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METHODS

Optimal Static Output Feedback Stabilization of Fractional-Order Systems With Caputo Derivative Order $1 \leq \alpha < 2$

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ABSTRACT The design of optimal static output feedback controllers for fractional-order linear systems with Caputo derivative order $1 \leq \alpha < 2$ is addressed in this paper. The cost function to be minimized is the Frobenius norm of the feedback matrix. First, based on the barrier method, we demonstrate how to create an auxiliary minimization problem that is easier to solve than the original optimization problem. The auxiliary problem is then solved numerically to obtain an approximate solution to the original problem. Necessary conditions for the optimal solution of the auxiliary minimization problem are derived using the Lagrange multiplier method. A numerical example is provided to validate the proposed method.

INDEX TERMS Barrier method, Caputo fractional derivative, fractional-order systems, optimal control, static output feedback.

I. INTRODUCTION

There has been a growing interest in fractional-order differential systems (see, for example, [1], [2], [3]). Although it is widely acknowledged as a promising mathematical tool for efficiently characterizing the historical memory and global correlation of complex dynamic systems, phenomena, or structures (see [4], [5], [6], [7], [8], [9], [10], [11]), there is some disagreement on how to define the fractional differential operation [12]. To provide researchers with a solid foundation and background in fractional-order differential systems, many textbooks and surveys have been published [13], [14], [15], [16]. The system architecture discussed in this paper is mentioned in these works. A key distinction in the management of fractional-order differential equations is the different stable regions for pole locations between the ranges $1 \leq \alpha < 2$ and $0 \leq \alpha < 1$. Despite the similarity in the techniques used across both ranges, few studies have focused on the $1 \leq \alpha < 2$ range. This gap motivates our research to explore this particular class of fractional-order derivatives. It is also worth noting that the design method we have developed is also applicable to the $0 \leq \alpha < 1$ range.

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In synthesizing controllers of linear control systems, the Frobenius norm of the designed feedback matrix is often targeted for minimization. Low-gain controllers have been demonstrated in many studies to be desirable due to their ability to reduce control signal saturation in practical control systems and offer several advantages including robustness to nonlinearity and uncertainty, energy efficiency, prevention of saturation, and enhanced control performance (see [17], [18], [19], [20], [21], [22], [23]). Recently, [20] and [24] have considered the low-gain condition in the design of controllers for fractional-order systems. In [20], the authors significantly advanced the field of fractional-order linear systems by developing a novel low-gain state feedback controller based on a unique eigenstructure assignment algorithm. In [24] the authors constructed the stability condition through the small gain theorem and used it to design the corresponding LMIs for controller design.

Over the past few years, the design of static output feedback (SOF) controllers for fractional-order systems has emerged as a challenging problem. The challenges include the non-convex nature of the problem, sensitivity to system parameters, limited information for feedback, difficulty in ensuring stability, and the dependence of many solution approaches on the initial guess (see [25], [26]). Despite these challenges, researchers have made significant strides in the stability analysis and control synthesis of fractional-order systems using the linear matrix inequality (LMI) approach (see [27], [28], [29]). Recently, there has been growing interest in designing controllers for fractional-order systems using various control theories. Sliding mode control was used in [30], and optimal algorithms for designing PID controllers for fractional-order systems were proposed in [31]. Furthermore, optimal control problems of fractional-order systems have been investigated. These problems can be solved by solving the corresponding fractional Euler-Lagrange equations, which were first introduced in [32] and have inspired subsequent research (see [33], [34]). Another solution method is to approximate the fractional-order derivative and fractional-order integration using various polynomials, such as Legendre and shifted Legendre polynomials (see [35], [36], [37]), and obtain the optimal solution. However, all these studies assume that the system state is available for feedback. The state of the system may not be measurable for a practical control system, and only the output signal can be obtained for feedback. However, few results on the static output feedback control of fractional-order control systems have been proposed.

The contributions of this work are as follows:

A. METHODOLOGY DEVELOPMENT

Our work focuses on the development of optimal static output feedback controllers for fractional-order systems. It is known that designing static output feedback controllers is very difficult, even for ordinary linear control systems [25]. Using a specific lemma, we transform the original problem into an equivalent problem for ordinary linear systems with a structural constraint on the controller. To address the inherent challenges of the design process, we innovate the use of the barrier method (see, [38], [39]), which significantly transforms the problem into a more manageable form. Importantly, our focus lies on the Caputo fractional-order system, addressing unique structural restrictions imposed by stability, and we prioritize the design of low-gain static output feedback controllers for these systems, an aspect relatively under-explored in the literature. Compared to [39], there are two significant differences. First, we distinguish our study from others by dealing with specific structural constraints on controllers caused by the stable region for the pole locations of fractional-order systems. Second, the required controller performance differs from that of [39], which focused on the simultaneous optimal stabilization of classical linear systems, whereas we mainly address the design problem of low-gain static output feedback controllers for fractional systems, a topic that has received little attention in fractional-order system research.

B. IMPACT AND IMPLICATIONS

Our methodology not only ensures that the minima of the transformed problem lie within the admissible solution set, but also enables us to derive an approximate optimal solution using numerical techniques. The resulting static output feedback low-gain controller stabilizes the original fractional-order system, representing a significant advancement in this field. This work improves system performance and paves the way for future research and practical applications.

This paper is organized as follows. Section II introduces the Caputo fractional differential linear systems, formulates the problem to be solved, and provides some preliminary information. The auxiliary minimization problem is introduced in Section III. The main findings are presented in Section IV. Section V provides a numerical verification example. Section VI concludes with some recommendations.

1) NOTATION

That $\mathbb{N}$ is the set of natural numbers, $\sigma(M)$ is the spectrum of the matrix M, $tr(M)$ is the trace of M, M^T is the transpose of M, and $\|M\|$ is the Frobenius norm of M. The complex conjugate of $\lambda \in \mathbb{C}$ is denoted as $\bar{\lambda}$ and the real part of λ is denoted as $Re(\lambda)$.

II. PROBLEM FORMULATION AND PRELIMINARIES

Fractional-order derivatives are defined in a variety of ways, including the Riemann–Liouville fractional-order derivative, the Grunwald-Letnikov fractional-order derivative, and the Caputo fractional-order derivative. In this paper, the Caputo fractional-order derivative is used.

Definition 1: Let $f(t)$ be an integrable piecewise continuous function on any finite subinterval of $[0, \infty)$. The fractional integral of $f(t)$ of order $\alpha > 0$ is defined as:

$$J^{\alpha}f(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)} * f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \; t > 0 \tag{1}$$

where Γ is the gamma function.

Definition 2: The Caputo fractional-order derivative of order $\alpha > 0$ is defined as:

$$D^{\alpha}f(t) = \frac{1}{\Gamma(M-\alpha)} \int_0^t \frac{f^{(M)}(\tau)}{(t-\tau)^{\alpha+1-M}} d\tau \tag{2}$$

where $f^{(M)}(\tau) = \frac{d^M f(\tau)}{d\tau^M}$ with $M-1 \leq \alpha < M, M \in N$.

Consider a fractional-order linear time-invariant system [1], [2], [3], [4]:

$$D^{\alpha}x(t) = Ax(t) + Bu(t) \tag{3}$$

$$y(t) = Cx(t) \tag{4}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $y(t) \in R^r$ is the output, $A \in R^{n\times n}$, $B \in R^{n\times m}$ and $C \in R^{r\times n}$ are constant matrices of appropriate dimensions, and $1 \leq \alpha < 2$. Assume that the pair (A, B) is controllable and the pair (A, C) is observable. Although these are recognized as necessary conditions for arbitrary pole placement using static output feedback, the necessary and sufficient conditions for arbitrary pole placement using static output feedback are still known (see [25], [26]).

In this paper, we want to design a static output feedback controller

$$u = Fy \tag{5}$$

with minimal $\|F\|$ to stabilize the fractional-order system (3).

Define

$$J(F)=\begin{cases} Tr\left(F^TF\right), & \text{if } F\in\Omega \\ \infty, & \text{otherwise}\end{cases} \tag{6}$$

where Ω is the set of all matrices $F \in R^{m\times r}$ such that the closed-loop system

$$D^\alpha x(t)=(A+BFC)x(t) \tag{7}$$

is asymptotically stable. By minimizing the cost function described in (6), we aim to select a controller, $u = Fy$, that requires the least "energy" or "effort" to stabilize the system. Such optimization, both in practical scenarios and in real-world applications, can lead to tangible benefits such as reduced power usage and improved control performance. Further details on this effort can be found in the Introduction. At its core, the performance function outlined in (6) offers a framework to identify the optimal controller for system (7), ensuring stability while minimizing the controller's energy expenditure. The problem considered can be equivalently expressed as

$$\inf_{F\epsilon\Omega} J(F)=\inf_{F\epsilon\Omega} Tr\left(F^TF\right). \tag{8}$$

The following lemma is needed in the following derivations.

Lemma 1: [28], [40] The fractional-order system $D^\alpha x(t)=Ax(t)$, with $1\le\alpha<2$, is asymptotically stable if and only if the matrix

$$\begin{bmatrix} A\sin\left(\frac{\alpha\pi}{2}\right) & A\cos\left(\frac{\alpha\pi}{2}\right) \\ -A\cos\left(\frac{\alpha\pi}{2}\right) & A\sin\left(\frac{\alpha\pi}{2}\right)\end{bmatrix}$$

is Hurwitz.

With Lemma 1, we know that the closed-loop system (7) is asymptotically stable if and only if

$$A_0+B_0F_0C_0 \tag{9}$$

is Hurwitz, where

$$A_0=\begin{bmatrix} A\sin\left(\frac{\alpha\pi}{2}\right) & A\cos\left(\frac{\alpha\pi}{2}\right) \\ -A\cos\left(\frac{\alpha\pi}{2}\right) & A\sin\left(\frac{\alpha\pi}{2}\right)\end{bmatrix},$$

$$B_0=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{bmatrix}\equiv\begin{bmatrix} B\sin\left(\frac{\alpha\pi}{2}\right) & B\cos\left(\frac{\alpha\pi}{2}\right) \\ -B\cos\left(\frac{\alpha\pi}{2}\right) & B\sin\left(\frac{\alpha\pi}{2}\right)\end{bmatrix},$$

$$C_0=\begin{bmatrix} C & 0 \\ 0 & C\end{bmatrix} and F_0=\begin{bmatrix} F & 0 \\ 0 & F\end{bmatrix}.$$

Therefore, finding an F to stabilize the closed-loop system (7) is equivalent to finding an F_0, having the structure $F_0=\mathrm{diag}(F,F)$, such that $A_0+B_0F_0C_0$ is Hurwitz.

Define

$$J_0(F)=\begin{cases} Tr\left(F_0^TF_0\right), \text{ if } F\in\Omega \\ \infty, \text{ otherwise}\end{cases}$$

Clearly, if $F\in\Omega$, $2J(F)=J_0(F)$. Because the feedback matrix F_0 has a specific structure, applying the LMI approach to find such an F_0 such that $A_0+B_0F_0C_0$ is Hurwitz is difficult. As a result, we employ the barrier method to approximate solve the problem. Note that the optimization problem (8), subject to the system (3), is equivalent to

$$\inf_{F\epsilon\Omega} J_0(F) \tag{10}$$

with the following auxiliary system

$$\dot{\xi}=(A_0+B_0F_0C_0)\,\xi \tag{11}$$

where $\xi\in R^{2n\times 2n}$ is the auxiliary system state, and A_0, B_0, C_0, and F_0 are defined in equation (9).

III. AUXILIARY MINIMIZATION PROBLEM

The minimization problem (10) is difficult to solve because it is a nonconvex-constrained optimization problem. The infimal solution may lie on the boundary of the admissible solution set Ω; furthermore, it might not be a stationary point either. To tackle this problem, we apply the barrier method ([38], [39]) to try to find an approximate solution to the original problem by solving an auxiliary minimization problem, which is formulated as

$$\min_{F\epsilon\Omega} J_{aux}(F)=\min_{F\epsilon\Omega} J_0(F)+\rho\cdot J_{add}(F) \tag{12}$$

where $\rho>0$ is a weighted factor, $J_{add}(F)$ is a barrier function defined as:

$$J_{add}(F)=\begin{cases} Tr(P_0), & \text{if} F\in\Omega \\ \infty, & \text{otherwise,}\end{cases} \tag{13}$$

and $P_0>0$ is the solution of (given $Q_0>0$)

$$(A_0+B_0F_0C_0)^T P_0+P_0(A_0+B_0F_0C_0)+Q_0=0. \tag{14}$$

Similar to [41], we have the following results.

Lemma 2: The function $J_{add}(F)$ satisfies the following

1. $J_{add}(F)$ is continuous in the set Ω;
2. $J_{add}(F)>0$ over the set Ω;
3. $J_{add}(F)$ approaches infinity as F approaches the boundary of the set Ω.

Proof: The proof is omitted because it is similar to that given in [41].

As demonstrated in [39], if the admissible set Ω is nonempty, the auxiliary cost function $J_{aux}(F)$ has a minimum point in the interior of set Ω.

IV. MAIN RESULTS

Because the minimum point of the auxiliary cost function $J_{aux}(F)$ is located within the admissible solution set Ω, it must be a stationary point. The Lagrange multiplier method can be used to derive the necessary conditions for local optimums of the cost function $J_{aux}(F)$. Importantly, it is worth noting that a key feature of the auxiliary minimization problem is that its cost function tends towards infinity at the boundaries, which the original problem does not have. This distinction is crucial because it allows the auxiliary minimization problem to be solved using techniques similar to unconstrained searching methods [38], which simplifies the algorithm's implementation.

Theorem 1: The optimal solution F of the auxiliary minimization problem (12) must satisfy

$$F=-\frac{1}{2}\left(\begin{bmatrix} I_m & 0_m\end{bmatrix}B_0^TP_0S_0\begin{bmatrix} I_n \\ 0_n\end{bmatrix}+\begin{bmatrix} 0_m & I_m\end{bmatrix}B_0^TP_0S_0\begin{bmatrix} 0_n \\ I_n\end{bmatrix}\right)C^T \tag{15}$$

where $P_0 > 0$ and $S_0 > 0$ are the solutions of the following Lyapunov equations:

$$(A_0 + B_0F_0C_0)^T P_0 + P_0 (A_0 + B_0F_0C_0) + Q_0 = 0 \quad (16)$$

$$(A_0 + B_0F_0C_0) S_0 + S_0 (A_0 + B_0F_0C_0)^T + \rho I_{2n} = 0 \quad (17)$$

with $F_0 = \operatorname{diag}(F, F)$.

Proof: The Hamiltonian is defined as:

$$H = Tr\left(F_0^T F_0\right) + \rho Tr(P_0) + Tr\left(S_0\left((A_0 + B_0F_0C_0)^T P_0 + P_0 (A_0 + B_0F_0C_0) + Q_0\right)\right)$$

where S_0 is the Lagrange multiplier. Let

$$P_0 = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \text{ and } S_0 = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}.$$

The necessary conditions for local optimal are $\frac{\partial H}{\partial S_0} = 0, \frac{\partial H}{\partial P_0} = 0$, and $\frac{\partial H}{\partial F} = 0$. By $\frac{\partial H}{\partial S_0} = 0$ and $\frac{\partial H}{\partial P_0} = 0$, we obtain (15) and (16), respectively. To derive $\frac{\partial H}{\partial F}$, note that

$$\begin{aligned} S_0 (B_0F_0C_0)^T P_0 = {} & S_{11}C^TF^TB_{11}^TP_{11} + S_{12}C^TF^TB_{12}^TP_{11} \\ & + S_{11}C^TF^TB_{21}^TP_{21} + S_{12}C^TF^TB_{22}^TP_{21} \\ & + S_{21}C^TF^TB_{11}^TP_{12} + S_{22}C^TF^TB_{12}^TP_{12} \\ & + S_{21}C^TF^TB_{21}^TP_{22} + S_{22}C^TF^TB_{22}^TP_{22}. \end{aligned}$$

Then, we can show that

$$\begin{aligned} \frac{\partial H}{\partial F} &= 2\left(\left(B_{11}^TP_{11} + B_{21}^TP_{21}\right)S_{11} + \left(B_{12}^TP_{11} + B_{22}^TP_{21}\right)S_{12} + \left(B_{11}^TP_{12} + B_{21}^TP_{22}\right)S_{21} + \left(B_{12}^TP_{12} + B_{22}^TP_{22}\right)S_{22}\right) C^T + 4F \\ &= 2\left(\begin{bmatrix} B_{11}^T & B_{21}^T \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} S_{11} \\ S_{21} \end{bmatrix} + \begin{bmatrix} B_{12}^T & B_{22}^T \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} S_{12} \\ S_{22} \end{bmatrix}\right) C^T + 4F \\ &= 2\left(\begin{bmatrix} I_m & 0_m \end{bmatrix} B_0^T P_0 S_0 \begin{bmatrix} I_n \\ 0_n \end{bmatrix} + \begin{bmatrix} 0_m & I_m \end{bmatrix} B_0^T P_0 S_0 \begin{bmatrix} 0_n \\ I_n \end{bmatrix}\right) C^T + 4F \end{aligned}$$

Hence, with $\frac{\partial H}{\partial F} = 0$, we have (15). ■

It is known that F is difficult to obtain from Theorem 1 using existing methods, so an algorithm based on the gradient descent method is developed to solve the auxiliary minimization problem.

A. MAIN-ALGORITHM

The goal is to find F_{opt}, the optimal solution to the auxiliary minimization problem (11).

Let ε be a small positive number.

1. Initialize the algorithm by selecting an $F(0) \in \Omega$. Set the iteration counter $k = 0$ and let $F_0(0) = \operatorname{diag}(F(0), F(0))$.
2. Solve equations (15) and (16) with F_0 being substituted by $F_0(k) = \operatorname{diag}(F(k), F(k))$, to yield $P_0(k)$ and $S_0(k)$.
3. Calculate.
$$\Delta F(k) = 2\left(\begin{bmatrix} I_m & 0_m \end{bmatrix} B_0^T P_0(k) S_0(k) \begin{bmatrix} I_n \\ 0_n \end{bmatrix} + \begin{bmatrix} 0_m & I_m \end{bmatrix} B_0^T P_0(k) S_0(k) \begin{bmatrix} 0_n \\ I_n \end{bmatrix}\right) C^T + 4F(k)$$
4. *If* $\|\Delta F(k)\| < \varepsilon$, *then* $F_{opt} = F_0(k)$, END; *else* find a $\eta > 0$, via a line search technique, such that $F(k+1) = F(k) - \eta\Delta F(k)$ minimizes $J_{aux}(F)$. Let $k = k + 1$ and go to Step 2.

The parameter ϵ serves as a small positive threshold that aids in determining the convergence of the algorithm. Setting it too large risks premature termination before F converges to its optimal value, whereas setting it too small may protract or even prevent convergence due to numerical errors. Meanwhile, another parameter, η, determines the iteration step size. Moreover, it is worth noting that finding an $F(0) \in \Omega$ in Step 1 is challenging, which can also affect the algorithm's performance.

We provide a pre-algorithm to find such an initial matrix.Let $P_0(\mu)$ be the positive defined solution of

$$(A_0 + B_0F_0C_0 - \mu I)^T P_0(\mu) + P_0(\mu)(A_0 + B_0F_0C_0 - \mu I) + Q_0 = 0 \quad (18)$$

and $S_0(\mu)$ be the solution of

$$S_0(\mu)(A_0 + B_0F_0C_0 - \mu I)^T + (A_0 + B_0F_0C_0 - \mu I) S_0(\mu) + \rho I = 0. \quad (19)$$

Let

$$\hat{J}_{aux}(F, \mu) = Tr\left(F_0^T F_0\right) + \rho Tr(P_0(\mu)).$$

We know that $\hat{J}_{aux}(F, \mu) \to \infty$ as F approaches the boundary of the following set:

$\hat{\Omega}(\mu) \equiv \{F \in R^{m\times r} \mid A_0 + B_0F_0C_0 - \mu I \text{ is Hurwitz}\}$.

Moreover, $\hat{J}_{aux}(F, 0) = J_{aux}(F)$, and $\hat{\Omega}(0) = \Omega$. Now, the pre-algorithm for searching for an $F \in \Omega$ is given below.

B. PRE-ALGORITHM

Find an $F \in \Omega$.

Start by setting $\epsilon > 0$ and $\epsilon_0 > 0$ as small positive numbers.

1. Choose an arbitrary $F(0) \in R^{m\times r}$ and let
$$F_0(0) = \operatorname{diag}F(0), F(0)\},$$
$$\mu(0) = Re\{\lambda_i(A_0 + B_0F_0(0)C_0)\} + \epsilon.$$
Set $i = 0$.
2. Solving equations (18) and (19), where F_0 is substituted by $F_0(i) = \operatorname{diag}F(i), F(i)\}$, yields $P_0(\mu(i))$ and $S_0(\mu(i))$.
3. Let
$$\Delta F(i) = 2\left(\begin{bmatrix} I_m & 0_m \end{bmatrix} B_0^T P_0(\mu(i)) S_0(\mu(i)) \begin{bmatrix} I_n \\ 0_n \end{bmatrix} + \begin{bmatrix} 0_m & I_m \end{bmatrix} B_0^T P_0(\mu(i)) S_0(\mu(i)) \begin{bmatrix} 0_n \\ I_n \end{bmatrix}\right) C^T + 4F(i).$$
4. Find an $\eta > 0$, via line search, such that $F(i+1) = F(i) - \eta\Delta F(i)$ minimizes $\hat{J}_{aux}(F, \mu(i))$.

5. Let $i = i + 1$. Let $\lambda_k(i)$, $k = 1, \ldots, n$, be the eigenvalues of $A_0 + B_0 F_0(i) C_0$, where $F_0(i) = \text{diag}F(i), F(i)\}$.
If $\hat{\mu}(i) = \max\{\{Re(\lambda_k(i)), 0\}\} = 0$, let $\mu(i) = 0$ and STOP, $F_0 = \text{diag}F(i), F(i)\} \in \Omega$;
else if $\mu(i-1) - \hat{\mu}(i) < \epsilon_0$, the algorithm fails, END;
else choose $0 < q(i) < 1$ and let
$$\mu(i) = \mu(i-1) - q(i)\left(\mu(i-1) - \hat{\mu}(i)\right),$$
and go to Step 2.

C. DISCUSSIONS ON CONVERGENCE OF ALGORITHMS

For brevity, the detailed proof is not included as it closely resembles those in [42] and [43]. Here, we only provide some conceptual discussions.

For the Main-algorithm, a line search technique is used to find a suitable step size η at each iteration, such that the updated solution $F(k) - \eta\Delta F(k)$ minimizes the cost function $J_{aux}(F)$. The updated solution is forced to lie within the interior of Ω due to the use of the barrier method. Furthermore, the newly updated solution $F(k+1)$ should result in a lower cost value compared to the previous solution $F(k)$, which means that $J_{aux}(F(k+1)) < J_{aux}(F(k))$. This leads to the fact that the value of the cost function $J_{aux}(F)$ is decreasing at each iteration. Since every level set of the cost function $J_{aux}(F)$ is compact, the convergence of Main-algorithm can be guaranteed by the Global Convergence Theorem [38].

For the Pre-algorithm, we know that $\mu(i) < \mu(i-1)$ when $\mu(i) \neq$ Ú0. By Step 5 of the Pre-algorithm, $\mu(i) - \mu(i-1) \leq -q\epsilon_0$ if the algorithm does not fail. The value of $\mu(i)$ is monotonically decreasing. Therefore, if the set $\hat{\Omega}(\mu)$ is connected for all $\mu \geq 0$ and $\Omega = \hat{\Omega}(0)$ is nonempty, $\mu(i) \rightarrow \cdot 0$ in finite steps.

Remark 1: For some multi-agent system control problems, the feedback matrix may need to be in a block diagonal structure, as shown in [19]. Our approach can be appropriately modified for these problems in order to find stabilizing feedback matrices in specific structures.

V. ILLUSTRATIVE EXAMPLE

Consider the following fractional-order system:
$$\begin{aligned} D^{\alpha}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{20}$$
where $\alpha = 1.2$, and
$$A = \begin{bmatrix} -4 & 5 & -4 \\ 4 & -21 & -18 \\ -32 & -4 & 34.5 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -5 & 3 \\ 0 & -4 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 1 \\ -2 & 0 & 3 \end{bmatrix}.$$

The goal is to find a static output feedback gain F with minimal $\|F\|$ to stabilize the system (20). Choose the weighting factor $\rho = 5$ in (16) and $Q_0 = I$ in (17). For the considered problem, we choose the initial guess $F(0) = \begin{bmatrix} 0 & 1000 \\ 1000 & 0 \end{bmatrix}$ and then run the Pre-algorithm. After some iterations, we obtain the solution
$$F = \begin{bmatrix} 326.238 & 479.443 \\ 415.831 & 921.997 \end{bmatrix} \in \Omega.$$

Then, take this solution as an initial guess and start the Main-algorithm with a stop condition $\|\Delta F(k)\| \leq \varepsilon = 10^{-5}$. We get the following solutions, as shown in the equation at the bottom of the page and the resultant optimal static output feedback gain is
$$F_{opt} = \begin{bmatrix} 0.0432 & -0.1608 \\ 0.8114 & 2.4764 \end{bmatrix} \in \Omega.$$

In this case, $J\left(F_{opt}\right) = 6.8186$, and
$$\sigma\left(A + BF_{opt}C\right) = \left\{ \begin{matrix} -8.4912 + 8.9872\mathrm{i} \\ -8.4912 - 8.9872\mathrm{i} \\ -8.8931 \end{matrix} \right\}$$
which implies that the closed-loop system is asymptotically stable.

Figure 1 depicts the locations of the closed-loop poles, while Figure 2 presents the iterative values of both $J_{aux}(F)$ and $J(F) = Tr(F^T F)$ throughout the iterations of the Main-algorithm. To highlight the variation in these two functions throughout the iterations, their initial values, which are significantly larger, are omitted. Notably, the value of $J_{aux}(F)$ demonstrates a monotonically decreasing trend with each iteration. Lastly, by integrating the approaches in [44], [45], and [46] for response simulations, Figure 3 illustrates the closed-loop system response with the initial condition $x_0 = \begin{bmatrix} 1.5 & 3.1 & 0 \end{bmatrix}^T$. The state of the closed-loop system converges to the origin asymptotically. In about 5 seconds, the state trajectory approaches the origin.

$$P_0 = \begin{bmatrix} 0.1368 & 0.0585 & -0.0966 & 0 & -0.0197 & -0.0320 \\ 0.0585 & 0.0776 & -0.0651 & 0.0197 & 0.0000 & -0.0345 \\ -0.0966 & -0.0651 & 0.1833 & 0.0320 & 0.0345 & 0 \\ 0 & 0.0197 & 0.0320 & 0.1368 & 0.0585 & -0.0966 \\ -0.0197 & 0 & 0.0345 & 0.0585 & 0.0776 & -0.0651 \\ -0.0320 & -0.0345 & 0 & -0.0966 & -0.0651 & 0.1833 \end{bmatrix},$$
$$S_0 = \begin{bmatrix} 0.4699 & -0.0074 & 0.2110 & 0 & -0.1788 & -0.2658 \\ -0.0074 & 0.3653 & 0.3905 & 0.1788 & 0.0000 & 0.0228 \\ 0.2110 & 0.3905 & 1.1535 & 0.2658 & -0.0228 & 0 \\ 0 & 0.1788 & 0.2658 & 0.4699 & -0.0074 & 0.2110 \\ -0.1788 & 0 & -0.0228 & -0.0074 & 0.3653 & 0.3905 \\ -0.2658 & 0.0228 & 0 & 0.2110 & 0.3905 & 1.1535 \end{bmatrix},$$

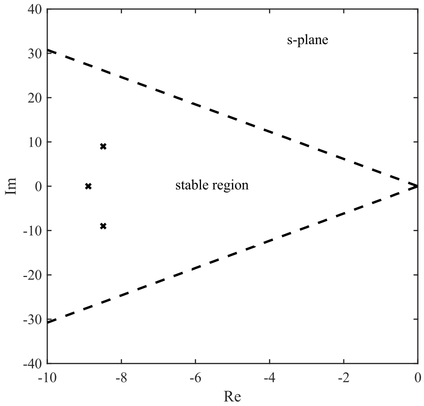

FIGURE 1. Locations of closed-loop poles.

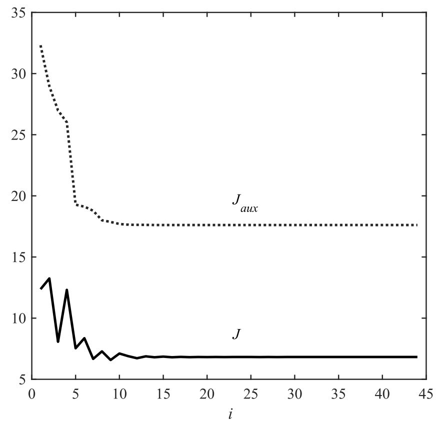

FIGURE 2. The values of J_{aux} and J in the iteration.

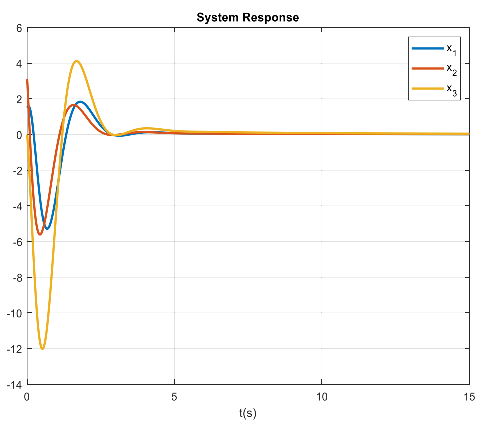

FIGURE 3. System response of the closed-loop system.

If we change the value of the weighting factor ρ to 0.01, the obtained optimal solution is

$$F_{opt} = \begin{bmatrix} -0.2867 & -0.5692 \\ 0.9511 & 1.8033 \end{bmatrix} \in \Omega$$

In this case, $J\left(F_{opt}\right) = 4.5625$, and

$$\sigma\left(A + BF_{opt}C\right) = \left\{ \begin{array}{c} -2.9395 + 2.1505i \\ -2.9395 + 2.1505i \\ -8.2458 \end{array} \right\}.$$

This result shows that as ρ decreases, the value of the resulting cost function decreases, and the suboptimal solution tends to move toward the infimal solution of the original optimization problem. The closed-loop poles remain within the stable region.

VI. CONCLUSION

Designing static output feedback stabilizing controllers for fractional-order control systems presents significant challenges. To address this, we propose an optimization-based approach based on the barrier method to solve this problem. We have developed numerical algorithms using the steepest descent method to find a suboptimal solution to the constrained optimization problem. Further studies include extending the current approach to design static output feedback L_2-gain controllers for fractional-order systems, design static output feedback stabilizing and L_2-gain controllers for fractional-order time-delay systems, and design static output feedback stabilizing and L_2-gain controllers for uncertain fractional-order systems. In addition, more efficient solution algorithms can be developed.

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