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RESEARCH ARTICLE

Intuitionistic Hesitant Fuzzy Partitioned Maclaurin Symmetric Mean Aggregation Operators-Based Algorithm and Its Application in Decision Making

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ABSTRACT Intuitionistic hesitant fuzzy sets, which enable the representation of an element's membership and non-membership as a set of multiple possible values, offer significant utility in describing uncertainty in various aspects of people's daily lives. Fundamental mathematical techniques known as intuitionistic hesitant fuzzy aggregation operators are employed to merge multiple inputs into a single result based on predetermined criteria. However, conventional approaches that rely on classic intuitionistic hesitant fuzzy aggregation operators have faced criticism due to their limited understanding of criteria characterization. We introduce two novel operators, namely the intuitionistic hesitant fuzzy partitioned Maclaurin symmetric mean (IHFPMSM) and intuitionistic hesitant fuzzy weighted partitioned Maclaurin symmetric mean (IHFWPMSM), which draw inspiration from the partitioned Maclaurin symmetric mean concept. Subsequently, we thoroughly examine various characteristics and special cases of these operators. Building upon the IHFWPMSM operator, we propose a novel multiple-criteria decision-making (MCDM) method that effectively selects the most suitable alternative from a set of options. To demonstrate the effectiveness of our proposed approach, we discuss a systematic methodology for selecting the optimal location for shoe company construction. Lastly, we demonstrate the superior prevalence and effectiveness of the developed approach through comprehensive comparative and sensitivity analyses, surpassing the capabilities of existing approaches.

INDEX TERMS Intuitionistic hesitant fuzzy set, Maclaurin symmetric mean, partitioned Maclaurin symmetric mean, multiple criteria decision-making.

I. INTRODUCTION

To make the best decision, people employ a cognitive process called multi-criteria decision-making (MCDM). It has been widely applied in a number of fields. A common need for decision-makers (DMs) in MCDM situations is to provide evaluative information on the criteria and alternatives in the form of classical data, which is inadequate for dealing with issues like ambiguity, imprecision, or fuzziness [1], [2], [3], [4], [5], [6], [7]. Zadeh [8] developed the fuzzy set (FS) as a solution, and it has since been the subject of extensive research. Membership (MS) functions are used in FS theory to quantify how strongly elements belong to a given set, enabling more flexible representations of ambiguous or inaccurate data. FS has garnered significant attention from various academics, and various scholars have conducted multiple studies. For instance, Aydin [9] presented the fuzzy MCDM technique based on fermatean FSs. John [10] explored the specific implications of type-2 FSs, Mandel and John [11] discussed the simplification of type-2 FSs, and

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Ali [12] introduced the concept of a probabilistic hesitant bipolar FS, described its Hamacher operational rules, and employed it to decision-making issues.

FS has attracted the interest of a number of scholars, and various academicians have implemented various applications. However, FS theory has limitations when it comes to expressing more complex fuzzy information that may require various levels of MS function or fuzzy interactions amongst multiple components. For example, if a person encounters information in the form of {0.8, 0.9, 0.7}, then the FS concept has been disregarded. To make up for this deficiency, Torra [13] introduced the well-known concept of hesitant FS (HFS) by transforming the strategy of FS into HFS, which includes the MS grade whose maximum value falls within the interval [0, 1]. HFS is an altered variant of FS that has attracted the focus of various experts. In particular, Meng and Chen [14] came up with correlation measures for HFSs, Li et al. [15] explored distance and similarity measurements for HFSs, and Wei et al. [16] studied entropy and a number of measures according to on HFSs. Pant and Kumar [17] put forward an integrated time series forecast approach using HFS, which allows the adoption of particle swarm optimization (PSO) and a support vector machine. Under hesitant fuzzy setting, Wang et al. [18] introduced a three-way decision strategy in accordance with regret theory; they also offered a three-way classification mechanism founded on the preference ranking organization approach for enrichment evaluations.

If an intellectual entity presented details within the form of "yes" or "no," the FS concept has been disregarded. For this purpose, Atanassov [19] introduced the well-known idea of intuitionistic FS (IFS) by improving an earlier form of FS into IFS, which incorporates the MS as well as non-membership (NMS) function, where the sum of both functions is limited to 1. The IFS theory has been frequently used in the resolution of MCDM problems due to the inclusion of this NMS function. An improved conceivable degree technique was put out by Garg et al. [20] to ordering intuitionistic fuzzy numbers (IFNs); in contrast, Joshi [21] developed a brand-new bi-parametric exponential data methodology based on IFS. A distinct relationship between two variables among IFSs was produced by Thao et al. [22], who also described their technique. In [23], Garg and Kumar looked into the application of an intuitionistic fuzzy MCDM approach using set pair evaluation to decision-making problems. Several academics have researched and utilized a wide range of additional methods that rely on intuitionistic fuzzy theory [24], [25], [26], [27], [28].

It was found that the prevalent data computed with the help of FSs, HFSs, and IFSs can be put to use in a wide range of disciplines, such as finance, information technology, engineering, and transportation. Still, it can't be denied that they're constrained in many ways. For example, we acknowledge that IFS has only ever operated with two-dimensional data that is in terms of a singleton set, i.e., each dimension of data can represent a single value. But what if a DM supplied two-dimensional information in the form of more than one value? In such a case, DMs observed the IFS theory could not accurately operate with the above data. To accomplish this, the well-known concept of an intuitionistic hesitant fuzzy (IHF) set (IHFS) was first proposed by Beg and Rashid [29], who adapted the IFS setup to create IHFS, which includes the MS and NMS values as a finite subset of [0, 1], with the sum of the supremum of the duplet falling within [0, 1]. IHFS is a hybrid of IFS and HFS that has attracted a lot of attention from academic circles because of its ability to handle complex and uncertain data in real-world problems. Various academics have implemented a number of applications. For instance, Peng et al. [30] used IHFSs to pioneer cross-entropy measures, and Zhai et al. [31] studied probabilistic interval-valued IHFSs. Chen et al. [32] developed the technique for order preference by similarity to an ideal solution (TOPSIS) algorithm based on their offered IHF distance measures. In [33], Mahmood et al. provided a number of power aggregation operators (AOs) with IHFSs and researched their basic properties. The authors of [34] linked IHFS to the theory of set pair analysis, which examines certainty and uncertainty as a cohesive system and expresses them from numerous viewpoints. Albaity and Mahmood [35] initiated the core theory of generalized dice similarity measures based on IHFS and discussed their specific cases via parameters. In addition, they addressed their proposed measures' applications to healthcare diagnosis and pattern analysis.

Data aggregation is a challenging endeavor in uncertain circumstances, and different AOs have been devised to cope with fuzzy contexts. A series of AOs were created by Xia and Xu [36] employing quasi-arithmetic techniques in hesitant fuzzy situations. Yu [37] formed a number of hesitant fuzzy aggregation operators based on Einstein triangular norms. Ali and his coauthors reported IHF Aczel-Alsina weighted averaging (geometric) operators and their desired results in numerous journals [38], [39], [40]. By pondering the prioritization of input data, numerous prioritized AOs [12], [41] have been explored for utilization in different fuzzy contexts. All the preceding AOs only aggregated data and considered only weights. None of them contemplate the connection between the input data. As a result, a number of other forms of AOs have been created, such as heronian mean operators [42], Bonferroni mean operators [43], power operators [33], and maclaurin symmetric mean operators [44], which link the data input to some extent used in the aggregate step. We additionally suggest [45], [46], [47], [48], [49] for more reading on MCDM and AOs analysis.

However, there may not always be connections between criteria in real-world decision-making situations. For example, we are comparing four varieties of coffee according to the following standards: aroma (S_1), bitterness (S_2), acidity (S_3), and aftertaste (S_4). $P_1 =$ "S_1, S_4" and $P_2 =$ "S_2, S_3" are two groups under which these requirements fall. We may infer that S_1 is linked to S_4 and that they all belong to P_1, and that S_2 is linked to S_3, and that they all reside in P_2. The subcategories P_1 and P_2, though, have no connection with one another. The partitioned Bonferroni mean (BM) operator by Dutta and Guha [50] and the partitioned Maclaurin symmetric mean (MSM) (PMSM) operator relying on IFNs was suggested by Liu et al. [51]. These two operators presuppose that overall criteria have been divided into a small number of categories as well as a connection among criteria within a single category but not among criteria of different categories. The MSM operator clearly differs from the BM operator, with the MSM operator capturing connections among the input arguments (or criteria) of any number specified by a parameter, while the BM operator is limited to describing the connections between only two arguments. To summarize, the MSM operator proves more generalized than the BM operator and outperforms it in addressing decision-making issues with numerous input arguments that have connections to each other. As a result, we focus on how the MSM operator handles IHF MCDM situations with both interaction and partitioning connections among criteria. By keeping the wide-ranging foundation of IHFSs, where both aspects (belongingness and non-belongingness) of several uncertain data can be characterized, reducing the possibility of data loss and the importance of MSM operator in mind, our goal in this work is to create the theory of PMSM operators for IHFSs. When dealing with cases in which criteria within the same partition are related to one another, but criteria within other partitions are unrelated to one another, the PMSM operator offers various benefits: i) It can show how various factors interact with one another; ii) It breaks the criteria down into a number of unrelated categories, each of which contains its own set of factors that are interconnected.

The main goals of the present research are as follows:

- To suggest the IHF PMSM (IHFPMSM) and IHF weighted PMSM (IHFWPMSM) operators to overcome the aforementioned review's drawback.
- To investigate several properties, theorems, and specific instances of the propound PMSM operators.
- To establish an MCDM strategy depending on the created IHFWPMSM operator.
- To conduct a case study and a comparative analysis to show the value and superiority of the conventional technique.

The subsequent sections of this research are organized as follows. Section II provides an exposition of the fundamental concepts of IHFS and the underlying principles of its operation. In Section III, we delve into the development of the IHFPMSM and IHFWPMSM operators and establish their core characteristics. Section IV introduces a method for decision-making utilizing the proposed operators. Section V presents a detailed case study to showcase the proposed approach's implementation and application. In Section VI, we conduct comparisons with existing methods by analyzing their ranking results and performing a validity test to assess the feasibility of our approach. Section VII contains some concluding remarks and future outlines.

II. SOME BASIC CONCEPTS

In this section, we allocate some basic concepts related to IHFS and partitioned Maclaurin symmetric mean.

Definition 1 [13]: Let S be a domain of discourse. Then, a HFS $\mathcal{D}$ on S is described as

$$\mathcal{D} = \{(s, \mathfrak{G}(s))\,|s \in S\}, \tag{1}$$

where $\mathfrak{G}$ is a function taking values in the unit interval [0, 1] which speak to the membership degree of $s \in S$ in $\mathcal{D}$.

For convenience, $\mathfrak{G} = \mathfrak{G}(s)$ is referred to as a hesitant fuzzy element (HFE), and $\mathcal{D}$ is the set of all HFEs.

Xia and Xu [52] established the following comparison procedure for two HFEs:

Definition 2 [53]: For an HFE $\mathfrak{G}$, the score function is defined as:

$$Sc(\mathfrak{G}) = \sum_{l=1}^{\#\mathfrak{G}} \mu_l / \#\mathfrak{G}, \tag{2}$$

where $\#\mathfrak{G}$ represent the number of elements in $\mathfrak{G}$.
For any two HFEs $\mathfrak{G}_1$ and $\mathfrak{G}_2$, if $Sc(\mathfrak{G}_1) > Sc(\mathfrak{G}_2)$, then $\mathfrak{G}_1 > \mathfrak{G}_2$; if $Sc(\mathfrak{G}_1) = Sc(\mathfrak{G}_2)$, then $\mathfrak{G}_1 = \mathfrak{G}_2$; if $Sc(\mathfrak{G}_1) < Sc(\mathfrak{G}_2)$, then $\mathfrak{G}_1 < \mathfrak{G}_2$. For any two HFEs $\mathfrak{G}_1$, and $\mathfrak{G}_2$, Torra [13] and Xia and Xu [36] described the following operation rules:

Definition 3: Let $\mathfrak{G}_1$, and $\mathfrak{G}_2$ be two HFEs and $\lambda > 0$, then

1. $\mathfrak{G}_1 \oplus \mathfrak{G}_2 = \bigcup_{\substack{k=1,2,\ldots,\#\mathfrak{G}_1,\\ l=1,2,\ldots,\#\mathfrak{G}_2}} \{\mu_k + \nu_l - \mu_k\mu_l\}$;
2. $\mathfrak{G}_1 \otimes \mathfrak{G}_2 = \bigcup_{\substack{k=1,2,\ldots,\#\mathfrak{G}_1,\\ l=1,2,\ldots,\#\mathfrak{G}_2}} \{\mu_k\mu_l\}$;
3. $\lambda\mathfrak{G}_1 = \bigcup_{k=1,2,\ldots,\#\mathfrak{G}_1} \left\{1 - (1-\mu_k)^{\lambda}\right\}$;
4. $\mathfrak{G}_1^{\lambda} = \bigcup_{k=1,2,\ldots,\#\mathfrak{G}_1} \left\{{\mu_k}^{\lambda}\right\}$;
5. $(\mathfrak{G}_1)^c = \bigcup_{k=1,2,\ldots,\#\mathfrak{G}_1} \{1 - \mu_k\}$.

Definition 4 [34]: Let S be a domain of discourse. Then, an IHFS $\mathcal{I}$ on S is described as

$$\mathcal{I} = \{(s, \mathfrak{G}(s), \mathfrak{H}(s))\,|s \in S\}, \tag{3}$$

where $\mathfrak{G}(s)$ and $\mathfrak{H}(s)$ represent the MS and NMS values, respectively ranging value from [0,1] for all $s \in S$ with given conditions:

$$0 \leq \max(\mathfrak{G}(s)) + \min(\mathfrak{H}(s)) \leq 1, \text{ and}$$
$$0 \leq \min(\mathfrak{G}(s)) + \max(\mathfrak{H}(s)) \leq 1,$$

For convenience, $\mathfrak{J} = (\mathfrak{G}(s), \mathfrak{H}(s))$ is named as a IHF element (IHFE).

Definition 5 [34]: For an IHFE $\mathfrak{J}$, the score function and accuracy functions are defined as:

$$Sc(\mathfrak{J}) = \frac{Sc(\mathfrak{G}) - Sc(\mathfrak{H})}{n}, \tag{4}$$

$$Ac(\mathfrak{J}) = \frac{Sc(\mathfrak{G}) + Sc(\mathfrak{H})}{n}, \tag{5}$$

where $Sc(\mathfrak{G}) = \frac{\text{Sum of all the element of}(\mathfrak{G})}{\text{order of}(\mathfrak{G})}$ and $Sc(\mathfrak{H}) = \frac{\text{Sum of all the element of}(\mathfrak{H})}{\text{order of}(\mathfrak{H})}$.

Definition 6: For any two IHFEs $\mathfrak{J}_1$ and $\mathfrak{J}_2$, denoted by $\mathfrak{J}_1 > \mathfrak{J}_2$ depend upon the following criteria:

(*i*) $Sc(\mathfrak{J}_1) > Sc(\mathfrak{J}_2)$;
(*ii*) $Sc(\mathfrak{J}_1) = Sc(\mathfrak{J}_2)$ and $Ac(\mathfrak{J}_1) > Ac(\mathfrak{J}_2)$.

MSM is characterized by the following characteristics:

i. If each $\phi_k = \phi\,(k = 1, 2, \ldots, n)$, then $MSM^\rho(\phi_1, \phi_2, \ldots, \phi_n) = \phi$.
ii. If $\varphi_k \leq \phi_k\,(k = 1, 2, \ldots, n)$, then $MSM^\rho(\varphi_1, \varphi_2, \ldots, \varphi_n) \leq MSM^\rho(\phi_1, \phi_2, \ldots, \phi_n)$.
iii. $\min_k \{\phi_k\} \leq MSM^\rho(\phi_1, \phi_2, \ldots, \phi_n) \leq \max_k \{\phi_k\}$.

III. PROPOSED IHFPMSM OPERATOR AND IHFWPMSM OPERATOR

In this section, we implement the PMSM operator into the IHF model in order to develop two clear synthetic operators, IHFPMSM and IHFWPMSM:

A. IHFPMSM AGGREGATION OPERATORS

Definition 7: Let $\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n$ be a range of n IHFEs. Then, IHFPMSM operator is characterized as

$$IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) = \frac{1}{t} \oplus_{\Upsilon=1}^{t} \left(\frac{\oplus_{1\leq k_1<\ldots<k_l\leq o_\Upsilon}\left(\otimes_{l=1}^{\rho} \mathfrak{J}_{kl}\right)}{C_{o_\Upsilon}^{\rho}} \right)^{\frac{1}{\rho}}, \tag{6}$$

where t denotes the number of categories, ρ is a parameter, $\rho = 1, 2, \ldots, o_\Upsilon$, o_Υ denotes the number of criteria in category t_Υ, $(k_1, k_2, \ldots, k_\rho)$ includes all the ρ-tuples of $(1, 2, \ldots, o_\Upsilon)$, $C_{o_\Upsilon}^{\rho}$ expresses the binomial coefficient, whose expression is $C_{o_\Upsilon}^{\rho} = \frac{o_\Upsilon!}{\rho!(o_\Upsilon-\rho)!}$.

Theorem 1: For given IHFEs $\mathfrak{J}_i$ $(i = 1, 2, \ldots, n)$. The aggregated result of formula citation is still *IHFEs*, characterized as in Eq. (7), shown at the bottom of the page:

Proof: By the operational laws of *IHFEs*, we can write ■

Likewise, we can also find the IHFPMSM operator has some features, including idempotency, monotonicity, and boundedness.

Theorem 2: (Idempotincy) If the given, *IHFEs* $\mathfrak{J}_i (i = 1, 2, 3, \ldots, n)$ are equal i.e., $\mathfrak{J}_i = \mathfrak{J} (i = 1, 2, \ldots, n)$.

$$IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) = \mathfrak{J}. \tag{8}$$

Proof: By Eq. (7), we have ■

Theorem 3: (Monotonicity) let $\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n$ be *IHFs* where $\mathfrak{J}_i = (u_i, v_i)$and i=1,2,…,n and let $\mathfrak{J}_1', \mathfrak{J}_2', \ldots, \mathfrak{J}_n'$ where $\mathfrak{J}_i' = (u_i', v_i')$ and i=1,2,…,n which meet the condition $u_i \geq u_i'$ and $v_i \leq v_i'$ for all i=1,2,…,n. Then

$$IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) \geq IHFPMSM^{(\rho)}(\mathfrak{J}_1', \mathfrak{J}_2', \ldots, \mathfrak{J}_n). \tag{9}$$

Proof: We can capture that $\rho \geq 1$ and $C_{o_\Upsilon}^{\rho} \geq 1$ easily. Firstly, let us consider the membership part. Because $u_i \geq u_i'$ for all i, we have $\prod_{l=1}^{\rho} u_{il} \geq \prod_{l=1}^{\rho} u_{il}'$. Thus

$$\bigcup_{u_{il}\in\mathfrak{J}_{il}} \prod_{1<i_1<\ldots<0_\Upsilon} \left(1 - \prod_{l=1}^{\rho} u_{il}\right) \leq \bigcup_{u_{il}'\in\mathfrak{J}_{il}'} \prod_{1<i_1<\ldots<0_\Upsilon} \left(1 - \prod_{l=1}^{\rho} u_{il}'\right)$$

$$\Longrightarrow \bigcup_{u_{il}\in\mathfrak{J}_{il}} \left(\prod_{1<i_1<\ldots<0_\Upsilon} \left(1 - \prod_{l=1}^{\rho} u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}} \leq \bigcup_{u_{il}'\in\mathfrak{J}_{il}'} \left(\prod_{1<i_1<\ldots<0_\Upsilon} \left(1 - \prod_{l=1}^{\rho} u_{il}'\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}$$

$$\Longrightarrow \bigcup_{u_{il}\in\mathfrak{J}_{il}} \left(1 - \left(\prod_{1<i_1<\ldots<0_\Upsilon} \left(1 - \prod_{l=1}^{\rho} u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}} \geq \bigcup_{u_{il}'\in\mathfrak{J}_{il}'} \left(1 - \left(\prod_{1<i_1<\ldots<0_\Upsilon} \left(1 - \prod_{l=1}^{\rho} u_{il}'\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}$$

Now, we consider the non-memberships part. Because $v_i \leq v_i'$ for all i, we have $\prod_{l=1}^{\rho}(1 - v_{il}) \geq \prod_{l=1}^{\rho}(1 - v_{il}')$. Thus, we can get

$$\bigcup_{v_{il}\in\mathfrak{J}_{il}} \prod_{1<i_1<\ldots<0_\Upsilon} \left(1 - \prod_{l=1}^{\rho}(1 - v_{il})\right) \leq \bigcup_{v_{il}'\in\mathfrak{J}_{il}'} \prod_{1<i_1<\ldots<0_\Upsilon} \left(1 - \prod_{l=1}^{\rho}(1 - v_{il}')\right)$$

Now compare the values of $IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n)$ with the value of $IHFPMSM^{(\rho)}(\mathfrak{J}_1', \mathfrak{J}_2', \ldots, \mathfrak{J}_n)$ let $\mathfrak{J}=(u_{\mathfrak{J}}, v_{\mathfrak{J}})= IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n)$ and let $\mathfrak{J}'=(u_{\mathfrak{J}'}, v_{\mathfrak{J}'})= IHFPMSM^{(\rho)}(\mathfrak{J}_1', \mathfrak{J}_2', \ldots, \mathfrak{J}_n)$. Then, we can get As a result to $u_{\mathfrak{J}} \geq u_{\mathfrak{J}'}$ in the aforementioned analysis, then $\mathfrak{J} \geq \mathfrak{J}'$, that is $IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) \geq$

$$IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) = \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{ \begin{array}{c} 1 - \left(\prod_{\Upsilon=1}^{t} \left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<o_\Upsilon} \left(1 - \prod_{l=1}^{\rho} u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}, \\ \left(\prod_{\Upsilon=1}^{t} \left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<o_\Upsilon} \left(1 - \prod_{l=1}^{\rho} \left(1 - v_{il}\right)\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}} \end{array} \right\}. \tag{7}$$

IHFPMSM$^{(\rho)}\left(\mathfrak{J}'_1, \mathfrak{J}'_2, \ldots, \mathfrak{J}_n\right)$, which complete the proof of this property. ■

Theorem 4 (Boundedness): For a given *IHFEs* suppose that $\mathfrak{J}^- = \min_i \mathfrak{J}_i$ and $\mathfrak{J}^+ = \max_i \mathfrak{J}_i$ and (i=1,2,…,n), then

$$\mathfrak{J}^- \leq IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) \leq \mathfrak{J}^+. \quad (10)$$

Proof: As given that $\mathfrak{J}^- = \min_i \mathfrak{J}_i \leq \mathfrak{J}_i$ from Theorem 2 and 3, we can write

$$\begin{aligned}\mathfrak{J}^- &= IHFPMSM^{(\rho)}\left(\mathfrak{J}^-, \mathfrak{J}^-, \ldots, \mathfrak{J}^-\right)\\ &\leq IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n)\end{aligned}$$

Same as above

$$\begin{aligned}&IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n)\\ &\quad\leq IHFPMSM^{(\rho)}\left(\mathfrak{J}^+, \mathfrak{J}^+, \ldots, \mathfrak{J}^+\right) = \mathfrak{J}^+.\end{aligned}$$

Thus, we have

$$\mathfrak{J}^- \leq IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) \leq \mathfrak{J}^+.$$

■

Then, we explore the effects of parameter ρ on the developed IHFPMSM operator.

Theorem 5: Suppose that $\mathfrak{J}_1$,$\mathfrak{J}_2$,…and $\mathfrak{J}_n$ are the *IHFs*, where $\mathfrak{J}_i$=(u_i, v_i) and i=1,2,…,n and let ρ = 1,2,…,$\min_{\Upsilon} o_{\Upsilon}$. As ρ decreases, the *IHFPMSM* operator increases monotonically.

Theorem 6: Suppose that $\mathfrak{J}_1$,$\mathfrak{J}_2$,…and $\mathfrak{J}_n$ are the *IHFEs*, where $\mathfrak{J}_i$=(u_i, v_i) and i=1,2,…,n and let ρ = 1,2,…,$\min_{\Upsilon} o_{\Upsilon}$. As ρ decreases, the *IHFPMSM* operator increases monotonically.

Proof: From Eq. (7) we have

Now we prove that the function $\mathfrak{Q}_u(\rho)$ increases monotonically as the parameter ρ decreases. According to the Maclaurin inequality

$$\begin{aligned}&\left(\prod_{1<i_1<\ldots<0_{\Upsilon}}\left(1-\prod_{l=1}^{\rho}u_{il}\right)\right)^{\frac{1}{C^{\rho}_{o_{\Upsilon}}}}\\ &\quad\leq \sum_{1<i_1<\ldots<o_{\Upsilon}} \frac{1-\prod_{l=1}^{\rho}u_{il}}{C^{\rho}_{o_{\Upsilon}}}\end{aligned}$$

$$\otimes^{\rho}_{l=1}\mathfrak{J}_{il} = \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{il}\in\mathfrak{J}_{il}}\left\{\prod_{l=1}^{\rho}u_{il}, 1-\prod_{l=1}^{\rho}\left(1-v_{il}\right)\right\}$$

$$\oplus_{1<i_1<\ldots.<i_k<o_{\Upsilon}}\left(\otimes^{\rho}_{l=1}\mathfrak{J}_{il}\right) = \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{il}\in\mathfrak{J}_{il}}\left\{\begin{array}{c}1-\prod_{1<i_1<\ldots<o_{\Upsilon}}\left(1-\prod_{l=1}^{\rho}u_{il}\right),\\ \prod_{1<i_1<\ldots<0_{\Upsilon}}\left(1-\prod_{l=1}^{\rho}\left(1-v_{il}\right)\right)\end{array}\right\}$$

$$\left(\frac{\oplus_{1<i_1<\ldots.<i_k<0_{\Upsilon}}(\otimes^{\rho}_{l=1}\mathfrak{J}_{il})}{C^{\rho}_{o_{\Upsilon}}}\right)^{\frac{1}{\rho}}$$

$$= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}}\left\{\begin{array}{c}\left(1-\left(\prod_{1<i_1<\ldots<o_{\Upsilon}}\left(1-\prod_{l=1}^{\rho}u_{il}\right)\right)^{\frac{1}{C^{\rho}_{o_{\Upsilon}}}}\right)^{\frac{1}{\rho}},\\ 1-\left(1-\left(\prod_{1<i_1<\ldots<o_{\Upsilon}}\left(1-\prod_{l=1}^{\rho}\left(1-v_{il}\right)\right)\right)^{\frac{1}{C^{\rho}_{o_{\Upsilon}}}}\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$\otimes^{t}_{\Upsilon=1}\left(\frac{\oplus_{1<i_1<\ldots.<i_k<0_{\Upsilon}}(\otimes^{\rho}_{l=1}\mathfrak{J}_{il})}{C^{\rho}_{o_{\Upsilon}}}\right)^{\frac{1}{\rho}}$$

$$= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}}\left\{\begin{array}{c}1-\prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_{\Upsilon}}\left(1-\prod_{l=1}^{\rho}u_{il}\right)\right)^{\frac{1}{C^{\rho}_{o_{\Upsilon}}}}\right)^{\frac{1}{\rho}}\right),\\ \prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_{\Upsilon}}\left(1-\prod_{l=1}^{\rho}\left(1-v_{il}\right)\right)\right)^{\frac{1}{C^{\rho}_{o_{\Upsilon}}}}\right)^{\frac{1}{\rho}}\right)\end{array}\right\}$$

$$\frac{1}{t}\otimes^{t}_{\Upsilon=1}\left(\frac{\oplus_{1<i_1<\ldots.<i_k<0_{\Upsilon}}(\otimes^{\rho}_{l=1}\mathfrak{J}_{il})}{C^{\rho}_{o_{\Upsilon}}}\right)^{\frac{1}{\rho}}$$

$$= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}}\left\{\begin{array}{c}1-\left(\prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_{\Upsilon}}\left(1-\prod_{l=1}^{\rho}u_{il}\right)\right)^{\frac{1}{C^{\rho}_{o_{\Upsilon}}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}},\\ \left(\prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_{\Upsilon}}\left(1-\prod_{l=1}^{\rho}\left(1-v_{il}\right)\right)\right)^{\frac{1}{C^{\rho}_{o_{\Upsilon}}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\end{array}\right\}.$$

This completes the verification.

$$\Longrightarrow 1-\left(\prod_{1<i_1<\ldots<0_\Upsilon}\left(1-\prod_{l=1}^{\rho}u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}} \geq \sum_{1<i_1<\ldots<o_\Upsilon}\frac{\prod_{l=1}^{\rho}u_{il}}{C_{o_\Upsilon}^{\rho}}$$

$$\Longrightarrow \left(1-\left(\prod_{1<i_1<\ldots<0_\Upsilon}\left(1-\prod_{l=1}^{\rho}u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}} \geq \left(\sum_{1<i_1<\ldots<o_\Upsilon}\frac{\prod_{l=1}^{\rho}u_{il}}{C_{o_\Upsilon}^{\rho}}\right)^{\frac{1}{\rho}}$$

$$\begin{aligned}
&IHFPMSM^{(\rho)}\left(\mathfrak{I}_1,\mathfrak{I}_2,\ldots,\mathfrak{I}_n\right)\\
&=\bigcup_{u_{il}\in\mathfrak{I}_{il},v_{ij}\in\mathfrak{I}_{il}}\left\{\begin{array}{l}1-\left(\prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{\rho}u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}},\\ \left(\prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{\rho}\left(1-v_{il}\right)\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\end{array}\right\}\\
&=\bigcup_{u_{il}\in\mathfrak{I}_{il},v_{ij}\in\mathfrak{I}_{il}}\left\{\begin{array}{l}1-\left(\prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-u^{\rho}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}},\\ \left(\prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\left(1-v\right)^{\rho}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\end{array}\right\}\\
&=\bigcup_{u_{il}\in\mathfrak{I}_{il},v_{il}\in\mathfrak{I}_{il}}\left\{\begin{array}{l}1-\left(\prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(\left(1-u^{\rho}\right)^{C_{o_\Upsilon}^{\rho}}\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}},\\ \left(\prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(\left(1-\left(1-v\right)^{\rho}\right)^{C_{o_\Upsilon}^{\rho}}\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\end{array}\right\}\\
&=\bigcup_{u_{il}\in\mathfrak{I}_{il},v_{ij}\in\mathfrak{I}_{il}}\left\{\begin{array}{l}1-\left(\prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(1-u^{\rho}\right)\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}},\\ \left(\prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(1-\left(1-v\right)^{\rho}\right)\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\end{array}\right\}\\
&=\bigcup_{u_{il}\in\mathfrak{I}_{il},v_{ij}\in\mathfrak{I}_{il}}\left\{1-\left(\prod_{\Upsilon=1}^{t}\left(1-u\right)\right)^{\frac{1}{t}},\left(\prod_{\Upsilon=1}^{t}\left(1-\left(1-v\right)\right)\right)^{\frac{1}{t}}\right\}=\bigcup_{u\in\mathfrak{I},v\in\mathfrak{I}}(u,v)=\mathfrak{I}
\end{aligned}$$

$$\begin{aligned}
&\Longrightarrow \bigcup_{u_{il}\in\mathfrak{I}_{il}}\prod_{o_\Upsilon}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<0_\Upsilon}\left(1-\prod_{l=1}^{\rho}u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\\
&\leq \bigcup_{u'_{il}\in\mathfrak{I}'_{il}}\prod_{o_\Upsilon}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<0_\Upsilon}\left(1-\prod_{l=1}^{\rho}u'_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\\
&\Longrightarrow \bigcup_{u_{il}\in\mathfrak{I}_{il}}\left(1-\left(\prod_{o_\Upsilon}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<0_\Upsilon}\left(1-\prod_{l=1}^{\rho}u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\right)\\
&\geq \bigcup_{u'_{il}\in\mathfrak{I}'_{il}}\left(1-\left(\prod_{o_\Upsilon}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<0_\Upsilon}\left(1-\prod_{l=1}^{\rho}u'_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\right).
\end{aligned}$$

Next, we solve by using the contradiction method. Suppose that it is monotonically decreasing as the ρ decrease, $\mathfrak{Q}_{\mathfrak{u}}\left(\min_{\curlyvee}(o_{\curlyvee})\right) \geq \ldots \geq \mathfrak{Q}_{\mathfrak{u}}(2) \geq \mathfrak{Q}_{\mathfrak{u}}(1)$ thus we have

$$\mathfrak{Q}_{\mathfrak{u}}(1) \geq 1-\left(\prod_{o_{\curlyvee}=1}^{t}\left(1-\left(\sum_{1<i_1<\ldots<o_{\curlyvee}}\frac{\prod_{l=1}^{1}u_{il}}{C_{o_{\curlyvee}}^{1}}\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{t}}$$

$$=1-\left(\prod_{o_{\curlyvee}}^{t}\left(1-\left(\frac{\sum_{i_l=1}^{o_{\curlyvee}}u_{i_l}}{o_{\curlyvee}}\right)\right)\right)^{\frac{1}{t}}.$$

Now we suppose that every criterion has the same category, namely $o_{\curlyvee} = o(\curlyvee = 1, 2, \ldots, t)$ because $\min_{\curlyvee}(o_{\curlyvee}) = o$, we can get Thus, based on assumption $\mathfrak{Q}(o_{\curlyvee}) \geq \mathfrak{Q}(1)$. We can get

$$\mathfrak{Q}(o) = 1-\left(\prod_{o}^{t}\left(1-\left(\prod_{l=1}^{o}u_{il}\right)^{\frac{1}{o}}\right)\right)^{\frac{1}{t}}$$

$$\geq \mathfrak{Q}(1) \geq 1-\left(\prod_{o_{\curlyvee}=1}^{t}\left(1-\left(\frac{\sum_{i_l=1}^{o_{\curlyvee}}u_{i_l}}{o_{\curlyvee}}\right)\right)\right)^{\frac{1}{t}}$$

$$\left(\prod_{l=1}^{o}u_{i_l}\right)^{\frac{1}{o}} \geq \left(\frac{\sum_{i_l=1}^{o}u_{i_l}}{o}\right)$$

$$\Longrightarrow \bigcup_{v_{il}\in\mathfrak{I}_{il}}\left(\prod_{1<i_1<\ldots<0_{\curlyvee}}\left(1-\prod_{l=1}^{\rho}(1-v_{il})\right)\right)^{\frac{1}{C_{o_{\curlyvee}}^{\rho}}}$$

$$\leq \bigcup_{v'_{il}\in\mathfrak{I}'_{il}}\left(\prod_{1<i_1<\ldots<0_{\curlyvee}}\left(1-\prod_{l=1}^{\rho}(1-v'_{il})\right)\right)^{\frac{1}{C_{o_{\curlyvee}}^{\rho}}}$$

$$\Longrightarrow \bigcup_{v_{il}\in\mathfrak{I}_{il}}\left(1-\left(\prod_{1<i_1<\ldots<0_{\curlyvee}}\left(1-\prod_{l=1}^{\rho}(1-v_{il})\right)\right)^{\frac{1}{C_{o_{\curlyvee}}^{\rho}}}\right)^{\frac{1}{\rho}}$$

$$\geq \bigcup_{v'_{il}\in\mathfrak{I}_{il}}\left(1-\left(\prod_{1<i_1<\ldots<0_{\curlyvee}}\left(1-\prod_{l=1}^{\rho}(1-v'_{il})\right)\right)^{\frac{1}{C_{o_{\curlyvee}}^{\rho}}}\right)^{\frac{1}{\rho}}$$

$$\Longrightarrow \bigcup_{v_{il}\in\mathfrak{I}_{il}}\prod_{l=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<0_{\curlyvee}}\left(1-\prod_{l=1}^{\rho}(1-v_{il})\right)\right)^{\frac{1}{C_{o_{\curlyvee}}^{\rho}}}\right)^{\frac{1}{\rho}}\right)$$

$$\leq \bigcup_{v'_{il}\in\mathfrak{I}'_{il}}\prod_{l=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<0_{\curlyvee}}\left(1-\prod_{l=1}^{\rho}(1-v'_{il})\right)\right)^{\frac{1}{C_{o_{\curlyvee}}^{\rho}}}\right)^{\frac{1}{\rho}}\right)$$

$$\Longrightarrow \bigcup_{v_{il}\in\mathfrak{I}_{il}}\left(\prod_{l=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<0_{\curlyvee}}\left(1-\prod_{l=1}^{\rho}(1-v_{il})\right)\right)^{\frac{1}{C_{o_{\curlyvee}}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}$$

$$\leq \bigcup_{v'_{il}\in\mathfrak{I}'_{il}}\left(\prod_{l=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<0_{\curlyvee}}\left(1-\prod_{l=1}^{\rho}(1-v'_{il})\right)\right)^{\frac{1}{C_{o_{\curlyvee}}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}.$$

Thus, $\left(\prod_{i_l=1}^{o} u_{i_l}\right)^{\frac{1}{o}} \geq \left(\frac{\sum_{i_l=1}^{o} u_{i_l}}{o}\right)$ is the contradiction to the Theorem 1. Thus as ρ decrease the function $\mathfrak{Q}(\rho)$ increases monotonically. Likewise, we can prove that the function $\mathfrak{P}(\rho)$ decreases monotonically. According to the above analysis, we have

$$IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) \geq IHFPMSM^{(\rho+1)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n).$$

$$u_{\mathfrak{J}} = \bigcup_{u_{il} \in \mathfrak{J}_{il}} \left(1 - \left(\prod_{o_\Upsilon}^{t}\left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<0_\Upsilon}\left(1 - \prod_{l=1}^{\rho} u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\right)$$

$$v_{\mathfrak{J}} = \bigcup_{v_{il} \in \mathfrak{J}_{il}} \left(\prod_{l=1}^{t}\left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<0_\Upsilon}\left(1 - \prod_{l=1}^{\rho}(1 - v_{il})\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}$$

$$u_{\mathfrak{J}'} = \bigcup_{u'_{il} \in \mathfrak{J}'_{il}} \left(1 - \left(\prod_{o_\Upsilon}^{t}\left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<0_\Upsilon}\left(1 - \prod_{l=1}^{\rho} u'_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\right)$$

$$v_{\mathfrak{J}'} = \bigcup_{v'_{il} \in \mathfrak{J}'_{il}} \left(\prod_{l=1}^{t}\left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<0_\Upsilon}\left(1 - \prod_{l=1}^{\rho}(1 - v'_{il})\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}.$$

$$\frac{1}{t} \otimes_{\Upsilon=1}^{t} \left(\frac{\oplus_{1<i_1<\ldots.<i_k<0_\Upsilon}(\otimes_{l=1}^{\rho} \mathfrak{J}_{il})}{C_{o_\Upsilon}^{\rho}}\right)^{\frac{1}{\rho}}$$

$$= \bigcup_{u_{il} \in \mathfrak{J}_{il}, v_{ij} \in \mathfrak{J}_{il}} \left\{1 - \left(\prod_{\Upsilon=1}^{t}\left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1 - \prod_{l=1}^{\rho} u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}, \left(\prod_{\Upsilon=1}^{t}\left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1 - \prod_{l=1}^{\rho}(1 - v_{il})\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\right\}.$$

Let

$$\mathfrak{Q}_u(\rho) = \left\{1 - \left(\prod_{\Upsilon=1}^{t}\left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1 - \prod_{l=1}^{\rho} u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\right\}$$

and

$$\mathfrak{P}_V(\rho) = \left\{\left(\prod_{\Upsilon=1}^{t}\left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1 - \prod_{l=1}^{\rho}(1 - v_{il})\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\right\}$$

Thus, the *IHFPMSM* operator is monotonically increased with respect to the parameter ρ. ■

Theorem 7: For given range of *IHFEs* $\mathfrak{J}_i = (i = 1, 2, \ldots, n)$, $L = 1, 2, \ldots, \min_{\curlyvee} o_{\curlyvee}$. Then

$$\max\left\{IHFPMSM^{(\rho)}\left(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n\right)\right\} = IHFPMSM^{(1)}\left(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n\right).$$

We now examine numerous peculiar instances of the *IHFPMSM* operator concerning various parameter l values.

Case 1: Consider there is only one category t_1 and number of sets in t_1 = n and parameter ρ based on the *IHFPMSM* operator, we have This is the IHF MSM operator.

Case 2: Now if $t = 1$ and $\rho = 1$ then the definition of *IHFPMSMS* operator, we have which is reduced to an IHF averaging operator.

Case 3: Now if $t = 1$ and $\rho = 2$ then the definition of *IHFPMSMS* operator, we have which is reduced to IHF Bonferroni mean operator $DHFBM^{(1,1)}\left(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n\right)$ [43].

Case 4: Now if $t = 1$ and $\rho = n$ then the definition of *IHFPMSMS* operator, we have
This is the IHF geometric mean operator.

Example 1: Let $\mathfrak{J}_1 = \{\{0.3, 0.5\}, \{0.2\}\}$, $\mathfrak{J}_2 = \{\{0.3, 0.4\}, \{0.1, 0.6\}\}$, $\mathfrak{J}_3 = \{\{0.2, 0.4\}, \{0.2, 0.5\}\}$ and $\mathfrak{J}_4 = \{\{0.4, 0.8\}, \{0.1, 0.2\}\}$ be four *IHFEs*. Suppose these four *IHFEs* are classified into two categories L_1 and L_2 with $L_1 = \{\mathfrak{J}_1, \mathfrak{J}_2\}$ and $L_2 = \{\mathfrak{J}_3, \mathfrak{J}_4\}$. Here, we apply the *IHFPMSM* operator to aggregate this *IHFEs*. In general, we let $\rho = 2$, then

B. IHFWPMSM AGGREGATION OPERATORS

In this section, we present the *IHFWPMSM* operator. It is considered that all the attributes have the same importance. However, their weights are not equal in applicable decision-making. Then, it is important to consider that each attribute has its own weight. Let the weight of each attribute $\mathfrak{J}_i (i = 1, 2, \ldots, n)$ is $\mathfrak{w}_i$ that fulfills $0 \leq \mathfrak{w}_i \leq 1$ and $\sum_{i=1}^{n} \mathfrak{w}_i = 1$. Thus, the developed *IHFWPMSM* operator for *IHFEs* is characterized as follows.

Definition 8: Let $\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_m$ be the set of m *IHFEs*. Then, the *IHFWPMSM* operator is characterized as

$$IHFPMSM^{(\rho)}\left(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n\right) = \frac{1}{t} \oplus_{\curlyvee=1}^{t} \left(\frac{\oplus_{1 \leq k_1 < \ldots < k_l \leq o_{\curlyvee}} \left(\otimes_{l=1}^{\rho} \left(\mathfrak{J}_{kl}\right)^{\mathfrak{W}_{kl}}\right)}{C_{o_{\curlyvee}}^{\rho}}\right)^{\frac{1}{\rho}}, \quad (15)$$

$$\Longrightarrow \prod_{o_{\curlyvee}}^{t} \left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<0_{\curlyvee}} \left(1 - \prod_{l=1}^{\rho} u_{il}\right)\right)^{\frac{1}{C_{o_{\curlyvee}}^{\rho}}}\right)^{\frac{1}{\rho}}\right) \leq \prod_{o_{\curlyvee}}^{t} \left(1 - \left(\sum_{1<i_1<\ldots<o_{\curlyvee}} \frac{\prod_{l=1}^{\rho} u_{il}}{C_{o_{\curlyvee}}^{\rho}}\right)^{\frac{1}{\rho}}\right)$$

Now, we can get

$$\mathfrak{Q}_u(\rho) = \left\{1 - \left(\prod_{\curlyvee=1}^{t} \left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<o_{\curlyvee}} \left(1 - \prod_{l=1}^{\rho} u_{il}\right)\right)^{\frac{1}{C_{o_{\curlyvee}}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\right\}$$
$$\geq \left\{1 - \left(\prod_{o_{\curlyvee}}^{t} \left(1 - \left(\sum_{1<i_1<\ldots<o_{\curlyvee}} \frac{\prod_{l=1}^{\rho} u_{il}}{C_{o_{\curlyvee}}^{\rho}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}\right\}.$$

$$\mathfrak{Q}\left(\min_{\curlyvee} o_{\curlyvee}\right) = \mathfrak{Q}(o) = 1 - \left(\prod_{o_{\curlyvee}=t}^{t} \left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<0_{\curlyvee}} \left(1 - \prod_{l=1}^{o} u_{il}\right)\right)^{\frac{1}{C_{o_{\curlyvee}}^{o_{\curlyvee}}}}\right)^{\frac{1}{\o_{\curlyvee}}}\right)\right)^{\frac{1}{t}}$$

$$\mathfrak{Q}(o) = 1 - \left(\prod_{o_{\curlyvee}=1}^{t} \left(1 - \left(\prod_{l=1}^{o} u_{il}\right)^{\frac{1}{o}}\right)\right)^{\frac{1}{t}}.$$

where t denotes the number of categories, ρ is a parameter, $\rho = 1, 2, \ldots, o_{\curlyvee}$, $o_{\curlyvee}$ denotes the number of criteria in category $t_{\curlyvee}$, $\left(k_1, k_2, \ldots, k_\rho\right)$ includes all the ρ-tuples of $(1, 2, \ldots, o_{\curlyvee})$, $C^{\rho}_{o_{\curlyvee}}$ expresses the binomial coefficient, whose expression is $C^{\rho}_{o_{\curlyvee}} = \frac{o_{\curlyvee}!}{\rho!(o_{\curlyvee}-\rho)!}$ and $\mathfrak{w}_l \geq 0$ show the weight satisfying $\sum_{l=1}^{m} \mathfrak{w}_{\mathfrak{l}}=1$.

Theorem 8: For given IHFEs $\mathfrak{J}_i$ $(1, 2, \ldots, n)$. The aggregated result of formula citation is still *IHFEs*, characterized as below:

Proof: Based on the lines of Theorem 1, one can easily prove it. ■

Theorem 9 (Idempotincy): If the domain of discourse of given *IHFEs* $\mathfrak{J}_i$ (i=1,2,…,n) are same i.e., $\mathfrak{J}_i = \mathfrak{J}$

$$\begin{aligned}
&\min\left\{IHFPMSM^{(\rho)}\left(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n\right)\right\} \\
&\quad = IHFPMSM^{\left(\min_{\curlyvee}\{o_{\curlyvee}\}\right)}\left(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n\right). \\
&\quad = \bigcup_{\substack{u_{il}\in\mathfrak{J}_{il},\\ v_{ij}\in\mathfrak{J}_{il}}} \left\{\begin{array}{c} 1-\left(\prod_{\curlyvee=1}^{t}\left(\prod_{1<i_1<\ldots<o_{\curlyvee}}\left(1-u_{i_j}\right)\right)^{\frac{1}{o_{\curlyvee}}}\right)^{\frac{1}{t}}, \\ \left(\prod_{\curlyvee=1}^{t}\left(\prod_{1<i_1<\ldots<0_{\curlyvee}}\left(v_{i_j}\right)\right)^{\frac{1}{\curlyvee}}\right)^{\frac{1}{t}} \end{array}\right\}
\end{aligned}$$

$$\begin{aligned}
&IHFPMSM^{(\rho)}\left(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n\right) \\
&\quad = \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{\begin{array}{c} 1-\left(\prod_{\curlyvee=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_{\curlyvee}}\left(1-\prod_{l=1}^{\rho}u_{il}\right)\right)^{\frac{1}{C^{\rho}_{o_{\curlyvee}}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{1}}, \\ \left(\prod_{\curlyvee=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_{\curlyvee}}\left(1-\prod_{l=1}^{\rho}(1-v_{il})\right)\right)^{\frac{1}{C^{\rho}_{o_{\curlyvee}}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{1}} \end{array}\right\} \\
&\quad = \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{\begin{array}{c} \left(1-\left(\prod_{1<i_1<\ldots<o_{\curlyvee}}\left(1-\prod_{l=1}^{\rho}u_{il}\right)\right)^{\frac{1}{C^{\rho}_{o_{\curlyvee}}}}\right)^{\frac{1}{\rho}}, \\ 1-\left(1-\left(\prod_{1<i_1<\ldots<o_{\curlyvee}}\left(1-\prod_{l=1}^{\rho}(1-v_{il})\right)\right)^{\frac{1}{C^{\rho}_{o_{\curlyvee}}}}\right)^{\frac{1}{\rho}} \end{array}\right\}
\end{aligned} \tag{11}$$

$$\begin{aligned}
&IHFPMSM^{(1)}\left(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n\right) \\
&\quad = \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{\begin{array}{c} 1-\left(\prod_{\curlyvee=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_{\curlyvee}}\left(1-\prod_{l=1}^{1}u_{il}\right)\right)^{\frac{1}{C^{1}_{o_{\curlyvee}}}}\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{1}}, \\ \left(\prod_{\curlyvee=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_{\curlyvee}}\left(1-\prod_{l=1}^{1}(1-v_{il})\right)\right)^{\frac{1}{C^{1}_{o_{\curlyvee}}}}\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{1}} \end{array}\right\} \\
&\quad = \bigcup_{\substack{u_{i_1},\mathfrak{p}_{i_1}\in\mathfrak{J}_{i_1},\\ v_{i_1},\mathfrak{p}_{i_1}\in\mathfrak{J}_{i_1}}} \left\{1-\left(\prod_{1<i_1<o_{\curlyvee}}\left(1-u_{i_1}\right)\right)^{\frac{1}{C^{1}_{o_{\curlyvee}}}}, \left(\prod_{1<i_1<o_{\curlyvee}}\left(1-\left(1-v_{i_1}\right)\right)\right)^{\frac{1}{C^{1}_{o_{\curlyvee}}}}\right\} \; let\ (i_1 = k) \\
&\quad = \bigcup_{\substack{u_{i_1},\mathfrak{p}_{i_1}\in\mathfrak{J}_{i_1},\\ v_{i_1},\mathfrak{p}_{i_1}\in\mathfrak{J}_{i_1}}} \left\{1-\left(\prod_{k=1}^{o_{\curlyvee}}\left(1-u_k\right)\right)^{\frac{1}{o_{\curlyvee}}}, \left(\prod_{k=1}^{o_{\curlyvee}}v_k\right)^{\frac{1}{o_{\curlyvee}}}\right\}
\end{aligned} \tag{12}$$

(i=1,2,...,n), Then

$$IHFWPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) = \mathfrak{J} \tag{17}$$

Theorem 10 (Monotonicity): let $\mathfrak{J}_i = (u_i, v_i)$ and $\mathfrak{J}_i' = (u_i', v_i')$ are the two domain of discourse of *IHFEs*, for (i=1,2,...,n) such that $\mathfrak{u}_i \geq \mathfrak{u}_i'$ and $\mathfrak{v}_i \leq \mathfrak{v}_i'$ all $(u_i, v_i) \in \mathfrak{J}_i$ and $(u_i', v_i') \in \mathfrak{J}_i'$, then

$$IHFWPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) \geq IHFWPMSM^{(\rho)}\left(\mathfrak{J}_1', \mathfrak{J}_2', \ldots, \mathfrak{J}_n\right) \tag{18}$$

Theorem 11 (Boundedness): Let $\mathfrak{J}_i$ be the domain of discourse of *IHFEs* for i=1,2,...,n and $\mathfrak{J}^- = \min_i \mathfrak{J}_i$ and $\mathfrak{J}^+ = \max_i \mathfrak{J}_i$

$$\mathfrak{J}^- \leq IHFWPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) \leq \mathfrak{J}^+. \tag{19}$$

Theorem 12 (Parameter Monotonicity): Let $\mathfrak{J}_i$ be the domain of discourse of *IHFEs* for i=1,2,...,n and ρ = 1,2,...,$\min_{\Upsilon} o_{\Upsilon}$ As ρ decrease the *IHFWPMSM* operator increase monotonically.

Theorem 13: For given range of *IHFEs* $\mathfrak{J}_i(i = 1, 2, \ldots, n), l = 1, 2, \ldots, \min_{\Upsilon} o_{\Upsilon}$. Then

$$\max\left\{IHFWPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n)\right\} = IHFWPMSM^{(1)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n)$$

We now examine numerous peculiar instances of the *IHFPMSM* operator concerning various parameter l values.

Case 1: Consider there is only one category t_1 and number of sets in t_1 = n and parameter ρ based on the *IHFWPMSM* operator, we have which is reduced to the IHF MSM operator.

$$\begin{aligned}
&IHFPMSM^{(2)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) \\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{ \begin{array}{l} 1 - \left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{2} u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{1}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{1}}, \\ \left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{2}(1-v_{il})\right)\right)^{\frac{1}{C_{o_\Upsilon}^{1}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{1}} \end{array} \right\} \\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{ \begin{array}{l} \left(1-\left(\prod_{1<i_1<i_2<o_\Upsilon}\left(1-u_{i_1}u_{i_2}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{2}}}\right)^{\frac{1}{2}}, \\ 1-\left(1-\left(\prod_{1<i_1<i_2<o_\Upsilon}\left(1-\left(1-v_{i_1}v_{i_2}\right)\right)\right)^{\frac{1}{C_{o_\Upsilon}^{2}}}\right)^{\frac{1}{2}} \end{array} \right\} \\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{ \begin{array}{l} \left(1-\left(\prod_{1<i_1<i_2<o_\Upsilon}\left(1-u_{i_1}u_{i_2}\right)\right)^{\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}, \\ 1-\left(1-\left(\prod_{1<i_1<i_2<o_\Upsilon}\left(v_{i_1}v_{i_2}\right)\right)^{\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}} \end{array} \right\} \\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{ \begin{array}{l} \left(1-\left(\prod_{i_1,i_2=1;i_1=i_2}^{o_\Upsilon}\left(1-u_{i_1}u_{i_2}\right)\right)^{\frac{1}{2}\cdot\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}, \\ 1-\left(1-\left(\prod_{i_1,i_2=1;i_1=i_2}^{o_\Upsilon}\left(v_{i_1}v_{i_2}\right)\right)^{\frac{1}{2}\cdot\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}} \end{array} \right\} \\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{ \begin{array}{l} \left(1-\left(\prod_{i_1,i_2=1;i_1=i_2}^{o_\Upsilon}\left(1-u_{i_1}u_{i_2}\right)\right)^{\frac{1}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}, \\ 1-\left(1-\left(\prod_{i_1,i_2=1;i_1=i_2}^{o_\Upsilon}\left(v_{i_1}v_{i_2}\right)\right)^{\frac{1}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}} \end{array} \right\},
\end{aligned} \tag{13}$$

Case 2: Consider there is only one category $t = 1$ and $\rho = 1$ based on the *IHFWPMSM* operator, we have which is reduced to intuitionistic hesitant weighted averaging operator.

Case 3: Consider there is only one category $t = 1$ and $\rho = 2$ based on the *IHFWPMSM* operator, we have which is reduced to IHF weighted Bonferroni mean operator $DHFWBM^{(1,1)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n)$.

Case 4: Consider there is only one category $t = 1$ and $\rho = o_\Upsilon$ based on the *IHFWPMSM* operator, we have which is reduced to dual hesitant fuzzy weighted geometric mean operator.

IV. TECHNIQUE FOR SOLVING DECISION-MAKING PROBLEMS BASED ON IHFWPMSM OPERATORS

This part is intended to organize the MCDM approach around newly emerging *IHF* operators. Let R$= \{R_1, R_2, \ldots, R_m\}$ be the alternatives set for each choice, $S = \{S_1, S_2, \ldots, S_n\}$ be the set of choices, and $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_n)$ be the weight vector of the criteria set S. For $\sum_{k=1}^{n} \mathfrak{w}_k = 1$ and

$$
\begin{aligned}
&IHFPMSM^{(n)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n)\\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left[\begin{array}{c} 1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{o_\Upsilon} u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{o_\Upsilon}}}\right)^{\frac{1}{o_\Upsilon}}\right)\right)^{\frac{1}{1}}, \\ \left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{o_\Upsilon}(1-v_{il})\right)\right)^{\frac{1}{C_{o_\Upsilon}^{o_\Upsilon}}}\right)^{\frac{1}{o_\Upsilon}}\right)\right)^{\frac{1}{1}} \end{array}\right]\\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{\left(1-\left(1-\prod_{l=1}^{o_\Upsilon} u_{il}\right)\right)^{\frac{1}{o_\Upsilon}}, 1-\left(1-\left(1-\prod_{l=1}^{\rho}(1-v_{il})\right)\right)^{\frac{1}{\rho}}\right\}\\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{\left(1-\left(1-\prod_{l=1}^{o_\Upsilon} u_{il}\right)\right)^{\frac{1}{o_\Upsilon}}, 1-\left(\prod_{l=1}^{\rho}(1-v_{il})\right)^{\frac{1}{o_\Upsilon}}\right\}.
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n)\\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left[\begin{array}{c} 1-\left(\prod_{\Upsilon=1}^{2}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{2} u_{il}\right)\right)^{\frac{1}{C_{o_\Upsilon}^{2}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{2}}, \\ \left(\prod_{\Upsilon=1}^{2}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{2}(1-v_{il})\right)\right)^{\frac{1}{C_{o_\Upsilon}^{2}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{2}} \end{array}\right]\\
&= \left\{\begin{array}{c} \left\{\begin{array}{c} 0.3737, 0.7373.0.4672, 0.3737, 0.8246, 0.9264, 0.8507, 0.8246, \\ 0.4241, 0.7584, 0.5100, 0.4241, 0.3937, 0.7456, 0.4846, 0.3937 \end{array}\right\}, \\ \{0.0563, 0.1346, 0.1181, 0.0808, 0.0736, 0.1759, 0.1543, 0.1055\} \end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&IHFWPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n)\\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{\begin{array}{c} 1-\left(\prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{\rho}\left(1-(1-u_{il})^{\mathfrak{w}_{i_l}}\right)\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}}, \\ \left(\prod_{\Upsilon=1}^{t}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{\rho}\left(1-v_{il}^{\mathfrak{w}_{i_l}}\right)\right)\right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{t}} \end{array}\right\}.
\end{aligned}
\tag{16}
$$

$\varpi_k \in [0,1]$, the weight vector ϖ is used to indicate the relative weights of several criteria in the decision-making process. DMs assess each choice R_i under the criteria S_l in terms of IHFE $\mathfrak{J}_{il}$. Suppose that $S = \{S_1, S_2, \ldots, S_m\}$ is partitioned into t distinct categories $\{P_1, P_2, \ldots, P_t\}$. There are connections among criteria within the same category and none between criteria within other categories. The instances that follow illustrate how the main phases of the proposed approach are demonstrated as follows:

Step 1: Formation of IHF conclusion matrix: With regard to the aforementioned circumstance, the MCDM problem may be formulated in the ensuing creation of a conclusion matrix:

$$
\begin{aligned}
&\min\left\{IHFWPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n)\right\} \\
&\quad = IHFWPMSM^{(\min_{\Upsilon}\{o_\Upsilon\})}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) \\
&\quad = \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{ \begin{array}{l} \left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{\rho}\left(1-(1-u_{il})^{\varpi_{i_l}}\right)\right)\right)^{\frac{1}{C^{\rho}_{o_\Upsilon}}}\right)^{\frac{1}{\rho}}, \\ \left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{\rho}\left(1-v_{il}^{\varpi_{i_l}}\right)\right)\right)^{\frac{1}{C^{\rho}_{o_\Upsilon}}}\right)^{\frac{1}{\rho}}\right) \end{array} \right\},
\end{aligned}
$$

$$
\begin{aligned}
&IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) \\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{ \begin{array}{l} 1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{\rho}\left(1-(1-u_{il})^{\varpi_{i_l}}\right)\right)\right)^{\frac{1}{C^{\rho}_{o_\Upsilon}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{1}}, \\ \left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{\rho}\left(1-v_{il}^{\varpi_{i_l}}\right)\right)\right)^{\frac{1}{C^{\rho}_{o_\Upsilon}}}\right)^{\frac{1}{\rho}}\right)\right)^{\frac{1}{1}} \end{array} \right\}. \\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{ \begin{array}{l} \left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{\rho}\left(1-(1-u_{il})^{\varpi_{i_l}}\right)\right)\right)^{\frac{1}{C^{\rho}_{o_\Upsilon}}}\right)^{\frac{1}{\rho}}, \\ 1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{\rho}\left(1-v_{il}^{\varpi_{i_l}}\right)\right)\right)^{\frac{1}{C^{\rho}_{o_\Upsilon}}}\right)^{\frac{1}{\rho}} \end{array} \right\},
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
&IHFPMSM^{(\rho)}(\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_n) \\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{ \begin{array}{l} 1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{1}\left(1-(1-u_{il})^{\varpi_{i_l}}\right)\right)\right)^{\frac{1}{C^{1}_{o_\Upsilon}}}\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{1}}, \\ \left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{1}\left(1-v_{il}^{\varpi_{i_l}}\right)\right)\right)^{\frac{1}{C^{1}_{o_\Upsilon}}}\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{1}} \end{array} \right\}. \\
&= \bigcup_{u_{il}\in\mathfrak{J}_{il}, v_{ij}\in\mathfrak{J}_{il}} \left\{ 1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}(1-u_{il})^{\varpi_{i_l}}\right)^{\frac{1}{o_\Upsilon}}, \left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(v_{il}^{\varpi_{i_l}}\right)\right)^{\frac{1}{o_\Upsilon}} \right\},
\end{aligned} \tag{21}
$$

$$\mathbf{K}_{m\times n}=\begin{pmatrix}\Im_{11} & \cdots & \Im_{1g} & \cdots & \Im_{1n}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \Im_{i1} & \cdots & \Im_{ig} & \cdots & \Im_{in}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \Im_{m1} & \cdots & \Im_{mg} & \cdots & \Im_{mn}\end{pmatrix} \tag{24}$$

Step 2:Normality: At this step, the IHF assessment matrix $K_{p\times r}$ is changed in accordance with a few benefit and expense criteria. These two criteria respond in opposing ways, meaning that when the value increases, the benefit criterion performs best, and the cost criterion performs badly. As a result,

$$\begin{aligned}
&IHFPMSM^{(\rho)}\left(\Im_1,\Im_2,\ldots,\Im_n\right)\\
&=\bigcup_{u_{il}\in\Im_{il},v_{ij}\in\Im_{il}}\left\{\begin{array}{c}1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{2}\left(1-\left(1-u_{il}\right)^{\mathfrak{w}_{i_l}}\right)\right)\right)^{\frac{1}{C^2_{o_\Upsilon}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{1}},\\ \left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{2}\left(1-v_{il}^{\mathfrak{w}_{i_l}}\right)\right)\right)^{\frac{1}{C^2_{o_\Upsilon}}}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{1}}\end{array}\right\}.\\
&=\bigcup_{u_{il}\in\Im_{il},v_{ij}\in\Im_{il}}\left\{\begin{array}{c}\left(1-\left(\prod_{1<i_1<i_2<o_\Upsilon}\left(\left(1-u_{i_1}\right)^{\mathfrak{w}_{i_1}}\left(1-u_{i_2}\right)^{\mathfrak{w}_{i_2}}\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}},\\ 1-\left(1-\left(\prod_{1<i_1<i_2<o_\Upsilon}\left(v_{i_1}^{\mathfrak{w}_{i_1}}v_{i_2}^{\mathfrak{w}_{i_2}}\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}\end{array}\right\}\\
&=\bigcup_{u_{il}\in\Im_{il},v_{ij}\in\Im_{il}}\left\{\begin{array}{c}\left(1-\left(\prod_{i_1,i_2=1;i_1=i_2}^{o_\Upsilon}\left(\left(1-u_{i_1}\right)^{\mathfrak{w}_{i_1}}\left(1-u_{i_2}\right)^{\mathfrak{w}_{i_2}}\right)\right)^{\frac{1}{2}\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}},\\ 1-\left(1-\left(\prod_{i_1,i_2=1;i_1=i_2}^{o_\Upsilon}\left(v_{i_1}^{\mathfrak{w}_{i_1}}v_{i_2}^{\mathfrak{w}_{i_2}}\right)\right)^{\frac{1}{2}\cdot\frac{2}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}\end{array}\right\}\\
&=\bigcup_{u_{il}\in\Im_{il},v_{ij}\in\Im_{il}}\left\{\begin{array}{c}\left(1-\left(\prod_{i_1,i_2=1;i_1=i_2}^{o_\Upsilon}\left(\left(1-u_{i_1}\right)^{\mathfrak{w}_{i_1}}\left(1-u_{i_2}\right)^{\mathfrak{w}_{i_2}}\right)\right)^{\frac{1}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}},\\ 1-\left(1-\left(\prod_{i_1,i_2=1;i_1=i_2}^{o_\Upsilon}\left(v_{i_1}^{\mathfrak{w}_{i_1}}v_{i_2}^{\mathfrak{w}_{i_2}}\right)\right)^{\frac{1}{o_\Upsilon(o_\Upsilon-1)}}\right)^{\frac{1}{2}}\end{array}\right\},
\end{aligned}\tag{22}$$

$$\begin{aligned}
&IHFPMSM^{(\rho)}\left(\Im_1,\Im_2,\ldots,\Im_n\right)\\
&=\bigcup_{u_{il}\in\Im_{il},v_{ij}\in\Im_{il}}\left\{\begin{array}{c}1-\left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{o_\Upsilon}\left(1-\left(1-u_{il}\right)^{\mathfrak{w}_{i_l}}\right)\right)\right)^{\frac{1}{C^{o_\Upsilon}_{o_\Upsilon}}}\right)^{\frac{1}{o_\Upsilon}}\right)\right)^{\frac{1}{1}},\\ \left(\prod_{\Upsilon=1}^{1}\left(1-\left(1-\left(\prod_{1<i_1<\ldots<o_\Upsilon}\left(1-\prod_{l=1}^{o_\Upsilon}\left(1-v_{il}^{\mathfrak{w}_{i_l}}\right)\right)\right)^{\frac{1}{C^{o_\Upsilon}_{o_\Upsilon}}}\right)^{\frac{1}{o_\Upsilon}}\right)\right)^{\frac{1}{1}}\end{array}\right\}.\\
&=\bigcup_{u_{il}\in\Im_{il},v_{ij}\in\Im_{il}}\left\{\left(\prod_{l=1}^{o_\Upsilon}\left(1-\left(1-u_{il}\right)^{\mathfrak{w}_{i_l}}\right)\right)^{\frac{1}{o_\Upsilon}},1-\left(\prod_{l=1}^{o_\Upsilon}\left(1-v_{il}^{\mathfrak{w}_{i_l}}\right)\right)^{\frac{1}{o_\Upsilon}}\right\}\\
&=\bigcup_{u_{il}\in\Im_{il},v_{ij}\in\Im_{il}}\left\{\left(\prod_{l=1}^{o_\Upsilon}\left(1-\left(1-u_{il}\right)^{\mathfrak{w}_{i_l}}\right)\right)^{\frac{1}{o_\Upsilon}},1-\left(\prod_{l=1}^{o_\Upsilon}\left(1-v_{il}^{\mathfrak{w}_{i_l}}\right)\right)^{\frac{1}{o_\Upsilon}}\right\},
\end{aligned}\tag{23}$$

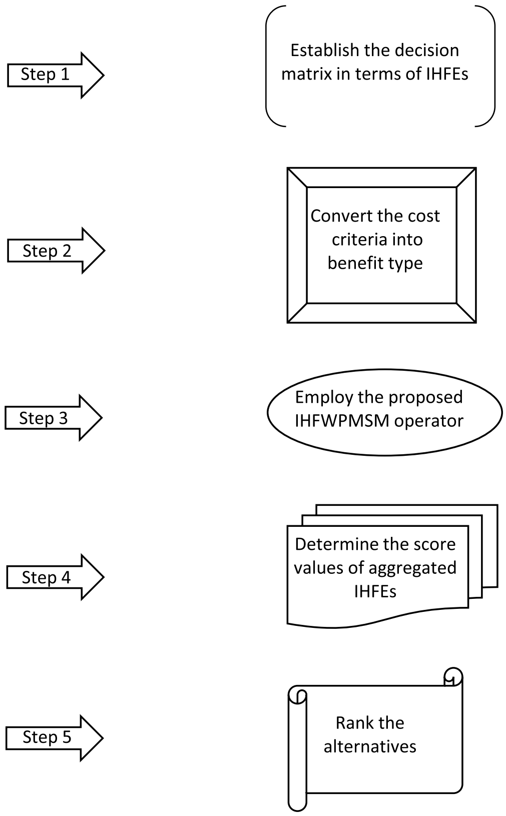


FIGURE 1. Flowchart of the proposed method.

we convert the cost criteria into the benefit criteria. employing the next normality method to make sure all requirements are met.

$$\mathfrak{J}'_{ig} = \begin{cases} \mathfrak{J}_{ig}, & S_g \text{ is a befit criterion,} \\ \left(\mathfrak{J}_{ig}\right)^c, & S_g \text{ is cost criteria.} \end{cases} \tag{25}$$

Step 3: Aggregation: The presented IHFWPMSM method is used to obtain the aggregated assessment score of every alternative $R_i (i = 1, 2, \ldots, m)$ as illustrated below:

where t denotes the number of categories, ρ is a parameter, $\rho = 1, 2, \ldots, o_\Upsilon$, o_Υ denotes the number of criteria in category t_Υ, $\left(k_1, k_2, \ldots, k_\rho\right)$ includes all the ρ-tuples of $(1, 2, \ldots, o_\Upsilon)$, $C^\rho_{o_\Upsilon}$ expresses the binomial coefficient, whose expression is $C^\rho_{o_\Upsilon} = \frac{o_\Upsilon!}{\rho!(o_\Upsilon - \rho)!}$ and $\mathfrak{w}_g \geq 0$ is the weight of the criteria S_g (g=1,2,…,n) and $\mathfrak{J}'_{ig}$ normalized value of $\mathfrak{J}_{ig}$ (g=1,2,…,n) which we take from Step 2.

Step 4: Score values: From Eq. (2), the score value of the concluding IHFEs $\mathfrak{J}_{ig}$ (g=1,2,…,n) are found.

Step 5: Ordering of alternatives: In this final stage, the options are sorted as per their score values, and the matching best solution is chosen.

The flowchart of the propound approach is depicted in Fig. 1.

V. ILLUSTRATIVE EXAMPLE

The practical prime example of choosing a good location for establishing a new shoe company is described in this part, followed by an essential for optimizing.

A. EXAMPLE

Suppose a shoe company intends to introduce a new brand within a specific city. In that case, it is recommended that the company identifies four distinct locations that could serve as potential alternatives, denoted as $R_i(i = 1, 2, \ldots, 4)$. The company decides to evaluate the choices based on the following four aspects, all of which are present, taking into account the company's strategic advantages: the price of an item S_1, location of the building S_2, people living standard S_3, quality of an item S_4. The equivalent weighted vector for the criterion is $\mathfrak{w}$=(0.3, 0.2, 0.3, 0.2). If the criteria S_1, S_2, S_3, and S_4, are separated into two distinct groups, T_1 and T_2, where $T_1 = \{S_1, S_3\}$ and $T_2 = \{S_2, S_4\}$, then any two criteria within each category are related to one another, with $l_1 = 2$ and $l_2 = 2$. The authorized specialists use the four aforementioned criteria and their accompanying weightings to assess four options regarding IHFEs. The DM's assessment information is listed in Table 1.

The following lists the stages that make up the proposed method.

Step 1: The provided IHF assessment information is given in Table 1.

Step 2: According to Eq. (25), the original data is normalized and is shown in Table 2.

Step 3: Now in Eq. (26), if we let $\rho = 2$ then the aggregated values of each alternative $R_i(i = 1, 2, \ldots, m)$ can be obtain as:

Step 4: Using Eq. (4), the score value of each alternatives $R_i(i = 1, 2, \ldots 4)$ can be fined as presented below: $S(R_1) = -0.6086$, $S(R_2) = -0.6448$, $S(R_3) = -0.7007$, $S(R_4) = -0.5405$.

Step 5: According to the derived values, the final ranking can be determined $R_4 > R_1 > R_2 > R_3$. Thus, the best place for launching a shoe company is R_4.

The derived ranking results are shown graphically in Fig. 2.

VI. COMPARATIVE STUDY

In the following section, comparison analysis with other existent aggregation operators including hesitant fuzzy weighted partition Maclaurin symmetric mean (HFWPMSM) operator [54], Einstein dual hesitant fuzzy weighted averaging (EDHFWA) operator [55], Einstein dual hesitant fuzzy weighted geometric (EDHFWG) operator [55], and Frank hybrid weighted arithmatic average (FHWAA) [56] operator are undertaken to illustrate the benefits of the presented method. The final results obtained from employing all of these aggregation operators to the preceding example are displayed in Table 3 and Fig. 3.

According to the results presented in Table 3, the approach provided in the present article and the approaches established on EDHFWA [55] and FHWAA [56] operators yield the

TABLE 1. IHF evaluation matrix.

	S_1	S_2	S_3	S_4
R_1	$\{\{0.4, 0.6\}, \{0.1, 0.3\}\}$	$\{\{0.7, 0.4, 0.3\}, \{0.1, 0.2\}\}$	$\{\{0.2, 0.3\}, \{0.5\}\}$	$\{\{0.2, 0.7\}, \{0.1, 0.3\}\}$
R_2	$\{\{0.3, 0.4, 0.5\}, \{0.1, 0.2\}\}$	$\{\{0.4, 0.6\}, \{0.2, 0.3\}\}$	$\{\{0.1, 0.4\}, \{0.2, 0.6\}\}$	$\{\{0.3, 0.6\}, \{0.2, 0.4\}\}$
R_3	$\{\{0.5, 0.6\}, \{0.1, 0.2, 0.3\}\}$	$\{\{0.1, 0.2\}, \{0.4, 0.5, 0.6\}\}$	$\{\{0.4\}, \{0.3, 0.6\}\}$	$\{\{0.5, 0.6, 0.7\}, \{0.1, 0.2\}\}$
R_4	$\{\{0.3\}, \{0.4, 0.7\}\}$	$\{\{0.3, 0.4\}, \{0.2, 0.5\}\}$	$\{\{0.1, 0.3\}, \{0.2, 0.6, 0.5\}\}$	$\{\{0.6, 0.8\}, \{0.1, 0.2\}\}$

identical ranking of the four alternatives. This demonstrates the reliability of the operator designed in the present research. The list of choices obtained from HFWPMSM [54] and EDHFWG [55] operators is $R_4 > R_2 > R_1 > R_3$, where the positions of R_2 and R_1 varied from those obtained by the suggested approach, but R_4 is still the most suitable option.

$$IHFWPMSM^{(\rho)}(\mathfrak{I}_1, \mathfrak{I}_2, \ldots, \mathfrak{I}_n) = \bigcup_{u_{il}\in\mathfrak{I}_{il}, v_{ij}\in\mathfrak{I}_{il}} \left\{ \begin{array}{c} 1 - \left(\prod_{\Upsilon=1}^{t} \left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<o_\Upsilon} \left(1 - \prod_{l=1}^{\rho} \left(1 - (1 - u_{il})^{\mathfrak{w}_{i_l}} \right) \right) \right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}} \right)^{\frac{1}{\rho}} \right) \right)^{\frac{1}{t}}, \\ \left(\prod_{\Upsilon=1}^{t} \left(1 - \left(1 - \left(\prod_{1<i_1<\ldots<o_\Upsilon} \left(1 - \prod_{l=1}^{\rho} \left(1 - v_{il}^{\mathfrak{w}_{i_l}} \right) \right) \right)^{\frac{1}{C_{o_\Upsilon}^{\rho}}} \right)^{\frac{1}{\rho}} \right) \right)^{\frac{1}{t}} \end{array} \right\}. \tag{26}$$

$$R_1 = \left\{ \begin{array}{c} \left\{ \begin{array}{c} 0.0657, 0.0708, 0.0865, 0.0973, 0.0992, 0.1023, \\ 0.1110, 0.1176, 0.1297, 0.1411, 0.1480, 0.1769 \end{array} \right\}, \\ \left\{ \begin{array}{c} 0.6629, 0.7077, 0.6890, 0.7263, 0.6787, \\ 0.7245, 0.7053, 0.7436, 0.6955, 0.7425, \\ 0.7228, 0.7620, 0.7070, 0.7548, 0.7348, \\ 0.7747 \end{array} \right\} \end{array} \right\}$$

$$R_2 = \left\{ \begin{array}{c} \left\{ \begin{array}{c} 0.0635, 0.0871, 0.0767, 0.1083, 0.0841, 0.1072, \\ 0.0969, 0.1278, 0.0733, 0.0967, 0.0863, 0.1176, \\ 0.1036, 0.1262, 0.1161, 0.1464 \end{array} \right\}, \\ \left\{ \begin{array}{c} 0.7219, 0.7515, 0.7379, 0.7635, 0.7275, 0.7572, \\ 0.7436, 0.7694, 0.6883, 0.7165, 0.7036, 0.7280, 0.7420, \\ 0.7725, 0.7585, 0.7848, 0.7092, 0.7383, 0.7250, 0.7501, \\ 0.7556, 0.7865, 0.7723, 0.7991 \end{array} \right\} \end{array} \right\}$$

$$R_3 = \left\{ \begin{array}{c} \left\{ \begin{array}{c} 0.0541, 0.0577, 0.0615, 0.0657, 0.0709, 0.0766, 0.0694, 0.0729, \\ 0.0767, 0.0808, 0.0859, 0.0915, 0.0667, 0.0702, 0.0740, 0.0781, \\ 0.0833, 0.0889, 0.0891, 0.0926, 0.0963, 0.1003, 0.1053, 0.1108, \\ 0.0771, 0.0806, 0.0844, 0.0884, 0.0935, 0.0991, 0.1056, 0.1090, \\ 0.1127, 0.1166, 0.1215, 0.1269 \end{array} \right\}, \\ \left\{ \begin{array}{c} 0.7694, 0.7865, 0.7845, 0.7994, 0.7991, 0.8117, \\ 0.7827, 0.8002, 0.7982, 0.8133, 0.8130, 0.8258 \end{array} \right\} \end{array} \right\}$$

$$R_4 = \left\{ \begin{array}{c} \left\{ \begin{array}{c} 0.1017, 0.1171, 0.1119, 0.1304, 0.1358, 0.1506, 0.1456, 0.1634, \\ 0.1470, 0.1616, 0.1566, 0.1742, 0.1239, 0.1390, 0.1339, 0.1519, \\ 0.1756, 0.1897, 0.1850, 0.2019, 0.1927, 0.2065, 0.2019, 0.2185 \end{array} \right\}, \\ \{0.6454, 0.6656, 0.6912, 0.7042, 0.6891, 0.7106, 0.7379, 0.7519\} \end{array} \right\}$$

TABLE 2. Normalized IHF decision matrix.

	S_1	S_2	S_3	S_4
R_1	$\{\{0.1, 0.3\}, \{0.4, 0.6\}\}$	$\{\{0.7, 0.4, 0.3\}, \{0.1, 0.2\}\}$	$\{\{0.5\}, \{0.2, 0.3\}\}$	$\{\{0.2, 0.7\}, \{0.1, 0.3\}\}$
R_2	$\{\{0.1, 0.2\}, \{0.3, 0.4, 0.5\}\}$	$\{\{0.4, 0.6\}, \{0.2, 0.3\}\}$	$\{\{0.2, 0.6\}, \{0.1, 0.4\}\}$	$\{\{0.3, 0.6\}, \{0.2, 0.4\}\}$
R_3	$\{\{0.1, 0.2, 0.3\}, \{0.5, 0.6\}\}$	$\{\{0.1, 0.2\}, \{0.4, 0.5, 0.6\}\}$	$\{\{0.3, 0.6\}, \{0.4\}\}$	$\{\{0.5, 0.6, 0.7\}, \{0.1, 0.2\}\}$
R_4	$\{\{0.4, 0.7\}, \{0.3\}\}$	$\{\{0.3, 0.4\}, \{0.2, 0.5\}\}$	$\{\{0.2, 0.6, 0.5\}, \{0.1, 0.3\}\}$	$\{\{0.6, 0.8\}, \{0.1, 0.2\}\}$

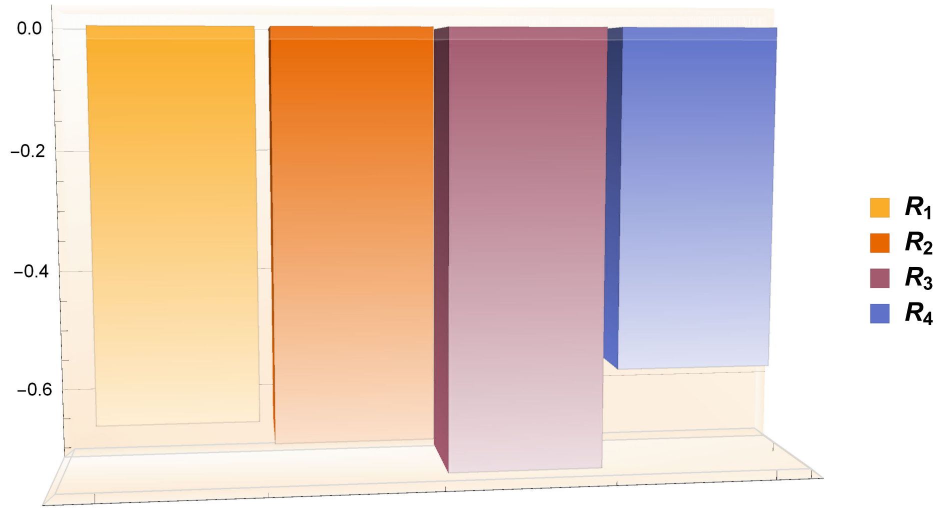


FIGURE 2. Ranking results obtained via proposed aggregation operator.

TABLE 3. Comparison with the existing AOs.

Aggregation operator	$S(R_1)$	$S(R_2)$	$S(R_3)$	$S(R_4)$	Ranking
Proposed IHFWPMSM	-0.6086	-0.6448	-0.7007	-0.5405	$R_4 > R_1 > R_2 > R_3$
HFWPMSM [54]	0.7290	0.7720	0.6449	0.8017	$R_4 > R_2 > R_1 > R_3$
EDHFWA [55]	0.1029	0.08639	-0.01965	0.2981	$R_4 > R_1 > R_2 > R_3$
EDHFWG [55]	-0.009162	0.003004	-0.1289	0.2312	$R_4 > R_2 > R_1 > R_3$
FHWAA [56]	0.1142	0.09395	-0.007500	0.3032	$R_4 > R_1 > R_2 > R_3$

TABLE 4. Characteristic comparison of different AOs.

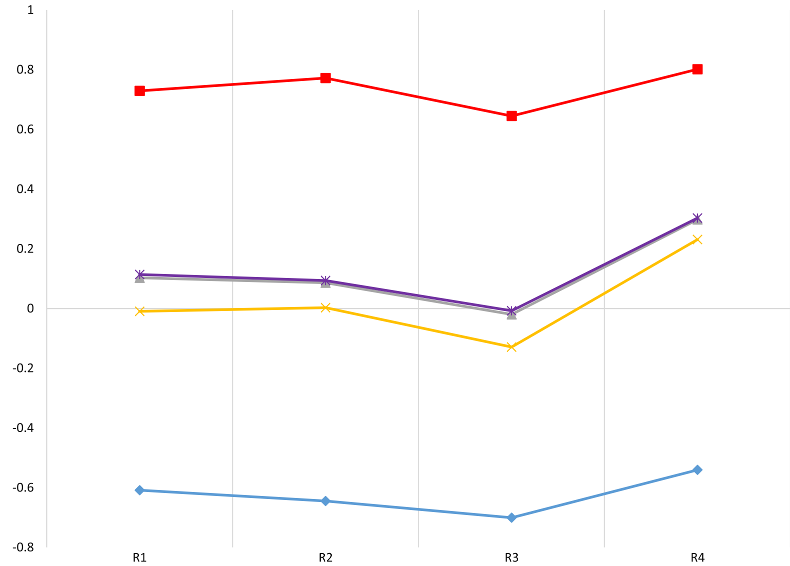
Aggregation operators	Evaluation criteria			
	NMS function	Correlation between two criteria	Correlation between multiple criteria	Partition of the input arguments
Proposed IHFWPMSM	✔	✔	✔	✔
HFWPMSM [54]	✘	✔	✔	✔
EDHFWA [55]	✔	✘	✘	✘
EDHFWG [55]	✔	✘	✘	✘
FHWAA [56]	✘	✘	✘	✘

According to the conducted analysis, the propound operators have the following merits over the existing ones:

i). Our devised MCDM method and Ali's MCDM approach [54] can capture interrelationships between criteria and can divide criteria into several groups, whereas the existing approaches [55], [56] does not divide criteria into groups. In addition, these methods do not account for correlations between criteria.

ii). Although Ali's [54] approach is capable of dividing the criteria into partitions, it only considers the membership portion and ignores the non-membership portion, resulting in a significant loss of information. The developed operators, on the other hand, are based on a dual hesitant fuzzy context and are able to capture all conceivable information.

iii). The proposed operators have two parameters t and ρ, which gives DMs greater flexibility and robustness in selecting the appropriate ρ based on their risk preferences. Additionally, several existing operators are special cases of the developed operator for specific parameter values. Thus,



	R1	R2	R3	R4
Proposed DHFWPMSM	-0.6086	-0.6448	-0.7007	-0.5405
HFWPMSM	0.729	0.772	0.6449	0.8017
EDHFWA	0.1029	0.08639	-0.01965	0.2981
EDHFWG	-0.009162	0.003004	-0.1289	0.2312
FHWAA	0.1142	0.09395	-0.0075	0.3032

FIGURE 3. **Ranking results obtained via different aggregation operators.**

the developed operators are more comprehensive and more capable of efficiently tackling MCDM problems.

iv). The scores $S(R_i)(i = 1, 2, \ldots, 4)$ presented in Table 3 indicate that the proposed operator has greater ability for discrimination among alternatives than the prevailing operators [55], [56].

In addition, Table 4 compares the characteristics of our designed AO with those of the existing ones.

A. VALIDITY TEST

The following test requirements, supported by Wang and Triantaphyllou [57], are met to show the feasibility of our technique in an evolving working environment.

Test criterion 1: "If we substitute the estimation ratings of the non-optimal choice with a worse choice, then the most favorable choice ought to remain the same, assuming the relative weighted condition stays constant."

Test criterion 2: "Procedure ought to be transitive in nature."

Test criterion 3: "The overall ordering of the possibilities ought to be similar to the ordering of the un-decomposed one when a given problem has been split into fewer alternatives and a comparable MCDM methodology has been used."

1) VALIDITY CHECK WITH CRITERION 1

Given that the suggested strategy's ranking is $R_4 > R_1 > R_2 > R_3$, the worst alternative R_3' is used in place of the suboptimal option R_3 to assess if our approach is similar under evaluate Criteria 1 And the four consideration criteria are written as follows for the rating value of

$$R_3' = \begin{Bmatrix} (\{0.1, 0.3\}, \{0.5, 0.6\}), \\ (\{0.1\}, \{0.4, 0.5, 0.6\}), \\ (\{0.3, 0.6\}, \{0.2, 0.4\}), \\ (\{0.5, 0.6\}, \{0.1, 0.2\}) \end{Bmatrix},$$

These findings led to the use of the suggested strategy, which resulted in the final score values of the options being as follows: $S(R_3') = -0.7071$ $S(R_1) = -0.6086$, $S(R_2) = -0.6448$, $S(R_4) = -0.5405$. As a result, the ranking order is $R_4 > R_1 > R_2 > R_3'$, with the best alternative remaining the same as the suggested strategy. Consequently, our strategy produces consistent outcomes in relation to test criterion 1.

2) VALIDITY CHECK WITH CRITERIA 2 AND 3

The splintered MCDM sub-problems are designated as $A_1 = \{R_2, R_3, R_4\}$, $A_2 = \{R_1, R_3, R_2\}$ and $A_3 = \{R_2, R_4, R_1\}$ the score values of each alternatives are $S(R_1) = -0.6086$, $S(R_2) = -0.6448$,$S(R_3) = -0.7007$, $S(R_4) = -0.5405$ to determine validity in accordance with criteria 2 and 3. Then, using the stated technique, their ranking is determined as follows: As according to the subset A_1, A_2 and A_3 have the ranking order $R_4 > R_2 > R_3$, $R_1 > R_2 > R_3$ and $R_4 > R_1 > R_2$, respectively. When all of them are combined, the total ranking is $R_4 > R_1 > R_2 > R_3$, which is identical to the outcomes of the initial MADM issue that was suggested. Hence it satisfies the transitive property. The proposed strategy therefore, proves effective when tested against test criteria 2 and 3.

VII. CONCLUSION

Since IHFS is one of the best tools for coping with complex and uncertain information, we studied an MCDM strategy within an IHF setting in the present research. We originated IHFPMSM and IHFWPMSM operators to combine IHF data precisely. Several characteristics of the framed operators are examined in depth, and it is demonstrated that some present operators are special instances of the provided operators. The presented operators consider not only the connection between criteria but also their partitioned relation. Moreover, we devised an MCDM strategy relying on the introduced IHF operators. Using the aforementioned operators, we can divide the criteria into various groups that measure the relationship among various criteria that fall within the same group. In addition, a numerical illustration of a shoe company problem is provided to demonstrate the Validity and applicability of the proposed method. The devised methodology was then compared to existing methods. From the comparative analysis, we determined that the proposed method is more accurate, appropriate for handling uncertain data, and capable of applying multiple partitions among criteria.

In the future, we will use the developed method for other real-world decision challenges, such as two-sided matching problems [58], medical diagnostics [59], and shipping industry 4.0 domains [60], etc.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

DATA SET

No data sets have been used in this article.

FUNDING INFORMATION

Not applicable.

AVAILABILITY OF DATA AND MATERIALS

All data generated or analysed during this study are included in this published article.

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