IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 6 September 2023, accepted 22 September 2023, date of publication 27 September 2023, date of current version 5 October 2023.

Digital Object Identifier 10.1109/ACCESS.2023.3320032

RESEARCH ARTICLE

A Supplier Selection Model Using the Triangular Fuzzy-Grey Numbers

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The research presented in this paper was conducted with the support of the DOC MOBILITY FELLOWSHIP GRANT provided by the La Commission de recherche de l'Université de Genève (COREC) from February to August 2023.

ABSTRACT Selecting suppliers involves making decisions based on multiple criteria and options, making it a multi-criteria decision-making (MCDM) problem. Optimal decisions can help the entire supply chain to reduce costs and increase efficiency. However, uncertainty in supplier selection can increase the risk of incorrect choices and unforeseen consequences, which may stem from criteria weights or supplier performance. To address these challenges, this paper presents the Triangular Fuzzy-Grey (TFG) system, an innovative approach for MCDM problems. Integrating grey numbers and triangular fuzzy numbers, the TFG system extends fuzzy logic. Grey systems and fuzzy numbers are valuable tools in MCDM, each with their own advantages and limitations. Grey systems exhibit robustness in handling uncertain and incomplete information, aided by their intuitive models. They offer prediction capabilities and adaptability, although they may provide approximate solutions and rely on expert judgment, lacking a comprehensive theoretical foundation. Fuzzy numbers excel in handling uncertainty, accommodating vague data, and expressing linguistic preferences. They facilitate criteria aggregation and accommodate various decision variables. Combining grey systems and fuzzy numbers enhances decision-making, leveraging their strengths to address uncertainty and improve accuracy. The TFG system effectively handles uncertainty by assigning higher probabilities to smaller, more certain areas. To demonstrate its effectiveness, an integrated TFG WLD-SAW model is proposed for green supplier selection, where Supplier No.2 with $hat\, \mathbb{SS}_i = 3.3202$ is selected as the best green supplier. Comparative analysis using the Zakeri-Konstantas weighted rankings similarity measure shows that TFG WLD achieves similar results to fuzzy and grey WLD methods in solving the green supplier selection problem.

INDEX TERMS Supplier selection, multi-criteria decision-making (MCDM), grey numbers, triangular fuzzy numbers, triangular fuzzy-grey system, the Zakeri–Konstantas weighted rankings similarity measure.

NOMENCLATURE

This paper adheres to certain conventions for representing vectors, matrices, and random variables, as is customary in scientific literature. These conventions are presented in the following Table 1.

The associate editor coordinating the review of this manuscript and approving it for publication was Yiqi Liu.

I. INTRODUCTION

Supply Chain Management (SCM) refers to the coordination and management of activities involved in producing and delivering products and services to customers. It encompasses all processes from procurement of raw materials, production, and manufacturing, to the final delivery of the products to the end customer [1]. Practical supply chain management aims to ensure the efficient and timely delivery of goods while

TABLE 1. The list of symbols and notations.

Symbol	Description
$\tilde{A}$	A triangular fuzzy number
$\otimes G$	A grey number
$\underline{G}, \overline{G}$	The higher and lower bounds of a grey number
$\widetilde{\otimes A}_{\mathrm{F}}$	A triangular grey-fuzzy number
$\widetilde{\otimes A}$	A triangular fuzzy-grey number (TFGN)
$\widetilde{\otimes A}^{\alpha}, \widetilde{\otimes A}^{\beta}$	Lower and higher boundaries of an TFGN
$\mathbb{Z}$	The integer numbers
$\mathbb{N}$	The natural numbers
$\underline{l_1}$,	The lower bound
$\overline{l_1}$,	The inner bound
$\underline{m_1}$,	The inner center
$\overline{m_1}$,	The outer center
$\underline{u_1}$	The outer bound
$\overline{u_1}$	The higher bound
$\Upsilon_{\widetilde{\otimes A}}$	The grey area
$-\Upsilon_{\widetilde{\otimes A}}$	The uncertain area
m	the number of alternatives in an MCDM problem
n	the number of in an MCDM problem
$\widehat{\mathbb{SS}}_{\iota}$	The relative magnitude of a TFGN
X_j^P	The matrix related to the WLD method
WW_j	Win in the WLD method
DD_j	Draw in the WLD method
LL_j	lose in the WLD method
$\mathbb{S}_j$	the scores of criteria in the WLD method
w_j'	The weights determined by Decision Makers
w_j	The weights of criteria
$\hat{A}$	The corresponding crisp number of $\tilde{A}$
$\widetilde{\otimes S}_{\iota}$	The TFG scores of suppliers
$\mathbb{A}$	The area of the blocks of information
$\mathbb{ZK}$	The Zakeri-Konstantas weighted rankings similarity measure (WRSM)
P_m^*	The maximum WRSM between two rankings

reducing costs and improving overall quality. Giri et al. [2] defined SCM as managing the movement of the firm's supplies, products, and services most efficiently and economically. SCM involves the collaboration and communication between various parties, including suppliers, manufacturers, distributors, and customers. This collaboration aids in ensuring that the supply chain operates smoothly and effectively, with minimal disruptions. Advanced technologies, such as artificial intelligence [3], [4], [5] and the Internet of Things [6], [7], [8], are being integrated into supply chain management to improve decision-making, increase visibility, and enhance overall performance. One of the most influential factors in increasing the performance of a supply chain is appropriate suppliers, which reduce the costs of materials and the time of product development by about 20% and improve the quality of materials by the same degree [9]. Moreover, appropriate suppliers participate in the four paramount competitive primacy: quality, delivery, flexibility, and cost [10]. As a non-stop process, supplier evaluation is an essential industrial problem in which various criteria are taken into account simultaneously to identify, assess, investigate, and deal with suppliers [11]. The selection of the best supplier is one of the critical factors that contribute to the firms' operational success, yet it consumes time and resources, which makes it vital for decreasing the overall cost of the supply chain [12]. Saputro et al. [13] argue that to extend the decision scope in selecting appropriate suppliers, the formulation of the supplier selection problems should incorporate sourcing strategy and selection criteria as the two primary dimensions.

The supplier selection problem could be well formulated into a complex multi-criteria decision-making (MCDM) problem. This complexity arises from a number of quantitative and qualitative factors affecting supplier alternatives as well as the inherent difficulty of making numerous trade-offs amongst these factors. There exists a rich literature on using MCDM methods for solving supplier selection problems. Various studies have investigated the role of MCDM methods in solving supplier selection problems, such as [14], [15], [16], [17], [18], [19], [20], [21], [22], and [23].

One of the main concerns in the selection of an appropriate supplier is the uncertainty which mainly emanates from the information regarding the suppliers and the firms' environments. In general, uncertainty in supplier selection is a common challenge that firms confront in procurement. As mentioned, the process of an appropriate supplier selection involves making critical decisions about the long-term health and success of the firm. The wrong choice can lead to costly consequences, such as delayed deliveries, subpar product quality, and bankruptcy. Given the high stakes involved, it is not surprising that many firms struggle with uncertainty in selecting suppliers. Lack of reliable information or access to complete/perfect information is one of the primary sources of uncertainty in supplier selection, which can be due to several reasons, including insufficient information provided by suppliers, limited data on supplier performance, and the dynamic nature of the market. Another source of uncertainty in supplier selection is the constant changes in the business environment, *e.g.*, market conditions, regulatory changes, and shifts in consumer demand which can all significantly impact suppliers' performance. This could be well correlated to the first source, which regards the lack of access to complete/perfect information about the firm's external factors. The third source is indeed the political and economic instability which increases noisy information and uncertainty in supplier selection. This uncertainty is reflected in solving the supplier selection problems using MCDM methods as well [24], [25], [26], [27], [28], [29], [30], [31].

Uncertainty is an inherent aspect of most MCDM problems and is distributed from the inputs of the MCDM algorithms to the outputs. A variety of sources could be addressed where the uncertainty emanates. The DMs may have limited information about the alternatives and the criteria being used, or the criteria may have vague definitions or conflicting values. Additionally, criteria values for the alternatives may be subject to random fluctuations or measurement errors. In general, six sources can be counted as the major sources of uncertainty in solving MCDM problems. The first source is the uncertainty in the inputs of MCDM algorithms which originates primarily from human involvement as the decision-maker (DM). This type of uncertainty injects uncertainty into solving MCDM problems through DM's behavior in the evaluation of situations/options, DM's expectations, judgments, opinions, levels of knowledge, and his/her perception of the environment and the reality (*e.g.,* see Dubois et al. [32] - using non-additive measures instead of using additive measures such as probability functions). Time also adds another layer of uncertainty in the inputs by incrementing Entropy [33]. The second source of uncertainty is the missing information in the complex MCDM problems in which the problem deals with several criteria with different natures (beneficial and non-beneficial criteria), mostly more than seven criteria [34]. The third type of uncertainty is generated by the MCDM methods due to *A:*using different philosophies and policies for extracting the best option and *B:*using diverse mathematical approaches for normalizing the decision matrix [35]. Another source that increases the uncertainty of using MCDM methods for MCDM problems analysis is the uncertainty in the validation of MCDM algorithms' outputs. It happens due to differences in the generated rankings for a same problem, *i.e.,* the decision-making paradox [36], [37], differences in selection of the best choice/alternative, Inherent deficiency in the conventional statistical measures for validation of results., *e.g.,* the adjusting the weights based on subjective judgments is a significant issue with sensitivity analysis or Spearman correlation needs more than a single case to validate the MCDM methods' results [38]. The fifth type of uncertainty comes from the decision-making's goals. When the decision-making is architected around more than a single goal, the interaction between goals and the degree of relationships of each criterion for reaching goals might involve subjective information, which adds uncertainty to the process. Finally, the last type of uncertainty generates from using different mathematical tools, *e.g.,* different extensions of fuzzy logic, probability functions, or using grey systems theory, etc., to formulate the uncertainty of MCDM algorithms' inputs, which leads to 1. Providing dissimilar values for the uncertain data; 2. extracting dissimilar values of information form the decision matrix; 3. increasing the complexity of decision matrix due to the various philosophies and approaches, with different complexity, they employ to interpret the uncertainty. This escalates the risk of missing information during the analysis of the decision matrix consequently (*see the second type of uncertainty*); 4. delivering dissimilar outputs for a samethe same problem using a same MCDM algorithm due to the same motive that increases the uncertainty in the previous product of sixth type of uncertainty. Some of the aforesaid uncertainty types and the possible ways for dealing with them are discussed in Zakeri and Konstantas [12] work. Dealing with uncertainty of the inputs derived from the MCDM problems requires the use of additive methods that can account for it, such as stochastic programming [39], fuzzy set theory, grey systems theory [40], [41], rough set [42], [43], or probabilistic methods [44]. However, the choice of method will depend on the nature and extent of uncertainty, as well as the DMs' goals and preferences. Despite the use of these methods, uncertainty may still play a role in the final decision.

In spite of the fact that fuzzy logic revolutionized the formulation of uncertainty in solving MCDM problems and also the fact that their applications for dealing with uncertainty in solving supplier selection/MCDM problems are extremely popular, they have received some criticisms in mathematics and computer science [45], [46], [47], [48], [49]. To put it in a nutshell, according to the mentioned studies, four main criticisms are collected as follows: 1. fuzzy logic and fuzzy numbers lack the mathematical rigor of traditional mathematical concepts such as binary logic and real numbers; 2. the mathematical representation of fuzzy concepts can be complex and challenging to understand, especially for those without a background in mathematics; 3. fuzzy logic and fuzzy numbers can be ambiguous and open to interpretation, leading to inconsistent results in some applications; and 4. critics argue that fuzzy logic and fuzzy numbers may not accurately represent real-world phenomena and may be too limited to provide valuable solutions in many situations.

On the other hand, the applications of grey systems theory and grey numbers in solving supplier selection received attention in the past two decades. Although they are relatively new concepts, they have shown some advantages in dealing with uncertainty. Grey numbers and grey systems theory provide a number of advantages for data analysis and decision-making in situations where incomplete or uncertain information is present [50], [51], [52], [53], [54], [55], [56], [57]. Respecting to the above studies, the advantages of employing grey systems theory and grey numbers could be addressed as: 1. grey numbers and grey systems theory are specifically designed to handle situations where data is uncertain or incomplete, making them ideal for real-world applications where information is often imperfect; 2. the use of grey numbers and grey systems can lead to improved accuracy in predictions and decision-making compared to traditional methods that do not account for uncertainty; 3. grey numbers and grey systems theory allow for the integration of subjective information, such as expert opinions, into the decision-making process, providing a more comprehensive view of the situation; 4. grey numbers and grey systems are flexible and can be applied to a wide range of applications, including engineering, economics, finance, and social sciences; and 5. by incorporating uncertainty into the decision-making process, grey numbers and grey systems can lead to better and more informed decisions, increasing the chances of success.

Grey systems bring robustness, simplicity, prediction, and adaptability to MCDM, while fuzzy logic excels in uncertainty handling, aggregation, linguistic expression, and flexibility. By taking into account the benefits of grey systems and fuzzy logic, as well as the limitations associated with using them independently to address uncertainty in supplier selection problems, integrating these approaches presents a solution that capitalizes on the advantages of both theories while mitigating their respective shortcomings. Integrating grey systems and fuzzy logic enhances decision-making capabilities by leveraging their strengths and mitigating individual limitations. This combination effectively handles uncertainty and imprecision, realistically representing decision problems. Decision outcomes become more accurate by improving precision while considering uncertain information. Furthermore, integration offers greater flexibility in modeling complex decision scenarios, accommodating diverse criteria and decision variables. This paper aims to make utilization of the benefits of grey numbers in a combined approach fashioned on a numeric platform provided by triangular fuzzy numbers. The present article is a continuation of the Zakeri and Keramati [58] work, in which they combined fuzzy numbers with grey numbers in order to propose a supplier evaluation model when uncertainty is involved in the problem. The rest of the paper is composed of seven sections. The approach that Zakeri and Keramati [58] is represented in the second section. The new combination of grey numbers and fuzzy numbers is proposed in the third section. In the fourth section, the WLD method is represented to compute the weights of criteria. The new uncertainty model is applied to a real-world case in the fifth. The sixth section is devoted to discussion, and finally, the conclusion and future work are denoted in the seventh section.

II. TRIANGULAR GREY-FUZZY NUMBERS (TGFNs)

Grey Systems Theory (GST) and Fuzzy Logic are two major mathematical theories used to model and control complex and uncertain systems. While they share some similarities, they also have key differences. GST is a mathematical framework that deals with incomplete or uncertain information in system modeling and analysis. It provides tools and methods to handle such situations and gain insights into real-world system behavior. GST finds applications in engineering, economics, and decision-making. On the other hand, Fuzzy Logic is a mathematical framework that allows for the representation and manipulation of uncertain or vague information. It extends classical set theory by introducing degrees of membership for elements in sets. Fuzzy Logic is commonly used in control engineering, artificial intelligence, and decision-making. The main distinction between GST and Fuzzy Logic lies in their approaches to handling uncertainty. GST focuses on modeling and analyzing systems with incomplete or uncertain information, using grey numbers to represent uncertainty. On the other hand, Fuzzy Logic focuses on representing and manipulating uncertain or vague information using fuzzy sets and membership degrees. Furthermore, GST is widely applied in engineering, economics, and decision-making domains, where information is limited. Fuzzy Logic, on the other hand, is particularly useful in fields where information is uncertain or vague. In terms of implementation, GST requires more mathematical sophistication and complexity compared to Fuzzy Logic. GST involves the use of mathematical models and algorithms to analyze systems and make predictions. Fuzzy Logic, on the other hand, relies on fuzzy sets and membership degrees to represent uncertainty.

The triangular grey-fuzzy (TGF) numbers were first proposed in Zakeri and Keramati [58] work, where the fuzzy numbers and grey numbers were combined to provide an extended interval of numbers to convert decision-makers' (DMs) opinions, which mostly appear as the linguistic variables/terms into numbers. They presented a systematic combination of grey and fuzzy numbers to provide more range of numbers with sets of certain numbers that were numerically higher than what both grey systems and fuzzy math are proposing to capture more certain information in solving MCDM problems under uncertain information, compared to its roots. Their original work can be found as the following, in which the fuzzy membership functions with their corresponding numeric grey intervals have been combined to approximate what a decision-maker intends to their actual number in the corresponding scale. The proposed numbers are architected on a triangular fuzzy number for formulating uncertainty.

TABLE 2. The scale of attributes of rating of $\otimes G$.

Linguistic variable	Grey number
Very Poor (VP)	[0, 1]
Poor (P)	[1, 3]
Medium Poor (MP)	[3, 4]
Fair (F)	[4, 5]
Medium Good (MG)	[5, 6]
Good (G)	[6, 9]
Very Good (VG)	[9,10]

To covert the triangular fuzzy numbers into the triangular fuzzy-grey numbers (TGFN) for the translation of the linguistic variables, mainly for scoring the values of the alternatives against criteria and to determine the importance weights of criteria, the original work used two different scales of. In Table 2, the development of TGFNs for rating alternatives (scoring) based on the scale of attributes of rating of $\otimes G$ is exhibited, where $\otimes G$ represents a grey number.

To compute the $\widetilde{\otimes A}_{\mathrm{F}}$, the several transformation equations were employed. An example of the equations is provided in Equation 1, where $\otimes G_1 = \left[\underline{G_1}, \overline{G_1}\right] = [0, 1]$. In order to compute and convert linguistic variables for weighing the criteria, the same process is executed.

$$\widetilde{\otimes A}_{VP} = \begin{cases} \mu_{\tilde{A}}\left(\underline{G_1}\right) = \begin{cases} \dfrac{\left(\underline{G_1} - 0\right)}{(0-0)}, & 0 \ll \underline{G_1} \ll 0 \\ \dfrac{\left(1 - \underline{G_1}\right)}{(1-0)}, & 0 \ll \underline{G_1} \ll 1 \\ 0, & \text{otherwise} \end{cases} \\ \mu_{\tilde{A}}\left(\overline{G_1}\right) = \begin{cases} \dfrac{\left(\overline{G_1} - 0\right)}{(0-0)}, & 0 \ll \overline{G_1} \ll 1, \\ \dfrac{\left(1 - \overline{G_1}\right)}{(3-1)}, & 1 \ll \overline{G_1} \ll 3, \\ 0, & \text{otherwise. ;} \end{cases} \end{cases} \tag{1}$$

III. THE TRIANGULAR FUZZY-GREY SYSTEM

The work done by Zakeri and Keramati struggles with some issues. In the article, the focus was merely on transforming linguistic terms into a combined system of fuzzy and grey numbers, in which the functions were not developed. The second issue is the incompleteness of the proposed system, in which no solutions were provided for the α cut in their combination process. As the third issue, the uncertainty distribution was not explained in the work. The final issue was the purpose of their research, where they used two probability functions instead of two certain boundaries of grey numbers in order to expand the range but did not consider the probability distribution of the certain information in which instead of two certain values and linear distance, the proposed model was defined based on six uncertain values provided by two fuzzy probability functions. Maintaining the advantages and premises of the proposed model in Zakeri and Keramati, a new model is proposed to address the mentioned issues in this section.

Developed on the basis of TFN and using advantages of both fuzzy numbers and grey numbers g a triangular fuzzygrey system (TFG) is a numeric system for formulating uncertainty in a decision-making process. TGF is a modified version of TGF which provides a more accurate and dynamic framework for interpretation of uncertainty and facilitates its translation into a set of numbers. The definition of TFG is provided as follows: if U stands for a universe of discourse, $\widetilde{\otimes A}^{\alpha}$, $\widetilde{\otimes A}^{\beta} \subseteq \widetilde{\otimes A}$, and the characteristic function value of x with respect to $\widetilde{\otimes A}$ states with a grey number $v^{\pm}$, then a $\widetilde{\otimes A}$ is a TFG set, $\chi_G : U \rightarrow D[0, 1]^{\pm}$. $\widetilde{\otimes A}$ could be also defined as a TFG number (see Equation 2), where:

$$\widetilde{\otimes A}^{\pm} \in \left[\widetilde{\otimes A}_{\alpha}^{-}, \widetilde{\otimes A}_{\beta}^{+}\right] = \left\{y \in \widetilde{\otimes A}^{\pm} \middle| \widetilde{\otimes A}_{\alpha}^{-} \leq y \leq \widetilde{\otimes A}_{\beta}^{+}\right\}, \\ \times \widetilde{\otimes A}^{\pm} \in U; \tag{2}$$

where y is a random/natural number and $y \in \mathbb{Z}$, $y \in \mathbb{N}$, and $y \neq 0$.

A TFG number is bounded between two fuzzy number, where $\widetilde{\otimes A}_{\alpha}^{-}$ and $\widetilde{\otimes A}_{\beta}^{+}$ are the lower and higher bounds, respectively, therefore $\widetilde{\otimes A}^{\alpha}$ and $\widetilde{\otimes A}^{\beta}$ could be defined as sets of ordered pairs, where $\mu_{\widetilde{\otimes A}^{\alpha}}(x) : X \rightarrow [0, 1]$ and $\mu_{\widetilde{\otimes A}^{\beta}}(x) : X \rightarrow [0, 1]$ are the fuzzy membership functions

(Equations 3-15), then:

$$\widetilde{\otimes A}^{\alpha} = \left\{\left\langle x, \mu_{\widetilde{\otimes A}}\alpha(x)\right\rangle \mid x \in \mathbb{U}\right\}; \tag{3}$$

$$\widetilde{\otimes A}^{\beta} = \left\{\left\langle x, \mu_{\widetilde{\otimes A}}(x)\right\rangle \mid x \in \mathbb{U}\right\}; \tag{4}$$

$$\widetilde{\otimes A} = \left[\widetilde{\otimes A}^{\alpha}, \widetilde{\otimes A}^{\beta}\right]; \tag{5}$$

$$\widetilde{\otimes A}, \widetilde{\otimes A}^{\beta} = (\underline{G}, C, \bar{G}), \quad \underline{G} = \left[\underline{l_1}, \overline{l_1}\right], \quad C \in \left\{\underline{m_1}, \overline{m_1}\right\}, \quad \bar{G} = \left[\underline{u_1}, \overline{u_1}\right]; \tag{6}$$

$$\widetilde{\otimes A}^{\alpha} = \left(\left[\underline{l_1}, \overline{l_1}\right], \underline{m_1}, \left[\underline{u_1}, \overline{u_1}\right]\right); \tag{7}$$

$$\widetilde{\otimes A}^{\beta} = \left(\left[\underline{l_1}, \overline{l_1}\right], \overline{m_1}, \left[\underline{u_1}, \overline{u_1}\right]\right); \tag{8}$$

$$\mu^{\alpha}_{\widetilde{\otimes A}}(x) = \begin{cases} \dfrac{\left[x-\overline{l_1}, x-\underline{l_1}\right]}{\left[m_1\overline{l_1}, \underline{m_1}-\underline{l_1}\right]}, & x \in \left[\underline{l_1}, \overline{l_1}\right], x \geq \underline{m_1} \\ \dfrac{\left[\underline{u_1}-x, \overline{u_1}-x\right]}{\left[\underline{u_1}-\underline{m_1}, \overline{u_1}-\underline{m_1}\right]}, & x \in \left[\underline{u_1}, \overline{u_1}\right], x \leq \underline{m_1} \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

$$\mu_{\widetilde{\otimes A}^{\alpha}}(x) = \begin{cases} \dfrac{\left[x-\overline{l_1}, x-\underline{l_1}\right]}{\left[\overline{m_1}-\overline{l_1}, \overline{m_1}-\underline{l_1}\right]}, & x \in \left[\underline{l_1}, \overline{l_1}\right], x \geq \overline{m_1} \\ \dfrac{\left[\underline{u_1}-x, \overline{u_1}-x\right]}{\left[\underline{u_1}-\overline{m_1}, \overline{u_1}-\overline{m_1}\right]}, & x \in \left[\underline{u_1}, \overline{u_1}\right], x \leq \overline{m_1} \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

$$\mu_{\widetilde{\otimes A}} = \begin{cases} \dfrac{\left[x-\overline{l_1}, x-\underline{l_1}\right]}{\left[\underline{m_1}-\overline{l_1}, \underline{m_1}-\underline{l_1}\right]}, & x \in \left[\underline{l_1}, \overline{l_1}\right], x \geq \underline{m_1} \\ \dfrac{\left[\underline{u_1}-x, \overline{u_1}-x\right]}{\left[\underline{u_1}-\overline{m_1}, \overline{u_1}-\underline{m_1}\right]}, & x \in \left[\underline{u_1}, \overline{u_1}\right], x \leq \underline{m_1} \\ \dfrac{\left[x-\overline{l_1}, x-\underline{l_1}\right]}{\left[\overline{m_1}-\overline{l_1}, \overline{m_1}-\underline{l_1}\right]} & x \in \left[\underline{l_1}, \overline{l_1}\right], x \geq \overline{m_1}; \\ \dfrac{\left[\underline{u_1}-x, \overline{u_1}-x\right]}{\left[\underline{u_1}-\overline{m_1}, \overline{u_1}-\overline{m_1}\right]}, & x \in \left[\underline{u_1}, \overline{u_1}\right], x \leq \overline{m_1} \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

where

$$\underline{m_1} = \min \begin{cases} \dfrac{\underline{l_1}+\overline{u_1}}{2} \\ \dfrac{\overline{l_1}+\underline{u_1}}{2} \end{cases}, \quad \underline{m_1} > \overline{m_1}; \tag{12}$$

$$\overline{m_1} = \max \begin{cases} \dfrac{\underline{l_1}+\overline{u_1}}{2} \\ \dfrac{\overline{l_1}+\underline{u_1}}{2} \end{cases}, \quad \underline{m_1} > \overline{m_1}; \tag{13}$$

$$\overline{u_1} - \underline{l_1} \cong 1, \ \ \underline{l_1} \cong 0, \ \overline{u_1} \cong 1, \ \overline{u_1} \cong 1+\xi, \ \underline{l_1} \cong 0+\xi; \tag{14}$$

$$\overline{u_1} - \underline{l_1} \cong 1 \mp \xi, \ \ \underline{l_1} \cong 0, \ \overline{u_1} \cong 1, \ \overline{u_1} \cong 1+\xi, \ \underline{l_1} \cong 0+\xi; \tag{15}$$

$\underline{l_1}$ stands for the lower bound, $\overline{l_1}$ is inner bound, $\underline{m_1}$ denotes inner center, $\overline{m_1}$ expresses outer center, $\underline{u_1}$ is outer bound, $\overline{u_1}$ stands higher bound. A TFG number, and its boundaries are illustrated in Figures 1-3.

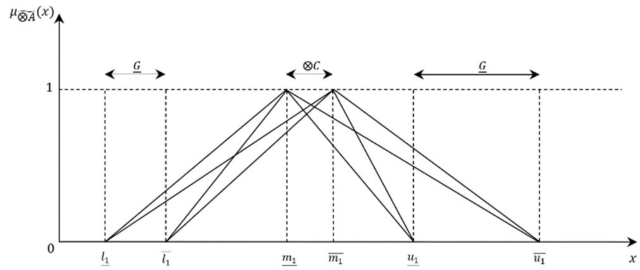

FIGURE 1. A triangular fuzzy-grey system.

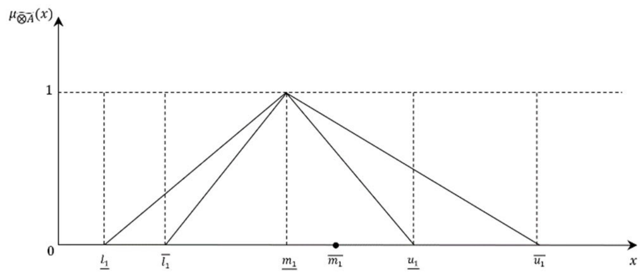

FIGURE 2. The lower bound of a triangular fuzzy-grey system.

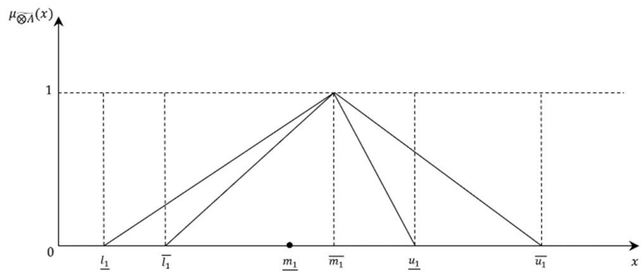

FIGURE 3. The upper bound of a triangular fuzzy-grey system.

A. THE OPERATIONS OF THE TFGNS

The operations of TFG numbers are shown in the following equations (16)-(33). These equations rely on the utilization of grey and TFN operations. The fundamental operations for fuzzy triangular numbers are detailed in [59], [60] while the grey operations can be found in [61]. Each operation' proof is provided in Appendix A.

if

$$\begin{aligned} \widetilde{\otimes A} &= \left[\widetilde{\otimes A}^{\alpha}, \widetilde{\otimes A}^{\beta}\right], \\ \widetilde{\otimes A}^{\alpha} &= \left(\left[\underline{l_1}, \overline{l_1}\right], \underline{m_1}, \left[\underline{u_1}, \overline{u_1}\right]\right), \\ \widetilde{\otimes A}^{\beta} &= \left(\left[\underline{l_1}, \overline{l_1}\right], \overline{m_1}, \left[\underline{u_1}, \overline{u_1}\right]\right); \end{aligned} \tag{16}$$

then

- *For addition*

$$\begin{aligned} &\widetilde{\otimes A_1} + \widetilde{\otimes A_2} \\ &= \left(\left[\underline{l_1}, \overline{l_1}\right], \overline{m_1}, \left[\underline{u_1}, \overline{u_1}\right]\right) + \left(\left[\underline{l_2}, \overline{l_2}\right], \overline{m_2}, \left[\underline{u_2}, \overline{u_2}\right]\right); \end{aligned} \tag{17}$$

$$\widetilde{\otimes A_1} + \widetilde{\otimes A_2} = \begin{cases} \left(\left[\underline{l_1}+\underline{l_2}, \overline{l_1}+\overline{l_2}\right], \underline{m_1}+\underline{m_2}, \left[\underline{u_1}+\underline{u_2}, \overline{u_1}+\overline{u_2}\right]\right) \\ \left(\left[\underline{l_1}+\underline{l_2}, \overline{l_1}+\overline{l_2}\right], \overline{m_1}+\overline{m_2}, \left[\underline{u_1}+\underline{u_2}, \overline{u_1}+\overline{u_2}\right]\right); \end{cases} \tag{18}$$

- *For Additive inverse:*

$$-\widetilde{\otimes A} = \left[-\widetilde{\otimes A^{\beta}}, -\otimes A^{\alpha}\right]; \tag{19}$$

$$-\widetilde{\otimes A} = \begin{cases} \left(\left[-\overline{u_1}, -\underline{u_1}\right], -\overline{m_1}, -\left[-\overline{l_1}, -\underline{l_1}\right]\right) \\ \left(\left[-\overline{u_1}, -\underline{u_1}\right], -\underline{m_1}, -\left[-\overline{l_1}, -\underline{l_1}\right]\right); \end{cases} \tag{20}$$

- *For Substraction:*

$$\otimes \widetilde{A_1} - \widetilde{\otimes} A_2 = \begin{cases} \left(\left[\underline{l_2}-\overline{u_1}, \overline{l_2}-\underline{u_1}\right], \underline{m_2}-\overline{m_1}, \left[\underline{u_2}-\overline{l_1}, \overline{u_2}-\underline{l_1}\right]\right) \\ \left(\left[\underline{l_2}-\overline{u_1}, \overline{l_2}-\underline{u_1}\right], \overline{m_2}-\underline{m_1}, \left[\underline{u_2}-\overline{l_1}, \overline{u_2}-\underline{l_1}\right]\right); \end{cases} \tag{21}$$

- *For Multiplication:*

$$\widetilde{\otimes A_1} \times \widetilde{\otimes A_2} \cong \Big[\min\Big\{\widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\alpha}}, \widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\beta}}, \widetilde{\otimes A_1^{\beta}} \times \widetilde{\otimes A_2^{\alpha}}, \widetilde{\otimes A_1^{\beta}} \times \widetilde{\otimes A_2^{\beta}}\Big\}, \max\Big\{\widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\alpha}}, \widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\beta}}, \widetilde{\otimes A_1^{\beta}} \times \widetilde{\otimes A_2^{\alpha}}, \widetilde{\otimes A_1^{\beta}} \times \widetilde{\otimes A_2^{\beta}}\Big\}\Big] \tag{22}$$

- *For Division:*

$$\widetilde{\otimes A_1} \times \widetilde{\otimes A_2}^{-1} = \left[\widetilde{\otimes A_1^{\alpha}}, \widetilde{\otimes A_1^{\beta}}\right] \times \left[\frac{1}{\widetilde{\otimes A_2^{\alpha}}}, \frac{1}{\widetilde{\otimes A_2^{\beta}}}\right]; \tag{23}$$

- *Crisp numbers - Multiplication:*

$$\widetilde{\otimes A_1} \times r = \left[\left(\left[r \times \underline{l_1}, r \times \overline{l_1}\right], r \times \underline{m_1}, \left[r \times \underline{u_1}, r \times \overline{u_1}\right]\right), \left(\left[r \times \underline{l_1}, r \times \overline{l_1}\right], r \times \overline{m_1}, \left[r \times \underline{u_1}, r \times \overline{u_1}\right]\right)\right]; \tag{24}$$

- *Crisp numbers - Division:*

$$\widetilde{\otimes A_1} \times r^{-1} = \left[\left(\left[\frac{1}{r} \times \underline{l_1}, r \times \overline{l_1}\right], \frac{1}{r} \times \underline{m_1}, \left[\frac{1}{r} \times \underline{u_1}, r \times \overline{u_1}\right]\right), \left(\left[\frac{1}{r} \times \underline{l_1}, \frac{1}{r} \times \overline{l_1}\right], \frac{1}{r} \times \overline{m_1}, \left[\frac{1}{r} \times \underline{u_1}, \frac{1}{r} \times \overline{u_1}\right]\right)\right]; \tag{25}$$

B. CONVERTING TFN INTO TFGN

Converting an TFN to an TFGN follows the Equations 26-57, where $\tilde{A}$ is an TFN. ξ refers to the amount of uncertain information ignored through the process and ξ value could be assigned by the DMs.

$$\underline{l} \cong \frac{l}{\max\{l, m, u\}} - \xi; \tag{26}$$

$$\overline{l} \cong \frac{l}{\max\{l, m, u\}} + \frac{\xi}{\underline{l}}; \tag{27}$$

$$\underline{m} \cong \frac{m}{\max\{l, m, u\}} + \frac{\xi}{\overline{l_L}} + \xi; \tag{28}$$

$$\overline{m} \cong \underline{m_1} + \xi; \tag{29}$$

$$\underline{u} \cong \frac{u}{\max\{l, m, u\}} - \frac{\xi}{\overline{m}}; \tag{30}$$

$$\overline{u} \cong \underline{u} + \xi; \tag{31}$$

if
$m = u$
then

$$\underline{m} \cong \frac{m}{\max\{l, m, u\}} - \frac{\xi}{\overline{l_L}}; \tag{32}$$

$$\underline{u}, \overline{u} \cong 1; \tag{33}$$

Example:
if $\tilde{A} = (1.812, 2.673, 5.034)$,
then

$$\widetilde{\otimes A} = [([0.356, 0.364], 0.520, [0.527, 1]), ([0.356, 0.364], 0.524, [0.527, 1])], \quad \xi = 0.004.$$

Assume $\otimes G = [G, \bar{G}]$ is a grey number, to transform it into an TFGN, the center has to be determined first (see Equation 34) to transform it to an TFN, where $\widetilde{G} = [\underline{G}, G, \bar{G}]$, and the probability of the existence of the certain information is more around the upper bound, $\bar{G}$.

$$G \cong (0.3 \times \underline{G}) + (0.6 \times \overline{G}); \tag{34}$$

Example:
if $\otimes G = [0.91, 1.12]$
then
$\tilde{G} = [0.91, 1.057, 1.12]$
then

$$\widetilde{\otimes A} = [([0.804, 0.821], 0.936, [0.952, 1]), ([0.804, 0.821], 0.944, [0.952, 1])], \xi = 0.008.$$

C. UNCERTAINTY DISTRIBUTION

In this section, the distribution of uncertainty in an TFG system is discussed. If U be the universe where x belongs to and the degree of its membership defines by Equations 3-15, the value of the block of information a TFG system generated from equals 1. The block is illustrated in Figure 4, where the black block shows the symmetric uncertain information block, whose value equals -1. In the figure, an TFGN is bounded between $[\underline{l}, \overline{l}]$, $[\overline{l}, \underline{m}]$, $[\underline{m}, \overline{m}]$, $[\overline{m}, \underline{u}]$, and $[\underline{u}, \overline{u}]$, embedded into a block of information which its value approximately equals 1. The computation process of the TFGNs runs in this block and certain information is located among the aforesaid intervals with different probabilities.

The value of the grey area where information is fluctuating is computed by following Equation 35, and the calculation of uncertain area follows Equation 36, in which $\Upsilon_{\widetilde{\otimes A}}$ and $-\Upsilon_{\widetilde{\otimes A}}$

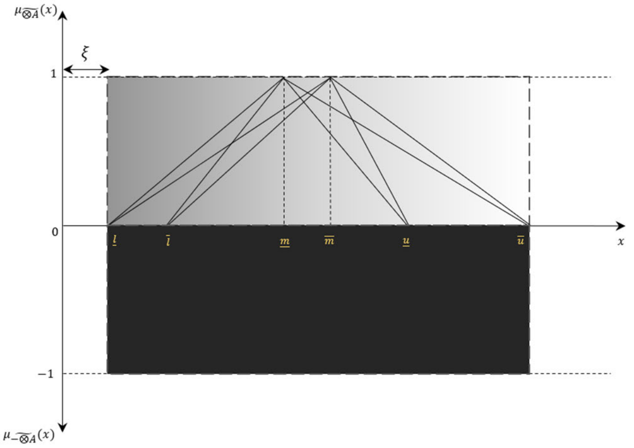

FIGURE 4. The information block where a TFG system belongs to and its symmetric uncertain block.

stand for the grey and uncertain areas.

$$\Upsilon_{\widetilde{\otimes A}} = \frac{\overline{u_1} - \underline{l_1}}{2}; \tag{35}$$

$$-\Upsilon_{\widetilde{\otimes A}} = \frac{\overline{u_1} - \underline{l_1}}{2} - 1; \tag{36}$$

The distribution of the probability in an TFG system, the spectrums the certain information could be probably located, and their symmetric uncertain reflection spectrums are shown in Figure 5.

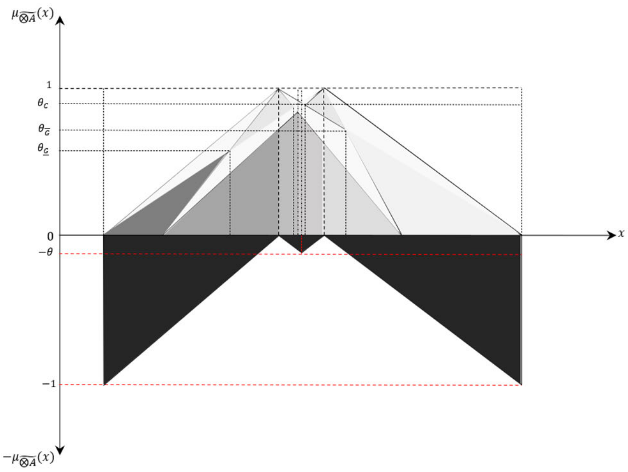

FIGURE 5. The distribution of uncertainty in a triangular fuzzy-grey system.

Similar to every dynamic system constructed on information, there is a missing amount of information in a TFG system as well, which increases the uncertainty of embedded information. Each TFG system is architected on eight different information spectrums. The missing information can be computed by the entropy of probable fluctuation of information between these eight spectrums (see Appendix B).

D. COMPARING TFG NUMBERS

Two TFG numbers cannot be compared independently since both numbers' higher bounds are equal; therefore, they ought to be from the same set, such as a set of options' ranks obtained from solving an MCDM problem.

The results of TFG MCDM methods are includes $(n \times m)$ TFGNs for the whole set and n TFGNs for the individuals, where m and n stands for the number of alternatives and criteria in an MCDM problem respectively. To compare them, the comparison processes out to implement on their roots (TFNs). Let $\widetilde{A}_1 = (l_1, m_1, u_1)$ and $\widetilde{A}_2 = (l_2, m_2, u_2)$ be two TFNs, then the two TFGNs could be computed as Equations 37-42:

$$\oplus\underline{l}_z = \frac{\oplus\underline{l}_z}{\max\{\oplus\overline{u_i}\}}, \quad z \in i, \ i = \{1, 2, \ldots, m\}; \tag{37}$$

$$\oplus\overline{l}_z = \frac{\oplus\overline{l}_1}{\max\left\{\overline{\oplus u_i}\right\}}, \quad z \in i, \ i = \{1, 2, \ldots, m\}; \tag{38}$$

$$\oplus\underline{m}_z = \frac{\oplus\underline{m}_1}{\max\{\oplus\overline{u_i}\}}, \quad z \in i, \ i = \{1, 2, \ldots, m\}; \tag{39}$$

$$\oplus\overline{m}_z = \frac{\oplus\overline{m}_z}{\max\{\oplus\overline{u_i}\}}, \quad z \in i, \ i = \{1, 2, \ldots, m\}; \tag{40}$$

$$\oplus\underline{u}_z = \frac{\oplus\underline{u}_z}{\max\{\oplus\overline{u_i}\}}, \quad z \in i, \ i = \{1, 2, \ldots, m\}; \tag{41}$$

$$\oplus\overline{u}_1 = \frac{\oplus\overline{u}_1}{\max\{\oplus\overline{u_i}\}}, \quad z \in i, \ i = \{1, 2, \ldots, m\}; \tag{42}$$

Hence the comparison and its result follow the following conditions:

Condition 1

if $\overline{u}_2 > \overline{u}_1$ and $\underline{u}_2 > \underline{u}_1$

then $\widetilde{\otimes A_2} > \widetilde{\otimes A_1}$, or vice versa.

Condition 2

if $\overline{u}_2 = \overline{u}_1$ and $\underline{u}_2 > \underline{u}_1$

then $\widetilde{\otimes A_2} > \widetilde{\otimes A_1}$, or vice versa.

Condition 3

if $\overline{u}_2 = \overline{u}_1$, $\underline{u}_2 = \underline{u}_1$, and $\overline{u}_2 - \overline{m}_2 < \overline{u}_1 - \overline{m}_1$

then $\widetilde{\otimes A_2} > \widetilde{\otimes A_1}$, or vice versa.

Condition 4

if $\overline{u}_2 = \overline{u}_1$, $\underline{u}_2 = \underline{u}_1$, $\overline{u}_2 - \overline{m}_2 = \overline{u}_1 - \overline{m}_1$, and $\overline{u}_2 - \underline{m}_2 < \overline{u}_1 - \underline{m}_1$

then $\widetilde{\otimes A_2} > \widetilde{\otimes A_1}$, or vice versa.

Condition 5

- if $\overline{u}_2 = \overline{u}_1$, $\underline{u}_2 = \underline{u}_1$, $\overline{u}_2 - \overline{m}_2 = \overline{u}_1 - \overline{m}_1$, $\overline{u}_2 - \underline{m}_2 = \overline{u}_1 - \underline{m}_1$, and $\overline{u}_2 - \overline{l}_2 < \overline{u}_1 - \overline{l}_1$

then $\widetilde{\otimes A_2} > \widetilde{\otimes A_1}$, or vice versa.

Condition 6

if $\overline{u}_2 = \overline{u}_1$, $\underline{u}_2 = \underline{u}_1$, $\overline{u}_2 - \overline{m}_2 = \overline{u}_1 - \overline{m}_1$, $\overline{u}_2 - \underline{m}_2 = \overline{u}_1 - \underline{m}_1$, $\overline{u}_2 - \overline{l}_2 = \overline{u}_1 - \overline{l}_1$, and $\overline{u}_2 - \underline{l}_2 < \overline{u}_1 - \underline{l}_1$

then $\widetilde{\otimes A_2} > \widetilde{\otimes A_1}$, or vice versa.

The comparison could also be conducted by Equation 43 based on the higher value of $\widehat{\mathbb{SS}}_l$.

$$\hat{\mathbb{SS}}_i = \sum_i \oplus\mathbb{A}_i, \ \tilde{\oplus}\mathbb{A}_i = \left\{\oplus\overline{l}_i, \oplus\overline{m}_i, \oplus\overline{u}_i\right\}, \quad i = \{1, 2, \ldots, m\}; \tag{43}$$

IV. THE WLD METHOD

Introduced by Zakeri et al., [11], the WLD method is a powerful MCDM subjective weighting method that extracts weights of criteria from DMs' opinions with a simple process functioning on three concepts of winning, losing, and drawing. WLD uses pairwise comparison and DMs' opinions in its operation. In the process, each concept receives a unique value, where the winner criterion collects 3, each criterion obtains 1 in a draw competition, and 0 is the value the loser criterion receives. The following steps conduct computing weights through the WLD algorithm.

Step 1. Evaluating criteria using a scale where the upper and lower bounds are 1,10, and the center is 5, in which DM can select any rational numbers between 1 to 5, and 5 to 10.

TABLE 3. The pairwise comparison of criteria.

	c_{11}		c_{nq}
c_{11}	1	$\cdots$	$(WW_{1,n} \vee DD_{1,n} \vee LL_{1,n})$
	$\vdots$	$\ddots$	$\vdots$
c_{nz}	$(WW_{n,1} \vee DD_{n,1} \vee LL_{n,1})$	$\cdots$	1

Step 2. Establishing the pairwise comparison matrix (see Table 3) presented displayed in Equations 44-47).

$$X_{j\ QZ}^{P} = \left(WW_{zq} \vee DD_{zq} \vee LL_{zq}\right), \quad Q = \{1, \ldots, q\}, \quad Z = \{1, \ldots, Z\}, \quad c_{nz} = c_{nq} \in c_j,\ j = \{1, \ldots, n\}; \tag{44}$$

$$WW_j = \sum_{Q=1}^{q} WW_{zq}; \tag{45}$$

$$DD_j = \left\langle \sum_{Q=1}^{q} DD_{zq} \right\rangle - 1; \tag{46}$$

$$LL_j = \sum_{Q=1}^{q} LL_{zq}; \tag{47}$$

where X_j^P stands for the matrix, where WW_j, DD_j, and LL_j stand for win, lose, and draw, respectively; WW_j describes the situation where one criterion is more important than the other one. In contrast, LL_j stands for the situation where a criterion is less important compared to another criterion. Finally, DD_j shows the equal importance between two criteria.

Step 3. Computing the final weights using Equations 48 and 49, where $w_j^{'}$ is the weights determined by DM and $\mathbb{S}_j$ denotes the scores of criteria.

$$\mathbb{S}_j = WW_j + DD_j + LL_j; \tag{48}$$

$$w_j = w_j^{'} \mathbb{S}_j \left\langle \sum_j w_j^{'} \mathbb{S}_j \right\rangle^{-1}; \tag{49}$$

V. THE APPLICATION

In this section, a complex green supplier selection case has been adopted from [62] comprising ten suppliers and twelve criteria in order to select the best supplier for suppliers to support the firm' adoption of GSCM procedures.

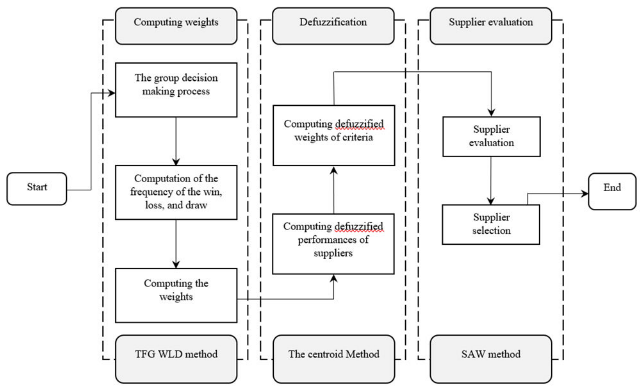

FIGURE 6. The green supplier selection process workflow.

A. THE METHODOLOGY

The process of solving the green supplier selection is demonstrated in Figure 6, in which the green supplier selection is divided into three sections, including 1. Computing weights; 2. Defuzzicaiton; and 3. Supplier evaluation. Each section has been briefly described as follows:

- **COMPUTING WEIGHTS**: In this section, two main processes are executed to extract the criteria weights. Using the WLD method's concepts, a group of experts evaluates the criteria in terms of their importance in reaching the objective, putting each criterion in one of three groups of W, L, or D group. WLD method runs by the inputs provided by the group decision-making in order to compute the weights of criteria.
- **DEFUZZIFICAITON**: The centroid method has been applied to convert the fuzzy green supplier selection decision matrix into a crisp decision matrix. Two processes arrange this section, including defuzzification of the weights provided in the original work and defuzzification of the performance of suppliers against the criteria.
- **SUPPLIER EVALUATION**: Using the weights computed by the TFG WLD method, the original criteria weights, and the crisp suppliers' performance, the SAW method is employed to evaluate the suppliers. The supplier that receives the first rank is selected as the best green supplier.

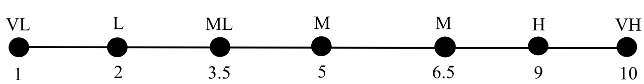

FIGURE 7. The linguist pattern for TFG WLD method.

B. THE TFG WLD METHOD

The following linguistic pattern (see Figure 7) is employed to convert the WLD method's scale to a TFGN. First, DM selects any number between [1, 10], and its corresponding TFGN is established on its higher and lower vicinities, where the chosen number is the center; then, transformation of the obtained weight into a TFGN follows Equations 26-33.

For instance

if $w_j^{'} = 7.5$then $\tilde{A}_{w_j^{'}} = (6.5, 7.3, 9)$, where $l = 0.6$, $m = 7.3$, and $l = 9$.

Then

$$\widetilde{\otimes} V_D = \begin{bmatrix} ([0.101, 0.432], 0.844, [0.988, 0.998]), \\ ([0.101, 0.432], 0.854, [0.988, 0.998]) \end{bmatrix}, \quad \xi = 0.01;$$

To convert win, loss, and draw variables into TFG numbers, we followed the following steps, Where V stands for the value:

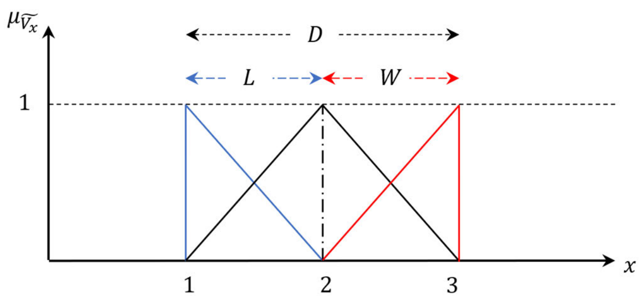

FIGURE 8. The win, loss, and draw variables' corresponding TFNs.

Step 1. Converting each variable into a TFN with respecting Equation 50. The converted variables are shown in Figure 8.

$$\mu_{\widetilde{V_L}} = \begin{cases} x-1, & 1 \leq x \\ 2-x, & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases} \tag{50}$$

then

$$V_L = 1 \rightarrow \widetilde{V_L} = (1, 1, 2); \tag{51}$$

The same process executes for D and W, where $V_D = 2 \rightarrow \widetilde{V_D} = (1, 2, 3)$and $V_W = 3 \rightarrow \widetilde{V_W} = (2, 3, 3)$.

Step 2. According to Equations 26-33, converting the obtained TFN into TFGN is the second step of the process. Since there exist three sets of number, corresponding to three variables, and the variables are evaluated based on their relationship with each other, $\max\{l, m, u\}$ computes by taking all sets into account, then:

$$\begin{aligned} &\max\{l, m, u\} \\ &\quad = \max\{V_L, V_D, V_W\}; \\ &\max\{V_L, V_D, V_W\} \\ &\quad = \max\{l_L, l_D, l_W, \ m_L, m_D, m_W, \ u_L, u_D, u_W\}; \\ &\max\{l_L, l_D, l_W, \ m_L, m_D, m_W, \ u_L, u_D, u_W\} \\ &\quad = 3; \\ &V_L = 1 \rightarrow \widetilde{V_L} = (1, 1, 2); \\ &V_D = 2 \rightarrow \widetilde{V_D} = (1, 2, 3); \\ &V_W = 3 \rightarrow \widetilde{V_W} = (2, 3, 3); \end{aligned}$$

therefore

$$\begin{aligned} &\widetilde{\otimes V_L} = \left[\widetilde{\otimes V_L^{\alpha}}, \widetilde{\otimes V_L^{\beta}}\right], \quad \widetilde{V_L} = (1, 1, 2); \\ &\widetilde{\otimes V_L^{\alpha}} = \left(\left[\underline{l_L}, \overline{l_L}\right], \underline{m_L}, \left[\underline{u_L}, \overline{u_L}\right]\right) \\ &\widetilde{\otimes V_L^{\beta}} = \left(\left[\underline{l_1}, \overline{l_1}\right], \overline{m_1}, \left[\underline{u_1}, \overline{u_1}\right]\right) \\ &\underline{l_L} \cong 0.323, \quad \xi = 0.01 \end{aligned}$$

then

$$\begin{aligned} &\overline{l_L} \cong 0.364, \ \underline{m_L} \cong 0.371, \ \overline{m_L} \cong 0.381, \\ &\underline{u_L} \cong 0.640, \ \overline{u_L} \cong 0.650; \end{aligned}$$

therefore

$$\begin{aligned} \widetilde{\otimes V_L} &= [([0.323, 0.364], 0.371, [0.640, 0.650]), \\ &\quad ([0.323, 0.364], 0.381, [0.640, 0.650])]; \\ \widetilde{\otimes V_D} &= \left[\widetilde{\otimes V_D^{\alpha}}, \widetilde{\otimes V_L^{\beta}}\right], \quad \widetilde{V_D} = (1, 2, 3); \\ &\underline{l} \cong 0.323, \bar{l} \cong 0.364, \underline{m} \cong 0.704, \\ &\bar{m} \cong 0.714, \underline{u} \cong 0.986, \bar{u} \cong 0.996; \end{aligned}$$

therefore

$$\begin{aligned} \widetilde{\otimes V_D} &= [([0.323, 0.364], 0.704, [0.986, 0.996]), \\ &\quad ([0.323, 0.364], 0.714, [0.986, 0.996])] \\ \widetilde{\otimes V_W} &= \left[\widetilde{\otimes V_W^{\alpha}}, \widetilde{\otimes V_W^{\beta}}\right], \quad \widetilde{V_W} = (2, 3, 3) \\ &\underline{l} \cong 0657, \bar{l} \cong 0.682, \underline{m} \cong 0.985, \bar{m} \cong 0.995 \\ &\underline{u} \cong 1, \bar{u} \cong 1; \\ \widetilde{\otimes V_W} &= [([0.657, 0.682], 0.985, [1, 1]), ([0.657, 0.682], \\ &\quad 0.995, [1, 1])] \end{aligned}$$

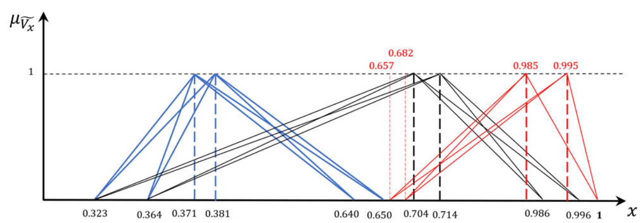

FIGURE 9. The win, loss, and draw variables' corresponding TFGNs.

The corresponding TFGNs of the variables are displayed in Figure 9.

The remainder of the steps follows the classic WLD algorithm's steps.

C. THE GREEN SUPPLIER SELECTION

The decision matrix for the green supplier evaluation is shown in Table 4, in which ten suppliers are evaluated with twelve criteria. In the decision matrix, w_j stands for the weights of criteria.

TABLE 4. The green supplier selection decision matrix.

w_j	(0.8,0.9,1)	(0.6,0.833,1)	(0.4,0.7,1)	(0.4,0.767,1)	(0.6,0.8,1)	(0.6,0.867,1)
	C_1	C_2	C_3	C_4	C_5	C_6
$Supplier_1$	(7,9.667,10)	(5,8.333,10)	(7,9.667,10)	(0,2.333,7)	(3,6.333,10)	(1,5,9)
$Supplier_2$	(7,9.667,10)	(7,9.333,10)	(5,8.667,10)	(7,9.333,10)	(7,9.667,10)	(7,9.333,10)
$Supplier_3$	(9,10,10)	(7,9.333,10)	(5,8.333,10)	(9,10,10)	(7,9.667,10)	(3,7,10)
$Supplier_4$	(3,7.667,10)	(7,9,10)	(1,7.333,10)	(0,3,7)	(0,2.333,5)	(0,1.667,5)
$Supplier_5$	(9,10,10)	(9,10,10)	(7,9.667,10)	(7,9.667,10)	(3,10,10)	(7,9.333,10)
$Supplier_6$	(3,6.333,10)	(3,6.333,10)	(1,4.333,7)	(0,4,9)	(0,2.667,9)	(1,3.667,7)
$Supplier_7$	(0,3,5)	(0,3,5)	(1,3,5)	(0,2.333,5)	(0,2,7)	(0,2.333,7)
$Supplier_8$	(9,10,10)	(9,10,10)	(1,3,5)	(0,2.333,7)	(0,3,7)	(0,3,7)
$Supplier_9$	(7,9.667,10)	(7,9.667,10)	(9,10,10)	(7,9.667,10)	(7,9.667,10)	(7,9.333,10)
$Supplier_{10}$	(0,4.333,9)	(0,4.333,9)	(0,1.667,3)	(0,0,1)	(1,3,5)	(0,1.333,5)
w_j	(0.6,0.833,1)	(0.4,0.7,1)	(0.6,0.8,1)	(0.2,0.533,0.8)	(0.6,0.8,1)	(0.4,0.7,1)
	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
$Supplier_1$	(1,4.333,7)	(0,3,7)	(5,8.333,10)	(7,9.333,10)	(7,9,10)	(5,8.333,10)
$Supplier_2$	(7,9,10)	(7,9.667,10)	(7,9,10)	(7,9,10)	(7,9.333,10)	(3,6.333,9)
$Supplier_3$	(3,5.667,9)	(3,5.667,9)	(5,8.333,10)	(7,9.333,10)	(3,7.667,9)	(7,9,10)
$Supplier_4$	(0,1.667,5)	(0,0.667,3)	(0,2,7)	(0.0.333,3)	(0,2,7)	(0.0.333,3)
$Supplier_5$	(7,9.333,10)	(5,7.667,10)	(0,3.667,7)	(0,2.333,7)	(3,5.667,10)	(3,7,10)
$Supplier_6$	(1,3.667,7)	(0,2.333,5)	(0,1.667,5)	(0,1.667,5)	(0,3,7)	(1,4.333,7)
$Supplier_7$	(3,5,7)	(3,5.667,9)	(0,4.333,9)	(1,5,9)	(3,5,7)	(0.3.667,7)
$Supplier_8$	(7,9.667,10)	(0,1.333,5)	(1,3.667,7)	(3,5,7)	(5,8.333,10)	(3,7,10)
$Supplier_9$	(5,8.667,10)	(5,7.667,10)	(7,9.333,10)	(7,9.333,10)	(7,9.667,10)	(7,9.667,10)
$Supplier_{10}$	(0,1.333,5)	(0,1.333,5)	(0,3.667,7)	(0,3,7)	(0,1.667,5)	(1,3,5)

1) COMPUTING THE WEIGHTS USING TFG WLD METHOD

Keeping the original weights of criteria, to evaluate the weights with the TFG WLD method, the weights have been reevaluated using experts' opinions. The evaluation of criteria through WLD's process and the experts' opinions are displayed in Table 6, and its corresponding analysis's results are shown in Table 6 and 7, respecting the corresponding values of W, L, and D obtained from the second step of the TFG WLD method's process (see Figure 8).

TABLE 5. The criteria pairwise comparsion.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
C_1	D	W	W	W	W	W	W	W	W	W	W	W
C_2	L	D	W	W	W	L	D	W	W	W	W	W
C_3	L	L	D	L	L	L	L	D	L	W	L	D
C_4	L	L	W	D	L	L	L	W	L	W	L	W
C_5	L	L	W	W	D	L	L	W	D	W	D	W
C_6	L	W	W	W	W	D	W	W	W	W	W	W
C_7	L	D	W	W	W	L	D	W	W	W	W	W
C_8	L	L	D	L	L	L	L	D	L	W	L	D
C_9	L	L	W	W	D	L	L	W	D	W	D	W
C_{10}	L	L	L	L	L	L	L	L	L	D	L	L
C_{11}	L	L	W	W	D	L	L	W	D	W	D	W
C_{12}	L	L	D	L	L	L	L	D	L	W	L	D

TABLE 6. The frequency of the WIN, LOSS, and draw in each criterion against other criteria.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
W	12	8	1	4	5	10	8	1	5	0	5	1
L	0	2	8	7	4	1	2	8	4	11	4	8
D	1	2	3	1	3	1	2	3	3	1	3	3

TABLE 7. The corresponding TFG values for each criterion's WIN, LOSS, and draw frequencies.

	C_1	C_2
$\otimes\widetilde{WW}_j$	$\left[\left([7.880,8.183], 11.824, [12,12]\right), \left([7.880,8.183], 11.944, [12,12]\right)\right]$	$\left[\left([5.253,5.455], 7.883, [8,8]\right), \left([5.253,5.455], 7.963, [8,8]\right)\right]$
$\otimes\widetilde{DD}_j$	0	0
$\otimes\widetilde{LL}_j$	$\left[\left([0.323,0.364], 0.704, [0.986,0.996]\right), \left([0.323,0.364], 0.714, [0.986,0.996]\right)\right]$	$\left[\left([0.647,0.729], 1.408, [1.972,1.992]\right), \left([0.647,0.729], 1.428, [1.972,1.992]\right)\right]$
	C_3	C_4
$\otimes\widetilde{WW}_j$	$\left[\left([0.657,0.682], 0.985, [1,1]\right), \left([0.657,0.682], 0.995, [1,1]\right)\right]$	$\left[\left([2.627,2.728], 3.941, [4,4]\right), \left([2.627,2.728], 3.981, [4,4]\right)\right]$
$\otimes\widetilde{DD}_j$	$\left[\left([0.647,0.729], 1.408, [1.972,1.992]\right), \left([0.647,0.729], 1.428, [1.972,1.992]\right)\right]$	0
$\otimes\widetilde{LL}_j$	$\left[\left([2.587,2.914], 2.966, [5.123,5.203]\right), \left([2.587,2.914], 3.046, [5.123,5.203]\right)\right]$	$\left[\left([2.263,2.550], 2.596, [4.483,4.553]\right), \left([2.263,2.550], 2.666, [4.483,4.553]\right)\right]$
	C_5	C_6
$\otimes\widetilde{WW}_j$	$\left[\left([3.283,3.409], 4.927, [5,5]\right), \left([3.283,3.409], 4.977, [5,5]\right)\right]$	$\left[\left([6.567,6.819], 9.853, [10,10]\right), \left([6.567,6.819], 9.953, [10,10]\right)\right]$
$\otimes\widetilde{DD}_j$	$\left[\left([0.647,0.729], 1.408, [1.972,1.992]\right), \left([0.647,0.729], 1.428, [1.972,1.992]\right)\right]$	0
$\otimes\widetilde{LL}_j$	$\left[\left([1.293,1.457], 1.483, [2.562,2.602]\right), \left([1.293,1.457], 1.523, [2.562,2.602]\right)\right]$	$\left[\left([0.323,0.364], 0.371, [0.640,0.650]\right), \left([0.323,0.364], 0.381, [0.640,0.650]\right)\right]$
	C_7	C_8
$\otimes\widetilde{WW}_j$	$\left[\left([5.253,5.455], 7.883, [8,8]\right), \left([5.253,5.455], 7.963, [8,8]\right)\right]$	$\left[\left([0.657,0.682], 0.985, [1,1]\right), \left([0.657,0.682], 0.995, [1,1]\right)\right]$
$\otimes\widetilde{DD}_j$	$\left[\left([0.323,0.364], 0.704, [0.986,0.996]\right), \left([0.323,0.364], 0.714, [0.986,0.996]\right)\right]$	$\left[\left([0.647,0.729], 1.408, [1.972,1.992]\right), \left([0.647,0.729], 1.428, [1.972,1.992]\right)\right]$
$\otimes\widetilde{LL}_j$	$\left[\left([0.647,0.729], 0.742, [1.281,1.301]\right), \left([0.647,0.729], 0.762, [1.281,1.301]\right)\right]$	$\left[\left([2.587,2.914], 2.966, [5.123,5.203]\right), \left([2.587,2.914], 3.046, [5.123,5.203]\right)\right]$
	C_9	C_{10}
$\otimes\widetilde{WW}_j$	$\left[\left([3.283,3.409], 4.927, [5,5]\right), \left([3.283,3.409], 4.977, [5,5]\right)\right]$	0
$\otimes\widetilde{DD}_j$	$\left[\left([0.647,0.729], 1.408, [1.972,1.992]\right), \left([0.647,0.729], 1.428, [1.972,1.992]\right)\right]$	0
$\otimes\widetilde{LL}_j$	$\left[\left([1.293,1.457], 1.483, [2.562,2.602]\right), \left([1.293,1.457], 1.523, [2.562,2.602]\right)\right]$	$\left[\left([3.557,4.007], 4.079, [7.044,7.154]\right), \left([3.557,4.007], 4.189, [7.044,7.154]\right)\right]$
	C_{11}	C_{12}
$\otimes\widetilde{WW}_j$	$\left[\left([3.283,3.409], 4.927, [5,5]\right), \left([3.283,3.409], 4.977, [5,5]\right)\right]$	$\left[\left([0.657,0.682], 0.985, [1,1]\right), \left([0.657,0.682], 0.995, [1,1]\right)\right]$
$\otimes\widetilde{DD}_j$	$\left[\left([0.647,0.729], 1.408, [1.972,1.992]\right), \left([0.647,0.729], 1.428, [1.972,1.992]\right)\right]$	$\left[\left([0.647,0.729], 1.408, [1.972,1.992]\right), \left([0.647,0.729], 1.428, [1.972,1.992]\right)\right]$
$\otimes\widetilde{LL}_j$	$\left[\left([1.293,1.457], 1.483, [2.562,2.602]\right), \left([1.293,1.457], 1.523, [2.562,2.602]\right)\right]$	$\left[\left([2.587,2.914], 2.966, [5.123,5.203]\right), \left([2.587,2.914], 3.046, [5.123,5.203]\right)\right]$

By following Equation 51, the scores of criteria are displayed in Table 8.

TABLE 8. The scores of criteria of the green supplier evaluation problem.

	C_1	C_2
$\widetilde{\otimes\mathbb{S}}_j$	$\left[\begin{pmatrix}[8.203,8.547],\\12.528,\\ [12.986,12.996]\end{pmatrix},\begin{pmatrix}[8.203,8.547],\\12.658,\\ [12.986,12.996]\end{pmatrix}\right]$	$\left[\begin{pmatrix}[5.900,6.184],\\8.624,\\ [9.281,9.301]\end{pmatrix},\begin{pmatrix}[5.900,6.184],\\8.724,\\ [9.281,9.301]\end{pmatrix}\right]$
	C_3	C_4
$\widetilde{\otimes\mathbb{S}}_j$	$\left[\begin{pmatrix}[3.890,4.325],\\5.360,\\ [8.095,8.195]\end{pmatrix},\begin{pmatrix}[3.890,4.325],\\5.470,\\ [8.095,8.195]\end{pmatrix}\right]$	$\left[\begin{pmatrix}[4.890,5.277],\\6.537,\\ [8.483,8.553]\end{pmatrix},\begin{pmatrix}[4.890,5.277],\\6.547,\\ [8.483,8.553]\end{pmatrix}\right]$
	C_5	C_6
$\widetilde{\otimes\mathbb{S}}_j$	$\left[\begin{pmatrix}[5.223,5.595],\\7.818,\\ [9.534,9.594]\end{pmatrix},\begin{pmatrix}[5.223,5.595],\\7.928,\\ [9.534,9.594]\end{pmatrix}\right]$	$\left[\begin{pmatrix}[6.890,7.183],\\10.224,\\ [10.640,10.650]\end{pmatrix},\begin{pmatrix}[6.890,7.183],\\10.334,\\ [10.640,10.650]\end{pmatrix}\right]$
	C_7	C_8
$\widetilde{\otimes\mathbb{S}}_j$	$\left[\begin{pmatrix}[6.223,6.548],\\9.328,\\ [10.267,10.297]\end{pmatrix},\begin{pmatrix}[6.223,6.548],\\9.438,\\ [10.267,10.297]\end{pmatrix}\right]$	$\left[\begin{pmatrix}[3.890,4.325],\\5.360,\\ [8.095,8.195]\end{pmatrix},\begin{pmatrix}[3.890,4.325],\\5.470,\\ [8.095,8.195]\end{pmatrix}\right]$
	C_9	C_{10}
$\widetilde{\otimes\mathbb{S}}_j$	$\left[\begin{pmatrix}[5.223,5.595],\\7.818,\\ [9.534,9.594]\end{pmatrix},\begin{pmatrix}[5.223,5.595],\\7.928,\\ [9.534,9.594]\end{pmatrix}\right]$	$\left[\begin{pmatrix}[3.557,4.007],\\4.079,\\ [7.044,7.154]\end{pmatrix},\begin{pmatrix}[3.557,4.007],\\4.189,\\ [7.044,7.154]\end{pmatrix}\right]$
	C_{11}	C_{12}
$\widetilde{\otimes\mathbb{S}}_j$	$\left[\begin{pmatrix}[5.595,6.548],\\7.818,\\ [9.534,9.594]\end{pmatrix},\begin{pmatrix}[5.595,6.548],\\7.928,\\ [9.534,9.594]\end{pmatrix}\right]$	$\left[\begin{pmatrix}[3.890,4.325],\\5.360,\\ [8.095,8.195]\end{pmatrix},\begin{pmatrix}[3.890,4.325],\\5.470,\\ [8.095,8.195]\end{pmatrix}\right]$

2) DEFUZZICATION OF THE DECISION MATRIX: THE CENTROID METHOD APPLICATION

In order to reduce the complexity of the process and to show the TFG applications clearly, the fuzzy decision matrix is converted to a crisp decision matrix through the application of the centroid method [63]. The method is shown in Equation 52, where $\hat{A}$ stands for the corresponding crisp number of $\tilde{A}$.

$$\hat{A} = \int_u^l x\mu_{\tilde{A}}(x)dx\langle\int_u^l x\mu_{\tilde{A}}(x)dx\rangle^{-1}, \quad \tilde{A} = (l, m, u); \tag{52}$$

The converted decision matrix including the performance of suppliers against the criteria and the normalized defuzzied weights of criteria is illustrated in Table 9.

By following Equations 50 and 51 and using w_j, adopted from Table 9, the obtained TFG weights of criteria are displayed in Table 10.

3) SUPPLIER SELECTION USING SAW METHOD

Simple additive weighting (SAW) method is a simple MCDM method using a weighted summation process to rank the decision' alternatives in MCDM problems. The method's process and application can be found in [63]. The obtained scores of suppliers using the SAW method are displayed in Table 11, where $\widetilde{\otimes S_i}$ stands for the TFG scores of suppliers obtained from SAW method application.

Table 12 displays the ranking of each supplier based on the utilization of Equation 43, where the second supplier is assigned the top rank.

VI. DISCUSSION

This paper introduces a novel approach that combines fuzzy and grey numbers to capture input uncertainty in MCDM algorithms. The first sub-section examines the properties of

TABLE 9. The supplier evaluation decision matrix with crisp values.

	C_1	C_2	C_3	C_4	C_5	C_6
w_j	0.099	0.089	0.077	0.080	0.088	0.091
$Supplier_1$	8.889	7.778	8.889	3.111	6.444	5.000
$Supplier_2$	8.889	8.778	7.889	8.778	8.889	8.778
$Supplier_3$	9.667	8.778	7.778	9.667	8.889	6.667
$Supplier_4$	6.889	8.667	6.111	3.333	2.444	2.222
$Supplier_5$	9.667	9.667	8.889	8.889	7.667	8.778
$Supplier_6$	6.444	6.444	4.111	4.333	3.889	3.889
$Supplier_7$	2.667	2.667	3.000	2.444	3.000	3.111
$Supplier_8$	9.667	9.667	3.000	3.111	3.333	3.333
$Supplier_9$	8.889	8.889	9.667	8.889	6.444	8.778
$Supplier_{10}$	4.444	4.444	8.889	3.111	8.889	2.111
	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
w_j	0.089	0.077	0.088	0.056	0.088	0.077
$Supplier_1$	4.111	3.333	7.778	8.778	8.667	7.778
$Supplier_2$	8.667	8.889	8.667	8.667	8.778	6.111
$Supplier_3$	5.889	5.889	7.778	8.778	6.556	8.667
$Supplier_4$	2.222	1.222	3.000	1.111	3.000	1.111
$Supplier_5$	8.778	7.556	3.556	3.111	6.222	6.667
$Supplier_6$	3.889	2.444	2.222	2.222	3.333	4.111
$Supplier_7$	5.000	5.889	4.444	5.000	5.000	3.556
$Supplier_8$	8.889	2.111	3.889	5.000	7.778	6.667
$Supplier_9$	7.889	7.556	8.778	8.778	8.778	8.889
$Supplier_{10}$	2.111	2.111	3.556	3.333	2.222	3.000

this new extension, while the second sub-section presents the application results and compares them with alternative uncertainty formulations.

A. THE PROPERTIES

A TFG system is proposed in this study to enhance the quantification of uncertainty in solving MCDM problems by using probability functions within upper and lower bounds. This system combines the advantages of grey systems theory and fuzzy logic while addressing their respective limitations.

In the TFG system, the probability distribution of specific information is divided into four distinct blocks (refer to Figures 10-13). Each block is defined by two points on the x vector and one on the $\mu_{\widetilde{\otimes A}}(x)$ vector, representing

TABLE 10. The TFG weights of criteria obtained from WLD method.

	$\widetilde{\otimes w_j}$
C_1	[([0.813,0.847], 1.242, [1.287,11.288]), ([0.813,0.847], 1.255, [1.287,11.288])]
C_2	[([0.527,0.552], 0.770, [0.829,0.831]), ([0.527,0.552], 0.779, [0.829,0.831])]
C_3	[([0.300,0.333], 0.413, [0.624,0.632]), ([0.300,0.333], 0.422, [0.624,0.632])]
C_4	[([0.389,0.420], 0.520, [0.675,0.681]), ([0.389,0.420], 0.529, [0.675,0.681])]
C_5	[([0.460,0.493], 0.689, [0.840,0.845]), ([0.460,0.493], 0.699, [0.840,0.845])]
C_6	[([0.624,0.651], 0.926, [0.964,0.965]), ([0.624,0.651], 0.936, [0.964,0.965])]
C_7	[([0.556,0.585], 0.833, [0.917,0.920]), ([0.556,0.585], 0.843, [0.917,0.920])]
C_8	[([0.300,0.333], 0.413, [0.624,0.632]), ([0.300,0.333], 0.422, [0.624,0.632])]
C_9	[([0.460,0.493], 0.689, [0.840,0.845]), ([0.460,0.493], 0.699, [0.840,0.845])]
C_{10}	[([0.200,0.226], 0.230, [0.397,0.403]), ([0.200,0.226], 0.236, [0.397,0.403])]
C_{11}	[([0.460,0.493], 0.689, [0.840,0.845]), ([0.460,0.493], 0.699, [0.840,0.845])]
C_{12}	[([0.300,0.333], 0.413, [0.624,0.632]), ([0.300,0.333], 0.422, [0.624,0.632])]

TABLE 11. The obtained scores of suppliers using saw method.

	$\widetilde{\otimes S_i}$
$Supplier_1$	[([0.5859,0.6272], 0.8505, [1.0382,1.0448]), ([0.5859,0.6272], 0.8628, [1.0382,1.0448])]
$Supplier_2$	[([0.7705,0.8238], 1.1172, [1.3550,1.3633]), ([0.7705,0.8238], 1.1331, [1.3550,1.3633])]
$Supplier_3$	[([0.7028,0.7522], 1.0160, [1.2392,1.2470]), ([0.7028,0.7522], 1.0307, [1.2392,1.2470])]
$Supplier_4$	[([0.3171,0.3370], 0.4644, [0.5434,0.5461]), ([0.3171,0.3370], 0.4706, [0.5434,0.5461])]
$Supplier_5$	[([0.6926,0.7388], 1.0078, [1.2056,1.2125]), ([0.6926,0.7388], 1.0218, [1.2056,1.2125])]
$Supplier_6$	[([0.3657,0.3896], 0.5331, [0.6326,0.6361]), ([0.3657,0.3896], 0.5404, [0.6326,0.6361])]
$Supplier_7$	[([0.3366,0.3613], 0.4863, [0.6031,0.6072]), ([0.3366,0.3613], 0.4935, [0.6031,0.6072])]
$Supplier_8$	[([0.5164,0.5496], 0.7562, [0.8912,0.8958]), ([0.5164,0.5496], 0.7664, [0.8912,0.8958])]
$Supplier_9$	[([0.7611,0.8145], 1.1012, [1.3426,1.3510]), ([0.7611,0.8145], 1.1170, [1.3426,1.3510])]
$Supplier_{10}$	[([0.3420,0.3668], 0.4961, [0.6118,0.6159]), ([0.3420,0.3668], 0.4961, [0.6118,0.6159])]

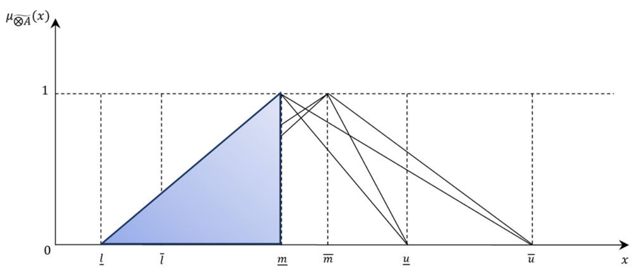

FIGURE 10. An information block bounded in [$\underline{l}$, $\underline{m}$] where the uncertain value is probably located.

its height. When comparing TFGNs, the likelihood of certain information being located in the bounded regions

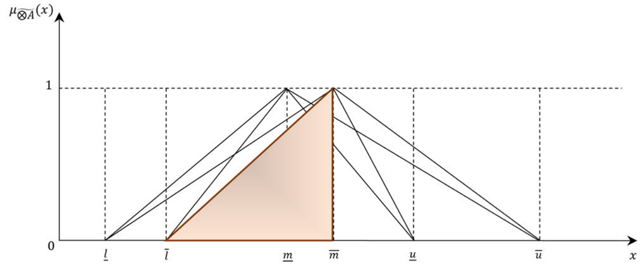

FIGURE 11. An information block bounded in [$\bar{l}$, $\bar{m}$] where the uncertain value is probably located.

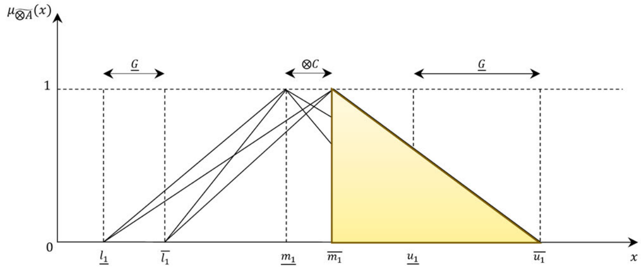

FIGURE 12. An information block bounded in [$\overline{m}$, $\overline{u}$] where the uncertain value is probably located.

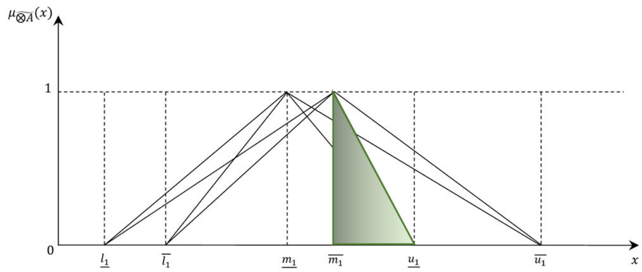

FIGURE 13. An information block bounded in [$\underline{m}$, $\underline{u}$] where the uncertain value is probably located.

is associated with the area of the triangles depicted in Figures 10-13. The bounded regions with smaller triangle areas have higher probabilities for the location of the certain information (as defined in Equations 53-56), in which $\mathbb{A}$ expresses the area of the blocks of information.

TABLE 12. The obtained rankings of green suppliers.

	$\widetilde{\mathbb{SS}}_i$	Rank
$Supplier_1$	2.5348	5
$Supplier_2$	3.3202	1
$Supplier_3$	3.0299	3
$Supplier_4$	1.3537	10
$Supplier_5$	2.9731	4
$Supplier_6$	1.5660	7
$Supplier_7$	1.4621	9
$Supplier_8$	2.2118	6
$Supplier_9$	3.2826	2
$Supplier_{10}$	1.4861	8

TABLE 13. The areas of regions associated with the WLD method.

	$[\underline{l}, \underline{m}]$	$[\bar{l}, \bar{m}]$	$[\underline{m}, \underline{u}]$	$[\bar{m}, \bar{u}]$
L	0.024	0.018	0.135	0.135
D	0.191	0.184	0.141	0.141
W	0.155	0.157	0.008	0.003

For example, see Figure 8, in which different areas located in different regions are illustrated. In Table 13, each region's area, associated with L, D, and W of the WLD method, is computed. As showillustrated in the Table 13, for L, the area bounded between $[\bar{l}, \bar{m}]$ has the highest probability of carrying certain information. For D, $[\underline{m}, \underline{u}]$ and $[\bar{m}, \bar{u}]$, and for W, $[\bar{m}, \bar{u}]$ has the highest probability of containing certain information (see Figure 10).

$$\mathbb{A}_{[\underline{l}, \underline{m}]} = \frac{(\underline{m} - \underline{l}) \times h}{2} \times \sin\theta, \quad \theta = 90, \quad h = 1; \tag{53}$$

$$\mathbb{A}_{[\bar{l}, \bar{m}]} = \frac{(\bar{m} - \bar{l}) \times h}{2} \times \sin\theta, \quad \theta = 90, \quad h = 1; \tag{54}$$

$$\mathbb{A}_{[\underline{m}, \underline{u}]} = \frac{(\underline{u} - \underline{m}) \times h}{2} \times \sin\theta, \quad \theta = 90, \quad h = 1; \tag{55}$$

$$\mathbb{A}_{[\bar{m}, \bar{u}]} = \frac{(\bar{u} - \bar{m}) \times h}{2} \times \sin\theta, \quad \theta = 90, \quad h = 1; \tag{56}$$

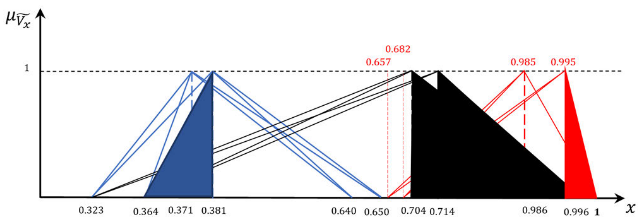

FIGURE 14. The local regions that have the highest probabilities of carrying certain information.

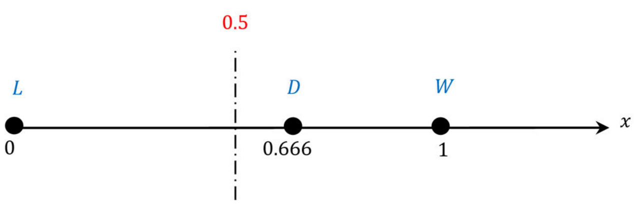

FIGURE 15. The tendency of W, L, and D to different values.

Based on the findings presented in Table 13, Figure 14, and the outcomes derived from Equations 53-56, the TFG system demonstrates a tendency for the TFG system of L to approach 0, the TFG system of D to converge towards W, and the TFG system of W to approach 1. These results not only indicate and forecast the behavior of the TFG system but also align with real-world observations. To validate the anticipated behavior, Figure 15 showcases the normalized tendencies of W, L, and D variables. The ability to predict certain information within an uncertain system is a notable advantage of the TFG system, distinguishing it from both grey systems theory and fuzzy logic.

TABLE 14. The areas of four different regions of suppliers' corresponding TFG systems.

	$[\underline{l}, \underline{m}]$	$[\bar{l}, \bar{m}]$	$[\underline{m}, \underline{u}]$	$[\bar{m}, \bar{u}]$
$Supplier_1$	0.1323	0.1178	0.0938	0.182
$Supplier_2$	0.1734	0.1547	0.1189	0.2302
$Supplier_3$	0.1566	0.1393	0.1116	0.2163
$Supplier_4$	0.0737	0.0668	0.0395	0.0755
$Supplier_5$	0.1576	0.1415	0.0989	0.1906
$Supplier_6$	0.0837	0.0754	0.0498	0.0957
$Supplier_7$	0.0748	0.0661	0.0584	0.1137
$Supplier_8$	0.1199	0.1084	0.0675	0.1294
$Supplier_9$	0.17	0.1513	0.1207	0.234
$Supplier_{10}$	0.0771	0.0683	0.0578	0.1125

TABLE 15. The each region's rank, based on the lower area.

	$[\underline{l}, \underline{m}]$	$[\bar{l}, \bar{m}]$	$[\underline{m}, \underline{u}]$	$[\bar{m}, \bar{u}]$
$Supplier_1$	3	2	1	4
$Supplier_2$	3	2	1	4
$Supplier_3$	3	2	1	4
$Supplier_4$	3	2	1	4
$Supplier_5$	3	2	1	4
$Supplier_6$	3	2	1	4
$Supplier_7$	3	2	1	4
$Supplier_8$	3	2	1	4
$Supplier_9$	3	2	1	4
$Supplier_{10}$	3	2	1	4

The same process has been done on the green supplier example, where the results are demonstrated in Table 14. The distributions also show that $[\underline{m}, \underline{u}]$ region has the highest probability of carrying certain information regarding the performance of suppliers against the criteria. The ranks of regions according to the lowest area are shown displayed in Table 15. To find the correlation between the areas of the regions with the highest probability of carrying information and the ranking obtained from the application of TFG WLD and SAW method, two different rankings are shown in Table 16: the Rank* and the Rank. The Rank* stands for the ranking of suppliers based on the lowest areas of their corresponding $[\underline{m}, \underline{u}]$ regions, and the Rank denotes the ranking obtained from the previous section. Using Excel, the correlation between the two rankings equals 0.939, indicating they are highly correlated.

TABLE 16. The suppliers' ranking based on the lowest area of [m, u] and their original ranking.

	$[m, u]$	Rank*	Rank
Supplier$_1$	0.093828	5	5
Supplier$_2$	0.118896	2	1
Supplier$_3$	0.111577	3	3
Supplier$_4$	0.039485	10	10
Supplier$_5$	0.098888	4	4
Supplier$_6$	0.049779	9	7
Supplier$_7$	0.058403	7	9
Supplier$_8$	0.067488	6	6
Supplier$_9$	0.120702	1	2
Supplier$_{10}$	0.057833	8	8

B. COMPARISON

To show the comparison between the application of TFGN, TFN, and grey numbers in solving the green supplier selection problem, we used the fuzzy WLD method, grey WLD method, and WLD method using crisp values with the results obtained from the TFG WLD application. To conduct the comparisons, we used the decision matrix presented in Table 9. The rankings obtained from different forms of WLD are shown in Table 17.

TABLE 17. The suppliers ranking using different forms of WLD application.

	WLD	F-WLD	G-WLD	TFG WLD
Supplier$_1$	5	5	5	5
Supplier$_2$	1	1	1	1
Supplier$_3$	4	4	3	3
Supplier$_4$	8	9	10	10
Supplier$_5$	3	3	4	4
Supplier$_6$	7	7	7	7
Supplier$_7$	10	10	9	9
Supplier$_8$	6	6	6	6
Supplier$_9$	2	2	2	2
Supplier$_{10}$	9	8	8	8

TABLE 18. The constant values of P^*_m per number of members of the data sets/ranks.

m	3	4	5	6	7	8	9
P^*_m	3.771	5.433	7.112	8.807	10.514	12.231	13.957
m	10	11	12	13	14	15	16
P^*_m	15.690	17.429	19.173	20.921	22.674	24.430	26.189

1) THE ZAKERI-KONSTANTAS WEIGHTED RANKINGS SIMILARITY MEASURE

To compare them through the similarities of the rankings obtained from different forms of WLD, a new similarity measure is introduced called the Zakeri-Konstantas weighted rankings similarity measure (WRSM), which considers different values for each rank generated by two different algorithms (see Equation 57, in which $\mathbb{ZK}$ stands for the WRSM; the extended version can be also found in Appendix C). In the equation, P stands for WRSM, X and B are two different algorithms, and P^*_m denote the maximum WRSM between two rankings' data sets with m members (see Table 18).

$$\mathbb{ZK}^{P}_{X,B} = 1 - \left\langle \sum_{i=1}^{m} \left(\frac{\left\langle (m+1) \left(R^X_{A_i} - R^B_{A_i} \right) \right\rangle}{R^B_{A_i} R^X_{A_i}} \right)^2 \right\rangle^{0.5} \times {P^*_m}^{-1},$$
$$i = \{1, \ldots, m\}, \; 0 \leq \mathbb{ZK}^{P}_{X,B} \leq 1; \qquad (57)$$

The results attained from the application of WRSM are shown pictured in Table 19.

TABLE 19. The weighted rankings similarity measure of the different rankings using different WLD method.

	WLD	F-WLD	G-WLD
TFG WLD	0.915	0.917	1.000

FIGURE 16. The comparison of different WLD applications with TFG WLD method using WRSM.

The WRSM results show that the TGF WLD application generated the same results as G-WLD in this paper's supplier selection example. It is also more similar to F-WLD than the WLD application with crisp values. The comparison is illustrated in Figure 16.

VII. CONCLUSION AND FUTURE RESEARCH

Supplier evaluation and selection pose a challenging MCDM (Multiple Criteria Decision Making) problem that often involves inherent uncertainty. The uncertainty arises from various sources, such as the weights assigned to evaluation criteria or the performance of suppliers against those criteria. In order to address this concern, the present study introduces a novel extension of fuzzy logic known as the triangular fuzzy grey (TFG) system. This system is specifically designed to effectively capture and manage uncertainty in the context of multi-criteria decision-making (MCDM) problem-solving. The TFG system operates based on a new type of number called the triangular fuzzy grey number (TFGN), which is employed to quantify uncertainty.

Each TFGN is constructed within a grey number framework, incorporating upper and lower bounds. These bounds are defined as triangular fuzzy numbers (TFN), allowing for a more comprehensive representation of uncertainty. Grey systems theory provides notable advantages in handling incomplete information, thus facilitating decision-making by offering a framework for analyzing systems with incomplete data. This theory exhibits versatility and robustness, making it applicable across various domains. On the other hand, the utilization of fuzzy triangular numbers brings certain benefits. The triangular membership function associated with these numbers enables straightforward visualization and comprehension, making it easier for decision-makers to understand the outcomes of TFN-based models.

Furthermore, fuzzy triangular numbers possess a simple and intuitive mathematical structure, enhancing their practical usability compared to other extensions of fuzzy numbers. Their computational efficiency is also advantageous due to the straightforward mathematical structure, simplifying manipulation, and algorithm integration. Moreover, established algorithms specifically designed for working with fuzzy triangular numbers further facilitate the development of decision-making models utilizing these numbers.

TFNs serve as valuable instruments for representing uncertainty in decision-making problems. Their straightforward mathematical structure, interpretability, and well-defined properties make them highly suitable for decision-making and optimization tasks. Leveraging the advantages previously discussed, TFGNs offer an expanded range of numbers compared to other fuzzy extensions with precise distribution regions that encapsulate crucial information. The introduction of the TFG system, incorporating TFGNs, extends the capabilities of fuzzy logic in effectively managing uncertainty within the realm of MCDM, providing decision-makers with enhanced tools for decision-making and optimization tasks.

In this study, we introduce diverse operations, comparisons, and interpretations of TFGNs, all of which can be applied to address MCDM problems. In order to address uncertainty in a green supplier evaluation case, a novel extension has been introduced. The TFG (Triangular Fuzzy Grey) WLD method is proposed to determine criteria weights within an uncertain environment, while the SAW (Simple Additive Weighting) method is employed for supplier ranking. Through the analysis of results, two main properties of the TFG systems have been identified. Firstly, when considering four primary regions of information distribution, the region with a smaller area has the highest probability of containing certain information. Secondly, the ranking of different TFGNs is associated with the areas where certain information is located with the highest probabilities. These properties provide decision-makers with the ability to predict system behavior and estimate probable values for uncertain variables, representing significant advantages of the TFG systems over grey systems theory and TFN. Notably, the results indicate similarities in behavior between the TFG system, grey systems, and fuzzy systems.

This paper serves as an initial endeavor to showcase the practicality, benefits, and effectiveness of the newly developed systems, while acknowledging the potential for further advancements in the field. Moving forward, it is recommended to investigate the limitations of the TFG system in managing uncertainty, conduct comparative studies with other fuzzy extensions, and explore the development of MCDM methods incorporating the application of TFGNs. These research directions will contribute to a deeper understanding and refinement of the proposed systems, leading to their continued evolution and enhancement.

APPENDIX A

TFGNs operations and proofs

if

$$\widetilde{\otimes A} = \left[\widetilde{\otimes A^{\alpha}}, \widetilde{\otimes A^{\beta}}\right],$$
$$\widetilde{\otimes A} = \left(\left[\underline{l_1}, \overline{l_1}\right], \underline{m_1}, \left[\underline{u_1}, \overline{u_1}\right]\right), \widetilde{\otimes A\beta}$$
$$= \left(\left[\underline{l_1}, \overline{l_1}\right], \overline{m_1}, \left[\underline{u_1}, \overline{u_1}\right]\right);$$

Then:

- For Addition:

$$\widetilde{\otimes A_1} + \widetilde{\otimes A_2}$$
$$= \left(\left[\underline{l_1}, \overline{l_1}\right], \overline{m_1}, \left[\underline{u_1}, \overline{u_1}\right]\right)$$
$$+ \left(\left[\underline{l_2}, \overline{l_2}\right], \overline{m_2}, \left[\underline{u_2}, \overline{u_2}\right]\right);$$
$$\widetilde{\otimes A_1} + \widetilde{\otimes A_2}$$
$$= \begin{cases} \left(\left[\underline{l_1} + \underline{l_2}, \overline{l_1} + \overline{l_2}\right], \underline{m_1} + \underline{m_2}, \left[\underline{u_1} + \underline{u_2}, \overline{u_1} + \overline{u_2}\right]\right) \\ \left(\left[\underline{l_1} + \underline{l_2}, \overline{l_1} + \overline{l_2}\right], \overline{m_1} + \overline{m_2}, \left[\underline{u_1} + \underline{u_2}, \overline{u_1} + \overline{u_2}\right]\right); \end{cases}$$

Proof:

$$\widehat{\otimes A_1} + \overline{\otimes A_2}$$
$$= \left[\bar{\otimes} A_1^{\alpha}, \bar{\otimes}_1^{\beta}\right] + \left[\widehat{\otimes A}_2^{\alpha}, \widehat{\otimes A}_2^{\beta}\right]$$
$$= \left[\widetilde{\otimes A}_1^{\alpha} + \widetilde{\otimes A}_1^{\alpha}, \widetilde{\otimes A}_1^{\beta} + \widetilde{\otimes A}_2^{\beta}\right]$$
$$= \left[\left(\left[\underline{l_1}, \overline{l_1}\right], \underline{m_1}, \left[\underline{u_1}, \overline{u_1}\right]\right)\right.$$
$$+ \left(\left[\underline{l_2}, \overline{l_2}\right], \underline{m_2}, \left[\underline{u_2}, \overline{u_2}\right]\right), \left(\left[\underline{l_1}, \overline{l_1}\right], \overline{m_1}, \left[\underline{u_1}, \overline{u_1}\right]\right)$$
$$\left.+ \left(\left[\underline{l_2}, \overline{l_2}\right], \overline{m_2}, \left[\underline{u_2}, \overline{u_2}\right]\right)\right]$$
$$= \left(\left[\underline{l_1}, \overline{l_1}\right], \underline{m_1}, \left[\underline{u_1}, \overline{u_1}\right]\right) + \left(\left[\underline{l_2}, \overline{l_2}\right], \underline{m_2}, \left[\underline{u_2}, \overline{u_2}\right]\right)$$
$$= \left(\left[\underline{l_1} + \underline{l_2}, \overline{l_1} + \overline{l_2}\right], \underline{m_1} + \underline{m_2}, \left[\underline{u_1} + \underline{u_2}, \overline{u_1} + \overline{u_2}\right]\right);$$

- For Additive inverse:

$$-\widetilde{\otimes A} = \left[\widetilde{- \otimes A^{\beta}}, \widetilde{- \otimes A^{\alpha}}\right];$$
$$-\widetilde{\otimes A} = \begin{cases} \left(\left[-\overline{u_1}, -\underline{u_1}\right], -\overline{m_1}, -\left[-\overline{l_1}, -\underline{l_1}\right]\right) \\ \left(\left[-\overline{u_1}, \underline{-\underline{u_1}}\right], -\underline{m_1}, -\left[-\overline{l_1}, -\underline{l_1}\right]\right); \end{cases}$$

Proof:

$$\begin{aligned}-\widetilde{\otimes A^{\beta}} &= \left(-\left[\underline{u_1},\overline{u_1}\right], -\overline{m_1}, -\left[\underline{l_1},\overline{l_1}\right]\right)\\ &= \left(\left[-\overline{u_1}, -\underline{u_1}\right], -\overline{m_1}, -\left[-\overline{l_1}, -\underline{l_1}\right]\right);\\ -\widetilde{\otimes A^{\beta}} &= \left(-\left[\underline{u_1},\overline{u_1}\right], -\underline{m_1}, -\left[\underline{l_1},\overline{l_1}\right]\right)\\ &= \left(\left[-\overline{u_1}, -\underline{u_1}\right], -\underline{m_1}, -\left[-\overline{l_1}, -\underline{l_1}\right]\right);\end{aligned}$$

- For Subtraction:

$$\otimes\tilde{A}_1 - \otimes\tilde{A}_2 = \begin{cases} \left(\left[\underline{l_2}-\overline{u_1}, \overline{l_2}-\underline{u_1}\right], \underline{m_2}-\overline{m_1}, \left[\underline{u_2}-\overline{l_1}, \overline{u_2}-\underline{l_1}\right]\right)\\ \left(\left[\underline{l_2}-\overline{u_1}, \overline{l_2}-\underline{u_1}\right], \overline{m_2}-\underline{m_1}, \left[\underline{u_2}-\overline{l_1}, \overline{u_2}-\underline{l_1}\right]\right);\end{cases}$$

Proof:

$$\begin{aligned}&\otimes\tilde{A}_2 - \otimes\tilde{A}_1\\ &= \left[\left(\left[\underline{l_2},\overline{l_2}\right], \underline{m_2}, \left[\underline{u_2},\overline{u_2}\right]\right), \left(\left[\underline{l_2},\overline{l_2}\right], \overline{m_2}, \left[\underline{u_2},\overline{u_2}\right]\right)\right]\\ &\quad+\left[\begin{array}{l}\left(\left[-\overline{u_1}, -\underline{u_1}\right], -\overline{m_1}, -\left[-\overline{l_1}, -\underline{l_1}\right]\right),\\ \left(\left[-\overline{u_1}, -\underline{u_1}\right], -\underline{m_1}, -\left[-\overline{l_1}, -\underline{l_1}\right]\right)\end{array}\right]\\ &= \left[\left(\left[\underline{l_2},\overline{l_2}\right], \underline{m_2}, \left[\underline{u_2},\overline{u_2}\right]\right)\right.\\ &\quad+\left(\left[-\overline{u_1}, -\underline{u_1}\right], -\overline{m_1}, -\left[-\overline{l_1}, -\underline{l_1}\right]\right),\\ &\quad\left(\left[\underline{l_2},\overline{l_2}\right], \overline{m_2}, \left[\underline{u_2},\overline{u_2}\right]\right)\\ &\quad\left.+\left(\left[-\overline{u_1}, -\underline{u_1}\right], -\underline{m_1}, -\left[-\overline{l_1}, -\underline{l_1}\right]\right)\right];\\ &\times\left(\left[\underline{l_2},\overline{l_2}\right], \underline{m_2}, \left[\underline{u_2},\overline{u_2}\right]\right)+\left(\left[-\overline{u_1}, -\underline{u_1}\right], -\overline{m_1},\right.\\ &\left.-\left[-\overline{l_1}, -\underline{l_1}\right]\right)\\ &= \left(\left[\underline{l_2}-\overline{u_1}, \overline{l_2}-\underline{u_1}\right], \underline{m_2}\right.\\ &\left.-\overline{m_1}, \left[\underline{u_2}-\overline{l_1}, \overline{u_2}-\underline{l_1}\right]\right);\\ &\times\left(\left[\underline{l_2},\overline{l_2}\right], \overline{m_2}, \left[\underline{u_2},\overline{u_2}\right]\right)+\left(\left[-\overline{u_1},\right.\right.\\ &\left.\left.-\underline{u_1}\right], -\underline{m_1}, -\left[-\overline{l_1}, -\underline{l_1}\right]\right)\\ &= \left(\left[\underline{l_2}-\overline{u_1}, \overline{l_2}-\underline{u_1}\right], \overline{m_2}-\underline{m_1}, \left[\underline{u_2}-\overline{l_1}, \overline{u_2}-\underline{l_1}\right]\right);\end{aligned}$$

- For Multiplication:

$$\begin{aligned}&\widetilde{\otimes A_1} \times \widetilde{\otimes A_2}\\ &\cong \left[\min\left\{\widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\alpha}}, \widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\beta}}, \widetilde{\otimes A_1^{\beta}} \times \widetilde{\otimes A_2^{\alpha}}, \widetilde{\otimes A_1^{\beta}}\right.\right.\\ &\quad\left.\times\widetilde{\otimes A_2^{\beta}}\right\},\\ &\quad\max\left\{\widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\alpha}}, \widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\beta}}, \widetilde{\otimes A_1^{\beta}} \times \widetilde{\otimes A_2^{\alpha}}, \widetilde{\otimes A_1^{\beta}}\right.\\ &\quad\left.\left.\times\widetilde{\otimes A_2^{\beta}}\right\}\right];\end{aligned}$$

Proof:

$$\begin{aligned}&\widetilde{\otimes A_1} \times \widetilde{\otimes A_2}\\ &= \left[\widetilde{\otimes A_1^{\alpha}}, \widetilde{\otimes A_1^{\beta}}\right] \times \left[\widetilde{\otimes A_2^{\alpha}}, \widetilde{\otimes A_2^{\beta}}\right] = \left[\min\left\{\widetilde{\otimes A_1^{\alpha}}\right.\right.\\ &\quad\left.\times\widetilde{\otimes A_2^{\alpha}}, \widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\beta}}, \widetilde{\otimes A_1^{\beta}} \times \widetilde{\otimes A_2^{\alpha}}, \widetilde{\otimes A_1^{\beta}} \times \widetilde{\otimes A_2^{\beta}}\right\},\ \max\\ &\quad\times\left\{\widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\alpha}}, \widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\beta}}, \widetilde{\otimes A_1^{\beta}} \times \widetilde{\otimes A_2^{\alpha}}, \widetilde{\otimes A_1^{\beta}}\right.\\ &\quad\left.\left.\times\widetilde{\otimes A_2^{\beta}}\right\}\right];\end{aligned}$$

$$\begin{aligned}&\widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\alpha}}\\ &= \left(\left[\underline{l_1},\overline{l_1}\right], \underline{m_1}, \left[\underline{u_1},\overline{u_1}\right]\right)\\ &\quad\times\left(\left[\underline{l_2},\overline{l_2}\right], \underline{m_2}, \left[\underline{u_2},\overline{u_2}\right]\right) = \left(\left[\underline{l_1},\overline{l_1}\right] \times \left[\underline{l_2},\overline{l_2}\right], \underline{m_1}\right.\\ &\quad\times\underline{m_2}, \left.\left[\underline{u_1},\overline{u_1}\right] \times \left[\underline{u_2},\overline{u_2}\right]\right) = \left[\min\left\{\underline{l_1} \times \underline{l_2}, \underline{l_1} \times \overline{l_2}, \overline{l_1}\right.\right.\\ &\quad\left.\times\underline{l_2}, \overline{l_1}\times\overline{l_2}\right\}, \max\left\{\underline{l_1}\times\underline{l_2}, \underline{l_1}\times\overline{l_2}, \overline{l_1} \times \underline{l_2}, \overline{l_1} \times \overline{l_2}\right\}\Big], \underline{m_1}\\ &\quad\times\underline{m_2}, \left[\min\left\{\underline{u_1}\times\underline{u_2}, \underline{u_1} \times \overline{u_2}, \overline{u_1} \times \underline{u_2}, \overline{u_1}\times\overline{u_2}\right\}, \max\left\{\underline{u_1}\right.\right.\\ &\quad\left.\left.\times\underline{u_2}, \underline{u_1} \times \overline{u_2}, \overline{u_1} \times \underline{u_2}, \overline{u_1} \times \overline{u_2}\right\}\right];\end{aligned}$$

$$\begin{aligned}&\widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_{2\,2}^{A^{\beta}}}\\ &= \left(\left[\underline{l_1},\overline{l_1}\right], \underline{m_1}, \left[\underline{u_1},\overline{u_1}\right]\right)\\ &\quad\times\left(\left[\underline{l_2},\overline{l_2}\right], \overline{m_2}, \left[\underline{u_2},\overline{u_2}\right]\right) = \left(\left[\underline{l_1},\overline{l_1}\right] \times \left[\underline{l_2},\overline{l_2}\right], \underline{m_1}\right.\\ &\quad\times\overline{m_2}, \left.\left[\underline{u_1},\overline{u_1}\right] \times \left[\underline{u_2},\overline{u_2}\right]\right) = \left(\left[\min\left\{\underline{l_1} \times \underline{l_2}, \underline{l_1} \times \overline{l_2}, \overline{l_1}\right.\right.\right.\\ &\quad\left.\times\underline{l_2}, \overline{l_1}\times\overline{l_2}\right\}, \max\left\{\underline{l_1}\times\underline{l_2}, \underline{l_1}\times\overline{l_2}, \overline{l_1} \times \underline{l_2}, \overline{l_1} \times \overline{l_2}\right\}\Big], \underline{m_1}\\ &\quad\times\overline{m_2}, \left[\min\left\{\underline{u_1}\times\underline{u_2}, \underline{u_1}\times\overline{u_2}, \overline{u_1} \times \underline{u_2}, \overline{u_1}\times\overline{u_2}\right\}, \max\left\{\underline{u_1}\right.\right.\\ &\quad\left.\left.\left.\times\underline{u_2}, \underline{u_1} \times \overline{u_2}, \overline{u_1} \times \underline{u_2}, \overline{u_1} \times \overline{u_2}\right\}\right]\right);\end{aligned}$$

$$\begin{aligned}&\widetilde{\otimes A_1^{\beta}} \times \widetilde{\otimes A_2^{\alpha}}\\ &= \left(\left[\underline{l_1},\overline{l_1}\right], \overline{m_1}, \left[\underline{u_1},\overline{u_1}\right]\right)\\ &\quad\times\left(\left[\underline{l_2},\overline{l_2}\right], \underline{m_2}, \left[\underline{u_2},\overline{u_2}\right]\right) = \left(\left[\underline{l_1},\overline{l_1}\right] \times \left[\underline{l_2},\overline{l_2}\right], \overline{m_1}\right.\\ &\quad\times\underline{m_2}, \left.\left[\underline{u_1},\overline{u_1}\right] \times \left[\underline{u_2},\overline{u_2}\right]\right) = \left(\left[\min\left\{\underline{l_1} \times \underline{l_2}, \underline{l_1} \times \overline{l_2}, \overline{l_1}\right.\right.\right.\\ &\quad\left.\times\underline{l_2}, \overline{l_1}\times\overline{l_2}\right\}, \max\left\{\underline{l_1}\times\underline{l_2}, \underline{l_1} \times \overline{l_2}, \overline{l_1}\times\underline{l_2}, \overline{l_1} \times \overline{l_2}\right\}\Big], \overline{m_1}\\ &\quad\times\underline{m_2}, \left[\min\left\{\underline{u_1}\times\underline{u_2}, \underline{u_1}\times\overline{u_2}, \overline{u_1}\times\underline{u_2}, \overline{u_1} \times \overline{u_2}\right\}, \max\left\{\underline{u_1}\right.\right.\\ &\quad\left.\left.\left.\times\underline{u_2}, \underline{u_1} \times \overline{u_2}, \overline{u_1} \times \underline{u_2}, \overline{u_1} \times \overline{u_2}\right\}\right]\right);\end{aligned}$$

$$\begin{aligned}&\widetilde{\otimes \boldsymbol{A_1^{\beta}}} \times \widetilde{\otimes \boldsymbol{A_2^{\beta}}}\\ &= \left(\left[l_1,\overline{l_1}\right], \overline{m_1}, \left[\underline{u_1},\overline{u_1}\right]\right)\\ &\quad\times\left(\left[\underline{l_2},\overline{l_2}\right], \overline{m_2}, \left[\underline{u_2},\overline{u_2}\right]\right) = \left(\left[\underline{l_1},\overline{l_1}\right] \times \left[\underline{l_2},\overline{l_2}\right], \overline{m_1}\right.\\ &\quad\times\overline{m_2}, \left.\left[\underline{u_1},\overline{u_1}\right] \times \left[\underline{u_2},\overline{u_2}\right]\right) = \left(\left[\min\left\{\underline{l_1} \times \underline{l_2}, \underline{l_1} \times \overline{l_2}, \overline{l_1}\right.\right.\right.\\ &\quad\left.\times\underline{l_2}, \overline{l_1}\times\overline{l_2}\right\}, \max\left\{\underline{l_1}\times\underline{l_2}, \underline{l_1}\times\overline{l_2}, \overline{l_1} \times \underline{l_2}, \overline{l_1} \times \overline{l_2}\right\}\Big], \overline{m_1}\\ &\quad\times\overline{m_2}, \left[\min\left\{\underline{u_1}\times\underline{u_2}, \underline{u_1}\times\overline{u_2}, \overline{u_1}\times\underline{u_2}, \overline{u_1} \times \overline{u_2}\right\}, \max\left\{\underline{u_1}\right.\right.\\ &\quad\left.\left.\left.\times\underline{u_2}, \underline{u_1} \times \overline{u_2}, \overline{u_1} \times \underline{u_2}, \overline{u_1} \times \overline{u_2}\right\}\right]\right);\end{aligned}$$

- For Division:

$$\widetilde{\otimes A_1} \times \widetilde{\otimes A_2}^{-1} = \left[\widetilde{\otimes A_1^{\alpha}}, \widetilde{\otimes A_1^{\beta}}\right] \times \left[\frac{1}{\widetilde{\otimes A_2^{\alpha}}}, \frac{1}{\widetilde{\otimes A_2^{\beta}}}\right];$$

Proof:

$$\begin{aligned}&\widetilde{\otimes A_1} \times \widetilde{\otimes A_2}^{-1}\\ &= \left[\widetilde{\otimes A_1^{\alpha}}, \widetilde{\otimes A_1^{\beta}}\right] \times \left[\frac{1}{\widetilde{\otimes A_2^{\alpha}}}, \frac{1}{\widetilde{\otimes A_2^{\beta}}}\right]\\ &= \left[\min\left\{\widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\alpha}}^{-1}, \widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\beta^{-1}}}, \widetilde{\otimes A_1^{\beta}}\right.\right.\\ &\quad\left.\times\widetilde{\otimes A_2^{\alpha}}^{-1}, \widetilde{\otimes A_1^{\beta}} \times \widetilde{\otimes A_2^{\beta}}^{-1}\right\}, \max\left\{\widetilde{\otimes A_1^{\alpha}} \times \widetilde{\otimes A_2^{\alpha}}^{-1}, \widetilde{\otimes A_1^{\alpha}}\right.\\ &\quad\left.\left.\times\widetilde{\otimes A_2^{\beta^{-1}}}, \widetilde{\otimes A_1^{\beta}} \times \widetilde{\otimes A_2^{\alpha}}^{-1}, \widetilde{\otimes A_1^{\beta}} \times \widetilde{\otimes A_2^{\beta}}^{-1}\right\}\right];\end{aligned}$$

$$\widetilde{\otimes \boldsymbol{A_1^{\alpha}}} \times \widetilde{\otimes \boldsymbol{A_2^{\alpha}}}^{\boldsymbol{-1}}$$

$$= \left(\left[\underline{l_1},\overline{l_1}\right],\underline{m_1},\left[\underline{u_1},\overline{u_1}\right]\right)\times\left(\left[\underline{l_2},\overline{l_2}\right],\underline{m_1},\left[\underline{u_2},\overline{u_2}\right]\right)^{-1}$$

$$= \left(\frac{\left[\underline{l_1},\overline{l_1}\right]}{\left[\underline{u_2},\overline{u_2}\right]},\frac{\underline{m_1}}{\underline{m_1}},\frac{\left[\underline{u_1},\overline{u_1}\right]}{\left[\underline{l_2},\overline{l_2}\right]}\right)$$

$$= \left(\begin{array}{c}\left[\min\left\{\frac{\underline{l_1}}{\underline{u_2}},\frac{\underline{l_1}}{\overline{u_2}},\frac{\overline{l_1}}{\underline{u_2}},\frac{\overline{l_1}}{\overline{u_2}}\right\},\max\left\{\frac{\underline{l_1}}{\underline{u_2}},\frac{\underline{l_1}}{\overline{u_2}},\frac{\overline{l_1}}{\underline{u_2}},\frac{\overline{l_1}}{\overline{u_2}}\right\}\right],\\ \frac{\underline{m_1}}{\underline{m_1}},\\ \left[\min\left\{\frac{\underline{u_1}}{\underline{l_2}},\frac{\underline{u_1}}{\overline{l_2}},\frac{\overline{u_1}}{\underline{l_2}},\frac{\overline{u_1}}{\overline{l_2}}\right\},\max\left\{\frac{\underline{l_1}}{\underline{u_2}},\frac{\underline{l_1}}{\overline{u_2}},\frac{\overline{l_1}}{\underline{u_2}},\frac{\overline{l_1}}{\overline{u_2}}\right\}\right]\end{array}\right);$$

$$\widetilde{\otimes \boldsymbol{A}_{\mathbf{1}}^{\boldsymbol{\alpha}}}\times\widetilde{\otimes \boldsymbol{A}_{\mathbf{2}}^{\boldsymbol{\beta}}}^{\mathbf{-1}}$$

$$= \left(\left[\underline{l_1},\overline{l_1}\right],\underline{m_1},\left[\underline{u_1},\overline{u_1}\right]\right)\times\left(\left[\underline{l_2},\overline{l_2}\right],\overline{m_2},\left[\underline{u_2},\overline{u_2}\right]\right)^{-1}$$

$$= \left(\frac{\left[\underline{l_1},\overline{l_1}\right]}{\left[\underline{u_2},\overline{u_2}\right]},\frac{\underline{m_1}}{\overline{m_2}},\frac{\left[\underline{u_1},\overline{u_1}\right]}{\left[\underline{l_2},\overline{l_2}\right]}\right)$$

$$= \left(\begin{array}{c}\left[\min\left\{\frac{\underline{l_1}}{\underline{u_2}},\frac{\underline{l_1}}{\overline{u_2}},\frac{\overline{l_1}}{\underline{u_2}},\frac{\overline{l_1}}{\overline{u_2}}\right\},\max\left\{\frac{\underline{l_1}}{\underline{u_2}},\frac{\underline{l_1}}{\overline{u_2}},\frac{\overline{l_1}}{\underline{u_2}},\frac{\overline{l_1}}{\overline{u_2}}\right\}\right],\\ \frac{\underline{m_1}}{\overline{m_2}},\\ \left[\min\left\{\frac{\underline{u_1}}{\underline{l_2}},\frac{\underline{u_1}}{\overline{l_2}},\frac{\overline{u_1}}{\underline{l_2}},\frac{\overline{u_1}}{\overline{l_2}}\right\},\max\left\{\frac{\underline{l_1}}{\underline{u_2}},\frac{\underline{l_1}}{\overline{u_2}},\frac{\overline{l_1}}{\underline{u_2}},\frac{\overline{l_1}}{\overline{u_2}}\right\}\right]\end{array}\right);$$

$$\widetilde{\otimes \boldsymbol{A}_{\mathbf{1}}^{\boldsymbol{\beta}}}\times\widetilde{\otimes \boldsymbol{A}_{\mathbf{2}}^{\boldsymbol{\alpha}}}^{\mathbf{-1}}$$

$$= \left(\left[\underline{l_1},\overline{l_1}\right],\overline{m_1},\left[\underline{u_1},\overline{u_1}\right]\right)\times\left(\left[\underline{l_2},\overline{l_2}\right],\underline{m_2},\left[\underline{u_2},\overline{u_2}\right]\right)^{-1}$$

$$= \left(\frac{\left[\underline{l_1},\overline{l_1}\right]}{\left[\underline{u_2},\overline{u_2}\right]},\frac{\overline{m_1}}{\underline{m_2}},\frac{\left[\underline{u_1},\overline{u_1}\right]}{\left[\underline{l_2},\overline{l_2}\right]}\right)$$

$$= \left(\begin{array}{c}\left[\min\left\{\frac{\underline{l_1}}{\underline{u_2}},\frac{\underline{l_1}}{\overline{u_2}},\frac{\overline{l_1}}{\underline{u_2}},\frac{\overline{l_1}}{\overline{u_2}}\right\},\max\left\{\frac{\underline{l_1}}{\underline{u_2}},\frac{\underline{l_1}}{\overline{u_2}},\frac{\overline{l_1}}{\underline{u_2}},\frac{\overline{l_1}}{\overline{u_2}}\right\}\right],\\ \frac{\overline{m_1}}{\underline{m_2}},\\ \left[\min\left\{\frac{\underline{u_1}}{\underline{l_2}},\frac{\underline{u_1}}{\overline{l_2}},\frac{\overline{u_1}}{\underline{l_2}},\frac{\overline{u_1}}{\overline{l_2}}\right\},\max\left\{\frac{\underline{l_1}}{\underline{u_2}},\frac{\underline{l_1}}{\overline{u_2}},\frac{\overline{l_1}}{\underline{u_2}},\frac{\overline{l_1}}{\overline{u_2}}\right\}\right]\end{array}\right);$$

$$\widetilde{\otimes \boldsymbol{A}_{\mathbf{1}}^{\boldsymbol{\beta}}}\times\widetilde{\otimes \boldsymbol{A}_{\mathbf{2}}^{\boldsymbol{\beta}}}$$

$$= \left(\left[\underline{l_1},\overline{l_1}\right],\overline{m_1},\left[\underline{u_1},\overline{u_1}\right]\right)\times\left(\left[\underline{l_2},\overline{l_2}\right],\overline{m_2},\left[\underline{u_2},\overline{u_2}\right]\right)^{-1}$$

$$= \left(\frac{\left[\underline{l_1},\overline{l_1}\right]}{\left[\underline{u_2},\overline{u_2}\right]},\frac{\overline{m_1}}{\overline{m_2}},\frac{\left[\underline{u_1},\overline{u_1}\right]}{\left[\underline{l_2},\overline{l_2}\right]}\right)$$

$$= \left(\begin{array}{c}\left[\min\left\{\frac{\underline{l_1}}{\underline{u_2}},\frac{\underline{l_1}}{\overline{u_2}},\frac{\overline{l_1}}{\underline{u_2}},\frac{\overline{l_1}}{\overline{u_2}}\right\},\max\left\{\frac{\underline{l_1}}{\underline{u_2}},\frac{\underline{l_1}}{\overline{u_2}},\frac{\overline{l_1}}{\underline{u_2}},\frac{\overline{l_1}}{\overline{u_2}}\right\}\right],\\ \frac{\overline{m_1}}{\overline{m_2}},\\ \left[\min\left\{\frac{\underline{u_1}}{\underline{l_2}},\frac{\underline{u_1}}{\overline{l_2}},\frac{\overline{u_1}}{\underline{l_2}},\frac{\overline{u_1}}{\overline{l_2}}\right\},\max\left\{\frac{\underline{l_1}}{\underline{u_2}},\frac{\underline{l_1}}{\overline{u_2}},\frac{\overline{l_1}}{\underline{u_2}},\frac{\overline{l_1}}{\overline{u_2}}\right\}\right]\end{array}\right);$$

- With Crisp number
- For Multiplication:

$$\widetilde{\otimes A_1}\times r=\left[\left(\left[r\times\underline{l_1},r\times\overline{l_1}\right],r\times\underline{m_1},\left[r\times\underline{u_1},r\times\overline{u_1}\right]\right),\left(\left[r\times\underline{l_1},r\times\overline{l_1}\right],r\times\overline{m_1},\left[r\times\underline{u_1},r\times\overline{u_1}\right]\right)\right];$$

Proof:

$$\widetilde{A}_1\times r$$

$$= \left[\widetilde{\otimes A}^{\alpha},\widetilde{\otimes A}^{\beta}\right]\times r=\left[\widetilde{\otimes A}^{\alpha}\times r,\widetilde{\otimes A}^{\beta}\times r\right]$$

$$= \left[r\times\left(\left[\underline{l_1},\overline{l_1}\right],\underline{m_1},\left[\underline{u_1},\overline{u_1}\right]\right),r\times\left(\left[\underline{l_1},\overline{l_1}\right],\overline{m_1},\left[\underline{u_1},\overline{u_1}\right]\right)\right];$$

$$\widetilde{\otimes \boldsymbol{A}^{\boldsymbol{\alpha}}}\times\boldsymbol{r}$$

$$= r\times\left(\left[\underline{l_1},\overline{l_1}\right],\underline{m_1},\left[\underline{u_1},\overline{u_1}\right]\right)=\left(\left[r\times\underline{l_1},r\times\overline{l_1}\right],r\times\underline{m_1},\left[r\times\underline{u_1},r\times\overline{u_1}\right]\right);$$

$$\widetilde{\otimes \boldsymbol{A}^{\boldsymbol{\beta}}}\times\boldsymbol{r}$$

$$= r\times\left(\left[\underline{l_1},\overline{l_1}\right],\overline{m_1},\left[\underline{u_1},\overline{u_1}\right]\right)=\left(\left[r\times\underline{l_1},r\times\overline{l_1}\right],r\times\overline{m_1},\left[r\times\underline{u_1},r\times\overline{u_1}\right]\right);$$

- Division

$$\widetilde{\otimes A_1}\times r^{-1}$$

$$= \left[\left(\left[\frac{1}{r}\times\underline{l_1},r\times\overline{l_1}\right],\frac{1}{r}\times\underline{m_1},\left[\frac{1}{r}\times\underline{u_1},r\times\overline{u_1}\right]\right),\left(\left[\frac{1}{r}\times\underline{l_1},\frac{1}{r}\times\overline{l_1}\right],\frac{1}{r}\times\overline{m_1},\left[\frac{1}{r}\times\underline{u_1},\frac{1}{r}\times\overline{u_1}\right]\right)\right];$$

Proof:

$$\widetilde{\otimes A_1}\times\frac{1}{r}$$

$$= \left[\widetilde{\otimes A}^{\alpha},\widetilde{\otimes A}^{\beta}\right]\times\frac{1}{r}=\left[\widetilde{\otimes A}^{\alpha}\times\frac{1}{r},\widetilde{\otimes A}^{\beta}\times\frac{1}{r}\right]$$

$$= \left[\frac{1}{r}\times\left(\left[\underline{l_1},\overline{l_1}\right],\underline{m_1},\left[\underline{u_1},\overline{u_1}\right]\right),\frac{1}{r}\times\left(\left[\underline{l_1},\overline{l_1}\right],\overline{m_1},\left[\underline{u_1},\overline{u_1}\right]\right)\right];$$

$$\widetilde{\otimes \boldsymbol{A}^{\boldsymbol{\alpha}}}\times\frac{\mathbf{1}}{\boldsymbol{r}}$$

$$= \frac{1}{r}\times\left(\left[\underline{l_1},\overline{l_1}\right],\underline{m_1},\left[\underline{u_1},\overline{u_1}\right]\right)=\left(\left[\frac{1}{r}\times\underline{l_1},r\times\overline{l_1}\right],\frac{1}{r}\times\underline{m_1},\left[\frac{1}{r}\times\underline{u_1},r\times\overline{u_1}\right]\right);$$

$$\widetilde{\otimes \boldsymbol{A}^{\boldsymbol{\beta}}}\times\frac{\mathbf{1}}{\boldsymbol{r}}$$

$$= \frac{1}{r}\times\left(\left[\underline{l_1},\overline{l_1}\right],\overline{m_1},\left[\underline{u_1},\overline{u_1}\right]\right)=\left(\left[\frac{1}{r}\times\underline{l_1},\frac{1}{r}\times\overline{l_1}\right],\frac{1}{r}\times\overline{m_1},\left[\frac{1}{r}\times\underline{u_1},\frac{1}{r}\times\overline{u_1}\right]\right);$$

APPENDIX B

The spectrums are shown in the following equations, where $S_{\mho}$stands for the set of spectrums, $S_{\mho}=\{S_1,\ S_2,\ldots,S_7,S_8\}$.

$$S_{\mho}=\left\{\begin{array}{l}S_1=\left(\underline{l},\underline{m},\underline{u}\right)\\ S_2=\left(\underline{l},\overline{m},\underline{u}\right)\\ S_3=\left(\underline{l},\underline{m},\overline{u}\right)\\ S_4=\left(\underline{l},\overline{m},\overline{u}\right)\\ S_5=\left(\overline{l},\underline{m},\underline{u}\right)\\ S_6=\left(\overline{l},\overline{m},\underline{u}\right)\\ S_7=\left(\overline{l},\underline{m},\overline{u}\right)\\ S_8=\left(\overline{l},\overline{m},\overline{u}\right)\end{array}\right.,\ \mho=\{1,2,\ldots,7,8\};$$

TABLE 20. The information matrix - $\mathcal{M}^{\alpha}$.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
$\mathcal{E}_1$	σ_{11}	σ_{12}	σ_{13}	σ_{14}	σ_{15}	σ_{16}	σ_{17}	σ_{18}
$\mathcal{E}_2$	σ_{12}	σ_{22}	σ_{23}	σ_{24}	σ_{25}	σ_{26}	σ_{27}	σ_{28}
$\mathcal{E}_3$	σ_{13}	σ_{23}	σ_{33}	σ_{34}	σ_{35}	σ_{36}	σ_{37}	σ_{38}
$\mathcal{E}_4$	σ_{14}	σ_{24}	σ_{34}	σ_{44}	σ_{45}	σ_{46}	σ_{47}	σ_{48}
$\mathcal{E}_5$	σ_{15}	σ_{25}	σ_{35}	σ_{45}	σ_{55}	σ_{56}	σ_{57}	σ_{58}
$\mathcal{E}_6$	σ_{16}	σ_{26}	σ_{36}	σ_{46}	σ_{56}	σ_{66}	σ_{67}	σ_{68}
$\mathcal{E}_7$	σ_{17}	σ_{27}	σ_{37}	σ_{47}	σ_{57}	σ_{67}	σ_{77}	σ_{78}
$\mathcal{E}_8$	σ_{18}	σ_{28}	σ_{38}	σ_{48}	σ_{58}	σ_{68}	σ_{78}	σ_{88}

To calculate the entropy of the system, the following information matrix, constructed on the eight spectrums and their values, is established (Table 20, 21), where $\mathcal{M}^{\alpha}$ and $\mathcal{M}^{\beta}$ stand two information matrices, and $\sigma^{\alpha}_{\aleph\mho}$ and $\partial^{\beta}_{\aleph\mho}$ represent the value of information in each spectrum (see the following equations).

$$\mathcal{M}^{\alpha} = \sigma^{\alpha}_{\aleph\mho},\ \aleph = \{1, 2, \ldots, 7, 8\},\ \mho = \{1, 2, \ldots, 7, 8\};$$
$$\mathcal{M}^{\beta} = \partial^{\beta}_{\aleph\mho},\ \aleph = \{1, 2, \ldots, 7, 8\},\ \mho = \{1, 2, \ldots, 7, 8\};$$

To calculate $\sigma^{\alpha}_{\aleph\mho}$ and $\sigma^{\beta}_{\aleph\mho}$, the common values and union values of spectrums are considered (see Table 22, 23).

TABLE 21. The information matrix - $\mathcal{M}^{\beta}$.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
$\mathcal{E}_1$	∂_{11}	∂_{12}	∂_{13}	∂_{14}	∂_{15}	∂_{16}	∂_{17}	∂_{18}
$\mathcal{E}_2$	∂_{12}	∂_{22}	∂_{23}	∂_{24}	∂_{25}	∂_{26}	∂_{27}	∂_{28}
$\mathcal{E}_3$	∂_{13}	∂_{23}	∂_{33}	∂_{34}	∂_{35}	∂_{36}	∂_{37}	∂_{38}
$\mathcal{E}_4$	∂_{14}	∂_{24}	∂_{34}	∂_{44}	∂_{45}	∂_{46}	∂_{47}	∂_{48}
$\mathcal{E}_5$	∂_{15}	∂_{25}	∂_{35}	∂_{45}	∂_{55}	∂_{56}	∂_{57}	∂_{58}
$\mathcal{E}_6$	∂_{16}	∂_{26}	∂_{36}	∂_{46}	∂_{56}	∂_{66}	∂_{67}	∂_{68}
$\mathcal{E}_7$	∂_{17}	∂_{27}	∂_{37}	∂_{47}	∂_{57}	∂_{67}	∂_{77}	∂_{78}
$\mathcal{E}_8$	∂_{18}	∂_{28}	∂_{38}	∂_{48}	∂_{58}	∂_{68}	∂_{78}	∂_{88}

The computation of missing information by entropy is conducted by the following equation, where $E^{\alpha}_{\mho}$ and $E^{\beta}_{\mho}$ stand

TABLE 22. The common values and union values of spectrums regarding - $\mathcal{M}^{\alpha}$.

$\boldsymbol{\sigma_{11}\sigma_{22}}$	$\boldsymbol{\sigma_{33}, \sigma_{44}}$	$\boldsymbol{\sigma_{55}, \sigma_{66}}$	$\boldsymbol{\sigma_{77}, \sigma_{88}}$	$\boldsymbol{\sigma_{12}}$	$\boldsymbol{\sigma_{11} \cup \sigma_{22}}$	$\boldsymbol{\sigma_{13}}$	$\boldsymbol{\sigma_{11} \cup \sigma_{33}}$
$\frac{\underline{u}-\underline{l}}{2}$	$\frac{\bar{u}-\underline{l}}{2}$	$\frac{\underline{u}-\bar{l}}{2}$	$\frac{\bar{u}-\bar{l}}{2}$	$\sigma_{11} \cup \sigma_{22}$	$\frac{\underline{u}-\underline{l}}{2}$	$\sigma_{11} \cup \sigma_{33}$	$\frac{\bar{u}-\underline{l}}{2}$
$\boldsymbol{\sigma_{14}}$	$\boldsymbol{\sigma_{11} \cup \sigma_{44}}$	$\boldsymbol{\sigma_{15}}$	$\boldsymbol{\sigma_{11} \cup \sigma_{55}}$	$\boldsymbol{\sigma_{16}}$	$\boldsymbol{\sigma_{11} \cup \sigma_{66}}$	$\boldsymbol{\sigma_{17}}$	$\boldsymbol{\sigma_{11} \cup \sigma_{77}}$
$\sigma_{11} \cup \sigma_{44}$	$\frac{\bar{u}-\underline{l}}{2}$	$\sigma_{11} \cup \sigma_{55}$	$\frac{\underline{u}-\underline{l}}{2}$	$\sigma_{11} \cup \sigma_{66}$	$\frac{\underline{u}-\underline{l}}{2}$	$\sigma_{11} \cup \sigma_{77}$	$\frac{\bar{u}-\underline{l}}{2}$
$\boldsymbol{\sigma_{18}}$	$\boldsymbol{\sigma_{11} \cup \sigma_{88}}$	$\boldsymbol{\sigma_{23}}$	$\boldsymbol{\sigma_{22} \cup \sigma_{33}}$	$\boldsymbol{\sigma_{24}}$	$\boldsymbol{\sigma_{22} \cup \sigma_{44}}$	$\boldsymbol{\sigma_{25}}$	$\boldsymbol{\sigma_{22} \cup \sigma_{55}}$
$\sigma_{11} \cup \sigma_{88}$	$\frac{\bar{u}-\underline{l}}{2}$	$\sigma_{22} \cup \sigma_{33}$	$\frac{\bar{u}-\underline{l}}{2}$	$\sigma_{22} \cap \sigma_{44}$	$\frac{\bar{u}-\underline{l}}{2}$	$\sigma_{22} \cup \sigma_{55}$	$\frac{\bar{u}-\underline{l}}{2}$
$\boldsymbol{\sigma_{26}}$	$\boldsymbol{\sigma_{22} \cup \sigma_{66}}$	$\boldsymbol{\sigma_{27}}$	$\boldsymbol{\sigma_{28}}$	$\boldsymbol{\sigma_{34}, \sigma_{35}, \sigma_{36}, \sigma_{37}, \sigma_{38}}$	$\boldsymbol{\sigma_{45}, \sigma_{46}, \sigma_{47}, \sigma_{48}}$	$\boldsymbol{\sigma_{56}}$	$\boldsymbol{\sigma_{55} \cap \sigma_{66}}$
$\sigma_{22} \cup \sigma_{66}$	$\frac{\bar{u}-\underline{l}}{2}$	$\sigma_{22} \cap \sigma_{77}$	$\frac{\bar{u}-\underline{l}}{2}$	$\frac{\bar{u}-\underline{l}}{2}$	$\frac{\bar{u}-\underline{l}}{2}$	$\sigma_{55} \cap \sigma_{66}$	$\frac{\underline{u}-\bar{l}}{2}$
$\boldsymbol{\sigma_{57}}$	$\boldsymbol{\sigma_{57}}$	$\boldsymbol{\sigma_{58}}$	$\boldsymbol{\sigma_{58}}$	$\boldsymbol{\sigma_{67}}$	$\boldsymbol{\sigma_{68}}$	$\boldsymbol{\sigma_{78}}$	$\boldsymbol{\sigma_{78}}$
$\sigma_{55} \cap \sigma_{77}$	$\frac{\bar{u}-\bar{l}}{2}$	$\sigma_{55} \cap \sigma_{88}$	$\frac{\bar{u}-\bar{l}}{2}$	$\frac{\bar{u}-\bar{l}}{2}$	$\frac{\bar{u}-\bar{l}}{2}$	$\sigma_{77} \cap \sigma_{88}$	$\frac{\bar{u}-\bar{l}}{2}$

TABLE 23. The common values and union values of spectrums regarding - $\mathcal{M}^{\beta}$.

$\boldsymbol{\partial_{12}}$	$\boldsymbol{\sigma_{13}}$	$\boldsymbol{\sigma_{14}}$	$\boldsymbol{\sigma_{15}}$	$\boldsymbol{\sigma_{16}}$
$\partial_{11} \cap \partial_{22}$	$\partial_{11} \cap \partial_{33}$	$\partial_{11} \cap \partial_{44}$	$\partial_{11} \cap \partial_{55}$	$\partial_{11} \cap \partial_{66}$
$\boldsymbol{\sigma_{17}}$	$\boldsymbol{\sigma_{18}}$	$\boldsymbol{\sigma_{23}}$	$\boldsymbol{\sigma_{24}}$	$\boldsymbol{\sigma_{25}}$
$\partial_{11} \cap \partial_{77}$	$\partial_{11} \cap \partial_{88}$	$\partial_{22} \cap \partial_{33}$	$\partial_{22} \cap \partial_{44}$	$\partial_{22} \cap \partial_{55}$
$\boldsymbol{\sigma_{26}}$	$\boldsymbol{\sigma_{27}}$	$\boldsymbol{\partial_{34}, \partial_{35}, \partial_{36}, \partial_{37}, \partial_{38}}$	$\boldsymbol{\partial_{45}, \partial_{46}, \partial_{47}, \partial_{48}}$	$\boldsymbol{\partial_{56}}$
$\partial_{22} \cap \partial_{66}$	$\partial_{22} \cap \partial_{77}$	$\partial_{33} \cap \partial_{44}$	$\partial_{44} \cap \partial_{55}$	$\partial_{55} \cap \partial_{66}$
$\boldsymbol{\partial_{57}}$	$\boldsymbol{\partial_{58}}$	$\boldsymbol{\partial_{67}}$	$\boldsymbol{\partial_{68}}$	$\boldsymbol{\partial_{78}}$
$\partial_{55} \cap \partial_{77}$	$\partial_{55} \cap \partial_{88}$	$\partial_{66} \cap \partial_{77}$	$\partial_{66} \cap \partial_{77}$	$\partial_{77} \cap \partial_{88}$

for the entropy.

$$E^{\alpha}_{\mho} = -\frac{1}{\log 8} \sum_{\aleph=1}^{8} \sigma^{\alpha}_{\aleph\mho} \log \sigma^{\alpha}_{\aleph\mho} \times \left(\sum_{\mho=1}^{8} -\frac{1}{\log 8} \sum_{\aleph=1}^{8} \sigma^{\alpha}_{\aleph\mho} \log \sigma^{\alpha}_{\aleph\mho} \right)^{-1};$$
$$\aleph = \{1, 2, \ldots, 7, 8\},\ \mho = \{1, 2, \ldots, 7, 8\};$$
$$\sum_{\mho=1}^{8} E^{\alpha}_{\mho} = -1;$$
$$E^{\beta}_{\mho} = \frac{-1}{\log 8} \sum_{\aleph=1}^{8} \partial^{\beta}_{\aleph\mho} \log \partial^{\beta}_{\aleph\mho} \times \left(\sum_{\mho=1}^{8} \frac{-1}{\log 8} \sum_{\aleph=1}^{8} \partial^{\beta}_{\aleph\mho} \log \partial^{\beta}_{\aleph\mho} \right)^{-1};$$
$$\aleph = \{1, 2, \ldots, 7, 8\},\ \mho = \{1, 2, \ldots, 7, 8\};$$

$$\sum_{\mho=1}^{8} E_{\mho}^{\beta} = -1;$$

$$\Delta_{S_{\mho}} = \max_{1 \leq \mho \leq 8} E_{\mho}^{\alpha}, E_{\mho}^{\beta};$$

APPENDIX C

The extended form of the Zakeri-Konstantas weighted rankings similarity measure:

$$\mathbb{ZK}_{X,B}^{\mathrm{P}} = 1 - \left\langle \sum_{i=1}^{m} \left(\frac{\left\langle (m+1) - R_{A_i}^{X} \right\rangle}{R_{A_i}^{X}} - \frac{\left\langle (m+1) - R_{A_i}^{B} \right\rangle}{R_{A_i}^{B}} \right)^{2} \times \mathrm{P}_{m}^{*\,-1},\ i = \{1, \ldots, m\}; \right\rangle^{0.5}$$

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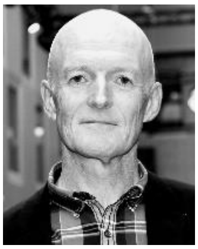

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