*, *b RE*:*RSE c* and *a RE c*, where $a, b, c \in T_{box}$, please determine whether this constraint network is consistent.

Firstly, by using our algorithm, the cardinal direction relation constraint network *RE*, *RSE*:*RE* and *RE* is transformed into an equivalent three dimensional block algebra network by means of the mapping between 3D rectangular cardinal direction and 3-block algebra shown in Fig. 6.

Equivalently, we have:

$$f(RE) = \{pi, mi\} \times \{eq, s, d, f\} \times \{eq, s, d, f\} \quad (5)$$

$$f(RE : RSE) = \{pi, mi\} \times \{fi, o\} \times \{eq, s, d, f\} \quad (6)$$

According to Definition 6, we have that the interval algebra relations $\{pi, mi\}$,$\{eq, s, d, f\}$,$\{eq, s, d, f\}$ and $\{fi, o\}$ are all convex relations, which appeared in the Cartesian product corresponding to the cardinal direction relations of the given constraint network. On this occasion, it will return true by calling algorithm CheckConvex().

According to theorem 9, it is necessary to further verify whether the network is path consistent. By Definition 8, we need to determine whether $R_{ac} \subseteq R_{ab} \circ R_{bc}$ holds. The composition of R_{ab} and R_{bc} can be computed by the composition of the equivalent algebra relations of R_{ab} and R_{bc}, as follows.

$$\{pi, mi\} \circ \{pi, mi\} = \{pi\}, \{eq, s, d, f\} \circ \{fi, o\} = \{p, m, o, fi, eq, s, d, f\}, \{eq, s, d, f\} \circ \{eq, s, d, f\} = \{eq, s, d, f\}$$

According to the definition of block algebra,we have that

$$(\{pi, mi\} \times \{eq, s, d, f\} \times \{eq, s, d, f\}) \circ (\{pi, mi\} \times \{fi, o\} \times \{eq, s, d, f\}) = \{\{pi, mi\} \circ \{pi, mi\}\} \times \{\{eq, s, d, f\} \circ \{fi, o\}\} \times \{\{eq, s, d, f\} \circ \{eq, s, d, f\}\} = \{pi\} \times \{p, m, o, fi, eq, s, d, f\} \times \{eq, s, d, f\}$$

Then, we have the following equation holds by means of the composition table of interval algebra shown as Table 1.

$$(\{pi, mi\} \times \{eq, s, d, f\} \times \{eq, s, d, f\}) \circ (\{pi, mi\} \times \{fi, o\} \times \{eq, s, d, f\}) = \{pi\} \times \{p, m, o, fi, eq, s, d, f\} \times \{eq, s, d, f\}$$

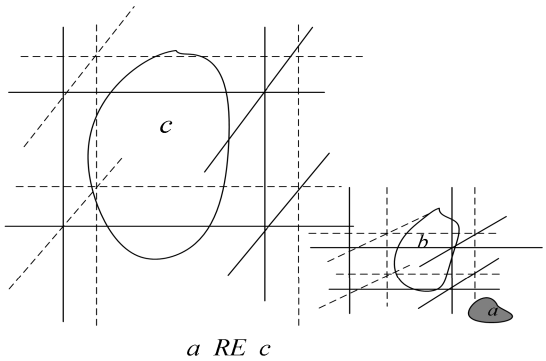

FIGURE 9. Example 1.

By looking up the mapping table between 3D rectangular cardinal direction and 3-block algebra shown as Fig. 4, we know that the 3-block algebra $\{pi\}\times\{p, m, o, fi, eq, s, d, f\}\times\{eq, s, d, f\}$ correspond to the cardinal direction relation {*RSE*, *RE:RSE*, *RE*}.

Therefore, we have $R_{ab} \circ R_{bc}$={*RSE*, *RE:RSE,RE*} and thus $R_{ac} \subseteq R_{ab} \circ R_{bc}$ holds. Then,we have that the given constraint network is path consistence and thus we get that the given constraint network is consistence by using our algorithm.

In reality, there exists objects $a, b, c \in T_{box}$, such that a *RE* $b \wedge b$ *RE:RSE* $c \wedge a$ *RE c* holds which is shown as Fig. 9. By the definition of the consistence of constraints network, we have that the direction relation constraint network formed by the spatial constraints *a RE b*, *b RE*:*RSE c* and *a RE c* is consistence, and thus the result of our algorithm is correct for this spatial constraint network.

Then, let us consider the another problem. Given a direction relation constraint network *a UW b*, *b USW*:*UW*:*UNW c* and *a RSE c*, where $a, b, c \in T_{box}$, please determine whether this constraint network is consistent by Algorithm PathConsistency(n, R).

We first use our algorithm to analyze the three cardinal direction relations that are not empty and the determination of the convex relations is carried out by transforming *UW*, *USW*:*UW*:*UNW* and *RSE* into 3D rectangular direction relations. Then by Theorem 1 and Fig. 6, we transform the mapping between 3D rectangular cardinal direction and 3-block algebra. Let us project them separately on the x, y and z axes, and we give a representation of the Cartesian product. Equivalently, we have:

$$f(UW) = \{p, m\} \times \{eq, s, d, f\} \times \{pi, mi\} \quad (7)$$

$$f(USW : UW : UNW) = \{p, m\} \times \{di\} \times \{pi, mi\} \quad (8)$$

$$f(RSE) = \{pi, mi\} \times \{p, m\} \times \{eq, s, d, f\} \quad (9)$$

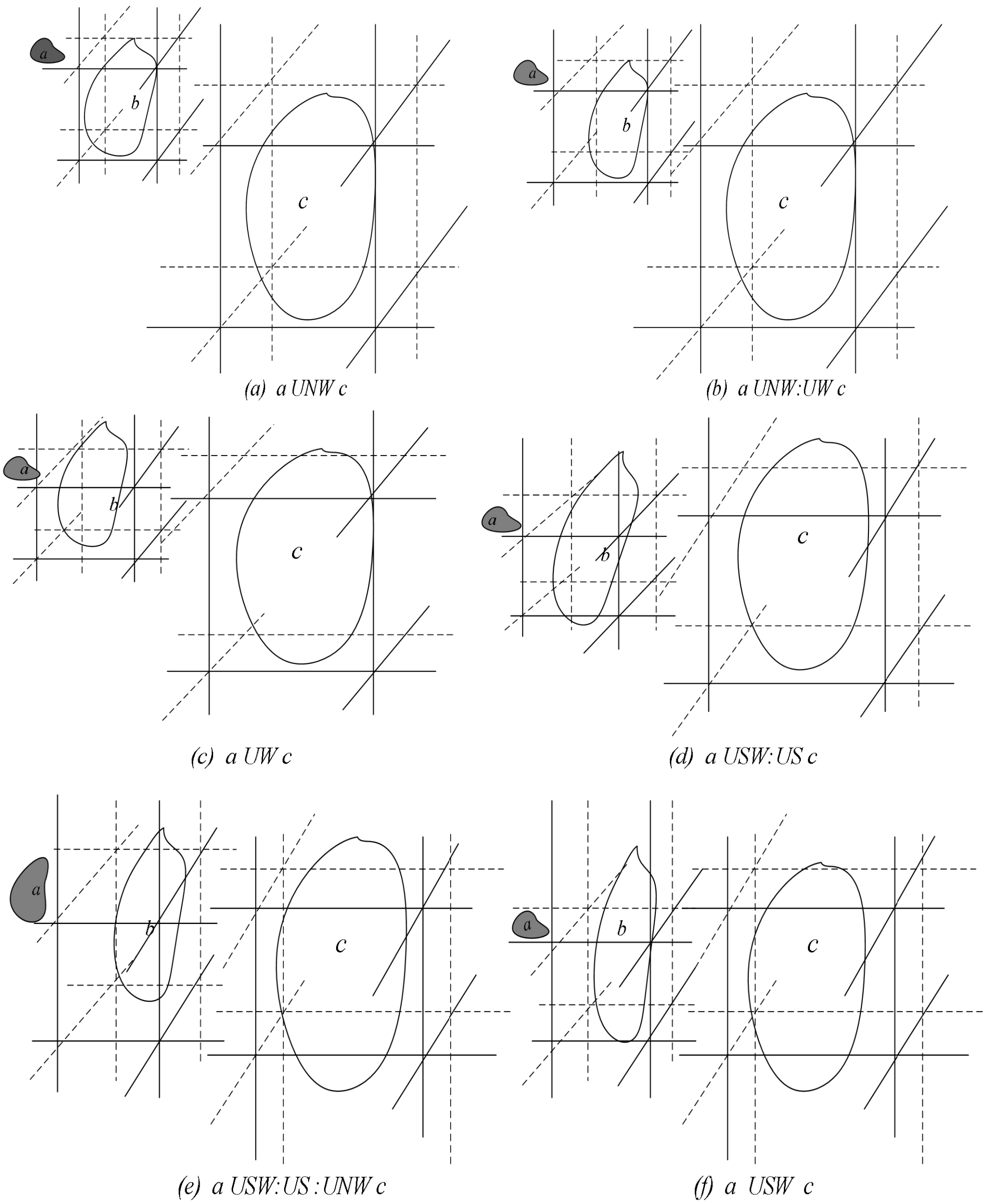

FIGURE 10. Example 2.

By Definition 6, we know the convex relation of the one-dimensional interval is satisfied on each axis. While output the *judge*=true, the R_{ac} into the stack.

In addition, the stack is not empty, R_{ac} is delivered from the stack. We combine the 6 interval relations composition in Table 1 for each axis on $R_{ab} \circ R_{bc}$. We know that:

$$\{p, m\} \circ \{p, m\} = \{p\}, \{eq, s, d, f\} \circ \{di\} = all$$
$$\{pi, mi\} \circ \{pi, mi\} = \{pi\}$$

Then by Definition 5, therefore we have that:

$$\begin{aligned}&\{p, m\} \times \{eq, s, d, f\} \times \{pi, mi\} \circ \{p, m\} \times \{di\} \times \{pi, mi\}\\ &= \{\{p, m\} \circ \{p, m\} \times \{eq, s, d, f\} \circ \{di\} \times \{pi, mi\} \circ \{pi, mi\}\}\\ &= \{\{p\} \times all \times \{pi\}\} = \{\{p, m\} \times \{eq, s, d, f\}\\ &\quad \times \{pi, mi\} \cup \{p, m\} \times \{p, m\} \times \{pi, mi\} \cup \{p, m\} \times \{pi, mi\}\\ &\quad \times \{pi, mi\} \cup \{p, m\} \times \{fi, o\} \times \{pi, mi\} \cup \{p, m\} \times \{si, oi\}\\ &\quad \times \{pi, mi\} \cup \{p, m\} \times \{di\} \times \{pi, mi\}\}\end{aligned}$$

Then by Theorem 2, we have the mapping between 3D rectangular cardinal direction and 3-block algebra. We have:

$$\begin{aligned}&f(\{p, m\} \times \{eq, s, d, f\} \times \{pi, mi\}) \cup f(\{p, m\} \times \{p, m\}\\ &\quad \times \{pi, mi\}) \cup f(\{p, m\} \times \{pi, mi\} \times \{pi, mi\}) \cup f(\{p, m\}\\ &\quad \times \{fi, o\} \times \{pi, mi\}) \cup f(\{p, m\} \times \{si, oi\}\\ &\quad \times \{pi, mi\} \cup f(\{p, m\} \times \{di\} \times \{pi, mi\})\\ &= \{\{UW\} \cup \{USW\} \cup \{UNW\} \cup \{UW : USW\}\\ &\quad \cup \{UW : UNW\} \cup \{UW : UNW : USW\}\}\text{holds.}\end{aligned}$$

Finally, we compare with the path consistency checking $f(Q) = f(R_{ij}) \cap f(R_{ik}) \circ f(R_{kj})$. By Theorem 2, the 3D rectangular algebra is transformed with the 3D cardinal direction relation, we can get $Q = \varnothing$, while $Q = \varnothing$ jumps out of the loop and outputs false. By performing our algorithm, we have that the consistency network is inconsistent.

In reality, we showed that the spatial objects $a, b, c \in T_{box}$, such that a *UW b*∧*b USW*:*UW*:*UNW c* which is shown as Fig. 10. But according to the definition of consistency network, there are none of these 6 cases belong to the above *a RSE c* of spatial layout. Therefore, we decide that the constraint network does not satisfy the consistency checking of the 3D spatial direction relations. The results of the algorithm execution are compared with the results of manual reasoning, we find that the results are consistent, so the algorithm is also correct.

In summary, these examples prove that the consistency checking algorithm for 3D spatial cardinal direction relations is correct, and the algorithm combines the good computational properties of block algebra with easy formal reasoning. Although we only quote two aspects of the algorithm path consistency (n, R) in the example analysis, the results are consistent in both theoretical analysis and algorithm verification. Therefore, the algorithm Consistency (n, R) is used to solve the problem of consistency checking with the constraint network of cardinal direction relations based on the 3DR27 model.

VII. CONCLUSION

In this paper, we discussed these questions, including describing the 3D cardinal direction relations model, the three-dimensional block algebra relations and the rectangular direction relations defined by the 3DR27 model, and building the mapping between 3D rectangular cardinal direction and 3-block algebra. Then, we provided several theorems for determining convex relation networks. In addition, block algebra has good computational abilities. Finally, we provided our algorithm for verifying the convex relation network and consistency checking by means of the interval algebra and the method of the spatial projection based on these theorems and definitions. The verification was carried out by the comparing the result of our algorithm with that of manual reasoning. The results of comparison also demonstrate that our algorithm can work correctly.

Notice that our algorithm used the spatial projection method to decide whether a network of 3D rectangular direction relations is a convex relation. The algorithm improved the ability of intelligent reasoning and prediction of the 3DR27 model and then enhanced the usability of this model. This work is of great significance for enhancing the practical application requirements of consistency checking.

However, due to the inherent complexity of 3D spatial direction relations, the spatial target objects considered in this paper are only closed and connected regions. In future research, the algorithm will be improved so that it can handle spatial objects with holes and non-connections, and this paper only considers a single spatial relation, and combining multiple spatial relations for consistency checking will be the top priority in the future. What's more, it is worth observing that we have used interval algebra only to represent the 3D rectangular direction relation. For future work, we can use algebraic theory to express the 3D cardinal direction relation so that the representation of spatial relations is more special and comprehensive.

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