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RESEARCH ARTICLE

Finite-Time Robust Stabilization for a Class of High-Order Nonlinear Systems With Multiple Uncertainties and External Disturbances

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ABSTRACT The paper investigates the problem of adaptive finite-time robust stabilization for a set of high-order uncertain nonlinear systems in the presence of asymmetric output constraint, dynamic uncertainty and complicated external disturbances. Via effectively integrating the artful Barrier Lyapunov Function (BLF) in conjunction with the continuous control armed with a serial of integral functions consisting of embedded sign operations, a continuous controller is generated, which promises that the closed-loop system's states converge to a compact set within finite time whilst preserving the validity of the output constraint. Preferable to the current techniques, the suggested methodology unifies the construct and theoretical evaluation for the constrained and unconstrained output as well as being able to concurrently handle the output asymmetric constraints, zero dynamics and complex external disturbances. Finally, an instance of simulation is included to illustrate the validity of the established methodology.

INDEX TERMS High-order nonlinear systems, finite-time convergence, robust, asymmetric output constraint, zero dynamics, external disturbances.

I. INTRODUCTION

It has been generally accepted that adaptive strategies are capable of resolving the issue of nonlinear systems' stabilization with operational uncertainties [1], [2]. Despite a number of strategies, such as the backstepping procedure and feedback linearization, possess the capacity to be utilized in adaptive fashion for nonlinear systems, they cannot be appropriate for p-normal form nonlinear systems because of the inherent nonlinearities initiated by the uncontrollability of the Jacobian linearization. Thankfully, the notion of adding a power integrator first put forward by [3] and [4], paved the way for substantial improvements and encouraged lots of research on the adaptive stabilization of high-order uncertain nonlinear systems; such as [5], [6], [7], [8], and [9]. In conjunction with adaptive technologies for addressing uncertainties, the finite-time stabilization for uncertain nonlinear systems has additionally captured an extensive amounts of focus; see, e.g., [5], [8]. As opposed to those with asymptotic state convergence, systems with finite-time state convergence benefit from desirable qualities like excellent control precision and powerful, reliable resistance to disturbances; as a result, research into finite-time stabilization is important from both a theoretical and practical standpoint [10], [11]. The key is that intriguing solutions which includes fuzzy techniques, neural networks, homogenous dominating procedure and filters [12], [13], [14], [15], are placed forth for addressing increasing levels of nonlinearities.

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The theme of dynamic uncertainty/zero dynamic, has been receiving considerable amount of interest alongside adaptive tactics for controlling uncertainties [16], [17], [18], [19], [20]. In actuality, partly because of the restricted capabilities of measuring methods and devices, management implemented in real-life scenarios generally possess zero dynamics. The changing supply-function theory [17] and small gain theory [16] proved capable to successfully address this concern by adding additional limitations on zero dynamic. Specifically, the investigation [18] addressed the asymptotical stability of certain kinds of tight feedback cascade systems by employing the small-gain principle and parameter separation mechanism. The finite-time stabilisation of a group of high-order nonlinear systems in the presence of zero dynamic was investigated in [19] by utilizing the identical approaches. Further, the backstepping algorithm was employed in [20] when dealing with the development of adaptive stabilizer for specific nonlinear systems equipped with dynamic uncertainty. Additionally, managing external disturbances has been identified as one of the major problems with regards to science and technology during a period of decades [21], [22]. Consequently, the process is essential to find a construction to the stabilization subject for a class of high-order uncertain nonlinear systems in the presence of both uncertainties, dynamic uncertainties and external disturbances, and these serve as one of the driving forces behind this study. Remarkably, focusing on temporarily behavior of system states, particularly the output all during the stabilization task, likewise qualifies as an important and worth noting subject [23], [24], [25], [26], [27], [28], [29], given that any violation of the limitations on output might result in a not preferred tendency to diminish system productivity or potentially trigger breakdown. As an illustration, a marine vessel's position should be constrained by its broadest possible range of travel [28]; similarly, an adaptable crane system's output should be restricted in order to ensure accuracy and safety [29]. As a result, throughout the stabilizing procedure, it can often be expected that the system output adhere to certain established limitation. This demand could also be deduced from the numerous advantageous strategies that have developed for managing various restrictions, such as [30], [31], [32], and [33]. Over the years, many kinds of solutions to this challenge have been proposed, which involves reference leaders, invariability control, and model-predictable control [34], [35]. As a whole, the BLF described in [23] and [24] has evolved into a useful method for dealing with output restrictions, where a log-type BLF was suggested for a specific category of strict-feedback nonlinear systems containing asymmetric or symmetric output limit. There are currently fairly several creative solutions to cope with the multiple restrictions; for example, [30], [31], [32], [33]. However, the findings in [30], [31], [32], and [33] display three items which should be noted. (i) Considering the time derivative of the BLF is just not greater than zero, the procedure described in [23] is not suitable for a system with particularly accurate control demands. (ii) In circumstances where there are no limitations at all, management systems may excessively manipulate the limitation. (iii) In cases where parameter uncertainties are compensated by online estimations, the procedure negatively affects the Barrier functions, such as logarithm or tangent types.

The aforementioned fact prompts the investigation to concentrate on developing a single explicit stability guidelines for assuring the feasibility of a generic control algorithm for both limited and unlimited output. As a result, an attractive issue is put forth concurrently: *Can the problem of adaptive stabilization for a set of high-order uncertain nonlinear systems endowed with parameter uncertainty and zero dynamics be solved in anew approach that is acceptable for both limited and unlimited output?* Our recent result [26] has taken this problem into account. However, [26] also leaves an unsolved problem: it remains unknown how to reduce the conservatism of the settling time while dominating complicated external disturbances effectively.

Thanks to our research and the resources offered previously, we are going to tackle the topic in question and propose a suitable reply. Actually, the lack of particular theoretical backing and strict instructions renders this type of strategy a challenge. In this investigation, we create an architecture that combines a skilled BLF with a continuous feedback dominance that is outfitted with an array of integral functions.

Three categories are employed to classify the study's main accomplishments and improvements:

(i) The presented research offers a completely new guidelines for gaining the adaptive convergence for a type of uncertain high-order nonlinear systems containing zero dynamic, parameter uncertainties and complicated external disturbances.

(ii)As opposed to earlier attempts [27], [28], [29], [30], [31], the development and evaluation procedures for limited and unlimited output are unified without modifying the controller's construction in this paper. In other words, it is possible to prevent the situation where the control techniques overpower the constraint when there is none.

(iii) Innovative mathematical tactics, like the barrier function and other inventive transformation procedures, are utilized to avoid zero division and to improve stability analysis.

Notations: The notations below will be applied during the remainder. $\mathbb{R}$ displays the series of real numbers, $\mathbb{R}^+$ denotes the series of nonnegative real numbers, and $\mathbb{R}^n$ denotes Euclidean space with dimension n, and $\mathbb{R}^{>i}_{\text{odd}} \triangleq \{q_1/q_2 > i \,|\, q_1 \text{ and } q_2 \text{ are positive odd integers}\}$ with $i = 0, 1$. Presented vector $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ and three real positive numbers c_1, c_2, c_3, for $i = 1, \ldots, n$, $\bar{x}_i \triangleq [x_1, \ldots, x_i]^T \in \mathbb{R}^i$, $\mathbb{S}_i(c_1, c_2) \triangleq \{\bar{x}_i | \bar{x}_i \in \mathbb{R}^i \text{ with } -c_1 < x_1 < c_2\} \subset \mathbb{R}^i$, and $\partial\mathbb{S}_i(c_1, c_2)$ represents the boundary of $\mathbb{S}_i(c_1, c_2)$; $\lceil s \rfloor^{c_3} = |c|^{c_3} sign(c)$ for all $c \in \mathbb{R}$ with $sign(\cdot)$ being the sign function that meets $sign(c) = -1$ if $c < 0$, $sign(c) = 1$ if $c > 0$, and $sign(c) = 0$ if $c = 0$. $\|A\| = \sqrt{\lambda_{max}(A^T A)}$ denotes the norm of $A \in \mathbb{R}^n$, where $\lambda_{max}(A^T A)$ denotes maximum eigenvalue of square matrix $A^T A$.

II. PRELIMINARIES AND PROBLEM FORMULATION

In this paper, we study the high-order uncertain nonlinear systems depicted by:

$$\begin{cases} \dot{z}(t) = f_0(z(t), y(t), d_0(t)), \\ \dot{x}_i(t) = \beta_i(\bar{x}_i(t)) x_{i+1}^{p_i}(t) + f_i(z(t), x(t), d_i(t)) + q_i(t), \\ \quad i = 1, \ldots, n-1, \\ \dot{x}_n(t) = \beta_n(x(t)) u^{p_n}(t) + f_n(z(t), x(t), d_n(t)) + q_n(t), \\ y(t) = x_1(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ and $z(t) \in \mathbb{R}^m$ are system state, control input, and unmeasured state, respectively. For each $i = 1, \cdots, n$, $q_i(t)$ represents the unknown bounded time-varying vector; $q_i(t)$ represents the unknown bounded disturbances with the unknown upper bound; $p_i \in \mathbb{R}_{odd}^{\geq 1}$ is called high-order of the systems and $p_n = 1$, $f_i(\cdot), f_n(\cdot)$ and $\beta_i(\cdot) \neq 0$ are continuous functions. The initial condition is $x(0) = 0$, $z(0) = 0$. $y(t) \in \mathbb{R}$ is the output that is restricted by $-a < y(t) < b, \forall t \geq 0$ and a, b are predetermined positive constants.

The control objective of this paper is to develop a continuous adaptive controller for system (1) such that (i) all the states of the closed-loop system are globally uniformly bounded and the system output satisfies the constraint $-a < y(t) < b, \forall t \geq 0$. (ii) $x(t)$ converges to a compact set within finite time.

The subsequent assumptions are necessary for us to achieve the control goal.

Assumption 1: There is an unknown constant $\bar{\theta} > 0$ and continuous nonnegative functions $\bar{f}_o(\cdot), \bar{f}_i(\cdot)$ and $\bar{f}_i(0) = 0$ such that

$$|f_i(\cdot)| \leq \bar{f}_0(\|z\|) + \bar{\theta} \sum_{j=1}^{i} |x_j|^{\delta_j + \mu_{ij}} \bar{f}_i(\bar{x}_i), \; i = 1, \ldots, n, \tag{2}$$

where $\mu_{ij} \geq 0$, $\delta_j = \frac{h_i + \eta}{h_j}$, h_i are iteratively defined by $h_1 = 1, h_j = \frac{h_{j-1}+\eta}{p_{j-1}}$, $j = 2, \ldots, n+1$, and η meets $\eta \in (-\frac{1}{\sum_{i=1}^{n} p_0 \cdots p_{i-1}}, 0)$. Notably, (2) can be rewritten as:

$$|f_i(\cdot)| \leq \bar{f}_0(\|z\|) + \bar{\theta} \sum_{j=1}^{i} |x_j|^{\delta_j} l_i(\bar{x}_i(t)). \tag{3}$$

where $l_i(\bar{x}_i(t)) \triangleq \sum_{j=1}^{i} |x_j|^{\mu_{ij}} \bar{f}_i(\bar{x}_i)$ is nonnegative continuous differential and $l_i(0) = 0$.

Assumption 2: There is a continuous differentiable and positive definite function $U_0(z)$ meets:

$$\begin{cases} \underline{\pi}(\|z\|) \leq U_0(z) \leq \overline{\pi}(\|z\|), \\ \dfrac{\partial U_0(z)}{\partial z} f_0(x_1, z, d_0(t)) \leq -\pi(\|z\|) + \sigma \tau(|x_1|), \end{cases} \tag{4}$$

where $\overline{\pi}(\cdot)$, $\underline{\pi}(\cdot)$, $\pi(\cdot)$, $\tau(\cdot) \in K_\infty$, $\pi(\|z\|) = k_0 U_0^{\alpha}(z)$, k_0, $\alpha < 1$ are positive constants, and $\sigma > 0$ is an unknown constant.

Assumption 3: The unknown disturbance $q_i(t)$ satisfies

$$|q_i(t)| \leq \Theta, \forall t \geq 0, \tag{5}$$

with Θ being an unknown constant.

Below are a couple of the lemmas that are crucial to the demonstration of the main conclusion.

Lemma 1 [4]: For $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, there are smooth functions $a(x) \geq 0, b(y) \geq 0, c(x) \geq 1$ and $d(y) \geq 1$ such that $|f(x, y)| \leq a(x) + b(y)$, $|f(x, y)| \leq c(x)d(y)$ and $f(x, y)$ is continuous.

Lemma 2 [4]: There exist a function $a(x, y)$ such $|ax^m y^n| \leq c(x, y)|x|^{m+n} + \frac{n}{m+n} |a(x, y)|^{\frac{m+n}{n}} \left(\frac{m}{(m+n)c(x,y)}\right)^{\frac{m}{n}} |y|^{m+n}$ holds for any $x \in \mathbb{R}$ and any $y \in \mathbb{R}$, where $c(x, y) > 0$, $m > 0$, $n > 0$ are given constants.

Lemma 3 [4]: For any $x \in \mathbb{R}$ and any $y \in \mathbb{R}$, the inequalities $|x+y|^p \leq 2^{p-1}(x^p + y^p)$, $|x-y|^p \leq 2^{p-1}(x^p - y^p)$, $(|x| + |y|)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}} ||x| + |y||^{\frac{1}{p}}$, $|x^{\frac{1}{p}} - y^{\frac{1}{p}}| \leq 2^{\frac{p-1}{p}} |x - y|^{\frac{1}{p}}$ hold for given $p \in \mathbb{R}_{\text{odd}}^{\geq 1}$, and $(x_1 + \cdots + x_n)^p \leq \max(n^{p-1}, 1)(x_1^p + \cdots + x_n^p)$ hold for given $p \in \mathbb{R}_{\text{odd}}^{>0}$ and any $x_1, \ldots, x_n \in \mathbb{R}$.

Lemma 4 [8]: Consider the following autonomous system

$$\dot{x} = f(x), f(0) = 0, x \in U \subseteq \mathbb{R}^q,$$

where $f : U_0 \to \mathbb{R}^n$ is continuous is continuous in domain U_0 containing $x = 0$. Assume the positive continuous function $V(x)$ is defined on U and satisfies $\dot{V}(x) \leq -\kappa_1 V^p(x) - \kappa_2 V^q(x) + \zeta$, where $\kappa_1 > 0, \kappa_2 > 0, 0 < p < 1, q \geq 1$ and $0 < \zeta < \infty$, then the following system is finite-time stable. Further, the states will converge to the following compact set

$$\Omega = \left\{ \lim_{t \to T} x \middle| V \leq \min\left\{ \left(\frac{\zeta}{\kappa_1(1-\varepsilon)}\right)^{\frac{1}{p}}, \left(\frac{\zeta}{\kappa_2(1-\varepsilon)}\right)^{\frac{1}{q}} \right\} \right\},$$

where $0 < \varepsilon < 1$, $T \leq \frac{1}{\kappa_1 \varepsilon (1-p)} + \frac{1}{\kappa_2 \varepsilon (1-q)}$.

III. CONTROL DESIGN PROCEDURE

The developer first established the following Proposition with to minimize the impact of zero dynamics.

Proposition 1: For a given continuous and monotone nondecreasing function $K : \mathbb{R}^+ \to [1, \infty)$ and the function $V_0(z) = \int_0^{U_0(z)} K(s) ds$ with $U_0(z)$ being continuously differential, positive and radially unbounded, there exist a smooth nondecreasing function $\bar{\tau}(\cdot)$, an unknown constant $\bar{\sigma}$ and a positive constant $\epsilon \in (0, 1)$ such that

$$\frac{\partial U_0(z)}{\partial z} f_0 \leq -(1-\epsilon) K(s) \circ \underline{\pi}(\|z\|) \pi(\|z\|) + \bar{\sigma} x_1^2 \bar{\tau}(x_1), \tag{6}$$

Proof: Check Appendix A. □

We next offer coordinate transformation:

$$\begin{cases} \xi_i = \lceil x_i \rceil^{\frac{1}{h_i}} - \lceil \alpha_{i-1}(\bar{x}_{i-1}, \hat{\theta}, \hat{\Theta}) \rceil^{\frac{1}{h_i}}, \\ u = \alpha_n(x, \hat{\theta}, \hat{\Theta}), \\ \alpha_i(\bar{x}_i, \hat{\theta}, \hat{\Theta}) = -g_i^{h_{i+1}}(\bar{x}_i, \hat{\theta}, \hat{\Theta}) \lceil \xi_i \rceil^{h_{i+1}}, \end{cases} \tag{7}$$

where $i = 1, \ldots, n$, $\hat{\Theta}$ is the estimation of Θ, $\hat{\theta}$ is the estimation of $\theta \triangleq \max\{\bar{\sigma}, \bar{\theta}, \bar{\theta}^{\frac{2}{1-\eta}}\}$. $g_1(\cdot), \cdots, g_n(\cdot)$ are positive smooth functions to be specified later. For ease of use, let $g_0 = \bar{x}_0 = \alpha_0 = 0$. Based on Assumption 1, one has

$$\frac{1}{h_i} \geq 1, 2-h_i+h_{i+1}p_i=2+\eta, 0<h_{i+1}p_i<1, i=1,\cdots,n, \tag{8}$$

according to (8), we introduce an integral function equipped with nested sign functions $W_k : \mathbb{R}^i \times \mathbb{R} \rightarrow \mathbb{R}, i = 1, \cdots, n$ as

$$W_i(\cdot)=\int_{\alpha_{i-1}}^{x_i} \left\lceil \lceil s \rceil^{\frac{1}{h_i}} - \lceil \alpha_{i-1} \rceil^{\frac{1}{h_i}} \right\rceil^{2-h_{i+1}p_i} ds, \tag{9}$$

Repeat the procedures in [5], it can prove that $W_k(\cdot)$ is continuously and satisfie

$$\begin{cases} \frac{\partial W_i}{\partial x_i} = \lceil \xi_i \rceil^{2-h_i-\eta}, \\ \frac{\partial W_i}{\partial \chi_k} =-\int_{\alpha_{i-1}}^{x_i} \left| \lceil s \rceil^{\frac{1}{h_i}} - \lceil \alpha_{i-1} \rceil^{\frac{1}{h_i}} \right|^{1-h_i-\eta} ds \, (2-h_{i+1}p_i) \\ \qquad \frac{\partial}{\partial \chi_k}(\lceil \alpha_{i-1} \rceil^{\frac{1}{h_i}}), \\ c_{i1} \mid x_i - \alpha_{i-1} |^{\frac{2-\eta}{h_i}} \leq W_i \leq c_{i2} |\xi_i|^{2-\eta}, \end{cases} \tag{10}$$

where $\chi_k = x_k$ for $k = 2, \cdots, i-1$, $\chi_i = \hat{\theta}$, $c_{i1} = \frac{h_i}{2-\eta} 2^{(2-h_{i+1}p_i)(h_i-1)/h_i}$ and $c_{i2} = 2^{1-h_i}$. Relying on (7), there holds

$$u = \alpha_n = -\left\lceil \sum_{l=1}^{n} \Big(\prod_{j=l}^{n} g_j(\bar{x}_j, \hat{\theta}, \hat{\Theta}) \Big) \lceil x_l \rceil^{\frac{1}{h_l}} \right\rceil^{h_{n+1}}. \tag{11}$$

The goal is to subsequent to iteratively establish the specific configuration of g_i.

step 1 Indeed, symmetric constraints are an advanced form of asymmetric constraints. The designer tries to investigate a more comprehensive and versatile BLF with the goal to offer flexbility and diversity in the control development process. In other words, the BLF should be configured to cope with both symmetric and asymmetric instances and to fully exploit the nonlinear properties of the system. Ths, we build the BLF as:

$$V_{blf} = \frac{a^{2-\eta}b^{2-\eta}|x_1|^{2-\eta}}{(2-\eta)(b-x_1)^{2-\eta}(a+x_1)^{2-\eta}}, \tag{12}$$

if $x_1 \rightarrow \partial \mathbb{S}_1(a, b)$, then $V_{blf} \rightarrow \infty$ holds. $y(t) \rightarrow -a$ or $y(t) \rightarrow b$ implies $V_{blf} \rightarrow \infty$ if$-a < y(0) < b$ and $y(t)$ is bounded, for $-a < x_1(0) = y(0) < b$; in other words, the output constraint $-a < y(t) < b$ can not be broken if V_{blf} and $y(t)$ are bounded. Moreover, it follows from (12) that

$$\begin{aligned} \frac{\partial V_{blf}(x_1)}{\partial x_1} &= \frac{a^{2-\eta}b^{2-\eta}(x_1{}^2+ab)}{(b-x_1)^{3-\eta}(a+x_1)^{3-\eta}} \lceil x_1 \rceil^{1-\eta} \\ &\triangleq \rho(x_1)\lceil x_1 \rceil^{1-\eta}, -a < x_1 < b, \end{aligned} \tag{13}$$

where $\rho(x_1)$ is a positive smooth function. To obtain the goal of (1) with output limitation, specify

$$V_1 = V_{blf} + V_0 + \frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}\tilde{\Theta}^2, \tag{14}$$

where $\tilde{\theta} \triangleq \theta - \hat{\theta}$, $\tilde{\Theta} \triangleq \Theta - \hat{\Theta}$. The time derivative of V_1 as

$$\begin{aligned} \dot{V}_1 &= \rho\lceil x_1 \rceil^{1-\eta}(\beta_1(x_1,t)x_2^{p_1} + f_1 + q_1) - \tilde{\theta}\dot{\hat{\theta}} \\ &\quad - (1-\epsilon)K(s) \circ \underline{\pi}(\|z\|)\pi(\|z\|) + \bar{\sigma}x_1^2\bar{\tau}(x_1) - \tilde{\Theta}\dot{\hat{\Theta}} \\ &= \rho\lceil x_1 \rceil^{1-\eta}\beta_1(x_2^{p_1} - \alpha_1^{p_1}) + \rho\beta_1\lceil x_1 \rceil^{1-\eta}\alpha_1^{p_1} - \tilde{\theta}\dot{\hat{\theta}} - \tilde{\Theta}\dot{\hat{\Theta}} \\ &\quad - (1-\epsilon)K \circ \underline{\pi}\pi + \bar{\sigma}x_1^2\bar{\tau} + \rho\lceil x_1 \rceil^{1-\eta}[f_1 + q_1], \end{aligned} \tag{15}$$

based on (3), (8) and Lemma 2, there has

$$\begin{aligned} \rho\lceil x_1 \rceil^{1-\eta}q_1 &\leq |\rho|\lceil x_1 \rceil^{1-\eta}\Theta \\ &\leq \varepsilon\Theta + \frac{\rho^2 x_1^2 x_1^{-2\eta}\Theta}{\sqrt{\rho^2 x_1^{2-2\eta} + \varepsilon^2}} \\ &\triangleq \varepsilon\Theta + \xi_1^2\Theta Q_1 \end{aligned} \tag{16}$$

where $Q_1 = \frac{\rho^2 x_1^{-2\eta}}{\sqrt{\rho^2 x_1^{2-2\eta}+\varepsilon^2}}$. The next task is estimate the last two terms on the right-hand of (15). Depending on (3), (8) and Lemma 2, one has

$$\begin{aligned} \rho\lceil x_1 \rceil^{1-\eta}f_1 &\leq \rho\,|x_1|^{1-\eta}\left(\bar{f}_0 + \tilde{\theta}\,|x_1|^{p_1h_2}\,l_1 + \hat{\theta}\,|x_1|^{p_1h_2}\,\bar{l}_1\right) \\ &\leq \phi_1\xi_1^2 + \bar{f}_0^{\frac{2}{\eta+1}} + \bar{f}_0^{\frac{2}{p_1h_2}} + \tilde{\theta}\rho l_1\xi_1^2, \end{aligned} \tag{17}$$

where $\phi_1 = \rho\bar{l}\hat{\theta} + \frac{1-\eta}{2}(\frac{4}{1+\eta})^{\frac{1+\eta}{\eta-1}}\rho^{\frac{2}{1-\eta}}$, ϕ_1 and $\bar{l}_1$ are positive smooth functions. Additionally, there holds

$$\bar{\sigma}x_1^2\bar{\tau}(x_1) \leq \tilde{\theta}\xi_1^2\bar{\tau}(x_1) + \hat{\theta}\xi_1^2\bar{\tau}(x_1) \tag{18}$$

substituting (17) and (18) into (15), it can be observed from $1-\eta+p_1h_2 = 2$ and $h_1 = 1$ that $\lceil x_1 \rceil^{1-\eta}\alpha_1^{p_1} = -g_1^{p_1h_2}\xi_1^2$. Then, (15) can be simplified as

$$\begin{aligned} \dot{V}_1 &\leq -(n+1)\xi_1^2 + \rho\beta_1\lceil x_1 \rceil^{1-\eta}(x_2^{p_1} - \alpha_1^{p_1}) + \bar{f}_0^{\frac{2}{1+\eta}} + \bar{f}_0^{\frac{2h_2}{p1}} \\ &\quad + \xi_1^2(\phi_1 + n + 1 + \hat{\theta}\bar{\tau} + Q_1\hat{\Theta} - \rho\beta_1 g_1^{p_1h_2}(x_1)) \\ &\quad - (1-\epsilon)K(s) \circ \underline{\pi}\pi - \tilde{\theta}\dot{\hat{\theta}} + \tilde{\theta}\bar{\tau}\xi_1^2 \\ &\quad + \tilde{\theta}\rho l_1\xi_1^2 + \tilde{\Theta}(Q_1\xi_1^2 - \dot{\hat{\Theta}}) + \varepsilon\Theta. \end{aligned} \tag{19}$$

So far, one can choose

$$g_1 = \Big(\frac{\phi_1 + n + 1 + \hat{\theta}\bar{\tau}(x_1) + Q_1\hat{\Theta}}{\rho\beta_1(x_1,t)}\Big)^{\frac{1}{p_1h_2}}. \tag{20}$$

(19) takes the form

$$\begin{aligned} \dot{V}_1 &\leq -(n+1)\xi_1^2 + \rho\beta_1\lceil x_1 \rceil^{1-\eta}(x_2^{p_1} - \alpha_1^{p_1}) + \bar{f}_0^{\frac{2}{1+\eta}} + \bar{f}_0^{\frac{2}{p_1h_2}} \\ &\quad - (1-\epsilon)K \circ \underline{\pi}\pi + \tilde{\theta}(\varpi_1(x_1)\xi_1^2 - \dot{\hat{\theta}}) \\ &\quad + \tilde{\Theta}(Q_1\xi_1^2 - \dot{\hat{\Theta}}) + \varepsilon\Theta, \end{aligned} \tag{21}$$

where $\varpi_1(x_1) = \rho l_1 + \bar{\tau}(x_1)$.

step 2 Choose $V_2 = V_1 + W_2$. Taking (10) and (21) into account, one has

$$\begin{aligned}\dot{V}_2 \leq& -(n+1)\xi_1^2 + \rho\beta_1\lceil x_1\rceil^{1-\eta}(x_2^{p_1} - \alpha_1^{p_1}) + \bar{f}_0^{\frac{2}{1+\eta}} + \bar{f}_0^{\frac{2}{p_1h_2}} \\ &+ \varepsilon\Theta - (1-\epsilon)K \circ \underline{\pi}\pi + \tilde{\theta}(\varpi_1\xi_1^2 - \dot{\hat{\theta}}) + \tilde{\Theta}(Q_1\xi_1^2 - \dot{\hat{\Theta}}) \\ &- W_2 + c_{22}(1+\xi_2^2)^{\frac{-\eta}{2}}\xi_2^2 + \beta_2\lceil\xi_2\rceil^{2-\eta-h_2}\alpha_2^{p_2} \\ &+ \beta_2\lceil\xi_2\rceil^{2-\eta-h_2}(x_3^{p_2} - \alpha_2^{p_2}) + \frac{\partial W_2}{\partial\hat{\theta}}\dot{\hat{\theta}} \\ &+ \frac{\partial W_2}{\partial\hat{\Theta}}\dot{\hat{\Theta}} + \lceil\xi_2\rceil^{2-\eta-h_2}(f_2 + q_2) + \frac{\partial W_2}{\partial x_1}\dot{x}_1. \end{aligned} \tag{22}$$

we next simplify the indefinite terms of (22). The selection of $m = \frac{1}{h_2}$, $b = \frac{1}{p_1h_2}$, $x = x_2$, $y = \alpha_1$ in Lemma 3 together with Lemma 2, (7) and (8), one can get

$$\begin{aligned}&\rho\beta_1\lceil x_1\rceil^{1-\eta}(x_2^{p_1} - \alpha_1^{p_1}) \\ &\quad\leq \rho\beta_1|x_1|^{1-\eta}\cdot 2^{1-p_1h_2}|\lceil x_2\rceil^{\frac{1}{h_2}} - \lceil\alpha_1\rceil^{\frac{1}{h_2}}|^{p_2h_2} \\ &\quad\leq 2^{-\eta}\rho\beta_1|\xi_1|^{1-\eta}|\xi_2|^{1+\eta} \\ &\quad\leq \frac{1}{5}\xi_1^2 + \phi_{21}\xi_2^2, \end{aligned} \tag{23}$$

where ϕ_{21} is positive smooth function. It deduces from (7) and (8) that $|x_i|^{\frac{h_3p_2}{h_i}} \leq |\xi_i|^{p_2h_3} + |g_{i-1}|^{p_2h_3}|\xi_{i-1}|^{p_2h_3}$, $i = 1, 2$. Next, based on (3) and Lemma 2, there holds

$$\begin{aligned}&\lceil\xi_2\rceil^{2-\eta-h_2}f_2 \\ &\quad\leq |\xi_2|^{2-\eta-h_2}(\bar{f}_0 + \bar{\theta}\sum_{i=1}^{2}|x_1|^{\frac{p_2h_3}{h_i}}l_2) \\ &\quad\leq |\xi_2|^{2-\eta-h_2}\bar{f}_0 + \bar{\theta}l_2\bar{g}_1|\xi_2|^{2-\eta-h_2}\sum_{i=1}^{2}|\xi_i|^{p_2h_3} \\ &\quad\leq \phi_{22}\xi_2^2 + \bar{f}_0^{\frac{2}{p_2h_3}} + \frac{1}{5}\xi_1^2 + \tilde{\theta}\varpi_{21}\xi_2^2, \end{aligned} \tag{24}$$

where ϕ_{22}, ϖ_{21} and $\bar{g_1} \geq 1 + g_1^{p_2h_3}$ are positive smooth functions. By employing (3), (7) and Lemma 1, the developer is able to calculate the following estimate:

$$\begin{aligned}\left|\frac{\partial\lceil\alpha_1\rceil^{\frac{1}{h_2}}}{\partial x_1}f_1\right| &\leq \left(g_1 + |\frac{\partial g_1}{\partial\xi_1}|\cdot|\xi_1|\right)(\bar{f}_0 + \bar{\theta}|\xi_1|^{p_1h_2}l_1) \\ &\leq \gamma_{21}(x_1)(\bar{f}_0 + \bar{\theta}|\xi_1|^{1+\eta}), \end{aligned} \tag{25}$$

where $\gamma_{21} \geq \left(g_1 + \left|\frac{\partial g_1}{\partial\xi_1}\xi_1\right|\right)(1+l_1)$ is a smooth positive function. Further, since $|x_2^{p_1}| \leq (|\xi_2| + g_1|\xi_1|)^{\eta+1} \leq (1 + g_1^{\eta+1})(|\xi_2|^{\eta+1} + |\xi_1|^{\eta+1})$, we have

$$\begin{aligned}&\left|\frac{\partial\lceil\alpha_1\rceil^{\frac{1}{h_2}}}{\partial x_1}\beta_1x_2^{p_1}\right| \\ &\quad\leq |g_1 + |\frac{\partial g_1}{\partial\xi_1}\xi_1|\cdot|\beta_1|\cdot(1+g_1^{\eta+1})(|\xi_2|^{\eta+1}+|\xi_1|^{\eta+1})| \\ &\quad\leq \varrho_{21}(x_1)\sum_{j=1}^{2}|\xi_j|^{\eta+1}, \end{aligned} \tag{26}$$

where $\varrho_{21} \geq |g_1 + |\frac{\partial g_1}{\partial\xi_1}\xi_1||\beta_1(x_1,t)\cdot(1+g_1^{\eta+1})$ is smooth and positive. Moreover, using Lemma 3 results in

$$\begin{aligned}&-(2-\eta-h_2)\int_{\alpha_1}^{x_2}\left|\lceil s\rceil^{\frac{1}{h_2}} - \lceil\alpha_1(x_1)\rceil^{\frac{1}{h_2}}\right|^{1-h_2-\eta}ds \\ &\quad\leq (2-\eta-h_2)|x_2-\alpha_1|\cdot|\xi_2|^{1-h_2-\eta} \\ &\quad\leq 2^{1-h_2}(2-\eta-h_2)|\xi_2|^{h_2}|\xi_2|^{1-h_2-\eta} \\ &\quad\leq \tilde{c}_2|\xi_2|^{1-\eta}, \end{aligned} \tag{27}$$

where $\tilde{c}_2 = 2^{1-h_2}(2-\eta-h_2) > 0$ is a constant. Subsequently, according to (3), (8), (25)-(27) and Lemma 2, one has

$$\begin{aligned}\frac{\partial W_2}{\partial x_1}\dot{x}_1 =& -(2-\eta-h_2)\int_{\alpha_1}^{x_2}|\lceil s\rceil^{\frac{1}{h_2}} - \lceil\alpha_1(x_1)\rceil^{\frac{1}{h_2}}|^{1-\eta-h_2}ds \\ &\cdot\frac{\partial}{\partial x_1}(\lceil\alpha_1(x_1)\rceil^{\frac{1}{h_2}})(\beta_1x_2^{p_1} + f_1 + q_1) \\ &\leq \tilde{c}_2(\varrho_{21}+\gamma_{21})|\xi_2|^{1-\eta}\Big(\bar{f}_0 + \sum_{j=1}^{2}|\xi_j|^{\eta+1} + \bar{\theta}|\xi_1|^{\eta+1}\Big) \\ &+ \tilde{c}_2(\varrho_{21}+\gamma_{21})|\xi_2|^{1-\eta}q_1 \\ &\leq \frac{1}{5}\xi_1^2 + \phi_{23}\xi_2^2 + f_0^{\frac{2}{\eta+1}} + \tilde{\theta}\varpi_{22}\xi_2^2 \\ &+ \tilde{c}_2(\varrho_{21}+\gamma_{21})|\xi_2|^{1-\eta}q_1, \end{aligned} \tag{28}$$

where ϖ_{22} and ϕ_{23} are positive smooth functions. According to Assumption 3, there has

$$\begin{aligned}&\tilde{c}_2(\varrho_{21}+\gamma_{21})|\xi_2|^{1-\eta}q_1 + |\xi_2|^{2-\eta-h_2}q_2 \\ &\quad\leq|\tilde{c}_2(\varrho_{21}+\gamma_{21})||\xi_2|^{1-\eta}\Theta + |\xi_2|^{2-\eta-h_2}\Theta \triangleq \Theta|\xi_2|\tilde{Q}_2 \\ &\quad\leq \varepsilon\Theta + \frac{\xi_2^2\tilde{Q}_2^2\Theta}{\sqrt{\xi_2^2\tilde{Q}_2^2+\varepsilon^2}} \triangleq \varepsilon\Theta + Q_2\xi_2^2\Theta, \end{aligned} \tag{29}$$

where $\tilde{Q}_2 = \tilde{c}_2(\varrho_{21}+\gamma_{21})|\xi_2|^{-\eta} + \lceil\xi_2\rceil^{1-\eta-h_2}$, $Q_2 = \frac{\tilde{Q}_2^2}{\sqrt{\xi_2^2\tilde{Q}_2^2+\varepsilon^2}}$ are smooth functions. Define $\mu_1 = Q_1\xi_1^2$, it is easy to deduce that:

$$\begin{aligned}&\frac{\partial W_2}{\partial\hat{\Theta}}\dot{\hat{\Theta}} + \tilde{\Theta}(\mu_1 - \dot{\hat{\Theta}}) + Q_2\xi_2^2\tilde{\Theta} \\ &\quad= (\tilde{\Theta} - \frac{\partial W_2}{\partial\hat{\Theta}})(\mu_2 - \dot{\hat{\Theta}}) + \mu_2\frac{\partial W_2}{\partial\hat{\Theta}}, \end{aligned} \tag{30}$$

where $\mu_2 = \mu_1 + Q_2\xi_2^2 = Q_1\xi_1^2 + Q_2\xi_2^2$ is a smooth function. It should be noted that:

$$\frac{\partial W_2}{\partial\hat{\Theta}}\mu_2 = \frac{\partial W_2}{\partial\hat{\Theta}}(Q_1\xi_1^2 + Q_2\xi_2^2) \leq \frac{1}{5}\xi_1^2 + \phi_{24}\xi_2^2, \tag{31}$$

where ϕ_{24} is a smooth function. On the other hand,

$$\frac{\partial W_2}{\partial\hat{\theta}}\left(\varpi_1\xi_1^2 + \varpi_2\xi_2^2\right) \leq \frac{1}{5}\xi_1^2 + \phi_{25}\xi_2^2, \tag{32}$$

where ϕ_{25} is a smooth function, and $\varpi_{21} + \varpi_{22} = \varpi_2$. Let $\phi_2(\bar{x}_2) = \phi_{21} + \phi_{22} + \phi_{23} + \phi_{24} + \phi_{25}$. Substituting (24)-(32)

into (22) and considering $\lceil\xi_2\rceil^{2-\eta-h_2}\alpha_2^{p_2} = -g_2^{p_2h_3}\xi_2^2$, then (22) takes the form of

$$\begin{aligned}\dot{V}_2 \le& -(n-1)(\xi_1^2+\xi_2^2)-\xi_1^2+2\bar{f}_0^{\frac{2}{1+\eta}}+\sum_{i=1}^{2}\bar{f}_0^{\frac{2}{p_ih_{i+1}}}-W_2\\&+\beta_2\lceil\xi_2\rceil^{2-\eta-h_2}(x_3^{p_2}-\alpha_2^{p_2})+(\tilde{\theta}-\frac{\partial W_2}{\partial\hat{\theta}})(\sum_{i=1}^{2}\varpi_i\xi_i^2-\dot{\hat{\theta}})\\&+\xi_2^2(\phi_2+n-1-\beta_2g_2^{p_2h_3}+c_{22}(1+\xi_1^2)^{-\frac{2}{\eta}}+Q_2\hat{\Theta})\\&+2\varepsilon\Theta+\Big(\tilde{\Theta}-\frac{\partial W_2}{\partial\hat{\Theta}}\Big)(\mu_2-\dot{\hat{\Theta}})-(1-\epsilon)K\circ\underline{\pi}\pi.\end{aligned}\tag{33}$$

Choose

$$g_2=\Big(\frac{\phi_2+n-1+c_{22}\left(1+\xi_2^2\right)^{-\frac{\eta}{2}}+Q_2\hat{\Theta}}{\beta_2}\Big)^{\frac{1}{p_2h_3}}.\tag{34}$$

Finally, (33) takes the form

$$\begin{aligned}\dot{V}_2 \le& -(n-1)(\xi_1^2+\xi_2^2)-\xi_1^2+2\bar{f}_0^{\frac{2}{1+\eta}}+\sum_{i=1}^{2}\bar{f}_0^{\frac{2}{p_ih_{i+1}}}-W_2\\&-(1-\epsilon)\circ\underline{\pi}\pi+\beta_2\lceil\xi_2\rceil^{2-\eta-h_2}\left(x_3^{p_2}-\alpha_2^{p_2}\right)+2\varepsilon\Theta\\&+\Big(\tilde{\theta}-\frac{\partial W_2}{\partial\hat{\theta}}\Big)\Big(\sum_{i=1}^{2}\varpi_i\xi_i^2-\dot{\hat{\theta}}\Big)+\Big(\tilde{\Theta}-\frac{\partial W_2}{\partial\hat{\Theta}}\Big)\Big(\mu_2-\dot{\hat{\Theta}}\Big).\end{aligned}\tag{35}$$

step k ($k=3,\ldots,n$) Given that at step $k-1$, we have created a continuously differential function $V_{k-1}(\bar{x}_{k-1})$ along with smooth positive functions $g_1,\cdots,g_{k-1}$ such that

$$\begin{aligned}\dot{V}_{k-1} \le& -(n-k+2)\sum_{i=1}^{k-1}\xi_i^2-\xi_1^2+(k-1)\bar{f}_0^{\frac{2}{\eta+1}}+\sum_{i=1}^{k-1}\bar{f}_0^{\frac{2}{p_ih_{i+1}}}\\&-\sum_{i=2}^{k-1}W_i+(k-1)\varepsilon\Theta-(1-\epsilon)K\circ\underline{\pi}\pi\\&+\beta_{k-1}\lceil\xi_{k-1}\rceil^{2-\eta-h_{k-1}}\left(x_k^{p_{k-1}}-\alpha_{k-1}^{p_{k-1}}\right)\\&+\Big(\tilde{\theta}-\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial\hat{\theta}}\Big)\Big(\sum_{i=1}^{k-1}\varpi_i\xi_i^2-\dot{\hat{\theta}}\Big)\\&+\Big(\tilde{\Theta}-\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial\hat{\Theta}}\Big)(\mu_{k-1}-\dot{\hat{\Theta}}),\end{aligned}\tag{36}$$

where $\varpi_i(\bar{x}_i,\hat{\theta})$ is a nonnegative continuous function and $\varpi_i(0,\hat{\theta})=0$, $\mu_{k-1}=\sum_{i=1}^{k-1}Q_i\xi_i^2$. Subsequently, what we should to prove is that (36) also holds in step k. So choose Subsequently, what we should do is to prove that (36) still holds in step i. So select $V_i=V_{i-1}+W_i$. Making use of (10), (36) could be rewritten as

$$\begin{aligned}\dot{V}_k \le& -(n-k+2)\sum_{i=1}^{k-1}\xi_i^2-\xi_1^2+(k-1)\bar{f}_0^{\frac{2}{\eta+1}}+\sum_{i=1}^{k-1}\bar{f}_0^{\frac{2}{p_ih_{i+1}}}\\&+\beta_k\lceil\xi_k\rceil^{2-\eta-h_k}\alpha_k^{p_k}+\beta_k\lceil\xi_k\rceil^{2-\eta-h_k}(x_{k+1}^{p_k}-\alpha_k^{p_k})\\&-\sum_{i=2}^{k}W_i+W_k-(1-\epsilon)K(s)\circ\underline{\pi}(\|z\|)\pi(\|z\|)\\&+\Big(\tilde{\theta}-\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial\hat{\theta}}\Big)\Big(\sum_{i=1}^{k-1}\varpi_i\xi_i^2-\dot{\hat{\theta}}\Big)+\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}\dot{x}_i\\&+\frac{\partial W_k}{\partial\hat{\theta}}\dot{\hat{\theta}}+\lceil\xi_k\rceil^{2-\eta-h_k}(f_k+q_k)+(k-1)\varepsilon\Theta\\&+\Big(\tilde{\Theta}-\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial\hat{\Theta}}\Big)(\mu_{k-1}-\dot{\hat{\Theta}})+\frac{\partial W_k}{\partial\hat{\Theta}}\dot{\hat{\Theta}}\\&+\beta_{k-1}\lceil\xi_{k-1}\rceil^{2-\eta-h_k}(x_k^{p_{k-1}}-\alpha_{k-1}^{p_{k-1}}).\end{aligned}\tag{37}$$

The estimate of the last five terms of (37) is included in Appendix B to prevent tiresome calculating. Or to put it another way, we achieve the inequality shown below after tedious calcuations:

$$\begin{aligned}&\beta_{k-1}\lceil\xi_{k-1}\rceil^{2-\eta-h_k}\left(x_k^{p_{k-1}}-\alpha_{k-1}^{p_{k-1}}\right)+\lceil\xi_k\rceil^{2-\eta-h_k}f_k\\&\quad+\frac{\partial W_k}{\partial x_i}\dot{x}_i+\frac{\partial W_k}{\partial\hat{\Theta}}\mu_k+\frac{\partial W_k}{\partial\hat{\theta}}\Big(\sum_{i=1}^{k-1}\varpi_i\xi_i^2+\sum_{i=2}^{k}\varpi_k\xi_k^2\Big)\\&\le\sum_{i=1}^{k-1}\xi_i^2+\phi_k\xi_k^2+\bar{f}_0^{\frac{2}{p_kh_{k+1}}}+\bar{f}_0^{\frac{2}{\eta+1}}+\tilde{\theta}\varpi_k\xi_k^2\\&\quad+\tilde{c}_k|\xi_k|^{1-\eta}\sum_{i=1}^{k-1}(\gamma_{k_i}+\varrho_{k_i})q_i,\end{aligned}\tag{38}$$

next, the designer can develop the smooth positive function g_k with $k\ge 2$ as:

$$g_k=\Big(\frac{\phi_k+n-k+1+c_{k2}(1+\xi_k^2)^{\frac{-\eta}{2}}+Q_k\hat{\Theta}}{\beta_k}\Big)^{\frac{1}{p_kh_{k+1}}},\tag{39}$$

where $c_{k2}>0$ is design parameters which directly shows the rate of convergence, control cost and CPU time. substituting (38) and (39) into (37) yields

$$\begin{aligned}\dot{V}_k \le& -(n-k+1)\sum_{i=1}^{k}\xi_i^2-\xi_1^2+k\bar{f}_0^{\frac{2}{\eta+1}}+\sum_{i=1}^{k}\bar{f}_0^{\frac{2}{p_ih_{i+1}}}+k\varepsilon\Theta\\&+\beta_k\lceil\xi_k\rceil^{2-\eta-h_k}(x_{k+1}^{p_k}-\alpha_k^{p_k})-(1-\epsilon)K\circ\underline{\pi}\pi\\&-\sum_{i=2}^{k}W_i+\Big(\tilde{\theta}-\sum_{i=2}^{k}\frac{\partial W_i}{\partial\hat{\theta}}\Big)\Big(\sum_{i=1}^{k}\varpi_i\xi_i^2-\dot{\hat{\theta}}\Big)\\&+\Big(\tilde{\Theta}-\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial\hat{\Theta}}\Big)(\mu_k-\dot{\hat{\Theta}}).\end{aligned}\tag{40}$$

It is worthy pointing out that (40) is holds for $k=n$ with $\xi_{n+1}=0$. Thus, we can develop an adaptive stabilizer as:

$$\begin{cases}\dot{\hat{\theta}}=\sum_{i=1}^{n}\varpi_i\xi_i^2,\ \hat{\theta}(t_0)=\hat{\theta}_0,\\\dot{\hat{\Theta}}=\mu_n=\sum_{i=1}^{n}Q_i\xi_i^2,\ \hat{\Theta}(t_0)=\hat{\Theta}_0,\end{cases}\tag{41}$$

$$u=-\Big\lceil\sum_{l=1}^{n}\Big(\prod_{j=l}^{n}g_j\Big(\bar{x}_j,\hat{\theta},\hat{\Theta}\Big)\Big)\lceil x_l\rfloor^{\frac{1}{h_l}}\Big\rfloor^{h_{n+1}}.\tag{42}$$

It should be pointed out there is no constraints on the control signal. At last, applying (40) and setting $k = n$, one has

$$\dot{V}_n \le -\sum_{i=1}^{n}\xi_i^2 - \xi_1^2 + n\bar{f}_0^{\frac{2}{\eta+1}} + \sum_{i=1}^{n}\bar{f}_0^{\frac{2}{p_ih_{i+1}}} - \sum_{i=2}^{n}W_i - (1-\epsilon)K(s)\circ\underline{\pi}(\|z\|)\pi(\|z\|) + n\varepsilon\Theta, \quad (43)$$

where $V_n = V_1 + \Sigma_{i=2}^{n}W_i$. Currently, the design process is completed. Ultimately, we highlight V_{blf}'s unique qualities from two distinct perspectives.

Remark 1: (i) The process for design is capable of being integrated by V_{blf} while confronted with both limited and unlimited systems, In deed, we define $b = a \to \infty$ and gain

$$\lim_{a\to\infty} V_{blf}(x_1) = \lim_{a\to\infty}\frac{a^{4-2\eta}|x_1|^{2\eta}}{(2-\eta)(a-x_1)^{2-\eta}(a+x_1)^{2-\eta}} = \frac{|x_1|^{2-\eta}}{2-\eta}.$$

Contrarily, some of the most recent studies [5], [6], [7], [8] similarly make use of the subsequent Lyapunov function:

$$W_1(x_1) = \int_0^{x_1}\lceil s\rceil^{2-r_2p_1}ds = \int_0^{x_1}\lceil s\rceil^{1-\eta}ds = \frac{|x_1|^{2-\eta}}{2-\eta}.$$

It turns out that the case in which $a = b \to \infty$ and x_1 possesses no limit are equivalent. Therefore, the barrier function becomes identical as well, further the rest of development and evaluation employ the comparable procedures as [5], [6], [7], and [8].

(ii) By successfully working with the properties of nonlinearities, V_{blf} is generated. The fact that η goes into the powers of V_{blf} nicely demonstrates the nonlinear characteristics of functions f_i's. Nevertheless, when constructing barrier functions in [23], [24], and [30], this information is neglected. It provides additional insight why using tangent or logarithm functions to regulate design is not practicable from an alternative angle.

IV. MAIN RESULTS

The following is a summary of the paper's key finding.

Theorem 1: For the high-order uncertain nonlinear system (1) under Assumptions 1 and 2, if (1) meets:

$$\lim_{s\to0^+}\sup\frac{\bar{\tau}(s)}{s^2} < +\infty,\ \lim_{s\to0^+}\sup\frac{\bar{f}_0^2(s)}{\underline{\pi}(s)} < +\infty. \quad (44)$$

there is a continuous stabilizer assures that the states of system (1) be drived to a compact set within finite time while keeping the output limitation unbroken.

Proof: The whole proof can be divided into two parts.

(a) Verification of finite-time convergence.

At the very begining, since $\frac{2}{\eta+1} > 2$, $\frac{2}{p_ih_{i+1}} > 2$, (44), $\bar{f}_0^{\frac{2}{\eta+1}}$ and $\bar{f}_0^{\frac{2}{p_ih_{i+1}}}$, it follows from the boundedness near the zero that $\lim_{s\to0^+}\sup\frac{\hat{j}_1(s)}{\underline{\pi}(s)} < \infty$, and

$$\hat{j}_1(\|z\|) = n\bar{f}_0^{\frac{2}{\eta+1}}(\|z\|) + \sum_{i=1}^{n}\bar{f}_0^{\frac{2}{p_ih_{i+1}}}(\|z\|). \quad (45)$$

define:

$$K(s) = \begin{cases} \frac{2}{(1-e)(1-\epsilon)}\lim\sup_{s\to0+}\frac{\hat{j}_1(s)}{\underline{\pi}(s)} + 1,\ s = 0,\\ \frac{2}{(1-e)(1-\epsilon)}\sup_{0<s'\le s}\frac{\hat{j}_1(s')}{\underline{\pi}(s')} + 1,\ s > 0,\end{cases} \quad (46)$$

where $0 < e < 1$ is a specified constant. $K(s)$ is nondecreasing, continuous and positive on $[0, \infty)$. Utilizing (44),

$$-\frac{(1-e)(1-\epsilon)}{2}K(s)\underline{\pi}(\|z\|)\pi(\|z\|) + \hat{j}_1(\|z\|) \le 0, \quad (47)$$

which blends with $-(1-\epsilon)K(s)\underline{\pi}(\|z\|)\pi(\|z\|) \le 0$ causes

$$\hat{j}_1(\|z\|) - (1-\epsilon)K(s)\underline{\pi}(\|z\|)\pi(\|z\|) \le -\frac{(1-e)(1-\epsilon)}{2}K(s)\underline{\pi}(\|z\|)\pi(\|z\|). \quad (48)$$

Combining (48), (4) with (43), one can obtain

$$\dot{V}_n \le -\frac{(1-e)(1-\epsilon)\bar{k}_0}{2}KU_0^{\alpha} - \sum_{i=1}^{n}\xi_i^2 - \xi_1^2 - \sum_{i=2}^{n}W_i + n\varepsilon\Theta \triangleq -V_n^*(x,z) + n\varepsilon\Theta. \quad (49)$$

To facilitate calculating, let $V = W + V_0$ with $W = \sum_{i=1}^{n}W_i$, $W_1 = \frac{a^{2-\eta}b^{2-\eta}|x_1|^{2-\eta}}{(2-\eta)(b-x_1)^{2-\eta}(a+x_1)^{2-\eta}}$, then V_n can be redefined as $V_n = V + \frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}\tilde{\Theta}^2$. In this position, define a continuous function $L(x,\hat{\theta}) = (\theta + |\hat{\theta}|)\sum_{i=1}^{n}\varpi_i$, and it is not hard to prove that there has a positive parameter λ such that $L(x,z,\hat{\theta}) < \frac{1}{2}$, $\forall\|Y\| \le \lambda$ with $Y = [z,x]^{\mathrm{T}}$. Utilizing (36) and (49) yields

$$\begin{aligned}\dot{V} &= \dot{V}_n + \tilde{\theta}\dot{\hat{\theta}} \le \dot{V}_n + (\theta + |\hat{\theta}|)\sum_{i=1}^{n}\varpi_i\xi_i^2 + n\varepsilon\Theta\\ &\le -\frac{(1-e)(1-\epsilon)\bar{k}_0}{2}K(s)U_0^{\alpha} - \frac{1}{2}\sum_{i=1}^{n}\xi_i^2 - \xi_1^2\\ &\quad - \sum_{i=2}^{n}W_i - (\frac{1}{2} - K)\sum_{i=1}^{n}\xi_i^2 + n\varepsilon\Theta\\ &\le -\frac{(1-e)(1-\epsilon)\bar{k}_0}{2}K(s)U_0^{\alpha} - \xi_1^2 - \frac{1}{2}\sum_{i=1}^{n}\xi_i^2\\ &\quad - \sum_{i=2}^{n}W_i + n\varepsilon\Theta. \end{aligned} \quad (50)$$

By means of Lemma 2 and (10), one has

$$\begin{aligned}-\frac{1}{2}\sum_{i=1}^{n}\xi_i^2 &= -\frac{1}{2}(W_1^{\frac{2}{4-\eta}} + \sum_{i=2}^{n}|\xi_i^{4-\eta}|^{\frac{2}{4-\eta}}) + \frac{1}{2}W_1^{\frac{2}{4-\eta}} - \frac{1}{2}\xi_1^2\\ &\le -\frac{1}{2}(W_1 + \sum_{i=2}^{n}|\xi_i^{4-\eta}|)^{\frac{2}{4-\eta}} + \frac{1}{2}W_1^{\frac{2}{4-\eta}} - \frac{1}{2}\xi_1^2\\ &\le -(\frac{1}{2})^{\frac{6-\eta}{4-\eta}}W^{\frac{2}{4-\eta}} + \frac{1}{2}W_1^{\frac{2}{4-\eta}} - \frac{1}{2}\xi_1^2, \end{aligned} \quad (51)$$

thus, (50) takes the form

$$\dot{V} \leq -\frac{(1-e)(1-\epsilon)\bar{k}_0}{2}K(s)U_0^{\alpha} - (\frac{1}{2})^{\frac{6-\eta}{4-\eta}}W^{\frac{2}{4-\eta}} + \frac{1}{2}W_1^{\frac{2}{4-\eta}} - \frac{3}{2}\xi_1^2 - \sum_{i=2}^{n}W_i + n\varepsilon\Theta. \quad (52)$$

Next, define $U(\lambda) = \{Y|\|Y\|\} \leq \lambda$. Applying mean value theorem of integrals leads to

$$V_0(z) = \int_0^{U_0(z)} K(s)ds = K(\theta U_0)U_0, \quad (53)$$

where $0 < \theta < 1$ is a constant. Then there holds

$$\lim_{\|z\|\to 0}\frac{V_0(z)}{U_0} = \lim_{\|z\|\to 0} K(\theta U_0) < \infty, \quad (54)$$

therefore, for any $\|z\| < \lambda_1$, there is $\lambda_1 > 0$ such that $V_0(z) \leq k_1 U_0$ holds with k_1 being a positive constant. And we have

$$\lim_{\|z\|\to 0}\frac{U_0^{\alpha}}{K(U_0)U_0^{\alpha}} = \lim_{\|z\|\to 0}\frac{1}{K(U_0(z))} = \frac{1}{K(0)} < \infty, \quad (55)$$

which shows there exists $\lambda_2 > 0$ such that $U_0^{\alpha} \leq k_2 K(U_0)U_0^{\alpha}, \forall \|z\| < \lambda_2$ and k_2 is a positive constant. As can be seen from above, there exists $V_0^{\alpha} \leq k_1^{\alpha}k_2 K(U_0)U_0^{\alpha}, \forall \|z\| < \min\{\lambda_1, \lambda_2\}$ such that

$$-\frac{(1-e)(1-\epsilon)\bar{k}_0}{2}K(s)U_0^{\alpha} \leq -\frac{(1-e)(1-\epsilon)\bar{k}_0}{2k_1^{\alpha}k_2}V_0^{\alpha}. \quad (56)$$

Besides, according to the continuity of $V_0^{1-\alpha}$ and $V_0 k^{1-\alpha}(0) = 0$, one can deduce that there is a positive constant λ_3 such that

$$V_0^{1-\alpha} < \frac{(1-e)(1-\epsilon)\bar{k}_0}{4k_1^{\alpha}k_2}. \quad (57)$$

Based on $V(0) = 0$ and the continuity of V, it is uncomplicated to conclude that there is a positive constant $\lambda_2 > 0$ such that $V < 1$ holds for any given $\|y(t)\| < \lambda_4$. In addition, there also exists $\lambda_5 = \min\{1, (a+b+ab-ab(2-\eta)^{\frac{1}{\eta-2}})^{\frac{1}{2}}\}$. For any given $|x_1| < \lambda_5$, we have

$$\frac{a^{2-\eta}b^{2-\eta}|x_1|^{2-\eta}}{(2-\eta)(b-x_1)^{2-\eta}(a+x_1)^{2-\eta}} < 1. \quad (58)$$

where $\lambda = \min\{\bar{\lambda}, \lambda_1, \cdots, \lambda_5, a, b\}$. It should be noted that $V < 1$ means $W_1 < 1$. Combining this and (58) leads to

$$-\frac{3}{2}\xi_1^2 + \frac{1}{2}W_1^{\frac{2}{2-\eta}} + W_1 \leq -\frac{3}{2}\xi_1^2 + \frac{3}{2}W_1^{\frac{2}{2-\eta}} = -\frac{3}{2}\xi_1^2 + \frac{3}{2}\Big(\frac{a^{2-\eta}b^{2-\eta}|x_1|^{2-\eta}}{(2-\eta)(b-x_1)^{2-\eta}(a+x_1)^{2-\eta}}\Big)^{\frac{2}{2-\eta}} \leq 0. \quad (59)$$

Then substituting (56), (57) and (59) into (54) yields

$$\begin{aligned}\dot{V} &\leq -\frac{(1-e)(1-\epsilon)\bar{k}_0}{2k_1^{\alpha}k_2}V_0^{\alpha} - (\frac{1}{2})^{\frac{6-\eta}{4-\eta}}W^{\frac{2}{4-\eta}} + \frac{1}{2}W_1^{\frac{2}{4-\eta}} - \frac{3}{2}\xi_1^2 - \sum_{i=2}^{n}W_i + n\varepsilon\Theta \\ &= -\frac{(1-e)(1-\epsilon)\bar{k}_0}{4k_1^{\alpha}k_2}V_0^{\alpha} - (\frac{1}{2})^{\frac{6-\eta}{4-\eta}}W^{\frac{2}{4-\eta}} - V + V_0^{\alpha}(V_0^{1-\alpha} - \frac{(1-e)(1-\epsilon)\bar{k}_0}{4k_1^{\alpha}k_2}) + n\varepsilon\Theta \\ &\leq -n_2(V_0^{\alpha} + W^{\frac{2}{4-\eta}}) - n_1 V^{m_1} + n\varepsilon\Theta, \end{aligned} \quad (60)$$

where $n_1 = 1, m_1 = 1, n_2 = \max\{\frac{(1-e)(1-\epsilon)\bar{k}_0 k_4}{4k_1^{\alpha}k_2}, (\frac{1}{2})^{\frac{6-\eta}{4-\eta}}\}$. Since $V < 1$, so $W < 1$ and $V_0 < 1$. Combining the definition of V with Lemma 2 leads to

$$V_0^{\alpha} + W^{\frac{2}{4-\eta}} \geq V_0^{m_2} + W^{m_2} \geq (V_0 + W)^{m_2} = V^{m_2}, \quad (61)$$

where $0 < m_2 = \max\{\alpha, \frac{2}{4-\eta}\} < 1$. Let $n\varepsilon\Theta = \zeta$, thus (60) can be rewritten as

$$\dot{V} \leq -n_1 V^{m_1} - n_2 V^{m_2} + \zeta. \quad (62)$$

To sum up, According to Lemma 4, the designer can obtain the states of the closed-loop system are capable of converging to a compact set

$$\Omega = \Big\{\lim_{t\to T}\xi_i\Big|V \leq \min\Big\{\Big(\frac{\bar{\zeta}}{n_1(1-\iota)}\Big)^{\frac{1}{m_1}}, \Big(\frac{\bar{\zeta}}{n_2(1-\iota)}\Big)^{\frac{1}{m_2}}\Big\}\Big\}, \quad (63)$$

where $0 < \iota < 1$. Additionally, it follows from Lemma 4 that the states of the closed-loop system are steered into a compact set within finite time T, and $T \leq \frac{1}{n_1\iota(1-m_1)} + \frac{1}{n_2\iota(1-m_2)}$.

(b)Verification of output constraints. What follows is a test to see whether $a < |y(t)| < b$ holds for all $t \geq 0$. Firstly, we define the initial condition $x(0) \in \mathbb{S}_n^{\lambda}$. By making use of (49), we can conclude that $0 \leq V_n(x(t)) \leq V_n(x(0))$ for all $t \geq 0$, which illustrates that

$$\frac{a^{2-\eta}b^{2-\eta}|x_1|^{2-\eta}}{(2-\eta)(b-x_1)^{2-\eta}(a+x_1)^{2-\eta}} \leq V_n(x(0)). \quad (64)$$

On the basis of $V(0) = 0$ and the continuous property of V, there exists a constant $\lambda_2 > 0$ such that for any given $\|y(t)\| < \lambda_2, V < 1$ holds. Besides, there also exists a positive constant $\lambda_3 = \min\{1, (a+b+ab-ab(2-\eta)^{\frac{1}{\eta-2}})^{\frac{1}{2}}\}$ such that for any $|x_1| < \lambda_3$, the following holds

$$\frac{a^{2-\eta}b^{2-\eta}|x_1|^{2-\eta}}{(2-\eta)(b-x_1)^{2-\eta}(a+x_1)^{2-\eta}} < 1. \quad (65)$$

Let $\lambda = \min\{\lambda_1, \lambda_2, \lambda_3, a, b\}$, $|y| < \lambda$ holds. That is, for all $t \geq 0$, $|y(t)| = |x_1(t)| < \lambda$ holds, which implies $\mathbb{S}_n^{\lambda} \subset \mathbb{R}^n$ is an estimation of attractive region. □

Remark 2: It is important to note that challenges faced and innovations created from two angles.

(i) In this study, we incorporate the asymmetric restriction, which poses barriers with regard to both practical scenarios as well as control theories. In light of this, the initial challenge of this study may be seen as how to construct a new barrier Lyapunov function to unify the control design to handle both restricted and unrestricted systems devoid of altering the basic framework of the stabilizer. The asymmetric output limitation is kept unbroken by an inventive barrier Lyapunov

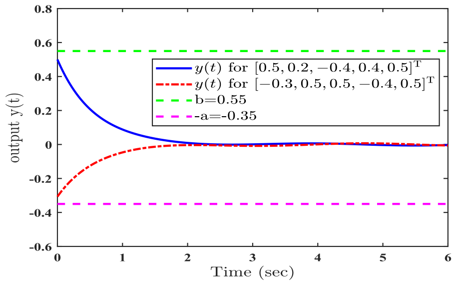

FIGURE 1. Trajectories of y.

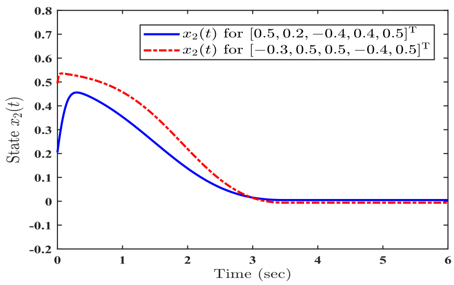

FIGURE 2. Trajectories of x_2.

function that is delicately built via successfully incorporating the properties of nonlinearities. Remark 1 provides supplementary information. (ii) The iterative construction ultimately becomes useless in controlling terms related to θ as well as possessing mismatched powers with other unstable nonlinear terms caused by the high-orders p_i's. On the other side, $p_i > 1$ undoubtedly induces distinct homogeneous degrees in every formula of system (1). A stabilizer generating efficient responses is created to compensate for the impact of fundamental nonlinearities whilst controlling high-orders p_i's through the creation of delicate state transformations along with enhancing the continuous domination methodology armed with a series of integral functions which includes embedded sign functions.

V. SIMULATION EXAMPLE

To illustrate the validity of the designed mechanism, we choose the following example:

$$\begin{cases} \dot z = -2z^{\frac{3}{5}} + \frac{1}{4}\theta x_1^{\frac{3}{5}}, \\ \dot x_1 = x_2^3 + \theta x_1 + z^2 + q_1, \\ \dot x_2 = u + \theta x_2^{\frac{4}{3}} + z^2 + q_2. \end{cases} \tag{66}$$

where $y = x_1, q_1, q_2$ are unknown disturbances. Choose $\beta_1 = \beta_2 = 1, \varepsilon = 0.01$. Besides, $p_1 = 3, p_2 = 1, \eta = -\frac{1}{25} \in (-\frac{1}{3}, 0), h_1 = 1, h_2 = \frac{h_1+\eta}{p_1} = \frac{8}{25}, h_3 = \frac{h_2+\eta}{p_2} = \frac{7}{25}$. By (3), we have

$$\begin{cases} |f_1| = |\theta x_1 + z^2| \le |\theta x_1| + |z^2| \le |\theta||x_1| + z^2 \\ \quad = |\theta||x_1|^{\frac{4}{5}}|x_1|^{\frac{1}{5}} + z^2, \\ |f_2| = |\theta x_2^{\frac{4}{3}} + z^2| \le |\theta||x_2|^{\frac{4}{3}} + |z^2| \le |\theta|(|x_1|^{\frac{7}{25}}|x_2|^{\frac{4}{3}} \\ \quad + |x_2|^{\frac{4}{3}}) + z^2 = |\theta|(|x_1|^{\frac{7}{25}} + |x_2|^{\frac{7}{12}})|x_2|^{\frac{3}{4}} + z^2, \end{cases} \tag{67}$$

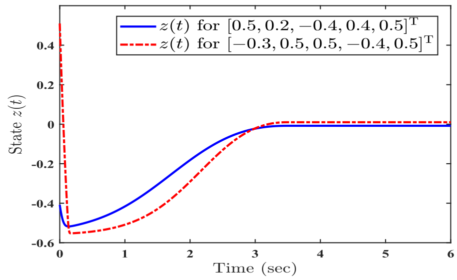

FIGURE 3. Trajectories of z.

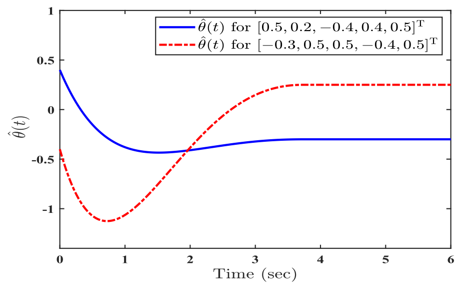

FIGURE 4. Trajectories of $\hat\theta$.

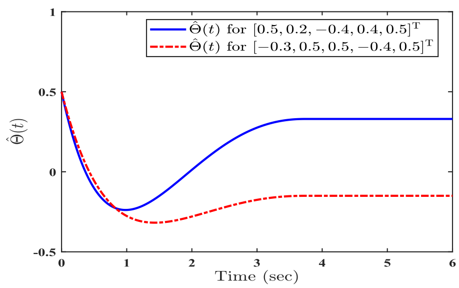

FIGURE 5. Trajectories of $\hat\theta$.

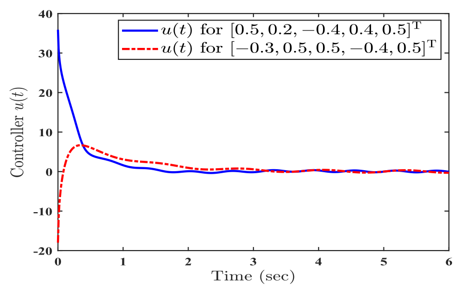

FIGURE 6. Trajectories of u.

some calculation illustrate that $\bar f_1 = |x_1|^{\frac{1}{5}}, \bar f_2 = |x_2|^{\frac{3}{4}}$, $\bar f_0 = z^2$, Here we select $U_0 = z^4$, thus $\frac{\partial U_0(z)}{\partial z} f_0(x_1, z, d_0) = 4z^3\left(-2z^{\frac{3}{5}} + \frac{1}{4}d_0 x_1^{\frac{3}{5}}\right) \le -\frac{1}{4}\left(z^4\right)^{\frac{9}{10}} + d_0^6|x_1|^{\frac{18}{5}}$,satisfies Assumption 2, where $\alpha = \frac{9}{10}, 1 - \epsilon = \frac{1}{4}, \epsilon = \frac{3}{4}, \sigma = 1$, $\tau(|x_1|) = d_0^6|x_1|^{\frac{18}{5}}, \Theta = \max\{1, \bar\Theta\}$. After complicated calculation, we provide the controller as $u = -g_2(g_1 x_1 + |x_2|^{\frac{25}{12}})^{\frac{7}{25}}$, where $g_1 = (\phi_1 + 3 + \hat\theta\tau_1(x_1) + Q_1\hat\Theta)^{\frac{5}{4}}$, $g_2 = (\phi_2 + 1 + 1.4(1 + \xi_2^2)^{\frac{1}{10}} + Q_2\hat\Theta)^{\frac{25}{7}}$,

$Q_1 = \frac{\rho^2 x_1^{\frac{2}{25}}}{\sqrt{\rho^2 x_1^{\frac{52}{25}}+0.01^2}}$, $\tilde{Q}_2 = \tilde{c}_2(\varrho_{21}+\gamma_{21})|\xi_2|^{\frac{1}{25}} + \lceil\xi_2\rceil^{\frac{26}{25}-h_2}$, $Q_2 = \frac{\tilde{Q}_2^2}{\sqrt{\xi_2^2\tilde{Q}_2^2+0.01^2}}$, $\phi_1 = \hat{\theta}\rho(1+x_1^2)^{\frac{1}{10}} + 0.2\rho^{\frac{5}{3}}$, $\phi_2 = \hat{\theta}(1+x_2^2)^{\frac{3}{8}} + 0.63\hat{\theta}(1+g_1^{\frac{7}{25}})^{\frac{50}{43}}(1+x_2^2)^{\frac{75}{172}} + 2.1\rho^{\frac{5}{2}} + 2.47(\varrho_{21}+\gamma_{21}) + (6+10.6)\hat{\theta}(\varrho_{21}+\gamma_{21})^{\frac{5}{3}} + 2.7((g_1+(1+(\frac{\partial g_1}{\partial x_1})^2 x_1^2)^{\frac{1}{2}}(\varpi_1+\varpi_2)(\xi_1^{\frac{6}{5}}+\xi_2^{\frac{6}{5}}))^{\frac{4}{3}} + 2.47(g_1 + (1+\frac{\partial g_1}{\partial x_1}x_1^2)^{\frac{1}{2}}(\varpi_1+\varpi_2)(\xi_1^{\frac{6}{5}}+\xi_2^{\frac{6}{5}})$.

To conduct the simulation, we assign $\theta = 1, a = 0.35, b = 0.55, d_0 = 0.2$, and select the initial values as $q_1 = q_2 = e^{-t}$, $[x_1(0), x_2(0), z(0), \hat{\theta}(0), \hat{\Theta}(0)]^T = [0.5, 0.2, -0.4, 0.4, 0.5]^T$, $[x_1(0),x_2(0),z(0),\hat{\theta}(0),\hat{\Theta}(0)]^T = [-0.3, 0.5, 0.5, -0.4, 0.5]^T$. Just as shown in Figs.1-6, the states of the closed-loop system can be drived to the compact set within finite time and the output constraint $-0.35 < y(t) < 0.55$ can not be broken.

It should be noted that the selection of parameters is independent, and each variable plays a different role. (i) Within the selected range, η and c_{k2} have an effect on the convergence speed, the control size and CPU time. (ii) The control strategy in this paper cannot guarantee that these parameters in simulation part are optimal and we only choose the relatively appropriate parameters in the simulation.

VI. CONCLUSION

The study provides a solution on the topic of adaptive finite-time stabilization for a kind of high-order uncertain nonlinear systems with zero dynamics, asymmetric output constraint and external disturbances. The construction of the continuous feedback controller is based on a novel Barrier Lyapunov Function (BLF) along with the tool of continuous state-feedback domination armed with a series of integral functions containing nested sign functions. Future research will need to address several difficulties, for example, (i) it is unknown whether method could be utilized to cope with the fast finite-time stabilization or even the prescribed-time stabilization, (ii) the question on how to tackle asymmetric time-varying output constraints by improving the proposed method is still open, (iii) Whether this strategy can be employed to tackle the stabilization of nonlinear stochastic systems. (iv) How to apply Theorem 1 in dealing with the practical problems?

APPENDIX

A. PROOF OF PROPOSITION 1

The specific supporting material for Proposition 1 is provided in this section. In the beginning, because $U_0(z)$ is positive, continuously differential and radially unbounded, it follows from Assumption 2 that

$$\begin{aligned} U_0(z) &= K(U_0(z))\dot{U}_0(z) \\ &\le -\pi(\|z\|)K(U_0(z)) + \sigma\tau(|x_1|)K(U_0(z)), \end{aligned}$$

Here, we'll talk about two instances:

Instance 1: if $\sigma\tau(|x_1|) < \epsilon\pi(\|z\|)$, there has

$$\begin{aligned} &-\pi(\|z\|)K(U_0(z)) + \sigma\tau(|x_1|)K(U_0(z)) \\ &\quad\le -\pi(\|z\|)K(U_0(z)) + \epsilon\pi(\|z\|)K(U_0(z)) \\ &\quad= -(1-\epsilon)\pi(\|z\|)K(U_0(z)). \end{aligned}$$

Case 2: if $\sigma\tau(|x_1|) \ge \epsilon\pi(\|z\|)$, $\pi(\|z\|) \le \frac{\sigma}{\epsilon}\tau(|x_1|)$ and $|z| \le \pi^{-1}\circ(\frac{\sigma}{\epsilon}\tau(|x_1|))$. Considering $U_0(z) \le \bar{\pi}(\|z\|)$ as well as K being nondecreasing, then the following inequality holds

$$K(U_0(z)) \le K\bar{\pi}\pi^{-1}(\frac{\sigma}{\epsilon}\tau(|x_1|)),$$

further,

$$\begin{aligned} &-\pi(\|z\|)K(U_0(z)) + \sigma\tau(|x_1|)K(U_0(z)) \\ &\le -(1-\epsilon)\pi(\|z\|)K(U_0(z)) + \sigma\tau(|x_1|)K\bar{\pi}\pi^{-1}\left(\frac{\sigma}{\epsilon}\tau(|x_1|)\right). \end{aligned}$$

On the basis of $U_0(z) \ge \underline{\pi}(\|z\|)$, we know

$$\begin{aligned} &K(U_0(z)) \\ &\ge K\circ\underline{\pi}(\|z\|), \\ &\quad -\pi(\|z\|)K(U_0(z)) + \sigma\tau(|x_1|)K(U_0(z)) \\ &\le -(1-\epsilon)\underline{\pi}(\|z\|)\pi(\|z\|) + \sigma\tau(|x_1|)K\bar{\pi}\pi^{-1}(\frac{\sigma}{\epsilon}\tau(|x_1|)). \end{aligned}$$

As Lemma 2.5 in [18], the designer find there is a constant $c(\sigma)$ relying on σ and positive smooth function $\hat{\tau}(|x_1|) \ge 1$ such that $K\bar{\pi}\pi^{-1}(\frac{\sigma}{\epsilon}\tau(|x_1|)) \le c(\sigma)\hat{\tau}(|x_1|)$. Let $\bar{\sigma} = \sigma c(\sigma)$ and $\tilde{\tau}(s) = \tau(s)\hat{\tau}(s)$, there holds

$$\frac{\partial U_0(z)}{\partial z}f_0 \le -(1-\epsilon)K(s)\underline{\pi}(\|z\|)\pi(\|z\|) + \bar{\sigma}\tilde{\tau}(|x_1|).$$

As a result, there is a smooth nondecreasing function $\bar{\tau}$ satisfies $\tilde{\tau}(|x_1|) = x_1^2\bar{\tau}(|x_1|)$. Finally, there holds

$$\frac{\partial U_0(z)}{\partial z}f_0 \le -(1-\epsilon)K(s)\underline{\pi}(\|z\|)\pi(\|z\|) + \bar{\sigma}x_1^2\bar{\tau}(x_1).$$

□

B. PROOF OF (39)

The specific proof of (38) is offered in this part. At beginning, based on Lemmas 2 and 3, one has

$$\begin{aligned} &\beta_{k-1}(\bar{x}_{k-1},t)\lceil\xi_{k-1}\rceil^{2-\eta-h_k}\left(x_k^{p_{k-1}} - \alpha_{k-1}^{p_{k-1}}\right) \\ &\quad\le \beta_{k-1}(\bar{x}_{k-1},t)\,|\xi_{k-1}|^{2-\eta-h_k}\left|x_k^{p_{k-1}} - \alpha_{k-1}^{p_{k-1}}\right| \\ &\quad\le \beta_{k-1}(\bar{x}_{k-1},t)2^{1-p_{k-1}h_k}\,|\xi_{k-1}|^{2-\eta-h_k}\,|\xi_k|^{p_{k-1}h_k} \\ &\quad\le \frac{1}{5}\xi_{k-1}^2 + \phi_{k1}\xi_k^2, \end{aligned} \tag{68}$$

where ϕ_{k1} is a constant. Then, based on $|x_i|^{\frac{h_{k+1}p_k}{h_i}} \le |\xi_i|^{p_k h_{k+1}} + |g_{i-1}|^{p_k h_{k+1}}\,|\xi_{i-1}|^{p_k h_{k+1}}$, $i = 1, \ldots, k$, Lemma 2, (3) and (8), one has

$$\begin{aligned} &\lceil\xi_k\rceil^{2-\eta-h_k}f_k \\ &\quad\le |\xi_k|^{2-\eta-h_k}\bar{f}_0 + |\xi_k|^{2-\eta-h_k}\bar{g}_{k-1}l_k\bar{\theta}\sum_{i=1}^{k}|\xi_i|^{p_k h_{k+1}} \\ &\quad\le \phi_{k2}\xi_k^2 + \bar{f}_0^{\frac{2}{p_k h_{k+1}}} + \frac{1}{5}\xi_{k-1}^2 + \frac{1}{4}\sum_{i=1}^{k-2}\xi_i^2 + \tilde{\theta}\varpi_{k1}\xi_k^2, \end{aligned} \tag{69}$$

where ϕ_{k2}, ϖ_{k1} and $\bar{g}_{k-1} \geq 1 + \sum_{i=1}^{k-1} g_i^{p_k h_{k+1}}$ are smooth positive functions. Then, it follows from (10) that

$$\sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i}\dot{x}_i \leq -(2-\eta-h_k)\int_{\alpha_{k-1}}^{x_k} |\lceil s \rceil^{\frac{1}{h_k}} - \lceil \alpha_{k-1}(\bar{x}_{k-1})\rceil^{\frac{1}{h_k}}|^{1-\eta-h_k} ds \cdot \sum_{i=1}^{k-1} \frac{\partial \lceil \alpha_{k-1}\rceil^{\frac{1}{h_k}}}{\partial x_i}(\beta_i(\bar{x}_i, t)x_{i+1}^{p_i} + f_i + q_i). \quad (70)$$

By conducting the same process in [5], for any $i = 1, \cdots, k-1$, $k = 2, \cdots, n$, it is not hard to get that

$$|\frac{\partial}{\partial x_i}(\lceil \alpha_{k-1}\rceil^{\frac{1}{h_k}})f_i| \leq \gamma_{ki}(\bar{x}_{k-1})(\bar{f}_0 + \bar{\theta}\sum_{j=1}^{k-1}|\xi_j|^{\eta+1}), \quad (71)$$

where $\gamma_{ki} > 0$ is a smooth function. Naturally, (25) is the instance that $k = 2$. Assuming when $k = m-1$, (71) holds, then when $k = m$, $i = 1, \cdots, m-2$, there has

$$\begin{aligned}
&|\frac{\partial \lceil \alpha_{m-1}\rceil^{\frac{1}{h_m}}}{\partial x_i} f_i| \\
&\leq |\frac{\partial \lceil \alpha_{m-2}\rceil^{\frac{1}{h_{m-1}}}}{\partial x_i} g_{m-1} f_i| + |\frac{\partial g_{m-1}}{\partial x_i}\xi_{m-1} f_i| \\
&\leq |\xi_{m-1}|^{-\eta} \cdot |\frac{\partial g_{m-1}}{\partial x_i}| \cdot \sum_{j=1}^{i} |x_j|^{\frac{p_i h_{i+1}}{h_j}} \bar{f}_i) \\
&\quad + \bar{\theta}\sum_{j=1}^{m-1}|\xi_j|^{\eta+1}(g_{m-1}\gamma_{m-1,i}) \\
&\quad + \bar{f}_0(g_{m-1}\gamma_{m-1,i} + |\xi_{m-1}| \cdot |\frac{\partial g_{m-1}}{\partial x_i}|) \\
&\leq \gamma_{mi}(\bar{x}_{m-1})(\bar{f}_0 + \bar{\theta}\sum_{j=1}^{m-1}|\xi_j|^{\eta+1}),
\end{aligned} \quad (72)$$

with $\gamma_{mi} \geq g_{m-1}\gamma_{m-1,i} + \bar{\theta}|\xi_{m-1}|^{-\eta} \cdot |\frac{\partial g_{m-1}}{\partial x_i}| \cdot \sum_{j=1}^{i}|x_j|^{\frac{p_i h_{i+1}}{h_j}}\bar{f}_i + |\xi_{m-1}| \cdot |\frac{\partial g_{m-1}}{\partial x_i}|$ being a smooth positive function. if $i = m-1$, one has

$$\begin{aligned}
&|\frac{\partial \lceil \alpha_{m-1}\rceil^{\frac{1}{h_m}}}{\partial x_{m-1}} f_{m-1}| \\
&\leq \Big(\bar{f}_0 + \bar{\theta}\sum_{j=1}^{m-1}|x_j|^{\frac{p_{m-1}h_m}{h_j}}\bar{f}_{m-1}\Big)\Big(|\xi_{m-1}\frac{\partial g_{m-1}}{\partial x_{m-1}}| \\
&\quad + \frac{g_{m-1}}{h_{m-1}}|\lceil x_{m-1}\rceil^{\frac{1}{h_{m-1}}-1}|\Big).
\end{aligned} \quad (73)$$

Utilizing Lemma 2, the following holds

$$\begin{aligned}
&\bar{\theta}\sum_{j=1}^{m-1}|x_j|^{\frac{p_{m-1}h_m}{h_j}} \cdot |\lceil x_{m-1}\rceil^{\frac{1}{h_{m-1}}-1}| \\
&\quad \cdot (\bar{\theta}\sum_{j=1}^{m-1}(|\xi_j|^{p_{m-1}h_m} + |g_{j-1}\xi_{j-1}|^{p_{m-1}h_m})) \\
&\leq \tilde{\gamma}_{m,m-1}(\bar{x}_{m-1})(|\xi_{m-1}|^{1-h_{m-1}} + |\xi_{m-2}|^{1-h_{m-1}}) \\
&\quad \cdot \bar{\theta}\sum_{j=1}^{m-1}(|\xi_j|^{p_{m-1}h_m} + |\xi_{j-1}|^{p_{m-1}h_m}) \\
&\leq \bar{\gamma}_{m,m-1}(\bar{x}_{m-1}) \cdot \bar{\theta}\sum_{j=1}^{m-1}|\xi_j|^{\eta+1},
\end{aligned} \quad (74)$$

where $\bar{\gamma}_{m,m-1} = (2m-3)[\frac{p_{m-1}h_m}{2+\eta} \cdot (\frac{\frac{1}{h_{m-1}}-1}{(2+\eta)\frac{1}{h_{m-1}}})^{\frac{(\frac{1}{h_{m-1}}-1)h_{m-1}}{h_m p_{m-1}}} + 1]\tilde{\gamma}_{m,m-1}$, $\tilde{\gamma}_{m,m-1} = (1 + g_{m-1}^{1-h_{m-1}})\bar{\theta}\sum_{j=1}^{m-1}(1 + g_{j-1}^{p_{m-1}h_m})$ are all smooth positive functions. Considering (73) and (74), we know

$$\begin{aligned}
&|\frac{\partial \lceil \alpha_{m-1}\rceil^{\frac{1}{h_m}}}{\partial x_{m-1}} f_{m-1}| \\
&\leq \bar{f}_0(|\xi_{m-1}| \cdot |\frac{\partial g_{m-1}}{\partial x_{m-1}}| + \frac{g_{m-1}}{h_{m-1}}|\lceil x_{m-1}\rceil^{\frac{1}{h_{m-1}}-1}|) \\
&\quad + |\xi_{m-1}| \cdot |\frac{\partial g_{m-1}}{\partial x_{m-1}}|^{-\eta} \cdot \bar{\theta}\sum_{j=1}^{m-1}|x_j|^{\frac{p_{m-1}h_m}{h_j}}\bar{f}_{m-1} \cdot |\xi_{m-1}|^{\eta+1} \\
&\quad + \frac{g_{m-1}}{h_{m-1}}\bar{f}_{m-1} \cdot \bar{\gamma}_{m,m-1}\sum_{j=1}^{m-1}|\xi_j|^{\eta+1} \\
&\leq \gamma_{m,m-1}(\bar{x}_{m-1})(\bar{f}_0 + \bar{\theta}\sum_{j=1}^{m-1}|\xi_j|^{\eta+1}),
\end{aligned} \quad (75)$$

where

$$\begin{aligned}
\gamma_{m,m-1} \geq\ & \frac{g_{m-1}}{h_{m-1}}(|x_{m-1}|^{\frac{1}{h_{m-1}}-1} + \bar{f}_{m-1}\bar{\gamma}_{m,m-1}) \\
& + \left|\frac{\partial g_{m-1}}{\partial x_{m-1}}\right|\Big(|\xi_{m-1}| + |\xi_{m-1}|^{-\eta}\bar{\theta}\sum_{j=1}^{m-1}|x_j|^{\frac{p_{m-1}h_m}{h_j}}\bar{f}_{m-1}\Big).
\end{aligned} \quad (76)$$

On the basis of (72) and (75), (71) holds. Additionally, following the same procedure in [5], one can deduce that there is a smooth positive function $\varrho_{ki}(\bar{x}_k) > 0$ such that:

$$\left|\frac{\partial(\lceil \alpha_{k-1}^{\frac{1}{h_k}}\rceil)}{\partial x_i}x_{i+1}^{p_i}\beta_i\right| \leq \varrho_{ki}(\bar{x}_k)\sum_{j=1}^{k}|\xi_j|^{\eta+1}, k = 2, \cdots, n. \quad (77)$$

Similar to (27), one has

$$-(2-\eta-h_k)\int_{\alpha_{k-1}}^{x_k}|\lceil s\rceil^{\frac{1}{h_k}} - \lceil \alpha_{k-1}\rceil^{\frac{1}{h_k}}|^{1-h_k-\eta}ds \leq \tilde{c}_k|\xi_k|^{1-\eta}, \quad (78)$$

where $\tilde{c}_k = (2-\eta-h_k)2^{1-h_k}$ is a constant. As a summary, (71) takes the form

$$\begin{aligned}
\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}\dot{x}_i \leq\ & \phi_{k_3}\xi_k^2 + \bar{f}_0^{\frac{2}{\eta+1}} + \frac{1}{4}\sum_{i=1}^{k-2}\xi_i^2 + \frac{1}{5}\xi_{k-1}^2 \\
& + \tilde{\theta}\varpi_{k2}\xi_k^2 + \tilde{c}_k|\xi_k|^{1-\eta}\sum_{i=1}^{k-1}(\gamma_{k_i} + \varrho_{k_i}) \cdot q_i,
\end{aligned} \quad (79)$$

where ϕ_{k3} and ϖ_{k2} are positive smooth functions. According to Assumption 3, there has

$$\begin{aligned}&\tilde{c}_k\,|\xi_k|^{1-\eta}\sum_{i=1}^{k-1}\left(\gamma_{k_i}+\varrho_{k_i}\right)\cdot q_i+|\xi_k|^{2-\eta-h_k}q_k\\&\quad\le\left|\tilde{c}_k\sum_{i=1}^{k-1}(\varrho_{k1}+\gamma_{k1})\right||\xi_k|^{1-\eta}\Theta\\&\quad\quad+|\xi_k|^{2-\eta-h_k}\Theta\le\varepsilon\Theta+Q_k\xi_k^2\Theta,\end{aligned}$$

where $\tilde{Q}_k=\tilde{c}_k\sum_{i=1}^{k-1}(\varrho_{k1}+\gamma_{k1})\,|\xi_k|^{-\eta}+\lceil\xi_k\rceil^{1-\eta-h_k}$, $Q_k=\frac{\tilde{Q}_k^2}{\sqrt{\xi_k^2\tilde{Q}_k^2+\varepsilon^2}}$ are smooth functions. Easily, one has

$$\begin{aligned}&\frac{\partial W_k}{\partial\hat{\Theta}}\dot{\hat{\Theta}}+\left(\tilde{\Theta}-\sum_{i=2}^{k-1}\frac{\partial W_i}{\partial\hat{\Theta}}\right)\left(\mu_{k-1}-\dot{\hat{\Theta}}\right)+Q_k\xi_k^2\tilde{\Theta}\\&\quad=(\tilde{\Theta}-\frac{\partial W_k}{\partial\hat{\Theta}})(\mu_k-\dot{\hat{\Theta}})+\frac{\partial W_k}{\partial\hat{\Theta}}\mu_k,\end{aligned}\tag{80}$$

where $\mu_k=\mu_{k-1}+Q_k\xi_k^2=\sum_{i=1}^{k}Q_i\xi_i^2$, it should be pointed out:

$$\frac{\partial W_k}{\partial\hat{\Theta}}\mu_k=\frac{\partial W_k}{\partial\hat{\Theta}}\sum_{i=1}^{k}Q_i\xi_i^2\le\frac{1}{4}\sum_{i=1}^{k-2}\xi_i^2+\frac{1}{5}\xi_{k-1}^2+\phi_{k4}\xi_k^2,\tag{81}$$

where ϕ_{k4} is a smooth function. On the other side, let $\varpi_k=\varpi_{k1}+\varpi_{k2}$, and taken Lemma 3 and (10) into account, one has

$$\frac{\partial W_k}{\partial\hat{\theta}}\Big(\sum_{i=1}^{k-1}\varpi_i\xi_i^2+\sum_{i=2}^{k}\varpi_k\xi_k^2\Big)\le\frac{1}{4}\sum_{i=1}^{k-2}\xi_i^2+\frac{1}{5}\xi_{k-1}^2+\phi_{k5}\xi_k^2,\tag{82}$$

where ϕ_{k5} is a positive function. Finally, letting $\phi_k=\phi_{k1}+\phi_{k2}+\phi_{k3}+\phi_{k4}+\phi_{k5}$ and conduting simple substitution operation, it is directly deduced from (68), (69), (79), (81) and (82) that the inequality (38) holds. □

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