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RESEARCH ARTICLE

Design of Observer-Based Feedback Controller for Multi-Rate Systems With Various Sampling Periods Using Cyclic Reformulation

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ABSTRACT Signal sensing periods typically vary depending on the sensor used and may differ even within a single control system that involves multiple sensors. Likewise, input periods can vary based on the actuator used. This paper discusses the design of observer-based feedback controllers for linear, time-invariant, discrete-time systems operating in a multi-rate sensing and actuating environment. The observation and control periods of the sensors and actuators in the plant are assumed to have mutually rational ratios. First, we reduce the multi-rate system to a periodically time-varying system and provide a linear matrix inequality (LMI) condition for analyzing the l_2 performance using cyclic reformulation, which is a type of time-invariant reformulation for periodic systems. Next, we extend the analysis method to design an observer-based feedback controller for the multi-rate system. This allows us to obtain multi-rate observer gains and feedback gains based on the l_2-induced norm from disturbances to outputs. Finally, we present numerical results to demonstrate the effectiveness of the observer-based feedback system in the multi-rate environment.

INDEX TERMS Multi-rate system, state observer, state feedback control, cyclic reformulation, LMI, l_2-induced norm.

I. INTRODUCTION

Practical control systems typically consist of multiple components, including various types of sensors and actuators. In a single control system, these sensors and actuators often have different specifications and can operate with different sampling periods. In recent years, IoT has been advancing [1], and we are transitioning to a society where objects are connected to the Internet. In such an IoT-driven society, control systems with various sensors and actuators will be increasingly used. Meanwhile, autonomous robots and self-driving cars have been actively researched, and accurate simultaneous localization and mapping (SLAM) [2], [3], [4] is crucial for their implementation. SLAM is built using various sensors, such as camera sensors, 3D-LiDAR sensors, and inertial measurement units (IMU), but the feasible sensing period varies for each sensor type. In addition, control of multi-vehicle systems will use more sensors, and management of time-series information for multiple sensors will become more important [5], [6]. Achieving appropriate "sensor fusion" is necessary for high localization and mapping performance. Utilizing multiple sensors allows a wide range of data to be gathered, leading to more precise control.

Diversified control issues, such as event-driven control, control security, and multi-rate control exist to solve the challenges of realizing an IoT society. The sensor network developed for the periodic system should be taken into account for the actual needs of monitoring or estimation. The problem of distributed filtering of periodic systems is dealt with over sensor networks under event-triggered communications for continuous-time settings [7], [8]. Additionally, state estimation in communication environments [9] and state estimation with outlier removal functions [10], [11] have also been extensively researched. In this paper, we focus on multi-rate control systems with a constant period.

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Various studies have been conducted on multi-rate systems. Multi-rate sampled-data stabilization for systems with time delay is proposed when the control sampling rate is faster than the observation rate in [12]. Perfect tracking control methods for multi-rate feedforward systems are explored for motors and electric vehicles in [13], [14], and [15]. A different multi-rate control scheme is introduced in [16], where multi-rate sampled-data measurements are employed to maintain the stability achieved by a slow sampled-data controller. Ripple-free conditions under a multi-rate environment are considered using the lifting method in [17]. State estimation methods focusing on the multi-rate sensing environment have been developed in [18], [19], [20], [21], and [22]. In [18], a state estimation method based on the moving horizon strategy was developed. Multi-rate control methods with asynchronous measurements have been studied in [19] and [20]. A multi-rate observer using the lifting method is discussed in [21]. While these earlier studies primarily focus on the difference between observation (sensing) and control periods, addressing a similar distinction for multiple sensors is also crucial. For example, the use of lifting techniques is difficult to design for multiple sensor environments because the product of design variables occurs in the design problem. A comprehensive multi-rate system design method, encompassing both sensing and actuator periods, is vital for managing various control systems.

This paper proposes a method for designing an observer-based feedback controller in a multi-rate sensing and actuation environment. Designing control systems where the control input period differs in addition to the observation period becomes increasingly challenging. The observer-based control is a practical approach for control systems and has been extensively utilized for several decades [23], [24], [25], [26]. Over the years, numerous studies on state estimation have been conducted, leading to the development of various approaches, such as those considering non-linear systems and robust estimation [27], [28], [29].

In our previous research [30], a state observer for a multi-rate system is handled as a periodically time-varying system, and the effects of process noise and observation noise on the estimated state are evaluated using a type of periodically time-varying Lyapunov function. This method can handle observer design problems with multiple sensing periods without occurring the product of design variables. In addition, [31] deals with multi-rate state feedback control for systems with multiple control input periods, where control input periods are assumed to be longer than the observation periods of sensors. For instance, in a system where the frequency of actuator drives directly affects operating costs, longer control command periods are more advantageous.

This paper presents a design method for the observer-based feedback controllers in linear time-invariant discrete-time systems operating within a multi-rate sensing and actuating environment based on [30] and [31]. We address a control problem that uses a state feedback controller with a state observer in a system where the control input and sensor observation periods differ with respect to the control period. First, we describe a multi-rate system, and the system is transformed into an equivalent periodically time-varying system. The cyclic reformulation [32], [33], [34], [35], [36], a time-invariant method used to deal with periodically time-varying systems as time-invariant systems, is applied to the observer-based feedback control system. Next, we discuss the l_2 performance analysis of the disturbance effects on the given observer-based feedback control system. A performance analysis method for both multi-rate observer gains and feedback gains in the observer-based feedback control system is introduced based on linear matrix inequalities (LMIs). Furthermore, we propose a design algorithm for both multi-rate observer gains and feedback gains based on the l_2 performance concerning disturbances. Finally, the effectiveness of the proposed design method is illustrated using numerical examples.

Notations: The set of real numbers is denoted by $\mathbb{R}$, and the set of positive integers is denoted by $\mathbb{N}$. A nonnegative integer k denotes the discrete time. I_n denotes an $n \times n$ identity matrix. $O_{n,m}$ denotes an $n \times m$ zero matrix. The l_2-induced norm of a discrete-time system G with an input u and an output y is given by

$$||G||_{l_2/l_2} = \sup_{u \in l_2} \frac{||y||_2}{||u||_2} \tag{1}$$

where $\| \cdot \|_2$ represents the l_2 norm of the signal.

II. FORMULATION OF PLANT UNDER MULTI-RATE ENVIRONMENT

A. LINEAR TIME-INVARIANT PLANT

In this section, we derive a system with different periods for each input and output channel. First, the plant P is assumed to be a discrete-time linear time-invariant multi-input multi-output system.

$$x(k+1) = Ax(k) + Bu_r(k) + B_2 d_u(k) \tag{2}$$

$$y_r(k) = Cx(k) + Dd_w(k) \tag{3}$$

Let $x(k) \in \mathbb{R}^n$ be the state of the plant at time k, $u_r(k) \in \mathbb{R}^m$ be the control input, $d_u(k) \in \mathbb{R}^{m_2}$ be the process noise, $d_w(k) \in \mathbb{R}^q$ be the observed noise, and $y_r(k) \in \mathbb{R}^q$ be the observed output. The matrices are given as $A \in \mathbb{R}^{n \times n}$, $B, B_2 \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{q \times n}$, $D \times \mathbb{R}^{q \times q}$. In this paper, we assume that (A, B) is a controllable pair and (C, A) is an observable pair.

B. STATE-SPACE REALIZATION OF MULTI-RATE SYSTEM

This paper assumes such limitations on the sensor devices that the i-th entry of $y_r(k)$ is periodically measured with the sensing period $\mathcal{M}_i \in \mathbb{N}$ for each $i = 1, \cdots, q$. Thus, the sensing of the underlying single-rate system is with multiple rates. In addition, the i-th entry of $u_r(k)$ is also periodically inputted with the input period $M_i \in \mathbb{N}$ for each $i = 1, \cdots, m$. Let M be the least common multiple of $M_1, M_2, \cdots, M_m$ and $\mathcal{M}_1, \mathcal{M}_2, \cdots, \mathcal{M}_q$. By using their least common multiple

M, all inputs and outputs can also be regarded as signals of period M, and (2) and (3) can be dealt with as a system of period M.

To describe the multi-rate sensing and actuation in more detail, we introduce the periodically time-varying matrices S_k and $T_k, (k = 0, 1, \cdots)$ as presented in [30](output) and [31](input). The matrix S_k, which characterizes the sampling timings of the observed outputs, is defined as follows.

$$S_k = \text{diag}\left[s_1(k), s_2(k), \cdots, s_q(k)\right] \tag{4}$$

Each element $s_i(k)$, $i = 1, \cdots, m$ corresponds to the i-th component of $y_r(k)$. Here, the elements $s_i(k), i = 1, \cdots, q$ are defined to take either 1 or 0 as follows: $s_i(k) = 1$ if the i-th component of $y_r(k)$ is observed at time k, while $s_i(k) = 0$ otherwise. The period of S_k as a whole is M, but that of $s_i(k)$ is $\mathcal{M}_i$ for each i. As indicated in the earlier part, since M is divisible by $\mathcal{M}_i$, each component can also be dealt with as an M-periodic signal.

Here, the matrix T_k, which characterizes the sampling timings of the control inputs, is defined as follows.

$$T_k = \text{diag}\left[t_1(k), t_2(k), \cdots, t_m(k)\right] \tag{5}$$

Each element $t_i(k)$, $i = 1, \cdots, m$ corresponds to the i-th component of $u_r(k)$. Here, the elements $t_i(k), i = 1, \cdots, m$ are defined to take either 1 or 0 as follows: $t_i(k) = 1$ if the i-th component of $u_r(k)$ is inputted at time k, while $t_i(k) = 0$ otherwise. In this case, since each element $t_i(k)$ has period M, T_k are periodic time-varying matrices with period M. Note that both T_k and S_k have period M, and $T_k = T_{M+k}$ and $S_k = S_{M+k}$ hold for all k. In this paper, the periodic matrices S_k and T_k are given based on the characteristics of the sensors and actuators.

More generally, once the mutual timing of the actions of the multiple sensors and actuators is determined by S_k and T_k, we can describe the plant with different input and output periods, instead of (2) and (3), as follows: (3) with the multi-rate environment can be written as follows

$$x(k+1) = Ax(k) + BT_k u(k) + B_2 d_u(k) \tag{6}$$

$$y(k) = S_k Cx(k) + S_k Dd_w(k) \tag{7}$$

From (6), (7), by using the periodic time-varying matrices T_k, S_k, the representation of the plant given as a multi-rate system can be regarded as a periodic time-varying system with period M. The above system is expressed as an M-periodic time-varying system in which periods of the control inputs and the observed outputs by the sensors are different from each other with respect to the control period. It is possible to address any case, not only the case where control and observation periods differ but also the control and observation periods are the same, by appropriately choosing the periodically time-varying matrices S_k and T_k.

Fig.1 shows a simple example of the input application and output observation periods and timing for a one-input, two-output system with the plant P. The input in Fig.1 is $u_1(k)$. $u_1(k)$ applies the input only when $k = M_1\kappa(\kappa = 0, 1, \cdots)$,

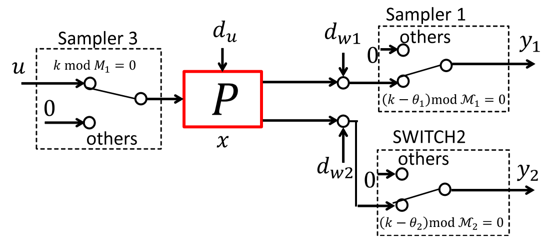


FIGURE 1. Multi-input multi-output plant with different sampling times.

i.e., every M_1 steps, and 0 at other times. The outputs are $y_1(k), y_2(k)$, where $y_1(k)$ is observed every $\mathcal{M}_1$ step and 0 at other times. Similarly, $y_2(k)$ observed every $\mathcal{M}_2$ step and 0 at other times. If the least common multiple of $M_1, \mathcal{M}_1$ and $\mathcal{M}_2$ is M, then the plant composed of (6), (7) can be regarded as a periodic time-varying system with period M.

C. PLANT WITH STEP-LIKE INPUTS

It is also possible to extend (6) and (7) for the case with a multi-rate system using a step-like input signal as follows: Consider a memory $u_m(k)$ for temporarily holding the input value u. Moreover, the state of the augmented system $x_*(k)$, which includes $u_m(k)$, is defined as follow:

$$x_*(k) = \begin{bmatrix} x(k) \\ u_m(k) \end{bmatrix}. \tag{8}$$

Then, the following augmented system can be derived as follow:

$$x_*(k+1) = A_k x_*(k) + B_k u(k) + B_2 w(k), \tag{9}$$

where A_k and B_k are periodic time-varying matrices, and B_2 is a time-invariant matrix. If all m inputs are updated simultaneously, A_k and B_k can be defined as follows when the input values are updated by:

$$A_k = \begin{bmatrix} A & O_{n,m} \\ O_{m,n} & O_{m,m} \end{bmatrix}, B_k = \begin{bmatrix} B \\ I_m \end{bmatrix}, \tag{10}$$

where I_m is the identity matrix with the same dimension as the number of inputs. In contrast, during the phase when input values are held, A_k and B_k can be defined as follows:

$$A_k = \begin{bmatrix} A & B \\ O_{m,n} & I_m \end{bmatrix}, \quad B_k = \begin{bmatrix} O_{n,m} \\ O_{m,m} \end{bmatrix}. \tag{11}$$

Moreover, $B2_2$ is given as

$$\mathcal{B}_2 = \begin{bmatrix} B_2 \\ O_{m,m_2} \end{bmatrix}. \tag{12}$$

Moreover, to consider a unified representation that encompasses both cases where all inputs are applied simultaneously, as shown in (10) and (11), and cases where the timing of input application differs among multiple inputs, we define the following matrices:

$$\mathcal{A} := \begin{bmatrix} A & B \\ O_{m,n} & I_m \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} B \\ I_m \end{bmatrix},$$
$$\mathcal{C} := \begin{bmatrix} C & O_{q,m} \end{bmatrix}, \quad \mathcal{D} := D.$$

We consider the following matrices as matrices characterizing multi-rate systems:

$$\bar{T}_k = \mathrm{diag}[1, \cdots, 1, \bar{t}_{k1}, \cdots, \bar{t}_{km}], \tag{13}$$
$$T_k = \mathrm{diag}[t_{k1}, \cdots, t_{km}]. \tag{14}$$

In this case, t_{ki} takes the value 1 at the timing of the i-th input application and 0 otherwise. Conversely, $\bar{t}_{ki}$ takes the opposite value, that is, it takes the value 0 at the timing of the i-th input application and 1 otherwise. Using these results, the system can be represented as:

$$x_*(k+1) = \mathcal{A}\bar{T}_k x_*(k) + \mathcal{B}T_k u(k) + \mathcal{B}_2 d_u(k), \tag{15}$$
$$y(k) = S_k \mathcal{C} x_*(k) + S_k \mathcal{D} d_w(k), \tag{16}$$

We can find that the system (15), (16) is represented as a M-periodic time-varying system. Consequently, we can see that it is possible to dealt with the multi-rate system using a step-like input signal if (15) is used instead of (6).

III. OBSERVER-BASED FEEDBACK CONTROLLER FOR MULTI-RATE SYSTEMS

In this section, we explore the implementation of state feedback control for the plant described by (6), (7) utilizing an observer-based approach. Due to multiple input application timings within the M-periodic system, a periodic time-varying system is also employed for state feedback. Concurrently, a periodic time-varying observer is utilized for state estimation to determine the state $x(k)$ of the plant, as outlined in (6). At first, a periodically time-varying observer and state feedback controller are introduced and can be regarded as a periodically time-varying system. Then, the observer-based feedback control system under multi-rate sensing and actuation environment is introduced in this section.

A. PERIODICALLY TIME-VARYING OBSERVER

The structure of an M-periodic time-varying observer for the plant, described by (6) and (7), is defined as follows.

$$x_{ob}(k+1) = (A - L_k S_k C)x_{ob}(k) + Bu(k) + L_k y(k) \tag{17}$$

Note that $L_k, k = 0, 1, \cdots, M-1, M, \cdots$ are the periodically time-varying observer gains. $L_{k+M} = L_k$ hold for all k, and we can rewrite L_k as $L_{k \bmod M}$. The initial value $x_{ob}(0)$ of the state estimate is assumed to be given.

B. PERIODICALLY TIME-VARYING STATE FEEDBACK CONTROL

Consider applying an M-periodic time-varying state feedback to the plant (6) in this section. By using the estimated state x_{ob}, a state feedback controller is given as $u(k) = -F_k x_{ob}(k)$. The control structure for a periodic time-varying system with period M and state feedback F_k is given by

$$x(k+1) = Ax(k) - BT_k F_k x_{ob}(k) + B_2 d_u(k), \tag{18}$$
$$y(k) = S_k C x(k) + S_k D d_w(k), \tag{19}$$

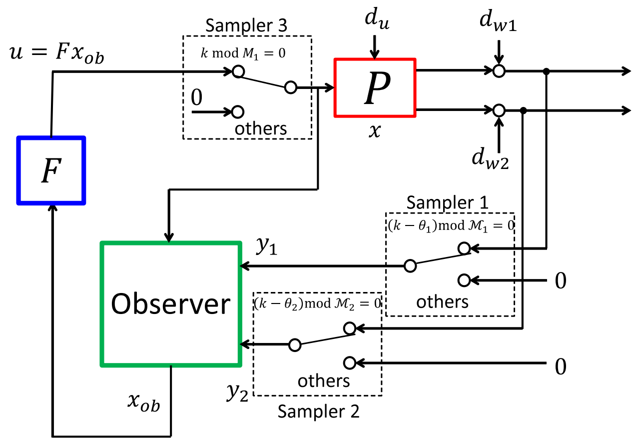


FIGURE 2. Observer-based feedback control system with different sampling times.

where, $F_k(k = 0, 1, \cdots, M-1)$ are an M-periodic time-varying feedback gain. $F_k = F_{k+M}$ is satisfied for any given k. We can rewrite F_k as $F_{k \bmod M}$ for convenience.

C. OBSERVER-BASED FEEDBACK CONTROL SYSTEM

Based on the results up to the previous section, we consider the estimation errors corresponding to (6), (7), and (17). An error system is derived with $d_u(k)$, $d_w(k)$ as the disturbance input for the estimation error $e(k) = x(k) - x_{ob}(k)$.

$$e(k+1) = (A - L_k S_k C)e(k) + B_2 d_u(k) - L_k S_k D d_w(k) \tag{20}$$

Using the state estimate x_{ob}, let the state feedback $u(k) = -F_k x_{ob}$. Using (18), (19), (20) in a periodic time-varying system with period M, the observer-based state feedback control system G is given by

$$\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} A - BT_kF_k & BT_kF_k \\ O_{n,n} & A - L_kS_kC \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix} + \begin{bmatrix} B_2 & O_{n,q} \\ B_2 & -L_kS_kD \end{bmatrix} \begin{bmatrix} d_u(k) \\ d_w(k) \end{bmatrix} \tag{21}$$

$$y(k) = \begin{bmatrix} S_kC & O_{q,n} \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix} + \begin{bmatrix} O_{q,m_2} & S_kD \end{bmatrix} \begin{bmatrix} d_u(k) \\ d_w(k) \end{bmatrix} \tag{22}$$

For example, to configure a state observer in a periodic time-varying system for a one-input, two-output system in which inputs are applied, and outputs are observed at the periods and timings shown in Fig.1. The observer-based feedback control system can be illustrated by Fig. 2

In (21) and (22), the feedback gains F_k and observer gains L_k should be designed to reduce the effects of the process noise d_u and the observation noise d_w. In addition to (21) and (22), an evaluation signal $z(k) \in \mathbb{R}^{q_2}$ is introduced for design the feedback gains F_k and observer gains L_k as follow:

$$z(k) = E_1 \begin{bmatrix} x(k) \\ e(k) \end{bmatrix} + E_2 \begin{bmatrix} d_u(k) \\ d_w(k) \end{bmatrix} \tag{23}$$
$$E_1 = \begin{bmatrix} E_{11} & E_{12} \end{bmatrix}, \quad E_2 = \begin{bmatrix} E_{21} & E_{22} \end{bmatrix}$$

We will design F_k and L_k based on the input-output relation from the disturbance signals $d(k) := \left[d_u(k)^T, d_w(k)^T \right]^T$ to $z(k)$. When we select $E_1 = \left[I_n \ O_{n,n} \right]$ and $E_2 = O_{1,q+m_2}$, the state of the plant $x(k)$ can be evaluated by (23). In addition, when it is allowed to use periodically time-varying evaluation signals, the evaluation signal can be set as $z(k) = y(t)$ by selecting periodically time-varying matrices $E_1 = \left[S_k C \ 0 \right]$ and $E_2 = \left[0 \ S_k D \right]$. We can use various types of evaluation signals by appropriate selection of E_1 and E_2. Specifically, the discrete-time system from d to z is denoted as G, and the optimal design of the feedback gains F_k, and the observer gains L_k can be done by dealing with the problem of stabilizing G and minimizing its l_2-induced norm.

IV. TIME-INVARIANT REPRESENTATION OF MULTI-RATE SYSTEM

A. CYCLIC REFORMULATION OF PLANT

In this section, we consider handling the system as a linear time-invariant discrete-time system by applying the cyclic reformulation to the periodic time-varying system (21), (22) and (23). Then, we define signals $\check{u}(k)$ which is cycled signal of $u(k)$ [30].

$$\check{u}(0) = \begin{bmatrix} u(0) \\ O_{1,m} \\ \vdots \\ O_{1,m} \end{bmatrix}, \quad \check{u}(1) = \begin{bmatrix} O_{1,m} \\ u(1) \\ O_{1,m} \\ \vdots \\ O_{1,m} \end{bmatrix}, \cdots,$$
$$\check{u}(M-1) = \begin{bmatrix} O_{1,m} \\ \vdots \\ O_{1,m} \\ u(M-1) \end{bmatrix}, \quad \check{u}(M) = \begin{bmatrix} u(M) \\ O_{1,m} \\ \vdots \\ O_{1,m} \end{bmatrix}, \cdots \tag{24}$$

$\check{u}(k)$ are column vectors with M elements. The signals $\check{u}(k)$ are shifted down by a non-zero element over time, and the shift from $M-1$ to M is repeated with the element at the top. The signal $\check{u}(k)$ is called the cycled signal of $u(k)$. Furthermore, $x(k), y(k), z(k), d_u(k)$, and $d_w(k)$ are also considered in cyclic reformulation by the same definition, yielding $\check{x}(k), \check{y}(k), \check{z}(k), \check{d}_u(k)$, and $\check{d}_w(k)$. In addition, we define signal $\check{d}(k)$ as $\check{d}(k) := \left[\check{d}_u(k)^T, \check{d}_w(k)^T \right]^T$.

Let $\check{P}$ be the cyclic reformulation of the system with period M represented by (6), (7), which is expressed as follows [30]:

$$\check{P} : \begin{cases} \check{x}(k+1) &= \check{A}\check{x}(k) + \check{B}\check{u}(k) + \check{B}_2\check{d}_u(k) \\ \check{y}(k) &= \check{C}\check{x}(k) + \check{D}\check{d}_w(k), \end{cases} \tag{25}$$

where the matrices $\check{A}$, $\check{B}$, $\check{B}_2$, $\check{C}$, and $\check{D}$ are given by

$$\check{A} = \begin{bmatrix} O_{n,n} & \cdots & \cdots & O_{n,n} & A \\ A & O_{n,n} & \cdots & O_{n,n} & O_{n,n} \\ O_{n,n} & A & \ddots & \vdots & \vdots \\ O_{n,n} & \ddots & \ddots & O_{n,n} & \vdots \\ O_{n,n} & \cdots & O_{n,n} & A & O_{n,n} \end{bmatrix} \tag{26}$$

$$\check{B} = \begin{bmatrix} O_{n,m} & \cdots & \cdots & O_{n,m} & BT_{M-1} \\ BT_0 & O_{n,m} & \cdots & O_{n,m} & O_{n,m} \\ O_{n,m} & BT_1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & O_{n,m} & \vdots \\ O_{n,m} & \cdots & O_{n,m} & BT_{M-2} & O_{n,m} \end{bmatrix} \tag{27}$$

$$\check{B}_2 = \begin{bmatrix} O_{n,m_2} & \cdots & \cdots & O_{n,m_2} & B_2 \\ B_2 & O_{n,m_2} & \cdots & O_{n,m_2} & O_{n,m_2} \\ O_{n,m_2} & B_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & O_{n,m_2} & \vdots \\ O_{n,m_2} & \cdots & O_{n,m_2} & B_2 & O_{n,m_2} \end{bmatrix} \tag{28}$$

$$\check{C} = \mathrm{diag}\left[S_0 C, S_1 C, \cdots, S_{M-1} C \right] \tag{29}$$
$$\check{D} = \mathrm{diag}\left[S_0 D, S_1 D, \cdots, S_{M-1} D \right] \tag{30}$$
$$\check{E}_{11} = \mathrm{diag}\left[E_{11}, E_{11}, \cdots, E_{11} \right] \tag{31}$$
$$\check{E}_{12} = \mathrm{diag}\left[E_{12}, E_{12}, \cdots, E_{12} \right] \tag{32}$$
$$\check{E}_{21} = \mathrm{diag}\left[E_{21}, E_{21}, \cdots, E_{21} \right] \tag{33}$$
$$\check{E}_{22} = \mathrm{diag}\left[E_{22}, E_{22}, \cdots, E_{22} \right] \tag{34}$$

As an example, if state $\check{x}(0) = [x(0)^T, 0, \cdots, 0]^T$, then using $\check{u}(0)$, $\check{d}_u(0)$, we obtain $\check{x}(1)$ as (35).

$$\check{x}(1) = \begin{bmatrix} O_{n,1} \\ Ax(0) + BT_0 u(0) + B_2 d_u(0) \\ O_{n,1} \\ \vdots \\ O_{n,1} \end{bmatrix} \tag{35}$$

In this case, the elements of $\check{x}(1)$ match those of $x(1)$ by (6). Furthermore, using (25) for time evolution, the non-zero elements of $\check{x}(k)$ always correspond to $x(k)$. This period is a full cycle in M steps. This shows that $\check{x}(k)$ obtained by (25) is a cycled signal of state $x(k)$. Similarly, $\check{y}(k)$ is a cycled signal of $y(k)$. As a result, the cycled system $\check{P}$ can be regarded as a linear time-invariant discrete-time system in spite of P being a time-varying system.

B. TIME-INVARIANT REPRESENTATION OF OBSERVER-BASED FEEDBACK CONTROL SYSTEM

Similar to the cycled system $\check{P}$, which is given in the previous section, the cyclic reformulation of the periodic time-varying state observer (17) can be formulated as

$$\check{x}_{ob}(k+1) = (\check{A} - \check{L}\check{C})\check{x}_{ob}(k) + \check{B}\check{u}(k) + \check{L}\check{y}(k), \tag{36}$$

where the matrix $\check{L}$ is given by

$$\check{L} = \begin{bmatrix} O_{n,q} & \cdots & \cdots & O_{n,q} & L_{M-1} \\ L_0 & O_{n,q} & \cdots & O_{n,q} & O_{n,q} \\ O_{n,q} & L_1 & \ddots & \vdots & \vdots \\ O_{n,q} & \ddots & \ddots & O_{n,q} & \vdots \\ O_{n,q} & \cdots & O_{n,q} & L_{M-2} & O_{n,q} \end{bmatrix} \tag{37}$$

It is obvious that (36) is a time-invariant representation of (17) if the cycled input $\check{u}(k)$ and cycled output $\check{y}(k)$ is given.

The cycled signals $\check{x}_{ob}(k)$ and $\check{e}(k)$ are obtained by $x_{ob}(k)$ and $e(k)$. By using the feedback gains F_k, the cyclic matrix $\check{F}$ is given as follow:

$$\check{F} = \mathrm{diag}\,[F_0, \cdots, F_{M-1}] \tag{38}$$

The plant $\check{P}$ with the input $\check{u}(k) = -\check{F}\check{x}_{ob}(k)$ is formulated as follow:

$$\check{x}(k+1) = \check{A}\check{x}(k) - \check{B}\check{F}\check{x}_{ob}(k) + \check{B}_2\check{d}_u(k) \tag{39}$$

Then, the cyclic reformulation of the state estimation error (20) is given by

$$\check{e}(k+1) = (\check{A} - \check{L}\check{C})\check{e}(k) + \check{B}_2\check{d}_u(k) - \check{L}\check{D}\check{d}_w(k) \tag{40}$$

In consequently, the cyclic reformulation of the observer-based feedback control system $\check{G}$ can be given by using the original system G as follows:

$$\begin{bmatrix} \check{x}(k+1) \\ \check{e}(k+1) \end{bmatrix} = \check{A}_G \begin{bmatrix} \check{x}(k) \\ \check{e}(k) \end{bmatrix} + \check{B}_G \begin{bmatrix} \check{d}_u(k) \\ \check{d}_w(k) \end{bmatrix}, \tag{41}$$

$$\check{y}(k) = \check{C}_G \begin{bmatrix} \check{x}(k) \\ \check{e}(k) \end{bmatrix} + \check{D}_G \begin{bmatrix} \check{d}_u(k) \\ \check{d}_w(k) \end{bmatrix}, \tag{42}$$

$$\check{z}(k) = \check{E}_1 \begin{bmatrix} \check{x}(k) \\ \check{e}(k) \end{bmatrix} + \check{E}_2 \begin{bmatrix} \check{d}_u(k) \\ \check{d}_w(k) \end{bmatrix}, \tag{43}$$

where $\check{A}_G$, $\check{B}_G$, $\check{C}_G$, $\check{D}_G$, $\check{E}_1$ and $\check{E}_2$ are given by

$$\check{A}_G = \begin{bmatrix} \check{A} - \check{B}\check{F} & \check{B}\check{F} \\ O_{n,n} & \check{A} - \check{L}\check{C} \end{bmatrix},$$
$$\check{B}_G = \begin{bmatrix} \check{B}_2 & O_{n,q} \\ \check{B}_2 & -\check{L}\check{D} \end{bmatrix},$$
$$\check{C}_G = \begin{bmatrix} \check{C} & O_{q,n} \end{bmatrix}, \quad \check{D}_G = \begin{bmatrix} O_{q,m_2} & \check{D} \end{bmatrix},$$
$$\check{E}_1 = \begin{bmatrix} \check{E}_{11} & \check{E}_{12} \end{bmatrix},$$
$$\check{E}_2 = \begin{bmatrix} \check{E}_{21} & \check{E}_{22} \end{bmatrix}.$$

We can see that the system $\check{G}$, which indicates an input-output relation from $\check{d}(k)$ to $\check{z}(k)$, is regarded as a linear time-invariant system.

V. ANALYSIS AND SYNTHESIS OF OBSERVER-BASED FEEDBACK CONTROL SYSTEM UNDER MULTI-RATE ENVIRONMENT

A. PERFORMANCE ANALYSIS BASED ON L_2-INDUCED NORM

In this section, the analysis method of G in the meaning of the l_2-induced norm is presented under an assumption that periodically time-varying observer gain L_k and feedback gain F_k are given. The l_2-induced norm is widely used as an evaluation signal in control engineering fields. It evaluates the worst-case impact amount for arbitrary disturbance or noise signals. Since $\check{G}$ is the equivalent system of G if $\check{d}_u$ and $\check{d}_w$ are given as cycled signals. A theorem about the l_2-induced norm for $\check{G}$ with given $\check{L}$ and $\check{F}$ is given as follows.

Theorem 5.1: Let $\check{G}$ be a linear time-invariant discrete-time system formulated as (41) and (42) with the disturbance $\check{d}(k)$, and the evaluation signal $\check{z}(k)$. For given γ, $\check{L}$ and $\check{F}$, the following conditions (i),(ii),(iii) are equivalent.

(i) The matrix $\check{A}_G$ is Schur stable and $\|\check{G}\|_{l_2/l_2} < \gamma$.

(ii) There exists $\check{X} > 0$ satisfying the following LMI.

$$\begin{bmatrix} -\check{X} + \check{A}_G\check{X}\check{A}_G^T + \check{B}_G\check{B}_G^T & \check{A}_G\check{X}\check{E}_1^T + \check{B}_G\check{E}_2^T \\ \check{E}_1\check{X}\check{A}_G^T + \check{E}_2\check{B}_G^T & \check{E}_1\check{X}\check{E}_1^T + \check{E}_2\check{E}_2^T - \gamma^2 I_{1*} \end{bmatrix} < 0 \tag{44}$$

The matrix size of the identity matrix I_{1*} is $Mq_2 \times Mq_2$.

(iii) There exists $\check{P} > 0$ satisfying the following LMI.

$$\begin{bmatrix} -\check{P} + \check{A}_G^T\check{P}\check{A}_G + \check{E}_1^T\check{E}_1 & \check{A}_G^T\check{P}\check{B}_G + \check{E}_1^T\check{E}_2 \\ \check{B}_G^T\check{P}\check{A}_G + \check{E}_2^T\check{E}_1 & \check{B}_G^T\check{P}\check{B}_G + \check{E}_2^T\check{E}_2 - \gamma^2 I_{2*} \end{bmatrix} < 0 \tag{45}$$

The matrix size of the identity matrix I_{2*} is $M(m+m_2) \times M(m+m_2)$.

Proof: For the case of the discrete-time linear time-invariant system, the necessary and sufficient conditions about LMIs for the l_2-induced norm have been proven in such as [37] and [38]. The system $(\check{A}_G, \check{B}_G, \check{E}_1, \check{E}_2)$, which is represented by using cyclic reformulation, is regarded as a discrete-time linear time-invariant system. Therefore, the equivalency of the conditions (i),(ii),(iii) is possible to prove in the same manner of [37] and [38]. □

The l_2-induced norm from $\check{d}_u(k)$ and $\check{d}_w(k)$ to $\check{z}(k)$ can be characterized by γ. The condition in (ii) is also LMIs in case γ^2 is given as a design variable. This fact also holds for (iii). When we can solve minimum value of γ by (ii) or (iii), the l_2-induced norm of $\check{G}$ can be characterized by Theorem 5.1-(i). The conditions (ii) and (iii) are easy to analyze numerically by using standard SDP-solvers.

Here, from the definition of the cyclic reformulation of signals, for any of the signals $\check{z}(k)$ and $\check{d}(k)$, all non-zero elements of the cycled signal and the signal in the original system match perfectly at all times. As a result, the l_2-norms of the signals in the original system and the cycled signal are equal. Therefore, with respect to the l_2-induced norms of the original system and the cyclic reformulated system, it holds that $\|G\|_{l_2/l_2} = \|\check{G}\|_{l_2/l_2}$ [39]. Consequently, we can analyze l_2-induced norm performance of the system (21), (22), (23) by Theorem 5.1.

B. DESIGN PROBLEM OF OBSERVER GAINS AND FEEDBACK GAINS

In this section, a design algorithm of F_k and L_k is introduced based on Theorem 5.1. We want to design F_k and L_k for $k = 0, \cdots, M-1$, which minify γ. When $\check{F}$ and $\check{L}$ are regarded as variable matrices in Theorem 5.1, the conditions in (ii) and (iii) are not regarded as LMI conditions.

An iterative design algorithm of $\check{F}$ and $\check{L}$ is presented based on (ii) in Theorem 5.1. Design problem of $\check{F}$ and $\check{L}$ can be written as follow:

At the first step for designing the observer-based state feedback controller, an initial solution of $\check{F}$ and $\check{L}$ is introduced by the following step.

Theorem 5.2: Let $\check{G}_e$ be a linear time-invariant discrete-time system with $\check{d}_u$ and $\check{d}_w$ as inputs and $\check{e}$ as a state estimated error. For a given γ_L and $\check{L}$, the following conditions are equivalent where $\check{A}_e = \check{A} - \check{L}\check{C}$.

(i) The matrix $\check{A}_e$ is Schur stable and $||\check{G}_e||_{l_2/l_2} < \gamma_L$.

(ii) There exists $\check{P}_L > 0$ satisfying the following LMI.

$$\begin{bmatrix} \check{P}_L & \check{P}_L\check{A} + \check{Y}\check{C} & \check{P}_L\check{B}_2 & \check{Y}\check{D} \\ (\check{P}_L\check{A} + \check{Y}\check{C})^T & \check{P}_L - I & O_{Mn,Mm_2} & O_{Mn,Mq} \\ (\check{P}_L\check{B}_2)^T & O_{Mm_2,Mn} & \gamma_L^2 I & O_{Mm_2,Mq} \\ (\check{Y}\check{D})^T & O_{Mq,Mn} & O_{Mq,Mm_2} & \gamma_L^2 I \end{bmatrix} > 0 \quad (46)$$

Proof: By applying Schur complement to (iii) of Theorem 5.1, and performing the variable transformation $\check{Y} = -\check{P}_L\check{L}$, we can prove this theorem using the same procedure as in Theorem 5.1. □

Here, constraining the structure to be $\check{P}_L = \mathrm{diag}[P_{L,M-1}, P_{L,0}, \cdots, P_{L,M-2}]$, the structure of $\check{Y}$ is given by

$$\check{Y} = \begin{bmatrix} O_{n,q} & \cdots & \cdots & O_{n,q} & Y_{M-1} \\ Y_0 & O_{n,q} & \cdots & O_{n,q} & O_{n,q} \\ O_{n,q} & Y_1 & \ddots & \vdots & \vdots \\ O_{n,q} & \ddots & \ddots & O_{n,q} & \vdots \\ O_{n,q} & \cdots & O_{n,q} & Y_{M-2} & O_{n,q} \end{bmatrix} \quad (47)$$

Under this constraint, if $\check{Y}$ and $\check{P}_L$ are found to minimize γ_L^2 in (46), $\check{L}$ can be obtained from $\check{L} = -\check{P}_L^{-1}\check{Y}$. When $\check{L}$ can be obtained, the periodic time-varying observer gains L_k can be obtained as their elements. In other words, the observer gains L_k are obtained by the following equation.

$$L_k = -P_{L,k}^{-1}Y_k, \quad k = 0, \cdots, M-1 \quad (48)$$

Consequently, an initial solution $\check{L}$ can be derived using Theorem 5.2.

Then, we address the problem of designing $\check{F}$ based on the l_2-induced norm from the input disturbance $\check{d}_u$ to the state $\check{x}$ by the following theorem.

Theorem 5.3: Let $\check{G}_F$ be a linear time-invariant discrete-time plant (25) with $\check{u}(k) = -\check{F}\check{x}(k)$. $\|\check{G}_F\|_{l_2/l_2}$ is the l_2-induced norm from the disturbance $\check{d}_u(k)$ to the state $\check{x}(k)$. For a given γ_F and $\check{F}$, the following conditions are equivalent where $\check{A}_F = \check{A} - \check{B}\check{F}$.

(i) The matrix $\check{A}_F$ is Schur stable and $\|\check{G}_F\|_{l_2/l_2} < \gamma_F$ holds.

(ii) There exists $\check{X}_F > 0$ satisfying the following LMI.

$$\begin{bmatrix} \check{X} & (\check{A}\check{X}_F - \check{B}\check{Y})^T & \check{X}_F\check{E}_1^T \\ \check{A}\check{X}_F - \check{B}\check{Y} & \check{X}_F - \check{B}_2\check{B}_2^T & -\check{B}_2\check{E}_2^T \\ \check{E}_1\check{X}_F & -\check{E}_2\check{B}_2^T & -\check{E}_2\check{E}_2^T + \gamma_F^2 I \end{bmatrix} > 0 \quad (49)$$

Proof: By applying Schur complement to (ii) of Theorem 5.1, and performing the variable transformation $\check{Y} = -\check{F}\check{X}_F$, we can prove this theorem using the same procedure as in Theorem 5.1. □

In addition, (49) also becomes an LMI condition when not only $\check{Y}$ and $\check{X}_F$ but also γ_F^2 are design variables. By further constraining the structure of $\check{Y}$ and $\check{X}_F$ to $\check{X}_F = \mathrm{diag}[X_{F,0}, X_{F,1}, \cdots, X_{F,M-1}]$ and $\check{Y} = \mathrm{diag}[Y_0, Y_1, \cdots, Y_{N-1}]$, respectively. The periodic time-varying feedback gains F_k can be obtained as elements of $\check{F}$. The feedback gain $\mathcal{F}_k$ is given by the following equation.

$$\mathcal{F}_k = -Y_k X_{F,k}^{-1} \quad (50)$$

Consequently, an initial solution $\check{F}$ can be derived using Theorem 5.3.

Then, the design algorithm of F_k and L_k is considered based on an analysis method of the observer-based feedback control system, shown in Theorem 5.1. First, an equivalent LMI condition of the condition in Theorem 5.1-(ii) is obtained using the Schur complement. When we fix $\check{P}$ in the LMI condition in Theorem 5.1-(ii), the following design problem of $\check{L}$ and $\check{F}$ can be dealt with.

Problem 1: For a given system (41), (42), (43), and fixed matrix $\check{P} > 0$. Find $\check{L}$ and $\check{F}$, which minimize γ under the following linear matrix inequality.

$$\begin{bmatrix} \check{P} & \check{P}\check{A}_G & \check{P}\check{B}_G \\ \check{A}_G^T\check{P} & \check{P} - \check{E}_1^T\check{E}_1 & -\check{E}_1^T\check{E}_2 \\ \check{B}_G^T\check{P} & -\check{E}_2^T\check{E}_1 & \gamma_L^2 I - \check{E}_2^T\check{E}_2 \end{bmatrix} > 0 \quad (51)$$

Based on the above results, the design algorithm for L_k and F_k using Theorem 5.1, Theorem 5.2, Theorem 5.3 and Problem 1 is given as follows. The number of iterations is set to be given as $i_{\max}$.

Algorithm 1 Gain Design of $\check{L}$ and $\check{F}$

[1.] Theorem 5.2 is used to give the initial observer gain $\check{L}$.
[2.] Theorem 5.2 is used to give the initial feedback gain $\check{F}$.
[3.] Assume initial value $i = 0$.
loop
  [2.] Find γ^2 and $\check{P}$, which minimize γ of the LMI in Theorem 5.1-(ii) under the condition of $\check{L}$ and $\check{F}$ be given. The solution $\check{P}$ is updated.
  [3.] Find $\check{L}$ and $\check{F}$ by solving LMIs in Problem 1 under the condition of $\check{P}$ is given by [2.]. Then, the solutions $\check{L}$ and $\check{F}$ are updated.
  [4.] Assume update count $i = i + 1$.
  [5.] If the number of updates $i = i_{\max}$ holds, the algorithm is finished.
end loop

In this algorithm, the value of γ decreases monotonically with each update. By setting a sufficient number of update iterations $i_{\max}$, we can obtain $\check{L}$ and $\check{F}$ for which the l_2-induced norm performance γ of $\check{G}$ can become small. The periodic time-varying gains L_k and F_k can be obtained as the elements of $\check{L}$ and $\check{F}$ by (37) and (38).

Since the LMI optimization problem is not difficult to solve numerically, we are able to obtain an appropriate solution for a system with multiple sensing and actuating periods by Algorithm 1.

VI. NUMERICAL EXAMPLE

In this section, we illustrate the effectiveness of the observer-based state feedback controller by numerical simulations. The parameters for the simulation are given as follows. Consider a discrete-time linear time-invariant plant with 2 inputs, 3 states, and 2 outputs. Plant parameters A, B, B_2, C, and D are given by

$$A = \begin{bmatrix} 0.92 & 0.1 & 0.2 \\ -0.15 & 1.1 & -0.25 \\ -0.1 & 0.3 & 0.95 \end{bmatrix}, \quad B = B_2 = \begin{bmatrix} 1 & -2 \\ -1 & 1.5 \\ 2 & 0.5 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0.5 & 0.2 \\ -0.3 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that the plant is an unstable system. Also, we assume that the periods of the inputs u_1, u_2 and outputs y_1, y_2 are $M_1 = 2$, $M_2 = 3$, $\mathcal{M}_1 = 2$ and $\mathcal{M}_2 = 4$, respectively. Since the least common multiple of $M_1, M_2, \mathcal{M}_1$, and $\mathcal{M}_2$ is 12, we consider constructing a periodic time-varying system with period $M = 12$. It seems that this plant appears to be challenging to control.

For simplicity, consider the case where u_1 and u_2 are inputted at times k, which are multiples of 2 and 3, respectively. In this case, the matrices $T_k = T_{k\,mod\,12}$, which characterize the timings of the application of the control input, are given at each time as follows.

$$\begin{aligned} T_0 &= \mathrm{diag}\,[1, 1], & T_1 &= \mathrm{diag}\,[0, 0], \\ T_2 &= \mathrm{diag}\,[1, 0], & T_3 &= \mathrm{diag}\,[0, 1], \\ T_4 &= \mathrm{diag}\,[1, 0], & T_5 &= \mathrm{diag}\,[0, 0], \\ T_6 &= \mathrm{diag}\,[1, 1], & T_7 &= \mathrm{diag}\,[0, 0], \\ T_8 &= \mathrm{diag}\,[1, 0], & T_9 &= \mathrm{diag}\,[0, 1], \\ T_{10} &= \mathrm{diag}\,[1, 0], & T_{11} &= \mathrm{diag}\,[0, 0] \end{aligned}$$

Similarly, consider the case where y_1 and y_2 are observed at times k, which are multiples of 2 and 4, respectively. In this case, the matrices $S_k = S_{k\,mod\,12}$, which characterize the observation timings, are given at each time as follows.

$$\begin{aligned} S_0 &= \mathrm{diag}\,[0, 1], & S_1 &= \mathrm{diag}\,[1, 0], \\ S_2 &= \mathrm{diag}\,[0, 0], & S_3 &= \mathrm{diag}\,[1, 0], \\ S_4 &= \mathrm{diag}\,[0, 1], & S_5 &= \mathrm{diag}\,[1, 0], \\ S_6 &= \mathrm{diag}\,[0, 0], & S_7 &= \mathrm{diag}\,[1, 0], \\ S_8 &= \mathrm{diag}\,[0, 1], & S_9 &= \mathrm{diag}\,[1, 0], \\ S_{10} &= \mathrm{diag}\,[0, 0], & S_{11} &= \mathrm{diag}\,[1, 0], \end{aligned}$$

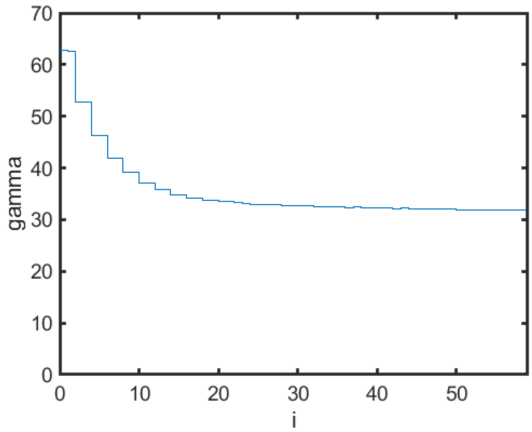


FIGURE 3. Transition of γ by iteration.

$E_1 = \begin{bmatrix} I_3 & O_{3,3} \end{bmatrix}$ and $E_2 = O_{1,4}$ are selected in (23) for minimizing the effect for the state of the plant. The time-varying observer gain $L_k = L_{k\,mod\,12}$ and state feedback gain $F_k = F_{k\,mod\,12}$ can be derived by solving Algorithm 1 in Section V-B. In this paper, $i_{\max} = 60$ is applied in the algorithm.

The transition of γ by iteration in Algorithm 1 is shown in Fig. 3. It can be seen that the value of γ decreases as the number of iterations increases. In this algorithm, $\gamma = 31.8283$ is obtained and γ characterize the l_2 induced norm from $d(k)$ to $x(k)$.

The resulting feedback gains are given by

$$F_1 = F_5 = F_7 = F_{11} = O_{2,3}$$
$$F_0 = \begin{bmatrix} -0.08 & -0.17 & 0.44 \\ -0.26 & 0.37 & 0.12 \end{bmatrix},$$
$$F_2 = \begin{bmatrix} 0.19 & 0.03 & -0.23 \\ 0 & 0 & 0 \end{bmatrix},$$
$$F_3 = \begin{bmatrix} 0 & 0 & 0 \\ -0.22 & -0.44 & 0.01 \end{bmatrix},$$
$$F_4 = \begin{bmatrix} 0.11 & -0.30 & 0.37 \\ 0 & 0 & 0 \end{bmatrix},$$
$$F_6 = \begin{bmatrix} -0.13 & -0.23 & 0.43 \\ -0.26 & 0.33 & 0.14 \end{bmatrix},$$
$$F_8 = \begin{bmatrix} 0.10 & -0.23 & 0.12 \\ 0 & 0 & 0 \end{bmatrix},$$
$$F_9 = \begin{bmatrix} 0 & 0 & 0 \\ -0.25 & 0.41 & -0.06 \end{bmatrix},$$
$$F_{10} = \begin{bmatrix} 0.12 & -0.41 & 0.35 \\ 0 & 0 & 0 \end{bmatrix},$$

Similarly, the obtained observer gains are given by

$$L_2 = L_6 = L_{10} = O_{3,2}$$
$$L_0 = \begin{bmatrix} 0 & -0.17 \\ 0 & 0.87 \\ 0 & 0.40 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0.98 & 0 \\ -0.63 & 0 \\ 0.04 & 0 \end{bmatrix},$$

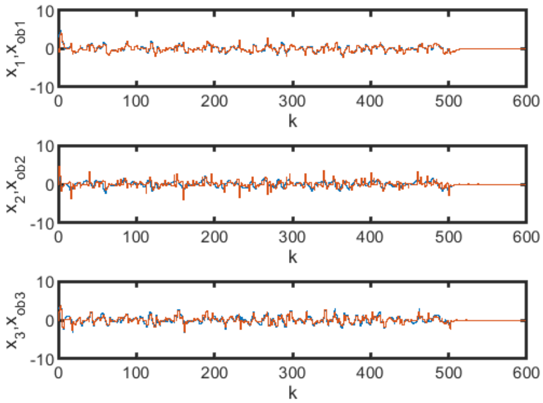


FIGURE 4. State and estimated state.

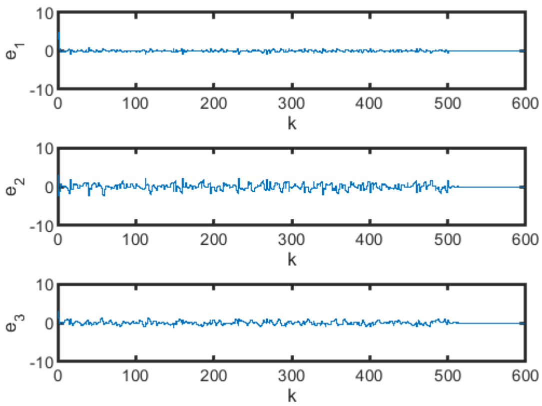


FIGURE 5. Estimation error.

$$L_3 = \begin{bmatrix} -1.08 & 0 \\ 5.71 & 0 \\ 2.38 & 0 \end{bmatrix}, \quad L_4 = \begin{bmatrix} 0 & -0.13 \\ 0 & 0.79 \\ 0 & 0.43 \end{bmatrix},$$
$$L_5 = \begin{bmatrix} 1.00 & 0 \\ -0.72 & 0 \\ 0.03 & 0 \end{bmatrix}, \quad L_7 = \begin{bmatrix} -0.51 & 0 \\ 4.44 & 0 \\ 1.78 & 0 \end{bmatrix},$$
$$L_8 = \begin{bmatrix} 0 & -0.11 \\ 0 & 0.79 \\ 0 & 0.43 \end{bmatrix}, \quad L_9 = \begin{bmatrix} 1.02 & 0 \\ -0.63 & 0 \\ 0.00 & 0 \end{bmatrix},$$
$$L_{11} = \begin{bmatrix} -0.26 & 0 \\ 1.82 & 0 \\ 1.04 & 0 \end{bmatrix}$$

We can find that different gains are obtained at different times within the period. It can be confirmed that $\check{A}_F$, and $\check{A}_L$ are all stable, and the entire system is stabilized by the above gains F_k and L_k.

Next, the initial states of the controlled object and observer are set to $x(0) = [3, 3, 3]^T$, $x_{ob}(0) = [0, 0, 0]^T$, $d_u(k)$ are random numbers taken from a normal distribution with mean 0 and standard deviation 0.1, and $d_w(k)$ are random numbers taken from the normal distribution with mean 0 and standard deviation 0.05. Fig.4 shows the state of the plant, the dashed line shows the estimated state, and Fig.5 shows the estimation error for each state value.

Fig.4 shows that all states of the plant are near to 0 (stabilized) by state feedback. Fig.5 shows that all estimation errors are near to 0, it indicates that all states are accurately estimated. In the simulation results, $d_u(k) = 0$, $d_w(k) = 0$ are set after $k = 500$, and the ratio of the l_2 norm of the noises to the state is measured up to $k = 600$. In $k = 600$, $x(k)$ becomes sufficiently small. The l_2-norm ratio is found to be $G_{\text{sim}} = 9.9733$. Therefore, we can find that G_{sim} is smaller than the value $\gamma = 31.8283$.

VII. CONCLUSION

This paper proposes an observer-based state feedback control system for a multi-rate system, where the control and observation periods are different. The multi-rate system can be represented as a periodically time-varying system, which can be treated as a linear time-invariant system using cyclic reformulation. The analytical conditions are obtained as LMI conditions based on the l_2-induced norm performance. In addition, the multi-rate observer gains and feedback gains are possible to be designed based on the analytical conditions with an iteration algorithm. One of the advantages of this paper compared to other previous research is its ability to deal with various combinations of different sampling periods using the appropriate choice of S_k and T_k. It is easy to design an observer-based feedback system for plants with multiple sensing and actuating periods.

Not only the l_2-induced norm but also other evaluation indexes are able to handle under the LMI framework of LTI systems. Our future works include the development of design methods for multi-rate systems based on other evaluation indexes.

REFERENCES

[1] A. Al-Fuqaha, M. Guizani, M. Mohammadi, M. Aledhari, and M. Ayyash, "Internet of Things: A survey on enabling technologies, protocols, and applications," *IEEE Commun. Surveys Tuts.*, vol. 17, no. 4, pp. 2347–2376, 4th Quart., 2015.

[2] S. Lowry, N. Sünderhauf, P. Newman, J. J. Leonard, D. Cox, P. Corke, and M. J. Milford, "Visual place recognition: A survey," *IEEE Trans. Robot.*, vol. 32, no. 1, pp. 1–19, Feb. 2016.

[3] R. Mur-Artal, J. M. M. Montiel, and J. D. Tardós, "Orb-SLAM: A versatile and accurate monocular SLAM system," *IEEE Trans. Robot.*, vol. 31, no. 5, pp. 1147–1163, Oct. 2015.

[4] D. E. Zlotnik and J. R. Forbes, "Gradient-based observer for simultaneous localization and mapping," *IEEE Trans. Autom. Control*, vol. 63, no. 12, pp. 4338–4344, Dec. 2018.

[5] Y. Cui, Y. Jia, Y. Li, J. Shen, T. Huang, and X. Gong, "Byzantine resilient joint localization and target tracking of multi-vehicle systems," *IEEE Trans. Intell. Vehicles*, vol. 8, no. 4, pp. 2899–2913, Apr. 2023.

[6] P. Yang, D. Duan, C. Chen, X. Cheng, and L. Yang, "Multi-sensor multi-vehicle (MSMV) localization and mobility tracking for autonomous driving," *IEEE Trans. Veh. Technol.*, vol. 69, no. 12, pp. 14355–14364, Dec. 2020.

[7] L. Yang, Y. Gao, Z. Feng, and L. Wu, "Event-based output feedback control of periodic systems: A piecewise impulsive method," *Int. J. Robust Nonlinear Control*, vol. 31, no. 16, pp. 7569–7586, Nov. 2021.

[8] L. Yang, Y. Gao, Z. Feng, and L. Wu, "Distributed filtering with dynamic event-triggered communication for networked periodic systems," *IEEE Trans. Control Netw. Syst.*, vol. 10, no. 1, pp. 26–37, Mar. 2023.

[9] A. F. Taha, A. Elmahdi, J. H. Panchal, and D. Sun, "Unknown input observer design and analysis for networked control systems," *Int. J. Control*, vol. 88, no. 5, pp. 920–934, 2015.

[10] H. Okajima, Y. Kaneda, and N. Matsunaga, "State estimation method using median of multiple candidates for observation signals including outliers," *SICE J. Control, Meas., Syst. Integr.*, vol. 14, no. 1, pp. 257–267, Jan. 2021.

[11] A. Alessandri and M. Awawdeh, "Moving-horizon estimation with guaranteed robustness for discrete-time linear systems and measurements subject to outliers," *Automatica*, vol. 67, pp. 85–93, May 2016.

[12] I. G. Polushin and H. J. Marquez, "Multirate versions of sampled-data stabilization of nonlinear systems," *Automatica*, vol. 40, no. 6, pp. 1035–1041, Jun. 2004.

[13] H. Fujimoto, Y. Hori, and A. Kawamura, "Perfect tracking control based on multirate feedforward control with generalized sampling periods," *IEEE Trans. Ind. Electron.*, vol. 48, no. 3, pp. 636–644, Jun. 2001.

[14] K. Nam, H. Fujimoto, and Y. Hori, "Advanced motion control of electric vehicles based on robust lateral tire force control via active front steering," *IEEE/ASME Trans. Mechatronics*, vol. 19, no. 1, pp. 289–299, Feb. 2014.

[15] Y. Wang, H. Fujimoto, and S. Hara, "Driving force distribution and control for EV with four in-wheel motors: A case study of acceleration on split-friction surfaces," *IEEE Trans. Ind. Electron.*, vol. 64, no. 4, pp. 3380–3388, Apr. 2017.

[16] J. H. Ahrens, X. Tan, and H. K. Khalil, "Multirate sampled-data output feedback control with application to smart material actuated systems," *IEEE Trans. Autom. Control*, vol. 54, no. 11, pp. 2518–2529, Nov. 2009.

[17] A. K. Tangirala, D. Li, R. S. Patwardhan, S. L. Shah, and T. Chen, "Ripple-free conditions for lifted multirate control systems," *Automatica*, vol. 37, no. 10, pp. 1637–1645, Oct. 2001.

[18] Y. Gu, Y. Chou, J. Liu, and Y. Ji, "Moving horizon estimation for multirate systems with time-varying time-delays," *J. Franklin Inst.*, vol. 356, no. 4, pp. 2325–2345, Mar. 2019.

[19] C. Ling and C. Kravaris, "Multi-rate observer design using asynchronous inter-sample output predictions," in *Proc. Amer. Control Conf. (ACC)*, May 2017, pp. 376–381.

[20] A. Sferlazza, S. Tarbouriech, and L. Zaccarian, "Time-varying sampled-data observer with asynchronous measurements," *IEEE Trans. Autom. Control*, vol. 64, no. 2, pp. 869–876, Feb. 2019.

[21] M. Zheng, L. Sun, and M. Tomizuka, "Multi-rate observer based sliding mode control with frequency shaping for vibration suppression beyond Nyquist frequency," *IFAC-PapersOnline*, vol. 49, no. 21, pp. 13–18, 2016.

[22] B. Liu, "Speed control for permanent magnet synchronous motor based on an improved extended state observer," *Adv. Mech. Eng.*, vol. 10, no. 1, Jan. 2018, Art. no. 168781401774766.

[23] C.-H. Lien, "Robust observer-based control of systems with state perturbations via LMI approach," *IEEE Trans. Autom. Control*, vol. 49, no. 8, pp. 1365–1370, Aug. 2004.

[24] Q. Hu, B. Jiang, and Y. Zhang, "Observer-based output feedback attitude stabilization for spacecraft with finite-time convergence," *IEEE Trans. Control Syst. Technol.*, vol. 27, no. 2, pp. 781–789, Mar. 2019.

[25] W. Shen and C. Shen, "An extended state observer-based control design for electro-hydraulic position servomechanism," *Control Eng. Pract.*, vol. 109, Apr. 2021, Art. no. 104730.

[26] F. Han, Q. He, Y. Song, and J. Song, "Outlier-resistant observer-based H_∞-consensus control for multi-rate multi-agent systems," *J. Franklin Inst.*, vol. 358, no. 17, pp. 8914–8928, Nov. 2021.

[27] D. Luenberger, "An introduction to observers," *IEEE Trans. Autom. Control*, vol. AC-16, no. 6, pp. 596–602, Dec. 1971.

[28] D. Chu, T. Chen, and H. J. Marquez, "Robust moving horizon state observer," *Int. J. Control*, vol. 80, no. 10, pp. 1636–1650, Oct. 2007.

[29] J. Kim, C. Lee, H. Shim, Y. Eun, and J. H. Seo, "Detection of sensor attack and resilient state estimation for uniformly observable nonlinear systems having redundant sensors," *IEEE Trans. Autom. Control*, vol. 64, no. 3, pp. 1162–1169, Mar. 2019.

[30] H. Okajima, Y. Hosoe, and T. Hagiwara, "State observer under multi-rate sensing environment and its design using l2-induced norm," *IEEE Access*, vol. 11, pp. 20079–20087, 2023.

[31] H. Okajima, "Design of a disturbance reduction control system for multi-rate systems with long input periods," (in Japanese), *Trans. SICE*, vol. 58, no. 10, pp. 451–458, 2022.

[32] S. Bittanti and P. Colaneri, "Invariant representations of discrete-time periodic systems," *Automatica*, vol. 36, no. 12, pp. 1777–1793, 2000.

[33] S. Bittanti and P. Colaneri, *Periodic Systems: Filtering and Control*. London, U.K.: Springer-Verlag, 2009.

[34] P. Colaneri and V. Kucera, "Model matching for periodic systems," in *Proc. Amer. Control Conf.*, Jun. 1997, pp. 3143–3144.

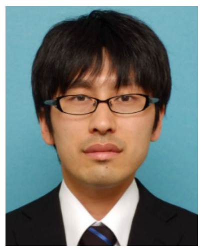
[35] Y. Hosoe, M. Miyamoto, and T. Hagiwara, "Robust output estimator synthesis based on cycling-based LPTV scaling," in *Proc. 20th IFAC World Congr.*, 2017, pp. 7578–7583.

[36] K. Fujimoto, Y. Oji, and K. Hamamoto, "On periodic Kalman filters and multi-rate estimation," in *Proc. IEEE Conf. Control Appl. (CCA)*, Sep. 2016, pp. 934–939.

[37] Y. Ebihara, *System Control With LMI*. Tokyo, Japan: Morikawa Publishing, 2012.

[38] M. C. De Oliveira, J. C. Geromel, and J. Bernussou, "Extended H_2 and H_∞ norm characterizations and controller parametrizations for discrete-time systems," *Int. J. Control*, vol. 75, no. 9, pp. 666–679, Jan. 2002.

[39] Y. Hosoe, M. Miyamoto, and T. Hagiwara, "Cycling-based synthesis of robust output estimators for uncertain LPTV systems," *SICE J. Control, Meas., Syst. Integr.*, vol. 12, no. 2, pp. 39–46, Mar. 2019.



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