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THEORY

Tracking Control Design for Takagi-Sugeno Fuzzy Systems Based on Membership Function-Dependent L_∞ Performance

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ABSTRACT The design of tracking control based on membership function-dependent L_∞ performance for the systems, which are described by the Takagi-Sugeno fuzzy models, is considered. In practice, most Takagi-Sugeno fuzzy systems have such a characteristic that they work on some local subsystems most of the time and on others less time. Taking advantage of this feature, an L_∞ performance index dependent on membership function is proposed, through which better performance may be guaranteed for those local subsystems that work most of the time. By means of the newly defined L_∞ performance index, the measurable premise variables, the estimation of immeasurable premise variables, and the estimation of nonlinear function, an observer-based tracking controller is designed in the form of linear matrix inequalities, which makes full use of the information of measurable premise variables. Finally, an example is provided to verify the effectiveness of the proposed approach. In the simulation, compared with the traditional L_∞ control method, the novel method has better robust tracking performance.

INDEX TERMS L_∞ control, membership function-dependent, observer-based tracking controller, Takagi-Sugeno fuzzy system.

I. INTRODUCTION

In recent years, how to address nonlinear systems has attracted considerable attention because in practical engineering, a large number of control systems are nonlinear [1]. As a powerful tool to address nonlinear systems, the Takagi-Sugeno (T-S) fuzzy model method can approach a complex nonlinear system by superimposing local linear systems through membership functions [2], [3]. Therefore, some theories of well-established linear systems can be adapted to nonlinear systems. Many interesting results with regard to the T-S fuzzy model approach have been reported in the articles [4], [5], [6], [7], [8], [9], [10], [11]. Reference [4] develops a T-S fuzzy model with nonlinear terms such that fewer fuzzy rules can be acquired. Less computational burden can be obtained based on this modelling method. In a framework of the fuzzy Lyapunov function, a steering control scheme of fuzzy observer-based output feedback for vehicle dynamics is proposed in [5]. In [6], the $L_2 - L_\infty/H_\infty$ optimization control of a kind of nonlinear system is studied by using the T-S fuzzy method. By using discrete-time T-S fuzzy models, [11] designs a sliding mode observer-based control scheme to estimate unmeasured states of the system.

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Many T-S fuzzy systems have a common potential property in that the system works under some particular fuzzy rules most of the time, which means that the system works on some specific local subsystems frequently and on others in less time [12]. In practical engineering applications, it is meaningful to design controllers by using this property of T-S fuzzy systems. However, most of the current methods ignore this characteristic of T-S fuzzy systems. In the current T-S fuzzy controller synthesis, all local linear subsystems have a fixed robust performance index [13], [14], [15], [16], [17], [18], [19], which makes the analysis of the control synthesis problem conservative. Fortunately, the membership functions are related to the property of the T-S fuzzy systems. Therefore, a robust performance index dependent on membership functions can be constructed to achieve better global T-S fuzzy system control effects. An H_∞ performance index dependent on membership functions is proposed by [12], through which a disturbance suppression scheme can obtain better system performance. This motivates us to construct a membership function-dependent robust performance index.

On the other hand, the system is inevitably subject to external disturbances [20], which adversely affect the stability of the system. Moreover, the disturbances are persistent and bounded [21]. Hence, it is important to attenuate the persistent bounded disturbance in controller synthesis. Fortunately, an L_∞ method is provided to effectively solve this problem. The peak value of the disturbance signal can be described by the L_∞ norm, which means that the L_∞ norm can be used as the performance criterion for control synthesis to minimize the upper bound of the continuously bounded disturbance when considering the control problem of a system with persistent disturbance [22]. At present, there are relevant studies on the L_∞ method [23], [24], [25]. In [23], the optimal L_∞-gain of the stabilization problem for T-S fuzzy systems is acquired. The problem of finite frequency $L_2 - L_\infty$ filtering for T-S fuzzy systems with unknown membership functions is considered in [24]. Therefore, an observer-based controller design condition for a T-S fuzzy system with persistent disturbance based on L_∞ performance is developed in this paper. The robustness of the system to external persistent disturbance is increased.

Additionally, the task of tracking is a typical design problem [26]. In recent years, many robust fuzzy tracking control schemes have been developed for tracking control design of nonlinear systems [27], [28], [29]. Reference [27] investigates fault-tolerant tracking control for near-space-vehicles (NSVs). Robust adaptive fuzzy tracking control for nonlinear systems is studied in [28]. In [29], the design of step tracking control is considered, which is aimed at discrete nonlinear systems with finite capacity.

On the basis of the previous discussions, it is of great significance to study how to reduce the effects of persistent disturbance on the system and how to improve system performance by the property that the T-S fuzzy system is in some specific local subsystems in most cases. These points are the driving force behind our current work. This paper investigates the problem of robust tracking control for T-S fuzzy systems on the basis of L_∞ performance dependent on the membership function. The main contributions of this paper are listed as follows:

(A) A novel observer-based tracking controller scheme is presented, which enables us to track the bounded reference input. The scheme is designed by the T-S fuzzy model with local nonlinear models, which reduces the number of fuzzy rules and decreases the computational burden. Then, the objective of tracking control can be realized even when the system premise variables are partly measurable.

(B) A membership function-dependent L_∞ performance index is proposed to address the persistent disturbance. Since many T-S fuzzy systems have a common potential property that they work under some specific local subsystems most of the time, an L_∞ performance index dependent on the membership function is developed. Compared with existing results, by the novel L_∞ performance index, the property of T-S fuzzy systems can be made better use of. Furthermore, the systems have better performance against persistent disturbance.

The remainder of this paper is arranged as follows. The system description is presented in Section II. In Section III, the corresponding linear matrix inequality (LMI) conditions of the observer-based tracking controller with membership function-dependent L_∞ performance are given, where the information of the measurable premise variables is fully used. Section IV employs an example to illustrate the effectiveness of the proposed method. The conclusion is drawn in Section V.

Notation: The sign "*" denotes an ellipsis as symmetry in a matrix. The "M^T" and "M^{-1}" stand for the transpose and inverse of matrix M, respectively. The notation $M > 0 (M < 0)$ means that the matrix M is real symmetric and positive (negative) definite. For a square matrix M, $He(M)$ is defined as $M + M^T$. For a two-point $x, y \in \mathcal{R}^n$, the convex hull of the two points is $\text{co}\{x, y\} = \{\theta_1 x + \theta_2 y : \theta_1 + \theta_2 = 1, \theta_i \geq 0\}$. I and 0 are the identity matrix and the zero matrix with appropriate dimensions, respectively. $diag\{\}$ denotes a block-diagonal matrix. The L_∞ norm of the signal $\xi(t)$ is defined as $\|\xi(t)\|_\infty \triangleq \sup_t \|\xi(t)\|$, where $\|\xi(t)\| \triangleq \sqrt{\xi^T(t)\xi(t)}$.

II. SYSTEM DESCRIPTION

A. SYSTEM MODEL

This article considers a kind of nonlinear continuous-time system, which is described as:

$$\begin{aligned} \dot{x}(t) &= g_1(x(t)) + g_2(x(t))\,u(t) + g_3(x(t))\,\tau(t) \\ &\quad + g_4(x(t))\,\phi(t) \\ y(t) &= g_5(x(t)) \end{aligned} \tag{1}$$

where $x(t)$ is the system state; $u(t)$ denotes the control input; $\tau(t)$ represents the bounded external disturbance, which is assumed to be satisfied with $\tau(t) \in L_\infty$; $y(t)$ stands for the measurable output; $\phi(t) = [\phi_1(t), \cdots, \phi_s(t)]^T$ refers to a nonlinear function, where s is the number of nonlinear terms; $g_1(\cdot)$, $g_2(\cdot)$ and $g_3(\cdot)$ are nonlinear functions to be linearized; and $g_4(\cdot)$ and $g_5(\cdot)$ are linear functions.

Remark 1: Referring to [4], a number of nonlinear terms in a nonlinear system are grouped into the nonlinear term $\phi(t)$. As a result, based on the T-S fuzzy model with local nonlinear models, the T-S fuzzy system has fewer fuzzy rules, the synthesis of the controllers or the observers is simplified, and the computational burden is reduced.

To illustrate the advantages of this modelling method, an example is given as follows:

$$\dot{x}(t) = \sin(x(t)) + x^3(t) + u(t) \tag{2}$$

For the above system, the T-S fuzzy model is described as:

PlantRule 1:
IF x^2 is Γ_1
THEN

$$\dot{x}(t) = A_1x(t) + B_1u(t) + G\phi(t)$$

PlantRule 2:
IF x^2 is Γ_2
THEN

$$\dot{x}(t) = A_2x(t) + B_2u(t) + G\phi(t)$$

where $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\Gamma_1 = 1 - \frac{4x^2}{\pi^2}$, $\Gamma_2 = \frac{4x^2}{\pi^2}$, $A_1 = \frac{2}{\pi}$, $A_2 = \frac{2}{\pi} + \frac{\pi^2}{4}$, $B_1 = B_2 = 1$, $G = 1$, $\phi(t) = \sin(x(t)) - \frac{2}{\pi}x(t)$.

System (2) has two nonlinear terms. If the traditional modelling method is used, there will be four fuzzy rules, while the method in [4] has two fuzzy rules. Therefore, the fuzzy rules of the system are reduced.

In addition, for the nonlinear term $\phi(t)$, the following assumption is made:

Assumption 1 ([4]): The nonlinearities $\phi_i(t), i \in \{1, \cdots, s\}$, are sector-bounded nonlinear functions and satisfy the property as follows:

$$\phi_i(t)\,(x(t)) \in \text{co}\{0, E_ix(t)\} \quad i = 1, \cdots, s \tag{3}$$

where E_i are constant vectors with appropriate dimensions, $E = [E_1^T \cdots E_s^T]^T$.

Next, the nonlinear system (1) is described by the following T-S fuzzy model:

Plant Rule i:
IF $z_1(t)$ is $\Gamma_{i1}, \cdots, z_p(t)$ is Γ_{ip}
THEN

$$\begin{aligned}\dot{x}(t) &= A_ix(t) + B_iu(t) + F_i\tau(t) + G\phi(t)\\ y(t) &= Cx(t)\end{aligned} \tag{4}$$

where A_i, B_i, C, F_i, G $(i = 1, \cdots, r)$ stand for constant real matrices with appropriate dimensions; $z(t) = [z_1(t), \cdots, z_p(t)]^T$, $z_g(t)(g = 1, \cdots, p)$ are the premise variables, where p represents the number of premise variables; Γ_{ig} denotes a $z_g(t)$-based fuzzy set, and they are linguistic terms characterized by fuzzy membership functions $\Gamma_{ig}(z_g(t))$, where r_g is the number of $z_g(t)$-based fuzzy sets. It is easy to obtain that the fuzzy rule base consists of $r = \prod_{g=1}^{p} r_g$ IF-THEN rules.

After employing the fuzzy inference method with a singleton fuzzifier, product inference, and center average defuzzifiers, the overall fuzzy model of system (4) can be inferred as follows:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r} h_i(z(t))(A_ix(t) + B_iu(t) + F_i\tau(t)) + G\phi(t)\\ y(t) &= Cx(t)\end{aligned} \tag{5}$$

with

$$h_i(z(t)) = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))}, \; w_i(z(t)) = \prod_{g=1}^{p} \Gamma_{ig}(z_g(t)) \tag{6}$$

where $\sum_{i=1}^{r} h_i(z(t)) = 1$, $h_i(z(t)) \geq 0$.

As the work in this paper considers a system with partly measurable premise variables, it is convenient to separate the expressions of the functions $g_i(\cdot)$ depending exclusively on measurable premise variables (z_μ) and depending on at least one immeasurable premise variable (z_λ). Following the sector nonlinearity approach in [30] together with this separation, the equivalent representation of (5) is obtained as follows:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{m}\sum_{j=1}^{n} \mu_i(z_\mu(t))\lambda_j(z_\lambda(t))(A_{ij}x(t) + B_{ij}u(t)\\ &\quad + F_{ij}\tau(t)) + G\phi(t)\\ y(t) &= Cx(t)\end{aligned} \tag{7}$$

with

$$\mu_i(z_\mu(t)) = \frac{u_i(z_\mu(t))}{\sum_{i=1}^{m} u_i(z_\mu(t))}, \; u_i(z_\mu(t)) = \prod_{g=1}^{p_0} \Gamma_{ig}(z_g(t)) \tag{8}$$

$$\lambda_j(z_\lambda(t)) = \frac{v_j(z_\lambda(t))}{\sum_{j=1}^{n} v_j(z_\lambda(t))}, \; v_j(z_\lambda(t)) = \prod_{g=p_0+1}^{p} \Gamma_{jg}(z_g(t)) \tag{9}$$

where $\mu_i(z_\mu(t)), i \in \{1, \cdots, m\}$ are positive functions depending on the measurable premise variables and $\lambda_j(z_\lambda(t)), j \in \{1, \cdots, n\}$, the positive functions depending on at least one immeasurable premise variable with $m = \prod_{g=1}^{p_0} r_g$, $n = \prod_{g=p_0+1}^{p} r_g$. When all the premise variables are measurable, $\mu_i(z_\mu(t)) = h_i(z(t))$, $\lambda_j(z_\lambda(t)) = 1$, while when none of them are measurable, $\mu_i(z_\mu(t)) = 1$, $\lambda_j(z_\lambda(t)) = h_i(z(t))$. It is noted that $\sum_{i=1}^{m}\sum_{j=1}^{n} \mu_i(z_\mu(t))\lambda_j(z_\lambda(t)) = 1$, $\sum_{i=1}^{m} \mu_i(z_\mu(t)) = 1$ and $\sum_{j=1}^{n} \lambda_j(z_\lambda(t)) = 1$. In this paper, it is assumed that the premise variables $z_g(t)$, $g = 1, \cdots, p_0$ are measurable and that the premise variables $z_g(t), g = p_0 + 1, \cdots, p$ are immeasurable.

Remark 2: Inspired by [31], the case where the premise variable $z(t)$ depends on the partially measurable states of the system is considered, which implies that some states are measurable and others are not. In this case, the premise variables $z_g(t), g = 1, \cdots, p_0$ are assumed to be measurable, and the premise variables $z_g(t), g = p_0 + 1, \cdots, p$ are assumed to be immeasurable. For example, when there are three premise variables, two of which are measurable, then z_1 and z_2 are measurable premise variables, and z_3 is an immeasurable premise variable.

B. OBSERVER-BASED CONTROLLER DESIGN

To design the observer, the equivalent form of the fuzzy model (7) is given as follows:

$$
\begin{aligned}
\dot{x}(t) = & \sum_{i=1}^{m} \mu_i(z_\mu(t))(\bar{A}_i x(t) + \bar{B}_i u(t)) \\
& + \sum_{i=1}^{m}\sum_{j=1}^{n} \mu_i(z_\mu(t))\lambda_j(z_\lambda(t))((A_{ij} - \bar{A}_i)x(t) \\
& + (B_{ij} - \bar{B}_i)u(t) + F_{ij}\tau(t)) + G\phi(t) \\
y(t) = & Cx(t)
\end{aligned} \tag{10}
$$

where $\bar{A}_i$ and $\bar{B}_i$ that can be given are matrices of the same dimensions as A_{ij} and B_{ij}, respectively.

Next, similar to [31], the measurable premise variables, the estimation of immeasurable premise variables and the estimation of the nonlinear function of the fuzzy model can be used to construct a fuzzy observer as follows:

$$
\begin{aligned}
\dot{\hat{x}}(t) = & \sum_{i=1}^{m} \mu_i(z_\mu(t))(\bar{A}_i \hat{x}(t) + \bar{B}_i u(t)) \\
& + \sum_{i=1}^{m}\sum_{j=1}^{n} \mu_i(z_\mu(t))\hat{\lambda}_j(z_\lambda(t))((A_{ij} - \bar{A}_i)\hat{x}(t) \\
& + (B_{ij} - \bar{B}_i)u(t)) + G\hat{\phi}(t) \\
& + \sum_{i=1}^{m} \mu_i(z_\mu(t))L_i(y(t) - \hat{y}(t)) \\
\hat{y}(t) = & C\hat{x}(t)
\end{aligned} \tag{11}
$$

where $\hat{x}(t)$ and $\hat{y}(t)$ are the estimated state and corresponding output, respectively. $\hat{\lambda}_j(\hat{z}_\lambda(t))(j = 1, \cdots, n)$ represents the membership function, which is dependent on $\hat{z}_\lambda(t)$; $\hat{\phi}(t)$ stands for the estimation of the nonlinear function $\phi(t)$; $L_i(i = 1, \cdots, m)$ denotes the observer gain matrices to be determined.

Moreover, consider the following reference model to be tracked [32]:

$$
\dot{x}_r(t) = \sum_{i=1}^{m}\sum_{j=1}^{n} \mu_i(z_\mu(t))\lambda_j(z_\lambda(t))\tilde{A}_{ij}x_r(t) + r(t) \tag{12}
$$

where $x_r(t)$ is the desired reference state to be tracked, $\tilde{A}_{ij}$ specifies asymptotically stable matrices and $r(t)$ is the bounded reference input.

Then, the following fuzzy control scheme is adopted for the T-S fuzzy system (10).

$$
u(t) = \sum_{i=1}^{m} \mu_i(z_\mu(t))K_i\left(\hat{x}(t) - x_r(t)\right) \tag{13}
$$

where $K_i(i = 1, \cdots, m)$ are the fuzzy control gains to be determined and only the measurable premise variables are used.

Remark 3: It should be noted that since some of the premise variables are immeasurable, the designed controller depends on measurable premise variables. In (13), $\mu_i(z_\mu(t))$ is known, which contains the information of measurable premise variables. Therefore, the controller design method effectively utilizes the information of measurable premise variables. At the same time, the controller (13) shares the same premise variables with the fuzzy model (10). Then, Lemma 1 can be used to obtain less conservative results.

Next, denote

$$
e_0(t) = x(t) - \hat{x}(t),\ e_r(t) = x(t) - x_r(t) \tag{14}
$$

where $e_0(t)$ and $e_r(t)$ represent the state estimation error and tracking error, respectively.

On the basis of (10) and (11), the derivative of the state estimation error is given by

$$
\begin{aligned}
\dot{e}_0(t) = & \dot{x}(t) - \dot{\hat{x}}(t) \\
= & \sum_{i=1}^{m} \mu_i(z_\mu(t))\left(\bar{A}_i - L_iC\right)e_0(t) + G\left(\phi(t) - \hat{\phi}(t)\right) \\
& + \sum_{i=1}^{m}\sum_{j=1}^{n} \mu_i(z_\mu(t))\lambda_j(z_\lambda(t))(F_{ij}\tau(t) + \left(A_{ij} - \bar{A}_i\right)\varpi_j(t) \\
& + \left(B_{ij} - \bar{B}_i\right)\Delta_j(t))
\end{aligned} \tag{15}
$$

where $\varpi_j(t) = \lambda_j(z_\lambda(t))x(t) - \hat{\lambda}_j(z_\lambda(t))\hat{x}(t)$, $\Delta_j(t) = \left(\lambda_j(z_\lambda(t)) - \hat{\lambda}_j(z_\lambda(t))\right)u(t)$.

Similarly, the tracking error dynamic can be expressed as

$$
\begin{aligned}
\dot{e}_r(t) = & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m} \mu_i(z_\mu(t))\lambda_j(z_\lambda(t))\mu_l(z_\mu(t))(-B_{ij}K_l e_0(t) \\
& + (A_{ij} + B_{ij}K_l)e_r(t) + (A_{ij} - \tilde{A}_{ij})x_r(t) + F_{ij}\tau(t)) \\
& + G\phi(t) - r(t)
\end{aligned} \tag{16}
$$

Combining (15) and (16), the fuzzy augmentation system can be expressed as follows:

$$
\begin{aligned}
\dot{\xi}(t) = & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m} \mu_i(z_\mu(t))\lambda_j(z_\lambda(t))\mu_l(z_\mu(t))\hat{A}_{ijl}\xi(t) \\
& + \sum_{i=1}^{m}\sum_{j=1}^{n} \mu_i(z_\mu(t))\lambda_j(z_\lambda(t))\hat{B}_{ij}\tilde{\tau}(t) + \begin{bmatrix} \omega_\Delta \\ 0 \end{bmatrix} \\
& + G\begin{bmatrix} \tilde{\phi}(t) \\ \phi(t) \end{bmatrix}
\end{aligned} \tag{17}
$$

where

$$\xi(t) = [e_0^T(t) \quad e_r^T(t)]^T, \tilde{\tau}(t) = [\tau^T(t) \quad x_r^T(t) \quad r^T(t)]^T,$$
$$\tilde{\phi}(t) = \phi(t) - \hat{\phi}(t),$$
$$\omega_\Delta = \sum_{i=1}^{m} \mu_i(z_\mu(t)) \sum_{j=1}^{n} \left((A_{ij} - \bar{A}_i)\varpi_j(t) + (B_{ij} - \bar{B}_i)\Delta_j(t)\right)$$
$$\hat{A}_{ijl} = \begin{bmatrix} \bar{A}_i - L_i C & 0 \\ -B_{ij}K_l & A_{ij} + B_{ij}K_l \end{bmatrix} \hat{B}_{ij} = \begin{bmatrix} F_{ij} & 0 & 0 \\ F_{ij} & A_{ij} - \tilde{A}_{ij} & -I \end{bmatrix}.$$

Assumption 2: 1) The input $u(t)$ is bounded $\|u(t)\| \le \eta$
2) $\|\varpi_j\| = \|\lambda_j x - \hat{\lambda}_j \hat{x}\| \le a_j \|x - \hat{x}\|$
3) $\|\lambda_j - \hat{\lambda}_j\| \le b_j \|x - \hat{x}\|$
4) $\|\tilde{\phi}\| = \|\phi - \hat{\phi}\| \le \varphi \|x - \hat{x}\|$
where a_j, b_j and φ are Lipschitz constants, which are known scalars.

C. PRELIMINARIES

Before obtaining the main result, the membership function-dependent L_∞ performance index will be given.

Definition 1: Under zero initial conditions, the L_∞ performance index γ can be defined as

$$\|\xi(t)\|_\infty \le \gamma(\mu_i, \lambda_j)\|\tau(t)\|_\infty, \forall \tau(t) \in L_\infty \quad (18)$$

where μ_i and λ_j are the membership functions. The definition of the L_∞ norm is given in the Notation. It is obvious that the L_∞ performance index γ depends on the membership functions μ_i and λ_j in Definition 1.

Remark 4: Inspired by [12], the membership function-dependent L_∞ performance index is developed, which can achieve better system performance. The specific analysis of the reasons is given as follows: in practice, many T-S fuzzy systems work under some fuzzy rules most of the time (since the system states will stay in a neighborhood of the origin) and do not work under other fuzzy rules frequently. In other words, these T-S fuzzy systems work on some subsystems most of the time, and others work at low frequencies. This case is illustrated by the following example (19). In this case, if some subsystems that work frequently have a relatively small disturbance attenuation index and others appropriately relax the index, better system performance will be achieved compared with the traditional L_∞ method in [33]. Additionally, the H_∞ method in [12] can only deal with energy-bounded disturbances but is not applicable for magnitude-bounded disturbances. However, the developed approach can deal with persistent disturbances, which overcomes the shortcomings of existing methods.

To illustrate the case above-mentioned, the following example is given:

$$\dot{x}(t) = \cos(x(t))x(t) + x(t) + u(t) \quad (19)$$

For the system (19), the following T-S fuzzy model is described as:

Rule 1:
IF $\cos(x)$ is Γ_1
THEN

$$\dot{x}(t) = A_1 x(t) + B_1 u(t)$$

Rule 2:
IF $\cos(x)$ is Γ_2
THEN

$$\dot{x}(t) = A_2 x(t) + B_2 u(t)$$

where $x \in [-\frac{\pi}{3}, \frac{\pi}{3}]$, $\Gamma_1 = 2 - 2\cos(x)$, $\Gamma_2 = 2\cos(x) - 1$, $A_1 = \frac{3}{2}$, $A_2 = 2$, and $B_1 = B_2 = 1$. When the system states stay near the origin, the system works on the subsystem corresponding to Rule 2.

In this case, if some subsystems that work frequently have a relatively small disturbance attenuation index and others appropriately relax the index, better system performance will be achieved.

In addition, the following lemma is also needed in the derivation of the main result.

Lemma 1 ([34]): If the following conditions hold for $i, l \in \{1, \cdots, m\}, j \in \{1, \cdots, n\}$:

$$\Phi_{iji} < 0$$
$$\frac{1}{r-1}\Phi_{iji} + \frac{1}{2}\left(\Phi_{ijl} + \Phi_{lji}\right) < 0 \quad i \ne l$$

then the following inequality holds:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m} \mu_i \lambda_j \mu_l \Phi_{ijl} < 0$$

where $\mu_i (i = 1, \cdots, m)$, $\lambda_j (j = 1, \cdots, n)$ satisfy $0 \le \mu_i \le 1$, $0 \le \lambda_j \le 1$, $\sum_{i=1}^{m} \mu_i = 1$, $\sum_{j=1}^{n} \lambda_j = 1$.

D. PROBLEM FORMULATION

In this paper, the aim is to design the gains of the controller and observer so that the following two requirements are satisfied simultaneously.

1) The fuzzy augmented system (17) is asymptotically stable when the disturbance $\tilde{\tau}(t) = 0$;
2) The fuzzy augmented system (17) satisfies the membership function-dependent L_∞ performance given below when the disturbance $\tilde{\tau}(t) \ne 0$.

$$\sup_{\tilde{\tau}(t) \in L_\infty} \frac{\|\xi(t)\|_\infty}{\|\tilde{\tau}(t)\|_\infty} < \gamma(\mu_i, \lambda_j) \quad (20)$$

III. MAIN RESULT

In this section, sufficient conditions for designing the observer-based tracking controller based on membership function-dependent L_∞ performance for a T-S fuzzy system with partly measurable premise variables are given.

Theorem 1: For given positive scalars a_j, b_j, φ, ε, ρ, κ, α and c, the fuzzy augmented system (17) is asymptotically stable with membership function-dependent L_∞ performance if there exists positive definite matrices $P_1 = P_1^T > 0$, $Q_2 = Q_2^T > 0$, symmetric matrices $X_l (i = 1, \cdots, m)$, M, $Z_i (i = 1, \cdots, m)$, diagonal matrix $\Lambda > 0$, such that the following inequalities hold for $i, l \in \{1, \cdots, m\}, j \in \{1, \cdots, n\}, x \in \{1, \cdots, m-1\}, y \in \{1, \cdots, n-1\}$:

$$\Theta_{iji} < 0$$

$$\frac{1}{r-1}\Theta iji + \frac{1}{2}(\Theta ijl + \Theta lji) < 0 \quad i \neq l \tag{21}$$

$$\Phi_i < 0 \tag{22}$$

$$\begin{bmatrix} I-\delta^2 P_1 & 0 & 0 \\ 0 & -\varepsilon\delta^2 Q_2 & Q_2 \\ 0 & Q_2 & -I \end{bmatrix} < 0 \tag{23}$$

where

$$\Theta_{ijl} = \begin{bmatrix} He(A_{ij}Q_2+B_{ij}X_l)+\alpha Q_2 & A_{ij}-Ar_{ij} & -I & F_{ij} & G\Lambda+Q_2E^T \\ * & -\frac{\Omega_{ij}}{\varepsilon} & 0 & 0 & E^T \\ * & * & -\frac{\Omega_{ij}}{\varepsilon} & 0 & 0 \\ * & * & * & -\frac{\Omega_{ij}}{\varepsilon} & 0 \\ * & * & * & * & -2\Lambda \end{bmatrix}$$

$$\Phi_i = \begin{bmatrix} \Upsilon_i & P_1+M-\kappa\bar{A}_i^T M^T+\kappa C^T Z_i^T & -MG & -M\Pi_A & -M\Pi_B \\ * & \kappa(M+M^T) & -\kappa MG & -\kappa M\Pi_A & -\kappa M\Pi_B \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -I \end{bmatrix}$$

$$\Omega_{ij} = \rho\mu_i\lambda_j, 1 \le i \le x, 1 \le j \le y$$

$$\Omega_{ij} = \mu_i\lambda_j, x < i \le m, y < j \le n$$

$$\Upsilon_i = He(-M\bar{A}_i + Z_iC) + \alpha P_1 + \varphi^2 I + \sum_{j=1}^{n}(a_j^2 + b_j^2\eta^2)I$$

$$\Pi_A = \begin{bmatrix} A_{i1}-\bar{A}_i & A_{i2}-\bar{A}_i & \cdots & A_{in}-\bar{A}_i \end{bmatrix}$$

$$\Pi_B = \begin{bmatrix} B_{i1}-\bar{B}_i & B_{i2}-\bar{B}_i & \cdots & B_{in}-\bar{B}_i \end{bmatrix}$$

The membership function-dependent L_∞ performance index is described as follows:

$$\|\xi\|_\infty < \gamma(\mu_i,\lambda_j)\|\tilde{\tau}\|_\infty \tag{24}$$

where $\gamma(\mu_i,\lambda_j) = \sqrt{\frac{\bar{c}}{\alpha}}\delta = \sqrt{\frac{\left(\sum_{i=1}^{x}\sum_{j=1}^{y}\rho\mu_i\lambda_j + \sum_{i=x+1}^{m}\sum_{j=y+1}^{n}\mu_i\lambda_j\right)c}{\alpha}}\delta$.

Next, the gains of the controller and observer are obtained by

$$K_l = X_l Q_2^{-1},\ L_i = M^{-1}Z_i, i,\ l = 1, \cdots, m$$

Proof: The proof is divided into two parts, which are proved by deriving Lyapunov conditions and solving them using the numerical techniques based on LMIs. The first part is to prove that the fuzzy augmented system (17) is asymptotically stable. In the second part, the augmented system (17) guarantees the membership function-dependent L_∞ performance index (24).

(A) Consider the following Lyapunov functional candidate:

$$V(\xi(t)) = \xi(t)^T P\xi(t) \tag{25}$$

where $P = \begin{bmatrix} P_1 & \\ & \varepsilon P_2 \end{bmatrix}$ with $P = P^T > 0$, $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$.

To guarantee the performance (24), the following condition needs to be demonstrated:

$$\dot{V} + \alpha V - \bar{c}\tilde{\tau}^T\tilde{\tau} < 0 \tag{26}$$

where $\bar{c} = \left(\sum_{i=1}^{x}\sum_{j=1}^{y}\rho\mu_i\lambda_j + \sum_{i=x+1}^{m}\sum_{j=y+1}^{n}\mu_i\lambda_j\right)c$.

Taking the derivative of the Lyapunov function, we have

$$\begin{aligned}\dot{V} &= \dot{\xi}^T P\xi + \xi^T P\dot{\xi} \\ &= \dot{e_0}^T P_1 e_0 + \varepsilon\dot{e_r}^T P_2 e_r + e_0^T P_1\dot{e_0} + \varepsilon e_r^T P_2\dot{e_r}\end{aligned} \tag{27}$$

It is clear from (15) that for the free-weighting matrices M_1 and M_2, the following zero-equation holds [35]:

$$\begin{aligned}0 = 2(e_0^T M_1 + \dot{e_0}^T M_2)(\dot{e_0} - \sum_{i=1}^{m}\mu_i(\bar{A}_i - L_iC)e_0 \\ - \sum_{i=1}^{m}\sum_{j=1}^{n}\mu_i\lambda_j F_{ij}\tau - G\tilde{\phi} - \omega_\Delta)\end{aligned} \tag{28}$$

where $M_1 = M$; $M_2 = \kappa M_1$; M is any symmetric matrix; and κ is an arbitrary constant.

Next, adding the right of (28) to the right of (27), we have

$$\begin{aligned}\dot{V} &= \dot{e_0}^T P_1 e_0 + \varepsilon\dot{e_r}^T P_2 e_r + e_0^T P_1\dot{e_0} + \varepsilon e_r^T P_2\dot{e_r} \\ &+ 2(e_0^T M_1 + \dot{e_0}^T M_2)(\dot{e_0} - \sum_{i=1}^{m}\mu_i(\bar{A}_i - L_iC)e_0 \\ &- \sum_{i=1}^{m}\sum_{j=1}^{n}\mu_i\lambda_j F_{ij}\tau - G\tilde{\phi} - \omega_\Delta)\end{aligned} \tag{29}$$

According to Assumption 1 and the method in [4], the following inequality can be derived:

$$\phi^T(t)\Lambda^{-1}Ex(t) - \phi^T(t)\Lambda^{-1}\phi(t) \ge 0 \tag{30}$$

where E is given in Assumption 1, Λ is a diagonal matrix and $\Lambda > 0$.

Based on Assumption 2, the following inequalities can be obtained:

$$a_j^2 e_0^T e_0 - \varpi_j^T\varpi_j \ge 0 \tag{31}$$

$$b_j^2\eta^2 e_0^T e_0 - \Delta_j^T\Delta_j \ge 0 \tag{32}$$

$$\varphi^2 e_0^T e_0 - \tilde{\phi}^T\tilde{\phi} \ge 0 \tag{33}$$

Combined with (30)-(33), it is clear that based on the S-procedure, the condition (26) is satisfied only if

$$\begin{aligned}\dot{V} + \alpha V - \bar{c}\tilde{\tau}^T\tilde{\tau} + \sum_{j=1}^{n}(a_j^2 e_0^T e_0 - \varpi_j^T\varpi_j + b_j^2\eta^2 e_0^T e_0 \\ - \Delta_j^T\Delta_j) + \varphi^2 e_0^T e_0 - \tilde{\phi}^T\tilde{\phi} + 2\varepsilon\phi^T\Lambda^{-1}Ex - 2\varepsilon\phi^T \\ \Lambda^{-1}\phi < 0\end{aligned} \tag{34}$$

Define $\chi^T = [e_r^T \quad x_r^T \quad r^T \quad \tau^T \quad \phi^T \quad e_0^T \quad \dot{e}_0^T \quad \tilde{\phi}^T \; \varpi^T \; \Delta^T]^T$, with (34), the following inequality holds:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m}\mu_i\lambda_j\mu_l\chi^T\Xi_{ijl}\chi < 0 \tag{35}$$

where, as shown in the equation at the bottom of the next page, $\Upsilon_i = He(-M\bar{A}_i + Z_iC) + \alpha P_1 + \varphi^2 I + \sum_{j=1}^{n}(a_j^2 + b_j^2\eta^2)I$, $Z_i = ML_i$.

We rewrite (35) as follows:

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m}\mu_i\lambda_j\mu_l\chi^T\Xi_{ijl}\chi \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m}\mu_i\lambda_j\mu_l(e_r^T(A_{ij}+B_{ij}K_l)^T\varepsilon P_2 e_r \\
&\quad + e_r^T\varepsilon P_2(A_{ij}+B_{ij}K_l)e_r + \alpha e_r^T\varepsilon P_2 e_r + 2e_r^T\varepsilon P_2(A_{ij} \\
&\quad - Ar_{ij})x_r - 2\,e_r^T\varepsilon P_2 r + 2\,e_r^T\varepsilon P_2 F_{ij}\tau + 2\,e_r^T\varepsilon P_2 G\psi \\
&\quad + 2\varepsilon e_r^T E^T\Lambda^{-1}\phi - \Omega_{ij}x_r^T x_r - \Omega_{ij}r^T r - \Omega_{ij}\tau^T\tau \\
&\quad - 2\varepsilon\phi^T\Lambda^{-1}\phi) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m}\mu_i\lambda_j\mu_l(2e_r^T\varepsilon P_2(-B_{ij}K_l)e_0 \\
&\quad - 2\tau^T F_{ij}^T M_1^T e_0 - 2\tau^T F_{ij}^T M_2^T\dot{e}_0) \\
&\quad + \sum_{i=1}^{m}\mu_i(e_0^T M_1(-\bar{A}_i+L_iC)e_0 + e_0^T(-\bar{A}_i+L_iC)^T M_1^T e_0 \\
&\quad + \alpha e_0^T P_1 e_0 + \varphi^2 e_0^T e_0 + e_0^T\sum_{j=1}^{n}a_j^2 e_0 + e_0^T\sum_{j=1}^{n}b_j^2\eta^2 e_0 \\
&\quad + 2\,e_0^T P_1\dot{e}_0 + 2\,e_0^T M_1\dot{e}_0 + 2\,e_0^T(-\bar{A}_i+L_iC)^T M_2^T\dot{e}_0 \\
&\quad - 2\,e_0^T M_1 G\tilde{\phi} - 2\,e_0^T M_1\Pi_A\varpi - 2\,e_0^T M_1\Pi_B\Delta - 2\dot{e}_0^T M_2 G\tilde{\phi} \\
&\quad - 2\dot{e}_0^T M_2\Pi_A\varpi - 2\dot{e}_0^T M_2\Pi_B\Delta \\
&\quad + 2\dot{e}_0^T M_2\dot{e}_0 - \tilde{\phi}^T\tilde{\phi} - \varpi^T\varpi - \Delta^T\Delta) = \dot{e}_0{}^T P_1 e_0 \\
&\quad + \varepsilon\dot{e}_r{}^T P_2 e_r + e_0^T P_1\dot{e}_0 + \varepsilon e_r^T P_2\dot{e}_r \\
&\quad + 2(e_0^T M_1 + \dot{e}_0{}^T M_2)(\dot{e}_0 - \sum_{i=1}^{m}\mu_i(\bar{A}_i - L_iC)e_0 \\
&\quad - \sum_{i=1}^{m}\sum_{j=1}^{n}\mu_i\lambda_j F_{ij}\tau - G\tilde{\phi} - \omega_\Delta) + \alpha e_0^T P_1 e_0 + \alpha e_0^T\varepsilon P_2 e_0 \\
&\quad - \bar{c}x_r^T x_r - \bar{c}r^T r - \bar{c}\tau^T\tau + \sum_{j=1}^{n}(a_j^2 e_0^T e_0 - \varpi_j^T\varpi_j \\
&\quad + b_j^2\eta^2 e_0^T e_0 - \Delta_j^T\Delta_j) + \varphi^2 e_0^T e_0 - \tilde{\phi}^T\tilde{\phi} + 2\varepsilon\phi^T\Lambda^{-1}Ex \\
&\quad - 2\varepsilon\phi^T\Lambda^{-1}\phi = \dot{V} + \alpha V - \bar{c}\tilde{\tau}^T\tilde{\tau} + \sum_{j=1}^{n}(a_j^2 e_0^T e_0 - \varpi_j^T\varpi_j \\
&\quad + b_j^2\eta^2 e_0^T e_0 - \Delta_j^T\Delta_j) + \varphi^2 e_0^T e_0 - \tilde{\phi}^T\tilde{\phi} \\
&\quad + 2\varepsilon\phi^T\Lambda^{-1}Ex - 2\varepsilon\phi^T\Lambda^{-1}\phi < 0
\end{aligned}$$

Furthermore, the inequality (35) holds only if

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m}\mu_i\lambda_j\mu_l\Xi_{ijl} < 0 \tag{36}$$

The Schur complement is applied to (36), which yields

$$\varepsilon\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m}\mu_i\lambda_j\mu_l\Xi_{ijl}^{11} < 0 \tag{37}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m}\mu_i\lambda_j\mu_l\left(\Xi_{ijl}^{22} - \varepsilon\Xi_{ijl}^{12}(\Xi_{ijl}^{11})^{-1}(\Xi_{ijl}^{12})^T\right) < 0 \tag{38}$$

The inequality (37) can be expressed in detail as follows:

$$\begin{aligned}
&\varepsilon\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m}\mu_i\lambda_j\mu_l\Xi_{ijl}^{11} = \varepsilon\sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{l=1}^{m}\mu_i\lambda_j\mu_l \\
&\begin{bmatrix} He(P_2A_{ij}+P_2B_{ij}K_l)+\alpha P_2 & P_2(A_{ij}-Ar_{ij}) & -P_2 & P_2F_{ij} & P_2G+E^T\Lambda^{-1} \\ * & -\frac{\rho\mu_i\lambda_j}{\varepsilon} & 0 & 0 & E^T\Lambda^{-1} \\ * & * & -\frac{\rho\mu_i\lambda_j}{\varepsilon} & 0 & 0 \\ * & * & * & -\frac{\rho\mu_i\lambda_j}{\varepsilon} & 0 \\ * & * & * & * & -2\Lambda^{-1} \end{bmatrix} \\
&+ \varepsilon\sum_{i=x+1}^{m}\sum_{j=y+1}^{n}\sum_{l=1}^{m}\mu_i\lambda_j\mu_l
\end{aligned}$$

$$\varpi^T = [\varpi_1^T \quad \varpi_2^T \quad \cdots \quad \varpi_n^T], \quad \Delta^T = [\Delta_1^T \quad \Delta_2^T \quad \cdots \quad \Delta_n^T]$$

$$\Xi_{ijl} = \begin{bmatrix} \varepsilon\Xi_{ijl}^{11} & \varepsilon\Xi_{ijl}^{12} \\ * & \Xi_{ijl}^{22} \end{bmatrix}$$

$$\Xi_{ijl}^{11} = \begin{bmatrix} He(P_2A_{ij}+P_2B_{ij}K_l)+\alpha P_2 & P_2(A_{ij}-Ar_{ij}) & -P_2 & P_2F_{ij} & P_2G+E^T\Lambda^{-1} \\ * & -\frac{\Omega_{ij}}{\varepsilon} & 0 & 0 & E^T\Lambda^{-1} \\ * & * & -\frac{\Omega_{ij}}{\varepsilon} & 0 & 0 \\ * & * & * & -\frac{\Omega_{ij}}{\varepsilon} & 0 \\ * & * & * & * & -2\Lambda^{-1} \end{bmatrix}$$

$$\Omega_{ij} = \rho\mu_i\lambda_j, 1 \le i \le x, 1 \le j \le y$$

$$\Omega_{ij} = \mu_i\lambda_j, x < i \le m, y < j \le n$$

$$\Xi_{ijl}^{12} = \begin{bmatrix} P_2(-B_{ij}K_l) & 0 & 0\ 0\ 0 \\ 0 & 0 & 0\ 0\ 0 \\ 0 & 0 & 0\ 0\ 0 \\ -F^TM_1^T/\varepsilon & -F^TM_2^T/\varepsilon & 0\ 0\ 0 \\ 0 & 0 & 0\ 0\ 0 \end{bmatrix}$$

$$\Xi_{ijl}^{22} = \begin{bmatrix} \Upsilon_i & P_1+M-\kappa\bar{A}_i^TM^T+\kappa C^TZ_i^T & -MG & -M\Pi_A & -M\Pi_B \\ * & \kappa(M+M^T) & -\kappa MG & -\kappa M\Pi_A & -\kappa M\Pi_B \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -I \end{bmatrix}$$

$$\begin{bmatrix} He(P_2A_{ij}+P_2B_{ij}K_l)+\alpha P_2 & P_2(A_{ij}-Ar_{ij}) & -P_2 & P_2F_{ij} & P_2G+E^T\Lambda^{-1} \\ * & -\frac{\mu_i\lambda_j}{\varepsilon} & 0 & 0 & E^T\Lambda^{-1} \\ * & * & -\frac{\mu_i\lambda_j}{\varepsilon} & 0 & 0 \\ * & * & * & -\frac{\mu_i\lambda_j}{\varepsilon} & 0 \\ * & * & * & * & -2\Lambda^{-1} \end{bmatrix}$$
$$= \varepsilon\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m}\mu_i\lambda_j\mu_l \begin{bmatrix} He(P_2A_{ij}+P_2B_{ij}K_l)+\alpha P_2 & P_2(A_{ij}-Ar_{ij}) & -P_2 & P_2F_{ij} & P_2G+E^T\Lambda^{-1} \\ * & -\frac{\Omega_{ij}}{\varepsilon} & 0 & 0 & E^T\Lambda^{-1} \\ * & * & -\frac{\Omega_{ij}}{\varepsilon} & 0 & 0 \\ * & * & * & -\frac{\Omega_{ij}}{\varepsilon} & 0 \\ * & * & * & * & -2\Lambda^{-1} \end{bmatrix} < 0$$

Furthermore, pre- and post-multiplying (37) by $diag\{P_2^{-1}, I, I, I, \Lambda\}$ and its transpose, it follows that

$$\varepsilon\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m}\mu_i\lambda_j\mu_l \begin{bmatrix} He(A_{ij}Q_2+B_{ij}X_l)+\alpha Q_2 & A_{ij}-Ar_{ij} & -I & F_{ij} & G\Lambda+Q_2E^T \\ * & -\frac{\Omega_{ij}}{\varepsilon} & 0 & 0 & E^T \\ * & * & -\frac{\Omega_{ij}}{\varepsilon} & 0 & 0 \\ * & * & * & -\frac{\Omega_{ij}}{\varepsilon} & 0 \\ * & * & * & * & -2\Lambda \end{bmatrix} < 0 \tag{39}$$

where $Q_2 = P_2^{-1}, X_l = K_lP_2^{-1}$.

Next, applying Lemma 1 to (39), the following can be obtained:

$$\begin{aligned} &\Theta_{iji} < 0 \quad i = 1, \cdots, m, j = 1, \cdots, n \\ &\frac{1}{r-1}\Theta iji + \frac{1}{2}\left(\Theta ijl + \Theta lji\right) < 0 \quad l = 1, \cdots, m, i \neq l \end{aligned} \tag{40}$$

where Θ_{iji} is given in (21).

Moreover, by selecting ε small enough, (38) can be written as follows [36]:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{m}\mu_i\lambda_j\mu_l\Xi_{ijl}^{22} < 0 \tag{41}$$

Since $\sum_{j=1}^{n}\lambda_j = 1$ and $\sum_{l=1}^{m}\mu_l = 1$, it can be derived from (41) that

$$\sum_{i=1}^{m}\mu_i\Phi_i < 0 \tag{42}$$

which can be guaranteed by the following inequality:

$$\Phi_i < 0 \tag{43}$$

where Φ_i is given in (22).

Therefore, it shows that once the conditions (21) - (23) hold, the inequality (26) is satisfied, which implies that the fuzzy augmented system (17) is asymptotically stable.

(B) Following [37], the proof process in Part B is as follows. Through mathematical operations, the inequality (26) can be derived as:

$$\dot{V} + \alpha V < \bar{c}\tilde{\tau}^T\tilde{\tau} \tag{44}$$

namely,

$$\dot{V} + \alpha V < \bar{c}\|\tilde{\tau}\|_\infty^2 \tag{45}$$

Then, multiplying both sides of inequality (45) by $e^{\alpha t}$ simultaneously, we obtain

$$e^{\alpha t}(\dot{V} + \alpha V) < \bar{c}e^{\alpha t}\|\tilde{\tau}\|_\infty^2 \tag{46}$$

Next, integrating inequality (46) from 0 to t, the following can be obtained:

$$\int_0^t \frac{d(e^{\alpha t}V)}{dt}\,dt < \frac{\bar{c}}{\alpha}(e^{\alpha t} - 1)\|\tilde{\tau}\|_\infty^2 \tag{47}$$

namely,

$$e^{\alpha t}V < V(0) + \frac{\bar{c}}{\alpha}(e^{\alpha t} - 1)\|\tilde{\tau}\|_\infty^2 \tag{48}$$

Then, multiplying the inequality (48) on the left and right by $e^{-\alpha t}$, we can obtain

$$V < e^{-\alpha t}V(0) + \frac{\bar{c}}{\alpha}(1 - e^{-\alpha t})\|\tilde{\tau}\|_\infty^2 < e^{-\alpha t}V(0) + \frac{\bar{c}}{\alpha}\|\tilde{\tau}\|_\infty^2 \tag{49}$$

Under the initial condition of zero, the following inequality can be obtained:

$$V < \frac{\bar{c}}{\alpha}\|\tilde{\tau}\|_\infty^2 \tag{50}$$

Substituting the definition of $V(t)$ into (50) yields

$$\xi^TP\xi < \frac{\bar{c}}{\alpha}\|\tilde{\tau}\|_\infty^2 \tag{51}$$

Next, let

$$I - \delta^2P < 0 \tag{52}$$

Then, pre- and post-multiplying (52) by $diag\{I, P_2^{-1}\}$ and its transpose yields

$$\begin{bmatrix} I - \delta^2P_1 & 0 \\ 0 & P_2^{-T}P_2^{-1} - \varepsilon\delta^2P_2^{-1} \end{bmatrix} < 0 \tag{53}$$

Applying the Schur complement to (53), the inequality can be obtained as follows:

$$\begin{bmatrix} I - \delta^2P_1 & 0 & 0 \\ 0 & -\varepsilon\delta^2Q_2 & Q_2 \\ 0 & Q_2 & -I \end{bmatrix} < 0 \tag{54}$$

where $Q_2 = P_2^{-1}$.

Moreover, substituting (52) into (51), we have

$$\xi^T\xi < \delta^2\xi^TP\xi < \frac{\bar{c}}{\alpha}\delta^2\|\tilde{\tau}\|_\infty^2 \tag{55}$$

namely,

$$\|\xi\|_\infty^2 < \frac{\bar{c}}{\alpha}\delta^2\|\tilde{\tau}\|_\infty^2 \tag{56}$$

Finally, it implies that the following L_∞ performance index is guaranteed:

$$\|\xi\|_\infty < \sqrt{\frac{\bar{c}}{\alpha}}\delta\|\tilde{\tau}\|_\infty \tag{57}$$

Thus, the proof is complete.

Remark 5: The conventional L_∞ performance index in [33] can be described as:

$$\|\xi\|_\infty < \gamma\|\tilde{\tau}\|_\infty \tag{58}$$

where $\gamma = \sqrt{\frac{c}{\alpha}}\delta$. The c in (58) is a constant, while $\bar{c}$ in (24) is a variable that depends on membership functions. In this paper, the subsystems that work most of the time obtain a smaller performance index, and the performance index for others are appropriately relaxed, which means that the performance of the subsystems that work most of the time is improved by losing the performance of others. It also makes sense in the control of a real system. In addition, the overall performance index $\gamma = \sqrt{\frac{\bar{c}}{\alpha}}\delta$ in Theorem 1 is reduced compared to all subsystems with the same performance index $\gamma = \sqrt{\frac{c}{\alpha}}\delta$ in [33]. Hence, the result obtained by Theorem 1 has less conservatism and better overall performance.

Remark 6: The observer-based tracking controller constructed in this paper adopts the T-S fuzzy control technology, which is one of the intelligent control technologies, and has human thinking of fuzzy reasoning and decision-making. This means that the tracking process is intelligent and smart. Meanwhile, the control scheme uses observer error and tracking error to realize closed-loop output feedback, which does not require manual participation and realizes automatic tracking control. Therefore, the tracking process is intelligent, smart, and automatic.

Remark 7: To highlight the advantages of the work in this paper, recent similar articles in the subject area are briefly compared as follows: first, in [37], the problem of L_∞ fault estimation and fault-tolerant control for T-S fuzzy systems with measurable premise variables is studied, but the case in which some premise variables are immeasurable is not considered. Due to the limitations of sensors in practice, some premise variables of the system are immeasurable. Therefore, T-S fuzzy systems with partly measurable premise variables are considered in this paper, which improves the application range of the method. Second, the local stabilization problem for T'S fuzzy systems with partly measurable premise variables and time-varying delay is investigated in [38]. However, it does not quantitatively describe the robustness. Therefore, the control scheme designed in this paper uses the robust performance index, which can quantitatively describe the robustness of the system and provide a direction for the optimization of the control scheme.

IV. EXAMPLE

This section provides an example to demonstrate the validity of the presented approach.

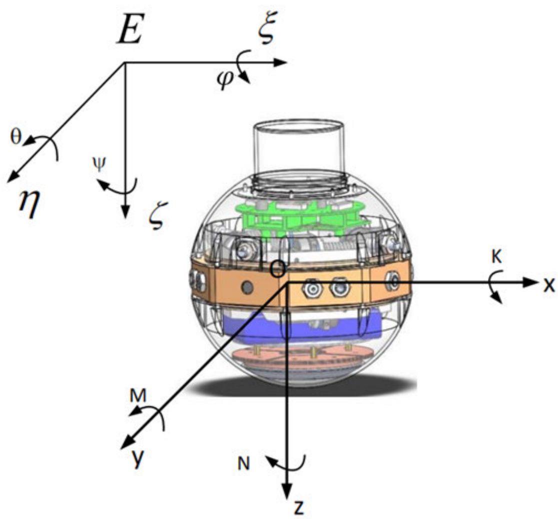

FIGURE 1. Coordinate system of the spherical robot.

A spherical robot system referred to in [39] is considered, whose coordinate system is shown in Fig. 1. In this paper, referring to the simplified kinematic and dynamic model of the spherical robot, the horizontal plane model is described as:

$$\begin{aligned}
\dot{\xi} &= \cos\psi\bar{u} - \sin\psi v\\
\dot{\eta} &= \sin\psi\bar{u} + \cos\psi v\\
\dot{\psi} &= r\\
\dot{\bar{u}} &= \frac{X_u}{m - X_{\dot{\bar{u}}}}\bar{u} + \frac{X_{|u|u}}{m - X_{\dot{u}}}|\bar{u}|\bar{u} + (m - Y_{\dot{v}})vr + \frac{1}{m - X_{\dot{u}}}X\\
&\quad + \frac{1}{m - X_{\dot{u}}}\tau_1\\
\dot{v} &= \frac{Y_v}{m - Y_{\dot{v}}}v + \frac{Y_{|v|v}}{m - Y_{\dot{v}}}|v|v - (m - X_{\dot{u}})\bar{u}r + \frac{1}{m - Y_{\dot{v}}}Y\\
&\quad + \frac{1}{m - Y_{\dot{v}}}\tau_2\\
\dot{r} &= \frac{N_r}{I_{zz} - N_{\dot{r}}}r - (m - Y_{\dot{v}})v\bar{u} + (m - X_{\dot{u}})\bar{u}v + \frac{N_{|r|r}}{I_{zz} - N_{\dot{r}}}|r|r\\
&\quad + \frac{1}{I_{zz} - N_{\dot{r}}}N + \frac{1}{I_{zz} - N_{\dot{r}}}\tau_3
\end{aligned} \tag{59}$$

where the relevant definition of spherical robot parameters can be found in [39]. In [39], only the values of some parameters are given. Since the robot structure is symmetrical about the Oxz-plane and Oyz-plane and appropriately symmetrical about the Oxy-plane, the values of the relevant parameters in the three directions x, y and z are the same. Hence, the values of each parameter are shown in Table 1.

Let $x(t) = [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t) \quad x_5(t) \quad x_6(t)]^T = [\xi \quad \eta \quad \psi \quad \bar{u} \quad v \quad r]^T$, $a_1 = \frac{X_u}{m - X_{\dot{u}}}$, $a_2 = \frac{Y_v}{m - Y_{\dot{v}}}$, $a_3 = \frac{N_r}{I_{zz} - N_{\dot{r}}}$, $b_1 = \frac{1}{m - X_{\dot{u}}}$, $b_2 = \frac{1}{m - Y_{\dot{v}}}$, $b_3 = \frac{1}{I_{zz} - N_{\dot{r}}}$, $c_1 = m - X_{\dot{u}}$, $c_2 = m - Y_{\dot{v}}$, $d_1 = \frac{X_{|u|u}}{m - X_{\dot{u}}}$, $d_2 = \frac{Y_{|v|v}}{m - Y_{\dot{v}}}$, $d_3 = \frac{N_{|r|r}}{I_{zz} - N_{\dot{r}}}$, $u = [X \quad Y \quad N]^T$. Since the robot is spherical, $c_1 = c_2$.

TABLE 1. **Values of spherical robot parameters.**

Parameter	Value	Parameter	Value
$X_{\dot{u}}$	-7.63 × 10^{-3} kg	$Y_{\dot{v}}$	-7.63 × 10^{-3} kg
X_u	-6.82 × 10^{-3} kg/s	Y_v	-6.82 × 10^{-3} kg/s
$X_{\|u\|u}$	-6.21 × 10^{-3} kg/s	$Y_{\|v\|v}$	-6.21 × 10^{-3} kg/s
$N_{\dot{r}}$	-4.33 × 10^{-5} kg	m	3.274 kg
N_r	-4.11 × 10^{-5} kg /s	I_{zz}	0.01 kg m^2
$N_{\|r\|r}$	-2.09 × 10^{-4} kg/s		

Therefore, the horizontal plane model is simplified as

$$\begin{aligned}
\dot{x}_1 &= \cos(x_3)x_4 - \sin(x_3)x_5 \\
\dot{x}_2 &= \sin(x_3)x_4 + \cos(x_3)x_5 \\
\dot{x}_3 &= x_6 \\
\dot{x}_4 &= a_1x_4 + b_1u_1 + b_1\tau_1 + c_2x_5x_6 + d_1|x_4|x_4 \\
\dot{x}_5 &= a_2x_5 + b_2u_2 + b_2\tau_2 - c_1x_4x_6 + d_2|x_5|x_5 \\
\dot{x}_6 &= a_3x_6 + b_3u_3 + b_3\tau_3 + d_3|x_6|x_6
\end{aligned} \tag{60}$$

Assume that $x_3 \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$, $x_4 \in [-0.1, 0.1]$, $x_5 \in [-0.1, 0.1]$, $x_6 \in [-0.1, 0.1]$ and x_4, x_5 are immeasurable. To decrease the number of fuzzy rules, the modelling method with local nonlinear functions is exploited.

Let $\phi_1 = x_5x_6 + 0.1x_5$, $\phi_2 = |x_4|x_4$, $\phi_3 = x_4x_6 + 0.1x_4$, $\phi_4 = |x_6|x_6$. Then, based on the method in [4] it follows that $\phi_1 \in \text{co}\{0, 0.2x_5\}$, $\phi_2 \in \text{co}\{0, 0.1\,x_4\}$, $\phi_3 \in \text{co}\{0, 0.2x_4\}$, $\phi_4 \in \text{co}\{0, 0.1\,x_6\}$. Furthermore, the model of the nonlinear system can be obtained as follows:

$$\dot{x}(t) = \begin{bmatrix} 0&0&0&\cos(x_3)&-\sin(x_3)&0 \\ 0&0&0&\sin(x_3)&\cos(x_3)&0 \\ 0&0&0&0&0&1 \\ 0&0&0&a_1&-0.1c_2&0 \\ 0&0&0&0.1c_1&a_2+d_2|x_5|&0 \\ 0&0&0&0&0&a_3 \end{bmatrix} x(t) + \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \\ b_1&0&0 \\ 0&b_2&0 \\ 0&0&b_3 \end{bmatrix} u(t)$$
$$+ \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \\ b_1&0&0 \\ 0&b_2&0 \\ 0&0&b_3 \end{bmatrix} \tau(t) + \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ c_2&d_1&0&0 \\ 0&0&-c_1&0 \\ 0&0&0&d_3 \end{bmatrix} \phi(t)$$
$$y(t) = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \end{bmatrix} x(t)$$

In addition, the matrix E given in Assumption 1 can be obtained as:

$$E = \begin{bmatrix} 0&0&0&0&r_{max}-r_{min}&0 \\ 0&0&0&u_{max}&0&0 \\ 0&0&0&r_{max}-r_{min}&0&0 \\ 0&0&0&0&0&r_{max} \end{bmatrix}$$

With the technique in [30], the nonlinear system can be accurately represented by the following T-S fuzzy model:

Rule(11):
IF $\cos(x_3)$ is Γ_{11} and $\sin(x_3)$ is Γ_{21} and $|x_5|$ is Γ_{31}
THEN

$$x(t) = A_{11}x(t) + B_{11}u(t) + F_{11}\tau + G\phi$$
$$y(t) = Cx(t)$$

Rule(12):
IF $\cos(x_3)$ is Γ_{11} and $\sin(x_3)$ is Γ_{21} and $|x_5|$ is Γ_{32}
THEN

$$x(t) = A_{12}x(t) + B_{12}u(t) + F_{12}\tau + G\phi$$
$$y(t) = Cx(t)$$

Rule(21):
IF $\cos(x_3)$ is Γ_{11} and $\sin(x_3)$ is Γ_{22} and $|x_5|$ is Γ_{31}
THEN

$$x(t) = A_{21}x(t) + B_{21}u(t) + F_{21}\tau + G\phi$$
$$y(t) = Cx(t)$$

Rule(22):
IF $\cos(x_3)$ is Γ_{11} and $\sin(x_3)$ is Γ_{22} and $|x_5|$ is Γ_{32}
THEN

$$x(t) = A_{22}x(t) + B_{22}u(t) + F_{22}\tau + G\phi$$
$$y(t) = Cx(t)$$

Rule(31):
IF $\cos(x_3)$ is Γ_{12} and $\sin(x_3)$ is Γ_{21} and $|x_5|$ is Γ_{31}
THEN

$$x(t) = A_{31}x(t) + B_{31}u(t) + F_{31}\tau + G\phi$$
$$y(t) = Cx(t)$$

Rule(32):
IF $\cos(x_3)$ is Γ_{12} and $\sin(x_3)$ is Γ_{21} and $|x_5|$ is Γ_{32}
THEN

$$x(t) = A_{32}x(t) + B_{32}u(t) + F_{32}\tau + G\phi$$
$$y(t) = Cx(t)$$

Rule(41):
IF $\cos(x_3)$ is Γ_{12} and $\sin(x_3)$ is Γ_{22} and $|x_5|$ is Γ_{31}
THEN

$$x(t) = A_{41}x(t) + B_{41}u(t) + F_{41}\tau + G\phi$$
$$y(t) = Cx(t)$$

Rule(42):
IF $\cos(x_3)$ is Γ_{12} and $\sin(x_3)$ is Γ_{22} and $|x_5|$ is Γ_{32}
THEN

$$x(t) = A_{42}x(t) + B_{42}u(t) + F_{42}\tau + G\phi$$
$$y(t) = Cx(t)$$

where the membership functions $\Gamma_{11}(\cos(x_3)) = 2 - 2\cos(x_3)$, $\Gamma_{12}(\cos(x_3)) = -1 + 2\cos(x_3)$, $\Gamma_{21}(\sin(x_3)) = \frac{1}{2} - \frac{\sqrt{3}}{3}\sin(x_3)$, $\Gamma_{22}(\sin(x_3)) = \frac{1}{2} + \frac{\sqrt{3}}{3}\sin(x_3)$, $\Gamma_{31}(|x_5|) = \frac{0.1-|x_5|}{0.1}$, $\Gamma_{32}(|x_5|) = \frac{|x_5|}{0.1}$, and

$$A_{11} = \begin{bmatrix} 0&0&0&\frac{1}{2}&\frac{\sqrt{3}}{2}&0 \\ 0&0&0&-\frac{\sqrt{3}}{2}&\frac{1}{2}&0 \\ 0&0&0&0&0&1 \\ 0&0&0&a_1&-0.1c_2&0 \\ 0&0&0&0.1c_1&a_2&0 \\ 0&0&0&0&0&a_3 \end{bmatrix} A_{12} = \begin{bmatrix} 0&0&0&\frac{1}{2}&\frac{\sqrt{3}}{2}&0 \\ 0&0&0&-\frac{\sqrt{3}}{2}&\frac{1}{2}&0 \\ 0&0&0&0&0&1 \\ 0&0&0&a_1&-0.1c_2&0 \\ 0&0&0&0.1c_1&a_2+0.1d_2&0 \\ 0&0&0&0&0&a_3 \end{bmatrix}$$

$$A_{21}=\begin{bmatrix} 0&0&0&\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\ 0&0&0&\frac{\sqrt{3}}{2}&\frac{1}{2}&0\\ 0&0&0&0&0&1\\ 0&0&0&a_1&-0.1c_2&0\\ 0&0&0&0.1c_1&a_2&0\\ 0&0&0&0&0&a_3 \end{bmatrix} A_{22}=\begin{bmatrix} 0&0&0&\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\ 0&0&0&\frac{\sqrt{3}}{2}&\frac{1}{2}&0\\ 0&0&0&0&0&1\\ 0&0&0&a_1&-0.1c_2&0\\ 0&0&0&0.1c_1&a_2+0.1\ d_2&0\\ 0&0&0&0&0&a_3 \end{bmatrix}$$

$$A_{31}=\begin{bmatrix} 0&0&0&1&\frac{\sqrt{3}}{2}&0\\ 0&0&0&-\frac{\sqrt{3}}{2}&1&0\\ 0&0&0&0&0&1\\ 0&0&0&a_1&-0.1c_2&0\\ 0&0&0&0.1c_1&a_2&0\\ 0&0&0&0&0&a_3 \end{bmatrix} A_{32}=\begin{bmatrix} 0&0&0&1&\frac{\sqrt{3}}{2}&0\\ 0&0&0&-\frac{\sqrt{3}}{2}&1&0\\ 0&0&0&0&0&1\\ 0&0&0&a_1&-0.1c_2&0\\ 0&0&0&0.1c_1&a_2+0.1d_2&0\\ 0&0&0&0&0&a_3 \end{bmatrix}$$

$$A_{41}=\begin{bmatrix} 0&0&0&1&-\frac{\sqrt{3}}{2}&0\\ 0&0&0&\frac{\sqrt{3}}{2}&1&0\\ 0&0&0&0&0&1\\ 0&0&0&a_1&-0.1c_2&0\\ 0&0&0&0.1c_1&a_2&0\\ 0&0&0&0&0&a_3 \end{bmatrix} A_{42}=\begin{bmatrix} 0&0&0&1&-\frac{\sqrt{3}}{2}&0\\ 0&0&0&\frac{\sqrt{3}}{2}&1&0\\ 0&0&0&0&0&1\\ 0&0&0&a_1&-0.1c_2&0\\ 0&0&0&0.1c_1&a_2+0.1d_2&0\\ 0&0&0&0&0&a_3 \end{bmatrix}$$

$$B_{ij}=\begin{bmatrix} 0&0&0\\ 0&0&0\\ b_1&0&0\\ 0&b_2&0\\ 0&0&b_3 \end{bmatrix}, F_{ij}=\begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&0\\ b_1&0&0\\ 0&b_2&0\\ 0&0&b_3 \end{bmatrix} (1\le i\le 4, 1\le j\le 2)$$

$$G=\begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ c_2&d_1&0&0\\ 0&0&-c_1&0\\ 0&0&0&d_3 \end{bmatrix}, C=\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1 \end{bmatrix}$$

From (8) and (9), the membership functions $\mu_1=(2-2\cos(x_3))*(\frac{1}{2}-\frac{\sqrt{3}}{3}\sin(x_3))$, $\mu_2=(2-2\cos(x_3))*(\frac{1}{2}+\frac{\sqrt{3}}{3}\sin(x_3))$, $\mu_3=(-1+2\cos(x_3))*(\frac{1}{2}-\frac{\sqrt{3}}{3}\sin(x_3))$, $\mu_4=(-1+2\cos(x_3))*(\frac{1}{2}+\frac{\sqrt{3}}{3}\sin(x_3))$, $\lambda_1=\frac{v_{max}-|x_5|}{v_{max}}$ and $\lambda_2=\frac{|x_5|}{v_{max}}$. Then, the correlation matrices in the solution process are set as follows:

$$\begin{aligned}
\tilde{A}_{11}&=diag\{-1,-1,-1,-1,-1,-1\},\\
\tilde{A}_{12}&=diag\{-1,-1,-1,-2,-2,-2\},\\
\tilde{A}_{21}&=diag\{-1,-1,-1,-3,-3,-3\},\\
\tilde{A}_{22}&=diag\{-1,-1,-1,-4,-4,-4\},\\
\tilde{A}_{31}&=diag\{-1,-1,-1,-5,-5,-5\},\\
\tilde{A}_{32}&=diag\{-1,-1,-1,-1,-1,-1\},\\
\tilde{A}_{41}&=diag\{-1,-1,-1,-2,-2,-2\},\\
\tilde{A}_{42}&=diag\{-1,-1,-1,-3,-3,-3\},\\
\bar{A}_1&=\frac{A_{11}+A_{12}}{2}, \bar{A}_2=\frac{A_{21}+A_{22}}{2},\\
\bar{A}_3&=\frac{A_{31}+A_{32}}{2}, \bar{A}_4=\frac{A_{41}+A_{42}}{2}.
\end{aligned}$$

Since $B_{11}=B_{12}=B_{21}=B_{22}=B_{31}=B_{32}=B_{41}=B_{42}$, set $\bar{B}_1=\bar{B}_2=\bar{B}_3=\bar{B}_4=B_{11}$.

Choosing $\kappa=0.01$, $\varepsilon=0.1$, $\varphi=1$, $\eta=3.2$, $a_1=a_2=1$, $b_1=b_2=1$, $\alpha=0.2$, $\delta=1.2$, $\rho=0.3$, and minimizing c via Theorem 1, we can obtain $c=15.1778$. Then, the designed gain matrices of the controller and observer are given by solving Theorem 1:

$$K_1=\begin{bmatrix} -449.301&13.22059&0&-467.6605&-0.6598695&0\\ -15.68869&-420.0979&0&-24.81019&-445.8279&0\\ 0&0&-421.5934&0&0&-439.883 \end{bmatrix}$$

$$K_2=\begin{bmatrix} -451.9029&-20.07844&0&-470.5679&-27.42098&0\\ 17.6034&-422.8807&0&1.319972&-447.4966&0\\ 0&0&-424.7268&0&0&-442.4602 \end{bmatrix}$$

$$K_3=\begin{bmatrix} -498.6646&34.17357&0&-506.0227&15.20297&0\\ -35.79208&-466.4813&0&-41.72023&-483.3413&0\\ 0&0&-461.793&0&0&-471.8011 \end{bmatrix}$$

$$K_4=\begin{bmatrix} -475.0507&-34.63238&0&-489.0229&-39.52&0\\ 36.95754&-447.3268&0&15.97687&-466.5661&0\\ 0&0&-441.0918&0&0&-455.4075 \end{bmatrix}$$

$$L_1=\begin{bmatrix} 724.8170&10.4245&0&-0.0399\\ -10.4245&724.81689&-0.00061&-0.1554\\ 0&0&0.2521&0.9933\\ 66980.8133&-10485.2693&-0.0001&1.6735\\ 10485.2678&66980.6792&-0.0813&-10.2646\\ -0.0031&-0.0201&0.1926&4.3150 \end{bmatrix}$$

$$L_2=\begin{bmatrix} 724.5420&-10.6037&0&0.0401\\ 10.6037&724.5419&-0.0006&-0.1552\\ 0&0&0.2521&0.9933\\ 67008.7533&10213.0661&0.0005&-1.5356\\ -10213.0635&67008.6193&-0.0813&-10.2564\\ 0.0032&-0.0201&0.1926&4.3150 \end{bmatrix}$$

$$L_3=\begin{bmatrix} 796.1203&25.7068&0&-0.0410\\ -25.7069&796.1199&-0.0006&-0.1845\\ 0&0&0.2521&0.9933\\ 75670.8278&-8615.4992&-0.0001&1.6575\\ 8615.4976&75670.6600&-0.0811&-11.7109\\ -0.0025&-0.0233&0.1926&4.3150 \end{bmatrix}$$

$$L_4=\begin{bmatrix} 795.8495&-26.0116&0&0.0413\\ 26.0117&795.8491&-0.0006&-0.1843\\ 0&0&0.2521&0.9933\\ 75693.6468&8319.7263&0.0005&-1.5231\\ -8319.7232&75693.4789&-0.0811&-11.7028\\ 0.0025&-0.0233&0.1926&4.3150 \end{bmatrix}$$

Furthermore, the performance index γ is optimized using Theorem 1 and the method presented in [33]. The comparison results are summarized in Table 2. Using the approach developed in this paper, the equivalent disturbance attenuation performance index is denoted by $\gamma=\sqrt{\frac{\bar{c}}{\alpha}}\delta\in[5.7257, 10.4537]$, where $\bar{c}=(\rho(\mu_3\lambda_1+\mu_4\lambda_1)+\mu_1\lambda_1+\mu_1\lambda_2+\mu_2\lambda_1+\mu_2\lambda_2+\mu_3\lambda_2+\mu_4\lambda_2)\times c$. In addition, the minimum allowable value of γ is $\sqrt{\frac{c}{\alpha}}\delta=7.5329$ by the approach in [33]. Fig. 2 illustrates the variation curves of the disturbance attenuation performance indexes obtained by Theorem 1 and the conventional L_∞ approach described in [33]. From Fig. 2, it can be observed that when $\mu_3\lambda_1+\mu_4\lambda_1>0.69$, the equivalent disturbance attenuation performance index obtained by Theorem 1 is smaller compared to the disturbance attenuation performance index obtained by [33]. Conversely, when $\mu_3\lambda_1+\mu_4\lambda_1<0.69$, the equivalent disturbance attenuation performance index obtained by Theorem 1 is larger than that obtained

TABLE 2. The minimum allowable values of γ.

Methods	Theorem 1	Conventional L_∞ method in [33]
L_∞ performance bound	[5.7257,10.4537]	7.5329

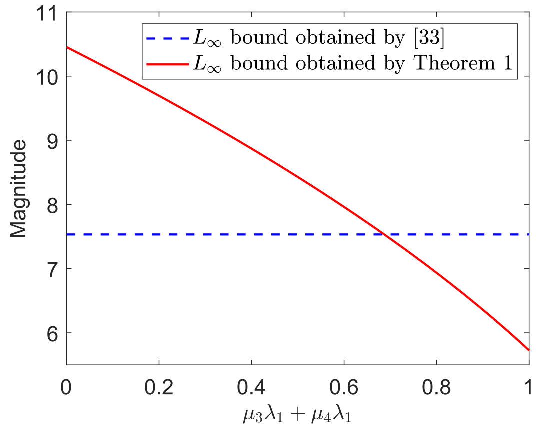

FIGURE 2. Trajectories of γ obtained by different methods.

by [33]. Therefore, this can be considered a compromise. A relatively small disturbance attenuation index is set for a class of fuzzy rules during which the system operates most of the time (corresponding to the fuzzy rules related to $\mu_3\lambda_1$ and $\mu_4\lambda_1$). However, as a trade-off, a relatively large disturbance attenuation index is obtained for other fuzzy rules that occur less frequently. This trade-off is also meaningful in the practical control of real systems. To enhance the overall system performance, the disturbance attenuation index can be relaxed for a short period of time to provide strong disturbance attenuation capability during the majority of the operating time.

First, there is a need to validate the effectiveness of the tracking control scheme. Since the actual system of the spherical robot can only control the position, not the speed, the first three components of the reference input $r(t)$ are set, namely, r_1, r_2 and r_3. First, the reference input is set as

$$r_1(t) = 0.5\sin(\frac{\pi}{50}t) \tag{61}$$

$$r_2(t) = 0.5\sin(\frac{\pi}{50}t) \tag{62}$$

$$r_3(t) = 0.5\sin(\frac{\pi}{50}t) \tag{63}$$

Fig. 3 - Fig. 5 are the tracking curves of the sinusoidal signal, respectively corresponding to (61) - (63). Then, the reference input is set respectively as

$$r_1(t) = 0.5 \tag{64}$$

$$r_2(t) = 0.5 \tag{65}$$

$$r_3(t) = 0.5 \tag{66}$$

Fig. 6 - Fig. 8 are the tracking curves of the step signal, corresponding to (64) - (66). From Fig. 3 - Fig. 8, it is clear that the proposed method can track the sinusoidal signal and step signal well. Therefore, the developed method in Theorem 1 allows us to ensure good tracking.

To prove the superiority of the method in Theorem 1, another comparison between the proposed method and the conventional L_∞ control method in [33] is made. The

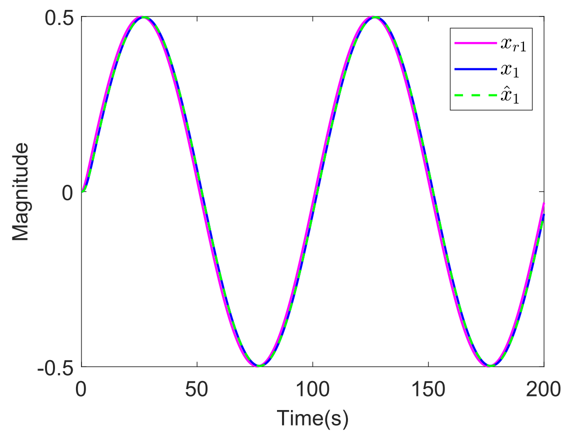

FIGURE 3. Desired state x_{r1}, state x_1 and observer state $\hat{x}_1$.

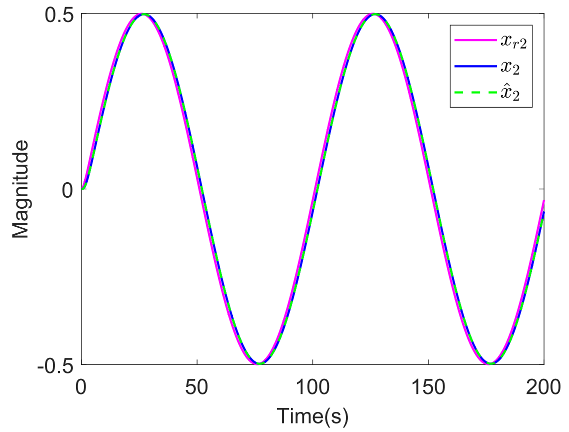

FIGURE 4. Desired state x_{r2}, state x_2 and observer state $\hat{x}_2$.

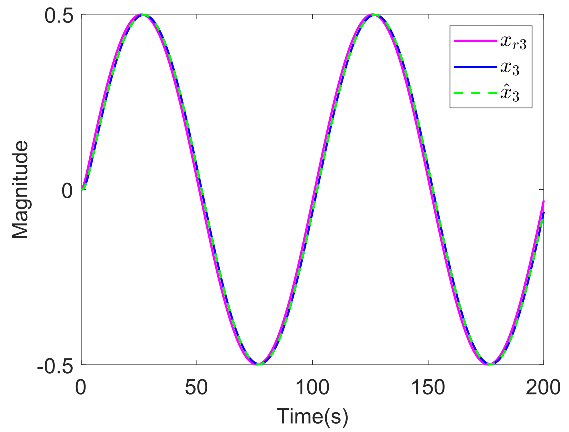

FIGURE 5. Desired state x_{r3}, state x_3 and observer state $\hat{x}_3$.

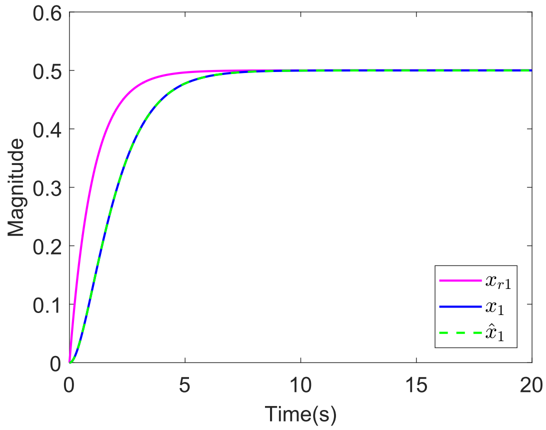

FIGURE 6. Desired state x_{r1}, state x_1 and observer state $\hat{x}_1$.

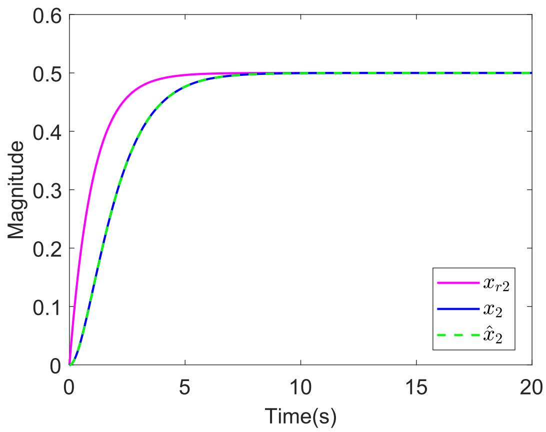

FIGURE 7. Desired state x_{r2}, state x_2 and observer state $\hat{x}_2$.

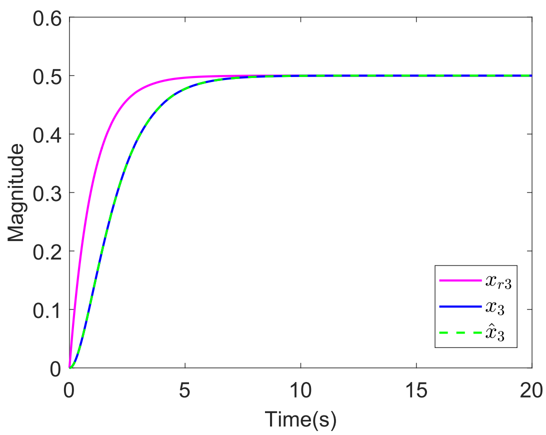

FIGURE 8. Desired state x_{r3}, state x_3 and observer state $\hat{x}_3$.

disturbance $\tau(t)$ is set as follows:

$$\tau(t) = \begin{cases} [1 \quad 0 \quad 0]^T, & 0 \le t \le 3 \\ [0 \quad 1 \quad 0]^T, & 8 \le t \le 11 \\ [0 \quad 0 \quad 0.05]^T, & 16 \le t \le 19 \\ [0 \quad 0 \quad 0]^T, & \text{others} \end{cases}$$

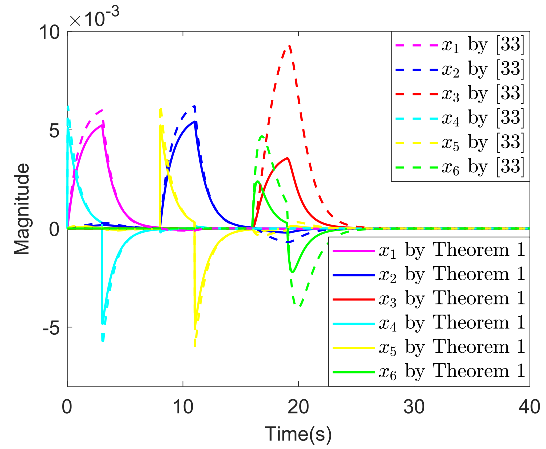

FIGURE 9. States $x(t)$.

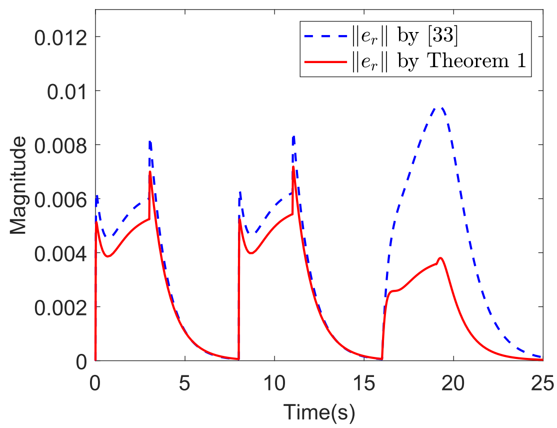

FIGURE 10. Tracking error $\|e_r\|$.

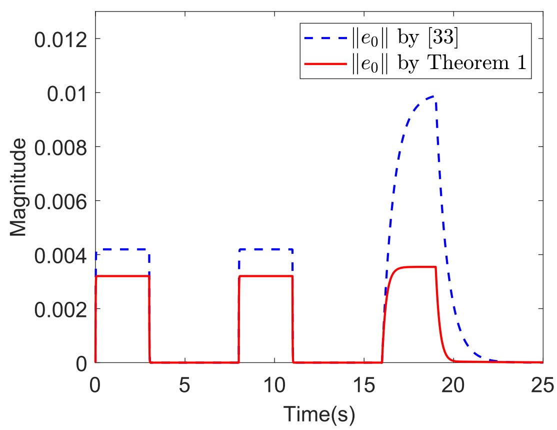

FIGURE 11. Observer error $\|e_0\|$.

The simulation results are shown in Fig. 9 - Fig. 13. The curves of the dynamic system states with disturbance are illustrated in Fig. 9. It is easy to see that the developed method in Theorem 1 has better robustness against disturbances compared with the conventional L_∞ control method in [33]. Additionally, the tracking error $\|e_r\|$, observer error $\|e_0\|$ and

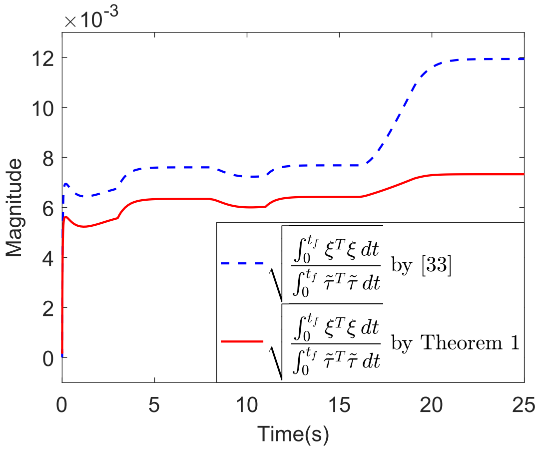

FIGURE 12. Squared root of ratio $\sqrt{\frac{\int_0^{t_f}\xi^T\xi\,dt}{\int_0^{t_f}\tau^T\tau\,dt}}$.

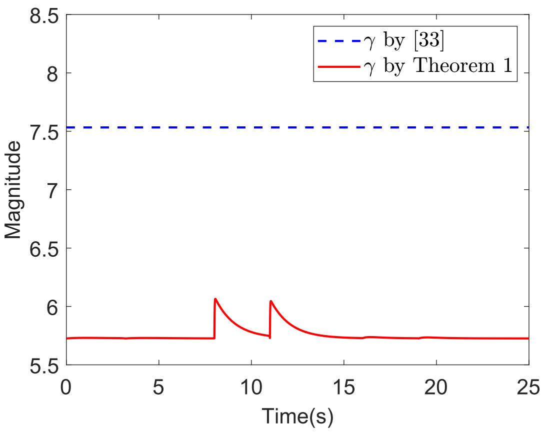

FIGURE 13. L_∞ performance index γ.

response of the ratio $\sqrt{\frac{\int_0^{t_f}\xi^T\xi\,dt}{\int_0^{t_f}\tau^T\tau\,dt}}$ are shown in Fig. 10 - Fig. 12, respectively. Since there is a case where $\|\tau\|$ is 0, the ratio $\frac{\|\xi\|}{\|\tau\|}$ cannot be shown continuously. Therefore, we plot the caves of the tracking error $\|e_r\|$ and observer error $\|e_0\|$ under the same disturbance and ratio $\sqrt{\frac{\int_0^{t_f}\xi^T\xi\,dt}{\int_0^{t_f}\tau^T\tau\,dt}}$. From Fig. 10 - Fig. 12, it is not difficult to see that the ratio $\frac{\|\xi\|}{\|\tau\|}$ by the approach in Theorem 1 is always less than the method based on the conventional L_∞ performance index in [33]. Furthermore, Fig. 13 shows the caves of the L_∞ performance index γ. Fig. 13 indicates that the L_∞ performance index γ by the method in Theorem 1 is obviously less than the conventional L_∞ performance index by [33]. At the same time, Fig. 13 also illustrates that $\mu_3\lambda_1+\mu_4\lambda_1 > 0.69$, which means that a lower L_∞ performance index and better system performance can be obtained by Theorem 1.

Finally, based on the above analysis, it is easy to conclude that the tracking control system is asymptotically stable with membership function-dependent L_∞ performance, and by using the proposed approach, it has better robustness and less conservatism.

V. CONCLUSION

In this paper, the problem of tracking control design for T-S fuzzy systems based on membership function-dependent L_∞ performance has been investigated. First, a novel membership function-dependent L_∞ performance index has been proposed by using the property of T-S fuzzy systems that work on some specific local subsystems most of the time. Then, by the proposed L_∞ performance index, an observer-based tracking controller is constructed. Finally, an example of a spherical robot horizontal plane model is provided to illustrate the effectiveness. Compared with traditional L_∞ control, the new tracking control strategy can obtain better robustness for the case of a specific class of fuzzy subsystems that work most of the time. However, the novel approach proposed is not very suitable for T-S fuzzy systems whose states change significantly. Therefore, to better improve the performance of such systems, we will focus on the control synthesis of T-S fuzzy systems whose states change significantly.

REFERENCES

[1] X. Cai, S. Zhong, J. Wang, and K. Shi, "Robust H_∞ control for uncertain delayed T-S fuzzy systems with stochastic packet dropouts," *Appl. Math. Comput.*, vol. 385, Nov. 2020, Art. no. 125432.

[2] Z. Lian, Y. He, C.-K. Zhang, and M. Wu, "Stability and stabilization of T–S fuzzy systems with time-varying delays via delay-product-type functional method," *IEEE Trans. Cybern.*, vol. 50, no. 6, pp. 2580–2589, Jun. 2020.

[3] H.-B. Zeng, K. L. Teo, Y. He, and W. Wang, "Sampled-data-based dissipative control of T-S fuzzy systems," *Appl. Math. Model.*, vol. 65, pp. 415–427, Jan. 2019.

[4] J. Dong, Y. Wang, and G.-H. Yang, "Control synthesis of continuous-time T-S fuzzy systems with local nonlinear models," *IEEE Trans. Syst., Man, Cybern., B Cybern.*, vol. 39, no. 5, pp. 1245–1258, Oct. 2009.

[5] C. Zhang, J. Hu, J. Qiu, W. Yang, H. Sun, and Q. Chen, "A novel fuzzy observer-based steering control approach for path tracking in autonomous vehicles," *IEEE Trans. Fuzzy Syst.*, vol. 27, no. 2, pp. 278–290, Feb. 2019.

[6] Y. Pan, Q. Li, H. Liang, and H.-K. Lam, "A novel mixed control approach for fuzzy systems via membership functions online learning policy," *IEEE Trans. Fuzzy Syst.*, vol. 30, no. 9, pp. 3812–3822, Sep. 2022.

[7] M. Chen, H. Yan, H. Zhang, M. Chi, and Z. Li, "Dynamic event-triggered asynchronous control for nonlinear multiagent systems based on T–S fuzzy models," *IEEE Trans. Fuzzy Syst.*, vol. 29, no. 9, pp. 2580–2592, Sep. 2021.

[8] S. Shi, Z. Fei, T. Wang, and Y. Xu, "Filtering for switched T–S fuzzy systems with persistent dwell time," *IEEE Trans. Cybern.*, vol. 49, no. 5, pp. 1923–1931, May 2019.

[9] L. Meng, W. Xie, L. Li, Y. Li, and Z. Liu, "Hesitant hierarchical T–S fuzzy system with fuzzily weighted recursive least square," *Eng. Appl. Artif. Intell.*, vol. 126, Nov. 2023, Art. no. 106812.

[10] S. Mao, X. Yang, M. Wang, and W. X. Zheng, "Secure stabilization of switched T–S fuzzy systems with mixed delay via mode-dependent event-triggered control," *IEEE Trans. Syst., Man, Cybern., Syst.*, early access, Sep. 7, 2023, doi: 10.1109/TSMC.2023.3306724.

[11] O. Alshammari, M. Kchaou, H. Jerbi, S. Ben Aoun, and V. Leiva, "A fuzzy design for a sliding mode observer-based control scheme of Takagi–Sugeno Markov jump systems under imperfect premise matching with bio-economic and industrial applications," *Mathematics*, vol. 10, no. 18, p. 3309, Sep. 2022, doi: 10.3390/math10183309.

[12] J. Dong, Q. Hou, and M. Ren, "Control synthesis for discrete-time T–S fuzzy systems based on membership function-dependent H_∞ performance," *IEEE Trans. Fuzzy Syst.*, vol. 28, no. 12, pp. 3360–3366, Dec. 2020.

[13] S.-W. Kau, H.-J. Lee, C.-M. Yang, C.-H. Lee, L. Hong, and C.-H. Fang, "Robust H_∞ fuzzy static output feedback control of T-S fuzzy systems with parametric uncertainties," *Fuzzy Sets Syst.*, vol. 158, no. 2, pp. 135–146, Jan. 2007.
[14] K. Zhang, B. Jiang, and M. Staroswiecki, "Dynamic output feedback-fault tolerant controller design for Takagi–Sugeno fuzzy systems with actuator faults," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 1, pp. 194–201, Feb. 2010.
[15] H. Li, C. Wu, S. Yin, and H.-K. Lam, "Observer-based fuzzy control for nonlinear networked systems under unmeasurable premise variables," *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 5, pp. 1233–1245, Oct. 2016.
[16] J. Lee, J. H. Moon, and H. J. Lee, "Robust H_∞ and L_∞-L_∞ sampled-data dynamic output-feedback control for nonlinear system in T-S form including singular perturbation," *Int. J. Syst. Sci.*, vol. 52, no. 7, pp. 1315–1328, May 2021.
[17] X. Ma, J. Dong, W. Tai, J. Zhou, and W. Paszke, "Asynchronous event-triggered H_∞ control for 2D Markov jump systems subject to networked random packet losses," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 126, Nov. 2023, Art. no. 107453.
[18] J. Xiong, J.-N. Li, and P. Du, "A novel non-fragile H_∞ fault-tolerant course-keeping control for uncertain unmanned surface vehicles with rudder failures," *Ocean Eng.*, vol. 280, Jul. 2023, Art. no. 114781.
[19] Y. Wu, J. Dong, and T. Li, "Sensor fault estimation in finite-frequency domain for nonlinear time-delayed systems by T–S fuzzy model approach with local nonlinear models," *Int. J. Syst. Sci.*, vol. 50, no. 11, pp. 2226–2247, Aug. 2019.
[20] H. Liang, G. Liu, H. Zhang, and T. Huang, "Neural-network-based event-triggered adaptive control of nonaffine nonlinear multiagent systems with dynamic uncertainties," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 32, no. 5, pp. 2239–2250, May 2021.
[21] S. Xu and Z. Huang, "Robust fuzzy attitude stabilization of networked spacecraft under actuator saturation and persistent disturbance," in *Proc. 21st IFAC World Congr.*, Berlin, Germany, Jul. 2020, pp. 14869–14875.
[22] M. Tabarraie, "Robust L_∞-induced deconvolution filtering for linear stochastic systems and its application to fault reconstruction," *Signal Process.*, vol. 93, no. 5, pp. 1379–1391, May 2013.
[23] J. X. Wang and J. M. Li, "Stabilization control with optimal L_1-gain and L_∞-gain for positive T-S fuzzy systems," *Int. J. Uncertainty Fuzziness Knowl.-Based Syst.*, vol. 27, no. 6, pp. 881–903, Dec. 2019.
[24] X.-J. Li and G.-H. Yang, "Finite frequency L_2-L_∞ filtering of T-S fuzzy systems with unknown membership functions," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 8, pp. 1884–1897, Aug. 2017.
[25] J. Pan, A.-T. Nguyen, W. Lai, X. Zhu, and H. Huang, "Joint estimation of nonlinear dynamics and resistance torque for integrated motor-transmission systems via switched ℓ_∞ observers with smoothness guarantee," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 53, no. 11, pp. 7280–7291, Nov. 2023, doi: 10.1109/TSMC.2023.3297064.
[26] B.-S. Chen and S.-J. Ho, "Multiobjective tracking control design of T–S fuzzy systems: Fuzzy Pareto optimal approach," *Fuzzy Sets Syst.*, vol. 290, pp. 39–55, May 2016.
[27] B. Jiang, Z. Gao, P. Shi, and Y. Xu, "Adaptive fault-tolerant tracking control of near-space vehicle using Takagi–Sugeno fuzzy models," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 5, pp. 1000–1007, Oct. 2010.
[28] T.-S. Li, S.-C. Tong, and G. Feng, "A novel robust adaptive-fuzzy-tracking control for a class of nonlinear multi-input/multi-output systems," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 1, pp. 150–160, Feb. 2010.
[29] H. Zhang, Y. Shi, and M. X. Liu, "H_∞ step tracking control for networked discrete-time nonlinear systems with integral and predictive actions," *IEEE Trans. Ind. Informat.*, vol. 9, no. 1, pp. 337–345, Feb. 2013.
[30] K. T. Wang, "Takagi–Sugeno fuzzy model and parallel distributed compensation," in *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*. New York, NY, USA: Wile, 2001, pp. 5–48.
[31] J. Dong, and S. Wang, "Robust H_∞-tracking control design for T-S fuzzy systems with partly immeasurable premise variables," *J. Franklin Inst.-Eng. Appl. Math.*, vol. 354, no. 10, pp. 3919–3944, Jul. 2017.
[32] B.-S. Chen, H.-J. Uang, and C.-S. Tseng, "Robust tracking enhancement of robot systems including motor dynamics: A fuzzy-based dynamic game approach," *IEEE Trans. Fuzzy Syst.*, vol. 6, no. 4, pp. 538–552, Nov. 1998.
[33] Y. Wu, Y. Wang, T. Li, and Z. Yang, "A new fault tolerant control scheme for non-linear systems by T-S fuzzy model approach," *IET Control Theory Appl.*, vol. 15, no. 15, pp. 1915–1930, Oct. 2021.
[34] J. C. Lo, and M. L. Lin, "Observer-based robust H_∞ control for fuzzy systems using two-step procedure," *IEEE Trans. Fuzzy Syst.*, vol. 12, no. 3, pp. 350–359, Jun. 2004.
[35] Y. He, M. Wu, J.-H. She, and G.-P. Liu, "Parameter-dependent Lyapunov functional for stability of time-delay systems with polytopic-type uncertainties," *IEEE Trans. Autom. Control*, vol. 49, no. 5, pp. 828–832, May 2004, doi: 10.1109/TAC.2004.828317.
[36] H. Moodi and M. Farrokhi, "On observer-based controller design for sugeno systems with unmeasurable premise variables," *ISA Trans.*, vol. 53, no. 2, pp. 305–316, Mar. 2014.
[37] Y. Wang, T. Li, Y. Wu, X. Yang, C. L. P. Chen, Y. Long, Z. Yang, and J. Ning, "L_∞ fault estimation and fault-tolerant control for nonlinear systems by T–S fuzzy model method with local nonlinear models," *Int. J. Fuzzy Syst.*, vol. 23, no. 6, pp. 1714–1727, Sep. 2021.
[38] Y. Wu and J. Dong, "Local stabilization of continuous-time T–S fuzzy systems with partly measurable premise variables and time-varying delay," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 51, no. 1, pp. 326–338, Jan. 2021.
[39] Y. Feng, Y. Liu, H. Gao, and Z. Ju, "Hovering control of submersible transformer inspection robot based on ASMBC method," *IEEE Access*, vol. 8, pp. 76287–76299, 2020.

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