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RESEARCH ARTICLE

Finite-Time Extended Dissipativity Analysis for Generalized Neural Networks With Discrete and Distributed Time-Varying Delays

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ABSTRACT This paper investigated the finite-time extended dissipativity for generalized neural networks with discrete and distributed time-varying delays via the improved Lyapunov-Krasovskii functional (LKF). We constructed an appropriate LKF by employing more neural network information and consisting of quadratic functions. By combining the proposed LKF, Jensen's integral inequality, orthogonal polynomials-based integral inequality, and extended Wirtinger's integral inequality, new delay-dependent conditions are achieved in the form of linear matrix inequalities (LMIs), which can be verified via MATLAB's LMI toolbox. In addition, we concentrate on the extended dissipative analysis problem, which is a unified formulation of $\mathcal{L}_2 - \mathcal{L}_\infty$, H_∞, passivity, and dissipative performance. This paper is less conservative delay bound than some recently published literature by stability criteria. In addition, we presented seven numerical examples to illustrate the effectiveness of the obtained results.

INDEX TERMS Extended dissipative, neural networks, time-varying delays, finite-time bounded, Lyapunov-Krasovskii functional.

I. INTRODUCTION

In the last two decades, neural networks (NNs) have been extensively investigated because of their successful applications in many practical systems, such as pattern recognition, signal processing, associative memories, and other engineering and scientific areas [1], [2], [3], [4], [5]. In the process of investigating neural networks, time delays are unavoidable as a result of the dynamical behaviors of networks generating instability, oscillation, divergence, the inherent communication time between neurons, and the finite switching speed of amplifiers [6], [7], [8]. Therefore, the stability of neural networks with a time-varying delay has received considerable attention from many researchers [9], [10], [11]. The stability criteria developed for DNNs can be divided into two categories: delay-independent ones and delay-dependent ones. Since the delay-dependent conditions, which include the size information of time-delayed are usually less conservative than delay-independent ones, especially for neural networks with small delays, more attentions have been paid to the delay-dependent stability analysis of time delay neural network [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11].

Recent studies have examined the dynamical behaviors of static neural networks (SNNs) [12] or local field neural networks (LFNNs) [13] separately due to differences in a neuron or local field state. In addition, these two models are not equivalent, but they can be combined into a more concise model by making reasonable assumptions. Thus, Zhang and Han [14] created the first unified system model, generalized neural networks (GNNs), which incorporated both LFNNs and SNNs. Furthermore, in recent years, there has been a

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heightened interest in analyzing the stability and performance of GNNs with time delay [15], [16], [17].

The neural network stability problem is to find a less conservative condition that guarantees the system's stability. The Lyapunov-Krasovskii functional (LKF) and many inequality techniques have been widely used to reduce the conservatism of stability criteria [4], [5]. For example, Jensen's integral inequality was presented to determine the new stability conditions for the NNs [5]. To obtain the conditions with the decline of conservatism, Wirtinger's integral inequality and reciprocally convex optimization are presented [18]. The free-weighting-based inequality has been shown as a powerful tool for analyzing the stability problem of NNs [11]. The orthogonal polynomials-based inequality was first introduced as an effective tool for analyzing the stability problem of NNs [19]. In addition, to derive better conditions, various types of LKF have been adopted, for instance, multiple integrals-based LKF [20], activation function-based LKF [21], and so on.

Recently, the performances of a neural network, which are usually characterized by an input-output relationship, played an important role in various science and engineering applications, such as H_∞ control problem, passivity, and passification problems, $\mathcal{L}_2 - \mathcal{L}_\infty$ performance, and dissipativity performance [22]. Up to now, a lot of researchers have paid increasing attention to the dissipativity analysis since it does not only linked with the H_∞ and passivity performance but also recommends a good comfortable control structure in many engineering applications, such as electrical networks, nonlinear control, power converters, and chemical process control [23]. Recently, the (Q, S, R)-dissipativity concept has been proposed in [9] and [22]. However, the $\mathcal{L}_2 - \mathcal{L}_\infty$ performance is not contained in the (Q, S, R)-dissipativity. In order to overcome this problem, Saravanakumar et al. [24] introduced a more general performance called extended dissipativity which can integrate several well-known performance indices such as passivity performance, (Q, S, R)-dissipativity performance, H_∞ performance, and $\mathcal{L}_2 - \mathcal{L}_\infty$ performance in a unified framework by setting the corresponding values of weighting matrices [9], [24], [25], [26]. More recently, the issue of the extended dissipative analysis has been applied to some NNs [9], [22], [23], [24].

In the previous decades, the existing literature has typically been concerned with asymptotic stability, which is defined over an infinite-time interval. Nonetheless, there is a bound for system trajectories over a fixed short time interval in some practical applications, such as rockets and airplanes, rather than asymptotic stability over an infinite-time interval. Our main objective lies in the behavior of dynamic systems over a given finite-time interval. More clearly, the state of dynamic systems does not exceed a special threshold of its state space for a given a priori bound of its initial state in a short time interval, which is called finite-time stability (FTS). In 1961, Dorato [27] first introduced the concept of FTS to the control framework. Subsequent work by Amato et al. [28] extends FTS to finite-time bound (FTB) by taking external disturbances into account. The FTS and FTB for NNs with time-varying delays have received a lot of attention [29], [30], [31], [32], [33].

In this paper, Jensen's integral inequality, orthogonal polynomials-based integral inequality, and extended Wirtinger's integral inequality are used to study finite-time extended dissipativity for generalized neural networks with mixed discrete and distributed time-varying delay problems. In addition, numerical examples are provided to demonstrate the efficiency of the theorems. Finally, numerical examples are presented to demonstrate the feasibility and effectiveness of the theorem. In addition, the major contributions and highlights of this paper are summarized in the following key points:

- We investigate finite-time extended dissipativity for generalized neural network problems with distributed and discrete time-varying delays.
- An enhanced LKF is constructed by optimizing the information of the time delay neural network as follows: firstly, the time-varying delay and its maximum are all employed, together with the activation function, the state, and its derivative. Secondly, the LKF includes more cross terms among the state, the integral of the state, the integral of the derivative of the state, terms among the state, the delayed state, the activation function, and the integral of the activation function.
- We estimate the bound of the time derivative of LKF using Jensen's integral inequality, an extended Wirtinger's integral inequality, and orthogonal polynomials-based integral inequality, which results in less conservatism than the other references, as demonstrated by numerical examples.

The framework of this paper is structured as follows: In Section II, the system model, definitions, assumptions, and lemmas are described. Section III presents the main results, which include finite-time stability, finite-time boundedness, and finite-time extended dissipativity. Section IV provides seven numerical examples to demonstrate the effectiveness of the obtained criteria. Finally, in Section V, we present the conclusion of our work.

Notations: This paper contains the following notations, $\mathbb{R}^n$ denotes the $n-$ dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ real matrices. $\mathbb{S}_n$, $\mathbb{S}_n^+$ are the set of symmetric and positive definite $n \times n$ real matrices, respectively. P^T and P^{-1} indicate the matrix P transport and matrix P inverse. The symmetric matrix P refers to $P = P^T$. The matrix P is positive definite that the symmetric matrix $P > 0$. $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimum and maximum eigenvalues for real symmetric matrix P, respectively. diag$\{\ldots\}$ denotes the block diagonal matrix. Sym$\{P\} = P + P^T$. $\star$ represents the symmetric forms in a symmetric matrix.

II. PRELIMINARIES

Consider the following generalized neural networks with discrete and distributed time-varying delays:

$$\dot{z}(t) = -A_0 z(t) + A_1 f(Wz(t)) + A_2 g(Wz(t-\tau(t)))$$

$$+A_3\int_{t-\gamma(t)}^{t} h(Wz(s))ds + A_4\omega(t),$$
$$y(t) = B_0 z(t),$$
$$z(t) = \phi(t), \quad t\in[-\tau,0], \tag{1}$$

where $z(t) = [z_1(t), z_2(t), \ldots, z_n(t)]^T \in \mathbb{R}^n$ is the neuron state vector; $A_0 = \text{diag}\{a_1, a_2, \ldots, a_n\}$ with $a_i > 0$ is a positive diagonal matrix; A_1, A_2, and A_3 are the connection weight matrices; A_4 is the connection disturbance; $f(Wz(\cdot)) = [f_1(W_1z(\cdot)), f_2(W_2z(\cdot)), \ldots, f_n(W_2z(\cdot))]^T$, $g(Wz(\cdot)) = [g_1(W_1z(\cdot)), g_2(W_2z(\cdot)), \ldots, g_n(W_2z(\cdot))]^T$ and $h(Wz(\cdot)) = [h_1(W_1z(\cdot)), h_2(W_2z(\cdot)), \ldots, h_n(W_2z(\cdot))]^T$ are the neuron activation functions with W_i denoting the ith row of W; $\omega(t)\in\mathbb{R}^n$ is the external disturbance vector that belongs to the class $\mathcal{L}_2[0,\infty)$; $y(t)$ is the output vector of the system; B_0 is known real constant matrices of suitable dimension; $\phi(t)$ is the initial function; The variable $\tau(t)$ and $\gamma(t)$ represent the discrete and distributed time-varying delays, respectively.

$\tau(t)$ is an discrete time-varying differentiable function satisfying

$$0 \le \tau(t) \le \tau, \; \dot{\tau}(t) \le \tau_d, \tag{2}$$

$\gamma(t)$ is an distributed time-varying satisfying

$$0 \le \gamma(t) \le \gamma_d. \tag{3}$$

Assumption 1 ([9]): The activation function $f_i(W_iz(\cdot))(i = 1, 2, \ldots, n)$ is continuous and bounded satisfying the following inequality

$$F_i^- \le \frac{f_i(u)-f_i(v)}{u-v} \le F_i^+,$$

$u, v\in\mathbb{R}$, $u\neq v$ where $f_i(0) = 0$, F_i^- and F_i^+ are known real scalars.

For the convenience of presentation, we denote

$$F_m = \text{diag}\{\frac{F_1^-+F_1^+}{2}, \frac{F_2^-+F_2^+}{2}, \ldots, \frac{F_n^-+F_n^+}{2}\},$$
$$F_p = \text{diag}\left\{F_1^+, F_2^+, \ldots, F_n^+\right\},$$
$$G_m = \text{diag}\left\{\frac{G_1^-+G_1^+}{2}, \frac{G_2^-+G_2^+}{2}, \ldots, \frac{G_n^-+G_n^+}{2}\right\},$$
$$G_p = \text{diag}\left\{G_1^+, G_2^+, \ldots, G_n^+\right\},$$
$$H_m = \text{diag}\left\{\frac{H_1^-+H_1^+}{2}, \frac{H_2^-+H_2^+}{2}, \ldots, \frac{H_n^-+H_n^+}{2}\right\},$$
$$H_p = \text{diag}\left\{H_1^+, H_2^+, \ldots, H_n^+\right\}.$$

Remark 1: The neuron activation functions may be non-differentiable, non-monotonic, and unbounded by the time-varying delay. The variables F_i^-, F_i^+, G_i^-, G_i^+, H_i^-, and H_i^+ can all be zero, positive, or negative. Notably, the assumption used in this study is weaker and more general than the usual Lipschitz condition, $|f(u) - f(v)| \le F|u - v|$. Therefore, our stability criteria with Assumption 1 are less conservative compared to the usual Lipschitz condition.

Assumption 2 ([26]): For any positive constant ω_f and time constant T_f, the external disturbance satisfies

$$\int_0^{T_f} \omega^T(t)\omega(t)dt \le \omega_f.$$

Assumption 3 ([26]): For any time constant T_f, the state vector of time-varying $z(t)$ satisfies

$$\int_0^{T_f} z^T(t)z(t)dt \le d,$$

where d denotes a sufficiently large fixed constant.

Assumption 4 ([26]): For any matrices Ω_1, Ω_2, Ω_3 and Ω_4 satisfy the following conditions:

1) $\Omega_1 = \Omega_1^T \le 0$,
2) $\Omega_3 = \Omega_3^T > 0$,
3) $\Omega_4 = \Omega_4^T \ge 0$,
4) $(\|\Omega_1\| + \|\Omega_2\|)\Omega_4 = 0$.

Definition 1 ([26]): For any matrices Ω_1, Ω_2, Ω_3 and Ω_4 satisfying Assumption 4, system (1) is said to be extended dissipativity performance if the following inequality holds for any $T_f > 0$ and for all $\omega(t)\in\mathcal{L}_2[0,\infty)$:

$$\int_0^{T_f} J(t)dt - \sup_{0\le t\le T_f} y^T(t)\Omega_4 y(t) \ge 0, \tag{4}$$

where $J(t) = y^T(t)\Omega_1y(t) + 2y^T(t)\Omega_2\omega(t) + \omega^T(t)\Omega_3\omega(t)$.

Remark 2: The concept of extended dissipativity performance proposed in Definition 1 contains some well-known performances as special cases by adjusting the weighting matrices Ω_1, Ω_2, Ω_3, Ω_4 and given constant matrices $Q\in\mathbb{R}^{n\times n}$, $S\in\mathbb{R}^{n\times n}$, and $R\in\mathbb{R}^{n\times n}$ with Q and R symmetric as follows:

- *If $\Omega_1 = -I$, $\Omega_2 = 0$, $\Omega_3 = \gamma^2 I$ and $\Omega_4 = 0$, then Definition 1 refers to the H_∞ performance;*
- *If $\Omega_1 = 0$, $\Omega_2 = 0$, $\Omega_3 = \gamma^2 I$ and $\Omega_4 = I$, then Definition 1 refers to the $\mathcal{L}_2 - \mathcal{L}_\infty$ performance;*
- *If $\Omega_1 = 0$, $\Omega_2 = I$, $\Omega_3 = \gamma I$ and $\Omega_4 = 0$, then Definition 1 refers to the passivity performance;*
- *If $\Omega_1 = Q$, $\Omega_2 = S$, $\Omega_3 = R - \beta I$ and $\Omega_4 = 0$, then Definition 1 refers to the (Q, S, R)-dissipativity performance.*

*Definition 2 (**Finite-Time Bounded** [10]): The system (1) is finite-time bounded with reference to $(c_1, c_2, T_f, V, \omega_f)$ with time constant $T_f > 0$, a matrix $V > 0$, and numbers $c_2 > c_1 > 0$, $\omega_f > 0$, if the following inequality holds:*

$$\sup_{-\tau\le s\le 0}\{z^T(s)Vz(s), \dot{z}^T(s)V\dot{z}(s)\} \le c_1$$
$$\Rightarrow z^T(t)Vz(t) < c_2, \forall t\in[0, T_f].$$

*Definition 3 (**Finite-Time Stable** [10]):* For a given time $T_f > 0$, numbers $c_2 > c_1 > 0$, and a matrix $V > 0$, the system (1) with $\omega(t) = 0$ is finite-time stable with respect to (c_1, c_2, T_f, V), if the following inequality holds:

$$\sup_{-\tau\le s\le 0}\{z^T(s)Vz(s), \dot{z}^T(s)V\dot{z}(s)\} \le c_1$$
$$\Rightarrow z^T(t)Vz(t) < c_2, \forall t\in[0, T_f].$$

Lemma 1 ([5]): For any matrix $R_1 \in \mathbb{S}^+_{n1}$, $R_2 \in \mathbb{S}^+_{n2}$, $\alpha \in (0,1)$ and any matrix $Z \in \mathbb{R}^{(n1+n2)\times(n1+n2)}$ the following inequality holds:

$$\begin{bmatrix} \frac{1}{\alpha}R_1 & 0 \\ 0 & \frac{1}{1-\alpha}R_2 \end{bmatrix} \geq Z + Z^T - Z\begin{bmatrix} \alpha R_1^{-1} & 0 \\ 0 & (1-\alpha)R_2^{-1} \end{bmatrix} Z^T.$$

*Lemma 2 (**Jensen's Integral Inequality** [5]): For a given matrix $R > 0$ scalar $\alpha_1 < \alpha_2$ and vector $z : [\alpha_1, \alpha_2] \to \mathbb{R}^n$ such that the following integrals are well defined, then the inequality holds*:

$$(\alpha_2 - \alpha_1)\int_{\alpha_1}^{\alpha_2} z^T(s)Rz(s)ds \geq \int_{\alpha_1}^{\alpha_2} z^T(s)dsR\int_{\alpha_1}^{\alpha_2} z(s)ds.$$

*Lemma 3 (**Orthogonal Polynomials-Based Integral Inequality** [34]:) Let $z(s)$ be a differentiable function $z : [\alpha_1, \alpha_2] \to \mathbb{R}^n$ for any matrices $R \in \mathbb{S}^+_n$, $M_i \in \mathbb{R}^{(k\times n)}(i = 1, 2, 3)$ and any vector $\xi \in \mathbb{R}^k$, the following inequality holds:*

$$-\int_{\alpha_1}^{\alpha_2} \dot{z}^T(s)R\dot{z}(s)ds \leq \xi^T\Bigg[\sum_{i=1}^{3}\frac{\alpha_2-\alpha_1}{2i-1}M_iR^{-1}M_i^T + \sum_{i=1}^{3}\text{Sym}\{M_iE_i\}\Bigg]\xi,$$

where

$$\begin{aligned} E_1\xi &= z(\alpha_2) - z(\alpha_1), \\ E_2\xi &= z(\alpha_2) + z(\alpha_1) - \frac{2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} z(s)ds, \\ E_3\xi &= z(\alpha_2) - z(\alpha_1) + \frac{6}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} z(s)ds \\ &\quad - \frac{12}{(\alpha_2-\alpha_1)^2}\int_{\alpha_1}^{\alpha_2}\int_{s}^{\alpha_2} z(u)duds. \end{aligned}$$

Lemma 4 ([34]): Let $z(s)$ be a differentiable function $z : [\alpha_1, \alpha_2] \to \mathbb{R}^n$ for any matrices $R \in \mathbb{S}^+_n$, $N_i \in \mathbb{R}^{(k\times n)}(i = 1, 2)$, any vector $\xi \in \mathbb{R}^k$ and all continuous function $z : [\alpha_1, \alpha_2] \to \mathbb{R}^n$, then the following holds:

$$\begin{aligned} -\int_{\alpha_1}^{\alpha_2} z^T(s)Rz(s)ds &\leq \xi^T\Big[(\alpha_2-\alpha_1)\Big(N_1R^{-1}N_1^T + \frac{1}{3}N_2R^{-1}N_2^T\Big) \\ &\quad + \text{Sym}\{N_1F_1 + N_2F_2\}\Big]\xi, \end{aligned}$$

where

$$\begin{aligned} F_1\xi &= \int_{\alpha_1}^{\alpha_2} z(s)ds, \\ F_2\xi &= -\int_{\alpha_1}^{\alpha_2} z(s)ds + \frac{2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\int_{s}^{\alpha_2} z(u)duds. \end{aligned}$$

*Lemma 5: (**Extended Wirtinger's Integral Inequality** [35]): For any matrix $R \in \mathbb{S}^+_n$, and any continuously differentiable function $z : [\alpha_1, \alpha_2] \to \mathbb{R}^n$, the following inequality holds*:

$$\int_{\alpha_1}^{\alpha_2}\int_{s}^{\alpha_2} \dot{z}^T(u)R\dot{z}(u)du \geq 2\chi_1^TR\chi_1 + 4\chi_2^TR\chi_2 + 6\chi_6^TR\chi_3,$$

where

$$\begin{aligned} \chi_1 &= z(\alpha_2) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} z(s)ds, \\ \chi_2 &= z(\alpha_2) + \frac{2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} z(s)ds \\ &\quad - \frac{6}{(\alpha_2-\alpha_1)^2}\int_{\alpha_1}^{\alpha_2}\int_{s}^{\alpha_2} z(u)duds, \\ \chi_3 &= z(\alpha_2) - \frac{3}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} z(s)ds \\ &\quad + \frac{24}{(\alpha_2-\alpha_1)^2}\int_{\alpha_1}^{\alpha_2}\int_{s}^{\alpha_2} z(u)duds \\ &\quad - \frac{60}{(\alpha_2-\alpha_1)^3}\int_{\alpha_1}^{\alpha_2}\int_{s}^{\alpha_2}\int_{u}^{\alpha_2} z(v)dvduds. \end{aligned}$$

Lemma 6 ([36]): For given real matrices R_1 and R_2 with appropriate dimensions, they satisfy $2R_1^TR_1 + R_2^TR_2$.

*Lemma 7 (**Schur Complement** [36]): Let R_1, R_2, and R_3 be given constant matrices with appropriate dimensions which satisfy $R_1 = R_1^T$, $R_2 = R_2^T > 0$, then $R_1 + R_3^TR_2^{-1}R_3 < 0$ if and only if*

$$\begin{bmatrix} R_1 & R_3^T \\ R_3 & -R_2 \end{bmatrix} < 0 \; or \begin{bmatrix} -R_2 & R_3 \\ R_3^T & R_1 \end{bmatrix} < 0.$$

Lemma 8 ([34]): For a quadratic function $f(z) = a_2z^2 + a_1z + a_0$ where $a_2, a_1, a_0 \in \mathbb{R}$. if the following inequalities hold

$$(i) f(0) < 0, \quad (ii) f(\tau) < 0, \quad (iii) - \tau^2 a_2 + f(0) < 0$$

then $f(z) < 0$, $\forall z \in [0, \tau]$.

Remark 3: Improved convex inequalities [54] and [55] can be reduced to Lemma 1. It is important to note that Lemma 2 in [57] is a special case of Lemma 1.

Remark 4: Lemma 3 in [53] with $N = 2$ is Lemma 3 in this work, and it can reduce the complexity of parameter calculations for obtaining sufficient conditions, making this work more efficient than other works.

Remark 5: Improved conditions for Lemma 8 have been proposed in Lemma 4 ([57]) with N=1. Lemma 8 in this work provides sufficient conditions and makes this work more efficient than others.

III. MAIN RESULTS

In this section, we will present the sufficient conditions of the main theorems for generalized neural networks with mixed time-varying delays. Firstly, the following notations for vectors and matrices are introduced to simplify the illustration:

$$e_i = \left[0_{n\times(i-1)n}\ I_{n\times n}\ 0_{n\times(21-i)n}\right]\quad (i=1,2,\cdots,21),$$
$$\hat{e}_j = \left[0_{n\times(j-1)n}\ I_{n\times n}\ 0_{n\times(5-j)n}\right]\ (j=1,2,\cdots,5),$$
$$e_s = -A_0e_1 + A_1e_7 + A_2e_{13} + A_3e_{16} + A_4e_{21},$$
$$e_0 = 0_{21n\times n},\quad f_a(s) = f(Wz(s)),$$
$$g_a(s) = g(Wz(s)),\ h_a(s) = h(Wz(s)),\ \tau_f = \frac{1}{\tau},$$
$$D_1 = e_1 - e_2,\quad D_2 = e_1 + e_2 - 2e_5,$$
$$D_3 = e_1 - e_2 + 6e_5 - 12e_{17},$$
$$E_1 = e_2 - e_3,\quad E_2 = e_2 + e_3 - 2e_6,$$
$$E_3 = e_2 - e_3 + 6e_6 - 12e_{18},$$
$$D_4 = \begin{bmatrix} e_1 - e_2 \\ \tau(t)e_5 \\ \tau(t)(e_1 - e_5)\end{bmatrix},\quad D_5 = \begin{bmatrix} e_1 + e_2 - 2e_5 \\ \tau(t)(-e_5 + 2e_{17}) \\ \tau(t)(e_5 - 2e_{17})\end{bmatrix},$$
$$E_4 = \begin{bmatrix} e_2 - e_3 \\ (\tau - \tau(t))e_6 \\ (\tau - \tau(t))(e_1 - e_6)\end{bmatrix},$$
$$E_5 = \begin{bmatrix} e_2 + e_3 - 2e_6 \\ (\tau - \tau(t))(-e_6 + 2e_{18}) \\ (\tau - \tau(t))(e_6 - 2e_{18})\end{bmatrix},$$
$$\varphi_1(t) = \Big[z^T(t)\ z^T(t-\tau)\ \int_{t-\tau}^{t} z^T(s)ds\ \int_{t-\tau}^{t} f_a^T(s)ds\ \int_{t-\tau}^{t}\int_s^t z^T(u)duds\Big]^T,$$
$$\varphi_2(t,s) = \Big[z^T(t)\ z^T(s)\ f_a^T(s)\ \int_s^t \dot{z}^T(u)du\ \int_s^t z^T(u)du\ z^T(t-\tau)\Big]^T,$$
$$\varphi_3(t,s) = \Big[z^T(t)\ z^T(s)\ \dot{z}^T(s)\ f_a^T(s)\ \int_s^t z^T(u)du\ z^T(t-\tau)\Big]^T,$$
$$\varphi_4(t,s) = \Big[\dot{z}^T(s)\ z^T(s)\ \int_s^t \dot{z}^T(u)du\Big]^T,$$
$$\xi_1(t) = \Big[z^T(t)\ z^T(t-\tau(t))\ z^T(t-\tau)\ \dot{z}^T(t-\tau)\Big]^T,$$
$$\xi_2(t) = \Big[\int_{t-\tau(t)}^{t} \frac{z^T(s)}{\tau(t)}ds\ \int_{t-\tau}^{t-\tau(t)} \frac{z^T(s)}{\tau-\tau(t)}\Big]^T,$$
$$\xi_3(t) = \Big[f_a^T(t)\ f_a^T(t-\tau(t))\ f_a^T(t-\tau)\Big]^T,$$
$$\xi_4(t) = \Big[\int_{t-\tau(t)}^{t} f_a^T(s)ds\ \int_{t-\tau}^{t-\tau(t)} f_a^T(s)ds\Big]^T,$$
$$\xi_5(t) = \Big[g_a^T(t)\ g_a^T(t-\tau(t))\ g_a^T(t-\tau)\ h_a^T(t)\Big]^T,$$
$$\xi_6(t) = \Big[\int_{t-\gamma(t)}^{t} h_a^T(s)ds\ \int_{t-\tau(t)}^{t}\int_s^t \frac{z^T(u)}{\tau^2(t)}duds\Big]^T,$$
$$\xi_7(t) = \Big[\int_{t-\tau}^{t-\tau(t)}\int_s^{t-\tau(t)} \frac{z^T(u)}{(\tau-\tau(t))^2}duds\Big]^T,$$
$$\xi_8(t) = \Big[\int_{t-\tau}^{t}\int_s^t z^T(u)duds\Big]^T,$$
$$\xi_9(t) = \Big[\int_{t-\tau}^{t}\int_s^t\int_u^t z^T(v)dvduds\ \ \omega^T(t)\Big]^T,$$
$$\xi(t) = \Big[\xi_1^T(t)\ \xi_2^T(t)\ \xi_3^T(t)\ \xi_4^T(t)\ \xi_5^T(t)\ \xi_6^T(t)\ \xi_7^T(t)\ \xi_8^T(t)\ \xi_9^T(t)\Big]^T,$$
$$\Xi_{[\tau(t)]} = \Phi_{1[\tau(t)]} + \Phi_2 + \Phi_{2[\tau(t)]} + \Phi_3 + \Phi_{3[\tau(t)]} + \Phi_{4[\tau(t)]} + \Phi_5 + \Phi_z + \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 - \tau^2\varrho - \alpha e_1^TP_1e_1 - e_{21}^TM_ae_{21},$$
$$\Phi_{1[\tau(t)]} = \mathrm{Sym}\left\{e_1^TP_1e_s\right\} + \mathrm{Sym}\left\{\begin{bmatrix} e_1 \\ e_3 \\ \tau(t)e_5 + (\tau-\tau(t))e_6 \\ e_{10}+e_{11} \\ e_{19}\end{bmatrix}^T \times P_2\begin{bmatrix} e_s \\ e_4 \\ e_1 - e_3 \\ e_7 - e_9 \\ \tau e_1 - \tau(t)e_5 - (\tau-\tau(t))e_6\end{bmatrix}\right\},$$
$$\Phi_2 = \begin{bmatrix} e_1 \\ e_1 \\ e_7 \\ e_0 \\ e_0 \\ e_3\end{bmatrix}^T Q_1 \begin{bmatrix} e_1 \\ e_1 \\ e_7 \\ e_0 \\ e_0 \\ e_3\end{bmatrix} + \begin{bmatrix} e_1 \\ e_1 \\ e_s \\ e_7 \\ e_0 \\ e_3\end{bmatrix}^T Q_2 \begin{bmatrix} e_1 \\ e_1 \\ e_s \\ e_7 \\ e_0 \\ e_3\end{bmatrix},$$
$$\Phi_{2[\tau(t)]} = \mathrm{Sym}\left\{\begin{bmatrix} \tau(t)e_1 \\ \tau(t)e_5 \\ e_{10} \\ \tau(t)(e_1-e_5) \\ \tau^2(t)e_{17} \\ \tau(t)e_{18}\end{bmatrix}^T Q_1 \begin{bmatrix} e_s \\ e_0 \\ e_0 \\ e_s \\ e_1 \\ e_4\end{bmatrix}\right\} - (1-\mu)\begin{bmatrix} e_1 \\ e_2 \\ e_8 \\ e_1 - e_2 \\ \tau(t)e_5 \\ e_3\end{bmatrix}^T Q_1 \begin{bmatrix} e_1 \\ e_2 \\ e_8 \\ e_1 - e_2 \\ \tau(t)e_5 \\ e_3\end{bmatrix} - \begin{bmatrix} e_1 \\ e_3 \\ e_4 \\ e_9 \\ \tau(t)e_5 + (\tau-\tau(t))e_6 \\ e_3\end{bmatrix}^T \times Q_2 \begin{bmatrix} e_1 \\ e_3 \\ e_4 \\ e_9 \\ \tau(t)e_5 + (\tau-\tau(t))e_6 \\ e_3\end{bmatrix}$$

$$+\,\mathrm{Sym}\left\{\begin{bmatrix}\tau e_1\\ \tau(t)e_5+(\tau-\tau(t))e_6\\ e_1-e_3\\ e_{10}+e_{11}\\ e_{19}\\ \tau e_3\end{bmatrix}^T Q_2\begin{bmatrix}e_s\\ e_0\\ e_0\\ e_0\\ e_1\\ e_4\end{bmatrix}\right\},$$

$$\begin{aligned}\Phi_3 &= \tau e_s^T R_1 e_s + \tau^2 e_7^T R_2 e_7 + e_1^T P_a e_1 - e_2^T P_a e_2\\ &\quad + e_2^T P_b e_2 - e_3^T P_b e_3\\ &\quad + \mathrm{Sym}\{N_1D_1+N_2D_2+N_3D_3\}\\ &\quad + \mathrm{Sym}\{M_1E_1+M_2E_2+M_3E_3\}\\ &\quad + \mathrm{Sym}\{N_4D_4+N_5D_5\}+\mathrm{Sym}\{M_4E_4+M_5E_5\}\\ &\quad + \tau\begin{bmatrix}e_s\\ e_1\\ e_0\end{bmatrix}^T R_3\begin{bmatrix}e_s\\ e_1\\ e_0\end{bmatrix}\\ &\quad - \begin{bmatrix}e_{10}\\ e_{11}\end{bmatrix}^T (\mathrm{Sym}\{[X_1\ \ X_2]\})\begin{bmatrix}e_{10}\\ e_{11}\end{bmatrix},\end{aligned}$$

$$\Phi_{3[\tau(t)]} = \mathrm{Sym}\left\{\begin{bmatrix}\tau e_1-\tau(t)e_5-(\tau-\tau(t))e_6\\ e_{19}\\ \frac{\tau^2}{2}e_1-e_{19}\end{bmatrix}^T R_3\begin{bmatrix}e_0\\ e_0\\ e_s\end{bmatrix}\right\},$$

$$\begin{aligned}\Phi_{4[\tau(t)]} &= \frac{\tau^2}{2}e_s^T S_1 e_s - 2\left[e_1-\tau_f(\tau(t)e_5+(\tau-\tau(t))e_6)\right]^T\\ &\quad\times S_1\left[e_1-\tau_f(\tau(t)e_5+(\tau-\tau(t))e_6)\right]\\ &\quad - 4\left[e_1+2\tau_f(\tau(t)e_5+(\tau-\tau(t))e_6)-6\tau_f^2 e_{19}\right]^T\\ &\quad\times S_1\left[e_1+2\tau_f(\tau(t)e_5+(\tau-\tau(t))e_6)-6\tau_f^2 e_{19}\right]\\ &\quad - 6\left[e_1-3\tau_f(\tau(t)e_5+(\tau-\tau(t))e_6)+24\tau_f^2 e_{19}\right.\\ &\quad \left.-60\tau_f^3 e_{20}\right]^T S_1\left[e_1-3\tau_f(\tau(t)e_5+(\tau-\tau(t))e_6)\right.\\ &\quad \left.+24\tau_f^2 e_{19}-60\tau_f^3 e_{20}\right],\end{aligned}$$

$$\Phi_5 = \gamma_d e_{15}^T Y e_{15} - e_{16}^T Y e_{16},$$

$$\begin{aligned}\Phi_z &= \tau^2(t)e_{17}+(\tau-\tau(t))^2 e_{18}+(\tau-\tau(t))\tau(t)e_5\\ &\quad - e_{19},\end{aligned}$$

$$\begin{aligned}\nu_1 &= \mathrm{Sym}\{[e_7-e_8-(F_mW(e_1-e_2))]^T L_{f1}\\ &\quad\times[(F_pW(e_1-e_2))-e_7+e_8]\\ &\quad+[e_8-e_9-(F_mW(e_2-e_3))]^T L_{f2}\\ &\quad\times[(F_pW(e_2-e_3))-e_8+e_9]\\ &\quad+[e_7-e_9-(F_mW(e_1-e_3))]^T L_{f3}\\ &\quad\times[(F_pW(e_1-e_3))-e_7+e_9]\},\end{aligned}$$

$$\begin{aligned}\nu_2 &= \mathrm{Sym}\{[e_7-F_mWe_1]^T V_{f1}[F_pWe_1-e_7]\}\\ &\quad+[e_8-F_mWe_2]^T V_{f2}[F_pWe_2-e_8]\\ &\quad+[e_9-F_mWe_3]^T V_{f3}[F_pWe_3-e_9]\},\end{aligned}$$

$$\begin{aligned}\nu_3 &= \mathrm{Sym}\{[e_{12}-e_{13}-(G_mW(e_1-e_2))]^T L_{g1}\\ &\quad\times[(G_pW(e_1-e_2))-e_{12}+e_{13}]\\ &\quad+[e_{13}-e_{14}-(G_mW(e_2-e_3))]^T L_{g2}\\ &\quad\times[(G_pW(e_2-e_3))-e_{13}+e_{14}]\\ &\quad+[e_{12}-e_{14}-(G_mW(e_1-e_3))]^T L_{g3}\\ &\quad\times[(G_pW(e_1-e_3))-e_{12}+e_{14}]\},\end{aligned}$$

$$\begin{aligned}\nu_4 &= \mathrm{Sym}\{[e_{12}-G_mWe_1]^T V_{g1}[G_pWe_1-e_{12}]\\ &\quad+[e_{13}-G_mWe_2]^T V_{g2}[G_pWe_2-e_{13}]\\ &\quad+[e_{14}-G_mWe_3]^T V_{g3}[G_pWe_3-e_{14}]\},\end{aligned}$$

$$\nu_5 = \mathrm{Sym}\{[e_{15}-H_mWe_1]^T V_{h1}[H_pWe_1-e_{15}]\},$$

$$\begin{aligned}\varrho &= \mathrm{Sym}\{[e_0^T\ \ e_0^T\ \ e_5^T-e_6^T\ \ e_0^T\ \ e_0^T]P_2\\ &\quad\times[e_0^T\ \ e_0^T\ \ e_0^T\ \ e_0^T\ \ e_6^T-e_5^T]^T\}\\ &\quad+\mathrm{Sym}\{[e_0^T\ \ e_0^T\ \ e_0^T\ \ e_0^T\ \ e_12^T\ \ e_0^T]Q_1\\ &\quad\times[e_s^T\ \ e_0^T\ \ e_0^T\ \ e_s^T\ \ e_1^T\ \ e_4^T]^T\}\\ &\quad-(1-\tau_d)[e_0^T\ \ e_0^T\ \ e_0^T\ \ e_0^T\ \ e_5^T\ \ e_0^T]Q_1\\ &\quad\times[e_0^T\ \ e_0^T\ \ e_0^T\ \ e_0^T\ \ e_5^T\ \ e_0^T]^T\\ &\quad-[e_0^T\ \ e_0^T\ \ e_0^T\ \ e_0^T\ \ e_5^T-e_6^T\ \ e_0^T]Q_2\\ &\quad\times[e_0^T\ \ e_0^T\ \ e_0^T\ \ e_0^T\ \ e_5^T-e_6^T\ \ e_0^T]^T\\ &\quad+\mathrm{Sym}\{L(e_{12}+e_{13}-e_5)\},\end{aligned}$$

$$\Pi_1 = \left[\tau N_1\ \ \tau N_2\ \ \tau N_3\ \ \tau N_4\ \ \tau N_5\ \ (\begin{bmatrix}e_{10}\\ e_{11}\end{bmatrix}^T X_1)\right],$$

$$\Pi_2 = \left[\tau M_1\ \ \tau M_2\ \ \tau M_3\ \ \tau M_4\ \ \tau M_5\ \ (\begin{bmatrix}e_{10}\\ e_{11}\end{bmatrix}^T X_2)\right],$$

$$\begin{aligned}\Upsilon_1 &= \mathrm{diag}\{-\tau R_1\ \ -3\tau R_1\ \ -5\tau R_1\\ &\qquad -\tau R_a\ \ -3\tau R_a\ \ -R_2\},\end{aligned}$$

$$\begin{aligned}\Upsilon_2 &= \mathrm{diag}\{-\tau R_1\ \ -3\tau R_1\ \ -5\tau R_1\\ &\qquad -\tau R_b\ \ -3\tau R_b\ \ -R_2\},\end{aligned}$$

$$\theta_p = \tau_f\begin{bmatrix}\tau\hat{e}_1\\ \hat{e}_3\\ \hat{e}_1-\hat{e}_2\\ \hat{e}_4\\ \hat{e}_5\end{bmatrix}^T Q_2\begin{bmatrix}\tau\hat{e}_1\\ \hat{e}_3\\ \hat{e}_1-\hat{e}_2\\ \hat{e}_4\\ \hat{e}_5\end{bmatrix},$$

$$R_a = R_3+\mathrm{Sym}\left\{\begin{bmatrix}0\\ I\\ 0\end{bmatrix}P_a\begin{bmatrix}I & 0 & 0\end{bmatrix}\right\},$$

$$R_b = R_3+\mathrm{Sym}\left\{\begin{bmatrix}0\\ I\\ 0\end{bmatrix}P_b\begin{bmatrix}I & 0 & 0\end{bmatrix}\right\},$$

$$\begin{aligned}&\epsilon_0=\lambda_{\min}(\hat{P}_1),\ \epsilon_1=\lambda_{\max}(\hat{P}_1),\ \epsilon_2=\lambda_{\max}(\hat{P}_2),\\ &\epsilon_3=\lambda_{\max}(\hat{Q}_1),\ \epsilon_4=\lambda_{\max}(\hat{Q}_2),\ \epsilon_5=\lambda_{\max}(\hat{R}_1),\\ &\epsilon_6=\lambda_{\max}(\hat{R}_2),\ \epsilon_7=\lambda_{\max}(\hat{R}_3),\ \epsilon_8=\lambda_{\max}(\hat{S}_1)\,,\\ &\epsilon_9=\lambda_{\max}(\hat{Y})\,,\epsilon_{10}=\lambda_{\max}(M_a).\end{aligned}$$

A. FINITE-TIME BOUNDEDNESS

In this subsection, we study finite-time boundedness for the generalized neural networks with mixed time-varying delays in the following form:

$$\begin{aligned}\dot{z}(t) &= -A_0z(t)+A_1f(Wz(t))+A_2g(Wz(t-\tau(t)))\\ &\quad+A_3\int_{t-\gamma(t)}^{t}h(Wz(s))ds+A_4\omega(t),\end{aligned}$$

$$z(t)=\phi(t), \quad t\in[-\tau,0]. \tag{5}$$

Theorem 1: For given positive scalars τ, τ_d and γ_d, the system (5) is finite-time bounded if there exist matrices $P_1\in\mathbb{S}_n^+$, $P_2\in\mathbb{S}_{5n}$, $Q_i(i=1,2)\in\mathbb{S}_{6n}^+$, $R_j(j=1,2)\in\mathbb{S}_n^+$, $R_3\in\mathbb{S}_{3n}^+$, S_1, Y, $M_a\in\mathbb{S}_n^+$, any matrices X_1, $X_2\in\mathbb{R}^{2n\times n}$, $L\in\mathbb{R}^{21n\times n}$ such that the following LMIs hold:

$$\begin{bmatrix}\Xi_{[\tau(t)=\tau]}\Pi_1\\ *\Upsilon_1\end{bmatrix}<0,\ \begin{bmatrix}\Xi_{[\tau(t)=0]}\Pi_2\\ *\Upsilon_2\end{bmatrix}<0,$$
$$\begin{bmatrix}\Xi_{[\tau(t)=0]}-\tau^2\varrho\Pi_2\\ *\Upsilon_2\end{bmatrix}<0, \tag{6}$$
$$P_2+\theta_p>0, \tag{7}$$
$$R_a>0,\ R_b>0, \tag{8}$$
$$\epsilon_0 I\le\hat{P}_1\le\epsilon_1 I,\ 0\le\hat{P}_2\le\epsilon_2 I,$$
$$0\le\hat{Q}_1\le\epsilon_3 I,\ 0\le\hat{Q}_2\le\epsilon_4 I,$$
$$0\le\hat{R}_1\le\epsilon_5 I,\ 0\le\hat{R}_2,\le\epsilon_6 I,$$
$$0\le\hat{R}_3\le\epsilon_7 I,\ 0\le\hat{S}_1\le\epsilon_8 I,$$
$$0\le\hat{Y}\le\epsilon_9 I,\ 0\le M_a\le\epsilon_{10}I, \tag{9}$$
$$e^{\alpha T_f}\left[\Pi c_1+\omega_f\epsilon_{10}(1-e^{-\alpha T_f})\right]<\epsilon_0 c_2. \tag{10}$$

Proof: We construct the Lyapunov–Krasovskii functional as follows:

$$V(z_t,t)=\sum_{j=1}^{5}V_j(z_t,t), \tag{11}$$

where

$$\begin{aligned}
V_1(z_t,t)&=z^T(t)P_1z(t)+\varphi_1^T(t)P_2\varphi_1(t),\\
V_2(z_t,t)&=\int_{t-\tau(t)}^{t}\varphi_2^T(t,s)Q_1\varphi_2(t,s)ds\\
&\quad+\int_{t-\tau}^{t}\varphi_3^T(t,s)Q_2\varphi_3(t,s)ds,\\
V_3(z_t,t)&=\int_{t-\tau}^{t}\int_{s}^{t}\dot{z}^T(u)R_1\dot{z}(u)duds\\
&\quad+\tau\int_{t-\tau}^{t}\int_{s}^{t}f_a^T(u)R_2f_a(u)duds\\
&\quad+\int_{t-\tau}^{t}\int_{s}^{t}\varphi_4^T(t,u)R_3\varphi_4(t,u)duds,\\
V_4(z_t,t)&=\int_{t-\tau}^{t}\int_{s}^{t}\int_{u}^{t}\dot{z}^T(v)S_1\dot{z}(v)dvduds,\\
V_5(z_t,t)&=\int_{t-\gamma_d}^{t}\int_{s}^{t}h_a^T(u)Yh_a(u)duds.
\end{aligned}$$

Then, the time derivatives of (10) are calculated as follows:

$$\begin{aligned}
\dot{V}_1(z_t,t)&=2z^T(t)P_1\dot{z}(t)+2\varphi_1^T(t)P_2\dot{\varphi}_1(t)\\
&=\xi^T(t)\Phi_{1[\tau(t)]}\xi(t),
\end{aligned} \tag{12}$$

$$\begin{aligned}
\dot{V}_2(z_t,t)&=\varphi_2^T(t,t)Q_1\varphi_2(t,t)\\
&\quad-(1-\dot{\tau}(t))\varphi_2^T(t,t-\tau(t))Q_1\varphi_2(t,t-\tau(t))\\
&\quad+2\int_{t-\tau(t)}^{t}\varphi_2^T(t,s)Q_1\dot{\varphi}_2(t,s)ds\\
&\quad+\varphi_3^T(t,t)Q_2\varphi_3(t,t)\\
&\quad-\varphi_3^T(t,t-\tau)Q_2\varphi_3(t,t-\tau)\\
&\quad+2\int_{t-\tau}^{t}\varphi_3^T(t,s)Q_2\dot{\varphi}_3(t,s)ds\\
&\le\xi^T(t)(\Phi_2+\Phi_{2[\tau(t)]})\xi(t).
\end{aligned} \tag{13}$$

Before calculating $\dot{V}_3(z_t,t)$, we present two zero equations with the symmetric matrices P_a and $P_b\in\mathbb{R}^{n\times n}$ inspired by the work of [5] as follows:

$$\begin{aligned}
0&=z^T(t)P_az(t)-z^T(t-\tau(t))P_az(t-\tau(t))\\
&\quad-2\int_{t-\tau(t)}^{t}z^T(s)P_a\dot{z}(s)ds,\\
0&=z^T(t-\tau(t))P_bz(t-\tau(t))-z^T(t-\tau)P_bz(t-\tau)\\
&\quad-2\int_{t-\tau}^{t-\tau(t)}z^T(s)P_b\dot{z}(s)ds.
\end{aligned}$$

As a result, the sum of $\dot{V}_3(z_t,t)$ and two zero items can be written as

$$\begin{aligned}
\dot{V}_3(z_t,t)&=\tau\dot{z}^T(t)R_1\dot{z}(t)-\int_{t-\tau(t)}^{t}\dot{z}^T(s)R_1\dot{z}(s)ds\\
&\quad-\int_{t-\tau}^{t-\tau(t)}\dot{z}^T(s)R_1\dot{z}(s)ds\\
&\quad+\tau^2f_a^T(t)R_2f_a(t)-\tau\int_{t-\tau(t)}^{t}f_a^T(s)R_2f_a(s)ds\\
&\quad-\tau\int_{t-\tau}^{t-\tau(t)}f_a^T(s)R_2f_a(s)ds\\
&\quad+\tau\begin{bmatrix}\dot{z}(t)\\ z(t)\\ 0\end{bmatrix}^T R_3\begin{bmatrix}\dot{z}(t)\\ z(t)\\ 0\end{bmatrix}\\
&\quad+2\begin{bmatrix}\tau z(t)-\int_{t-\tau}^{t}z(s)ds\\ \int_{t-\tau}^{t}\int_{s}^{t}z(u)duds\\ \frac{\tau^2}{2}z(t)-\int_{t-\tau}^{t}\int_{s}^{t}z(u)duds\end{bmatrix}^T R_3\begin{bmatrix}0\\ 0\\ \dot{z}(t)\end{bmatrix}\\
&\quad-\int_{t-\tau}^{t}\varphi_4^T(t,s)R_3\varphi_4(t,s)ds+z^T(t)P_az(t)\\
&\quad-z^T(t-\tau(t))P_az(t-\tau(t))\\
&\quad-2\int_{t-\tau(t)}^{t}z^T(s)P_a\dot{z}(s)ds\\
&\quad+z^T(t-\tau(t))P_bz(t-\tau(t))\\
&\quad-z^T(t-\tau)P_bz(t-\tau)\\
&\quad-2\int_{t-\tau(t-\tau)}^{t-\tau(t)}z^T(s)P_b\dot{z}(s)ds\\
&=\tau\dot{z}^T(t)R_1\dot{z}(t)-\int_{t-\tau(t)}^{t}\dot{z}^T(s)R_1\dot{z}(s)ds\\
&\quad-\int_{t-\tau}^{t-\tau(t)}\dot{z}^T(s)R_1\dot{z}(s)ds+\tau^2f_a^T(t)R_2f_a(t)\\
&\quad-\tau\int_{t-\tau(t)}^{t}f_a^T(s)R_2f_a(s)ds
\end{aligned}$$

$$
\begin{aligned}
&-\tau\int_{t-\tau}^{t-\tau(t)} f_a^T(s)R_2f_a(s)ds \\
&+\tau\begin{bmatrix}\dot z(t)\\ z(t)\\ 0\end{bmatrix}^T R_3\begin{bmatrix}\dot z(t)\\ z(t)\\ 0\end{bmatrix} \\
&+2\begin{bmatrix}\tau z(t)-\int_{t-\tau}^t z(s)ds\\ \int_{t-\tau}^t\int_s^t z(u)duds\\ \frac{\tau^2}{2}z(t)-\int_{t-\tau}^t\int_s^t z(u)duds\end{bmatrix}^T R_3\begin{bmatrix}0\\0\\ \dot z(t)\end{bmatrix} \\
&+z^T(t)P_az(t)-z^T(t-\tau(t))P_az(t-\tau(t)) \\
&+z^T(t-\tau(t))P_bz(t-\tau(t)) \\
&-z^T(t-\tau)P_bz(t-\tau) \\
&-\int_{t-\tau(t)}^t \varphi_4^T(t,s)R_a\varphi_4(t,s)ds \\
&-\int_{t-\tau}^{t-\tau(t)} \varphi_4^T(t,s)R_b\varphi_4(t,s)ds.
\end{aligned}
$$

Using Lemma 3, we have

$$
\begin{aligned}
&-\int_{t-\tau(t)}^t \dot z^T(s)R_1\dot z(s)ds-\int_{t-\tau}^{t-\tau(t)} \dot z^T(s)R_1\dot z(s)ds \\
&\le \xi^T(t)\Big\{\tau(t)\Big(N_1R_1^{-1}N_1^T+\frac{1}{3}N_2R_1^{-1}N_2^T \\
&+\frac{1}{5}N_3R_1^{-1}N_3^T\Big)+\mathrm{Sym}\{N_1D_1+N_2D_2+N_3D_3\} \\
&+(\tau-\tau(t))\Big(M_1R_1^{-1}M_1^T+\frac{1}{3}M_2R_1^{-1}M_2^T \\
&+\frac{1}{5}M_3R_1^{-1}M_3^T\Big)+\mathrm{Sym}\{M_1E_1+M_2E_2 \\
&+M_3E_3\}\Big\}\xi(t).
\end{aligned}
$$

By applying Lemma 1 and Lemma 2, we obtain

$$
\begin{aligned}
&-\tau\int_{t-\tau(t)}^t f_a^T(s)R_2f_a(s)ds-\tau\int_{t-\tau}^{t-\tau(t)} f_a^T(s)R_2f_a(s)ds \\
&\le -\frac{\tau}{\tau(t)}\int_{t-\tau(t)}^t f_a^T(s)dsR_2\int_{t-\tau(t)}^t f_a(s)ds \\
&\quad -\frac{\tau}{\tau-\tau(t)}\int_{t-\tau}^{t-\tau(t)} f_a^T(s)dsR_2\int_{t-\tau}^{t-\tau(t)} f_a(s)ds \\
&=-\begin{bmatrix}\int_{t-\tau(t)}^t f_a(s)ds\\ \int_{t-\tau}^{t-\tau(t)} f_a(s)ds\end{bmatrix}^T\begin{bmatrix}\frac{\tau}{\tau(t)}R_2 & 0\\ 0 & \frac{\tau}{\tau-\tau(t)}R_2\end{bmatrix} \\
&\quad\times\begin{bmatrix}\int_{t-\tau(t)}^t f_a(s)ds\\ \int_{t-\tau}^{t-\tau(t)} f_a(s)ds\end{bmatrix} \\
&\le\begin{bmatrix}\int_{t-\tau(t)}^t f_a(s)ds\\ \int_{t-\tau}^{t-\tau(t)} f_a(s)ds\end{bmatrix}^T \\
&\quad\times\Big(-\mathrm{Sym}\{[X_1\ \ X_2]\}+\frac{\tau(t)}{\tau}X_1R_2^{-1}X_1^T \\
&\quad+\frac{\tau-\tau(t)}{\tau}X_2R_2^{-1}X_2^T\Big)\begin{bmatrix}\int_{t-\tau(t)}^t f_a(s)ds\\ \int_{t-\tau}^{t-\tau(t)} f_a(s)ds\end{bmatrix}.
\end{aligned}
$$

By utilizing Lemma 4, we get

$$
\begin{aligned}
&-\int_{t-\tau(t)}^t \varphi_4^T(t,s)R_a\varphi_4(t,s)ds \\
&\quad-\int_{t-\tau}^{t-\tau(t)} \varphi_4^T(t,s)R_b\varphi_4(t,s)ds \\
&\quad\le \xi^T(t)\Big\{\tau(t)\Big(N_4R_a^{-1}N_4^T+\frac{1}{3}N_5R_a^{-1}N_5^T\Big) \\
&\quad+\mathrm{Sym}\{N_4D_4+N_5D_5\} \\
&\quad+(\tau-\tau(t))\Big(M_4R_b^{-1}M_4^T+\frac{1}{3}M_5R_b^{-1}M_5^T\Big) \\
&\quad+\mathrm{Sym}\{M_4E_4+M_5E_5\}\Big\}\xi(t).
\end{aligned}
$$

Therefore, we obtain

$$
\dot V_3\le \xi^T(t)\{\Phi_3+\Phi_{3[\tau(t)]}\}\xi(t). \tag{14}
$$

Further, the calculation of $\dot V_4(z_t,t)$ can be presented as

$$
\dot V_4(z_t,t)=\frac{\tau^2}{2}\dot z^T(t)S_1\dot z(t)-\int_{t-\tau}^t\int_s^t \dot z^T(u)S_1\dot z(u)duds.
$$

By applying Lemma 5, we deduce

$$
\begin{aligned}
&-\int_{t-\tau}^t\int_s^t \dot z^T(u)S_1\dot z(u)duds \\
&\le -2\Big[z(t)-\frac{1}{\tau}\int_{t-\tau}^t z(s)ds\Big]^T S_1\Big[z(t)-\frac{1}{\tau}\int_{t-\tau}^t z(s)ds\Big] \\
&\quad-4\Big[z(t)+\frac{2}{\tau}\int_{t-\tau}^t z(s)ds-\frac{6}{\tau^2}\int_{t-\tau}^t\int_s^t z(u)duds\Big]^T \\
&\quad\times S_1\Big[z(t)+\frac{2}{\tau}\int_{t-\tau}^t z(s)ds-\frac{6}{\tau^2}\int_{t-\tau}^t\int_s^t z(u)duds\Big] \\
&\quad-6\Big[z(t)-\frac{3}{\tau}\int_{t-\tau}^t z(s)ds+\frac{24}{\tau^2}\int_{t-\tau}^t\int_s^t z(u)duds \\
&\quad-\frac{60}{\tau^3}\int_{t-\tau}^t\int_s^t\int_u^t z(v)dvduds\Big]S_1\Big[z(t)-\frac{3}{\tau}\int_{t-\tau}^t z(s)ds \\
&\quad+\frac{24}{\tau^2}\int_{t-\tau}^t\int_s^t z(u)duds-\frac{60}{\tau^3}\int_{t-\tau}^t\int_s^t\int_u^t z(v)dvduds\Big].
\end{aligned}
$$

Then, we obtain

$$
\dot V_4(z_t,t)\le \xi^T(t)\Phi_{4[\tau(t)]}\xi(t). \tag{15}
$$

Calculation of $\dot V_5(z_t,t)$ is

$$
\begin{aligned}
\dot V_5(z_t,t)&=\gamma_dh_a^T(t)Yh_a(t)-\int_{t-\gamma_d}^t h_a^T(s)Yh_a(s)ds \\
&\le \gamma_dh_a^T(t)Yh_a(t)-\int_{t-\gamma(t)}^t h_a^T(s)Yh_a(s)ds.
\end{aligned}
$$

By Lemma2, we obtain

$$
\begin{aligned}
&-\int_{t-\gamma(t)}^t h_a^T(s)Yh_a(s)ds \\
&\qquad\le -\int_{t-\gamma(t)}^t h_a^T(s)dsY\int_{t-\gamma(t)}^t h_a(s)ds.
\end{aligned}
$$

Then, we get

$$\dot{V}_5(z_t,t) \le \xi^T(t)\Phi_5\xi(t). \tag{16}$$

By utilizing Assumption 1, we get

$$\begin{aligned}
&l_{fi}(v_1,v_2):\\
&\ = 2[f_a(v_1)-f_a(v_2)-F_mW(z(v_1)-z(v_2))]^T L_{fi}\\
&\quad \times [F_pW(z(v_1)-z(v_2))-f_a(v_1)+f_a(v_2)] \ge 0,\\
&v_{fi}(v):\\
&\ = 2[f_a(v)-F_mWz(v)]^T V_{fi}[F_pWz(v)-f_a(v)] \ge 0,\\
&l_{gi}(v_1,v_2):\\
&\ = 2[g_a(v_1)-g_a(v_2)-G_mW(z(v_1)-z(v_2))]^T L_{gi}\\
&\quad \times [G_pW(z(v_1)-z(v_2))-g_a(v_1)+g_a(v_2)] \ge 0,\\
&v_{gi}(v) := 2[g_a(v)-G_mWz(v)]^T V_{gi}[G_pWz(v)-g_a(v)] \ge 0,\\
&v_h(v) := 2[h_a(v)-H_mWz(v)]^T V_h[H_pWz(v)-h_a(v)] \ge 0,
\end{aligned}$$

where

$$\begin{aligned}
L_{fi} &= \mathrm{diag}\{l_{1fi}, l_{2fi}, \dots, l_{nfi}\},\\
V_{fi} &= \mathrm{diag}\{v_{1fi}, v_{2fi}, \dots, v_{nfi}\},\\
L_{gi} &= \mathrm{diag}\{l_{1gi}, l_{2gi}, \dots, l_{ngi}\},\\
V_{gi} &= \mathrm{diag}\{v_{1gi}, v_{2gi}, \dots, v_{ngi}\},\\
V_h &= \mathrm{diag}\{v_{1h}, v_{2h}, \dots, v_{nh}\}, \quad i=1,2,3.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&l_{f1}(t,t-\tau(t)) + l_{f2}(t-\tau(t),t-\tau) + l_{f3}(t,t-\tau)\\
&\quad = \xi^T(t)v_1\xi(t) \ge 0, && (17)\\
&v_{f1}(t) + v_{f2}(t-\tau(t)) + v_{f3}(t-\tau)\\
&\quad = \xi^T(t)v_2\xi(t) \ge 0, && (18)\\
&l_{g1}(t,t-\tau(t)) + l_{g2}(t-\tau(t),t-\tau) + l_{g3}(t,t-\tau)\\
&\quad = \xi^T(t)v_3\xi(t) \ge 0, && (19)\\
&v_{g1}(t) + v_{g2}(t-\tau(t)) + v_{g3}(t-\tau)\\
&\quad = \xi^T(t)v_4\xi(t) \ge 0, && (20)\\
&v_h(t) = \xi^T(t)v_5\xi(t) \ge 0. && (21)
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{t-\tau}^{t}\int_{s}^{t} z(u)duds = &\int_{t-\tau(t)}^{t}\int_{s}^{t} z(u)duds\\
&+ (\tau-\tau(t))\int_{t-\tau}^{t} z(s)ds\\
&+ \int_{t-\tau}^{t-\tau(t)}\int_{s}^{t-\tau(t)} z(u)duds.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
0 = 2\xi^T(t)L\Bigg(&\int_{t-\tau(t)}^{t}\int_{s}^{t} z(u)duds + (\tau-\tau(t))\\
&\times \int_{t-\tau}^{t} z(s)ds + \int_{t-\tau}^{t-\tau(t)}\int_{s}^{t-\tau(t)} z(u)duds\\
&- \int_{t-\tau}^{t}\int_{s}^{t} z(u)duds\Bigg)\\
= \xi^T(t)&\Phi_z\xi(t). \qquad (22)
\end{aligned}$$

Combining (12)-(22), it can be inferred that

$$\dot{V}(z_t,t) - \alpha V(z_t,t) - \alpha\omega^T(t)M\omega(t) \le \xi^T(t)\Xi_{[\tau(t)]}\xi(t). \tag{23}$$

Obviously the equation (23) is quadratic. By Lemma 8 if

$$\Xi_{[\tau(t)}=\tau] < 0,\ \ \Xi_{[\tau(t)}=0] < 0,\ \ \Xi_{[\tau(t)}=0] - \tau^2\varrho < 0.$$

Therefore, we obtain

$$\Xi_{[\tau(t)]} < 0. \tag{24}$$

It follows from (23) and (24), we have

$$\dot{V}(z_t,t) - \alpha V(z_t,t) - \alpha\omega^T(t)M\omega(t) \le \xi^T(t)\Xi_{[\tau(t)]}\xi(t) < 0. \tag{25}$$

By multiplying of (25) with $e^{-\alpha t}$, then (25) becomes

$$\frac{d}{dt}\left(e^{-\alpha t}V(z_t,t)\right) < \alpha e^{-\alpha t}\omega^T(t)M\omega(t). \tag{26}$$

By integrating (26) on $[0,t]$ where $t \in [0,T_f]$ and Assumption 2, we obtain

$$\begin{aligned}
V(z_t,t) &< e^{\alpha T_f}\left[V(z_0,0) + \alpha\int_0^{T_f} e^{-\alpha s}\omega^T(s)M\omega(s)ds\right]\\
&< e^{\alpha T_f}\left[V(z_0,0) + \omega\epsilon_{10}(1-e^{-\alpha T_f})\right].
\end{aligned}$$

Next, we consider $V(z_0,0)$ by Assumption 1, we get

$$\begin{aligned}
V(z_0,0) \le\ & z^T(0)P_1z(0) + \varphi_1^T(0)P_2\varphi_1(0)\\
&+ \int_{-\tau(0)}^{0} \varphi_2^T(0,s)Q_1\varphi_2(0,s)ds\\
&+ \int_{-\tau}^{0} \varphi_3^T(0,s)Q_2\varphi_3(0,s)ds\\
&+ \int_{-\tau}^{0}\int_{s}^{0} \dot{z}^T(u)R_1\dot{z}(u)duds\\
&+ \tau\int_{-\tau}^{0}\int_{s}^{t} f_a^T(u)R_2f_a(u)duds\\
&+ \int_{\tau}^{0}\int_{s}^{0} \varphi_4^T(0,u)(s)R_3\varphi_4(0,u)duds\\
&+ \int_{-\tau}^{0}\int_{s}^{0}\int_{u}^{0} \dot{z}^T(v)S_1\dot{z}(v)dvduds\\
&+ \int_{-\gamma_d}^{0}\int_{s}^{0} \hat{H}^T(u)Y\hat{H}(u)duds,
\end{aligned}$$

where $\hat{H} = \mathrm{diag}\{H_1^+,\dots,H_n^+\}$. Furthermore, we let $\hat{P}_i = V^{\frac{-1}{2}}P_iV^{\frac{-1}{2}}$, $\hat{Q}_i = V^{\frac{-1}{2}}Q_iV^{\frac{-1}{2}}$, $i=1,2$, $\hat{R}_j = V^{\frac{-1}{2}}R_jV^{\frac{-1}{2}}$, $j=1,2,3$, $\hat{S} = V^{\frac{-1}{2}}SV^{\frac{-1}{2}}$, $\hat{Y} = V^{\frac{-1}{2}}\hat{H}^TY\hat{H}V^{\frac{-1}{2}}$. We can derive that

$$\begin{aligned}
V(z_0,0) \le\ & z^T(0)V^{\frac{1}{2}}P_1V^{\frac{1}{2}}z(0) + \varphi_1^T(0)V^{\frac{1}{2}}P_2V^{\frac{1}{2}}\varphi_1(0)\\
&+ \int_{-\tau(0)}^{0} \varphi_2^T(0,s)V^{\frac{1}{2}}Q_1V^{\frac{1}{2}}\varphi_2(0,s)ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\tau}^{0}\varphi_3^T(0,s)V^{\frac{1}{2}}Q_2V^{\frac{1}{2}}\varphi_3(0,s)ds\\
&+\int_{-\tau}^{0}\int_{s}^{0}\dot{z}^T(u)V^{\frac{1}{2}}R_1V^{\frac{1}{2}}\dot{z}(u)duds\\
&+\tau\int_{-\tau}^{0}\int_{s}^{t}f_a^T(u)V^{\frac{1}{2}}R_2V^{\frac{1}{2}}f_a(u)duds\\
&+\int_{\tau}^{0}\int_{s}^{0}\varphi_4^T(0,u)(s)V^{\frac{1}{2}}R_3V^{\frac{1}{2}}\varphi_4(0,u)duds\\
&+\int_{-\tau}^{0}\int_{s}^{0}\int_{u}^{0}\dot{z}^T(v)S_1\dot{z}(v)dvduds\\
&+\int_{-\gamma_d}^{0}\int_{s}^{0}\hat{H}^T(u)Y\hat{H}(u)duds.\\
\leq&\Big\{\lambda_{\max}(\hat{P}_1)+\lambda_{\max}(\hat{P}_2)+\tau\lambda_{\max}(\hat{Q}_1)\\
&+\tau\lambda_{\max}(\hat{Q}_2)+\frac{\tau^2}{2}\lambda_{\max}(\hat{R}_1)+\frac{\tau^3}{2}\lambda_{\max}(\hat{R}_2)\\
&+\frac{\tau^2}{2}\lambda_{\max}(\hat{R}_3)+\frac{\tau^3}{6}\lambda_{\max}(\hat{S}_1)+\frac{\gamma_d^2}{2}\lambda_{\max}(\hat{Y})\Big\}\\
&\times\sup_{\tau_2\leq s\leq 0}\{z^T(s)Vz(s),\dot{z}^T(s)V\dot{z}(s)\}\leq\Gamma c_1,
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma=&\,\epsilon_1+\epsilon_2+\tau\epsilon_3+\tau\epsilon_4+\frac{\tau^2}{2}\epsilon_5+\frac{\tau^3}{2}\epsilon_6\\
&+\frac{\tau^2}{2}\epsilon_7+\frac{\tau^3}{6}\epsilon_8+\frac{\gamma_d^2}{2}\epsilon_9. \qquad (27)
\end{aligned}
$$

In addition, it follows from (11) that

$$
\begin{aligned}
V(z_t,t)&\geq z^T(t)P_1z(t)\\
&\geq\lambda_{\min}(\hat{P}_1)z^T(t)Vz(t)=\epsilon_0z^T(t)Vz(t). \qquad (28)
\end{aligned}
$$

Then, from the inequalities (27)-(28) and the condition (10), we obtain

$$
z^T(t)Vz(t)\leq\frac{e^{\alpha T_f}}{\epsilon_0}\left[\Gamma c_1+\omega_f\epsilon_{10}(1-e^{-\alpha T_f})\right]<c_2.
$$

By definition (3), the system (5) is finite-time bounded. The proof is complete. □

Remark 6: In Assumption 1, select (v_1, v_2) as $(t, t-\tau(t))$, $(t-\tau(t)$, and $(t, t-\tau)$. As a result, we incorporated more information on cross terms between the terms t, $t-\tau$, and $t-\tau(t)$. Thus, our method leads to less conservative stability criteria.

Remark 7: In this research, the LKFs consist of single, double, and triple integral terms that make utilize additional information regarding the delays τ and γ_d, and a state variable. We improved LKFs and compared them to LKFs reported in recent publications [5], [34], [37], [38]. In addition, the LKFs consisting of the triple integral term $\int_{t-\tau}^{t}\int_{s}^{t}\int_{u}^{t}\dot{z}^T(v)S_1\dot{z}(v)dvduds$ that were not used in [5], [34], [37], and [38]. Moreover, the stability and performance analysis has employed more information on activation functions, as demonstrated by the inclusion of $f(z)$, $g(z)$, and $h(z)$ in the proof. Constructing improved LKFs and employing techniques for estimating the time derivatives, which result in less conservatism.

B. FINITE-TIME STABLE

In this subsection, we investigate finite-time stability for the GNNs with mixed time-varying delays and asymptotically stable for the NNs with discrete time-varying delays. We defined:

$$
\begin{aligned}
e_i&=\left[0_{n\times(i-1)n}\ I_{n\times n}\ 0_{n\times(20-i)n}\right],\ (i=1,2,\cdots,20)\\
e_0&=0_{20n\times n},\quad e_s=-A_0e_1+A_1e_7+A_2e_{13}+A_3e_{16},\\
\psi_1(t)&=\Big[z^T(t)\ z^T(t-\tau(t))\ z^T(t-\tau)\ \dot{z}^T(t-\tau)\\
&\qquad\int_{t-\tau(t)}^{t}\frac{z^T(s)}{\tau(t)}ds\ \int_{t-\tau}^{t-\tau(t)}\frac{z^T(s)}{\tau-\tau(t)}\Big]^T,\\
\psi_2(t)&=\Big[f_a^T(t)\ f_a^T(t-\tau(t))\ f_a^T(t-\tau)\\
&\qquad\int_{t-\tau(t)}^{t}f_a^T(s)ds\ \int_{t-\tau}^{t-\tau(t)}f_a^T(s)ds\Big]^T,\\
\psi_3(t)&=\Big[g_a^T(t)\ g_a^T(t-\tau(t))\ g_a^T(t-\tau)\\
&\qquad h_a^T(t)\ \int_{t-\gamma(t)}^{t}h_a^T(s)ds\Big]^T,\\
\psi_4(t)&=\Big[\int_{t-\tau(t)}^{t}\int_{s}^{t}\frac{z^T(u)}{\tau^2(t)}duds\\
&\qquad\int_{t-\tau}^{t-\tau(t)}\int_{s}^{t-\tau(t)}\frac{z^T(u)}{(\tau-\tau(t))^2}duds\Big]^T,\\
\psi_5(t)&=\Big[\int_{t-\tau}^{t}\int_{s}^{t}z^T(u)duds\ \int_{t-\tau}^{t}\int_{s}^{t}\int_{u}^{t}z^T(v)dvduds\Big]^T,\\
\Psi(t)&=\Big[\psi_1^T(t)\quad\psi_2^T(t)\quad\psi_3^T(t)\quad\psi_4^T(t)\quad\psi_5^T(t)\Big]^T,\\
\Theta_{[\tau(t)]}&=\Phi_{1[\tau(t)]}+\Phi_2+\Phi_{2[\tau(t)]}+\Phi_3+\Phi_{3[\tau(t)]}+\Phi_{4[\tau(t)]}\\
&\quad+\Phi_5+\Phi_z+\nu_1+\nu_2+\nu_3+\nu_4+\nu_5-\tau^2\varrho\\
&\quad-\alpha e_1^TP_1e_1,\\
\Sigma_{[\tau(t)]}&=\Phi_{1[\tau(t)]}+\Phi_2+\Phi_{2[\tau(t)]}+\Phi_3+\Phi_{3[\tau(t)]}+\Phi_{4[\tau(t)]}\\
&\quad+\Phi_5+\Phi_z+\nu_1+\nu_2-\tau^2\varrho.
\end{aligned}
$$

Remark 8: The generalized neural networks (5) without external disturbance ($\omega(t)=0$) satisfying (2)-(3) becomes

$$
\begin{aligned}
\dot{z}(t)&=-A_0z(t)+A_1f(Wz(t))+A_2g(Wz(t-\tau(t)))\\
&\quad+A_3\int_{t-\gamma(t)}^{t}h(Wz(s))ds,\\
z(t)&=\phi(t),\quad t\in[-\tau,0], \qquad (29)
\end{aligned}
$$

Corollary 1: For given positive scalars τ, τ_d and γ_d, the system (29) is finite-time stable if there exist matrices $P_1\in\mathbb{S}_n^+$, $P_2\in\mathbb{S}_{5n}$, $Q_i(i=1,2)\in\mathbb{S}_{6n}^+$, $R_j(j=1,2)\in\mathbb{S}_n^+$, $R_3\in\mathbb{S}_{3n}^+$ S_1, $Y\in\mathbb{S}_n^+$, any matrices X_1, $X_2\in\mathbb{R}^{2n\times n}$, $L\in\mathbb{R}^{20n\times n}$ such that the following LMIs hold:

$$
\begin{bmatrix}\Theta_{[\tau(t)=\tau]} & \Pi_1\\ * & \Upsilon_1\end{bmatrix}<0,\ \begin{bmatrix}\Theta_{[\tau(t)=0]} & \Pi_2\\ * & \Upsilon_2\end{bmatrix}<0,
$$

$$\begin{bmatrix} \Theta_{[\tau(t)=0]} - \tau^2 \varrho \Pi_2 \\ * \Upsilon_2 \end{bmatrix} < 0, \quad (30)$$
$$P_2 + \theta_p > 0, \quad (31)$$
$$R_a > 0,\ R_b > 0, \quad (32)$$
$$\begin{aligned} &\epsilon_0 I \le \hat{P}_1 \le \epsilon_1 I,\ 0 \le \hat{P}_2 \le \epsilon_2 I,\ 0 \le \hat{Q}_1 \le \epsilon_3 I, \\ &0 \le \hat{Q}_2 \le \epsilon_4 I,\ 0 \le \hat{R}_1 \le \epsilon_5 I,\ 0 \le \hat{R}_2, \le \epsilon_6 I, \\ &0 \le \hat{R}_3 \le \epsilon_7 I,\ 0 \le \hat{S}_1 \le \epsilon_8 I,\ 0 \le \hat{Y} \le \epsilon_9 I, \end{aligned} \quad (33)$$
$$e^{\alpha T_f} \Pi c_1 < \epsilon_0 c_2. \quad (34)$$

Proof: Similarly to the proof of Theorem 1, therefore, it is omitted here. □

Remark 9: The generalized neural networks (29) without distributed delay and W is identity matrix ($B_2 = 0$ and $W = I$) can be written as follows:

$$\begin{aligned} \dot{z}(t) &= -A_0 z(t) + A_1 f(z(t)) + A_2 g(z(t-\tau(t))) \\ z(t) &= \phi(t), \quad t \in [-\tau, 0] \end{aligned} \quad (35)$$

satisfying $0 \le \tau(t) \le \tau$ and $\dot{\tau}(t) \le \tau_d$, which mean that the system (35) becomes a special case of the system (29).

Corollary 2: For given positive scalars τ and τ_d, the system (35) is asymptotically stable if there exist matrices $P_1 \in \mathbb{S}_n^+$, $P_2 \in \mathbb{S}_{5n}$, $Q_i (i = 1, 2) \in \mathbb{S}_{6n}^+$, $R_j (j = 1, 2) \in \mathbb{S}_n^+$, $R_3 \in \mathbb{S}_{3n}^+$ S_1, $Y \in \mathbb{S}_n^+$, any matrices X_1, $X_2 \in \mathbb{R}^{2n\times n}$, $L \in \mathbb{R}^{20n\times n}$ such that the following LMIs hold:

$$\begin{bmatrix} \Sigma_{[\tau(t)=\tau]} \Pi_1 \\ * \Upsilon_1 \end{bmatrix} < 0, \quad \begin{bmatrix} \Sigma_{[\tau(t)=0]} \Pi_2 \\ * \Upsilon_2 \end{bmatrix} < 0,$$
$$\begin{bmatrix} \Sigma_{[\tau(t)=0]} - \tau^2 \varrho \Pi_2 \\ * \Upsilon_2 \end{bmatrix} < 0, \quad (36)$$
$$P_2 + \theta_p > 0, \quad (37)$$
$$R_a > 0,\ R_b > 0. \quad (38)$$

Proof: The proof of Corollary 2 is similar to the proof of Theorem 1, hence it is omitted here. □

Remark 10: As demonstrated previously, we can derive a stability criterion for neural networks with time-varying delay, even if the delay rate is τ. Our results are more effective, as illustrated by the numerical example section.

C. FINITE-TIME EXTENDED DISSIPATIVITY ANALYSIS

In this section, we look at the finite-time extended dissipativity performance of generalized neural networks with discrete and distributed time-varying delays as follows:

$$\begin{aligned} \dot{z}(t) &= -A_0 z(t) + A_1 f(Wz(t)) + A_2 g(Wz(t-\tau(t))) \\ &\quad + A_3 \int_{t-\gamma(t)}^{t} h(Wz(s))ds + A_4 \omega(t), \\ y(t) &= B_0 z(t), \\ z(t) &= \phi(t), \quad t \in [-\tau, 0]. \end{aligned} \quad (39)$$

We define:

$$\begin{aligned} \bar{\Xi}_{[\tau(t)]} &= \Phi_{1[\tau(t)]} + \Phi_2 + \Phi_{2[\tau(t)]} + \Phi_3 + \Phi_{3[\tau(t)]} \\ &\quad + \Phi_{4[\tau(t)]} + \Phi_5 + \Phi_z + \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 \\ &\quad - \tau^2 \varrho - \alpha e_1^T P_1 e_1 - e_1^T B_0^T \Omega_1 B_0 e_1 \\ &\quad - \mathrm{Sym}\{e_1^T B_0^T \Omega_2 e_{21}\} - e_{21}^T \Omega_3 e_{21}, \\ \epsilon_{11} &= \lambda_{\max}(B_0^T B_0),\ \epsilon_{12} = \lambda_{\max}(\Omega_2^T \Omega_2), \\ \epsilon_{13} &= \lambda_{\max}(\Omega_3). \end{aligned}$$

Theorem 2: For given positive scalars τ, τ_d and γ_d, the system (39) is finite-time extened dissipativity respecting $(c_1, c_2, T_f, V, \omega)$ if there exist matrices $P_1 \in \mathbb{S}_n^+$, $P_2 \in \mathbb{S}_{5n}$, $Q_i (i = 1, 2) \in \mathbb{S}_{6n}^+$, $R_j (j = 1, 2) \in \mathbb{S}_n^+$, $R_3 \in \mathbb{S}_{3n}^+$ S_1, Y, $M \in \mathbb{S}_n^+$, any matrices X_1, $X_2 \in \mathbb{R}^{2n\times n}$, $L \in \mathbb{R}^{21n\times n}$ such that the following LMIs hold:

$$\begin{bmatrix} \bar{\Xi}_{[\tau(t)=\tau]} \Pi_1 \\ * \Upsilon_1 \end{bmatrix} < 0, \quad \begin{bmatrix} \bar{\Xi}_{[\tau(t)=0]} \Pi_2 \\ * \Upsilon_2 \end{bmatrix} < 0,$$
$$\begin{bmatrix} \bar{\Xi}_{[\tau(t)=0]} - \tau^2 \varrho \Pi_2 \\ * \Upsilon_2 \end{bmatrix} < 0, \quad (40)$$
$$P_2 + \theta_p > 0, \quad (41)$$
$$R_a > 0,\ R_b > 0, \quad (42)$$
$$e^{-\alpha T_f} P_1 - B_0^T \omega_4 B_0 > 0, \quad (43)$$
$$e^{\alpha T_f} \left[\epsilon_{11} d + (\epsilon_{12} + \epsilon_{13}) \omega_f \right] < \epsilon_0 c_2. \quad (44)$$
$$\begin{aligned} &\epsilon_0 I \le \hat{P}_1 \le \epsilon_1 I,\ 0 \le \hat{P}_2 \le \epsilon_2 I, \\ &0 \le \hat{Q}_1 \le \epsilon_3 I,\ 0 \le \hat{Q}_2 \le \epsilon_4 I, \\ &0 \le \hat{R}_1 \le \epsilon_5 I,\ 0 \le \hat{R}_2, \le \epsilon_6 I, \\ &0 \le \hat{R}_3 \le \epsilon_7 I,\ 0 \le \hat{S}_1 \le \epsilon_8 I, \\ &0 \le \hat{Y} \le \epsilon_9 I,\ 0 \le M_a \le \epsilon_{10} I, \end{aligned} \quad (45)$$

Proof: By using LKF and the proof of Theorem 1, we have

$$\dot{V}(z_t, t) - \alpha V(z_t, t) - J(t) < \xi^T(t) \bar{\Xi}_{\tau(t)} \xi(t) < 0. \quad (46)$$

By multiplying of (46) with $e^{-\alpha t}$ and integrating on $[0, t]$, we obtain

$$V(z_t, t) < e^{\alpha t} \left[V(z_0, 0) + \int_0^t J(s)ds \right].$$

From the condition $V(z_0, 0) = 0$ and $0 < z^T(t) P_1 z(t) < V(z_t, t)$, we get

$$0 < e^{-\alpha t} z^T(t) P_1 z(t) < e^{-\alpha t} V(z_t, t) < \int_0^t J(s)ds. \quad (47)$$

According to Assumption 4, consider the two case $\Omega_4 = 0$ and $\Omega_4 > 0$.

case I When $\Omega_4 = 0$,

$$\int_0^{T_f} J(t)dt - \sup_{0\le t\le T_f} y^T(t) \Omega_4 y(t) = \int_0^{T_f} J(t)dt \ge 0.$$

case II When $\Omega_4 > 0$, we have $\Omega_1 = 0$, $\Omega_2 = 0$ and $\Omega_3 > 0$.

From (47) and for all $t \in [0, T_f]$, we can get

$$\int_0^{T_f} J(s)ds \ge \int_0^t J(s)ds > e^{-\alpha t} z^(t) P_1 z(t) > 0.$$

According to condition (43), we obtain

$$\int_0^{T_f} J(s)ds \ge z^T(t) B_0^T \Omega_4 B_0 z(t) = y^T(t) \Omega_4 y(t).$$

Hence, we get

$$\int_0^{T_f} J(s)ds - \sup_{0\le t\le T_f} y^T(t)\Omega_4 y(t) \ge 0.$$

So, the extended dissipativity performance proof is finished. Next, we prove the finite time boundedness as follows.

$$V(z_t, t) < e^{-\alpha t}\int_0^t J(s)ds.$$

For $\Omega_1 \le 0$, we get

$$V(z_t, t) < e^{-\alpha t}\int_0^t [2y^T(s)\Omega_2\omega(s) + \omega^T(s)\Omega_3\omega(s)]ds.$$

From $V(z_t, t) \ge z^T(t)P_1 z(t) \ge \lambda_{\min}(\hat{P})z^T(t)Vz(t) = \epsilon_0 z^T(t)Vz(t)$, it can be expressed as

$$\begin{aligned} z^T(t)Vz(t) &\le \frac{e^{\alpha T_f}}{\epsilon_0}\int_0^{T_f}[2y^T(s)\Omega_2\omega(s) + \omega^T(s)\Omega_3\omega(s)]ds \\ &= \frac{e^{\alpha T_f}}{\epsilon_0}\int_0^{T_f}[2z^T(s)B_0^T\Omega_2\omega(s) + \omega^T(s)\Omega_3\omega(s)]ds. \end{aligned}$$

By applying Lemma 6, we obtain

$$2z^T(t)B_0^T\Omega_2\omega(t) \le z^T(t)B_0^TB_0 z(t) + \omega^T(t)\Omega_2^T\Omega_2\omega(t).$$

From Assumption (2) and (3), we get

$$\begin{aligned} z^T(t)Vz(t) &\le \frac{e^{\alpha T_f}}{\epsilon_0}\int_0^{T_f}[z^T(t)B_0^TB_0 z(t) + \omega^T(t)\Omega_2^T\Omega_2\omega(t) \\ &\quad + \omega^T(s)\Omega_3\omega(s)]ds \\ &\le \frac{e^{\alpha T_f}}{\epsilon_0}\left[\epsilon_{11}d + (\epsilon_{12}+\epsilon_{13})\omega_f\right]. \end{aligned}$$

From condition (44), we obtain

$$z^T(t)Vz(t) < c_2.$$

As a result, the system (39) is finite-time bounded with an extended dissipativity. The proof is now complete. □

IV. NUMERICAL EXAMPLES

Next, we show numerical examples to demonstrate the efficientcy of the present results.

Example 1: Consider the generalized neural networks described in (5) with the following matrix parameters:

$$\begin{aligned} A_0 &= \text{diag}\{8.2345, 7.1258, 6.9563\}, \\ F_m &= G_m = H_m = \text{diag}\{0, 0, 0\}, \\ F_p &= \text{diag}\{0.3457, 0.5378, 0.1852\}, \\ G_P &= \text{diag}\{1.2539, 0.1258, 0.5971\}, \\ H_P &= \text{diag}\{1.7509, 0.0211, 0.0913\}, \\ A_1 &= \begin{bmatrix} 1.2357 & -1.5634 & 1.6938 \\ -1.5361 & 1.3208 & -1.7030 \\ 1.8239 & -1.4675 & 1.6998 \end{bmatrix}, \end{aligned}$$

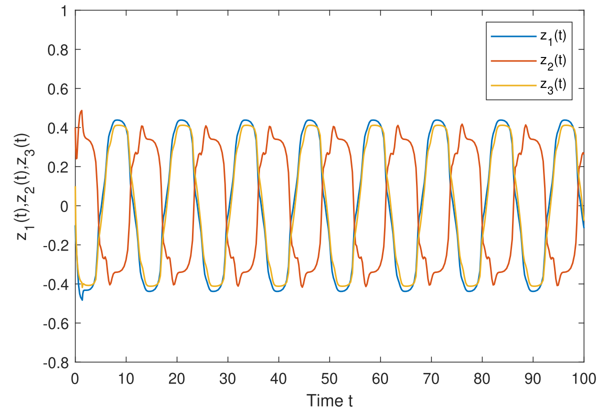

FIGURE 1. The trajectories of $z_1(t)$, $z_2(t)$ and $z_3(t)$ of system (5) in Example 1.

$$\begin{aligned} A_2 &= \begin{bmatrix} 0.88 & 1.22 & 1.02 \\ 1.57 & 1.07 & 0.33 \\ 1.55 & 0.92 & 1.11 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 1.35 & 0.25 & 0.64 \\ -1.82 & -0.29 & -0.12 \\ 0.36 & 0.87 & 1.11 \end{bmatrix}, \\ A_4 &= \begin{bmatrix} 0.2 & -0.6 & 0.8 \\ 0.3 & -0.2 & 0.2 \\ 0.1 & -0.5 & 0.7 \end{bmatrix}, \\ W &= \begin{bmatrix} 12.3654 & 2.5876 & -0.9782 \\ 7.5867 & 22.5513 & 3.5236 \\ 0.8562 & -2.7190 & -21.5037 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} f(z) &= [0.3457\tanh(z_1),\ 0.5378\tanh(z_2), 0.1852\tanh(z_3)]^T, \\ g(z) &= [1.2539\tanh(z_1),\ 0.1258\tanh(z_2),\ 0.5971\tanh(z_3)]^T, \\ h(z) &= [1.7509\tanh(z_1),\ 0.0211\tanh(z_2),\ 0.0913\tanh(z_3)]^T. \end{aligned}$$

Let the discrete time-varying is $\tau(t) = 0.8 + 0.5\sin(t)$, the distributed time-varying delays is $\gamma(t) = 0.4 + 0.2\sin(t)$ and the external disturbance is $\omega(t) = \frac{1}{1+e^t}$. For given scalars $\tau = 0.5$, $\omega_f = 0.1$, $c_1 = 1.12$, $T = 30$, $\alpha = 0.1$ and V is identity matrix. Solving LMIs (6)-(10) in Theorem 1, we obtain $c_2 = 3.56$.

For an initial condition $\phi(t) = [-0.1 \quad 0.4 \quad 0.1]^T$, figure 1 demonstrates the trajectories of solutions $z_1(t)$, $z_2(t)$, and $z_3(t)$ of generalized neural networks (5) with discrete time-varying delay ($\tau(t)$) and distributed time-varying delay ($\gamma(t)$) via various activation functions $f(z)$, $g(z)$, and $h(z)$. Figure 2 illustrates the time history of $z^T(t)z(t)$ for the delay generalized neural network system (5). In conclusion, system (5) is finite-time boundedness with respect to (1.12, 3.56, 30, I, 0.1). Thus, this proves the effectiveness of our obtained results in Theorem 1.

Example 2: Consider the generalized neural networks described in (29) with the following matrix parameters:

$$\begin{aligned} &A_0 = \text{diag}\{2, 2\},\ F_m = G_m = H_m = \text{diag}\{0, 0\}, \\ &F_p = \text{diag}\{0.2, 0.3\},\ G_P = \text{diag}\{0.4, 0.6\}, \end{aligned}$$

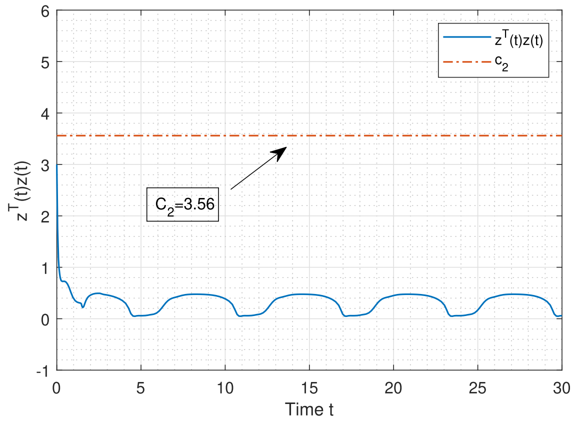

FIGURE 2. Time history of $z^T(t)z(t)$ for closed-loop system in Example 1.

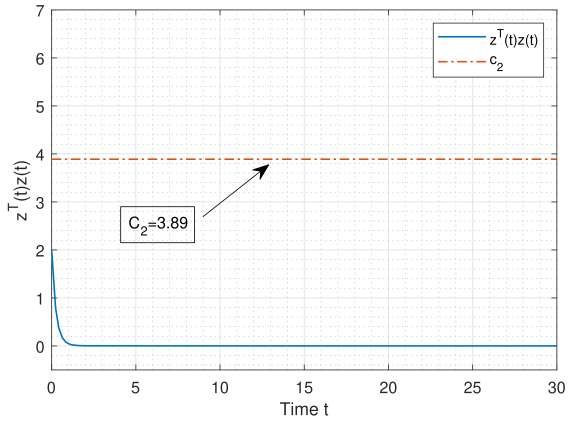

FIGURE 4. Time history of $z^T(t)z(t)$ for closed-loop system in Example 2.

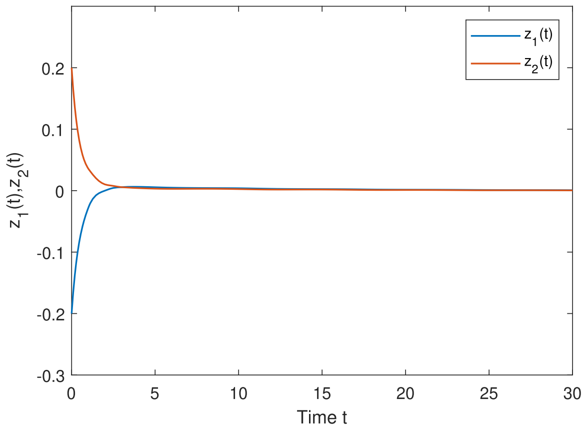

FIGURE 3. The trajectories of $z_1(t)$ and $z_2(t)$ of system (29) in Example 2.

$$H_P = \text{diag}\{1, 0.5\},\ A_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix},\ A_3 = \begin{bmatrix} 0.2 & -0.6 \\ 0.3 & 0.2 \end{bmatrix},\ W = \begin{bmatrix} 1.35 & 0.45 \\ 0.21 & 1.29 \end{bmatrix},$$

$$f(z) = [0.2\tanh(z_1),\ 0.3\tanh(z_2)]^T,$$

$$g(z) = [0.2(|z_1+1| - |z_1-1|),\ 0.3(|z_2+1| - |z_2-1|)]^T$$

$$h(z) = [\tanh(z_1),\ 0.5\tanh(z_2)]^T.$$

Let the discrete time-varying is $\tau(t) = 0.7 + 0.4\sin(t)$, the distributed time-varying delays is $\gamma(t) = 0.9+0.3\sin(t)$. For given scalars $\tau = 0.5$, $c_1 = 1$, $\omega_f = 0.1$, $T = 30$, $\alpha = 0.1$ and V is identity matrix. Solving LMIs (30)-(34) in Corollary 1, we obtain $c_2 = 3.89$.

For an initial condition $\phi(t) = [-0.2 \quad 0.2]^T$, figure 1 demonstrates the trajectories of solution $z_1(t)$ and $z_2(t)$ of generalized neural networks (29) with various activation functions and mixed time-varying. Figure 2 illustrates the time history of $z^T(t)z(t)$ for the delay generalized neural network system (29). In conclusion, system (29) is finite-time stable with respect to (1, 3.89, 30, I, 0.1). Thus, this proves the effectiveness of our obtained results in Corollary 1.

TABLE 1. Maximum allowable bounds of τ with different τ_d in Example 3.

τ	0.00	0.10	0.50
[39]	1.5575	0.9430	0.4417
[40]	1.6409	0.9962	0.4470
[11]	1.7250	1.0408	0.4480
[41]	1.7302	1.0453	0.4486
[42]	1.8898	1.1114	0.4514
[43]	1.8899	1.1194	0.4599
[5]	1.9349	1.1365	0.4678
[35]	3.1150	1.4410	1.0299
Corollary 2	3.9574	1.9521	1.8366

Example 3: Consider the neural networks described in (35) with the following matrix parameters:

$$A = \text{diag}\{7.3458, 6.9987, 5.5949\},\ W_0 = \text{diag}\{0, 0, 0\},$$

$$W_1 = \text{diag}\{1, 1, 1\},\ F_m = \text{diag}\{0, 0, 0\},$$

$$F_p = \begin{bmatrix} 0.3680 & 0 & 0 \\ 0 & 0.1795 & 0 \\ 0 & 0 & 0.2876 \end{bmatrix},$$

$$W = \begin{bmatrix} 13.6014 & -2.9616 & -0.6936 \\ 7.4736 & 21.6810 & 3.2100 \\ 0.7290 & -2.6334 & -20.1300 \end{bmatrix},$$

$$f(z) = [0.3680\tanh(z_1),\ 0.1795\tanh(z_2),\ 0.2876\tanh(z_3)]^T.$$

Table 1 lists the proposed criteria, the maximum delay bounds with τ calculated by the Corollary 2. Furthermore particular, we compare the obtained results to those that have already been published. The results demonstrate that the stability conditions given in this article are more efficient than those described in the previous literature.

Example 4: Consider the neural networks described in (35) with the matrix parameters in the following:

$$A = \text{diag}\{1.5, 0.7\},\ W = \text{diag}\{1, 1\},$$

$$F_p = \text{diag}\{0.3, 0.8\},\ F_m = \text{diag}\{0, 0\},$$

$$W_0 = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix},\ W_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix}.$$

Let the neuron activation function is taken as $f(z) = [0.3\tanh(z_1),\ 0.8\tanh(z_2)]^T$. Table 2 displays the proposed

TABLE 2. Maximum allowable bounds of τ with different τ_d in Example 4.

τ	0.40	0.45	0.50	0.55
[44]	4.6569	3.7268	3.4076	3.2841
[45]	4.5543	3.8364	3.5583	3.4110
[46]	7.6697	6.7287	6.4126	3.2569
[41]	8.3498	7.3817	7.0219	6.8156
[38]	10.1095	8.6732	8.1733	7.8993
[34]	10.5730	9.3566	8.8467	8.5176
[43]	16.8020	11.6745	9.9098	9.0062
[5]	17.2697	12.0698	10.2903	9.3879
[35]	19.5194	12.2110	12.4201	10.3990
Corollary 2	20.0598	13.0115	13.2116	11.3612

TABLE 3. Maximum allowable bounds of τ with different τ_d in Example 5.

τ	0.80	0.90
[47]	4.5940	3.4671
[11]	4.8167	3.4245
[48]	5.4428	3.6482
[34]	5.6384	3.7718
[5]	6.7186	3.9623
[35]	8.5200	4.0979
Corollary 2	9.2613	4.9861

conditions and maximum delay bounds computed by Corollary 2. In addition, we compare the obtained results to those of previously published studies. The results demonstrate that the stability conditions presented in this paper are greater than those found in the existing literature.

Example 5: Consider the neural networks described in (35) with the matrix parameters as follows:

$$A = \text{diag}\{2, 2\},\ W = \text{diag}\{1, 1\},$$
$$F_p = \text{diag}\{0.4, 0.8\},\ F_m = \text{diag}\{0, 0\},$$
$$W_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},\ W_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let the neuron activation function is taken as $f(z) = [0.4\tanh(z_1),\quad 0.8\tanh(z_2)]^T$. *The proposed criteria, the maximum delay bounds with τ estimated by the Corollary 2 are shown in Table 3. Furthermore, we compare the results with previously published research. The results suggest that the stability conditions shown in this paper are superior to those previously outlined in the literature.*

Example 6: Consider the neural networks described in (35) with the matrix parameters as follows:

$$A_0 = \text{diag}\{1.2769, 0.6231, 0.9230, 0.4480\},$$
$$W = \text{diag}\{1, 1, 1, 1\},\ F_m = \text{diag}\{0, 0, 0, 0\},$$
$$F_p = \text{diag}\{0.1137, 0.1278, 0.7994, 0.2368\},$$
$$A_1 = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}.$$

TABLE 4. Maximum allowable bounds of τ with different τ_d in Example 6.

τ_d	0.10	0.50	0.90
[49]	3.65	3.32	3.26
[50]	3.78	3.45	3.39
[51]	4.19	3.62	3.59
[52]	5.45	4.65	4.57
[36]	30.22	29.03	28.02
Corollary 2	31.06	30.92	29.16

TABLE 5. Matrices for each case of extend dissipativity performance.

Method	Ω_1	Ω_2	Ω_3	Ω_4
H_∞ performance	-I	0	$\gamma_d^2 I$	0
$\mathcal{L}_2 - \mathcal{L}_\infty$ performance	0	0	$\gamma_d^2 I$	I
Passivity performance	0	I	$\gamma_d I$	0
Dissipativity performance	-I	I	$(2-\beta)I$	0

Table 4 displays the proposed criteria, the maximum delay bounds with τ computed by Corollary 2. Also, we compare the obtained results to previously published research. The results suggest that this paper's stability conditions are better than those stated in previous publications.

Remark 11: This paper extends the proof by incorporating Jensen's integral inequality, extended Wirtinger's integral inequalities, and orthogonal polynomials-based integral inequality with improved LKFs. Consequently, our maximum delay outperforms the existing literature, as presented in Tables 1–4.

Example 7: Consider the generalized neural networks described in (39) with the following matrix parameters:

$$A_0 = \text{diag}\{1, 1\},\ W = \text{diag}\{1, 1\},$$
$$F_m = G_m = H_m = \text{diag}\{0, 0\},$$
$$F_p = \text{diag}\{0.12, 0.28\},\ G_P = \text{diag}\{0.24, 0.38\},$$
$$H_P = \text{diag}\{0.35, 0.49\},\ A_1 = \begin{bmatrix} 1.188 & 0.09 \\ 0.09 & 1.188 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.09 & 0.14 \\ 0.05 & 0.09 \end{bmatrix},\ A_3 = \begin{bmatrix} 0.44 & -0.21 \\ 0.29 & 0.41 \end{bmatrix},$$

Let the discrete time-varying is $\tau(t) = 0.7 + 0.5\sin(t)$, *the distributed time-varying delays is* $\gamma(t) = 0.9 + 0.5\sin(t)$ *and the external disturbance is* $\omega(t) = \sqrt{0.1}cos(t)$. *For given scalars* $\tau = 0.5$, $c_1 = 1.2$, $\omega_f = 0.1$, $T = 30$, $\alpha = 0.1$ *and V is identity matrix. Solving LMIs (40)-(44) in Theorem 2, we obtain* $c_2 = 4.12$.

For an initial condition $\phi(t) = [-1 \quad 1]^T$, *figure 5 demonstrates the trajectories of solution $z_1(t)$ and $z_2(t)$ of generalized neural networks (39) with discrete time-varying delay ($\tau(t)$) and distributed time-varying delay ($\gamma(t)$) via various activation functions* $f(z) = [0.12\tanh(z_1),\ 0.28\tanh(z_2)]^T$, $g(z) = [0.12(|z_1+1|-|z_1-1|),\ 0.19(|z_2+1|-|z_2-1|)]^T$, *and* $h(z) = [0.35\tanh(z_1),\ 0.49\tanh(z_2)]^T$. *Figure 6 illustrates the time history of $z^T(t)z(t)$ for the delay generalized neural network system (39). In conclusion, system (39) is finite-time stable with respect to (1.2, 4.12, 30, I, 0.1). Thus, this proves the effectiveness of our obtained results in Theorem 2.*

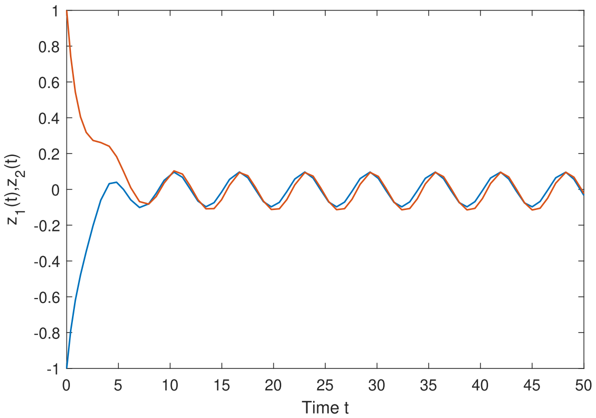

FIGURE 5. The trajectories of $z_1(t)$ and $z_2(t)$ of system (5) in Example 7.

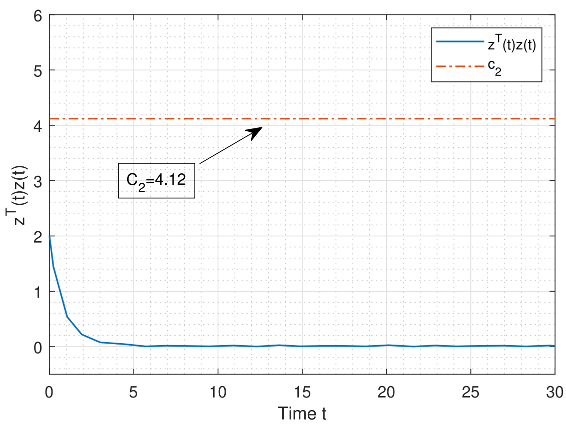

FIGURE 6. Time history of $z^T(t)z(t)$ for closed-loop system in Example 7.

TABLE 6. Minimum γ_d and Maximum β for different values of τ_d in Example 6.

Method	$\tau_d = 0.10$	$\tau_d = 0.50$	$\tau_d = 0.90$
H_∞ performance	0.0871	0.1521	0.1856
$\mathcal{L}_2 - \mathcal{L}_\infty$ performance	0.2511	0.3678	0.5013
Passivity performance	0.0197	0.0427	0.0599
Dissipativity performance	1.9895	1.9912	1.9976

V. CONCLUSION

This paper employed the improved LKF to investigate the problem of finite-time extended dissipativity for generalized neural networks with time-varying delays. To estimate the bound of the time derivative, we constructed a suitable LKF and utilized effective inequalities, including orthogonal polynomials-based integral inequality, Jensen's integral inequality, and extended Wirtinger's integral inequality. This allowed us to obtain several sufficient conditions as linear matrix inequalities (LMIs). This article is less conservative than other recently published publications by stability criteria. However, there are numerical examples to demonstrate that the presented results work and are better compared to [5], [11], [34], [35], [36], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], and [52]. In future work, this work can be extended to many dynamical systems, such as neutral-type neural networks and T-Sfuzzy neural networks, with more efficient techniques [53], [54], [55], [56], [57], [58], [59], [60], [61], [62].

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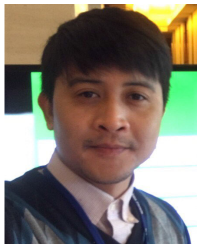

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