IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 27 July 2023, accepted 3 September 2023, date of publication 6 September 2023, date of current version 12 September 2023.

Digital Object Identifier 10.1109/ACCESS.2023.3312561

THEORY

Ellipsoidal State Estimation for UAV Real Time Target Tracking

YUE GUONING[1] AND WANG JIANHONG[2]
[1]School of Management Science and Engineering, Henan University of Engineering, Zhengzhou 451191, China
[2]School of Engineering and Sciences, Tecnológico de Monterrey, Monterrey 64849, Mexico

Corresponding author: Wang Jianhong (wangjianhong@nuaa.edu.cn)

This work was supported in part by the Chinese National Science Foundation under Grant 50627865.

ABSTRACT Due to the equivalent relation between UAV target tracking in practice and state estimation in academy, this paper concentrates on state estimation and its application in UAV target tracking. To relax the strict assumption on white noise for classical Kalman filter, we consider the more general bounded noise, being included in an ellipsoid. Then for better understanding our proposed ellipsoidal state estimation, three continuous processes are shown sequently, i.e. ellipsoidal approximation of arithmetic sum, ellipsoidal approximations of intersections between ellipsoid and strip, real time recursive form for generating a sequence of ellipsoids, including each state estimation at each time instant. To combine the theoretic result and engineering application, the detailed simulation example is given to prove the efficiency of our real time ellipsoidal state estimation algorithm.

INDEX TERMS UAV target tracking, state estimation, ellipsoidal algorithm, real time.

I. INTRODUCTION

Unmanned aerial vehicle (UAV) is one kind of machine, equipped with flight control chips. Compared with traditional aircraft, UAV has some better characters or properties, such as small size, strong maneuverability, simple structure, low cost and convenient maintenance. Moreover, UAV does not require onboard personnel, so it can perform tasks through remote control or autonomous flight without causing driver casualties during the task. As UAV can replace manned aircraft in carrying out some dangerous tasks around narrow areas, so it has received significant attention and widespread application for military.

Traditional UAV remotely issues tasks or missions through control terminals and operators,located on the ground station, then UAV only responsible for executing control commands. However in actual combat, due to the limitations of transmission distance, transmission delay, communication technology, and electromagnetic interference in some urgent regions, UAV controlled remotely does not work well, thus making autonomous flight control be important. Generally, autonomous flight technology for UAV will be a major research direction in future. By the way, autonomous slight technology for UAV can be divided into path planning and tracking control. More Specifically, path planning refers to the ability to meet constraints and avoid surrounding obstacles in a given three dimensional space,while reformulating a path curve that connects the original point and terminal position. To deal with path planning, there are various optimal algorithms to plan the optimal path form one expected goal, for example, artificial potential field algorithm, star algorithm etc. On the other hand, tracking control means one process of finding or locating a tracked target, while keeping UAV within the view field and then continuously following the target motion. On this basis, various state estimation and control algorithms exists. Specifically, state estimation includes UAV's own state estimation and also state estimation for other tracking targets. As some important physical variables are reformulated as some corresponding elements for the constructed state vector. More information are yielded from state estimation process about its own and other tracking target, thus it is possible to achieve reasonable and optimal controllable planning by virtue of some advanced control algorithms for example, model predictive control, adaptive control and optimal control etc. As different control algorithms are usually adopted to satisfy different flight performance, therefore our main mission in this paper is state estimation to guarantee UAV track target well. State estimation is an important tool

The associate editor coordinating the review of this manuscript and approving it for publication was Engang Tian.

for identifying, understanding and controlling system. State estimation, also named as filtering, is mainly used to estimate and calculate some unmeasurable variables from the measured data, thus applying these unmeasurable variables for latter research, such as power network, battery parameter and the working state of robot. Generally, that about estimating the motion state of ship, vehicle, and aircraft (UAV) will make their movements more reasonable, so the research on state estimation is the future development trend. As early in 1960s, classical Kalman filter was proposed to estimate the unknown state only for linear discrete system. Then its extended form includes unscented Kalman filter for nonlinear discrete system. But classical Kalman filter and its extended form work well only under the condition that the white noise with Gaussian distribution and the known original state with known mean and variance. To relax above strict requirements for classical Kalman filter and consider more actual situation, research about setmember filter appeared in 1990s for unknown noise. Generally, setmember filter is to predict the range of unknown state into one interval for the unknown but bounded noise, existing in real life. Compared with Kalman filter and setmember filter, the strict requirement on noise is relaxed, i.e. unknown but bounded noise, not the white noise.

State estimation problem has been studied for many years, and there exist lots of references about it for the case of white noise. For example, reference [1] presents an improved extended Kalman filter to identify the unknown state for battery, whose unknown parameters in one state space model are estimated through above introduce classical least squares method [2]. Reference [3] proposes one square root cubature Kalman filter form to approximate the mean value for state variable. To improve the efficiency about state estimation, an improved adaptive cubature Kalman filter is given in reference [4], where the model parameters are estimated from the real time forgetting factor recursive least squares method. Then adaptation is combined with least squares method to analyze the estimation accuracy and stability [5], while achieving online adaptive modifications [6].To reduce the iterative computational complexity, a two stage recursive least squares is yielded to estimate the unknown state [7], and a multi-scale parameter adaptive method based on dual Kalman filter is introduced to identify multiple parameters [8]. Moreover, a variational Bayesian approximation based adaptive dual extended Kalman filter is given in reference [9], where the measurement noise variances are all simultaneously identified within the case of statistical noise [10]. Generally, above description is around Kalman filter by virtue of least squares method and its extended forms.

It is well known that extended noise is described by two kinds, i.e. statistical description and deterministic description. Specifically, statistical description corresponds to the white noise with zero mean and unit variance, then above introduced Kalman filter is benefit for state estimation problem with statistical noise [11]. As white noise is an ideal case, not exist in practice, so this condition is very strict, thus making Kalman filter method not suited for practical engineering. To relax this condition about white noise and propose more advanced filter method for practice, deterministic noise, i.e. unknown but bounded noise is considered, corresponding to bounded amplitude on noise, then it means the considered noise is in one interval or domain. Consider this filter problem for bounded noise, setmember filter method appears to achieve state estimation within bounded noise. During our previous research, we get some new contributions about setmember filter, i.e. bounded state estimation. For example, reference [12] considers one ellipsoidal method for UAVs target tracking and recognition, meaning the bounded noise is included in one known ellipsoid. Setmember filter is applied to identify state of charge estimation for Lithium-ion battery [13], and its iterative multiple form is studied in [14], while combining adjustable scaling parameters. Furthermore, reference [15] considers ellipsoidal approximation into target tracking for UAVs formation,and an improved ellipsoidal optimization algorithm is used in subspace predictive control, achieving the filter problem and controller design together. Generally, our above previous contributions are around state estimation problem with ellipsoidal description on unknown but bounded noise [15].

By far, consider the problem of data driven estimation for state estimation, in case of the number of observations be more exceed this sample size, then the input is persistent excitation, while the identification model satisfies the expected accuracy. From the knowledge of system identification theory, the situation with observed disturbance or noise in the output corresponds to the robust system identification [16], which being also extended to robust optimal control. When using the probabilistic or statistical inference in system identification theory in [17] to measure the asymptotic accuracy about the final identification model. Furthermore in recent years, risk sensitive theory and reinforce learning are all introduced in system theory and advanced control theory [18], i.e. the risk decision and limitations of policies were considered during the whole process of identification and controller design. Then the final identification system or plant is more realistic then classical theoretical result. From these ongoing subjects about applying risk theory, dynamic programming and probabilistic limitation for system identification and control theory, we are thinking to extend graph theory and topology to system identification. More specifically, the second step-model structure choice is related with graph theory, i.e. the chosen model is constructed as one network system, being formulated as graph theory. System identification theory is not only for our considered aircraft system identification, but also for robot system identification in [19], where the detailed identification steps are all similar with each other, and only the considered plants are different. As lots of identification processed are transformed into their corresponding constrain optimization problems, so some existed optimization results can be applied directly, for example, convex optimization [20], scenario optimization [21], and scenario robust control [22], etc. Consider the last step for system identification-model validation, some nice properties are satisfied for the final identification model or designed controller, such as controllability, stochastic chance constraints, robustness and nonlinearity, which are seen in reference [23]. For that nonlinearity in system identification and control, nonlinear identification and nonlinear control are our ongoing work, whose plant and system is nonlinear form, not the simple linear form. Roughly the research on nonlinear identification depends on neural network and other mathematical tools, being used to change the considered nonlinear plant to its approximated linear form, then the existed results about linear identification are all applied directly [24]. In our opinions, this linearized process is not good in practice, as it is the linear form that can not be used to replace the original nonlinear form. Can we find out one direct method to identify or design the nonlinear plant without the above linearized process? This problem is our studying case through topology. Due to the closed relation between system identification theory and UAV system identification [25], generally the step of experiment design concerns determining which physical quantities will be measured, how those quantities will be measured, what the test conditions will be and how the system being studied will be excited. For aircraft system identification, selecting the aircraft configurations and flight conditions, this translates into specifying the instrumentation and data acquisition system, and designing inputs for the maneuvers. The goal of experiment design is to maximize the information content in the data, subject to practical constraints [26], for example, limits on input or output amplitude to ensure that a linear model structure can be used to estimate parameters from the measured data.

Based on above descriptions about state estimation problem and our previous research on ellipsoidal state estimation, this new paper continues to derive more innovative findings on ellipsoidal state estimation and also apply them into UAV target tracking with one real time form. For the sake of completeness, firstly after reviewing UAV target tracking in some interesting backgrounds, some important physical variables, such as position, velocity and acceleration etc, are combined together to construct one state vector. Then our mission is changed to use some measurements or observed output data to estimate this unknown state vector, while considering the external noise be in one ellipsoid, i.e. corresponding to above defined setmember filter or ellipsoidal state estimation. Secondly, our contributions about ellipsoidal state estimation are shown to generate one sequence of ellipsoids, which terminal ellipsoid corresponds to the final state estimation. Furthermore, three aspects or properties are derived through our own mathematical derivations, for example, ellipsoidal approximation of arithmetic sum, ellipsoidal approximation of intersections between ellipsoid and strip, real time ellipsoid algorithm etc. Thirdly, to do the combination about theoretical analysis and engineering application, our theoretical results on ellipsoidal state estimation are applied into UAV target tracking to estimate some physical variables, thus giving more information to UAV in real time way.

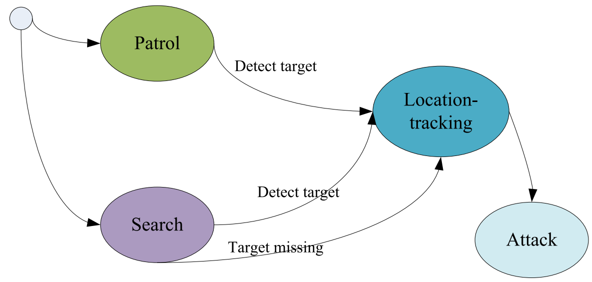

FIGURE 1. Main mission for UAV.

This paper is organized as follows. In section II, preliminaries about UAV target tracking is reviewed, and the detailed connection between UAV target tracking and state estimation is explained under the circumstance of bounded external noise. Section III gives our new theoretical contributions on ellipsoidal state estimation form three different aspects, corresponding to the setmember filter in section II. Section IV is to prove the efficiency of our new theoretical results through applying into one practical example. Section V ends the paper with a final conclusion and proposes the next subject.

II. UAV TARGET TRACKING

The main missions for UAV include intelligence reconnaissance and surveillance, target attack, communication relay, and electronic disturbance etc. UAV mission process can be divided into patrol, search, location-tracking and attack. The transformation relationships for each above phase are shown in following Figure 1, where the location-tracking phase occurs after the target is detected, then more accurate and continuous information from the tracking target are provided.

From above Figure 1, UAV target tracking problem is defined as follows. Assume one UAV flying in sky, and one target moving on ground. UAV wants to acquire more accurate information about target in order to attack it. Roughly UAV applies its radar or other physical devices to collect lots of measurements or data, corrupted with external noises. Then the flight controller starts to deal with these data to get the accurate position, velocity, and acceleration about the moving target. The above principle is plotted in following Figure 2.

From Figure 2, to attack that moving target on ground, UAV flying in sky needs to obtain the position, velocity and acceleration about the moving target. After combining above three physical variables into one state vector, constructing within the framework of state space model, then UAV target tracking problem is transformed into our mentioned state estimation problem.

III. ELLIPSOIDAL STATE ESTIMATION

A. PRELIMINARY

Radar and cameras, installed ahead of UAV, scan the moving target to generate one sequence of images, so we extract some

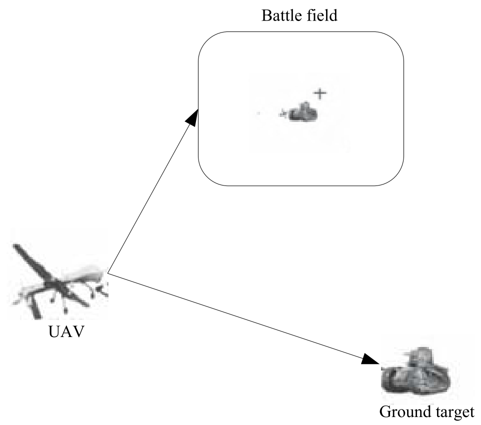

FIGURE 2. **UAV target tracking.**

interesting physical variables from images through some estimation of filter methods. Before to de estimation of filter problem, one mathematical model about that moving target on ground is constructed based on model control theory as one state space form. Date driven strategies are benefit for modeling the moving target into one state space form in [27], where we have directly.

$$\begin{cases} x_{k+1} = f_k(x_k) \\ y_k = h_k(x_k), \quad k = 1, \cdots, n \end{cases} \tag{1}$$

where in above equation (1), $x_k \in R^{n_x}$ and $y_k \in R^{n_y}$ are state vector and measurement output at time instant k respectively. Two maps $f_k : R^{n_x} \to R^{n_x}; h_k : R^{n_x} \to R^{n_y}$ correspond to two unknown functions, n is the number of state variable.

State vector x_k includes some physical variables for that moving target, i.e.

$$x_k = [positon, velocity, acceleration]^T \tag{2}$$

so to get above three physical variables at time instant k, we want to estimate that state vector x_k from only the measurement output y_k, corresponding to blind state estimation,due to noe input exists.

For convergence, one linearized equation of equation (1) is yielded only through the linearized process, i.e.

$$\begin{cases} x_{k+1} = Ax_k + w_k \\ y_k = Cx_k + v_k, \quad k = 1, \cdots, n \end{cases} \tag{3}$$

where

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(0,0,\cdots,0)};$$

$$C = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_n}{\partial x_1} & \frac{\partial h_n}{\partial x_2} & \cdots & \frac{\partial h_n}{\partial x_n} \end{bmatrix}_{(0,0,\cdots,0)} \tag{4}$$

$\{w_k, v_k\}$ are two external noises. If $\{w_k, v_k\}$ are all assumed to be white noises with zero mean and unit variance, then classical Kalman filter work well to estimate state vector x_k only from measurement output y_k.

Comment: During above linearized process, we assume the equilibrium state be the constant vector $(0, 0, \cdots, 0)$, satisfying $f(0, 0, \cdots, 0) = 0$. Moreover the above linearized process will generate one linear time invariant system, i.e. matrices A and C are all constant matrices through substituting the equilibrium state into matrix operation.

Observing that state space form (1) again, as no control input u_k exist, so state space form (1) ia an autonomous system. Applying the basic mathematical analysis, we have

$$\begin{aligned} f(x_1, x_2, \cdots, x_n) &= \begin{bmatrix} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) \end{bmatrix} \\ &\approx \begin{bmatrix} f_1(0, 0, \cdots, 0) \\ f_2(0, 0, \cdots, 0) \\ \vdots \\ f_n(0, 0, \cdots, 0) \end{bmatrix} + A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &\quad + \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \frac{\partial^2 f(x_1, x_2, \cdots, x_n)}{\partial x^2} \\ &\quad \times \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &\quad + high \ \ order \ \ term \\ x &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \end{aligned}$$

Using the property of the equilibrium state, i.e.

$f_k(0, 0, \cdots, 0) = 0, k = 1, 2, \cdots n$ and denoting that high order term as $v = [v_1, v_2, \cdots, v_n]$, the condition about guaranteeing the linearization error be small as possible it the following case.

$$\|\frac{\partial^2 f(x_1, x_2, \cdots, x_n)}{\partial x^2}\| \leq \epsilon$$

where ϵ is one small positive value, for example, $\epsilon = 0.1$, and $\|.\|$ is one norm.

Based on above inequality, then above third term is neglected to generate our considered autonomous linearized process (3).

B. ELLIPSOIDAL APPROXIMATION OF ARITHMETIC SUM

Observing equation (3) again, the initial state vector x_0 and initial external noise w_0 are assumed to be in two different ellipsoids, i.e.

$$x_0 \in X_0; w_0 \in W_0 \tag{5}$$

where above two ellipsoids are defined as follows

$$X_0 = \{x_0 + A_0 u | u^T u \leq 1\}$$

$$W_0 = \{w_0 + A_1 u | u^T u \leq 1\} \tag{6}$$

Then

$$X_0 + W_0 = (x_0 + w_0) + (A_0 u + A_1 u) \tag{7}$$

where x_0 and w_0 are two centers of ellipsoids X_0 and W_0, u is one virtual variable. Given two ellipsoids X_0 and W_0, find the best inner ellipsoid approximations of their arithmetic sum $X_0 + W_0$.

$$\hat{X}_0 = \{x_0 + w_0, x_0 \in X_0; w_0 \in W_0\} \tag{8}$$

Consider this ellipsoidal approximation of arithmetic sum, we need to construct one new ellipsoid to approximate two ellipsoidal arithmetic sum, i.e. $X_0 + W_0$. From our published book [26], we directly have the following results, reformulating as two Theorems.

Theorem 1: An ellipsoid $E[Z_0] = \{x = Z_0 u | u^T u \leq 1\}$ is contained in the sum $X_0 + W_0$ of the ellipsoid X_0 and W_0, if and only if one has

$$\forall x : \|Z_0^T x\|_2 \leq \|A_0^T x\|_2 + \|A_1^T x\|_2 \tag{9}$$

Theorem 2: Let A_0 and A_1 be two nonsingular matrices with approximated dimension, and let $X_0 = \{A_0 u | u^T u \leq 1\}$, and $W_0 = \{A_1 u | u^T u \leq 1\}$ be two associated ellipsoids. Set $\Delta = \{\lambda \in R_+ | \lambda_1 + \lambda_2 = 1\}$, then

(1) Whenever $\lambda \in \Delta$ and Z_0 is such that

$$Z_0 Z_0^T \geq F(\lambda) = \lambda_1 A_0 A_0^T + \lambda_2 A_1 A_1^T \tag{10}$$

the ellipsoid $E[Z_0] = \{x = Z_0 u | u^T u \leq 1\}$ contains $X_0 + W_0$.

(2) Whenever Z_0 is such that

$$AA^T \leq F(\lambda) = \lambda_1 A_0 A_0^T + \lambda_2 A_1 A_1^T \tag{11}$$

the ellipsoid $E[Z_0] = \{x = Z_0 u | u^T u \leq 1\}$ contained in $X_0 + W_0$, and vice versa.

The detailed proofs about two Theorems can be referred to our previous contribution [11]. The goals of about two Theorems are to construct a new ellipsoid $\hat{X}_0$, approximating the arithmetic sum $X_0 + W_0$, i.e.

$$\hat{X}_0 = \{x_0 + w_0, x_0 \in X_0; w_0 \in W_0\} = \{X_0 + W_0\} \tag{12}$$

where two ellipsoids X_0 and W_0 are given in priori, used to include the initial state vector x_0 and initial external noise w_0 at the initial time instant.

C. ELLIPSOIDAL APPROXIMATIONS OF INTERSECTIONS BETWEEN ELLIPSOID AND STRIP

After given two initial ellipsoids X_0 and W_0, we apply above two Theorems to construct a new ellipsoid $\hat{X}_0$, while considering that state equation in equation (3) as one arithmetic sum operation. As section III-B only considers that state equation $x_{k+1} = Ax_k + w_k$,but output equation $y_k = h_k(x_k)$ exists still, not neglecting it, or information is incomplete. Before to consider that output equation, one definition is needed.

Definition 1 (Information State Set [11]): Given the observed output $y_k, k = 1, 2 \cdots N$ at time instant k, where N is the total number of observed data. information state set I_k is a set of all feasible states, being consistent with the observed equation in equation (3) and one upper bound at time instant k, i.e.

$$\begin{aligned} I_k &= \{x_k : -\sigma \leq y_k - Cx_k \leq \sigma\} \\ &= \{x_k : |y_k - Cx_k| \leq \sigma\} \\ &= \{x_k : |Cx_k - y_k| \leq \sigma\} \end{aligned} \tag{13}$$

where σ is one bound for that observed noise v_k, i.e.

$$|v_k| \leq \sigma, \quad \forall k \tag{14}$$

From equation (13), we see information state set I_k at time instant k is a strip. Combining state equation and observed output equation together, we need to compute one intersection between ellipsoid and strip, i.e.

$$x_k \in I_k \cap X_k \tag{15}$$

where X_k denotes feasible state set at time instant k, expressed as an ellipsoid. For convergence, feasible state set X_k and information state set I_k are denoted as follows

$$\begin{aligned} X_k &= \{x_k = c + Bu | u^T u \leq 1\}; (DetB \neq 0); \\ I_k &= \{x_k : |Cx_k - y_k| \leq \sigma\} \\ &= \{x_k : -\sigma \leq y_k - Cx_k \leq \sigma\} \end{aligned} \tag{16}$$

where observing output y_k is known or collected in priori and state x_k must satisfy $x_k \in X_k$ and $x_k \in I_k$ simultaneously, i.e. equation (15).

As it holds that

$$I_k = \{x_k : |Cx_k - y_k| \leq \sigma\} \tag{17}$$

substituting $x_k = c + Bu$ into $Cx_k - y_k \leq \sigma$ and considering $u^T u \leq 1$ together, we have

$$u^T u \leq 1 \rightarrow C(c + Bu) \leq y_k + \sigma \tag{18}$$

i.e.

$$u^T u \leq 1 \rightarrow Cc + CBu - y_k - \sigma \leq 0 \tag{19}$$

Then there exists λ_1 such that

$$\begin{aligned} & y_k + \sigma - CBu - Cc - \lambda_1 [1 - u^T u] \geq 0 \\ & y_k + \sigma - CBu - Cc - \lambda_1 + \lambda_1 u^T u \geq 0 \end{aligned} \tag{20}$$

Rewriting above equation (20) into one linear matrix inequality as

$$\begin{bmatrix} u & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & -\frac{1}{2}CB \\ -\frac{1}{2}CB & y_k + \sigma - Cc - \lambda_1 \end{bmatrix} \begin{bmatrix} u \\ 1 \end{bmatrix} \geq 0 \tag{21}$$

Similarly substituting $x_k = c + Bu$ into $Cx_k - y_k \geq -\sigma$ and considering $u^T u \leq 1$ together, then it holds that

$$u^T u \leq 1 \rightarrow C(c + Bu) - y_k + \sigma \geq 0 \tag{22}$$

i.e. there exists λ_2 such that

$$\begin{aligned} & -y_k + \sigma + CBu + Cc - \lambda_2 [1 - u^T u] \geq 0 \\ & -y_k + \sigma + CBu + Cc - \lambda_2 + \lambda_2 u^T u \geq 0 \end{aligned} \tag{23}$$

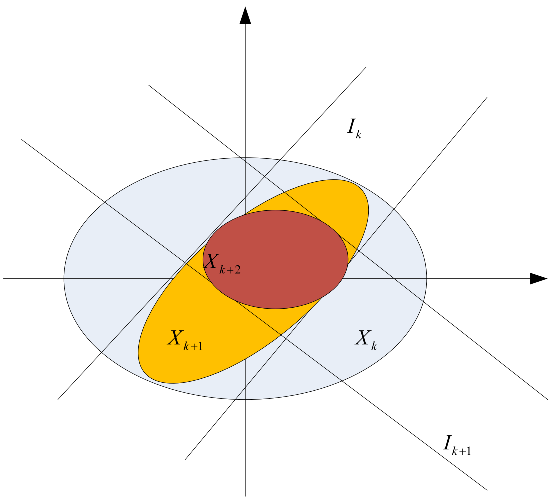

FIGURE 3. **Ellipsoidal approximation of intersection.**

Similarly one linear matrix inequality is yielded.

$$\begin{bmatrix} u & 1 \end{bmatrix}\begin{bmatrix} \lambda_2 & \frac{1}{2}CB \\ \frac{1}{2}CB & -y_k-\sigma+Cc+\lambda_2 \end{bmatrix}\begin{bmatrix} u \\ 1 \end{bmatrix} \geq 0 \tag{24}$$

Combining linear matrix inequalities (21) and (24), the problem about ellipsoidal approximations of intersections between ellipsoid and strip is solvable by satisfying the following Theorem 3.

Theorem 3: Given one ellipsoid $X_k = \{x_k = c+Bu|u^T u \leq 1\}$; ($DetB \neq 0$) and a strip $I_k = \{x_k : |Cx_k - y_k| \leq \sigma\}$, their intersection can be approximated by one new small ellipsoid on condition that there exist two scale values λ_1 and λ_2 such that two other linear matrix inequalities hold.

$$\begin{bmatrix} \lambda_1 & -\frac{1}{2}CB \\ -\frac{1}{2}CB & y_k+\sigma-Cc-\lambda_1 \end{bmatrix} \geq 0;$$
$$\begin{bmatrix} \lambda_2 & \frac{1}{2}CB \\ \frac{1}{2}CB & -y_k-\sigma+Cc+\lambda_2 \end{bmatrix} \geq 0 \tag{25}$$

Theorem 3 gives one condition about using one small ellipsoid to replace the intersection between ellipsoid and strip.

D. REAL TIME ELLIPSOIDAL ALGORITHM

Theorem 3 gives one condition about whether there exists a new small ellipsoid, using to include the intersection between one ellipsoid and a strip. Here we give the detailed form of this new small ellipsoid.

Rewriting equation (16) again, we need to give a new small ellipsoid X_{k+1} such that.

$$\begin{aligned} X_{k+1} &= X_k \cap I_k; \\ X_k &= \{x_k = c+Bu|u^T u \leq 1\}; \\ I_k &= \{x_k : -\sigma \leq y_k - Cx_k \leq \sigma\} \end{aligned} \tag{26}$$

then construct a new small ellipsoid such that

$$\begin{aligned} X_{k+1} &= X_k \cap I_k; \\ X_{k+1} &= \{x_{k+1} = c^+ + B^+ u|u^T u \leq 1\} \end{aligned} \tag{27}$$

where

$$\begin{aligned} c^+ &= c - \frac{1}{n_x}Bp; \\ B^+ &= \frac{n_x}{\sqrt{n_x^2-1}}B + (\frac{n_x}{n_x+1} - \frac{n_x}{\sqrt{n_x^2-1}})(Bp)p^T; \\ p &= \frac{B^T C}{\sqrt{C^T BB^T C}} \end{aligned} \tag{28}$$

where n_x is the dimension of state vector.

If $n_x = 1$,then the new ellipsoid is deemed as follows.

$$\begin{aligned} X_{k+1} &= \{x_{k+1} = c^+ + B^+ u|u^T u \leq 1\}; \\ c^+ &= c - \frac{1}{2}\frac{B\sigma}{|B\sigma|}; \\ B^+ &= \frac{1}{2}B \end{aligned} \tag{29}$$

Above construction process about new small ellipsoid is plotted in following Figure 3.

Finally, combining all of ellipsoidal approximations, our proposed ellipsoidal state estimation algorithm is reformulated as follows.

Ellipsoidal state estimation algorithm

Step 1: Collect observed output sequence $\{y_k\}_{k=0}^N$.

Step 2: Given two initial ellipsoids X_0 and W_0, including initial state vector x_0 and initial external noise w_0.

Step 3: Construct one ellipsoid $\hat{X}_0$ to approximate the arithmetic sum $X_0 + W_0$,i.e.

$$\hat{X}_0 = X_0 + W_0$$

Step 4: Build a strip that bounds the considered state set at initial time instant

$$I_0 = \{x_0 : -\sigma \leq y_0 - Cx_0 \leq \sigma\}$$

Step 5: Testify whether th0se two linear matrix inequalities hold. If yes, then construct a new small ellipsoid X_1 to satisfy

$$X_1 = \hat{X}_0 \cap I_0$$

or return back to step 4 and build a new strip.

Step 6: Compute ellipsoidal arithmetic sum

$$\hat{X}_1 = X_1 + W_1$$

Step 7: Build a strip at time instant 1

$$I_1 = \{x_1 : -\sigma \leq y_1 - Cx_1 \leq \sigma\}$$

Step 8: Construct ellipsoid intersection

$$X_2 = \hat{X}_1 \cap I_1$$

...

Step N: Construct ellipsoid intersection

$$X_N = \hat{X}_{N-1} \cap I_{N-1} \tag{30}$$

Terminate above repeated steps, then we have

$$x_0 \in X_0; x_1 \in X_1; \cdots, x_{N-1} \in X_{N-1}; x_N \in X_N$$

An easy way to choose the final state estimate x_N is to choose the center of the *Nth* ellipsoid X_N.

Equation (30) corresponds to the real time strategy due to its recursive form, so above algorithm is names as real time ellipsoidal state estimation, plotting in Figure 4.

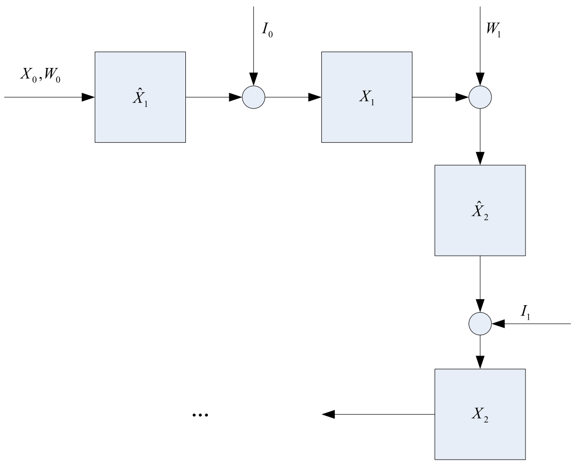

FIGURE 4. Recursive ellipsoidal state estimation.

IV. SIMULATION EXAMPLE

During this simulation example, one UAV is placed on top of one turn table, rotating around 360 angle. A camera is installed in front of this turn table, used to track one moving vehicle. Figure 5 shows our practical experiment setup, where that moving vehicle is the tracking target. Let that vehicle moves with one constant velocity, according to Newton law, we have

$$\begin{aligned} s &= s_0 + vt + \frac{1}{2}at^2 \\ v &= v_0 + at \end{aligned} \tag{31}$$

where s and s_0 denote the position and initial position, v and v_0 are velocity and initial velocity, t is the time instant.

Linearized equation (31) to get a linear discrete state space model for that moving vehicle, i.e.

$$x_{k+1} = \begin{bmatrix} 1 & \delta t \\ 0 & 1 \end{bmatrix} x_k + w_k;$$

(a) UAV platform

(b) Moving vehicle

FIGURE 5. Experiment setup.

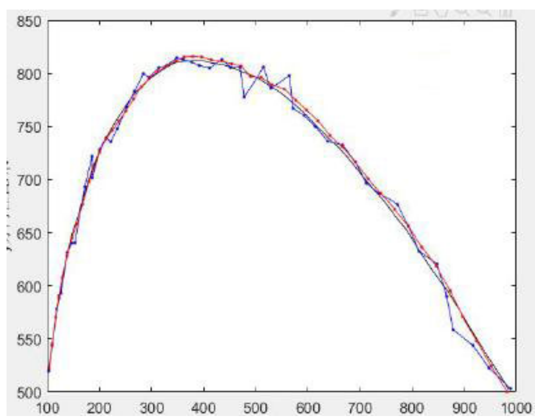

FIGURE 6. Comparison with positions.

$$\begin{aligned} y_k &= \begin{bmatrix} 1 & 1 \end{bmatrix} x_k + v_k; \\ s_k &= \begin{bmatrix} s & v \end{bmatrix}^T; \\ A &= \begin{bmatrix} 1 & \delta t \\ 0 & 1 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix} \end{aligned} \tag{32}$$

where δt is sampled time, state vector is constituted of position and velocity of that moving vehicle.

When to start our proposed ellipsoidal state estimation, i.e. identify or estimate the position and velocity of that moving vehicle. Simulation results are shown in Figure 6, where black line is the real trajectory, blue and red line correspond to moving trajectories for observed trajectory and estimated

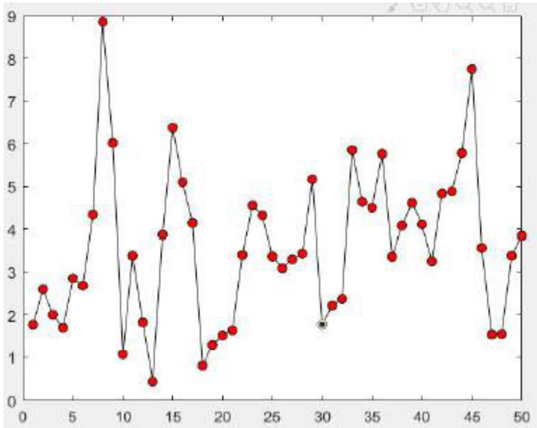

FIGURE 7. Comparison with error curves.

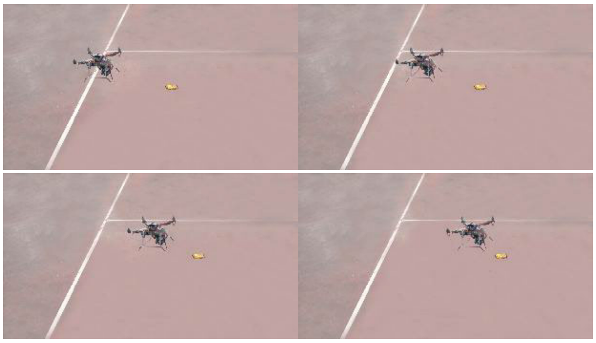

FIGURE 8. Practical scenario.

FIGURE 9. Detailed tracking process.

trajectory respectively. From Figure 6, we see error exists for the observed trajectory, and the estimated trajectory is closed to the real trajectory. Furthermore, the detailed error curves are also shown in Figure 7, where error lies within ten meters.

To let the readers understand our saying more easily, we do a true practical test in Figure 8, where a small UAV is flying in sky, and a yellow vehicle is moving on ground at the same time. The goal of that flying UAV is to track that moving vehicle, i.e. UAV flies while following vehicle with time increases. As the vehicle moves continuously, the camera, installed in front of UAV must grasp that moving vehicle, meaning the moving vehicle lies in the center of that red rectangle,plotting in Figure 9.

V. CONCLUSION

On the basis of our previous contributions about state estimation problem, we find UAV target tracking corresponds to one similar state estimation problem. To relax the strict assumption on white noise for classical Kalman filter,this paper turns to study the other setmember filter for one special case of external noise with unknown but bounded property. We use ellipsoid to reformulate the bounded initial state vector and initial external noise, then to get a sequence of ellipsoids, including each state vector at each time instant. For convenience to show our contributions, we divide into three main processes, i.e. ellipsoidal approximation of arithmetic sum, ellipsoidal approximations of intersections between ellipsoid and strip, real time recursive form. Generally, this paper proposes ellipsoidal algorithm into state estimation, next subject will be concentrated on ellipsoidal algorithm for controller design.

REFERENCES

[1] X. Zhang, S. Grammatico, G. Schildbach, P. Goulart, and J. Lygeros, "On the sample size of random convex programs with structured dependence on the uncertainty," *Automatica*, vol. 60, no. 10, pp. 182–188, 2015.

[2] X. Zhang, M. Kamgarpour, A. Georghiou, P. Goulart, and J. Lygeros, "Robust optimal control with adjustable uncertainty sets," *Automatica*, vol. 75, pp. 249–259, Jan. 2017.

[3] T. Alamo, R. Tempo, and E. F. Camacho, "Randomized strategies for probabilistic solutions of uncertain feasibility and optimization problems," *IEEE Trans. Autom. Control*, vol. 54, no. 11, pp. 2545–2559, Nov. 2009.

[4] D. P. Bertsekas, "Affine monotonic and risk-sensitive models in dynamic programming," *IEEE Trans. Autom. Control*, vol. 64, no. 8, pp. 3117–3128, Aug. 2019.

[5] D. Bertsimas and V. Goyal, "On the power and limitations of affine policies in two-stage adaptive optimization," *Math. Program.*, vol. 134, no. 2, pp. 491–531, Sep. 2012.

[6] L. Blackmore, M. Ono, and B. C. Williams, "Chance-constrained optimal path planning with obstacles," *IEEE Trans. Robot.*, vol. 27, no. 6, pp. 1080–1094, Dec. 2011.

[7] G. C. Calafiore, "Random convex programs," *SIAM J. Optim.*, vol. 20, no. 6, pp. 3029–3427, 2010.

[8] M. C. Campi and S. Garatti, "Wait and judge scenario optimization," *Automatica*, vol. 50, no. 12, pp. 3019–3029, 2016.

[9] M. C. Campi, S. Garatti, and M. Prandini, "The scenario approach for systems and control design," *Annu. Rev. Control*, vol. 33, no. 2, pp. 149–157, 2009.

[10] D. S. Callaway and I. A. Hiskens, "Achieving controllability of electric loads," *Proc. IEEE*, vol. 99, no. 1, pp. 184–199, Jan. 2011.

[11] D. Liushuan, "Ellipsoidal method for UAVs target tracking and recognition," *IEEE Access*, vol. 10, pp. 113935–113944, 2022.

[12] W. Jianhong and R. A. Ramirez-Mendoza, "Applying set membership strategy in state of charge estimation for lithium-ion battery," *WSEAS Trans. Syst.*, vol. 21, no. 1, pp. 1–19, Jan. 2022.

[13] H. Jianwang, R. A. Ramirez-Mendoza, and J. D. J. Lozoya-Santos, "Adjustable scaling parameters for state of charge estimation for lithium-ion batteries using iterative multiple UKFs," *Math. Problems Eng.*, vol. 2020, pp. 1–14, Apr. 2020.

[14] W. Jianhong, M. He, and R. A. Ramirez-Mendoza, "Application of ellipsoidal approximation into target tracking for multi UAVs formation cooperative detection," *Int. J. Model., Identificat. Control*, vol. 35, no. 2, pp. 108–119, 2020.

[15] W. Jianhong, "An improved ellipsoid optimization algorithm in subspace predictive control," *Aircr. Eng. Aerosp. Technol.*, vol. 93, no. 5, pp. 870–879, Aug. 2021.

[16] Y. Zhou and D. Chen, "Optimized state-dependent switching law design for a class of switched nonlinear systems with two unstable subsystems," *Appl. Math. Comput.*, vol. 397, no. 2, May 2021, Art. no. 125872.

[17] J. Wang, M. Feckan, and Y. Guan, "Local and global analysis for discontinuous atmospheric Ekman equations," *J. Dyn. Differ. Equc.*, vol. 36, no. 10, pp. 663–667, 2023.

[18] S. Li and W. Wang, "Rigorous justification of the uniaxial limit from the Qian-Sheng inertial Q-tensor theory to the Ericksen–Leslie theory," *SIAM J. Math. Anal.*, vol. 52, no. 5, pp. 4421–4468, Jan. 2020.

[19] D. Peng, N. Xiu, and J. Yu, ''Global optimality condition and fixed point continuation algorithm for non-Lipschitz ℓ_p regularized matrix minimization,'' *Sci. China Math.*, vol. 61, no. 6, pp. 1139–1152, Jun. 2018.
[20] X. M. Wang, ''Subgradient algorithms on Riemannian manifolds of lower bounded curvatures,'' *Optimization*, vol. 67, no. 1, pp. 179–194, Jan. 2018.
[21] S. Xiang, S. Xia, and Y. Yang, ''A direct proof of Shapley-Sperners lemma based on the KKMS lemma,'' *J. Math. Anal. Appl.*, vol. 496, no. 6, Apr. 2021, Art. no. 124789.
[22] H. Liu, J. Ma, and C. Peng, ''Shrinkage estimation for identification of linear components in composite quantile additive models,'' *Commun. Statist.-Simul. Comput.*, vol. 49, no. 7, pp. 2678–2692, 2022.
[23] H. Liu and X. Xia, ''Estimation and empirical likelihood for single-index multiplicative models,'' *J. Stat. Planning Inference*, vol. 193, no. 1, pp. 70–88, Feb. 2018.
[24] H. Liu, H. Yang, and X. Xia, ''Robust estimation and variable selection in censored partially linear additive models,'' *J. Korean Stat. Soc.*, vol. 46, no. 1, pp. 88–103, Mar. 2017.
[25] H. Liu and H. Yang, ''Estimation and variable selection in single-index composite quantile regression,'' *Commun. Statist. Simul. Comput.*, vol. 46, no. 9, pp. 7022–7039, Oct. 2017.
[26] D. Luo, J. Wang, and D. Shen, ''Learning formation control for fractional-order multi-agent systems,'' *Math. Methods Appl. Sci.*, vol. 41, no. 10, pp. 5003–5014, 2018.
[27] W. Jianhong, R. A. Ramirez-Mendoza, and R. Morales-Menendez, *Data Driven Strategies: Theory and Application*. Boca Raton, FL, USA: CRC Press, 2023.

YUE GUONING received the M.B.A. degree from Zhengzhou University, in 2008. He is currently a Faculty Member at the School of Management Engineering, Henan University of Engineering. His current research interests include information management and information systems.

WANG JIANHONG received the Diploma degree in engineering cybernetics from Yunnan University, China, in 2007, and the Dr.Sc. degree from the College of Automation Engineering, Nanjing University of Aeronautics and Astronautics, China, in 2011. From 2013 to 2015, he was a Postdoctoral Fellow with the Politecnico di Milano. From 2016 to 2018, he was a Professor at the University of Seville. Since 2019, he has been a Professor with Tecnológico de Monterrey. His current research interests include real-time and distributed control and optimization and system identification.