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RESEARCH ARTICLE

Almost Sure Exponential Consensus of Linear and Nonlinear Multi-Agent Systems Under Semi-Markovian Switching Topologies and Application to Synchronization of Chaos Systems

XIA XIAO[1], XIHUI WU[1], XIAOLI HOU[1], AND YUEHUA HE[2]

[1]College of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, Henan 450000, China
[2]School of Science, Xuchang University, Xuchang, Henan 461000, China

Corresponding author: Xia Xiao (xiaoxia-0124@163.com)

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ABSTRACT This paper researches the almost sure exponential leader-following consensus (ASELFC) of linear and nonlinear multi-agent systems (MASs) under semi-Markovian switching topologies (SMSTs). Differing from existing research works of MASs under SMSTs, we remove the restriction that every topology has a directed spanning tree (DST) rooted in the leader. Only partial switching topologies are requested to have a DST. By employing Lyapunov analysis method and stochastic technique, sufficient conditions for ASELFC are obtained. Finally, the theoretical results are applied to solve the synchronization of Chaos systems by using an example of Chua's circuits.

INDEX TERMS Almost sure exponential leader-following consensus, semi-Markovian switching topologies, multi-agent systems.

I. INTRODUCTION

In the past two decades, coordination problem of multi-agent systems (MASs) has drawn a large amount of attention in virtue of its broad applications in multi-mobile networks [1], the formation flying [2], communication networks [3], robot rendezvous [4], etc. The consensus is an important matter in addressing the issue of MASs [5], [6], [7], [8], [9]. The main thought of consensus is to design controler that derive all agents to a common state. The consensus in the context of a leader is called leader-following consensus, whose goal is to design protocols that derive each follower agent has the same state as the leader's [10], [11], [12].

The changes of communication topologies among the agents may be random in MASs, owning to packet dropouts, random link failures or the change of environments, etc.

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Accordingly, it is meaning to research MASs under stochastic switching topologies. Under Markov switching topologies, the consensus problems of MASs have been addressed [13], [14], [15]. However, there are some restrictions on the Markov switching topologies, because of each communication topology obeys exponential distribution and the transition rates are constant owning to the memoryless property of the exponential distribution. Therefore, semi-Markovian switching topologies (SMSTs) [16], [17], [18], [19], [20], [21] have been researched to overcome these restrictions, since the sojourn time of each possible topology is not necessary to be exponential distributed. The leader-following consensus problem of nonlinear MASs under SMSTs in the mean square sense is studied [16]. The containment control of stochastic MASs under SMSTs in the asymptotic mean square sense is researched [17]. The leader-following consensus of nonlinear MASs in the mean square sense under SMSTs and cyber attacks is studied [18]. The

H_∞ leader-following consensus problem in the mean square sense for nonlinear MASs under SMSTs is investigated [19]. Event-triggered leader-following consensus of nonlinear MASs under SMSTs in the mean sense is researched [20]. Impulsive consensus of stochastic MASs under semi-Markovian switching topologies is researched [21].

For MASs under SMSTs, most of previous works [16], [17], [18], [19], [20] require the each possible topology contains a directed spanning tree (DST). In fact, some topologies perhaps do not have any DST. When communication topology has not a DST, some properties of Laplacian matrix cannot be used (e.g. Lemma 1). That is, the methods [16], [17], [18], [19], [20] cannot be extended to the case that some topologies do not contain a DST. It is worthy to study the partial switching topologies have a DST with the leader as the root. Moreover, most of the existing works [16], [17], [18], [19], [20] on MASs under SMSTs focus on the consensus in the sense of mathematical expectation. As far as we know, no results have been reported on almost sure exponential consensus problem with semi-Markovian jump MASs.

Inspired by the above discussion, we aim at designing consensus protocol for MASs with SMSTs, in which partial switching topologies have not a DST. The dominant contributions include the following three aspects. (1) For the MASs under SMSTs, the existing works [16], [17], [18], [19], [20] focused on the consensus in the mean square sense. For the first time, the almost sure exponential leader-following consensus (ASELFC) is studied for MASs under SMSTs. (2) In contrast to the previous works [16], [17], [18], [19], [20], [22], which require that each possible topology contains a DST, we investigate the SMSTs that partial switching topologies have not a DST. (3) The control gains of designed communication protocol are topology-dependent, which is less conservative (see Remark 1).

In Section II, the MASs with SMSTs is presented, together with some useful preliminary results. Section III contains main results. Simulation results are provided in Section IV to illustrate the effectiveness of the proposed results. The conclusion is given in Section V.

Notations: $\mathbb{R}^n$ stands for the n-dimensional Euclidean space with real entries. The superscript T denotes the transpose of vector (or matrix). I_n represents an $n \times n$ identity matrix. $1_n = [1, \ldots, 1]^T \in \mathbb{R}^n$. $diag\{l_1, l_2, \ldots, l_n\}$ stands for a diagonal matrix that diagonal elements are $l_1, l_2, \ldots, l_n$. $P > 0$ means that P is a real symmetric positive definite matrix. For real symmetric matrix M, $\lambda_{min}(M)$ and $\lambda_{max}(M)$ represent the minimum and maximum eigenvalue of matrix M. $M \otimes N$ stands for the Kronecker product of matrices M and N. $\mathbb{P}$ represents the probability measure, $\mathbb{E}$ represents the mathematical expectation.

II. PRELIMINARIES

A. GRAPH THEORY

The directed graphs $\overline{\mathcal{G}}_{\varpi(t)} = (\overline{\mathcal{V}}, \overline{\mathcal{E}}_{\varpi(t)}, \overline{\mathcal{A}}_{\varpi(t)})$ denote semi-Markovian switching interaction topologies which consist of a leader 0 and N followers, where $\overline{\mathcal{V}} = 0 \cup \mathcal{V}$, $\mathcal{V} = \{1, 2, \ldots, N\}$, the switching signal $\varpi(t)$ is a right continuous semi-Markovian chain taking values in a given state space $S = \{1, 2, \ldots, s\}$. The kth switching instant is denoted by $t_k, k \geq 1$, which satisfies $0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ and $\lim_{k\to\infty} t_k = \infty$. The graphs $\mathcal{G}_{\varpi(t)} = (\mathcal{V}, \mathcal{E}_{\varpi(t)}, \mathcal{A}_{\varpi(t)})$ represent the switching interaction topologies which consist of N followers. The corresponding adjacency and Laplacian matrices of $\mathcal{G}_{\varpi(t)}$ are denoted as $\mathcal{A}_{\varpi(t)} = [a_{ij}^{\varpi(t)}]_{N\times N}, \mathcal{L}_{\varpi(t)} = [l_{ij}^{\varpi(t)}]_{N\times N}$, respectively. Note that $a_{ij}^{\varpi(t)} > 0$ if $(i, j) \in \mathcal{E}_{\varpi(t)}$ and $a_{ij}^{\varpi(t)} = 0$ otherwise. In general, $a_{ii}^{\varpi(t)} = 0$. Note that $l_{ij}^{\varpi(t)} = \sum_{j=1, j\neq i}^{N} a_{ij}^{\varpi(t)}$, $i = j$ and $l_{ij}^{\varpi(t)} = -a_{ij}^{\varpi(t)}, i \neq j$. The corresponding Laplacian matrix of $\overline{\mathcal{G}}_{\varpi(t)}$ is represented as $\mathcal{H}_{\varpi(t)}$, where $\mathcal{H}_{\varpi(t)} = \mathcal{L}_{\varpi(t)} + \mathcal{D}_{\varpi(t)}$. Define $\mathcal{D}_{\varpi(t)} = diag\{d_1^{\varpi(t)}, d_2^{\varpi(t)}, \ldots, d_N^{\varpi(t)}\}$ with $d_i^{\varpi(t)} > 0$ means there is a directed edge form the leader 0 to agent i at time t; otherwise, $d_i^{\varpi(t)} = 0$.

B. PROBLEM FORMULATION

Consider a multi-agent system composed of N followers and a leader. The dynamics of the ith follower can be described as

$$\dot{x}_i(t) = Ax_i(t) + f(x_i(t)) + Bu_i(t), \quad i = 1, 2, \ldots, N, \quad (1)$$

the $f(\cdot)$: $\mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function.

The leader is represented as

$$\dot{x}_0(t) = Ax_0(t) + f(x_0(t)). \quad (2)$$

The protocol for agent i can be designed as

$$u_i(t) = \alpha_{\varpi(t)} K_{\varpi(t)} [\sum_{j=1}^{N} a_{ij}^{\varpi(t)} (x_j(t) - x_i(t)) + d_i^{\varpi(t)} (x_0(t) - x_i(t))], \quad (3)$$

where $K_{\varpi(t)}$ is the control gain matrix will be devised later.

Definition 1 [23]: The switching signal $\varpi(t)$ is referred to as semi-Markovian switching, if we denote $\varpi(t) = \varpi(t_k) = \varpi_k$ for $t \in [t_k, t_{k+1}), k \geq 0$,

(i) the discrete-time process $\{\varpi(t_k), k \geq 0\}$ is a Markovian chain with transition probability matrix

$$P = [p_{rl}]_{s\times s}\ (r, l \in S),$$

where $p_{rl} = \mathbb{P}\{\varpi(t_{k+1}) = l | \varpi(t_k) = r\}$ and $p_{rr} = 0$.
(ii) the distribution function of $T(k+1) \triangleq t_{k+1} - t_k$ is defined by

$$F_{rl}(t) = \mathbb{P}\{T(k+1) \leq t | \varpi_k = r, \varpi_{k+1} = l\}, r, l \in S, t \geq 0,$$

which depends on ϖ_k and ϖ_{k+1}.

Assumption 1: Every graph $\overline{\mathcal{G}}_r, r \in S$ has a DST with the leader as the root.

Lemma 1 [24]: Assume that Assumption 1 holds. Let $z_r = [z_1^r, z_2^r, \ldots, z_N^r]^T = \mathcal{H}_r^{-1} 1_N$, $y_r = [y_1^r, y_2^r, \ldots, y_N^r]^T = \mathcal{H}_r^{-T} 1_N$, $\Theta_r = diag\{y_1^r/z_1^r, y_2^r/z_2^r, \ldots, y_N^r/z_N^r\}$, $Q_r = \mathcal{H}_r^T \Theta_r + \Theta_r \mathcal{H}_r$. Then $Q_r > 0$ and $\Theta_r > 0$.

Lemma 2 [25]: Let $\varpi(t)$ be a semi-Markovian process and $\bar{\kappa} = [\bar{\kappa}_1, \bar{\kappa}_2, \cdots, \bar{\kappa}_s]$ be the stationary distribution of its embedded Markov chain $\{\varpi_k, k \geq 0\}$, then,

$$\lim_{t\to\infty} \frac{\mathcal{T}_r(t)}{t} = \kappa_r, \ a.s. \tag{4}$$

$$\lim_{t\to\infty} \frac{\mathcal{N}_r(t)}{t} = \frac{\kappa_r}{m_r}, \ a.s. \tag{5}$$

$$\lim_{t\to\infty} \frac{\mathcal{N}_{rl}(t)}{t} = \kappa_r \frac{p_{rl}}{m_r}, \ a.s. \tag{6}$$

where $\kappa_r = \frac{\bar{\kappa}_r m_r}{\sum_{l\in S}\bar{\kappa}_l m_l}$, $m_r = \mathbb{E}[T_r(1)]$. $T_r(k)$ is the sojourn time of the rth topology for the kth visiting of $\varpi(t)$, $k \geq 1$, $\mathcal{T}_r(t)$ is the total active time of the rth topology over interval $[0, t]$, $\mathcal{N}_r(t)$ is the total active numbers of rth topology over interval $[0, t]$, $\mathcal{N}_{rl}(t)$ is the total numbers of the events of switching from the r-th topology to the l-th topology over interval $[0, t]$, $r, l \in S$, $a.s.$ means almost surely.

Lemma 3 [26]: Suppose that $D \in \mathbb{R}^{n\times n}$ is a positive definite matrix and $G \in \mathbb{R}^{n\times n}$ is a symmetric matrix. Then, for any given positive semi-definite matrix $W \in \mathbb{R}^{r\times r}$ and vector $\xi \in \mathbb{R}^{nr}$, the following inequality holds:

$$\xi^T(G \otimes W)\xi \leq \lambda_{min}(D^{-1}G)\xi^T(D \otimes W)\xi \tag{7}$$

Let $\epsilon_i(t) = x_i(t) - x_0(t)$ be the state error between the agent i and the leader, $i = 1, 2, \ldots, N$.

Definition 2 [27]: ASELFC of MASs (1) − (2) can be achieved via protocol (3), if for each agent $i \in \mathcal{V}$

$$\lim_{t\to\infty}\sup \frac{\ln\|\epsilon_i(t)\|}{t} = \lim_{t\to\infty}\sup \frac{\ln\|x_i(t) - x_0(t)\|}{t} < 0. \ a.s.$$

III. MAIN RESULTS

By using the multiple Lyapunov functions method and the stationary distribution of semi-Markovian process, we will study ASELFC of linear and nonlinear MASs with a general topology that only needs partial switching topologies have a DST under SMSTs in this section. Then two conclusions are provided.

Assumption 2: Let $S = S_T \bigcup S_U$, if $r \in S_T$, then $\overline{\mathcal{G}}_r$ has a DST with the leader as the root; otherwise, $r \in S_U$.

A. THE CONSENSUS OF LINEAR MULTI-AGENT SYSTEMS

For the $f(\cdot) = 0$ in (1) and (2), the MASs is linear. The dynamics of the ith follower can be described as

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \quad i = 1, 2, \ldots, N, \tag{8}$$

where A and B are constant real matrices.

The leader is represented as

$$\dot{x}_0(t) = Ax_0(t). \tag{9}$$

$\epsilon_i(t) = x_i(t) - x_0(t)$ is the state error between the agent i and the leader, $i = 1, 2, \ldots, N$. Then

$$\dot{\epsilon}(t) = [I_N \otimes A - \alpha_{\varpi(t)}(\mathcal{H}_{\varpi(t)} \otimes BK_{\varpi(t)}]\epsilon(t), \tag{10}$$

where $\epsilon(t) = [\epsilon_1^T(t), \epsilon_2^T(t), \ldots, \epsilon_N^T(t)]^T$.

Assumption 3: (A, B) is stabilizable.

Theorem 1: Consider the MASs with dynamics (8) and (9) satisfying Assumption 2 and Assumption 3, ASELFC is solved by (3) with $K_r = B^T P_r^{-1}$ for $r \in S$, if there exist constant $\beta_r > 0$, $\gamma_r > 0$ and matrices $P_r > 0$ such that

$$AP_r + P_rA^T - \alpha_r\bar{\lambda}_rBB^T + \beta_rP_r < 0, \ r \in S_T, \tag{11}$$

$$AP_r + P_rA^T - \alpha_r\tilde{\lambda}_rBB^T - \gamma_rP_r < 0, \ r \in S_U, \tag{12}$$

$$\sum_{r,l\in S} \kappa_r \frac{p_{rl}}{m_r} \ln\tilde{\mu}_{rl} - \sum_{r\in S_T} \kappa_r\beta_r + \sum_{r\in S_U} \kappa_r\gamma_r < 0 \tag{13}$$

where $\tilde{\lambda}_r = \lambda_{min}(\mathcal{H}_r + \mathcal{H}_r^T)$, $z_r = [z_1^r, z_2^r, \ldots, z_N^r]^T = \mathcal{H}_r^{-1}1_N$, $y_r = [y_1^r, y_2^r, \ldots, y_N^r]^T = \mathcal{H}_r^{-T}1_N$, $\Theta_r = diag\{y_1^r/z_1^r, y_2^r/z_2^r, \ldots, y_N^r/z_N^r\}$, $\tilde{\Psi}_r \triangleq \begin{cases} \Theta_r \otimes P_r^{-1}, & r \in S_T \\ I_N \otimes P_r^{-1}, & r \in S_U, \end{cases}$ $\bar{\lambda}_r \triangleq \lambda_{min}(\Theta_r^{-1}\mathcal{H}_r^T\Theta_r + \mathcal{H}_r)$, $\tilde{\mu}_{rl} = \frac{\lambda_{max}(\tilde{\Psi}_l)}{\lambda_{min}(\tilde{\Psi}_r)}(r \neq l), r, l \in S$.

Proof : Consider the multiple Lyapunov function candidate for system (10):

$$V_1(\epsilon(t), \varpi(t)) = \begin{cases} \epsilon^T(t)(\Theta_{\varpi(t)} \otimes P_{\varpi(t)}^{-1})\epsilon(t), & \varpi(t) \in S_T \\ \epsilon^T(t)(I_N \otimes P_{\varpi(t)}^{-1})\epsilon(t), & \varpi(t) \in S_U. \end{cases} \tag{14}$$

For $t \in [t_k, t_{k+1})$, if $\varpi(t) = l$, $l \in S_T$, then derivation of $V_1(\epsilon(t), l)$ can be obtained as

$$\begin{aligned}\dot{V}_1(\epsilon(t), l) &= \epsilon^T(t)[\Theta_l \otimes (P_l^{-1}A + A^TP_l^{-1})]\epsilon(t) \\ &\quad - \alpha_l\epsilon^T(t)[(\mathcal{H}_l^T\Theta_l + \Theta_l\mathcal{H}_l) \otimes P_l^{-1}BB^TP_l^{-1}]\epsilon(t).\end{aligned}$$

From Lemma3, we get

$$\begin{aligned}\dot{V}_1(\epsilon(t), l) &\leq \epsilon^T(t)[\Theta_l \otimes (P_l^{-1}A + A^TP_l^{-1})]\epsilon(t) \\ &\quad - \alpha_l\bar{\lambda}_l\epsilon^T(t)(\Theta_l \otimes P_l^{-1}BB^TP_l^{-1})\epsilon(t) \\ &= \epsilon^T(t)[\Theta_l \otimes (P_l^{-1}A + A^TP_l^{-1} \\ &\quad - \alpha_l\bar{\lambda}_lP_l^{-1}BB^TP_l^{-1})]\epsilon(t).\end{aligned} \tag{15}$$

Combining (15) with (11), we get

$$\dot{V}_1(\epsilon(t), l) \leq -\beta_lV_1(\epsilon(t), l). \tag{16}$$

If $\varpi(t) = l$, $l \in S_U$, taking the derivation of $V_1(\epsilon(t_k), l)$, $\dot{V}_1(\epsilon(t_k), l)$ can be obtained as

$$\begin{aligned}\dot{V}_1(\epsilon(t), l) &= \epsilon^T(t)[I_N \otimes (P_l^{-1}A + A^TP_l^{-1})]\epsilon(t) \\ &\quad - \epsilon^T(t)[(\mathcal{H}_l^T + \mathcal{H}_l) \otimes P_l^{-1}BB^TP_l^{-1}]\epsilon(t) \\ &\leq \epsilon^T(t)[I_N \otimes (P_l^{-1}A + A^TP_l^{-1}) \\ &\quad - \tilde{\lambda}_lP_l^{-1}BB^TP_l^{-1}]\epsilon(t).\end{aligned} \tag{17}$$

Combining (17) with (12) yields that

$$\dot{V}_1(\epsilon(t), l) \leq \gamma_lV_1(\epsilon(t), l), \quad l \in S_U. \tag{18}$$

Denote $\tilde{\beta}_{\varpi(t)} \triangleq \begin{cases} \beta_{\varpi(t)}, & \varpi(t) \in S_T \\ -\gamma_{\varpi(t)}, & \varpi(t) \in S_U. \end{cases}$

For $t \in [t_{k-1}, t_k)$, $\varpi(t) = r \in S, r \neq l$. Based on the earlier analysis, we obtain

$$V_1(\epsilon(t_k), l) \leq \tilde{\mu}_{rl} V_1(\epsilon(t_k^-), r). \tag{19}$$

For $t \in [t_k, t_{k+1})$, combining (14), (16), (18), (19), we get

$$\begin{aligned} V_1(\epsilon(t), \varpi(t)) &\leq V_1(\epsilon(t_k), l) e^{-\tilde{\beta}_l(t-t_k)} \\ &\leq \tilde{\mu}_{rl} V_1(\epsilon(t_k^-), r) e^{-\tilde{\beta}_l(t-t_k)}. \end{aligned} \tag{20}$$

Let $\mathcal{N}(t)$ is the total numbers of switching over interval $[0, t]$. By recursion,

$$\begin{aligned} &V_1(\epsilon(t), \varpi(t)) \\ &\leq V_1(\epsilon(0), \varpi(0)) \prod_{k=1}^{\mathcal{N}(t)} \tilde{\mu}_{\varpi_{k-1},\varpi_k} e^{\int_0^t -\tilde{\beta}_{\varpi(s)} ds} \\ &= V_1(\epsilon(0), \varpi(0)) \prod_{r,l \in S} \tilde{\mu}_{rl}^{\mathcal{N}_{rl}(t)} e^{\int_0^t -\tilde{\beta}_{\varpi(s)} ds}, \end{aligned} \tag{21}$$

then,

$$\begin{aligned} &\ln V_1(\epsilon(t), \varpi(t)) \\ &\leq \ln V_1(\epsilon(0), \varpi(0)) + \sum_{r,l \in S} \mathcal{N}_{rl}(t) \ln \tilde{\mu}_{rl} - \sum_{r \in S} \tilde{\beta}_r \mathcal{T}_r(t) \\ &= \ln V_1(\epsilon(0), \varpi(0)) + \sum_{r,l \in S} \mathcal{N}_{rl}(t) \ln \tilde{\mu}_{rl} \\ &\quad - \sum_{r \in S_T} \beta_r \mathcal{T}_r(t) + \sum_{r \in S_U} \gamma_r \mathcal{T}_r(t). \end{aligned} \tag{22}$$

Thus

$$\begin{aligned} &\lim_{t\to\infty} \sup \frac{\ln V_1(\epsilon(t), \varpi(t))}{t} \\ &\leq \lim_{t\to\infty} \sum_{r,l \in S} \frac{\mathcal{N}_{rl}(t)}{t} \ln \tilde{\mu}_{rl} - \lim_{t\to\infty} \sum_{r \in S_T} \beta_r \frac{\mathcal{T}_r(t)}{t} \\ &\quad + \lim_{t\to\infty} \sum_{r \in S_U} \gamma_r \frac{\mathcal{T}_r(t)}{t}. \end{aligned} \tag{23}$$

From Lemma 2, and (13), we get

$$\begin{aligned} &\lim_{t\to\infty} \sup \frac{\ln V_1(\epsilon(t), \varpi(t))}{t} \\ &\leq \sum_{r,l \in S} \kappa_r \frac{p_{rl}}{m_r} \ln \tilde{\mu}_{rl} - \sum_{r \in S_T} \kappa_r \beta_r \\ &\quad + \sum_{r \in S_U} \kappa_r \gamma_r < 0, \ a.s. \end{aligned} \tag{24}$$

so $\lim_{t\to\infty} \sup \frac{\ln \|\epsilon_i(t)\|}{t} < 0, i \in \mathcal{V}$. *a.s.* From Definition 2, we get that the MASs with a general SMSTs is ASELFC. ∎

Corollary 1: Consider the MASs with dynamics (8) – (9) satisfying Assumption 1 and Assumption 3, ASELFC is solved by (3) with $K_r = B^T P_r^{-1}$,$r \in S$, if there exist constant $\beta_r > 0$ and matrices $P_r > 0$ such that

$$AP_r + P_r A^T - \alpha_r \bar{\lambda}_r BB^T + \beta_r P_r < 0, \tag{25}$$

$$\sum_{r \in S} \kappa_r \left(\sum_{l \in S} \frac{p_{rl}}{m_r} \ln \mu_{rl} - \beta_r\right) < 0 \tag{26}$$

where $z_r = [z_1^r, z_2^r, \ldots, z_N^r]^T = \mathcal{H}_r^{-1} 1_N$, $y_r = [y_1^r, y_2^r, \ldots, y_N^r]^T = \mathcal{H}_r^{-T} 1_N$, $\Theta_r = diag\{y_1^r/z_1^r, y_2^r/z_2^r, \ldots, y_N^r/z_N^r\}$, $\bar{\lambda}_r \triangleq \lambda_{min}(\Theta_r^{-1} \mathcal{H}_r^T \Theta_r + \mathcal{H}_r)$, $\Psi_r \triangleq \Theta_r \otimes P_r^{-1}$, $\mu_{rl} = \frac{\lambda_{max}(\Psi_l)}{\lambda_{min}(\Psi_r)} (r \neq l), r, l \in S$.

Proof : Consider the Lyapunov functon candidate:

$$V_2(\epsilon(t), \varpi(t)) = \epsilon^T(t)(\Theta_{\varpi(t)} \otimes P_{\varpi(t)}^{-1})\epsilon(t). \tag{27}$$

By using Theorem 1, we get that the linear MASs with SMSTs is ASELFC.

B. CONSENSUS OF NONLINEAR MULTI-AGENT SYSTEMS

For the $f(\cdot)$: $\mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz nonlinear function with parameter matrix F, we consider the MASs with dynamics (1) and (2). Then

$$\dot{\epsilon}(t) = [I_N \otimes A - \alpha_{\varpi(t)}(\mathcal{H}_{\varpi(t)} \otimes BK_{\varpi(t)}]\epsilon(t) + \bar{F}(\epsilon(t)), \tag{28}$$

where $\bar{F}(\epsilon(t)) = [\bar{f}_1^T(\epsilon_1(t)), \bar{f}_2^T(\epsilon_2(t)), \ldots, \bar{f}_N^T(\epsilon_N(t))]^T$, $\bar{f}_i(\epsilon_i(t)) = f(\epsilon_i(t)) - f(\epsilon_0(t)), i = 1, 2, \ldots, N$.

Theorem 2: Consider the MASs with dynamics (1) and (2) satisfying Assumption 2 and Assumption 3, ASELFC is solved by (3) with $K_r = B^T P_r^{-1}$ for $r \in S$, if there exist constant $\beta_r > 0$, $\gamma_r > 0$ and matrices $P_r > 0$ such that

$$\begin{bmatrix} AP_r + P_r A^T - \alpha_r \tilde{\lambda}_r BB^T + I_n + \beta_r P_r & P_r F^T \\ FP_r & -I_n \end{bmatrix} \leq 0, \tag{29}$$

$(r \in S_T)$

$$\begin{bmatrix} AP_r + P_r A^T - \alpha_r \tilde{\lambda}_r BB^T + I_n - \gamma_r P_r & P_r F^T \\ FP_r & -I_n \end{bmatrix} \leq 0, \tag{30}$$

$(r \in S_U)$

$$\sum_{r,l \in S} \kappa_r \frac{p_{rl}}{m_r} \ln \tilde{\mu}_{rl} - \sum_{r \in S_T} \kappa_r \beta_r + \sum_{r \in S_U} \kappa_r \gamma_r < 0, \tag{31}$$

where $\tilde{\lambda}_r = \lambda_{min}(\mathcal{H}_r + \mathcal{H}_r^T)$, $z_r = [z_1^r, z_2^r, \ldots, z_N^r]^T = \mathcal{H}_r^{-1} 1_N$, $y_r = [y_1^r, y_2^r, \ldots, y_N^r]^T = \mathcal{H}_r^{-T} 1_N$, $\Theta_r = diag\{y_1^r/z_1^r, y_2^r/z_2^r, \ldots, y_N^r/z_N^r\}$, $\tilde{\Psi}_r \triangleq \begin{cases} \Theta_r \otimes P_r^{-1}, & r \in S_T \\ I_N \otimes P_r^{-1}, & r \in S_U, \end{cases}$ $\bar{\lambda}_r \triangleq \lambda_{min}(\Theta_r^{-1} \mathcal{H}_r^T \Theta_r + \mathcal{H}_r)$, $\tilde{\mu}_{rl} = \frac{\lambda_{max}(\tilde{\Psi}_l)}{\lambda_{min}(\tilde{\Psi}_r)} (r \neq l), r, l \in S$.

Proof : Consider the multiple Lyapunov function candidate for system (28):

$$V_3(\epsilon(t), \varpi(t)) = \begin{cases} \epsilon^T(t)(\Theta_{\varpi(t)} \otimes P_{\varpi(t)}^{-1})\epsilon(t), & \varpi(t) \in S_T \\ \epsilon^T(t)(I_N \otimes P_{\varpi(t)}^{-1})\epsilon(t), & \varpi(t) \in S_U. \end{cases} \tag{32}$$

For $t \in [t_k, t_{k+1})$, if $\varpi(t) = l, l \in S_T$, then derivation of $V_3(\epsilon(t), l)$ can be obtained as

$$\begin{aligned} &\dot{V}_3(\epsilon(t), l) \\ &= \epsilon^T(t)[\Theta_l \otimes (P_l^{-1} A + A^T P_l^{-1})]\epsilon(t) \end{aligned}$$

$$- \alpha_l \epsilon^T(t)[(\mathcal{H}_l^T \Theta_l + \Theta_l \mathcal{H}_l) \otimes P_l^{-1} BB^T P_l^{-1}]\epsilon(t)$$
$$+ 2\epsilon^T(t)(\Theta_l \otimes P_l^{-1})\bar{F}(\epsilon(t)).$$

From Lemma3, we get

$$\begin{aligned}
&\dot{V}_3(\epsilon(t), l) \\
&\quad \leq \epsilon^T(t)[\Theta_l \otimes (P_l^{-1}A + A^T P_l^{-1})]\epsilon(t) \\
&\qquad - \alpha_l \bar{\lambda}_l \epsilon^T(t)(\Theta_l \otimes P_l^{-1} BB^T P_l^{-1})\epsilon(t) \\
&\qquad + \epsilon^T(t)\Theta_l \otimes (F^T F + P_l^{-1} P_l^{-1})\epsilon(t) \\
&\quad \leq \epsilon^T(t)[\Theta_l \otimes (P_l^{-1}A + A^T P_l^{-1} + F^T F \\
&\qquad + (P_l^{-1})^2 - \alpha_l \bar{\lambda}_l P_l^{-1} BB^T P_l^{-1})]\epsilon(t). \qquad (33)
\end{aligned}$$

Combining (33) with (29), we get

$$\dot{V}_3(\epsilon(t), l) \leq -\beta_l V_3(\epsilon(t), l). \qquad (34)$$

If $\varpi(t) = l, l \in S_U$, taking the derivation of $V_3(\epsilon(t_k), l)$, $\dot{V}_3(\epsilon(t_k), l)$ can be got as

$$\begin{aligned}
&\dot{V}_3(\epsilon(t), l) \\
&\quad = \epsilon^T(t)[I_N \otimes (P_l^{-1}A + A^T P_l^{-1})]\epsilon(t) \\
&\qquad - \alpha_r \epsilon^T(t)[(\mathcal{H}_l^T + \mathcal{H}_l) \otimes P_l^{-1} BB^T P_l^{-1}]\epsilon(t) \\
&\qquad + 2\epsilon^T(t)(I_N \otimes P_l^{-1})\bar{F}(\epsilon(t)) \\
&\quad \leq \epsilon^T(t)[I_N \otimes (P_l^{-1}A + A^T P_l^{-1} + (P_l^{-1})^2 \\
&\qquad + F^T F - \alpha_r \tilde{\lambda}_l P_l^{-1} BB^T P_l^{-1})]\epsilon(t). \qquad (35)
\end{aligned}$$

Combining (35) with (30) yields that

$$\dot{V}_3(\epsilon(t), l) \leq \gamma_l V_3(\epsilon(t), l), \qquad l \in S_U. \qquad (36)$$

Denote $\tilde{\beta}_{\varpi(t)} \triangleq \begin{cases} \beta_{\varpi(t)}, & \varpi(t) \in S_T \\ -\gamma_{\varpi(t)}, & \varpi(t) \in S_U. \end{cases}$

According Theorem 1, we get that the nonlinear MASs with a general SMSTs is ASELFC. ■

Remark 1: Due to the topology $r, r \in S_U$ has not a DST, we cannot use Lemma 1 to obtain Θ_r. Thus, according to whether or not the communication topology has a DST, we design the Lyapunov function (32). To accurately estimate the growth of $V_3(\epsilon(t), r)$, we consider the topology-dependent control gain K_r to obtain the topology-dependent parameter γ_r (see Eq. (36)) which is less conservative than the topology-independent parameter γ in [22].

Corollary 2: Consider the MASs with dynamics (1) – (2) satisfying Assumption 1 and Assumption 3, ASELFC is solved by (3) with $K_r = B^T P_r^{-1}, r \in S$, if there exist constant $\beta_r > 0$ and matrices $P_r > 0$ such that

$$\begin{bmatrix} AP_r + P_r A^T - \alpha_r \bar{\lambda}_r BB^T + I_n + \beta_r P_r & P_r F^T \\ FP_r & -I_n \end{bmatrix} \leq 0, \qquad (37)$$

$$\sum_{r \in S} \kappa_r \Big(\sum_{l \in S} \frac{p_{rl}}{m_r} \ln \mu_{rl} - \beta_r\Big) < 0 \qquad (38)$$

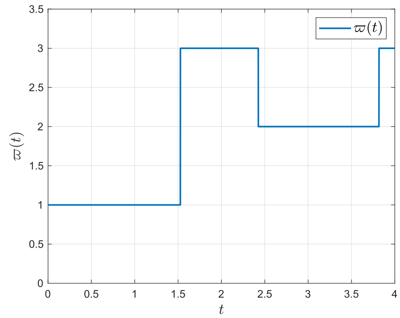

FIGURE 1. The switching signal $\varpi(t)$.

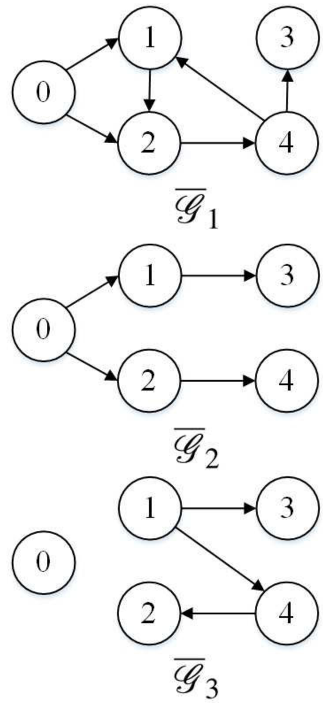

FIGURE 2. The communication topologies.

where $z_r = [z_1^r, z_2^r, \ldots, z_N^r]^T = \mathcal{H}_r^{-1} 1_N$, $y_r = [y_1^r, y_2^r, \ldots, y_N^r]^T = \mathcal{H}_r^{-T} 1_N$, $\Theta_r = diag\{y_1^r/z_1^r, y_2^r/z_2^r, \ldots, y_N^r/z_N^r\}$, $\bar{\lambda}_r \triangleq \lambda_{min}(\Theta_r^{-1}\mathcal{H}_r^T \Theta_r + \mathcal{H}_r)$, $\Psi_r \triangleq \Theta_r \otimes P_r^{-1}$, $\mu_{rl} = \frac{\lambda_{max}(\Psi_l)}{\lambda_{min}(\Psi_r)} (r \neq l), r, l \in S$.

Proof : Consider the Lyapunov functon candidate:

$$V_4(\epsilon(t), \varpi(t)) = \epsilon^T(t)(\Theta_{\varpi(t)} \otimes P_{\varpi(t)}^{-1})\epsilon(t). \qquad (39)$$

By using Theorem 2, we get that the MASs with SMSTs is ASELFC. ■

IV. SIMULATION EXAMPLE

To confirm the effectiveness of the theoretical results, synchronization of Chaos systems by using an example of Chua's circuits is provided in this section. Consider the MASs

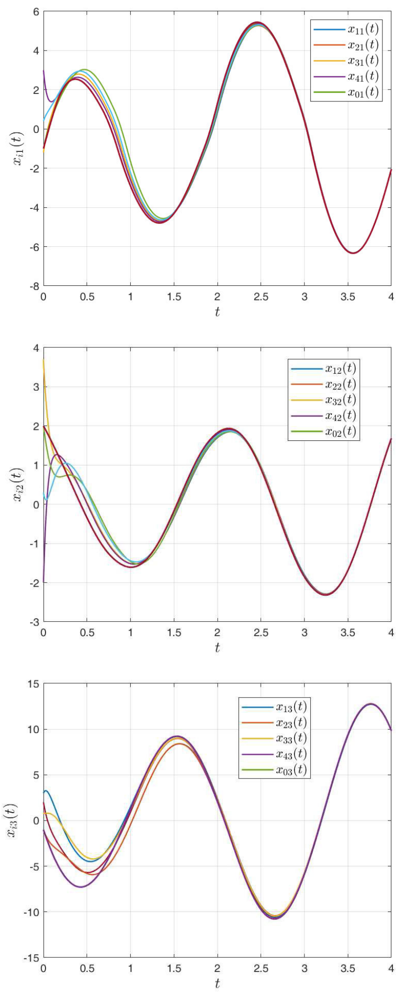

FIGURE 3. State trajectories of all agents.

with four followers and a leader, described by (1) and (2) with

$$A = \begin{bmatrix} -18/7 & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -14.286 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$f(x_i(t)) = \left[\tfrac{27}{14}(|\, x_{i1}(t)+1 \,| + |\, x_{i1}(t)-1 \,|) \; 0 \; 0 \right]^T,$$

$i \in \{0, 1, 2, 3, 4\}$. The Lipschitz matrix F is $diag\{\frac{27}{14}, 0, 0\}$.

In FIG. 1, $\varpi(t) \in S = \{1, 2, 3\}$ is the switching signal. The communication topologies are shown in FIG. 2.

The transition rate matrix of $\varpi(t)$ is assumed to be

$$P = \begin{bmatrix} 0 & 0.22 & 0.78 \\ 0.2 & 0 & 0.8 \\ 0.6 & 0.4 & 0 \end{bmatrix},$$

and the stationary distribution is $\bar{\kappa} = [0.3137, 0.2454, 0.4410]$. The expectations of sojourn time for graphs

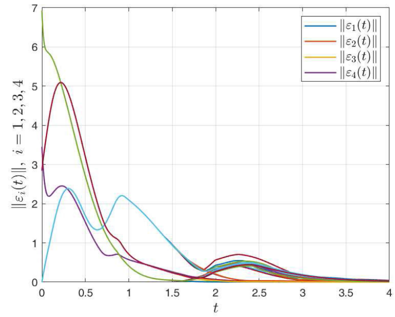

FIGURE 4. Error state.

$\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2, \bar{\mathcal{G}}_3$ is defined as $m_1 = 1.5, m_2 = 1.4, m_3 = 0.9$ seperately.

By solving the LMIs in the Theorem 2 with $\beta_1 = 0.1, \beta_2 = 0.1, \gamma_3 = 11.4, \alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 1$, we obtain

$$P_1 = \begin{bmatrix} 0.2052 & -0.0403 & -0.0790 \\ -0.0403 & 0.1255 & 0.0180 \\ -0.0790 & 0.0180 & 1.4183 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.1166 & -0.0018 & -0.0364 \\ -0.0018 & 0.0395 & 0.0071 \\ -0.0364 & 0.0071 & 0.5830 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1.5754 & -1.1558 & 0.1703 \\ -1.1558 & 1.8464 & -2.1788 \\ 0.1703 & -2.1788 & 8.0558 \end{bmatrix}.$$

Then, the controller gains matrices are got

$$K_1 = \begin{bmatrix} 5.3053 & 1.6656 & 0.2743 \\ 1.6656 & 8.5077 & -0.0151 \\ 0.2743 & -0.0151 & 0.7205 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 8.7520 & 0.3107 & 0.5428 \\ 0.3107 & 25.3569 & -0.2904 \\ 0.5428 & -0.2904 & 1.7526 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 1.6886 & 1.4906 & 0.3674 \\ 1.4906 & 2.1112 & 0.5395 \\ 0.3674 & 0.5395 & 0.2623 \end{bmatrix}.$$

The initial states of the leader and followers are given as $x_0(0) = [-1, 2, -1]^T, x_1(0) = [-1.2, 3.7, 2]^T, x_2(0) = [3, -2, 3]^T, x_3(0) = [-1, 2, -1]^T, x_4(0) = [0.4, 0.3, 0.8]^T$. State trajectories of all agents are shown in FIG. 3 and error state are shown in FIG. 4, from which it can be deduced that ASELFC of MASs is achieved.

V. CONCLUSION

In this paper, ASELFC of linear and nonlinear MASs under SMSTs that the partial switching topologies contain a DST is studied. Taking Lyapunov analysis method and stochastic technique, sufficient conditions for ASELFC are got. Then, two corollaries with SMSTs that any switching topology contains a DST with the leader as the root are provided. Notice that the proposed protocol is based on real-time data transmission. For future work, we will focus on the design of event-triggered control protocol for MASs under SMSTs.

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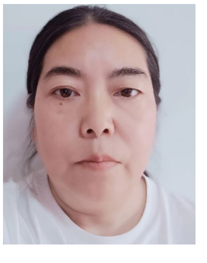
XIA XIAO received the B.S., M.S., and Ph.D. degrees from the School of Mathematics and Statistics, Zhengzhou University, in 2004, 2007, and 2017, respectively. Currently, she is with the College of Mathematics and Information Science, Zhengzhou University of Light Industry. Her research interests include nonlinear systems, stochastic systems, multi-agent systems, and network control systems.

XIHUI WU received the B.S., M.S., and Ph.D. degrees from the School of Mathematics and Statistics, Zhengzhou University, in 2015, 2017, and 2021, respectively. Currently, she is with the College of Mathematics and Information Science, Zhengzhou University of Light Industry. Her current research interests include multi-agent systems, event-triggered control, and sliding mode control.



XIAOLI HOU received the Ph.D. degree from the School of Automation, South China University of Technology. Currently, she is an Associate Professor in mathematics and information science with the Zhengzhou University of Light Industry. Her research interest includes nonlinear systems.

YUEHUA HE received the M.S. and Ph.D. degrees from the School of Mathematics and Statistics, Zhengzhou University. Currently, she is with the College of School of Science, Xuchang University. Her research interests include multi-agent flocking systems and network communication.