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RESEARCH ARTICLE

Uniform Stability of Linear Delay Impulsive Differential Systems With Impulse Time Windows and Logic Choice

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ABSTRACT This paper focuses on the uniform stability of linear delay impulsive differential systems with impulse time windows and logic choice. Firstly, a class of linear delay impulsive differential systems with impulse time windows and logic choice is constructed. The impulsive effects of this system have the following two properties: (i) The impulses do not appear at the fixed time points, but may occur at any points in a little range of time. (ii) The impulsive effects are affected by logic choice. Furthermore, by using the semi-tensor product method, we convert the logical functions contained in impulses into equivalent algebraic expressions. Next, based on Lyapunov functions and Razumikhin technique, the uniform stability criterion is obtained. Then, the uniform stability criterion is also applied to a class of linear uncertain delay impulsive differential systems with impulse time windows and logic choice. Finally, two illustrative examples are also discussed.

INDEX TERMS Uniform stability, impulsive differential systems, impulse time windows, logic choice, semi-tensor product, uncertain.

I. INTRODUCTION

In the past decades, impulsive systems have great advantages to model the abrupt change dynamics at discrete time, and have been widely used in the fields of biology, medicine, and communication security and so forth. A large number of scholars from different fields have conducted in-depth research on impulsive effects and achieved many research results [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]. At present, the impulsive effect has become one of the hot issues in the field of control and mathematics. As is known to all, time delay exists widely in various practical systems. Many results have been made in the analysis of time-delay systems [11], [12], [13], [14], [15], [16], [17], [33]. In recent years, the stability of impulsive delay differential equations has received much extensive attention from researchers. For example, [11] obtained some results on the stability of linear delay differential equations, [12] studied the exponential stability of a class of linear impulsive delay differential equations, [13] developed a new method regarding asymptotic behavior and stability of first order linear impulsive neutral delay differential equations with constant coefficients and constant delays, [14] obtained some sufficient conditions for local stability of nonlinear differential systems with state-dependent delayed impulsive control based on impulsive control theory, [15] studied the stability of time-delay systems with impulsive control involving stabilizing delays. Reference [34] studied the problems of stability and L2-gain for impulsive systems with time-delay.

The logic system has been applied in many fields such as game theory, information science, biological evolution and so on. In the early stage, it focused on the fixed point, attractor, period and other topological structures, and lacked the results of general qualitative research. Because at that time, there was a lack of mathematical tools that could effectively analyze logical relations. The emergence of the semi-tensor product proposed by [20] greatly changed this

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situation. It can transform logical functions into equivalent algebraic expressions, which facilitates the processing of logical factors, and has made significant progress in the fields of biological systems and life sciences, game theory, graph theory and information control.

From the above research results, we noticed two problems. Firstly, the impulses of the above studied systems all occurred at the fixed-time points, which has attracted the attention of some scholars. They found that the impulses may occur at any points in a little range of time, i.e., impulse time windows. For example, [18] investigated the uniform stability of linear delayed differential equations with impulse time windows. Reference [19] studied globally exponential stability of delayed impulsive functional differential systems with impulse time windows. Secondly, the impulsive effects can be affected by the logic effects in practical problems. Reference [21] first constructed a class of logic impulse systems, in which, impulsive control can be determined by logical choice. Thanks to the logic effects, impulsive control suffered by logic choice has better control effects. So far, the research on logic impulse systems has achieved some results. For example, [21] investigated the asymptotic stability of differential logic impulse systems, [22] worked out the problem of the finite-time stability of non-linear logic impulse systems, [23] studied time-delay discrete logic impulse systems, [24] obtained the result of the stability of delay differential logic impulse systems by the properties of a corresponding non-impulsive differential delay system. Reference [25] obtained the stability criteria for linear delay differential systems under logic impulsive control. Reference [26] investigated stability problems of stochastic delay differential logic impulse systems. But to our best of knowledge, at present, systems with these two properties have not been studied.

In fact, in the real world, sometimes there are some situations that have two characteristics at the same time. For example, the machine or computer may apply impulsive effects which are chosen from different functions according to some logical relationships among variable values [21], and it is well known that any machine or computer has errors in the input of impulses [31], [32], it is not easy to ensure that the impulsive input is exactly according to the fixed time points and thus the expected time is always different from the actual time, so one can set the machine to add the impulse in an impulse time window. Therefore, we think it is meaningful to construct and study a class of delay impulsive differential systems with impulse time windows and logic choice.

At the same time, we found that there are many scholars studying the uncertain systems. For example, [28] investigated the robust exponential stability of uncertain impulsive neural networks with time-varying delays and delayed impulses, [29] studied the robust control of a class of nonlinear systems with real time-varying parameter uncertainty. Therefore, we also considered the class of linear uncertain delay impulsive differential systems with impulse time windows and logic choice.

Based on the above considerations, the main purpose of this paper can be summarized as follows: (i) Construct a class of linear delay impulsive differential systems with impulse time windows and logic choice. (ii) By using the semi-tensor product, the logic functions contained in the impulsive effects are transformed into equivalent algebraic expressions. (iii) By using Lyapunov function and Razumikhin technique, the criterion of uniform stability is obtained. (iv) construct a class of linear uncertain delay impulsive differential systems with impulse time windows and logic choice, and apply the above criterion of uniform stability into the system. (v) Two numerical examples are discussed to illustrate the effectiveness of the results.

The structure of this paper is as follows: In Section II, we introduce some definitions and lemmas. Section III establishs a class of linear delay impulsive differential systems with impulse time windows and logic choice, and gives the main stability results. Section IV introduces a class of linear uncertain delay impulsive differential systems with impulse time windows and logic choice, and applies the stability results in Section III into it. Section V discusses two numerical examples to illustrate the effectiveness of the results. The conclusion is given in Section VI.

II. PRELIMINARIES

The following basic concepts, notations and lemmas will be used in the whole paper.

For a vector $x = (x_1, x_2, \cdots, x_n)^T \in R^n$, $\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$ denotes the Euclidean norm of x. For logical values, identify them with equivalent vectors: $T = 1 \sim \delta_2^1, F = 0 \sim \delta_2^2$. For an identity matrix I_k, δ_k^i denotes the ith cloumn, and $\Delta_k = \{\delta_k^i | i = 1, 2, \cdots, k\}$.

The block diagonal matrix is denoted by the shorthand $\mathrm{diag}\{\cdots\}$. The ith row (column) of matrix B is denoted by $Row_i(B)(Col_i(B))$. Moreover, the set of columns of matrix B is denoted by $Col(B)$.

For a $m \times n$ matrix $L \in R^{m\times n}$, if $Col(L) \subset \Delta_m$, then we call L the logical matrix. And $\mathcal{L}_{m\times n}$ denotes the set of $m \times n$ logical matrices. For a logical matrix $L \in \mathcal{L}_{m\times n}, L = \delta_m(i_1, i_2, \cdots, i_n)$ denotes $L = (\delta_m^{i_1}, \delta_m^{i_2}, \cdots, \delta_m^{i_n})$ for simplicity. Let '$\bigotimes$' denotes the Kronecker product of matrices. '$lcm(n, p)$' denotes the least common multiple of n and p.

For two symmetric matrices A and B, $A \leq B$ (respectively, $A < B$) means that $B - A$ is positive semi-definite (respectively, positive definite).

For $a, b \in R$ with $a < b$ and $S \subseteq R^n$, the following classes of functions are defined. $PC([a, b], S) = \{\phi : [a, b] \to S | \phi(t) = \phi(t^+), \forall t \in [a, b];\ \phi(t^-)$ exists in $S, \forall t \in [a, b]$ and $\phi(t^-) = \phi(t)$ for all but at most a finite number of points $t \in [a, b]\}$. We define $J_\tau = PC([-\tau, 0], R^n)$, for $\Phi \in J_\tau$. Let $|\Phi| = \sup_{-\tau\leq s\leq 0} \|\Phi(s)\|$ be the norm of Φ and we define

$x_t \in J_\tau$ with $x_t(s) = x(t+s)$ for $s \in [-\tau, 0]$, where $\tau > 0$ is a constant. For any $\alpha > 0$, let $PC(\alpha) = \{\phi \in J_\tau : |\phi| < \alpha\}$.

Definition 1 [20]: For two matrices $A \in R^{m\times n}$ and $B \in R^{p\times q}$, $\alpha = lcm(n, p)$, the semi-tensor product of A and B is

$$A \ltimes B = (A \bigotimes I_{\alpha/n})(B \bigotimes I_{\alpha/p})$$

Obviously, the semi-tensor product no longer has dimensional constraints on two multiplicative matrices, it is a generalization of the traditional product.

Definition 2 [18]: The function $V : [t_0, \infty) \times R^n \to R_+$ belongs to class v_0 if the following conditions are true.

(i) $V(t, x)$ is locally Lipschitzian in $x \in R^n$ and is continuous on each of the sets $[\tau_{k-1}, \tau_k)$, $V(t, 0) \equiv 0$ for all $t \geq t_0$.

(ii) For each $k = 1, 2, \ldots$, there exist finite limits

$$\lim_{(t,y)\to(\tau_k^-,x)} V(t, y) = V(\tau_k^-, x),$$
$$\lim_{(t,y)\to(\tau_k^+,x)} V(t, y) = V(\tau_k^+, x)$$

with $V(\tau_k^+, x) = V(\tau_k, x)$ being sastisfied.

Definition 3 [18]: Let $V \in v_0$, for $t \in (\tau_{k-1}, \tau_k)$, the upper right-hand derivarive of V is defined by

$$D^+V(t, x(t)) = \lim_{h\to 0^+} sup\frac{1}{h}\{V(t+h, x(t+h)) - V(t, x(t))\}$$

Lemma 1 [20]: For a logical function $f(p_1, p_2, \cdots, p_r) \in \Delta_2$, there exists a unique 2×2^r matrix $M_f \in \mathcal{L}_{m\times n}$, called the structure matrix of f, such that

$$f(p_1, p_2, \cdots, p_r) = M_f \ltimes p_1 \ltimes p_2 \ltimes \cdots \ltimes p_r = M_f \ltimes_{i=1}^r p_i,$$

where $p_1, p_2, \cdots, p_r \in \Delta_2$ are logical variables. Moreover, note that $\ltimes_{i=1}^r p_i \subset \Delta_{2^r}$

Lemma 2 [30]: If $P \in R^{n\times n}$ is a positive-define matrix, $Q \in R^{n\times n}$ is a symmetric matrix, $x \in R^n$ then

$$\lambda_{min}(P^{-1}Q)x^TPx \leq x^TQx \leq \lambda_{max}(P^{-1}Q)x^TPx.$$

Lemma 3 [28]: For any constant matrices with appropriate dimensions A and B and any positive matrix $Q > 0$, the following inequality is satisfied:

$$A^TB + B^TA \leq A^TQA + B^TQ^{-1}B.$$

III. STABILITY ANALYSIS OF A CLASS OF LINEAR DELAY IMPULSIVE DIFFERENTIAL SYSTEMS WITH IMPULSE TIME WINDOWS AND LOGIC CHOICE

A. SYSTEM MODEL

Inspired by [18], [21], and [24], we construct a class of linear delay impulsive differential systems with impulse time windows and logic choice, which has two properties: the first is that the impulses may occur at any points in a little range of time, the second is that the impulsive effects can be determined by logic effects.

For any given t_0 and $\varphi \in J_\tau$, consider a class of linear delay impulsive differential systems with impulse time windows and logic choice:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau), & t \geq t_0, t \neq \tau_k \\ \Delta x(t) = x(t) - x(t^-) = \Psi_k x(t^-), & t = \tau_k, k \in N_+ \\ x(t_0 + s) = \varphi(s), & s \in [-\tau, 0] \end{cases} \tag{1}$$

where $x \in R^n$, $\Delta x = (\Delta x_1, \Delta x_2, \cdots, \Delta x_n) \in R^n$, $A, B \in R^{n\times n}$, $\tau > 0$, $\Psi_k = diag\{\psi_k^1, \psi_k^2, \cdots, \psi_k^n\}$, $t_0 \geq \tau_0$ and the fixed points $\{\tau_k^l\}_{k=1}^\infty$, $\{\tau_k^r\}_{k=1}^\infty$ satisfying $0 \leq \tau_0^l = \tau_0 = \tau_0^r \leq \tau_1^l < \tau_1^r \leq \tau_2^l < \tau_2^r \leq \cdots \leq \tau_k^l < \tau_k^r \leq \cdots$, $\lim_{k\to\infty} \tau_k^r = \infty$. The impulsive time τ_k is any value of fixed-time windows $[\tau_k^l, \tau_k^r)$, N_+ denotes the set of positive integer.

At $t = \tau_k$, we choose the impulsive effects from the functions $c_{ik}x_i$ and $d_{ik}x_i$, where $c_{ik}, d_{ik} \in R$,

$$\begin{aligned} \Delta x_i(\tau_k) &= x_i(\tau_k) - x_i(\tau_k^-) \\ &= g_i(p_1(x_1(\tau_k^-)), \cdots, p_n(x_n(\tau_k^-)))c_{ik}x_i(\tau_k^-) \\ &\quad + \overline{g_i(p_1(x_1(\tau_k^-)), \cdots, p_n(x_n(\tau_k^-)))}d_{ik}x_i(\tau_k^-) \end{aligned}$$

where $g_i, \bar{g}_i : \{\delta_2^1, \delta_2^2\}^n \to \{0, 1\}$ are logical functions, the negation function of g_i is denoted by $\bar{g}_i$, and we define the piecewise functions $p_i : \mathcal{R} \to \{0, 1\}$ as follows:

$$p_i(u) = \begin{cases} \delta_2^2 \sim 0, & |u| \geq w_i \\ \delta_2^1 \sim 1, & |u| < w_i \end{cases}$$

where $w_i > 0$ is a threshold.

Then, we convert logic impulsive effects to the algebraic expressions. For the linear differential delayed systems with impulse time windows suffered by logic choice, we can express logical functions g_i and $\bar{g}_i$ in the following form according to the Lemma 1.

$$\begin{aligned} &g_i(p_1(x_1(\tau_k^-)), \cdots, p_n(x_n(\tau_k^-))) \\ &\qquad = Row_1(M_i) \ltimes_{i=1}^n p_i(x_i(\tau_k^-)), \\ &\overline{g_i(p_1(x_1(\tau_k^-)), \cdots, p_n(x_n(\tau_k^-)))} \\ &\qquad = Row_2(M_i) \ltimes_{i=1}^n p_i(x_i(\tau_k^-)). \end{aligned}$$

where the logical matrix $M_i = (m_{ql}^i)_{2\times 2^n}$ is the unique structure matrix. Let $p(x(\tau_k^-)) \triangleq \ltimes_{i=1}^n p_i(x_i(\tau_k^-))$, then we can rewrite the impulsive effects suffered by logic choice as

$$\begin{aligned} \Delta x_i(\tau_k) &= (c_{ik}x_i(\tau_k^-), d_{ik}x_i(\tau_k^-))M_i p(x(\tau_k^-)) \\ &= (c_{ik}, d_{ik})M_i p(x(\tau_k^-))x_i(\tau_k^-) \end{aligned}$$

According to the definition of $p_i(u)$, $p_i(x_i(\tau_k^-)) \in \Delta_2$, we can derive that $p(x(\tau_k^-)) \in \Delta_{2^n}$. Let $p(x(\tau_k^-)) = \delta_{2^n}^{j_k}$. Then we obtain

$$\begin{aligned} \Delta x_i(\tau_k^-) &= (c_{ik}, d_{ik})Col_{j_k}(M_i)x_i(\tau_k^-) \\ &= (c_{ik}, d_{ik})(m_{1,j_k}^i, m_{2,j_k}^i)^T x_i(\tau_k^-) \\ &= (c_{ik}m_{1,j_k}^i + d_{ik}m_{2,j_k}^i)x_i(\tau_k^-) \end{aligned}$$

Therefore, we can convert the impulsive effects into the following form:

$$\Delta x(\tau_k^-) = \Psi_k x(\tau_k^-) = diag\{\psi_k^1, \psi_k^2, \cdots, \psi_k^n\} x(\tau_k^-)$$

where $\psi_k^i = c_{ik} m_{1,j_k}^i + d_{ik} m_{2,j_k}^i, i = 1, 2, \cdots, n, k \in N_+$.

Therefore, a class of linear delay impulsive differential systems with impulse time windows and logic choice (1) is constructed.

Meanwhile, if $\Phi_k(t^-)$ is denoted by

$$\Phi_k(t^-) \triangleq \begin{pmatrix} c_{1k}x_1(t^-)\ d_{1k}x_1(t^-) & & \\ & \ddots & \\ & & c_{nk}x_n(t^-)\ d_{nk}x_n(t^-) \end{pmatrix}.$$

Then, impusive effects can be also converted into the following form:

$$\Delta x(\tau_k^-) = \Phi(\tau_k^-) M p(x(\tau_k^-)),$$

where $M = (M_1^T, M_2^T, \cdots, M_n^T)^T$.

Definition 4 [27]: The trival solution of (1) is said to be stable if for any $t_0 \geq \tau_0$ and $\varepsilon > 0$ there is a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\phi \in PC(\delta), t \geq t_0$ implies that

$$\|x(t, t_0, \phi)\| < \varepsilon, \quad t \geq t_0$$

If δ is independent of t_0, the trival solution of (1) is said to be uniformly stable.

B. STABILITY CRITERION

Theorem 1: For a symmetric and positive matrix P, $\lambda_1 > 0$, $\lambda_2 > 0$ denote the maximum and the minimum eigenvalues of P respectively, λ_3 and λ_4 denote the largest eigenvalues of $P^{-1}(A^TP + PA + PP)$ and $P^{-1}B^TB$ respectively, $\beta = \max_{\substack{i=1,\cdots,n\\k\in N_+}} \{(1+\psi_k^i)^2\}$, $\lambda_5 = \frac{\lambda_1\beta}{\lambda_2}$, $0 < \lambda_5 < 1$. Then the trival solution of (1) is uniformly sable if

$$(\lambda_3 + \frac{\lambda_4}{\lambda_5})(\tau_k^r - \tau_{k-1}^r) < -ln\lambda_5, \quad k \in N_+ \tag{2}$$

Proof of Theorem 1: For any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$, such that $\delta < \sqrt{\beta}\varepsilon$. Let $x(t) = x(t, t_0, \phi)$ be any solution of the linear impulsive systems (1) through (t_0, ϕ). Consider a Lyapunov function $V(t, x(t)) = x^T(t)Px(t) \in v_0, t \geq t_0 - \tau$, then, $\lambda_2\|x(t)\|^2 \leq V(t, x(t)) \leq \lambda_1\|x(t)\|^2$.

When $t \neq \tau_k, k \in N_+$ we have

$$\begin{aligned} &D^+V(t, x(t)) \\ &= (x^T(t))'Px(t) + x^T(t)Px(t)' \\ &= [Ax(t) + Bx(t-\tau)]^T Px(t) + x^T(t)P(Ax(t) + Bx(t-\tau)) \\ &= x^T(t)[A^TP + PA]x(t) + 2x^T(t-\tau)B^TPx(t) \\ &\leq x^T(t)[A^TP + PA]x(t) + x^T(t-\tau)B^TBx(t-\tau) \\ &\quad + x^T(t)PPx(t) \\ &= x^T(t)[A^TP + PA + PP]x(t) + x^T(t-\tau)B^TBx(t-\tau) \\ &\leq \lambda_3 V(t, x(t)) + \lambda_4 V(t-\tau, x(t-\tau)). \end{aligned} \tag{3}$$

Step 1: We first prove: for any $t_0 \geq \tau_0, \phi \in PC(\delta)$, $V(t, x(t)) \leq \frac{\lambda_1}{\lambda_5}\delta^2, \quad t \in [t_0 - \tau, t_0]$.

For any $t_0 \geq \tau_0, t \in [t_0 - \tau, t_0]$, and $\phi \in PC(\delta)$, there exists a $\theta \in [-\tau, 0]$ such that $t = t_0 + \theta$, and

$$\begin{aligned} V(t, x(t)) &= V(t_0 + \theta, x(t_0 + \theta)) \\ &\leq \lambda_1\|\phi(\theta)\|^2 \leq \lambda_1\delta^2 < \frac{\lambda_1}{\lambda_5}\delta^2. \end{aligned}$$

Step 2: Let $t_0 \in [\tau_{k-1}^r, \tau_k^r)$ for some $k \in N_+$. We then prove that

$$V(t, x(t)) \leq \frac{\lambda_1}{\lambda_5}\delta^2, \quad t_0 \leq t < \tau_k^r. \tag{4}$$

The value of t_0 has two possible cases: Case I ($t_0 \in [\tau_{k-1}^r, \tau_k)$) and Case II ($t_0 \in [\tau_k, \tau_k^r)$).

Case I: $t_0 \in [\tau_{k-1}^r, \tau_k)$

According to the value of t, we will consider two parts: Case I-1 ($t \in [t_0, \tau_k)$) and Case I-2 ($t \in [\tau_k, \tau_k^r)$) in this case.

Case I-1: $t \in [t_0, \tau_k)$, we can get

$$V(t, x(t)) \leq \frac{\lambda_1}{\lambda_5}\delta^2, \quad t_0 \leq t < \tau_k. \tag{5}$$

If inequality (5) is not true, then there is a $\hat{t} \in (t_0, \tau_k)$, such that

$$V(t_0, x(t_0)) \leq \lambda_1\delta^2 < \frac{\lambda_1}{\lambda_5}\delta^2 < V(\hat{t}, x(\hat{t})), \tag{6}$$

For $V(t, x(t)) \in V_0$, then from the continuity of $V(t, x(t)), t \in [\tau_{k-1}^r, \tau_k)$, there exists a $t_1 \in (t_0, \hat{t}]$ such that

$$\begin{aligned} V(t_1, x(t_1)) &= \frac{\lambda_1}{\lambda_5}\delta^2, \\ V(t, x(t)) &\leq \frac{\lambda_1}{\lambda_5}\delta^2, \quad t_0 - \tau \leq t \leq t_1. \end{aligned} \tag{7}$$

From the inequality (6), it follows that there exists a $t_2 \in [t_0, t_1)$ such that

$$\begin{aligned} V(t_2, x(t_2)) &= \lambda_1\delta^2, \\ V(t, x(t)) &\geq \lambda_1\delta^2, \quad t_2 \leq t \leq t_1. \end{aligned} \tag{8}$$

Therefore, from inequalities (7) and (8), we have that for $\theta \in [-\tau, 0]$,

$$V(t+\theta, x(t+\theta)) \leq \frac{\lambda_1}{\lambda_5}\delta^2 \leq \frac{1}{\lambda_5}V(t, x(t)), \quad t \in [t_2, t_1].$$

So,

$$V(t-\tau, x(t-\tau)) \leq (1/\lambda_5)V(t, x(t)).$$

Therefore, for $t \in [t_2, t_1]$

$$D^+V(t, x(t)) \leq (\lambda_3 + \frac{\lambda_4}{\lambda_5})V(t, x(t)). \tag{9}$$

Integrate (9) in $t \in [t_2, t_1]$, we get

$$\int_{t_2}^{t_1} \frac{D^+V(t, x(t))}{V(t, x(t))} dt = \int_{V(t_2, x(t_2))}^{V(t_1, x(t_1))} \frac{du}{u} = \int_{\lambda_1\delta^2}^{\frac{\lambda_1}{\lambda_5}\delta^2} \frac{1}{u} du$$

$$= ln\frac{\frac{\lambda_1}{\lambda_5}\delta^2}{\lambda_1\delta^2} = ln\frac{1}{\lambda 5} = -ln\lambda_5$$

And at the same time, we can conclude that

$$\int_{t_2}^{t_1}\frac{D^+V(t,x(t))}{V(t,x(t))}dt$$
$$\leq \int_{t_2}^{t_1}(\lambda_3+\frac{\lambda_4}{\lambda_5})dt$$
$$\leq \int_{\tau_{k-1}^r}^{\tau_k^r}(\lambda_3+\frac{\lambda_4}{\lambda_5})dt = (\lambda_3+\frac{\lambda_4}{\lambda_5})(\tau_k^r-\tau_{k-1}^r)$$
$$< -ln\lambda_5.$$

It is obvious that a contradiction occur. So (5) holds.

Case I-2: $t \in [\tau_k, \tau_k^r)$.

For $t = \tau_k$ and the given impulse conditions, we have

$$V(\tau_k, x(\tau_k)) = x^T(\tau_k^-)(I+\Psi_k)^T P(I+\Psi_k)x(\tau_k^-).$$

Let $D_k = I + \Psi_k = diag\{1+\psi_k^1, 1+\psi_k^2, \ldots, 1+\psi_k^n\}$, Then $D_k^T D_k = diag\{(1+\psi_k^1)^2, (1+\psi_k^2)^2, \ldots, (1+\psi_k^n)^2\}$, which implies that β is the maximum eigenvalue of $D_k^T D_k$, for $k \in N_+$. And since λ_2 denotes the minimum eigenvalue of the symmetric and positive matrix P, $1/\lambda_2$ denotes the maximum eigenvalue of matrix P^{-1}.

So, we can conclude that

$$V(\tau_k, x(\tau_k))$$
$$= x^T(\tau_k^-)D_k^T P D_k x(\tau_k^-) \leq \lambda_1 x^T(\tau_k^-)D_k^T D_k x(\tau_k^-)$$
$$\leq \lambda_1\beta x^T(\tau_k^-)x(\tau_k^-) = \lambda_1\beta x^T(\tau_k^-)P^{-1}Px(\tau_k^-)$$
$$\leq \lambda_1\beta\frac{1}{\lambda_2}x^T(\tau_k^-)Px(\tau_k^-) = \lambda_1\beta\frac{1}{\lambda_2}V(\tau_k^-, x(\tau_k^-))$$
$$= \lambda_5 V(\tau_k^-, x(\tau_k^-)) \leq \lambda_1\delta^2.$$

Then, by employing the same way of Case I-1, we can get that

$$V(t,x(t)) \leq \frac{\lambda_1}{\lambda_5}\delta^2, \quad \tau_k \leq t < \tau_k^r. \tag{10}$$

By combing inequalities (5) and (10), for $t_0 \in [\tau_{k-1}^r, \tau_k)$, the inequality (4) is concluded.

Case II: $t_0 \in [\tau_k, \tau_k^r)$.

We can get that

$$V(t,x(t)) \leq \frac{\lambda_1}{\lambda_5}\delta^2, \quad t_0 \leq t < \tau_k^r. \tag{11}$$

We can prove the inequality (11) using the same method of case I-1.

According to the discussion of the above two cases, we prove that inequality (4) is true.

Step 3: Next, we prove that

$$V(t,x(t)) \leq \frac{\lambda_1}{\lambda_5}\delta^2, \quad \tau_k^r \leq t < \tau_{k+1}^r. \tag{12}$$

We will also consider two cases, that is $t \in [\tau_k^r, \tau_{k+1})$ and $t \in [\tau_{k+1}, \tau_{k+1}^r)$.

Case A: If $t \in [\tau_k^r, \tau_{k+1})$, we can get the following inequality:

$$V(t,x(t)) \leq \frac{\lambda_1}{\lambda_5}\delta^2, \quad \tau_k \leq t < \tau_{k+1}. \tag{13}$$

According to the inequality (4), we can easily prove the inequality (13) by using the same method of the proof of Case I-1.

Note that $[\tau_k^r, \tau_{k+1}) \subset [\tau_k, \tau_{k+1})$, we can obtain that

$$V(t,x(t)) \leq \frac{\lambda_1}{\lambda_5}\delta^2, \quad \tau_k^r \leq t < \tau_{k+1}.$$

Case B: If $t \in [\tau_{k+1}, \tau_{k+1}^r)$, we first can get that for $t = \tau_{k+1}$,

$$V(\tau_{k+1}, x(\tau_{k+1}))$$
$$= x^T(\tau_{k+1}^-)(I+\Psi_{k+1})^T P(I+\Psi_{k+1})x(\tau_{k+1}^-)$$
$$= \lambda_5 V(\tau_{k+1}^-, x(\tau_{k+1}^-)) \leq \lambda_1\delta^2.$$

Next, for $t \in [\tau_{k+1}, \tau_{k+1}^r)$ using the same method of the proof of Case I-1, we can get

$$V(t,x(t)) \leq \frac{\lambda_1}{\lambda_5}\delta^2, \quad \tau_{k+1} \leq t < \tau_{k+1}^r.$$

Therefore, we have proved the inequality (12) by discussing the above two cases.

Step 4: By simple induction, for $m = 0, 1, 2, \cdots$,

$$V(t,x(t)) \leq \frac{\lambda_1}{\lambda_5}\delta^2, \quad \tau_{k+m} \leq t < \tau_{k+m+1}^r.$$

Therefore, for any $t \geq t_0$, the following inequality can be obtained

$$V(t,x(t)) \leq \frac{\lambda_1}{\lambda_5}\delta^2, \quad t \geq t_0.$$

Step 5: Now, we can get the following conclusion for any $t \geq t_0$, and $\phi \in PC(\delta)$,

$$\lambda_2\|x(t)\|^2 \leq V(t,x(t)) = x^T(t)Px(t) \leq \frac{\lambda_1}{\lambda_5}\delta^2, \quad t \geq t_0,$$

which implies

$$\|x(t)\| \leq \sqrt{\frac{\lambda_1}{\lambda_2\lambda_5}}\delta < \varepsilon, \quad t \geq t_0.$$

Then, the trival solution of (1) is uniformly stable.

IV. STABILITY ANALYSIS OF LINEAR UNCERTAIN DELAY IMPULSIVE DIFFERENTIAL SYSTEMS WITH IMPULSE TIME WINDOWS AND LOGIC CHOICE

Consider the following linear uncertain delay impulsive differential systems with impulse time windows and logic choice:

$$\begin{cases} \dot{x}(t) = (A+\Delta A(t))x(t) + (B+\Delta B(t)) & x(t-\tau), \\ & t \geq t_0, t \neq \tau_k \\ \Delta x(t) = x(t) - x(t^-) = \Psi_k x(t^-), & t = \tau_k, k \in N_+ \\ x(t_0+s) = \varphi(s), & s \in [-\tau, 0] \end{cases} \tag{14}$$

where $\Delta A(\cdot)$ and $\Delta B(\cdot)$ are real-valued matrix functions representing time-varying parameter uncertainties, which are assumed to be of the following forms:

$$(\Delta A(t) \ \ \Delta B(t)) = HF(t)(E_1 \ \ E_2),$$

here H, E_1, E_2 are known real constant matrices with appropriate dimensions, $F(t)$ is an unknown real matrix function satisfying

$$F^T(t)F(t) \leq I, \quad \forall t \geq 0,$$

and I is the identity matrix.

The specific meaning of other symbols are the same as system (1). Obviously, system (14) is an extension of system (1). Moreover, the uniform stability of system (14) is similar to that of system (1), which is omitted here.

Remark 1: This construction of uncertainty is common in the existing research results, such as [28] and [29].

In order to facilitate the subsequent stability analysis, the following conversions are made to system (14).

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau) + HQ(t), & t \geq t_0, t \neq \tau_k \\ Q(t) = F(t)q(t), & t \geq t_0, t \neq \tau_k \\ q(t) = E_1x(t) + E_2x(t-\tau), & t \geq t_0, t \neq \tau_k \\ \Delta x(t) = x(t) - x(t^-) = \Psi_k x(t^-), & t = \tau_k, k \in N_+ \\ x(t_0+s) = \varphi(s), & s \in [-\tau, 0] \end{cases} \tag{15}$$

Theorem 2: For a symmetric and positive matrix P, $\lambda_1 > 0$, $\lambda_2 > 0$ denote the maximum and the minimum eigenvalues of P respectively, λ_3 and λ_4 denote the largest eigenvalues of $P^{-1}(A^TP + PA + PP + PHH^TP + 2E_1^TE_1)$ and $P^{-1}(B^TB + 2E_2^TE_2)$ respectively, $\beta = \underset{\substack{i=1,\cdots,n \\ k\in N_+}}{max}\{(1+\psi_k^i)^2\}, \lambda_5 = \frac{\lambda_1\beta}{\lambda_2}, 0 < \lambda_5 < 1$. Then the trival solution of (14) is uniformly sable if

$$(\lambda_3 + \frac{\lambda_4}{\lambda_5})(\tau_k^r - \tau_{k-1}^r) < -ln\lambda_5, \quad k \in N_+ \tag{16}$$

Proof of Theorem 2: For any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$, such that $\delta < \sqrt{\beta}\varepsilon$. Let $x(t) = x(t, t_0, \phi)$ be any solution of the linear impulsive systems (14) through (t_0, ϕ). Consider a Lyapunov function $V(t, x(t)) = x^T(t)Px(t) \in v_0, t \geq t_0 - \tau$, then, $\lambda_2\|x(t)\|^2 \leq V(t, x(t)) \leq \lambda_1\|x(t)\|^2$.

When $t \neq \tau_k, k \in N_+$, From Lemma 2 and Lemma 3, we have

$$\begin{aligned}
&D^+V(t, x(t)) \\
&= (x^T(t))'Px(t) + x^T(t)Px(t)' \\
&= [Ax(t) + Bx(t-\tau) + HQ(t)]^TPx(t) \\
&\quad + x^T(t)P(Ax(t) + Bx(t-\tau) + HQ(t)) \\
&= x^T(t)[A^TP + PA]x(t) + 2x^T(t-\tau)B^TPx(t) \\
&\quad + x^T(t)PHQ(t) + Q^T(t)(x^T(t)PH)^T \\
&\leq x^T(t)[A^TP + PA]x(t) + x^T(t-\tau)B^TBx(t-\tau) \\
&\quad + x^T(t)PPx(t) + x^T(t)PH(x^T(t)PH)^T + Q^T(t)Q(t) \\
&= x^T(t)[A^TP + PA + PP]x(t) + x^T(t-\tau)B^TBx(t-\tau) \\
&\quad + x^T(t)PHH^TP^Tx(t) + q^T(t)F^T(t)F(t)q(t) \\
&\leq x^T(t)[A^TP + PA + PP]x(t) + x^T(t-\tau)B^TBx(t-\tau) \\
&\quad + x^T(t)PHH^TPx(t) + q^T(t)q(t) \\
&\leq x^T(t)[A^TP + PA + PP + PHH^TP]x(t) \\
&\quad + x^T(t-\tau)B^TBx(t-\tau) \\
&\quad + (x^T(t)E_1^T + x^T(t-\tau)E_2^T)(E_1x(t) + E_2x(t-\tau)) \\
&\leq x^T(t)[A^TP + PA + PP + PHH^TP]x(t) \\
&\quad + x^T(t-\tau)B^TBx(t-\tau) + x^T(t)E_1^TE_1x(t) \\
&\quad + x^T(t)E_1^TE_2x(t-\tau) + x^T(t-\tau)E_2^TE_1x(t) \\
&\quad + x^T(t-\tau)E_2^TE_2x(t-\tau) \\
&\leq x^T(t)[A^TP + PA + PP + PHH^TP]x(t) \\
&\quad + x^T(t-\tau)B^TBx(t-\tau) + 2x^T(t)E_1^TE_1x(t) \\
&\quad + 2x^T(t-\tau)E_2^TE_2x(t-\tau) \\
&\leq x^T(t)[A^TP + PA + PP + PHH^TP + 2E_1^TE_1]x(t) \\
&\quad + x^T(t-\tau)[B^TB + 2E_2^TE_2]x(t-\tau) \\
&\leq \lambda_3V(t, x(t)) + \lambda_4V(t-\tau, x(t-\tau)).
\end{aligned} \tag{17}$$

The proof of remainder is the same as Theorem 1, which is omitted here.

For system (14), if let $H = I, E_1 = e_1I, E_2 = e_2I$, that is to say, $(\Delta A(t) \quad \Delta B(t)) = F(t)(e_1I \quad e_2I)$ where e_1, e_2 are real constant. Then we obtain the following system.

$$\begin{cases} \dot{x}(t) = (A + e_1F(t))x(t) + (B + e_2F(t)) & x(t-\tau), \\ & t \geq t_0, t \neq \tau_k \\ \Delta x(t) = x(t) - x(t^-) = \Psi_k x(t^-), & t = \tau_k, k \in N_+ \\ x(t_0+s) = \varphi(s), & s \in [-\tau, 0] \end{cases} \tag{18}$$

According to Theorem 2, we can get the following corollory.

Corollory 1: For a symmetric and positive matrix P, $\lambda_1 > 0$, $\lambda_2 > 0$ denote the maximum and the minimum eigenvalues of P respectively, λ_3 and λ_4 denote the largest eigenvalues of $P^{-1}(A^TP + PA + 2PP + 2e_1^2I)$ and $P^{-1}(B^TB + 2e_2^2I)$ respectively, $\beta = \underset{\substack{i=1,\cdots,n \\ k\in N_+}}{max}\{(1+\psi_k^i)^2\}, \lambda_5 = \frac{\lambda_1\beta}{\lambda_2}, 0 < \lambda_5 < 1$. Then the trival solution of (18) is uniformly sable if

$$(\lambda_3 + \frac{\lambda_4}{\lambda_5})(\tau_k^r - \tau_{k-1}^r) < -ln\lambda_5, \quad k \in N_+ \tag{19}$$

V. NUMERICAL EXAMPLES

In this section, we discuss two numerical examples to illustrate the effectiveness of the stability result.

Example 1: consider the following linear delay differential system:

$$\dot{x}(t) = \begin{pmatrix} 1 & 0.6 & 0.7 \\ 0.5 & 2 & 0.6 \\ 0.2 & 0.3 & 1 \end{pmatrix} x(t)$$

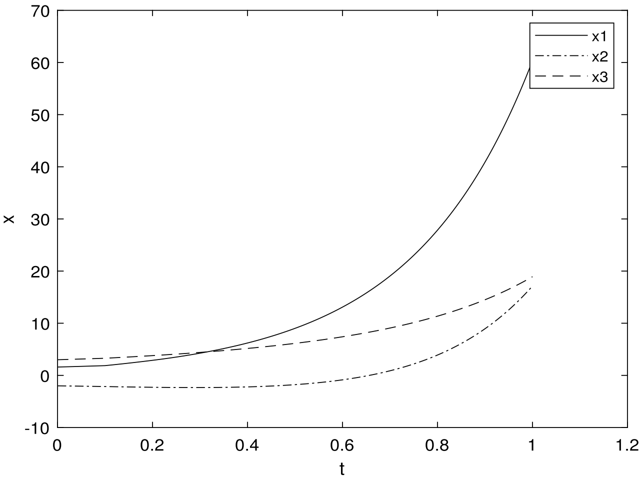

FIGURE 1. The trajectory of $x(t)$ in non-impulsive system (20).

$$+\begin{pmatrix} 2 & 3 & 3 \\ 0.3 & 1 & 0.4 \\ 0.2 & 0.1 & 0.5 \end{pmatrix} x(t-0.1), \tag{20}$$

$t \geq 0$ and with initial conditions:

$$\varphi_1(t) = \begin{cases} 0, & t \in [-0.1, 0), \\ 1.6, & t = 0, \end{cases}$$

$$\varphi_2(t) = \begin{cases} 0, & t \in [-0.1, 0), \\ -2, & t = 0, \end{cases}$$

$$\varphi_3(t) = \begin{cases} 0, & t \in [-0.1, 0), \\ 3, & t = 0. \end{cases}$$

From Fig 1, we can illustrate that the system (20) is unstable.

Then we add the impusive effects to system (20) by the way of impulse time windows and logic choice:

Let $\tau_0 = \tau_0^l = \tau_0^r = 0$, $\tau_{k+1}^r - \tau_{k+1}^l = 0.006$, $\tau_{k+1}^l - \tau_k^r = 0.002$, and the impulsive time τ_k is any value of fixed-time windows $[\tau_k^l, \tau_k^r)$.

The impulses are suffered by logic choice can be described as

$$\begin{cases} \Delta x_1(\tau_k) & = x_1(\tau_k) - x_1(\tau_k^-) \\ & = -0.4x_1(\tau_k^-)g_1(\tau_k^-) - 0.5x_1(\tau_k^-)\overline{g_1(\tau_k^-)} \\ \Delta x_2(\tau_k) & = x_2(\tau_k) - x_2(\tau_k^-) \\ & = -0.5x_2(\tau_k^-)g_2(\tau_k^-) - 0.45x_2(\tau_k^-)\overline{g_2(\tau_k^-)} \\ \Delta x_3(\tau_k) & = x_3(\tau_k) - x_3(\tau_k^-) \\ & = -0.2x_3(\tau_k^-)g_3(\tau_k^-) - 0.4x_3(\tau_k^-)\overline{g_3(\tau_k^-)} \end{cases}$$

where:

$$\begin{aligned} g_1(t) &= [p_1(x_1(t)) \wedge (p_2(x_2(t)) \vee p_3(x_3(t))) \\ &\quad \vee [\neg p_1(x_1(t)) \wedge (p_2(x_2(t)) \uparrow p_3(x_3(t)))], \\ g_2(t) &= [p_1(x_1(t)) \wedge (p_2(x_2(t)) \wedge p_3(x_3(t)))] \\ &\quad \vee [\neg p_1(x_1(t)) \wedge (p_2(x_2(t)) \downarrow p_3(x_3(t)))], \\ g_3(t) &= [p_1(x_1(t)) \wedge (p_2(x_2(t)) \rightarrow p_3(x_3(t)))] \\ &\quad \vee [\neg p_1(x_1(t)) \wedge (p_2(x_2(t)) \bar{\vee} p_3(x_3(t)))]. \end{aligned}$$

TABLE 1. The impulses suffered by logic choice for impulsive system (21).

$\|x_1(t^-)\|$	$\|x_2(t^-)\|$	$\|x_3(t^-)\|$	$\Delta x_1(t)$	$\Delta x_2(t)$	$\Delta x_3(t)$
< 0.7	< 0.6	< 0.8	$-0.4x_1(t^-)$	$-0.5x_2(t^-)$	$-0.2x_3(t^-)$
< 0.7	< 0.6	≥ 0.8	$-0.4x_1(t^-)$	$-0.45x_2(t^-)$	$-0.4x_3(t^-)$
< 0.7	≥ 0.6	< 0.8	$-0.4x_1(t^-)$	$-0.45x_2(t^-)$	$-0.2x_3(t^-)$
< 0.7	≥ 0.6	≥ 0.8	$-0.5x_1(t^-)$	$-0.45x_2(t^-)$	$-0.2x_3(t^-)$
≥ 0.7	< 0.6	< 0.8	$-0.5x_1(t^-)$	$-0.45x_2(t^-)$	$-0.4x_3(t^-)$
≥ 0.7	< 0.6	≥ 0.8	$-0.4x_1(t^-)$	$-0.45x_2(t^-)$	$-0.2x_3(t^-)$
≥ 0.7	≥ 0.6	< 0.8	$-0.4x_1(t^-)$	$-0.45x_2(t^-)$	$-0.2x_3(t^-)$
≥ 0.7	≥ 0.6	≥ 0.8	$-0.4x_1(t^-)$	$-0.5x_2(t^-)$	$-0.4x_3(t^-)$

Let $w_1 = 0.7$ in the piecewise function $p_1(u)$, $w_2 = 0.6$ in the piecewise function $p_2(u)$, $w_3 = 0.8$ in the piecewise function $p_3(u)$, namely,

$$p_1(u) = \begin{cases} \delta_2^2 \sim 0, & |u| \geq 0.7 \\ \delta_2^1 \sim 1, & |u| < 0.7 \end{cases}$$

$$p_2(u) = \begin{cases} \delta_2^2 \sim 0, & |u| \geq 0.6 \\ \delta_2^1 \sim 1, & |u| < 0.6 \end{cases}$$

$$p_3(u) = \begin{cases} \delta_2^2 \sim 0, & |u| \geq 0.8 \\ \delta_2^1 \sim 1, & |u| < 0.8 \end{cases}$$

We can descirbe the impulses suffered by logic choice as the Table1.

Now, we convert the logic impulsive effects above into the algebraic state space expressions. According to Lemma 1, we can carry out:

$$\begin{aligned} g_1(t) &= \delta_2(1, 1, 1, 2, 2, 1, 1, 1)p(x(t)), \\ g_2(t) &= \delta_2(1, 2, 2, 2, 2, 2, 2, 1)p(x(t)), \\ g_3(t) &= \delta_2(1, 2, 1, 1, 2, 1, 1, 2)p(x(t)). \end{aligned}$$

Where

$$p(x(t)) = \ltimes_{i=1}^{3} p_i(x_i(t)) \in \Delta_8.$$

Let

$$\begin{aligned} \Delta x(\tau_k) &= (\Delta x_1(\tau_k), \Delta x_2(\tau_k), \Delta x_3(\tau_k))^T, \\ M_1 &= \delta_2(1, 1, 1, 2, 2, 1, 1, 1), \\ M_2 &= \delta_2(1, 2, 2, 2, 2, 2, 2, 1), \\ M_3 &= \delta_2(1, 2, 1, 1, 2, 1, 1, 2), \\ M &= (M_1^T, M_2^T, M_3^T)^T. \end{aligned}$$

Therefore,we get the linear delay impulsive differential system (21) with impulse time windows and logic choice:

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 1 & 0.6 & 0.7 \\ 0.5 & 2 & 0.6 \\ 0.2 & 0.3 & 1 \end{pmatrix} x(t) & + \begin{pmatrix} 2 & 3 & 3 \\ 0.3 & 1 & 0.4 \\ 0.2 & 0.1 & 0.5 \end{pmatrix} \\ & x(t-\tau), \\ & t \geq 0, t \neq \tau_k \\ \Delta x(t) = \Phi_k(t^-)Mp(x(t^-)), & t = \tau_k, k \in N \end{cases} \tag{21}$$

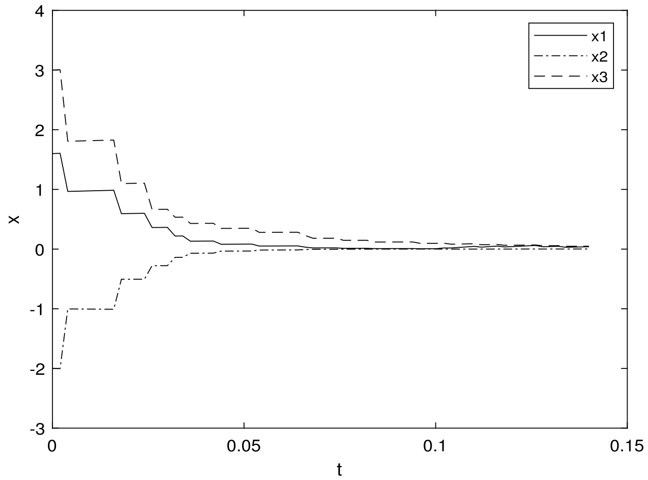

FIGURE 2. The trajectory of $x(t)$ in the impulsive system (21).

where

$$\Phi_k(t^-) = \begin{pmatrix} \Phi_{1k}(t^-) & & \\ & \Phi_{2k}(t^-) & \\ & & \Phi_{3k}(t^-) \end{pmatrix},$$

$$\Phi_{1k}t^-) = (-0.4x_1(t^-), -0.5x_1(t^-)),$$
$$\Phi_{2k}(t^-) = (-0.5x_2(t^-), -0.45x_2(t^-)),$$
$$\Phi_{3k}(t^-) = (-0.2x_2(t^-), -0.4x_2(t^-)).$$

and with the same initial conditions with system (20).

Let $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

According to the notations of Theorem 1,we get that $\lambda_3 = 5.9709$, $\lambda_4 = 23.2723$, $\lambda_5 = 0.64$ and $\tau^r_{k+1} - \tau^r_k = 0.008$, thus

$$(\lambda_3 + \frac{\lambda_4}{\lambda_5})(\tau^r_{k+1} - \tau^r_k) < -ln\lambda_5$$

Therefore, the system (21) is uniformly stable. It can be illustrated by Fig 2. From $\tau^r_{k+1} - \tau^l_{k+1} = 0.006$, $\tau^l_{k+1} - \tau^r_k = 0.002$, we can conclude $\tau^r_{k+1} - \tau^l_k = 0.014$, which indicates that the length of impulsive interval $\tau_{k+1} - \tau_k$ is any value of interval [0.002, 0.014].

Example 2: consider the following linear uncertain delay differential system:

$$\dot{x}(t) = (A + \Delta A(t))x(t) + (B + \Delta B(t))x(t - 0.1), \ t \geq 0 \tag{22}$$

with initial conditions:

$$\varphi_1(t) = \begin{cases} 0, & t \in [-0.1, 0), \\ 1.6, & t = 0, \end{cases}$$

$$\varphi_2(t) = \begin{cases} 0, & t \in [-0.1, 0), \\ -2, & t = 0, \end{cases}$$

where $A = \begin{pmatrix} 2 & 0.3 \\ 0.6 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 \\ 0.5 & -2 \end{pmatrix}$,

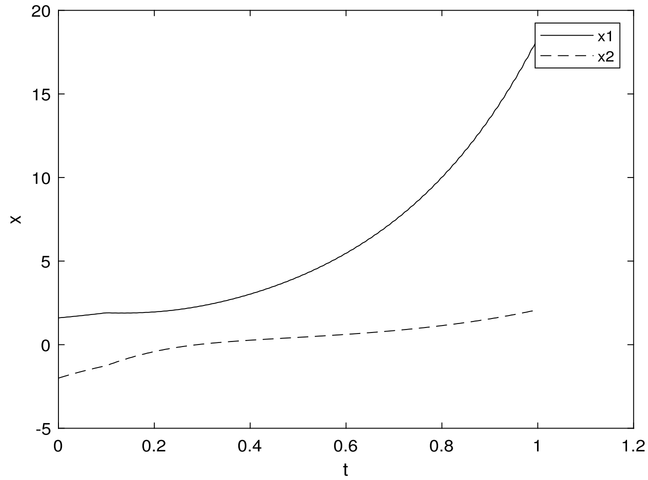

FIGURE 3. The trajectory of $x(t)$ in non-impulsive system (23).

$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $F(t) = \begin{pmatrix} sint & 0 \\ 0 & cost \end{pmatrix}$,

$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Then, we can rewrite the system (22) to the following form:

$$\dot{x}(t) = \begin{pmatrix} 2 + sint & 0.3 \\ 0.6 & -4 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 3 \\ 0.5 & cost - 2 \end{pmatrix} x(t - 0.1), \tag{23}$$

where $t \geq 0$, and with the same initial consitions as system (22).

From Fig 3, we can illustrate that the system (23) is unstable.

Then we add the impusive effects to system (23) by the way of impulse time windows and logic choice:

Let $\tau_0 = \tau^l_0 = \tau^r_0 = 0$, $\tau^r_{k+1} - \tau^l_{k+1} = (2 - \frac{1}{k+1})0.002$, $\tau^l_{k+1} - \tau^r_k = (2 + \frac{1}{k+1})0.002$, and the impulsive time τ_k is any value of fixed-time windows $[\tau^l_k, \tau^r_k)$.

The impulses are suffered by logic choice and can be described as follows:

$$\begin{cases} \Delta x_1(\tau_k) & = x_1(\tau_k) - x_1(\tau_k^-) \\ & = -0.4x_1(\tau_k^-)g_1(\tau_k^-) - 0.3x_1(\tau_k^-)\overline{g_1(\tau_k^-)} \\ \Delta x_2(\tau_k) & = x_2(\tau_k) - x_2(\tau_k^-) \\ & = -0.5x_2(\tau_k^-)g_2(\tau_k^-) - 0.4x_2(\tau_k^-)\overline{g_2(\tau_k^-)} \end{cases}$$

where

$$g_1(t) = p_1(x_1(t)) \rightarrow p_2(x_2(t)),$$
$$g_2(t) = p_1(x_1(t)) \uparrow p_2(x_2(t)).$$

Let $w_1 = 0.7$ in the piecewise function $p_1(u)$, $w_2 = 0.6$ in the piecewise function $p_2(u)$, namely,

$$p_1(u) = \begin{cases} \delta^2_2 \sim 0, & |u| \geq 0.7, \\ \delta^1_2 \sim 1, & |u| < 0.7, \end{cases}$$

$$p_2(u) = \begin{cases} \delta^2_2 \sim 0, & |u| \geq 0.6, \\ \delta^1_2 \sim 1, & |u| < 0.6. \end{cases}$$

TABLE 2. The impulses suffered by logic choice for impusive system (24) at $t = \tau_k, k \in N_+$.

$\|x_1(t^-)\|$	$\|x_2(t^-)\|$	$\Delta x_1(t)$	$\Delta x_2(t)$
< 0.7	< 0.6	$-0.4x_1(t^-)$	$-0.4x_2(t^-)$
< 0.7	≥ 0.6	$-0.3x_1(t^-)$	$-0.5x_2(t^-)$
≥ 0.7	< 0.6	$-0.4x_1(t^-)$	$-0.5x_2(t^-)$
≥ 0.7	≥ 0.6	$-0.4x_1(t^-)$	$-0.5x_2(t^-)$

We can descirbe the impulses suffered by logic choice as the Table 2.

Now, we convert the logic impulsive effects above into the algebraic state space expressions.

According to Lemma 1, we can carry out:

$$g_1(t) = \delta_2(1, 2, 1, 1)p(x(t)),$$
$$g_2(t) = \delta_2(2, 1, 1, 1)p(x(t)),$$

where

$$p(x(t)) = \ltimes_{i=1}^{2} p_i(x_i(t)) \in \Delta_4.$$

Let

$$\Delta x(\tau_k) = (\Delta x_1(\tau_k), \Delta x_2(\tau_k))^T, M_1 = \delta_2(1, 2, 1, 1),$$
$$M_2 = \delta_2(2, 1, 1, 1), M = (M_1^T, M_2^T)^T.$$

Therefore, we get the linear uncertain delay impulsive differential system (24) with impulse time windows and logic choice:

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 2 + sint & 0.3 \\ 0.6 & -4 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 3 \\ 0.5 & cost - 2 \end{pmatrix} x(t - \tau), & t \geq 0, t \neq \tau_k \\ \Delta x(t) = \Phi_k(t^-)Mp(x(t^-)), & t = \tau_k, k \in N \end{cases} \tag{24}$$

where $\Phi_k(t^-) =$

$$\begin{pmatrix} -0.4x_1(t^-) & -0.3x_1(t^-) & & \\ & & -0.5x_2(t^-) & -0.4x_2(t^-) \end{pmatrix},$$

and with the same initial conditions as system (23).

Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

According to the notations of Theorem 2,we get that $\lambda_3 = 8.0576$, $\lambda_4 = 15.2850$, $\lambda_5 = 0.49$ and $\tau_{k+1}^r - \tau_k^r = 0.008$, thus

$$(\lambda_3 + \frac{\lambda_4}{\lambda_5})(\tau_{k+1}^r - \tau_k^r) < -ln\lambda_5.$$

Therefore, the system (24) is uniformly stable. It can be illustrated by Fig 4. From $\tau_{k+1}^r - \tau_{k+1}^l = (2 - \frac{1}{k+1})0.002$, $\tau_{k+1}^l - \tau_k^r = (2 + \frac{1}{k+1})0.002$, we can conclude $\tau_{k+1}^r - \tau_k^l = 0.012 - \frac{0.002}{k+1}$, which indicates that the length of impulsive interval $\tau_{k+1} - \tau_k$ is any value of interval $[0.004 + \frac{0.002}{k+1}, 0.012 - \frac{0.002}{k+1}]$.

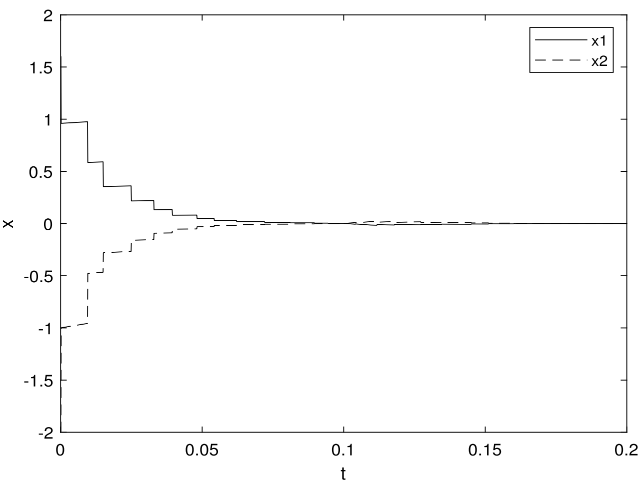

FIGURE 4. The trajectory of $x(t)$ in the impulsive system (24).

VI. CONCLUSION

In this paper, the uniform stability of linear delay impulsive differential systems with impulse time windows and logic choice is studied. The new systems we proposed in our paper have two properties: the first is that the impulses in our systems do not appear at the fixed time points, but may occur at any points in a little range of time, the second is that the impulsive effects in our systems are determined by logic choice. Based on Lyapunov functions and Razumikhin technique combined with the semi-tensor product method, we obtained the uniform stability criterion for the system, and we also applied the stability criterion to the linear uncertain delay impulsive differential systems with impulse time windows and logic choice. Finally, two numerical examples are discusssed to verify our results. In the future, we will carry out other stability studies for the linear delay impulsive differential systems with impulse time windows and logic choice, such as asymptotical stability and exponential stability.

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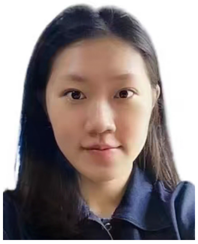

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