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RESEARCH ARTICLE

A Novel Framework of Pythagorean Fuzzy Dominance-Based Rough Sets and Analysis of Knowledge Reductions

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ABSTRACT The dominance–based rough set approach is crucial to the advancement of rough set theory. It gives a more thorough and adaptable framework for knowledge acquisition, information analysis, and DM. It is a means of expressing discrepancies resulting from the examination of the domains with specified preference rankings of the characteristics. This work seeks to extend the Rough set approach utilizing dominance relationships to a Pythagorean fuzzy setting. The lower and upper approximations of Pythagorean fuzzy dominance–based rough set are determined by using the constructive technique. Next, we examine the basic characteristics for the rough estimations relying on the Pythagorean fuzzy dominance. By combining Approximate Distribution Reductions with a Pythagorean fuzzy dominance–based rough set, reductions are prescribed in four distinct manners. Additionally, the discernibility matrices and theorems connected to these reductions are produced. Such findings are all Pythagorean fuzzy generalizations or extensions of the conventional rough set method relying on dominance. Finally, the conceptual ideas are supported with a numerical example.

INDEX TERMS Dominance–based rough set approach, dominance–based fuzzy rough set approach, Pythagorean fuzzy dominance–based rough set approach, approximate distribution reduct, approximate distribution consistent set.

LIST OF ABBREVIATIONS

Symbols	Description
DRSA	Dominance–based Rough Set Approach
DFRSA	Dominance–based Fuzzy Rough Set Approach
PFDRSA	Pythagorean Fuzzy Dominance–based Rough Set Approach
ADDM	Approximate Distribution Discernibility Matrix
ADCS	Approximate Distribution Consistent Set
ADR	Approximate Distribution Reduct
FDR	Fuzzy Dominance Relation
$\mathcal{L}/\mathcal{U}$ Approxs	Lower/Upper Approximations

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I. INTRODUCTION

To cope with ambiguous and unclear information, Pawlak's RST [1], [2], [3], [4] was devised. Essential ideas within rough set theory comprise operators for lower and upper approximations, these concepts were established based on an equivalence relation that possesses the characteristics of reflexivity, symmetry, and transitivity. With the aid of these two approximations, it is possible to uncover and produce in the shape of decision rules any knowledge concealed inside the information systems.

The indiscernibility relation is generally recognized to be excessively constrictive for classification analysis in real–world applications. Due to this, several writers have summarized the ideas of rough approximations by employing some more broad binary connections, like the tolerance connection [5], similarity connection [6], characteristic connection [7], and so forth. These enhancements for the rough approximations have practical applications in reasoning and knowledge development within incomplete systems [8], [9], [10], [11], continuing–valued systems [12], and some more advanced information systems. So the expansion of rough approximations to fuzzy settings has been a significant factor in the growth of RST.

By employing the vagueness connection to approximate a fuzzy idea, Dubios and Prade [13] introduced a model for a FRSs. On the other hand, a FS within a fuzzy estimation space is approximated through a fuzzy rough model. The interval–valued FRS was proposed by Gong et al. [14] by merging the interval–valued FS and rough set. In the context of the conversation, Cornelis et al. [15] established the notion of IFRSs, where they $\mathfrak{L}$ *and* $\mathfrak{U}$ Approxs are together IFSs. The IF rough approximation was put out by Zhou et al. [16] from the perspectives of constructive and axiomatic methods. The fuzzy-rough sets were presented by Bhatt and Gopal [17] on a small computational domain. To make a fuzzy rough set a particular instance, Zhao et al. [18] studied the rough sets with fuzzy variable precision. A fuzzy RS established on Gaussian Kernel was suggested by Hu et al. [19], by employing the Gaussian Kernel, the computation of the fuzzy T–equivalence relation facilitates objective approximation. The rough sets with fuzzy preferences were also proposed by the same authors [20]. The following articles include further information on current developments in fuzzy rough sets: [21], [22], [23], [24], [25], [26].

Although the rough set has proven its utility in various domains like Information retrieval and analysis, it cannot identify inconsistencies that may arise due to the deliberation of criteria. These standards consist of characteristics with prioritized realm, like product excellence, market position, and interest coverage ratio. To address this issue, Greco et al. suggested the DRSA [27], [28], [29], [30], which is a development of Pawlak's RST. The foundation of this innovation lies in the replacement of an indiscernible connection with a dominance relation. Currently, the development of a DRSA is also moving quickly. For instance, Shao et al. [31] and Yang et al. [32] extended the DRSA to handle partial scenarios by incorporating two distinct approaches to providing semantic explanations for unfamiliar values. Wei et al. [33] proposed the concept of valued dominance–based approximate approximations, which incorporated the notion of the RS with variable precision [34] in the direction of DRSA. Bszczynki et al. [35] offered the method of variable consistency for DRSA. Kotowski et al. [36] introduced a novel DRSA Approxs by utilizing a probabilistic model to address the ordinal regression issues. Greco et al. [37] extended the DRSA to a fuzzy setting and introduced the DFRSA by applying a fuzzy dominance relation.

Yager [38] proposed the notion of the PFS as an extension of the Zadeh's FS. In the PFS, the sum of the squares of the MG and the NMG is restricted from zero to one. Our research combines the Pythagorean fuzzy set with the DRSA technique to propose an innovative model known as PFDRS. The PFDRS is a fresh approach to the conventional DRSA by employing a Pythagorean FDR for the estimation of the ascending and descending unions of the decision classes rather than a crisp or FDR. In our research, we utilize a constructive method to define the PFDRS. The dominance relation in classical DRSA can only be used to determine if one object dominates another. Furthermore, the FDR is introduced to represent the plausibility that one object is superior to another. In a fuzzy dominance relationship, when an element x is superior to an other element y with a certain level of credibility, it implies that x does not superior than y completely. To broaden this concept, the Pythagorean fuzzy method is inherently incorporated into DRSA. Within our PFDRS, the Pythagorean FDR can depict not just the credibility of x dominating y but also the lack of credibility in x dominating y.

After introducing a novel rough set model, the initial challenge that arises is attribute reduction. This process involves identifying particular subsets of features that offer equivalent information, serving a specific objective, as the entire collection of current attributes. Their subsets are referred to as reductions. Pawlak introduced the positive–region based reduct in classic RST [39], which can be utilized to maintain the union of all L Approxs. Kryszkiewicz [40] examined and analyzed five data reduction theories in inconsistent informations after Pawlak's work, while the notions of distribution reducts and maximal distribution reducts were developed by Zhang et al. [41]. Furthermore, Wang et al. [42] introduced a methodical strategy for reducing knowledge by utilizing a relation and it associate with rough estimation. Chen et al. [43], on the other hand, explored the issue of knowledge reduction in decision systems by employing a covering based–rough estimation. Yang et al. [44] developed an innovative theory of reductions by modifying the approximation space and reducts in the scope of covering extended RSs. Beynon [45] introduced the notion of reducts using the RS environment with variable precision, while Mi et al. [46] presented lower and upper ADRs. In our research, we aim to incorporate Mi's approximate distribution reducts into our PFDRS. Since DRSA involves two sets of approximations, we provide four different notions of ADRs. Furthermore, we develop different theorems and discernibility matrices linked to these reduction, enabling us to derive practical techniques for computing ADRs within the framework of PFDRS.

The subsequent sections of this manuscript are arranged as follows: In section II, the basic ideas of PFS, classical DRSA, and DFRSs are concisely examined to ease our discussion. Section III presents a constructive method for defining the PFDRS model with some subsequent characteristics. Section IV introduces the probabilistic interpretation of PFDRS, we also present an example for illustrating the utilization of PFDRS in decision systems with a probabilistic interpretation. Section V investigates the ADRs for PFDRS. In section VI, this method is demonstrated through an example. Section VII provides the concluding remarks of this manuscript.

II. PRELIMINARIES

This passage provides a condensed overview of essential details related to IFSs, PFSs, DRSA, DFRSA, and DIFRSA, along with fundamental concepts pertinent to the article's content. Additionally, it covers prevalent ideas relevant to sequential analysis.

A. IFSs, PFSs

Atanassov [47] introduced IFS as an adaptation of Zadeh's FST [48].

Definition 1 [47]: For universe U, and for $\mathfrak{x} \in U$. The IFS '$\mathfrak{I}$' is described over U as:

$$\mathfrak{I} = \{(\mathfrak{x}, u_{\mathfrak{i}}(\mathfrak{x}), v_{\mathfrak{i}}(\mathfrak{x})) : \mathfrak{x} \in U\},$$

where, $u_{\mathfrak{i}}(\mathfrak{x}) \in [0, 1]$ denotes the MG and $v_{\mathfrak{i}}(\mathfrak{x}) \in [0, 1]$ denotes the NMG of $\mathfrak{x}$ to $\mathfrak{I}$, correspondingly, with $0 \leq u_{\mathfrak{i}}(\mathfrak{x}) + v_{\mathfrak{i}}(\mathfrak{x}) \leq 1, \forall \mathfrak{x} \in U$.

Yager [38] introduced the concept of PFS, indicating that it holds when the sum of MG and NMG surpasses 1, unlike IFSs which do not succeed under such circumstances.

Definition 2 [38]: For universe U, and for $\mathfrak{x} \in U$. The PFS 'P' is described over U as:

$$\mathrm{P} = \{(\mathfrak{x}, u_{\mathrm{P}}(\mathfrak{x}), v_{\mathrm{P}}(\mathfrak{x})) : \mathfrak{x} \in U\},$$

where, $u_{\mathrm{P}} \in [0, 1]$ denotes the MG and $v_{\mathrm{P}} \in [0, 1]$ denotes the NMG of $\mathfrak{x}$ to P, correspondingly, with $0 \leq (u_{\mathrm{P}}(\mathfrak{x}))^2 + (v_{\mathrm{P}}(\mathfrak{x}))^2 \leq 1$. The degree of indeterminacy is $\pi_{\mathrm{P}}(\mathfrak{x}) = \sqrt{1 - (u_{\mathrm{P}}(\mathfrak{x}))^2 - (v_{\mathrm{P}}(\mathfrak{x}))^2}$.

B. GRECO'S DRSA

Definition 3 [29]: A pair $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$ is a decision system, U refers to the universe is a non–empty set of finite objects. $\mathcal{AT}$ conditional attributes is a non–empty finite set, decision attribute is f with $\mathcal{AT} \cap \{f\} = \emptyset$. For all $\mathfrak{q} \in \mathcal{AT}$, $V_{\mathfrak{q}}$ is the realm of attribute $\mathfrak{q}$, therefore $V = V_{\mathcal{AT}} = \bigcup_{\mathfrak{q} \in \mathcal{AT}} V_{\mathfrak{q}}$ is the domain of all attributes. Furthermore, $\forall \mathfrak{x} \in U$, $\mathfrak{q}(\mathfrak{x})$ is the value that $\mathfrak{x}$ holds on $\mathfrak{q}$ ($\mathfrak{q} \in AT$).

For preference–ordered realm of attributes, the DRSA, which has been offered by Greco et al. [27] is a continuation of the classical rough set that can address the inconsistencies common for outstanding choices in MCDM situations. Consider $\succcurlyeq_{\mathfrak{q}}$ denotes a weak preference connection for the set U (commonly known as outranking) which represents a preference among objects based on a criterion $\mathfrak{q}(\mathfrak{q} \in \mathcal{AT})$, also called dominance relation; $\mathfrak{x} \succcurlyeq_{\mathfrak{q}} y$ means "$\mathfrak{x}$ is at least as good as y with respect to criterion $\mathfrak{q}$". We assert that $\mathfrak{x}$ dominates y w.r.t $\mathcal{A} \subseteq \mathcal{AT}$, if and only if $\mathfrak{x} \succcurlyeq_{\mathfrak{q}} y$ for every $\mathfrak{q} \in \mathcal{A}$.

Thus, we have the ability to establish two sets for every $\mathfrak{x}$ in U:

- "The set of objects that dominate $\mathfrak{x}$ *i.e.* $[\mathfrak{x}]_{\mathcal{A}}^{\succcurlyeq} = \{y \in U : \forall \mathfrak{q} \in \mathcal{A}, y \succcurlyeq_{\mathfrak{q}} \mathfrak{x}\}$"
- "The set of objects that dominated by $\mathfrak{x}$, *i.e.* $[\mathfrak{x}]_{\mathcal{A}}^{\preccurlyeq} = \{y \in U : \forall \mathfrak{q} \in \mathcal{A}, \mathfrak{x} \succcurlyeq_{\mathfrak{q}} y\}$"

In this context, we presume that the attribute f, within the traditional DRSA framework, defines the partition of U into a limited set of classes. Suppose $\mathcal{CL} = \{\mathcal{CL}_{\mathfrak{y}}, \mathfrak{y} \in \aleph\}, \aleph = \{1, 2, \ldots, \mathfrak{m}\}$, be a collection of ordered classes. Contrary to the Pawlak's rough estimation, the sets that need to be estimated in DRSA can be viewed as an ascending and descending combinations of decision classes. That are specified as:

$$\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq} = \bigcup_{\mathfrak{y}' \succcurlyeq \mathfrak{y}} \mathcal{CL}_{\mathfrak{y}'}, \quad \mathcal{CL}_{\mathfrak{y}}^{\preccurlyeq} = \bigcup_{\mathfrak{y}' \preccurlyeq \mathfrak{y}} \mathcal{CL}_{\mathfrak{y}'} \ \ for\ \mathfrak{y}, \mathfrak{y}' \in \aleph;$$

The $\mathcal{A} - \mathcal{L}$ *and* $\mathcal{U}$ Approxs of $\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}$ are defined as:

$$\underline{\mathcal{A}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}) = \left\{\mathfrak{x} \in U : [\mathfrak{x}]_{\mathcal{A}}^{\succcurlyeq} \subseteq \mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}\right\},$$
$$\overline{\mathcal{A}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}) = \left\{\mathfrak{x} \in U : [\mathfrak{x}]_{\mathcal{A}}^{\preccurlyeq} \cap \mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq} \neq \emptyset\right\};$$

The $\mathcal{A} - \mathcal{L}$ *and* $\mathcal{U}$ Approxs of $\mathcal{CL}_{\mathfrak{y}}$ are defined as:

$$\underline{\mathcal{A}}(\mathcal{CL}_{\mathfrak{y}}^{\preccurlyeq}) = \left\{\mathfrak{x} \in U : [\mathfrak{x}]_{\mathcal{A}}^{\preccurlyeq} \subseteq \mathcal{CL}_{\mathfrak{y}}^{\preccurlyeq}\right\},$$
$$\overline{\mathcal{A}}(\mathcal{CL}_{\mathfrak{y}}^{\preccurlyeq}) = \left\{\mathfrak{x} \in U : [\mathfrak{x}]_{\mathcal{A}}^{\succcurlyeq} \cap \mathcal{CL}_{\mathfrak{y}}^{\preccurlyeq} \neq \emptyset\right\};$$

The $\mathcal{A} - \mathcal{BND}$ of $\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}$ and $\mathcal{CL}_{\mathfrak{y}}^{\preccurlyeq}$ are defined as:

$$\mathcal{BND}_{\mathcal{A}}\left(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}\right) = \overline{\mathcal{A}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}) - \underline{\mathcal{A}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}),$$
$$\mathcal{BND}_{\mathcal{A}}\left(\mathcal{CL}_{\mathfrak{y}}^{\preccurlyeq}\right) = \overline{\mathcal{A}}(\mathcal{CL}_{\mathfrak{y}}^{\preccurlyeq}) - \underline{\mathcal{A}}(\mathcal{CL}_{\mathfrak{y}}^{\preccurlyeq}).$$

C. DFRSA

The dominance–based fuzzy rough set approach is an expanded version of DRSA that incorporates fuzzy concepts with DRSA. In the DFRSA, the traditional dominance relation is substituted with a fuzzy dominance relation.

Definition 4 [30]: Suppose that $R_{\mathfrak{q}}$ is a fuzzy dominance relation over U w.r.t attribute $\mathfrak{q}$, *i.e.* $R_{\mathfrak{q}} : U \times U \rightarrow [0, 1]$, $\forall \mathfrak{x}, y \in U$. Then, $R_{\mathfrak{q}}(\mathfrak{x}, y)$ demonstrates the reliability of the statement "$\mathfrak{x}$ is at least as good as y with respect to attribute $\mathfrak{q}$". For each $\mathcal{A} \subseteq \mathcal{AT}$, the FDR over U is expressed and established as:

$$R_{\mathcal{A}}(\mathfrak{x}, y) = \wedge \left\{R_{\mathfrak{q}}(\mathfrak{x}, y) : \mathfrak{q} \in \mathcal{A}\right\}.$$

Definition 5 [30]: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$ with $\mathcal{A} \subseteq \mathcal{AT}$ then for all $\mathfrak{y} \in \aleph$, the $\mathcal{A} - \mathcal{L}$ *and* $\mathcal{U}$ Approxs of $\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}$ with FDR are represented by $\underline{\mathcal{A}_R}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq})$ and $\overline{\mathcal{A}_R}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq})$

correspondingly, of whom memberships for every $\chi \in U$, are termed as:

$$u_{\underline{A_R}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi) = \wedge_{y \in U}\left(u_{\mathcal{CL}_\eta^{\succcurlyeq}}(y) \vee (1 - R_{\mathcal{A}}(y, \chi))\right);$$

$$u_{\overline{A_R}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi) = \vee_{y \in U}\left(u_{\mathcal{CL}_\eta^{\succcurlyeq}}(y) \wedge R_{\mathcal{A}}(\chi, y)\right);$$

Similarly, the $\mathcal{A}-\mathcal{L}$ *and* $\mathcal{U}$ Approxs of $\mathcal{CL}_\eta^{\preccurlyeq}$ are expressed by $\underline{A_R}(\mathcal{CL}_\eta^{\preccurlyeq})$ and $\overline{A_R}(\mathcal{CL}_\eta^{\preccurlyeq})$ respectively, of whom memberships for every $\chi \in U$, are termed as:

$$u_{\underline{A_R}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi) = \wedge_{y \in U}\left(u_{\mathcal{CL}_\eta^{\preccurlyeq}}(y) \vee (1 - R_{\mathcal{A}}(\chi, y))\right);$$

$$u_{\overline{A_R}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi) = \vee_{y \in U}\left(u_{\mathcal{CL}_\eta^{\preccurlyeq}}(y) \wedge R_{\mathcal{A}}(y, \chi)\right).$$

III. PROPOSED PYTHAGOREAN FUZZY DOMINANCE-BASED ROUGH SET APPROACH

In this portion, we construct an innovative Pythagorean Fuzzy Dominance–based Rough Set model.

Definition 6: For universe U, and for $\chi \in U$. The PFS '$\mathcal{P}$' is described over U as:

$$\mathcal{P} = \{(\chi, u_{\mathcal{P}}(\chi), \upsilon_{\mathcal{P}}(\chi)) : \chi \in U\},$$

where, $u_{\mathcal{P}} : U \to [0, 1]$ and $\upsilon_{\mathcal{P}} : U \to [0, 1]$ satisfying the condition $0 \preccurlyeq (u_{\mathcal{P}}(\chi))^2 + (\upsilon_{\mathcal{P}}(\chi))^2 \preccurlyeq 1$ for every $\chi \in U$. The combination of all Pythagorean fuzzy subsets on U is represented by $\mathcal{PF}(U)$.

Definition 7: The Pythagorean fuzzy connection $\mathcal{R}$ on U is a Pythagorean fuzzy subset of $U \times U$ and is specified as:

$$\mathcal{R} = \{\langle(\chi, y), u_{\mathcal{R}}(\chi, y), \upsilon_{\mathcal{R}}(\chi, y)\rangle : (\chi, y) \in U \times U\};$$

where, $u_{\mathcal{R}} : U \times U \to [0, 1]$ and $\upsilon_{\mathcal{R}} : U \times U \to [0, 1]$ satisfying that $0 \preccurlyeq (u_{\mathcal{R}}(\chi))^2 + (\upsilon_{\mathcal{R}}(\chi))^2 \preccurlyeq 1$ for each $(\chi, y) \in U \times U$. The set of all Pythagorean fuzzy relations on U is expressed by $\mathcal{PFR}(U \times U)$.

Definition 8: For universe U, and for all $\mathcal{R} \in \mathcal{PFR}(U \times U)$, if $u_{\mathcal{R}}(\chi, y)$ is the reliability of the statement "χ is at least as good as good as y in $\mathcal{R}$" and $\upsilon_{\mathcal{R}}(\chi, y)$ is the non-reliability of the statement "χ is at least as good as y in $\mathcal{R}$". Then, $\mathcal{R}$ is $\mathcal{R}$ is considered as Pythagorean FDR.

The Pythagorean FDR can express both the reliability and non-reliability of dominance principle between different objects. For a decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$, and for $\mathfrak{a} \in \mathcal{AT}$, then $\mathcal{R}_{\mathfrak{a}}$ is Pythagorean FDR w.r.t attribute $\mathfrak{a}$, for $\mathcal{AT}$ it is expressed by $\mathcal{R}_{\mathcal{AT}}$ and is described as:

$$\begin{aligned}\mathcal{R}_{\mathcal{AT}}(\chi, y) &= \left(u_{\mathcal{R}_{\mathcal{A}}}(\chi, y), \upsilon_{\mathcal{R}_{\mathcal{A}}}(\chi, y)\right)\\ &= \left(\wedge\left\{u_{\mathcal{R}_{\mathfrak{a}}}(\chi, y) : \mathfrak{a} \in \mathcal{AT}\right\},\right.\\ &\qquad \left.\vee\left\{\upsilon_{\mathcal{R}_{\mathfrak{a}}}(\chi, y) : \mathfrak{a} \in \mathcal{AT}\right\}\right);\end{aligned}$$

For every $(\chi, y) \in U \times U$, the Pythagorean FDR is reflexive, i.e. $\mathcal{R}_{\mathfrak{a}}(\chi, \chi) = 1$, $\left(u_{\mathcal{R}_{\mathfrak{a}}}(\chi, \chi) = 1, \upsilon_{\mathcal{R}_{\mathfrak{a}}}(\chi, \chi) = 0\right)$ for every $\chi \in U$ and $\mathfrak{a} \in \mathcal{AT}$.

Definition 9: For decision system $S = (U, \mathcal{AT} \cup \{f\})$, $\mathcal{A} \subseteq \mathcal{AT}$, Pythagorean FDR $\mathcal{R}_{\mathcal{A}}$ with respect to $\mathcal{A}$, and for all $\eta \in \aleph$, and then the $\mathcal{A}-\mathcal{L}$ *and* $\mathcal{U}$ Approxs of $\mathcal{CL}_\eta^{\succcurlyeq}$ w.r.t $\mathcal{R}_{\mathcal{A}}$ are represented and expressed as:

$$\begin{aligned}&\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})\\ &\quad = \left\{\left(\chi, u_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi), \upsilon_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi)\right) : \chi \in U\right\},\\ &\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})\\ &\quad = \left\{\left(\chi, u_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi), \upsilon_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi)\right) : \chi \in U\right\},\end{aligned}$$

where,

$$u_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi) = \wedge_{y \in U}\left(u_{\mathcal{CL}_\eta^{\succcurlyeq}}(y) \vee \upsilon_{\mathcal{R}_{\mathcal{A}}}(y, \chi)\right),$$

$$\upsilon_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi) = \vee_{y \in U}\left(\upsilon_{\mathcal{CL}_\eta^{\succcurlyeq}}(y) \wedge u_{\mathcal{R}_{\mathcal{A}}}(y, \chi)\right),$$

$$u_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi) = \vee_{y \in U}\left(u_{\mathcal{CL}_\eta^{\succcurlyeq}}(y) \wedge u_{\mathcal{R}_{\mathcal{A}}}(\chi, y)\right),$$

$$\upsilon_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi) = \wedge_{y \in U}\left(\upsilon_{\mathcal{CL}_\eta^{\succcurlyeq}}(y) \vee \upsilon_{\mathcal{R}_{\mathcal{A}}}(\chi, y)\right).$$

Similarly, for $\mathcal{CL}_\eta^{\preccurlyeq}$ the $\mathcal{A}-\mathcal{L}$ *and* $\mathcal{U}$ Approxs are represented and expressed as:

$$\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq}) = \left\{\left(\chi, u_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi), \upsilon_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi)\right) : \chi \in U\right\},$$

$$\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq}) = \left\{\left(\chi, u_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi), \upsilon_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi)\right) : \chi \in U\right\},$$

where,

$$u_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi) = \wedge_{y \in U}\left(u_{\mathcal{CL}_\eta^{\preccurlyeq}}(y) \vee \upsilon_{\mathcal{R}_{\mathcal{A}}}(\chi, y)\right),$$

$$\upsilon_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi) = \vee_{y \in U}\left(\upsilon_{\mathcal{CL}_\eta^{\preccurlyeq}}(y) \wedge u_{\mathcal{R}_{\mathcal{A}}}(\chi, y)\right),$$

$$u_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi) = \vee_{y \in U}\left(u_{\mathcal{CL}_\eta^{\preccurlyeq}}(y) \wedge u_{\mathcal{R}_{\mathcal{A}}}(y, \chi)\right),$$

$$\upsilon_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi) = \wedge_{y \in U}\left(\upsilon_{\mathcal{CL}_\eta^{\preccurlyeq}}(y) \vee \upsilon_{\mathcal{R}_{\mathcal{A}}}(y, \chi)\right).$$

The membership grade and non-membership grade of $\chi \in \underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})$ are $u_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi)$ and $\upsilon_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi)$ respectively. The MG and NMG of $\chi \in \overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})$ are $u_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi)$ and $\upsilon_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi)$ respectively. The MG and NMG of $\chi \in \underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})$ are $u_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi)$ and $\upsilon_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi)$ respectively. The MG and NMG of $\chi \in \overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})$ are $u_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi)$ and $\upsilon_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi)$ respectively.

Theorem 1: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$, $\mathcal{A} \subseteq \mathcal{AT}$, Pythagorean FDR $\mathcal{R}_{\mathcal{A}}$ w.r.t $\mathcal{A}$, and $\forall(\chi, y) \in U \times U$ if $u^2_{\mathcal{R}_{\mathcal{A}}}(\chi, y) + \upsilon^2_{\mathcal{R}_{\mathcal{A}}}(\chi, y) = 1$, then for each $\chi \in U$, we have

(a) $u^2_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi) + \upsilon^2_{\overline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi) = 1;$

(b) $u^2_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi) + \upsilon^2_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\succcurlyeq})}(\chi) = 1;$

(c) $u^2_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi) + \upsilon^2_{\underline{A_{\mathcal{R}}}(\mathcal{CL}_\eta^{\preccurlyeq})}(\chi) = 1;$

(d) $u^2_{\overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq})}(\mathfrak{x}) + v^2_{\overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq})}(\mathfrak{x}) = 1.$

Proof: Only (a) is proved, remaining can be proven similarly.

(a) For all $\mathfrak{x}, y \in U$, and by using definition 9

$$u^2_{\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}(\mathfrak{x}) + v^2_{\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}(\mathfrak{x}) = \bigwedge_{y\in U}\left(u^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \vee v^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})\right) + \bigvee_{y\in U}\left(v^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \wedge u^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})\right),$$

If $y \in \mathcal{CL}_{\eta}^{\succcurlyeq}$ *i.e.* $u^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) = 1$ and $v^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) = 0$,

Then $u^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \vee v^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) = 1$, $v^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \wedge u^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) = 0$.

If $y \notin \mathcal{CL}_{\eta}^{\succcurlyeq}$ *i.e.* $u^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) = 0$ and $v^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) = 1$,

Then $u^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \vee v^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) = v^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})$, $v^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \wedge u^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) = u^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})$.

Thus, if $\eta = 1$, then $\bigwedge_{y\in U}\left(u^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \vee v^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})\right) + \bigvee_{y\in U}\left(v^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \wedge u^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})\right) = 1$ holds. If $\eta \neq 1$ therefore, it implies that $y \notin \mathcal{CL}_{\eta}^{\succcurlyeq}$ s.t. $\bigwedge_{y\in U}\left(u^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \vee v^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})\right) = v^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})$, $\bigvee_{y\in U}\left(v^2_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \wedge u^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})\right) = u^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})$.

Since $u^2_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x},y) + v^2_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x},y) = 1$ then for each $(\mathfrak{x},y) \in U \times U$, $u^2_{\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}(\mathfrak{x}) + v^2_{\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}(\mathfrak{x}) = v^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) + u^2_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) = 1.$ ■

This indicates that when the Pythagorean FDR transitions to the fuzzy dominance relation, the PFDRS is also transitions to the FDRSA. Taking this approach, the PFDRSA can be seen as an extension of the conventional FDRSA.

Theorem 2: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$ with $\mathcal{A} \subseteq \mathcal{AT}$, then

(a) $u_{\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}(\mathfrak{x}) = \bigwedge\left\{v_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) : y \notin \mathcal{CL}_{\eta}^{\succcurlyeq}\right\}$ $(\eta = 2, \ldots, \mathfrak{m})$;

(b) $v_{\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}(\mathfrak{x}) = \bigvee\left\{u_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) : y \notin \mathcal{CL}_{\eta}^{\succcurlyeq}\right\}$ $(\eta = 2, \ldots, \mathfrak{m})$;

(c) $u_{\overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}(\mathfrak{x}) = \bigvee\left\{u_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x},y) : y \in \mathcal{CL}_{\eta}^{\succcurlyeq}\right\}$ $(\eta = 1, \ldots, \mathfrak{m})$;

(d) $v_{\overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}(\mathfrak{x}) = \bigwedge\left\{v_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x},y) : y \in \mathcal{CL}_{\eta}\right\}$ $(\eta = 1, \ldots, \mathfrak{m}^{\succcurlyeq})$;

(e) $u_{\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\preccurlyeq})}(\mathfrak{x}) = \bigwedge\left\{v_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x},y) : y \notin \mathcal{CL}_{\eta}^{\preccurlyeq}\right\}$ $(\eta = 1, \ldots, \mathfrak{m}-1)$;

(f) $v_{\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\preccurlyeq})}(\mathfrak{x}) = \bigvee\left\{u_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x},y) : y \notin \mathcal{CL}_{\eta}^{\preccurlyeq}\right\}$ $(\eta = 1, \ldots, \mathfrak{m}-1)$;

(g) $u_{\overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq})}(\mathfrak{x}) = \bigvee\left\{u_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) : y \in \mathcal{CL}_{\eta}^{\preccurlyeq}\right\}$ $(\eta = 1, \ldots, \mathfrak{m})$;

(h) $v_{\overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq})}(\mathfrak{x}) = \bigwedge\left\{v_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) : y \in \mathcal{CL}_{\eta}^{\preccurlyeq}\right\}$ $(\eta = 1, \ldots, \mathfrak{m})$.

Proof: Only (a) is proved, remaining can be proven similarly.

(a) For all $\eta = 2, \ldots, \mathfrak{m}$ and by definition 9, we have $u_{\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}(\mathfrak{x}) = \bigwedge_{y\in U}\left(u_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \vee v_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})\right)$. If $y \in \mathcal{CL}_{\eta}^{\succcurlyeq}$ then $u_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) = 1$ it implies that $u_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \vee v_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) = 1$. If $y \notin \mathcal{CL}_{\eta}^{\succcurlyeq}$ then $u_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) = 0$ it implies that $u_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \vee v_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) = v_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})$. Since $\eta = 2, \ldots, \mathfrak{m}$, then there must be $y \notin \mathcal{CL}_{\eta}^{\succcurlyeq}$ $s:t$ $u_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \vee v_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) = v_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})$. As a result $u_{\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}(\mathfrak{x}) = \bigwedge\left\{v_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x}) : y \notin \mathcal{CL}_{\eta}^{\succcurlyeq}\right\}.$ ■

Theorem 3: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$ with $\mathcal{A} \subseteq \mathcal{AT}$ then, Pythagorean fuzzy dominance–based rough Approxs possess the subsequent characteristics:

(C.1) Compression and Expansion:

$$\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq}) \subseteq \mathcal{CL}_{\eta}^{\succcurlyeq} \subseteq \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq});$$
$$\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq}) \subseteq \mathcal{CL}_{\eta}^{\succcurlyeq} \subseteq \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq});$$

(C.2) Complements:

$$\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq}) = U - \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta-1}^{\preccurlyeq}), \quad \eta = 2, \ldots, \mathfrak{m},$$
$$\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\preccurlyeq}) = U - \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta+1}^{\succcurlyeq}), \eta = 1, \ldots, \mathfrak{m}-1;$$
$$\overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq}) = U - \underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta-1}^{\preccurlyeq}), \quad \eta = 2, \ldots, \mathfrak{m};$$
$$\overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq}) = U - \underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta+1}^{\preccurlyeq}), \quad \eta = 1, \ldots, \mathfrak{m}-1;$$

(C.3) Monotonic with attributes:

$$\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq}) \subseteq \underline{\mathcal{AT}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq}); \quad \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq}) \supseteq \overline{\mathcal{AT}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq});$$
$$\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\preccurlyeq}) \subseteq \underline{\mathcal{AT}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\preccurlyeq}); \quad \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq}) \supseteq \overline{\mathcal{AT}}_{\mathfrak{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq});$$

(C.4) Monotonic with decision classes:

For $\eta_1, \eta_2 \in \aleph$ *such that* $\eta_1 \preccurlyeq \eta_2$, $\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta_1}^{\succcurlyeq})$

$$\supseteq \underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta_2}^{\succcurlyeq}); \quad \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta_1}^{\succcurlyeq}) \supseteq \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta_2}^{\succcurlyeq});$$
$$\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta_1}^{\preccurlyeq}) \subseteq \underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta_2}^{\preccurlyeq}); \quad \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta_1}^{\preccurlyeq}) \subseteq \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\eta_2}^{\preccurlyeq}).$$

Proof: (C.1) For every $\mathfrak{x} \notin \mathcal{CL}_{\eta}^{\succcurlyeq}$, *i.e.* $u_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(\mathfrak{x}) = 0$ and $v_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(\mathfrak{x}) = 1$, then

$$u_{\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}(\mathfrak{x}) = \bigwedge_{y\in U}\left(u_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \vee v_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})\right) \preccurlyeq u_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(\mathfrak{x}) \vee v_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x},\mathfrak{x}) = 0 = u_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(\mathfrak{x});$$

$$v_{\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta})}(\mathfrak{x}) = \bigvee_{y\in U}\left(v_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(y) \wedge u_{\mathcal{R}_{\mathcal{A}}}(y,\mathfrak{x})\right) \succcurlyeq v_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(\mathfrak{x}) \wedge u_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x},\mathfrak{x}) = 1 = v_{\mathcal{CL}_{\eta}^{\succcurlyeq}}(\mathfrak{x}).$$

Thus $\underline{\mathcal{A}_{\mathfrak{R}}}(\mathcal{CL}_{\eta}^{\succcurlyeq}) \subseteq \mathcal{CL}_{\eta}^{\preccurlyeq}$.

Now, for all $\mathfrak{x} \in U$, if $\mathfrak{x} \in C\L_n^{\succcurlyeq}$ *i.e.* $u_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(\mathfrak{x}) = 1$ and $v_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(\mathfrak{x}) = 0$, then

$$\begin{aligned} u_{\overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq})}(\mathfrak{x}) &= \vee_{y\in U}\left(u_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(y) \wedge u_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x}, y)\right) \\ &\succcurlyeq u_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(\mathfrak{x}) \wedge u_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x}, \mathfrak{x}) = u_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(\mathfrak{x}); \\ v_{\overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq})}(\mathfrak{x}) &= \wedge_{y\in U}\left(v_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(y) \vee v_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x}, y)\right) \\ &\preccurlyeq v_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(\mathfrak{x}) \vee v_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x}, \mathfrak{x}) = v_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(\mathfrak{x}); \end{aligned}$$

Thus, as a result $\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq} \subseteq \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq})$. Hence, $\underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}) \subseteq \mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq} \subseteq \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq})$. ■

Similarly, it can be prove that $\underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\preccurlyeq}) \subseteq \mathcal{CL}_{\mathfrak{y}}^{\preccurlyeq} \subseteq \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\preccurlyeq})$.

Proof: (C.2) For every $\mathfrak{x} \in U$, as $u_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(\mathfrak{x}) = v_{\mathcal{CL}_{\mathfrak{y}-1}^{\preccurlyeq}}(\mathfrak{x})$ and $v_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(\mathfrak{x}) = u_{\mathcal{CL}_{\mathfrak{y}-1}^{\preccurlyeq}}(\mathfrak{x})$ for $\mathfrak{y} = 2, \ldots, \mathfrak{m}$ then

$$\begin{aligned} u_{\underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq})}(\mathfrak{x}) &= \wedge_{y\in U}\left(u_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(y) \vee v_{\mathcal{R}_{\mathcal{A}}}(y, \mathfrak{x})\right) \\ &= \wedge_{y\in U}\left(v_{\mathcal{CL}_{\mathfrak{y}-1}^{\preccurlyeq}}(y) \vee v_{\mathcal{R}_{\mathcal{A}}}(y, \mathfrak{x})\right) \\ &= v_{\overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}-1}^{\preccurlyeq})}(\mathfrak{x}); \\ v_{\underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq})}(\mathfrak{x}) &= \vee_{y\in U}\left(v_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(y) \wedge u_{\mathcal{R}_{\mathcal{A}}}(y, \mathfrak{x})\right) \\ &= \vee_{y\in U}\left(u_{\mathcal{CL}_{\mathfrak{y}-1}^{\preccurlyeq}}(y) \wedge u_{\mathcal{R}_{\mathcal{A}}}(y, \mathfrak{x})\right) \\ &= u_{\overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}-1}^{\preccurlyeq})}(\mathfrak{x}); \end{aligned}$$

Thus, $\underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}) = U - \overline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}-1}^{\preccurlyeq})$ for $\mathfrak{y} = 2, \ldots, \mathfrak{m}$. In a similar manner, we can establish others. ■

Proof: (C.3) For $(\mathfrak{x}, y) \in U \times U$, we have $u_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x}, y) \geq u_{\mathcal{R}_{\mathcal{AT}}}(\mathfrak{x}, y)$ and $v_{\mathcal{R}_{\mathcal{A}}}(\mathfrak{x}, y) \leq v_{\mathcal{R}_{\mathcal{AT}}}(\mathfrak{x}, y)$ since $\mathcal{A} \subseteq \mathcal{AT}$, therefore

$$\begin{aligned} u_{\underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq})}(\mathfrak{x}) &= \wedge_{y\in U}\left(u_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(y) \vee v_{\mathcal{R}_{\mathcal{A}}}(y, \mathfrak{x})\right) \\ &\preccurlyeq \wedge_{y\in U}\left(u_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(y) \vee v_{\mathcal{R}_{\mathcal{AT}}}(y, \mathfrak{x})\right) \\ &= u_{\underline{\mathcal{AT}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq})}(\mathfrak{x}); \\ v_{\underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq})}(\mathfrak{x}) &= \vee_{y\in U}\left(v_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(y) \wedge u_{\mathcal{R}_{\mathcal{A}}}(y, \mathfrak{x})\right) \\ &\succcurlyeq \vee_{y\in U}\left(v_{\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}}(y) \wedge u_{\mathcal{R}_{\mathcal{AT}}}(y, \mathfrak{x})\right) \\ &= v_{\underline{\mathcal{AT}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq})}(\mathfrak{x}); \end{aligned}$$

Thus $\underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq}) \subseteq \underline{\mathcal{AT}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}}^{\succcurlyeq})$. Similarly, we can prove others. ■

Proof: (C.4) For $\mathfrak{y}_1 \leq \mathfrak{y}_2$ and for each $\mathfrak{x} \in U$, we obtain $\mathcal{CL}_{\mathfrak{y}_1}^{\succcurlyeq} \supseteq \mathcal{CL}_{\mathfrak{y}_2}^{\succcurlyeq}$, *i.e.* $u_{\mathcal{CL}_{\mathfrak{y}_1}^{\succcurlyeq}}(\mathfrak{x}) \succcurlyeq u_{\mathcal{CL}_{\mathfrak{y}_2}^{\succcurlyeq}}(\mathfrak{x})$ and $v_{\mathcal{CL}_{\mathfrak{y}_1}^{\succcurlyeq}}(\mathfrak{x}) \preccurlyeq v_{\mathcal{CL}_{\mathfrak{y}_2}^{\succcurlyeq}}(\mathfrak{x})$, so

$$\begin{aligned} u_{\underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}_1}^{\succcurlyeq})}(\mathfrak{x}) &= \wedge_{y\in U}\left(u_{\mathcal{CL}_{\mathfrak{y}_1}^{\succcurlyeq}}(y) \vee v_{\mathcal{R}_{\mathcal{A}}}(y, \mathfrak{x})\right) \\ &\succcurlyeq \wedge_{y\in U}\left(u_{\mathcal{CL}_{\mathfrak{y}_2}^{\succcurlyeq}}(y) \vee v_{\mathcal{R}_{\mathcal{A}}}(y, \mathfrak{x})\right) \\ &= u_{\underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}_2}^{\succcurlyeq})}(\mathfrak{x}); \\ v_{\underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}_1}^{\succcurlyeq})}(\mathfrak{x}) &= \vee_{y\in U}\left(v_{\mathcal{CL}_{\mathfrak{y}_1}^{\succcurlyeq}}(y) \wedge u_{\mathcal{R}_{\mathcal{A}}}(y, \mathfrak{x})\right) \\ &\preccurlyeq \vee_{y\in U}\left(v_{\mathcal{CL}_{\mathfrak{y}_2}^{\succcurlyeq}}(y) \wedge u_{\mathcal{R}_{\mathcal{A}}}(y, \mathfrak{x})\right) \\ &= v_{\underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}_2}^{\succcurlyeq})}(\mathfrak{x}); \end{aligned}$$

Thus $\underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}_1}^{\succcurlyeq}) \supseteq \underline{\mathcal{A}}_{\mathfrak{R}}(\mathcal{CL}_{\mathfrak{y}_2}^{\succcurlyeq})$. Similarly, we can prove others. ■

(C.1) states that the union of decision classes in an upward (or downward) manner contains its Pythagorean fuzzy rough $\mathcal{L}or\mathcal{U}$ Approx. (C.2) explains the complementary characteristics of the Pythagorean fuzzy dominance–based rough Approxs that are being proposed. (C.3) represents the monotonicity of the novel PFDRS based on conditional attributes. (C.4) specifies the monotonicity of the novel PFDRS in the form of the consistent growth patterns of unions of decision classes.

IV. PROPOSED PROBABILISTIC INTERPRETATION FOR PYTHAGOREAN FUZZY DOMINANCE–BASED ROUGH SET APPROACH

In this part, we will demonstrate the real–world application of the suggested PFDRSA in a decision–making system that incorporates probability interpretation.

Definition 10: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$ and for all $\mathfrak{x} \in U$ and $\mathfrak{a} \in \mathcal{AT}$, $\mathfrak{a}(\mathfrak{x}) \subseteq V_{\mathfrak{a}}$, *i.e.*

$$\mathfrak{a} : U \longrightarrow \mathcal{P}\left(V_{\mathfrak{a}}\right)$$

where, $\mathcal{P}\left(V_{\mathfrak{a}}\right)$ represents the collection of all nonempty subsets of $V_{\mathfrak{a}}$, therefore $\mathcal{S}$ is considered as a set–valued decision information, and $\mathfrak{x}$ represents a set of outcomes for every attribute rather than a single value.

For the probabilistic interpretation of set-valued decision system, $\forall \mu \in V_{\mathfrak{a}}$, $\mathfrak{a}(\mathfrak{x})(\mu) \in [0, 1]$ signifies the occurrence of μ. $\forall \mathfrak{x} \in U$, $\mathfrak{a} \in AT$ we consider

$$\sum_{\mu\in V_{\mathfrak{a}}} \mathfrak{a}(\mathfrak{x})(\mu) = 1$$

Every set value can be defined as a probability distribution encompassing the components within the assigned set. Consequently, the set value can be characterized as a probability distribution in this manner:

$$\mathfrak{a}(\mathfrak{x}) = \left\{\mu_1/\mathfrak{a}(\mathfrak{x})(\mu_1), \mu_2/\mathfrak{a}(\mathfrak{x})(\mu_2), \ldots, \mu_k/\mathfrak{a}(\mathfrak{x})(\mu_k)\right\},$$

where $\mu_1, \mu_2, \ldots, \mu_k \in V_{\mathfrak{a}}$.

By applying the RSA, we have examined the decision system with a set of values and a probabilistic perspective. For instance, in rough sets for insufficient information, the uncertain values are represented as a consistent probability distribution across the components within the realm of the associated attribute. This representation is achieved through the utilization of the valued–tolerance connection and the valued–dominance connection. Let $V_{ą} = \{ą_1, ą_2, ą_3, ą_4\}$, if $ą(ჯ) = *$ where $*$ represents unknown value like "do not care value" then the probability distribution can be defined as:

$$ą(ჯ) = \{ą_1/0.24, ą_2/0.26, ą_3/0.22, ą_4/0.28\}.$$

It means that if the value of $ჯ$ is not known for the attribute $ą$, then $ჯ$ may takes a single value from $V_{ą}$. Further, the level of probability that $ჯ$ possesses each value is identical. Although valued tolerance and dominance relations evaluate memberships in the tolerance and dominance degrees only, non–memberships are ignored. The Pythagorean fuzzy rough approach has become necessary to overcome this limitation.

Example 1: For universe $U = \{ჯ_1, ჯ_2, \ldots, ჯ_{10}\}$, conditional attributes $\mathcal{AT} = \{ą, ḫ, ç, ḑ, ę\}$, $V_{ą} = \{ą_0, ą_1, ą_2\}$, $V_{ḫ} = \{ḫ_0, ḫ_1, ḫ_2\}$, $V_{ç} = \{ç_0, ç_1, ç_2\}$, $V_{ḑ} = \{ḑ_0, ḑ_1, ḑ_2\}$, $V_{ę} = \{ę_0, ę_1, ę_2\}$, with $ą_0 \prec ą_1 \prec ą_2$, $ḫ_0 \prec ḫ_1 \prec ḫ_2$, $ç_0 \prec ç_1 \prec ç_2$, $ḑ_0 \prec ḑ_1 \prec ḑ_2$, $ę_0 \prec ę_1 \prec ę_2$, for f decision attribute $V_f = \{1, 2, 3\}$, then for all $(ჯ, y) \in U \times U$, $ą \in \mathcal{AT}$, Pythagorean fuzzy dominance relation is described as:

$$\mathcal{R}_{\mathcal{AT}}(ჯ, y) = \left\{ \begin{array}{ll} [1, 0]: & ჯ = y \\ \left(u_{\mathcal{R}_{\mathcal{AT}}}(ჯ, y), v_{\mathcal{R}_{\mathcal{AT}}}(ჯ, y)\right): & otherwise \end{array} \right\};$$

where, $\forall ą \in \mathcal{AT}$

$$u_{\mathcal{R}_{ą}}(ჯ, y) = \sum_{\mu_1 \succcurlyeq \mu_2, \mu_1, \mu_2 \in V_{ą}} ą(ჯ)(\mu_1)^2 \cdot ą(ჯ)(\mu_2)^2$$

$$v_{\mathcal{R}_{ą}}(ჯ, y) = \sum_{\mu_1 \prec \mu_2, \mu_1, \mu_2 \in V_{ą}} ą(ჯ)(\mu_1)^2 \cdot ą(ჯ)(\mu_2)^2$$

Here, $u_{\mathcal{R}_{\mathcal{A}}}(ჯ, y)$ represents the concept of dominance principle connecting with attributes $\mathcal{AT}$, while $v_{\mathcal{R}_{\mathcal{A}}}(ჯ, y)$ represents the degree of non-dominance principle concerning attributes $\mathcal{AT}$. **Table 1** represents information system with probability distribution.

For instance,

$$u_{\mathcal{R}_{ḫ}}(ჯ_7, ჯ_2) = \sum_{\mu_1 \succcurlyeq \mu_2, \mu_1, \mu_2 \in V_{ḫ}} ḫ(ჯ_7)(\mu_1)^2 . ḫ(ჯ_2)(\mu_2)^2$$

$$u_{\mathcal{R}_{ḫ}}(ჯ_7, ჯ_2) = ḫ(ჯ_7)(ḫ_0)^2 . ḫ(ჯ_2)(ḫ_0)^2 + ḫ(ჯ_7)(ḫ_1)^2 . ḫ(ჯ_2)(ḫ_0)^2 = (0.5)^2(0.6)^2 + (0.5)^2(0.6)^2 = 0.18$$

$$v_{\mathcal{R}_{b}}(ჯ_7, ჯ_2) = \sum_{\mu_1 \prec \mu_2, \mu_1, \mu_2 \in V_{ḫ}} ḫ(ჯ_7)(\mu_1)^2 . ḫ(ჯ_2)(\mu_2)^2$$

$$v_{\mathcal{R}_{ḫ}}(ჯ_7, ჯ_2) = ḫ(ჯ_7)(ḫ_0)^2 . ḫ(ჯ_2)(ḫ_2)^2 + ḫ(ჯ_7)(ḫ_1)^2 . ḫ(ჯ_2)(ḫ_2)^2$$

$$= (0.5)^2(0.4)^2 + (0.5)^2(0.4)^2 = 0.08$$

Similarly, the all outcomes of Pythagorean FDR represented in **Table 1** are displayed in **Table 2**.

Using the Pythagorean FDR, we can derive the respective rough approximations of MG and NMG based on Definition 3.4. By employing the decision attribute f, the U can be divided into the classes in this manner. $\mathfrak{CL} = \{\mathfrak{CL}_1, \mathfrak{CL}_2, \mathfrak{CL}_3\}$. Where $\mathfrak{CL}_1 = \{ჯ_2, ჯ_4, ჯ_8\}$, $\mathfrak{CL}_2 = \{ჯ_1, ჯ_5, ჯ_7, ჯ_9\}$ and $\mathfrak{CL}_3 = \{ჯ_3, ჯ_6, ჯ_{10}\}$. The upward and downward unions are $\mathfrak{CL}_1^{\succcurlyeq} = \{ჯ_1, ჯ_2, ჯ_3, ჯ_4, ჯ_5, ჯ_6, ჯ_7, ჯ_8, ჯ_9, ჯ_{10}\}$, $\mathfrak{CL}_2^{\succcurlyeq} = \{ჯ_1, ჯ_3, ჯ_5, ჯ_6, ჯ_7, ჯ_9, ჯ_{10}\}$, $\mathfrak{CL}_3^{\succcurlyeq} = \{ჯ_3, ჯ_6, ჯ_{10}\}$, $\mathfrak{CL}_1^{\preccurlyeq} = \{ჯ_2, ჯ_4, ჯ_8\}$, $\mathfrak{CL}_2^{\preccurlyeq} = \{ჯ_1, ჯ_2, ჯ_4, ჯ_5, ჯ_7, ჯ_8, ჯ_9\}$, and $\mathfrak{CL}_3^{\preccurlyeq} = \{ჯ_1, ჯ_2, ჯ_3, ჯ_4, ჯ_5, ჯ_6, ჯ_7, ჯ_8, ჯ_9, ჯ_{10}\}$. The outcomes of Pythagorean fuzzy dominance–based rough Approxs in **Table 1** are displayed in **Table 3.**

V. PROPOSED APPROXIMATE DISTRIBUTION REDUCTION FOR PYTHAGOREAN FUZZY DOMINANCE–BASED ROUGH SET

Mi [46] was the first to suggest the idea of ADR. Based on Mi's finding, we have included such reduction to DRSA for handling the information system's missing and unknowable values. The notion of ADR will be further generalized into our PFDRS model.

A. APPROACH TO APPROXIMATE DISTRIBUTION REDUCTION

Definition 11: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$ with $\mathcal{A} \subseteq \mathcal{AT}$, we denote

$$Ł_{\mathcal{AT}}^{\succcurlyeq} = \left\{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathfrak{CL}_1^{\succcurlyeq}), \underline{\mathcal{AT}_{\mathcal{R}}}(\mathfrak{CL}_2^{\succcurlyeq}), \ldots, \underline{\mathcal{AT}_{\mathcal{R}}}(\mathfrak{CL}_{\mathfrak{n}}^{\succcurlyeq})\right\};$$

$$Ł_{\mathcal{AT}}^{\preccurlyeq} = \left\{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathfrak{CL}_1^{\preccurlyeq}), \underline{\mathcal{AT}_{\mathcal{R}}}(\mathfrak{CL}_2^{\preccurlyeq}), \ldots, \underline{\mathcal{AT}_{\mathcal{R}}}(\mathfrak{CL}_{\mathfrak{n}}^{\preccurlyeq})\right\};$$

$$H_{\mathcal{AT}}^{\succcurlyeq} = \left\{\overline{\mathcal{AT}}_{\mathcal{R}}(\mathfrak{CL}_1^{\succcurlyeq}), \overline{\mathcal{AT}}_{\mathcal{R}}(\mathfrak{CL}_2^{\succcurlyeq}), \ldots, \overline{\mathcal{AT}}_{\mathcal{R}}(\mathfrak{CL}_{\mathfrak{n}})\right\};$$

$$H_{\mathcal{AT}}^{\preccurlyeq} = \left\{\overline{\mathcal{AT}}_{\mathcal{R}}(\mathfrak{CL}_1^{\preccurlyeq}), \overline{\mathcal{AT}}_{\mathcal{R}}(\mathfrak{CL}_2^{\preccurlyeq}), \ldots, \overline{\mathcal{AT}}_{\mathcal{R}}(\mathfrak{CL}_{\mathfrak{n}}^{\preccurlyeq})\right\};$$

(a) If $Ł_{\mathcal{A}}^{\succcurlyeq} = Ł_{\mathcal{AT}}^{\succcurlyeq}$, then $\mathcal{A}$ is known as the $\succcurlyeq$–lower ADCS; If $Ł_{\mathcal{A}}^{\succcurlyeq} = Ł_{\mathcal{AT}}^{\succcurlyeq}$ and $Ł_{\mathcal{B}}^{\succcurlyeq} \neq Ł_{\mathcal{A}}^{\succcurlyeq}\ \forall \mathcal{B} \subset \mathcal{A}$, then $\mathcal{A}$ is known as a $\succcurlyeq$–lower ADR of $\mathcal{S}$;

(b) If $Ł_{\mathcal{A}}^{\preccurlyeq} = Ł_{\mathcal{AT}}^{\preccurlyeq}$, then $\mathcal{A}$ is known as the $\preccurlyeq$–lower ADCS; If $Ł_{\mathcal{A}}^{\preccurlyeq} = Ł_{\mathcal{AT}}^{\preccurlyeq}$ and $Ł_{\mathcal{B}}^{\preccurlyeq} \neq Ł_{\mathcal{A}}^{\preccurlyeq}\ \forall \mathcal{B} \subset \mathcal{A}$, then $\mathcal{A}$ is known as a $\preccurlyeq$–lower ADR of $\mathcal{S}$;

(c) If $H_{\mathcal{A}}^{\succcurlyeq} = H_{\mathcal{AT}}^{\succcurlyeq}$, then $\mathcal{A}$ is known as the $\succcurlyeq$–upper ADCS; If $H_{\mathcal{A}}^{\succcurlyeq} = H_{\mathcal{AT}}^{\succcurlyeq}$ and $H_{\mathcal{B}}^{\succcurlyeq} \neq H_{\mathcal{A}}^{\succcurlyeq}\ \forall \mathcal{B} \subset \mathcal{A}$, then $\mathcal{A}$ is known as a $\succcurlyeq$–upper ADR of $\mathcal{S}$;

(d) If $H_{\mathcal{A}}^{\preccurlyeq} = H_{\mathcal{AT}}^{\preccurlyeq}$, then $\mathcal{A}$ is known as the $\preccurlyeq$ $\preccurlyeq$–upper ADCS; If $H_{\mathcal{A}}^{\preccurlyeq} = H_{\mathcal{AT}}^{\preccurlyeq}$ and $H_{\mathcal{B}}^{\preccurlyeq} \neq H_{\mathcal{A}}^{\preccurlyeq}\ \forall \mathcal{B} \subset \mathcal{A}$, then $\mathcal{A}$ is known as a $\preccurlyeq$–upper ADR of $\mathcal{S}$.

Hence, $\succcurlyeq -\mathcal{L}/\mathcal{U}$ *and* $\preccurlyeq -\mathcal{L}/\mathcal{U}$ ADCS refer to a set of attributes that retains the Pythagorean fuzzy dominance–based $\mathcal{L}$ *and* $\mathcal{U}$ Approxs for all the upward and downward unions of the decision classes, respectively.

TABLE 1. Probabilistic interpretation of information system.

$U/\mathcal{AT}$	ą	ḅ	ç	ḑ	ę	f
ҳ$_1$	{ą$_0$/0.7, ą$_1$/0.3}	{ḅ$_1$/1}	{ç$_0$/1}	{ḑ$_1$/0.5, ḑ$_2$/0.5}	{ę$_2$/1}	2
ҳ$_2$	{ą$_1$/1}	{ḅ$_0$/0.6, ḅ$_2$/0.4}	{ç$_1$/0.7, ç$_2$/0.3}	{ḑ$_1$/1}	{ę$_0$/0.7, ę$_1$/0.3}	1
ҳ$_3$	{ą$_1$/0.5, ą$_2$/0.5}	{ḅ$_1$/1}	{ç$_0$/0.8, ç$_1$/0.2}	{ḑ$_0$/1}	{ę$_0$/0.9, ę$_1$/0.1}	3
ҳ$_4$	{ą$_2$/1}	{ḅ$_1$/0.8, ḅ$_2$/0.2}	{ç$_1$/1}	{ḑ$_0$/0.7, ḑ$_1$/0.3}	{ę$_1$/1}	1
ҳ$_5$	{ą$_0$/0.9, ą$_1$/0.1}	{ḅ$_1$/1}	{ç$_0$/0.4, ç$_2$/0.6}	{ḑ$_1$/0.2, ḑ$_2$/0.8}	{ę$_0$/1}	2
ҳ$_6$	{ą$_0$/0.6, ą$_2$/0.4}	{ḅ$_1$/1}	{ç$_2$/1}	{ḑ$_1$/1}	{ę$_1$/0.5, ę$_2$/0.5}	3
ҳ$_7$	{ą$_2$/1}	{ḅ$_0$/0.5, ḅ$_1$/0.5}	{ç$_1$/1}	{ḑ$_1$/0.8, ḑ$_2$/0.2}	{ę$_1$/1}	2
ҳ$_8$	{ą$_0$/0.8, ą$_2$/0.2}	{ḅ$_1$/1}	{ç$_0$/0.9, ç$_1$/0.1}	{ḑ$_1$/1}	{ę$_1$/0.8, ę$_2$/0.2}	1
ҳ$_9$	{ą$_0$/1}	{ḅ$_0$/0.3, ḅ$_2$/0.7}	{ç$_1$/1}	{ḑ$_0$/0.6, ḑ$_1$/0.4}	{ę$_2$/1}	2
ҳ$_{10}$	{ą$_1$/1}	{ḅ$_1$/0.9, ḅ$_2$/0.1}	{ç$_0$/1}	{ḑ$_1$/1}	{ę$_0$/0.4, ę$_2$/0.6}	3

TABLE 2. Pythagorean Fuzzy Dominance Relation for Table 1.

ҳ/y	ҳ$_1$	ҳ$_2$	ҳ$_3$	ҳ$_4$	ҳ$_5$	ҳ$_6$	ҳ$_7$	ҳ$_8$	ҳ$_9$	ҳ$_{10}$
ҳ$_1$	[1.00, 0.00]	[0.00, 0.58]	[0.02, 0.27]	[0.00, 1.00]	[0.16, 0.36]	[0.00, 1.00]	[0.00, 1.00]	[0.37, 1.00]	[0.00, 1.00]	[0.09, 1.00]
ҳ$_2$	[0.09, 0.49]	[1.00, 0.00]	[0.16, 0.36]	[0.00, 1.00]	[0.04, 0.64]	[0.02, 0.49]	[0.00, 1.00]	[0.06, 0.36]	[0.00, 0.58]	[0.09, 0.30]
ҳ$_3$	[0.00, 0.82]	[0.00, 1.00]	[1.00, 0.00]	[0.01, 0.81]	[0.11, 0.68]	[0.00, 1.00]	[0.00, 0.81]	[0.00, 1.00]	[0.05, 0.82]	[0.00, 1.00]
ҳ$_4$	[0.00, 1.00]	[0.09, 0.49]	[0.50, 0.25]	[1.00, 0.00]	[0.01, 0.36]	[0.00, 1.00]	[0.06, 0.34]	[0.09, 0.49]	[0.00, 1.00]	[0.09, 0.64]
ҳ$_5$	[0.00, 1.00]	[0.01, 0.81]	[0.01, 0.41]	[0.00, 1.00]	[1.00, 0.00]	[0.00, 0.50]	[0.00, 1.00]	[0.00, 0.68]	[0.00, 1.00]	[0.01, 0.81]
ҳ$_6$	[0.25, 0.25]	[0.16, 0.36]	[0.08, 0.18]	[0.16, 0.36]	[0.04, 0.64]	[1.00, 0.00]	[0.16, 0.36]	[0.33, 0.01]	[0.09, 0.49]	[0.16, 0.36]
ҳ$_7$	[0.00, 0.25]	[0.18, 0.09]	[0.25, 0.25]	[0.16, 0.18]	[0.05, 0.41]	[0.25, 1.00]	[1.00, 0.00]	[0.25, 0.25]	[0.00, 0.81]	[0.16, 0.46]
ҳ$_8$	[0.04, 0.64]	[0.01, 0.64]	[0.02, 0.32]	[0.01, 0.81]	[0.04, 0.64]	[0.01, 0.82]	[0.01, 0.81]	[1.00, 0.00]	[0.01, 0.81]	[0.04, 0.64]
ҳ$_9$	[0.04, 0.22]	[0.00, 1.00]	[0.00, 0.50]	[0.00, 1.00]	[0.16, 0.35]	[0.00, 1.00]	[0.00, 1.00]	[0.16, 0.36]	[1.00, 0.00]	[0.00, 1.00]
ҳ$_{10}$	[0.25, 0.25]	[0.00, 0.58]	[0.25, 0.30]	[0.00, 1.00]	[0.04, 0.64]	[0.00, 1.00]	[0.00, 1.00]	[0.24, 0.11]	[0.00, 1.00]	[1.00, 0.00]

$-\mathcal{L}/\mathcal{U}$ *and* $\preccurlyeq -\mathcal{L}/\mathcal{U}$ ADR represent the minimal set of attributes required to maintain the Pythagorean fuzzy dominance–based $\mathcal{L}$ *and* $\mathcal{U}$ Approxs for all the upward and downward unions of the decision classes, respectively.

Theorem 4: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$ with $\mathcal{A} \subseteq \mathcal{AT}$, then

(a) $\mathcal{A}$ is $\succcurlyeq$ –lower ADCS $\Longleftrightarrow$ $\mathcal{A}$ is $\preccurlyeq$ –upper ADCS;
(b) $\mathcal{A}$ is $\preccurlyeq$ –lower ADCS $\Longleftrightarrow$ $\mathcal{A}$ is $\succcurlyeq$ –upper ADCS.

Proof: It can be proved by (C.2) of Theorem 3 and Definition 11. ■

Theorem 5: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$ with $\mathcal{A} \subseteq \mathcal{AT}$, then

(a) $\mathcal{A}$ is $\succcurlyeq$ –lower ADR $\Longleftrightarrow$ $\mathcal{A}$ is $\preccurlyeq$ –upper ADR;
(b) $\mathcal{A}$ is $\preccurlyeq$ –lower ADR $\Longleftrightarrow$ $\mathcal{A}$ is $\succcurlyeq$ –upper ADR.

Proof: It can be proved by Theorem 4 and Definition 11. ■

Theorem 6: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$ with $\mathcal{A} \subseteq \mathcal{AT}$, for all ҳ $\in U$ we denote

$$\mathcal{P}^{\succcurlyeq}_{\mathcal{AT}}(ҳ) = \left\{ \left(u_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}^{\succcurlyeq}_{\eta})}(ҳ), v_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}^{\succcurlyeq}_{\eta})}(ҳ) \right) : \eta \in \mathcal{N} \right\};$$

$$\mathcal{P}^{\preccurlyeq}_{\mathcal{AT}}(ҳ) = \left\{ \left(u_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}^{\preccurlyeq}_{\eta})}(ҳ), v_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}^{\preccurlyeq}_{\eta})}(ҳ) \right) : \eta \in \mathcal{N} \right\};$$

$$\mathcal{Q}^{\succcurlyeq}_{\mathcal{AT}}(ҳ) = \left\{ \left(u_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}^{\succcurlyeq}_{\eta})}(ҳ), v_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}^{\succcurlyeq}_{\eta})}(ҳ) \right) : \eta \in \mathcal{N} \right\};$$

$$\mathcal{Q}^{\preccurlyeq}_{\mathcal{AT}}(ҳ) = \left\{ \left(u_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}^{\preccurlyeq}_{\eta})}(ҳ), v_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}^{\preccurlyeq}_{\eta})}(ҳ) \right) : \eta \in \mathcal{N} \right\};$$

Then,

(a) $\mathcal{A}$ is $\succcurlyeq$ –lower ADCS $\Longleftrightarrow$ for all ҳ $\in U$, $\mathcal{P}^{\succcurlyeq}_{\mathcal{A}}(ҳ) = \mathcal{P}^{\succcurlyeq}_{\mathcal{AT}}(ҳ)$;
(b) $\mathcal{A}$ is $\preccurlyeq$ –lower ADCS $\Longleftrightarrow$ for all ҳ $\in U$, $\mathcal{P}^{\preccurlyeq}_{\mathcal{A}}(ҳ) = \mathcal{P}^{\preccurlyeq}_{\mathcal{AT}}(ҳ)$;
(c) $\mathcal{A}$ is $\succcurlyeq$ –upper ADCS $\Longleftrightarrow$ for all ҳ $\in U$, $\mathcal{Q}^{\succcurlyeq}_{\mathcal{A}}(ҳ) = \mathcal{Q}^{\succcurlyeq}_{\mathcal{AT}}(ҳ)$;
(d) $\mathcal{A}$ is $\preccurlyeq$ –upper ADCS $\Longleftrightarrow$ for all ҳ $\in U$, $\mathcal{Q}^{\preccurlyeq}_{\mathcal{A}}(ҳ) = \mathcal{Q}^{\preccurlyeq}_{\mathcal{AT}}(ҳ)$.

Proof: Only (a) is proved, remaining can be proven similarly.

$$\begin{aligned} Ł^{\succcurlyeq}_{\mathcal{A}} = Ł^{\succcurlyeq}_{\mathcal{AT}} &\Longleftrightarrow \underline{\mathcal{A}_{\mathcal{R}}}(\mathcal{CL}^{\succcurlyeq}_{\eta}) \\ &= \underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_{\eta}), (\eta \in \aleph) \\ \Longleftrightarrow u_{\underline{\mathcal{A}_{\mathcal{R}}}(\mathcal{CL}^{\succcurlyeq}_{\eta})}(ҳ) &= u_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}^{\succcurlyeq}_{\eta})}(ҳ), \\ v_{\underline{\mathcal{A}_{\mathcal{R}}}(\mathcal{CL}^{\succcurlyeq}_{\eta})}(ҳ) &= v_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}^{\succcurlyeq}_{\eta})}(ҳ), \quad (\forall ҳ \in U) \\ \Longleftrightarrow \mathcal{P}^{\succcurlyeq}_{\mathcal{A}}(ҳ) &= \mathcal{P}^{\succcurlyeq}_{\mathcal{AT}}(ҳ). \end{aligned}$$

■

TABLE 3. Pythagorean fuzzy dominance–based rough approxs for Table 1.

$\mathfrak{x}/y$	$\mathfrak{x}_1$	$\mathfrak{x}_2$	$\mathfrak{x}_3$	$\mathfrak{x}_4$	$\mathfrak{x}_5$	$\mathfrak{x}_6$	$\mathfrak{x}_7$	$\mathfrak{x}_8$	$\mathfrak{x}_9$	$\mathfrak{x}_{10}$
$u_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_1^{\succcurlyeq})}(\mathfrak{x})$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$u_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_2^{\succcurlyeq})}(\mathfrak{x})$	0.49	0.00	0.25	0.00	0.36	0.49	0.34	0.00	0.58	0.30
$u_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_3^{\succcurlyeq})}(\mathfrak{x})$	0.00	0.00	0.25	0.00	0.00	0.49	0.00	0.00	0.00	0.30
$v_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_1^{\succcurlyeq})}(\mathfrak{x})$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$v_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_2^{\succcurlyeq})}(\mathfrak{x})$	0.09	1.00	0.50	1.00	0.04	0.02	0.06	1.00	0.01	0.09
$v_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_3^{\succcurlyeq})}(\mathfrak{x})$	1.00	1.00	0.50	1.00	1.00	0.25	1.00	1.00	1.00	0.16
$u_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_1^{\succcurlyeq})}(\mathfrak{x})$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$u_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_2^{\succcurlyeq})}(\mathfrak{x})$	1.00	0.16	1.00	0.50	1.00	1.00	1.00	0.04	1.00	1.00
$u_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_3^{\succcurlyeq})}(\mathfrak{x})$	0.02	0.16	1.00	0.50	0.01	1.00	0.25	0.02	1.00	0.25
$v_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_1^{\succcurlyeq})}(\mathfrak{x})$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$v_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_2^{\succcurlyeq})}(\mathfrak{x})$	0.00	0.30	0.00	0.25	0.00	0.00	0.00	0.32	0.00	0.00
$v_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_3^{\succcurlyeq})}(\mathfrak{x})$	0.27	0.30	0.00	0.25	0.41	0.00	0.25	0.32	0.50	0.00
$u_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_1^{\preccurlyeq})}(\mathfrak{x})$	0.00	0.30	0.00	0.25	0.00	0.00	0.00	0.32	0.00	0.00
$u_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_2^{\preccurlyeq})}(\mathfrak{x})$	0.27	0.30	0.00	0.25	0.41	0.00	0.25	0.32	0.50	0.00
$u_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_3^{\preccurlyeq})}(\mathfrak{x})$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$v_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_1^{\preccurlyeq})}(\mathfrak{x})$	1.00	0.16	1.00	0.50	1.00	1.00	1.00	0.04	1.00	1.00
$v_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_2^{\preccurlyeq})}(\mathfrak{x})$	0.09	0.16	1.00	0.50	0.01	1.00	0.25	0.04	0.00	1.00
$v_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_3^{\preccurlyeq})}(\mathfrak{x})$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$u_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_1^{\preccurlyeq})}(\mathfrak{x})$	0.09	1.00	0.50	1.00	0.04	0.02	0.06	1.00	0.01	0.09
$u_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_2^{\preccurlyeq})}(\mathfrak{x})$	1.00	1.00	0.50	1.00	1.00	0.25	1.00	1.00	1.00	0.16
$u_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_3^{\preccurlyeq})}(\mathfrak{x})$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$v_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_1^{\preccurlyeq})}(\mathfrak{x})$	0.49	0.00	0.25	0.00	0.36	0.49	0.34	0.00	0.58	0.30
$v_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_2^{\preccurlyeq})}(\mathfrak{x})$	0.00	0.00	0.25	0.00	0.00	0.49	0.00	0.00	0.00	0.30
$v_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_3^{\preccurlyeq})}(\mathfrak{x})$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Definition 12: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$ and for $\mathcal{A} \subseteq \mathcal{AT}$, we define

$$\mathfrak{D}_{Ł}^{\succcurlyeq} = \left\{(\mathfrak{x}, y) \in U^2 : \mathfrak{x} \in U,\ y \notin \mathcal{CL}_{\mathfrak{n}}^{\succcurlyeq},\ \mathfrak{n} = 2, 3, \ldots, \mathfrak{m}\right\};$$
$$\mathfrak{D}_{Ł}^{\preccurlyeq} = \left\{(\mathfrak{x}, y) \in U^2 : \mathfrak{x} \in U,\ y \notin \mathcal{CL}_{\mathfrak{n}}^{\preccurlyeq},\ \mathfrak{n} = 1, 2, \ldots, \mathfrak{m}-1\right\};$$
$$\mathfrak{D}_{H}^{\succcurlyeq} = \left\{(\mathfrak{x}, y) \in U^2 : \mathfrak{x} \in U,\ y \in \mathcal{CL}_{\mathfrak{n}}^{\succcurlyeq},\ \mathfrak{n} = 2, 3, \ldots, \mathfrak{m}\right\};$$
$$\mathfrak{D}_{H}^{\preccurlyeq} = \left\{(\mathfrak{x}, y) \in U^2 : \mathfrak{x} \in U,\ y \in \mathcal{CL}_{\mathfrak{n}}^{\preccurlyeq},\ \mathfrak{n} = 1, 2, \ldots, \mathfrak{m}-1\right\}.$$

where,

(a) If $(\mathfrak{x}, y) \in \mathfrak{D}_{Ł}^{\succcurlyeq}$, *then* $\mathfrak{D}_{Ł}^{\succcurlyeq u}(\mathfrak{x}, y) = \left\{\mathfrak{a} \in \mathcal{AT} : u_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_{\mathfrak{n}}^{\succcurlyeq})}(\mathfrak{x}) \preccurlyeq v_{\mathcal{R}_{\mathfrak{a}}}(y, \mathfrak{x})\right\}$, otherwise, $\mathfrak{D}_{Ł}^{\succcurlyeq u}(\mathfrak{x}, y) = \emptyset$;

(b) If $(\mathfrak{x}, y) \in \mathfrak{D}_{Ł}^{\succcurlyeq}$, *then* $\mathfrak{D}_{Ł}^{\succcurlyeq v}(\mathfrak{x}, y) = \left\{\mathfrak{a} \in \mathcal{AT} : v_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_{\mathfrak{n}}^{\succcurlyeq})}(\mathfrak{x}) \succcurlyeq u_{\mathcal{R}_{\mathfrak{a}}}(y, \mathfrak{x})\right\}$, otherwise, $\mathfrak{D}_{Ł}^{\succcurlyeq v}(\mathfrak{x}, y) = \emptyset$;

(c) If $(\mathfrak{x}, y) \in \mathfrak{D}_{Ł}^{\preccurlyeq}$, *then* $\mathfrak{D}_{Ł}^{\preccurlyeq u}(\mathfrak{x}, y) = \left\{\mathfrak{a} \in \mathcal{AT} : u_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_{\mathfrak{n}}^{\preccurlyeq})}(\mathfrak{x}) \preccurlyeq v_{\mathcal{R}_{\mathfrak{a}}}(\mathfrak{x}, y)\right\}$, otherwise, $\mathfrak{D}_{Ł}^{\preccurlyeq u}(\mathfrak{x}, y) = \emptyset$;

(d) If $(\mathfrak{x}, y) \in \mathfrak{D}_{Ł}^{\preccurlyeq}$, *then* $\mathfrak{D}_{Ł}^{\preccurlyeq v}(\mathfrak{x}, y) = \left\{\mathfrak{a} \in \mathcal{AT} : v_{\underline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_{\mathfrak{n}}^{\preccurlyeq})}(\mathfrak{x}) \succcurlyeq u_{\mathcal{R}_{\mathfrak{a}}}(\mathfrak{x}, y)\right\}$, otherwise, $\mathfrak{D}_{Ł}^{\preccurlyeq v}(\mathfrak{x}, y) = \emptyset$;

(e) If $(\mathfrak{x}, y) \in \mathfrak{D}_{H}^{\succcurlyeq}$, *then* $\mathfrak{D}_{H}^{\succcurlyeq u}(\mathfrak{x}, y) = \left\{\mathfrak{a} \in \mathcal{AT} : u_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_{\mathfrak{n}}^{\succcurlyeq})}(\mathfrak{x}) \succcurlyeq u_{\mathcal{R}_{\mathfrak{a}}}(\mathfrak{x}, y)\right\}$, otherwise, $\mathfrak{D}_{H}^{\succcurlyeq u}(\mathfrak{x}, y) = \emptyset$;

(f) If $(\mathfrak{x}, y) \in \mathfrak{D}_{H}^{\succcurlyeq}$, *then* $\mathfrak{D}_{H}^{\succcurlyeq v}(\mathfrak{x}, y) = \left\{\mathfrak{a} \in \mathcal{AT} : v_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_{\mathfrak{n}}^{\succcurlyeq})}(\mathfrak{x}) \preccurlyeq v_{\mathcal{R}_{\mathfrak{a}}}(\mathfrak{x}, y)\right\}$, otherwise, $\mathfrak{D}_{H}^{\succcurlyeq v}(\mathfrak{x}, y) = \emptyset$;

(g) If $(\mathfrak{x}, y) \in \mathfrak{D}_{H}^{\preccurlyeq}$, *then* $\mathfrak{D}_{H}^{\preccurlyeq u}(\mathfrak{x}, y) = \left\{\mathfrak{a} \in \mathcal{AT} : u_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_{\mathfrak{n}}^{\preccurlyeq})}(\mathfrak{x}) \succcurlyeq u_{\mathcal{R}_{\mathfrak{a}}}(y, \mathfrak{x})\right\}$, otherwise, $\mathfrak{D}_{H}^{\preccurlyeq u}(\mathfrak{x}, y) = \emptyset$;

(h) If $(\mathfrak{x}, y) \in \mathfrak{D}_{H}^{\preccurlyeq}$, *then* $\mathfrak{D}_{H}^{\preccurlyeq v}(\mathfrak{x}, y) = \left\{\mathfrak{a} \in \mathcal{AT} : v_{\overline{\mathcal{AT}_{\mathcal{R}}}(\mathcal{CL}_{\mathfrak{n}}^{\preccurlyeq})}(\mathfrak{x}) \preccurlyeq v_{\mathcal{R}_{\mathfrak{a}}}(\mathfrak{x}, y)\right\}$, otherwise, $\mathfrak{D}_{H}^{\preccurlyeq v}(\mathfrak{x}, y) = \emptyset$.

$\mathfrak{D}_{Ł}^{\succcurlyeq u}(\mathfrak{x}, y)$, $\mathfrak{D}_{Ł}^{\succcurlyeq v}(\mathfrak{x}, y)$, $\mathfrak{D}_{Ł}^{\preccurlyeq u}(\mathfrak{x}, y)$, $\mathfrak{D}_{Ł}^{\preccurlyeq v}(\mathfrak{x}, y)$, $\mathfrak{D}_{H}^{\succcurlyeq u}(\mathfrak{x}, y)$, $\mathfrak{D}_{H}^{\succcurlyeq v}(\mathfrak{x}, y)$, $\mathfrak{D}_{H}^{\preccurlyeq u}(\mathfrak{x}, y)$, $\mathfrak{D}_{H}^{\preccurlyeq v}(\mathfrak{x}, y)$ are the $\succcurlyeq^{u}$–lower,

TABLE 4. $\succcurlyeq^{u}$–lower approximate distribution discernibility matrix.

$\mathfrak{D}_{Ł}^{\succcurlyeq u}(x,y)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	$\emptyset$	$\{e\}$	$\emptyset$	$\{e\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{e\}$	$\emptyset$	$\emptyset$
x_2	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\emptyset$
x_3	$\emptyset$	$\{a, b\}$	$\emptyset$	$\{a\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a\}$	$\emptyset$	$\emptyset$
x_4	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\emptyset$
x_5	$\emptyset$	$\{b, d\}$	$\emptyset$	$\{c\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a, d\}$	$\emptyset$	$\emptyset$
x_6	$\emptyset$	$\{c\}$	$\emptyset$	$\{c, d\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{c\}$	$\emptyset$	$\emptyset$
x_7	$\emptyset$	$\{a, e\}$	$\emptyset$	$\{d\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a, c\}$	$\emptyset$	$\emptyset$
x_8	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\emptyset$
x_9	$\emptyset$	$\{e\}$	$\emptyset$	$\{e\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{c\}$	$\emptyset$	$\emptyset$
x_{10}	$\emptyset$	$\{e\}$	$\emptyset$	$\{b, d, e\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a\}$	$\emptyset$	$\emptyset$

TABLE 5. $\succcurlyeq^{v}$–lower approximate distribution discernibility matrix.

$\mathfrak{D}_{Ł}^{\succcurlyeq v}(x,y)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	$\emptyset$	$\{e\}$	$\emptyset$	$\{d, e\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{e\}$	$\emptyset$	$\emptyset$
x_2	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\emptyset$
x_3	$\emptyset$	$\{a, b, c, e\}$	$\emptyset$	$\{a\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a\}$	$\emptyset$	$\emptyset$
x_4	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\emptyset$
x_5	$\emptyset$	$\{d\}$	$\emptyset$	$\{d\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{d\}$	$\emptyset$	$\emptyset$
x_6	$\emptyset$	$\{e\}$	$\emptyset$	$\{c\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{c\}$	$\emptyset$	$\emptyset$
x_7	$\emptyset$	$\{a\}$	$\emptyset$	$\{d\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a, c\}$	$\emptyset$	$\emptyset$
x_8	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\emptyset$
x_9	$\emptyset$	$\{e\}$	$\emptyset$	$\{e\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{c\}$	$\emptyset$	$\emptyset$
x_{10}	$\emptyset$	$\{e\}$	$\emptyset$	$\{e\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a\}$	$\emptyset$	$\emptyset$

$\succcurlyeq^{v}$–lower, $\preccurlyeq^{u}$–lower, $\preccurlyeq^{v}$–lower, $\succcurlyeq^{u}$–upper, $\succcurlyeq^{v}$ –upper, $\preccurlyeq^{u}$–upper and $\preccurlyeq^{v}$–upper approximate discernibility sets for pair of the objects (x, y) respectively.

The approximate discernibility matrixes are:

$$M_{Ł}^{\succcurlyeq u} = \left\{\mathfrak{D}_{Ł}^{\succcurlyeq u}(x, y) : (x, y) \in \mathfrak{D}_{Ł}^{\succcurlyeq}\right\},$$
$$M_{Ł}^{\succcurlyeq v} = \left\{\mathfrak{D}_{Ł}^{\succcurlyeq v}(x, y) : (x, y) \in \mathfrak{D}_{Ł}^{\succcurlyeq}\right\},$$
$$M_{Ł}^{\preccurlyeq u} = \left\{\mathfrak{D}_{Ł}^{\preccurlyeq u}(x, y) : (x, y) \in \mathfrak{D}_{Ł}^{\preccurlyeq}\right\},$$
$$M_{Ł}^{\preccurlyeq v} = \left\{\mathfrak{D}_{Ł}^{\preccurlyeq v}(x, y) : (x, y) \in \mathfrak{D}_{Ł}^{\preccurlyeq}\right\},$$
$$M_{H}^{\succcurlyeq u} = \left\{\mathfrak{D}_{H}^{\succcurlyeq u}(x, y) : (x, y) \in \mathfrak{D}_{H}^{\succcurlyeq}\right\},$$
$$M_{H}^{\succcurlyeq v} = \left\{\mathfrak{D}_{H}^{\succcurlyeq v}(x, y) : (x, y) \in \mathfrak{D}_{H}^{\succcurlyeq}\right\},$$
$$M_{H}^{\preccurlyeq u} = \left\{\mathfrak{D}_{H}^{\preccurlyeq u}(x, y) : (x, y) \in \mathfrak{D}_{H}^{\preccurlyeq}\right\},$$
$$M_{H}^{\preccurlyeq v} = \left\{\mathfrak{D}_{H}^{\preccurlyeq v}(x, y) : (x, y) \in \mathfrak{D}_{H}^{\preccurlyeq}\right\}.$$

These are the $\succcurlyeq^{u}$–lower, $\succcurlyeq^{v}$–lower, $\preccurlyeq^{u}$–lower, $\preccurlyeq^{v}$–lower, $\succcurlyeq^{u}$–upper, $\succcurlyeq^{v}$ –upper, $\preccurlyeq^{u}$–upper and $\preccurlyeq^{v}$–upper approximate discernibility matrices respectively.

Theorem 7: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$ with $\mathcal{A} \subseteq \mathcal{AT}$, then

(a) $u_{\underline{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}}(xx) = u_{\underline{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}}(), \forall x \in U$ *and* $\eta \in \aleph \iff \mathcal{A} \cap \mathfrak{D}_{Ł}^{\succcurlyeq u}(x, y) \neq \emptyset \, \forall (x, y) \in \mathfrak{D}_{Ł}^{\succcurlyeq}$;

(b) $v_{\underline{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}}(xx) = v_{\underline{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}}(), \forall x \in U$ *and* $\eta \in \aleph \iff \mathcal{A} \cap \mathfrak{D}_{Ł}^{\succcurlyeq v}(x, y) \neq \emptyset \, \forall (x, y) \in \mathfrak{D}_{Ł}^{\succcurlyeq}$;

(c) $u_{\underline{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq})}}(xx) = u_{\underline{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq})}}(), \forall x \in U$ *and* $\eta \in \aleph \iff \mathcal{A} \cap \mathfrak{D}_{Ł}^{\preccurlyeq u}(x, y) \neq \emptyset \, \forall (x, y) \in \mathfrak{D}_{Ł}^{\preccurlyeq}$;

(d) $v_{\underline{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq})}}(xx) = v_{\underline{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq})}}(), \forall x \in U$ *and* $\eta \in \aleph \iff \mathcal{A} \cap \mathfrak{D}_{Ł}^{\preccurlyeq v}(x, y) \neq \emptyset \, \forall (x, y) \in \mathfrak{D}_{Ł}^{\preccurlyeq}$;

(e) $u_{\overline{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}}(x) = u_{\overline{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}}(x), \forall x \in U$ *and* $\eta \in \aleph \iff \mathcal{A} \cap \mathfrak{D}_{H}^{\succcurlyeq u}(x, y) \neq \emptyset \, \forall (x, y) \in \mathfrak{D}_{H}^{\succcurlyeq}$;

(f) $v_{\overline{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}}(x) = v_{\overline{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\succcurlyeq})}}(x), \forall x \in U$ *and* $\eta \in \aleph \iff \mathcal{A} \cap \mathfrak{D}_{H}^{\succcurlyeq v}(x, y) \neq \emptyset \, \forall (x, y) \in \mathfrak{D}_{H}^{\succcurlyeq}$;

(g) $u_{\overline{\mathcal{AT}_{\mathcal{R}}^{\preccurlyeq}(\mathcal{CL}_{\eta}^{\preccurlyeq})}}(x) = u_{\overline{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq})}}(x), \forall x \in U$ *and* $\eta \in \aleph \iff \mathcal{A} \cap \mathfrak{D}_{H}^{\preccurlyeq u}(x, y) \neq \emptyset \, \forall (x, y) \in \mathfrak{D}_{H}^{\preccurlyeq}$;

(h) $v_{\overline{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq})}}(x) = v_{\overline{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_{\eta}^{\preccurlyeq})}}(x), \forall x \in U$ *and* $\eta \in \aleph \iff \mathcal{A} \cap \mathfrak{D}_{H}^{\preccurlyeq v}(x, y) \neq \emptyset \, \forall (x, y) \in \mathfrak{D}_{H}^{\preccurlyeq}$.

Proof: Only (a) is proved, remaining can be proven similarly.

For $\eta = 1$, $u_{\underline{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_{1}^{\succcurlyeq})}}(x) = u_{\underline{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_{1}^{\succcurlyeq})}}(x) = 1$ as $\mathcal{CL}_{1}^{\succcurlyeq} = U$.

TABLE 6. $\succcurlyeq^{u}$–upper approximate distribution discernibility matrix.

$\mathcal{D}_H^{\succcurlyeq u}(\chi, y)$	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
χ_1	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$
χ_2	{ḅ, ę}	$\emptyset$	{ḅ}	$\emptyset$	{ḅ, ç, ḍ}	{ḅ, ç, ę}	{ą, ę}	$\emptyset$	{ḅ, ę}	{ę}
χ_3	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$
χ_4	{ḅ, ḍ, ę}	$\emptyset$	{ą}	$\emptyset$	{ç, ḍ}	{ç, ḍ, ę}	{ḅ, ḍ}	$\emptyset$	{ḅ, ḍ, ę}	{ḍ, ę}
χ_5	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$
χ_6	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$
χ_7	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$
χ_8	{ę}	$\emptyset$	{ą}	$\emptyset$	{ḍ}	{ę}	{ą, ç}	$\emptyset$	{ç, ę}	{ą}
χ_9	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$
χ_{10}	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$

TABLE 7. $\succcurlyeq^{v}$–upper approximate distribution discernibility matrix.

$\mathcal{D}_H^{\succcurlyeq v}(\chi, y)$	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
χ_1	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$
χ_2	{ḅ, ę}	$\emptyset$	{ḅ}	$\emptyset$	{ḅ, ḍ}	{ḅ, ç}	{ą, ę}	$\emptyset$	{ę}	{ḅ}
χ_3	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$
χ_4	{ḍ, ę}	$\emptyset$	{ą}	$\emptyset$	{ç, ḍ}	{ç, ḍ, ę}	{ḍ}	$\emptyset$	{ḍ, ę}	{ḅ, ḍ, ę}
χ_5	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$
χ_6	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$
χ_7	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$
χ_8	{ę}	$\emptyset$	{ą}	$\emptyset$	{ą, ḍ}	{ę}	{ą, ç}	$\emptyset$	{ḅ, ç, ę}	{ą}
χ_9	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$
χ_{10}	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$	$\mathcal{AT}$	$\emptyset$	$\mathcal{AT}$	$\mathcal{AT}$

For ŋ > 1, suppose there exist $(\chi, y) \in \mathcal{D}_Ł^{\succcurlyeq}$ s.t $\mathcal{A} \cap \mathcal{D}_Ł^{\succcurlyeq u}(\chi, y) = \emptyset$, then for every ą $\in \mathcal{A}$, by Definition 12 $u_{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi) > v_{\mathcal{R}_ą}(y, \chi)$. By Definition 8 $v_{\mathcal{R}_{\mathcal{A}}}(y, \chi) = \bigvee \left\{v_{\mathcal{R}_ą}(y, \chi) : ą \in \mathcal{AT}\right\}$, thus $u_{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi) > v_{\mathcal{R}_{\mathcal{A}}}(y, \chi)$. Our assumption is $u_{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi) = u_{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi)$, therefore $u_{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi) > v_{\mathcal{R}_{\mathcal{A}}}(y, \chi)$ holds, it contradicts to $u_{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi) \leq v_{\mathcal{R}_ą}(y, \chi)$ because $u_{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi) = \bigwedge \left\{v_{\mathcal{R}_{\mathcal{A}}}(y, \chi) : y \notin \mathcal{CL}_ŋ^{\succcurlyeq}\right\}$ (ŋ $= 2, \ldots,$ m). So, $\mathcal{A} \cap \mathcal{D}_Ł^{\succcurlyeq v}(\chi, y) \neq \emptyset$.

Conversely, suppose there exist $\chi \in U$ and ŋ $\in \aleph$ with $u_{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi) \neq u_{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi)$, then ŋ $= 2, \ldots,$ m and by Theorem 3 $u_{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi) > u_{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi)$. So, there will be $y \notin \mathcal{CL}_ŋ^{\succcurlyeq}$ $s:t$ $v_{\mathcal{R}_ą}(y, \chi) < u_{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi)$, then for each ą $\in \mathcal{A}$, $v_{\mathcal{R}_ą}(y, \chi) < u_{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi)$ holds, that is $\mathcal{A} \cap \mathcal{D}_Ł^{\succcurlyeq u}(\chi, y) = \emptyset$, with $(\chi, y) \in \mathcal{D}_Ł^{\succcurlyeq}$. By this discussion we conclude the required result. ■

Theorem 8: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$ with $\mathcal{A} \subseteq \mathcal{AT}$, then

(a) $Ł_{\mathcal{A}}^{\succcurlyeq} = Ł_{\mathcal{AT}}^{\succcurlyeq} \iff \forall(\chi, y) \in \mathcal{D}_Ł^{\succcurlyeq}$ $s:t$ $\mathcal{A} \cap \left(\mathcal{D}_Ł^{\succcurlyeq u}(\chi, y) \cap \mathcal{D}_Ł^{\succcurlyeq v}(\chi, y)\right) \neq \emptyset$;

(b) $Ł_{\mathcal{A}}^{\preccurlyeq} = Ł_{\mathcal{AT}}^{\preccurlyeq} \iff \forall(\chi, y) \in \mathcal{D}_Ł^{\preccurlyeq}$ $s:t$ $\mathcal{A} \cap \left(\mathcal{D}_Ł^{\preccurlyeq u}(\chi, y) \cap \mathcal{D}_Ł^{\preccurlyeq v}(\chi, y)\right) \neq \emptyset$;

(c) $H_{\mathcal{A}}^{\succcurlyeq} = H_{\mathcal{AT}}^{\succcurlyeq} \iff \forall(\chi, y) \in \mathcal{D}_H^{\succcurlyeq}$ $s:t$ $\mathcal{A} \cap \left(\mathcal{D}_H^{\succcurlyeq u}(\chi, y) \cap \mathcal{D}_H^{\succcurlyeq v}(\chi, y)\right) \neq \emptyset$;

(d) $H_{\mathcal{A}}^{\preccurlyeq} = H_{\mathcal{AT}}^{\preccurlyeq} \iff \forall(\chi, y) \in \mathcal{D}_H^{\preccurlyeq}$ $s:t$ $\mathcal{A} \cap \left(\mathcal{D}_H^{\preccurlyeq u}(\chi, y) \cap \mathcal{D}_H^{\preccurlyeq v}(\chi, y)\right) \neq \emptyset$.

Proof: Only (a) is proved, remaining can be proven similarly.

If $Ł_{\mathcal{A}}^{\succcurlyeq} = Ł_{\mathcal{AT}}^{\succcurlyeq}$, then $\forall \chi \in U$ and ŋ $\in \aleph$, $u_{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi) = u_{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi)$ and $v_{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi) = v_{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi)$. By Theorem 7 $\mathcal{A} \cap \mathcal{D}_Ł^{\succcurlyeq u}(\chi, y) \neq \emptyset$ and $\mathcal{A} \cap \mathcal{D}_Ł^{\succcurlyeq v}(\chi, y) \neq \emptyset$ for each $(\chi, y) \in \mathcal{D}_Ł^{\succcurlyeq}$, it implies that $\mathcal{A} \cap \left(\mathcal{D}_Ł^{\succcurlyeq u}(\chi, y) \cap \mathcal{D}_Ł^{\succcurlyeq v}(\chi, y)\right) \neq \emptyset$.

Conversely, if $\mathcal{A} \cap \left(\mathcal{D}_Ł^{\succcurlyeq u}(\chi, y) \cap \mathcal{D}_Ł^{\succcurlyeq v}(\chi, y)\right) \neq \emptyset \forall(\chi, y) \in \mathcal{D}_Ł$, then $\mathcal{A} \cap \mathcal{D}_Ł^{\succcurlyeq u}(\chi, y) \neq \emptyset$ and $\mathcal{A} \cap \mathcal{D}_Ł^{\succcurlyeq v}(\chi, y) \neq \emptyset$. By Theorem 7, $u_{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi) = u_{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_ŋ)}(\chi)$ and $v_{\mathcal{AT}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi) = v_{\mathcal{A}_{\mathcal{R}}(\mathcal{CL}_ŋ^{\succcurlyeq})}(\chi)$, it implies that $Ł_{\mathcal{A}}^{\succcurlyeq} = Ł_{\mathcal{AT}}^{\succcurlyeq}$. ■

Definition 13: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$, define

$$\Delta_{\text{Ł}}^{\succsim} = \bigwedge_{(x,y)\in\mathfrak{D}_{\text{Ł}}^{\succsim}} \left(\left(\vee\mathfrak{D}_{\text{Ł}}^{\succsim u}(x,y)\right) \wedge \left(\vee\mathfrak{D}_{\text{Ł}}^{\succsim v}(x,y)\right)\right);$$
$$\Delta_{\text{Ł}}^{\precsim} = \bigwedge_{(x,y)\in\mathfrak{D}_{\text{Ł}}^{\precsim}} \left(\left(\vee\mathfrak{D}_{\text{Ł}}^{\precsim u}(x,y)\right) \wedge \left(\vee\mathfrak{D}_{\text{Ł}}^{\precsim v}(x,y)\right)\right);$$
$$\Delta_{H}^{\succsim} = \bigwedge_{(x,y)\in\mathfrak{D}_{H}^{\succsim}} \left(\left(\vee\mathfrak{D}_{H}^{\succsim u}(x,y)\right) \wedge \left(\vee\mathfrak{D}_{H}^{\succsim v}(x,y)\right)\right);$$
$$\Delta_{H}^{\precsim} = \bigwedge_{(x,y)\in\mathfrak{D}_{H}^{\precsim}} \left(\left(\vee\mathfrak{D}_{H}^{\precsim u}(x,y)\right) \wedge \left(\vee\mathfrak{D}_{H}^{\precsim v}(x,y)\right)\right).$$

$\Delta_{\text{Ł}}^{\succsim}$, $\Delta_{\text{Ł}}^{\precsim}$, $\Delta_{H}^{\succsim}$, $\Delta_{H}^{\precsim}$ are the $\succsim$–lower, $\precsim$– lower, $\succsim$–upper and $\precsim$–upper approximate discernibility functions, respectively.

By employing Boolean reasoning methods, we can derive the subsequent theorem.

Theorem 9: For decision system $\mathcal{S} = (U, \mathcal{AT} \cup \{f\})$ with $\mathcal{A} \subseteq \mathcal{AT}$, then

1) $\mathcal{A}$ is $\succsim$–lower ADR $\iff \bigwedge\mathcal{A}$ is a prime implicant of $\Delta_{\text{Ł}}^{\succsim}$;
2) $\mathcal{A}$ is $\precsim$–lower ADR $\iff \bigwedge\mathcal{A}$ is a prime implicant of $\Delta_{\text{Ł}}^{\precsim}$;
3) $\mathcal{A}$ is $\succsim$–upper ADR $\iff \bigwedge\mathcal{A}$ is a prime implicant of $\Delta_{H}^{\succsim}$;
4) $\mathcal{A}$ is $\precsim$–upper ADR $\iff \bigwedge\mathcal{A}$ is a prime implicant of $\Delta_{H}^{\precsim}$;

Proof: Only (a) is proved, remaining can be proven similarly.

(a). If $\mathcal{A}$ is $\succsim$–lower ADR, then $\mathcal{A}$ is also a $\succsim$–lower ADCS. Using Theorem 8, we have $\mathcal{A} \cap \left(\mathfrak{D}_{\text{Ł}}^{\succsim u}(x,y) \cap \mathfrak{D}_{\text{Ł}}^{\succsim v}(x,y)\right) \neq \emptyset, \forall (x,y) \in \mathfrak{D}_{\text{Ł}}^{\succsim}$. Then we say that for every ą $\in \mathcal{A}$, there exist $(x,y) \in \mathfrak{D}_{\text{Ł}}^{\succsim}$ s:t $\mathcal{A} \cap \left(\mathfrak{D}_{\text{Ł}}^{\succsim u}(x,y) \cap \mathfrak{D}_{\text{Ł}}^{\succsim v}(x,y)\right) = \{ą\}$. If for every $(x,y) \in \mathfrak{D}_{\text{Ł}}^{\succsim}$ there exist ą $\in \mathfrak{D}_{\text{Ł}}^{\succsim}(x,y)$ such that $card\left(\mathcal{A} \cap \left(\mathfrak{D}_{\text{Ł}}^{\succsim u}(x,y) \cap \mathfrak{D}_{\text{Ł}}^{\succsim v}(x,y)\right)\right) > 2$ where ą $\in \mathcal{A} \cap \left(\mathfrak{D}_{\text{Ł}}^{\succsim u}(x,y) \cap \mathfrak{D}_{\text{Ł}}^{\succsim v}(x,y)\right)$, let $\mathcal{A}' = \mathcal{A} - \{ą\}$, then by Theorem 8, $\mathcal{A}'$ is a $\succsim$–lower ADCS, which opposes the notion that $\mathcal{A}$ is $\succsim$–lower ADR. This implies that $\bigwedge\mathcal{A}$ is a prime implicant of $\mathfrak{D}_{\text{Ł}}$.

Conversely, if $\bigwedge\mathcal{A}$ is a prime implicant of $_{\text{Ł}}$, then by Theorem 8. $\mathcal{A} \cap \left(\mathfrak{D}_{\text{Ł}}^{\succsim u}(x,y) \cap \mathfrak{D}_{\text{Ł}}^{\succsim v}(x,y)\right) \neq \emptyset, \forall (x,y) \in \mathfrak{D}_{\text{Ł}}^{\succsim}$. For every ą $\in \mathcal{A}$ there exist $(x,y) \in \mathfrak{D}_{\text{Ł}}^{\succsim}$ $s : t$ $\mathcal{A} \cap \left(\mathfrak{D}_{\text{Ł}}^{\succsim u}(x,y) \cap \mathfrak{D}_{\text{Ł}}^{\succsim v}(x,y)\right) = \{ą\}$. Consequently, for all $\mathcal{A}'$ is not the –lower ADCS. Thus, it follows that $\mathcal{A}$ is a $\succsim$–lower ADR. ■

VI. ILLUSTRATIVE EXAMPLE

Following Example 1, we compute the $\succsim$–lower ADR, $\precsim$–lower ADR, $\succsim$–upper ADR and $\precsim$–upper ADR of **Table 1**. Based on Definition 12, we have the ability to derive eight distinct types of distribution discernibility matrices. In this context, we exclusively demonstrate $\succsim^{u}$–lower, $\succsim^{v}$–lower, $\succsim^{u}$–upper, $\succsim^{v}$–upper ADDM, as illustrated in **Table 4**, **Table 5**, **Table 6**, and **Table 7**, correspondingly.

By using Definition 13, we calculate $\succsim$–lower, $\succsim$–upper approximate discernibility functions:

$$\Delta_{\text{Ł}}^{\succsim} = \bigwedge_{(x,y)\in\mathfrak{D}_{\text{Ł}}^{\succsim}} \left(\left(\vee\mathfrak{D}_{\text{Ł}}^{\succsim u}(x,y)\right) \wedge \left(\vee\mathfrak{D}_{\text{Ł}}^{\succsim v}(x,y)\right)\right);$$
$$\Delta_{\text{Ł}}^{\succsim} = (ȩ) \wedge (ą) \wedge (ç \wedge ḑ) \wedge (ç \wedge ȩ) \wedge (ą \wedge ḑ) \wedge (ȩ \wedge ç) \wedge (ȩ \wedge ą);$$
$$\Delta_{\text{Ł}}^{\succsim} = (ą) \wedge (ç) \wedge (d) \wedge (e).$$
$$\Delta_{H}^{\succsim} = \bigwedge_{(x,y)\in\mathfrak{D}_{H}^{\succsim}} \left(\left(\vee\mathfrak{D}_{\text{Ł}}^{\succsim u}(x,y)\right) \wedge \left(\vee\mathfrak{D}_{H}^{\succsim v}(x,y)\right)\right);$$
$$\Delta_{H}^{\succsim} = (ḅ \wedge ȩ) \wedge (ą \wedge ḑ) \wedge (ą \wedge ḑ \wedge ȩ);$$
$$\Delta_{H}^{\succsim} = (ą) \wedge (ḅ) \wedge (ḑ) \wedge (ȩ).$$

Consequently by Theorem 9, {ą, ç, ḑ, ȩ} is the $\succsim$–lower and {ą, ḅ, ḑ, ȩ} is the $\succsim$–upper ADRs for **Table 1**. We can say that, to retain the lower approxs based on Pythagorean fuzzy dominance of all the ascending unions of the decision classes, attribute ḅ can be omitted. For upper approxs attribute ç is redundant. Similarly, it can be prove that {ą, ḅ, ḑ, ȩ} is the $\precsim$–lower and {ą, ç, ḑ, ȩ} is the $\precsim$–upper ADRs for **Table 1**. That illustrate the validity of Theorem 12.

VII. CONCLUSION AND FUTURE WORK

Within this manuscript, we have created a comprehensive structure aimed at facilitating the generalization of DRSA. Our approach involves combining the idea of the Pythagorean fuzzy set with the DRSA and establishing the notion of the PFDRSA. Additionally, we incorporated the notion of ADRs into the PFDRS model, four different forms of ADRs are introduced, and the pragmatic methods for computing these reducts are also deliberated upon. In contrast to the prior DRSA, our dominance-based rough set model employs a Pythagorean fuzzy dominance relation, rather than a crisp or fuzzy dominance relation. The Illustrative example yielded valuable rules for redundant attributes. Moreover, for the practical application of our PFDRSA, numerous experimental analyses will be essential in the future.

Plans include expanding our research to include FF, q-ROP, SF, and TSF settings. Additionally, we will make them using multi-criteria decision-making techniques [49], [50], [51], [52], which may employ to tackle diverse and complicated engineering challenges.

DECLARATION OF INTERESTS

The authors confirm that they do not possess any recognizable financial or interpersonal conflicts that could have potentially influenced the research outlined in this publication. Furthermore, they assert that no conflicting interests are associated with this paper's release.

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