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RESEARCH ARTICLE

Novel Linear Diophantine Fuzzy Information Measures Based Decision Making Approach Using Extended VIKOR Method

ALİ AYDOĞDU

Department of Basic Sciences, Air NCO Vocational HE School, Turkish National Defence University, 65897 İzmir, Turkey

e-mail: aaydogdu@msu.edu.tr

ABSTRACT One of the purposes of fuzzy set theory is to overcome the uncertainties in the problems of multi-criteria decision-making (MCDM) via membership functions. But the fuzzy set theory has own limitations. To remove the limitations on membership functions of fuzzy sets, such as intuitionistic fuzzy, Pythagorean fuzzy, q-rung orthopair fuzzy sets, the linear Diophantine fuzzy (LDF) concept is defined with the reference parameters. The benefit of this approach is that it is more flexible and efficient at handling uncertain data than other fuzzy sets. In this study, firstly, new information measures (distance, similarity, entropy) have been proposed for linear Diophantine fuzzy sets and their properties are studied. Secondly, the LDF-VIKOR method is given in as full detail. In the proposed method, the weights of criteria are calculated using the entropy-based objective weighting method. Thirdly, the effect of entropy measures in the LDF-VIKOR method on the best and compromise solutions is examined. Finally, an application of LDF-VIKOR on a healthcare management decision problem is given to show applicability of proposed method.

INDEX TERMS Distance measure, entropy measure, linear Diophantine fuzzy sets, similarity measure, VIKOR.

I. INTRODUCTION

In real-life decision-making problems, vague and uncertain data has become a major issue. To overcome the difficulties of the complexities and uncertainties of real-world problems, Zadeh [1] developed the concept of fuzzy set (FS). The fact that the fuzzy set definition did not consider the negative opinions of the experts caused some limitations in solving real-life problems. Then, Atanassov [2] proposed intuitionistic fuzzy set (IFS) as an extension of FS to handle this problem by using the condition that the sum of membership and non-membership degrees between 0 and 1. In some real-life problems, the sum of the degree of membership and non-membership, which is determined by experts, may be greater than one (e.g., $0.7 + 0.9 > 1$). Yager [3] introduced Pythagorean fuzzy set (PFS) to overcome these problems. PFS introduced the concept of membership and non-member degrees with the sum of the squares of them is equal or less than 1. In some cases, the sum of squares of these grades may also be larger than 1 (e.g., $0.7^2 + 0.9^2 > 1$). These situations cannot be eradicated by PFS theory. Then, Yager [4] proposed q-rung orthopair fuzzy set (q-ROFS) with the condition that the sum of q^{th} (where $q \geq 1$) power of membership degree and q^{th} power of non-membership degrees is equal or less than 1. This means that if $q = 1$, then q-ROFS is reduced to IFS and if $q = 2$, then q-ROFS is induced to PFS. In q-ROFS, the fact that the value of the q parameter can be determined by the decision makers (DMs) provides freedom in determining the values of the membership and non-membership degrees. Sometimes vagueness and uncertainties in real-life decision-making problems cannot be overcome by using q-ROF sets. For example, both membership and non-membership grades are equal to 1, it is obtained that $1^q + 1^q > 1$, which opposes the restriction of q-ROFS. In this case, the concept of q-ROFS restrains the judgment of DMs. This means that the

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concept of q-ROFS has own limitations. Riaz and Hashmi [5] presented the concept of linear Diophantine fuzzy set (LDFS) by adding reference parameters in the definition of FS to remove the limitations on membership and non-membership degrees of FSs. Thanks to its reference parameters, LDFS is more flexible and efficient in handling uncertain data than other fuzzy sets. Also, the addition of reference parameters made the valuation area of theoretical knowledge more inclusive.

Information measures (distance, similarity, entropy) have general application fields such as medical diagnosis, clustering, pattern recognition, etc. Distance measure is used to distinguish between two fuzzy sets. Similarity measure is a very useful tool for overcoming vague and uncertain data to reach the final decision by determining the degree of similarity between fuzzy concepts. Entropy measure determines the fuzziness degree of fuzzy sets besides being used for attribute weighting in the applications of MCDM problems. Over the past decades, distance, similarity, and entropy measures have received growing interest as a significant quantitative measures of uncertain information from many researchers.

Atanassov [6] first introduced distance measure for IFS. Burillo and Bustince [7] gave distance measures between IFSs and entropy measures for IFS. Szmidt and Kacprzyk [8] proposed new definitions of distance measures for IFSs. Grzegorzewski [9] defined new distance measures for IFSs based on the Hausdorff metric. Wang and Xin [10] introduced several intuitionistic fuzzy distance measures. Hung and Yang [11] developed several fuzzy similarity measures to IFSs and gave two new similarity measures for IFSs. Iancu [12] presented similarity measures for IFSs based on min-max operators. Gohain et al. [13] defined a novel distance measure based on the concept of the information held by the IFSs and applied it to pattern recognition, medical diagnosis, and the decision-making problem of face mask selection for the novel COVID-19 virus. Joshi and Kumar [14] proposed α-ordered entropy measure for IFS and gave an application of entropy measure in multi attribute decision-making (MADM) problem. Thao [15] obtained an entropy measure of IFS based on divergence measures and used the proposed entropy to determine criteria weight to solve the MCDM problem.

Zhang and Xu [16] introduced distance measure for PFSs and extended the TOPSIS method to PFS. Peng et al. [17] proposed distance, similarity, and entropy measures for PFSs. Also, they gave the relationship with each other and its application of pattern recognition, clustering analysis, and medical diagnosis. Biswas and Sarkar [18] defined entropy measure based on distance measure for PFSs and used proposed entropy measure for determining the weights of criteria. Hussian and Yang [19] introduced distance and similarity measures for PFSs based on the Hausdorff metric and gave an application to PF-TOPSIS with proposed measures. Ejegwa [20] presented axiomatic definition of distance and similarity measure for PFSs and extended some distance and similarity measures in IFS to PFS. Sarkar and Biswas [21] proposed Pythagorean fuzzy distance and entropy measures and designed an entropy weight model to determine criteria weight for PF-MCDM problems. Wan et al. [22] presented entropy measure of PFS to obtain attribute weights. Yang and Hussain [23] defined Pythagorean fuzzy entropy measures based on probabilistic-type, distance, and min–max operator. Thao and Smarandache [24] introduced entropy measure for PFS extension of IFS entropy measure and calculated weights by using proposed entropy measure in COPRAS method. Xu et al. [25] gave new definition of entropy measure for PFS and used entropy weight formula to solve MCDM problem.

Peng and Liu [26] proposed q-rung orthopair fuzzy information measures (distance, similarity, entropy, and inclusion measure) and gave the relationships between these measures. Also, they applied the similarity measure to pattern recognition, clustering analysis, and medical diagnosis. Peng and Dai [27] gave the formulae of distance and similarity measures and determined two algorithms for solving q-ROF decision-making problem with CODAS and multi-parametric similarity measures. Pinar and Boran [28] achieved a new distance measure for q-ROFSs and used it in q-ROF TOPSIS and q-ROF ELECTRE. Verma [29] introduced an α-ordered entropy measure for q-ROFS and determined weights of attributes based on the proposed entropy measure for solving MCDM problem. Liu et al. [30] defined entropy measures for q-ROFS to obtain the attribute weights.

In LDFS, Mohammad et al. [31] and Gül and Aydoğdu [32] introduced Euclidean and Hamming distance measures, simultaneously and unknowingly. Also, Mohammad et al. [31] gave the generalization of Euclidean and Hamming distance measures and several similarity measures for LDFS and applied proposed measures to medical diagnosis problem. Gül and Aydoğdu [32] introduced the first entropy measure for LDFS and used the proposed entropy measure to obtain weights of attributes in a novel extension of LDFS-TOPSIS. Furthermore, the extension of LDFSs have been studied by many researchers such as Riaz et al. [33] defined spherical LDFSs, Mahmood et al. [34] introduced interval-valued LDFS. Almagrabi et al. [35] proposed q-LDFSs and aggregation operators for q-LDSS. Kamacı [36] defined complex LDFSs and similarity measures for complex LDFSs. Ashraf et al. [37] gave a generalization of q-LDFSs that is named spherical q-LDFSs. Also, Riaz et al. [38] presented the prioritized aggregation operators for LDF for solving the best third party reverse logistic provider problem, Alnoor et al. [39] extended the linear Diophantine fuzzy rough sets (LDFRSs) into the MCDM and applied them to the problem of sustainable transportation.

Many MCDM methods have been developed to achieve successful outcomes in solving real-life problems contain different alternatives and multiple criteria in the decision-making process. Fuzzy logic is frequently used in the literature to overcome the uncertainties that occur in decision-making process in MCDM methods [40], [41], [42], [43], [44], [45], [46], [47]. In MCDM problems, determining the importance weight of each attribute is an important issue in the evaluation of the criteria. In literature, there are two basic types of weighting. In the subjective weighting methods, the weights are obtained by representing the decision makers' preferences and judgments with numbers. In the objective weighting method, the weights are determined by mathematical models without considering the decision makers' preferences. One of the weighting methods used to determine the attribute weights objectively is entropy. In the entropy-based objective weighting method, the importance of an attribute is determined by the dispersion that occurs in the evaluation of alternatives.

One of the well-known MCDM method is VIKOR (VIsekriterijumska optimizacija i KOmpromisno Resenje) is focuses on proposing compromise solutions that can help the decision makers to reach a final decision in MCDM problems with non-commensurable and conflicting criteria [48]. This method provides a maximum "group utility" of the "majority", and a minimum of the individual regret of the "opponent" and acquires more reasonable sorting results. In the Fuzzy-VIKOR method, some of the recent studies in the literature on the applications of the entropy-based objective weighting method are briefly given in Table 1.

To the best of our knowledge, linear Diophantine fuzzy VIKOR method has not yet been developed in the literature. Therefore, the information measures of the current study are used in extension of VIKOR to LDFS. The contributions of this research are listed as follows:

(i) New distance and similarity measures are defined on LDFSs, and their properties are investigated. The relationships between the distance measure and similarity measure are studied. Also, a new similarity measure based on distance measure is introduced.
(ii) A new entropy measure is proposed for LDFS. We discuss the relationship among the distance measure, similarity measure, and entropy measure. Besides two new entropy measures based on distance and similarity measures are obtained.
(iii) The proposed entropy measures are used to obtain the weights of criteria. The proposed distance measure is used for calculating the closeness of alternatives to the ideal solution in required by the new extension of VIKOR, namely LDF-VIKOR. Then the effects of entropy measures and distance measures on the compromise solution is showed.
(iv) The classical VIKOR method is extended to apply to a health management decision-making problem based on the proposed distance and entropy measures. Moreover, the comparison and sensitivity analyses are given to expose the advantages of the proposed method.

The rest of this paper is organized as follows. Section II explains the preliminaries of LDFS. Novel information measures for LDFS are given and shown its properties are exhibited in Section III. In section IV, entropy based LDF-VIKOR method is described. In section V, proposed LDF-VIKOR is applied in a healthcare management problem presented in [63]. Section VI includes a comparative analysis between LDF-VIKOR and LDF-TOPSIS. A sensitivity analysis for LDF- VIKOR is given in Section VII. Section VIII concludes the study with the results and future research agenda.

TABLE 1. Some of the recent fuzzy VIKOR applications.

Author(s)	Year	Type of FS	Application area
Shemshadi et al. [49]	2011	FS	Supplier selection problem
Girubha and Vinodh [50]	2012	FS	Material selection process
Wan et al. [51]	2013	IFS	Personnel selection problem
Liu et al. [52]	2015	FS	Analyzing the failure mode and effects
Mohsen and Fereshteh [53]	2017	FS	Analyzing the failure mode
Luo and Wang [54]	2017	IFS	Selection problem of enterprise resource planning
Tian et al. [55]	2018	FS	Analyzing the failure mode and effects
Ebadi Torkayesh et al. [56]	2019	FS	Selecting the waste-to-energy (WtE) technology
Rani et al. [57]	2019	PFS	Selecting renewable energy technologies
Fu et al. [58]	2020	IFS	Selecting energy service company
Alkafaas et al. [59]	2020	IFS	Selecting the facility location
Lam et al. [60]	2021	FS	Comparing financial performance of construction companies
Cheng et al. [61]	2021	q-ROF	Evaluating risk assessment of Sustainability Enterprise Risk Management
Razzaque et al. [62]	2023	spherical q-LDFS	Medical diagnosis problem

II. LINEAR DIOPHANTINE FUZZY SETS

Definition 1 [5]: Let $\mathfrak{R}$ be the reference set. A linear Diophantine fuzzy set $\mathfrak{D}$ is defined by

$$\mathfrak{D} = \{(\varrho, \langle \varphi_{\mathfrak{D}}(\varrho), \upsilon_{\mathfrak{D}}(\varrho)\rangle, \langle a_{\mathfrak{D}}(\varrho), b_{\mathfrak{D}}(\varrho)\rangle) : \varrho \in \mathfrak{R}\}$$

where $\varphi_{\mathfrak{D}}(\varrho)$ is membership grade, $\upsilon_{\mathfrak{D}}(\varrho)$ is non-membership grade, $a_{\mathfrak{D}}(\varrho)$ and $b_{\mathfrak{D}}(\varrho)$ are reference parameters where $\varphi_{\mathfrak{D}}(\varrho), \upsilon_{\mathfrak{D}}(\varrho), a_{\mathfrak{D}}(\varrho), b_{\mathfrak{D}}(\varrho) \in [0, 1]$, and for every $\varrho \in \mathfrak{R}$, $0 \leq a_{\mathfrak{D}}(\varrho) + b_{\mathfrak{D}}(\varrho) \leq 1$ and $0 \leq a_{\mathfrak{D}}(\varrho)\varphi_{\mathfrak{D}}(\varrho) + b_{\mathfrak{D}}(\varrho)\upsilon_{\mathfrak{D}}(\varrho) \leq 1$.

The hesitancy degree, $\iota_{\mathcal{L}}$, is evaluated as $c_{\mathcal{L}}(\varrho)\iota_{\mathcal{L}}(\varrho) = 1 - (a_{\mathfrak{D}}(\varrho)\varphi_{\mathfrak{D}}(\varrho) + b_{\mathfrak{D}}(\varrho)\upsilon_{\mathfrak{D}}(\varrho))$ where $c_{\mathcal{L}}(\varrho)$ is the reference parameter of hesitancy degree.

The following sets are called absolute and null LDF sets on $\mathfrak{R}$, respectively:

$$\mathfrak{D}^1 = \{(\varrho, \langle 1, 0\rangle, \langle 1, 0\rangle) : \varrho \in \mathfrak{R}\}$$
$$\mathfrak{D}^0 = \{(\varrho, \langle 0, 1\rangle, \langle 0, 1\rangle) : \varrho \in \mathfrak{R}\}$$

Definition 2 [5]: A linear Diophantine fuzzy number (LDFN) is defined by $\mathfrak{N} = (\langle\varphi_{\mathfrak{N}}, \upsilon_{\mathfrak{N}}\rangle, \langle a_{\mathfrak{N}}, b_{\mathfrak{N}}\rangle)$, where $\varphi_{\mathfrak{N}}, \upsilon_{\mathfrak{N}}, a_{\mathfrak{N}}, b_{\mathfrak{N}} \in [0, 1]$, with the conditions $0 \leq a_{\mathfrak{N}} + b_{\mathfrak{N}} \leq 1$ and $0 \leq a_{\mathfrak{N}}\varphi_{\mathfrak{N}} + b_{\mathfrak{N}}\upsilon_{\mathfrak{N}} \leq 1$.

We denote by $LDFN(\mathfrak{R})$ all LDFNs on $\mathfrak{R}$.

Definition 3 [5]: Let $\mathfrak{D} = \{(\varrho, \langle\varphi_{\mathfrak{D}}(\varrho), \upsilon_{\mathfrak{D}}(\varrho)\rangle, \langle a_{\mathfrak{D}}(\varrho), b_{\mathfrak{D}}(\varrho)\rangle) : \varrho \in \mathfrak{R}\}$ and $\mathfrak{E} = \{(\varrho, \langle\varphi_{\mathfrak{E}}(\varrho), \upsilon_{\mathfrak{E}}(\varrho)\rangle, \langle a_{\mathfrak{E}}(\varrho), b_{\mathfrak{E}}(\varrho)\rangle) : \varrho \in \mathfrak{R}\}$ be two LDFSs on $\mathfrak{R}$, then

1. $\mathfrak{D}^c = \{(\varrho, \langle\upsilon_D(\varrho), \varphi_D(\varrho)\rangle, \langle b_{\mathfrak{D}}(\varrho), a_{\mathfrak{D}}(\varrho)\rangle) : \varrho \in \mathfrak{R}\}$
2. $\mathfrak{D} = \mathfrak{E}$ if and only if $\varphi_{\mathfrak{D}}(\varrho) = \varphi_{\mathfrak{E}}(\varrho), \upsilon_{\mathfrak{D}}(\varrho) = \upsilon_{\mathfrak{E}}(\varrho), a_{\mathfrak{D}}(\varrho) = a_{\mathfrak{E}}(\varrho), b_{\mathfrak{D}}(\varrho) = b_{\mathfrak{E}}(\varrho)$
3. $\mathfrak{D} \subseteq \mathfrak{E}$ if and only if $\varphi_{\mathfrak{D}}(\varrho) \leq \varphi_{\mathfrak{E}}(\varrho), \upsilon_{\mathfrak{D}}(\varrho) \geq \upsilon_{\mathfrak{E}}(\varrho), a_{\mathfrak{D}}(\varrho) \leq a_{\mathfrak{E}}(\varrho), b_{\mathfrak{D}}(\varrho) \geq b_{\mathfrak{E}}(\varrho)$
4. $\mathfrak{D} \cup \mathfrak{E} = \left\{\left(\varrho, \begin{array}{c}\left\langle \begin{array}{c}\max(\varphi_{\mathfrak{D}}(\varrho), \varphi_{\mathfrak{E}}(\varrho)), \\ \min(\upsilon_{\mathfrak{D}}(\varrho), \upsilon_{\mathfrak{E}}(\varrho))\end{array}\right\rangle, \\ \left\langle \begin{array}{c}\max(a_{\mathfrak{D}}(\varrho), a_{\mathfrak{E}}(\varrho)), \\ \min(b_{\mathfrak{D}}(\varrho), b_{\mathfrak{E}}(\varrho))\end{array}\right\rangle\end{array}\right) : \varrho \in \mathfrak{R}\right\}$
5. $\mathfrak{D} \cap \mathfrak{E} = \left\{\left(\varrho, \begin{array}{c}\left\langle \begin{array}{c}\min(\varphi_{\mathfrak{D}}(\varrho), \varphi_{\mathfrak{E}}(\varrho)), \\ \max(\upsilon_{\mathfrak{D}}(\varrho), \upsilon_{\mathfrak{E}}(\varrho))\end{array}\right\rangle, \\ \left\langle \begin{array}{c}\min(a_{\mathfrak{D}}(\varrho), a_{\mathfrak{E}}(\varrho)), \\ \max(b_{\mathfrak{D}}(\varrho), b_{\mathfrak{E}}(\varrho))\end{array}\right\rangle\end{array}\right) : \varrho \in \mathfrak{R}\right\}$

Definition 4 [5]: Let $\mathfrak{N}_1 = \left(\langle\varphi_{\mathfrak{N}_1}, \upsilon_{\mathfrak{N}_1}\rangle, \langle a_{\mathfrak{N}_1}, b_{\mathfrak{N}_1}\rangle\right)$ and $\mathfrak{N}_2 = \left(\langle\varphi_{\mathfrak{N}_2}, \upsilon_{\mathfrak{N}_2}\rangle, \langle a_{\mathfrak{N}_2}, b_{\mathfrak{N}_2}\rangle\right)$ be two LDFNs on $\mathfrak{R}$ and $k > 0$ then

1) $\mathfrak{N}_1 \oplus \mathfrak{N}_2 = \left(\langle\varphi_{\mathfrak{N}_1} + \varphi_{\mathfrak{N}_2} - \varphi_{\mathfrak{N}_1}\varphi_{\mathfrak{N}_2}, \upsilon_{\mathfrak{N}_1}\upsilon_{\mathfrak{N}_2}\rangle, \langle a_{\mathfrak{N}_1} + a_{\mathfrak{N}_2} - a_{\mathfrak{N}_1}a_{\mathfrak{N}_2}, b_{\mathfrak{N}_1}b_{\mathfrak{N}_2}\rangle\right)$
2) $\mathfrak{N}_1 \otimes \mathfrak{N}_2 = \left(\langle\varphi_{\mathfrak{N}_1}\varphi_{\mathfrak{N}_2}, \upsilon_{\mathfrak{N}_1} + \upsilon_{\mathfrak{N}_2} - \upsilon_{\mathfrak{N}_1}\upsilon_{\mathfrak{N}_2}\rangle, \langle a_{\mathfrak{N}_1}a_{\mathfrak{N}_2}, b_{\mathfrak{N}_1} + b_{\mathfrak{N}_2} - b_{\mathfrak{N}_1}b_{\mathfrak{N}_2}\rangle\right)$
3) $k\mathfrak{N}_1 = \left(\langle 1 - (1 - \varphi_{\mathfrak{N}_1})^k, \upsilon_{\mathfrak{N}_1}^k\rangle, \langle 1 - (1 - a_{\mathfrak{N}_1})^k, b_{\mathfrak{N}_1}^k\rangle\right)$
4) $\mathfrak{N}_1^k = \left(\langle\varphi_{\mathfrak{N}_1}^k, 1 - (1 - \upsilon_{\mathfrak{N}_1})^k\rangle, \langle a_{\mathfrak{N}_1}^k, 1 - (1 - b_{\mathfrak{N}_1})^k\rangle\right)$.

Definition 5 [5]: Let $\mathfrak{N} = (\langle\varphi_{\mathfrak{N}}, \upsilon_{\mathfrak{N}}\rangle, \langle a_{\mathfrak{N}}, b_{\mathfrak{N}}\rangle)$ be an LDFN. The mapping $\mathfrak{s} : LDFN(\mathfrak{R}) \to [-1, 1]$ is called score function on LDFN $\mathfrak{N}$ defined by

$$\mathfrak{s}(\mathfrak{N}) = \frac{(\varphi_{\mathfrak{N}} - \upsilon_{\mathfrak{N}}) + (a_{\mathfrak{N}} - b_{\mathfrak{N}})}{2}$$

Definition 6 [5]: Let $\mathfrak{N} = (\langle\varphi_{\mathfrak{N}}, \upsilon_{\mathfrak{N}}\rangle, \langle a_{\mathfrak{N}}, b_{\mathfrak{N}}\rangle)$ be an LDFN. The accuracy function $a : LDFN(\mathfrak{R}) \to [0, 1]$ is defined by

$$a(\mathfrak{N}) = \frac{(\varphi_{\mathfrak{N}} + \upsilon_{\mathfrak{N}}) + 2(a_{\mathfrak{N}} + b_{\mathfrak{N}})}{4}.$$

Mohammad et al. [31] introduced generalized distance measure as follow:

$$\delta_M(\mathfrak{D}, \mathfrak{E}) = \left(\frac{1}{4n}\sum_{i=1}^{n}\left[\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|^p \\ + |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|^p \\ + |a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|^p \\ + |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|^p\end{array}\right]\right)^{\frac{1}{p}}$$

Here $\mathfrak{D}$ and $\mathfrak{E}$ are two LDFSs. For $p = 1, 2$ we get normalized Hamming distance and normalized Euclidean distance between two LDFSs $\mathfrak{D}$ and $\mathfrak{E}$ as follow, respectively:

$$\delta_M^H(\mathfrak{D}, \mathfrak{E}) = \frac{1}{4n}\sum_{i=1}^{n}\left[\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)| \\ + |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)| \\ + |a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)| \\ + |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right],$$

$$\delta_M^E(\mathfrak{D}, \mathfrak{E}) = \left(\frac{1}{4n}\sum_{i=1}^{n}\left[\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|^2 \\ + |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|^2 \\ + |a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|^2 \\ + |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|^2\end{array}\right]\right)^{\frac{1}{2}}.$$

Gül and Aydoğdu [32] presented two entropy measures for LDFS $\mathfrak{D}$ as follow:

$$\varepsilon_{GA_1}(\mathfrak{D}) = 1 - \frac{1}{2n}\sum_{i=1}^{n}\left[\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{D}}(\varrho_i)| \\ + |a_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{D}}(\varrho_i)|\end{array}\right],$$

$$\varepsilon_{GA_2}(\mathfrak{D}) = 1 - \left\{\frac{1}{2n}\sum_{i=1}^{n}\left[\begin{array}{c}(\varphi_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{D}}(\varrho_i))^2 \\ + (a_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{D}}(\varrho_i))^2\end{array}\right]\right\}^{\frac{1}{2}}.$$

III. NEW INFORMATION MEASURES FOR LDFS

In this part, we establish the axiomatic skeleton of LDF information measures and present the corresponding formula.

A. DISTANCE MEASURE FOR LDFS

Definition 7 [31]: A distance measure δ is a mapping $\delta : LDFS(\mathfrak{R}) \times LDFS(\mathfrak{R}) \to [0, 1]$, providing the following properties, for all $\mathfrak{D}, \mathfrak{E}, \mathfrak{F} \in LDFS(\mathfrak{R})$,

D1) $0 \leq \delta(\mathfrak{D}, \mathfrak{E}) \leq 1$,
D2) $\delta(\mathfrak{D}, \mathfrak{E}) = 0$ if $\mathfrak{D} = \mathfrak{E}$,
D3) $\delta(\mathfrak{D}, \mathfrak{E}) = \delta(\mathfrak{E}, \mathfrak{D})$,
D4) If $\mathfrak{D} \subseteq \mathfrak{E} \subseteq \mathfrak{F}$, then $\delta(\mathfrak{D}, \mathfrak{F}) \geq \delta(\mathfrak{D}, \mathfrak{E})$ and $\delta(\mathfrak{D}, \mathfrak{F}) \geq \delta(\mathfrak{E}, \mathfrak{F})$.

Let $\mathfrak{D}$ and $\mathfrak{E}$ be two LDFSs on reference set $\mathfrak{R}$. We define a mapping on $LDFS(\mathfrak{R})$ as follows:

$$\delta(\mathfrak{D}, \mathfrak{E}) = \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)| \\ + |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)| \\ + |a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)| \\ + |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right) \\ + 4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|, \\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|, \\ |a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|, \\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right].$$

Theorem 1: Let $\mathfrak{D}$ and $\mathfrak{E}$ be two LDFSs, then $\delta(\mathfrak{D}, \mathfrak{E})$ is a distance measure.

Proof:

D1) By the definition of LDFS, we get $\varphi_{\mathfrak{D}}(\varrho), \varphi_{\mathfrak{E}}(\varrho), \upsilon_{\mathfrak{D}}(\varrho), \upsilon_{\mathfrak{E}}(\varrho), a_{\mathfrak{D}}(\varrho), a_{\mathfrak{E}}(\varrho), b_{\mathfrak{D}}(\varrho), b_{\mathfrak{E}}(\varrho) \in [0,1]$ for all $\varrho \in \mathfrak{R}$. Then,

$$
\begin{aligned}
&0 \le \varphi_{\mathfrak{D}}(\varrho) \le 1, \quad 0 \le \varphi_{\mathfrak{E}}(\varrho) \le 1 \Rightarrow 0 \\
&\qquad\qquad \le |\varphi_{\mathfrak{D}}(\varrho) - \varphi_{\mathfrak{E}}(\varrho)| \le 1 \\
&0 \le \upsilon_{\mathfrak{D}}(\varrho) \le 1, \quad 0 \le \upsilon_{\mathfrak{E}}(\varrho) \le 1 \Rightarrow 0 \\
&\qquad\qquad \le |\upsilon_{\mathfrak{D}}(\varrho) - \upsilon_{\mathfrak{E}}(\varrho)| \le 1 \\
&0 \le a_{\mathfrak{D}}(\varrho) \le 1, \quad 0 \le a_{\mathfrak{E}}(\varrho) \le 1 \Rightarrow 0 \\
&\qquad\qquad \le |a_{\mathfrak{D}}(\varrho) - a_{\mathfrak{E}}(\varrho)| \le 1 \\
&0 \le b_{\mathfrak{D}}(\varrho) \le 1, \quad 0 \le b_{\mathfrak{E}}(\varrho) \le 1 \Rightarrow 0 \\
&\qquad\qquad \le |b_{\mathfrak{D}}(\varrho) - b_{\mathfrak{E}}(\varrho)| \le 1
\end{aligned}
$$

$$
\begin{aligned}
\delta(\mathfrak{D},\mathfrak{E}) &= \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)| \\ +|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)| \\ +|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)| \\ +|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|, \\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|, \\ |a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|, \\ |b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right] \\
&\le \frac{1}{8n}\sum_{i=1}^{n}\left[\left(1+1+1+1\right)+4\cdot 1\right] \\
&= 1.
\end{aligned}
$$

It is obviously that $\delta(\mathfrak{D},\mathfrak{E}) \ge 0$.

D2) If $\mathfrak{D} = \mathfrak{E}$, then $\varphi_{\mathfrak{D}}(\varrho) = \varphi_{\mathfrak{E}}(\varrho), \upsilon_{\mathfrak{D}}(\varrho) = \upsilon_{\mathfrak{E}}(\varrho), a_{\mathfrak{D}}(\varrho) = a_{\mathfrak{E}}(\varrho), b_{\mathfrak{D}}(\varrho) = b_{\mathfrak{E}}(\varrho)$ for all $\varrho \in \mathfrak{R}$. So, $\delta(\mathfrak{D},\mathfrak{E}) = 0$.

D3)

$$
\begin{aligned}
&\delta(\mathfrak{D},\mathfrak{E}) \\
&= \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)| \\ +|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)| \\ +|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)| \\ +|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|, \\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|, \\ |a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|, \\ |b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right] \\
&= \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}|\varphi_{\mathfrak{E}}(\varrho_i)-\varphi_{\mathfrak{D}}(\varrho_i)| \\ +|\upsilon_{\mathfrak{E}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)| \\ +|a_{\mathfrak{E}}(\varrho_i)-a_{\mathfrak{D}}(\varrho_i)| \\ +|b_{\mathfrak{E}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{E}}(\varrho_i)-\varphi_{\mathfrak{D}}(\varrho_i)|, \\ |\upsilon_{\mathfrak{E}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)|, \\ |a_{\mathfrak{E}}(\varrho_i)-a_{\mathfrak{D}}(\varrho_i)|, \\ |b_{\mathfrak{E}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\end{array}\right)\end{array}\right] \\
&= \delta(\mathfrak{E},\mathfrak{D}).
\end{aligned}
$$

D4) For any LDFS $\mathfrak{F} = \{(\varrho, \langle\varphi_{\mathfrak{F}}(\varrho), \upsilon_{\mathfrak{F}}(\varrho)\rangle, \langle a_{\mathfrak{F}}(\varrho), b_{\mathfrak{F}}(\varrho)\rangle) : \varrho \in \mathfrak{R}\}$, if $\mathfrak{D} \subseteq \mathfrak{E} \subseteq \mathfrak{F}$, then $\varphi_{\mathfrak{D}}(\varrho) \le \varphi_{\mathfrak{E}}(\varrho) \le \varphi_{\mathfrak{F}}(\varrho), \upsilon_{\mathfrak{D}}(\varrho) \ge \upsilon_{\mathfrak{E}}(\varrho) \ge \upsilon_{\mathfrak{F}}(\varrho), a_{\mathfrak{D}}(\varrho) \le a_{\mathfrak{E}}(\varrho) \le a_{\mathfrak{F}}(\varrho), b_{\mathfrak{D}}(\varrho) \ge b_{\mathfrak{E}}(\varrho) \ge b_{\mathfrak{F}}(\varrho)$ for all $\varrho \in \mathfrak{R}$. So, we have $|\varphi_{\mathfrak{D}}(\varrho) - \varphi_{\mathfrak{E}}(\varrho)| \le |\varphi_{\mathfrak{D}}(\varrho) - \varphi_{\mathfrak{F}}(\varrho)|$, $|\upsilon_{\mathfrak{D}}(\varrho) - \upsilon_{\mathfrak{E}}(\varrho)| \le |\upsilon_{\mathfrak{D}}(\varrho) - \upsilon_{\mathfrak{F}}(\varrho)|$, $|a_{\mathfrak{D}}(\varrho) - a_{\mathfrak{E}}(\varrho)| \le |a_{\mathfrak{D}}(\varrho) - a_{\mathfrak{F}}(\varrho)|$ and $|b_{\mathfrak{D}}(\varrho) - b_{\mathfrak{E}}(\varrho)| \le |b_{\mathfrak{D}}(\varrho) - b_{\mathfrak{F}}(\varrho)|$. Thus,

$$
\begin{aligned}
&\delta(\mathfrak{D},\mathfrak{E}) \\
&= \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)| \\ +|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)| \\ +|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)| \\ +|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|, \\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|, \\ |a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|, \\ |b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right] \\
&\le \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{F}}(\varrho_i)| \\ +|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{F}}(\varrho_i)| \\ +|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{F}}(\varrho_i)| \\ +|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{F}}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{F}}(\varrho_i)|, \\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{F}}(\varrho_i)|, \\ |a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{F}}(\varrho_i)|, \\ |b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{F}}(\varrho_i)|\end{array}\right)\end{array}\right] \\
&= \delta(\mathfrak{D},\mathfrak{F}).
\end{aligned}
$$

It can be shown $\delta(\mathfrak{D},\mathfrak{F}) \ge \delta(\mathfrak{E},\mathfrak{F})$ in the same way, which completes the proof.

The following propositions give the various features of the proposed distance measure under the complement of LDFS.

Proposition 1: Let $\mathfrak{D}$ and $\mathfrak{E}$ be two LDFSs, then we get

1. $\delta(\mathfrak{D},\mathfrak{E}^c) = \delta(\mathfrak{D}^c,\mathfrak{E})$;
2. $\delta(\mathfrak{D},\mathfrak{E}) = \delta(\mathfrak{D}^c,\mathfrak{E}^c)$.

Proof:

1. By the definition of complement of LDFS, we get

$$
\begin{aligned}
&\delta(\mathfrak{D},\mathfrak{E}^c) \\
&= \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}^c}(\varrho_i)| \\ +|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}^c}(\varrho_i)| \\ +|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}^c}(\varrho_i)| \\ +|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}^c}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}^c}(\varrho_i)|, \\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}^c}(\varrho_i)|, \\ |a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}^c}(\varrho_i)|, \\ |b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}^c}(\varrho_i)|\end{array}\right)\end{array}\right] \\
&= \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|+ \\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)| \\ +|a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|+ \\ |b_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|, \\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|, \\ |a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|, \\ |b_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right]
\end{aligned}
$$

$$= \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}|\upsilon_{\mathfrak{D}^c}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)| \\ +|\varphi_{\mathfrak{D}^c}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)| \\ +|b_{\mathfrak{D}^c}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)| \\ +|a_{\mathfrak{D}^c}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\upsilon_{\mathfrak{D}^c}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|, \\ |\varphi_{\mathfrak{D}^c}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|, \\ |b_{\mathfrak{D}^c}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|, \\ |a_{\mathfrak{D}^c}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right]$$
$$= \delta\left(\mathfrak{D}^c, \mathfrak{E}\right).$$

2.

$$\delta(\mathfrak{D}, \mathfrak{E})$$
$$= \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)| \\ +|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)| \\ +|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)| \\ +|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|, \\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|, \\ |a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|, \\ |b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right]$$
$$= \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}|\upsilon_{\mathfrak{D}^c}(\varrho_i)-\upsilon_{\mathfrak{E}^c}(\varrho_i)| \\ +|\varphi_{\mathfrak{D}^c}(\varrho_i)-\varphi_{\mathfrak{E}^c}(\varrho_i)| \\ +|b_{\mathfrak{D}^c}(\varrho_i)-b_{\mathfrak{E}^c}(\varrho_i)| \\ +|a_{\mathfrak{D}^c}(\varrho_i)-a_{\mathfrak{E}^c}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\upsilon_{\mathfrak{D}^c}(\varrho_i)-\upsilon_{\mathfrak{E}^c}(\varrho_i)|, \\ |\varphi_{\mathfrak{D}^c}(\varrho_i)-\varphi_{\mathfrak{E}^c}(\varrho_i)|, \\ |b_{\mathfrak{D}^c}(\varrho_i)-b_{\mathfrak{E}^c}(\varrho_i)|, \\ |a_{\mathfrak{D}^c}(\varrho_i)-a_{\mathfrak{E}^c}(\varrho_i)|\end{array}\right)\end{array}\right]$$
$$= \delta\left(\mathfrak{D}^c, \mathfrak{E}^c\right).$$

Proposition 2: Let $\mathfrak{D}$ be an LDFSs, then $\delta(\mathfrak{D}, \mathfrak{D}^c) = 1$ iff $\mathfrak{D}$ is crisp set.

Proof: If $\mathfrak{D}$ is a crisp set, then $\varphi_{\mathfrak{D}}(\varrho) = 1$, $\upsilon_{\mathfrak{D}}(\varrho) = 0$, $a_{\mathfrak{D}}(\varrho) = 1$ and $b_{\mathfrak{D}}(\varrho) = 0$ or $\varphi_{\mathfrak{D}}(\varrho) = 0$, $\upsilon_{\mathfrak{D}}(\varrho) = 1$, $a_{\mathfrak{D}}(\varrho) = 0$ and $b_{\mathfrak{D}}(\varrho) = 1$. In both cases

$$\delta\left(\mathfrak{D}, \mathfrak{D}^c\right)$$
$$= \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{D}^c}(\varrho_i)| \\ +|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}^c}(\varrho_i)| \\ +|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{D}^c}(\varrho_i)| \\ +|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}^c}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{D}^c}(\varrho_i)|, \\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}^c}(\varrho_i)|, \\ |a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{D}^c}(\varrho_i)|, \\ |b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}^c}(\varrho_i)|\end{array}\right)\end{array}\right]$$
$$= \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}2|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)| \\ +2|a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)|, \\ |a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\end{array}\right)\end{array}\right]$$
$$= \frac{1}{8n}\sum_{i=1}^{n}\left[\left(2\cdot 1+2\cdot 1\right)+4\cdot 1\right]$$
$$= 1.$$

Otherwise, if $\delta(\mathfrak{D}, \mathfrak{D}^c) = 1$, then we have

$$\left[\begin{array}{c}\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho)-\varphi_{\mathfrak{D}^c}(\varrho)| \\ +|\upsilon_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{D}^c}(\varrho)| \\ +|a_{\mathfrak{D}}(\varrho)-a_{\mathfrak{D}^c}(\varrho)| \\ +|b_{\mathfrak{D}}(\varrho)-b_{\mathfrak{D}^c}(\varrho)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho)-\varphi_{\mathfrak{D}^c}(\varrho)|, \\ |\upsilon_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{D}^c}(\varrho)|, \\ |a_{\mathfrak{D}}(\varrho)-a_{\mathfrak{D}^c}(\varrho)|, \\ |b_{\mathfrak{D}}(\varrho)-b_{\mathfrak{D}^c}(\varrho)|\end{array}\right)\end{array}\right] = 8$$
$$\left[\begin{array}{c}\left(\begin{array}{c}2|\varphi_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{D}}(\varrho)| \\ +2|a_{\mathfrak{D}}(\varrho)-b_{\mathfrak{D}}(\varrho)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{D}}(\varrho)|, \\ |a_{\mathfrak{D}}(\varrho)-b_{\mathfrak{D}}(\varrho)|\end{array}\right)\end{array}\right] = 8$$
$$\left[\begin{array}{c}\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{D}}(\varrho)| \\ +2|a_{\mathfrak{D}}(\varrho)-b_{\mathfrak{D}}(\varrho)|\end{array}\right) \\ +2\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{D}}(\varrho)|, \\ |a_{\mathfrak{D}}(\varrho)-b_{\mathfrak{D}}(\varrho)|\end{array}\right)\end{array}\right] = 4.$$

In this case, either $|\varphi_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{D}}(\varrho)| > |a_{\mathfrak{D}}(\varrho)-b_{\mathfrak{D}}(\varrho)|$, or $|\varphi_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{D}}(\varrho)| < |a_{\mathfrak{D}}(\varrho)-b_{\mathfrak{D}}(\varrho)|$, since $0 \le |\varphi_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{D}}(\varrho)| \le 1$ and $0 \le |a_{\mathfrak{D}}(\varrho)-b_{\mathfrak{D}}(\varrho)| \le 1$, we get $0 \le 3|\varphi_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{D}}(\varrho)| + |a_{\mathfrak{D}}(\varrho)-b_{\mathfrak{D}}(\varrho)| \le 4$ or $0 \le |\varphi_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{D}}(\varrho)| + 3|a_{\mathfrak{D}}(\varrho)-b_{\mathfrak{D}}(\varrho)| \le 4$, respectively. In both cases, it can be seen clearly $\mathfrak{D}$ is a crisp set.

Proposition 3: Let $\mathfrak{D}$ be an LDFSs, then $\delta(\mathfrak{D}, \mathfrak{D}^c) = 0$ if and only if $\varphi_{\mathfrak{D}}(\varrho) = \upsilon_{\mathfrak{D}}(\varrho)$ and $a_{\mathfrak{D}}(\varrho) = b_{\mathfrak{D}}(\varrho)$ for all $\varrho \in \mathfrak{R}$.

Proof: If $\varphi_{\mathfrak{D}}(\varrho) = \upsilon_{\mathfrak{D}}(\varrho)$ and $a_{\mathfrak{D}}(\varrho) = b_{\mathfrak{D}}(\varrho)$ for all $\varrho \in \mathfrak{R}$, then

$$\delta\left(\mathfrak{D}, \mathfrak{D}^c\right) = \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{D}^c}(\varrho_i)| \\ +|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}^c}(\varrho_i)| \\ +|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{D}^c}(\varrho_i)| \\ +|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}^c}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{D}^c}(\varrho_i)|, \\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}^c}(\varrho_i)|, \\ |a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{D}^c}(\varrho_i)|, \\ |b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}^c}(\varrho_i)|\end{array}\right)\end{array}\right]$$
$$= \frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}2|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)| \\ +2|a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\end{array}\right) \\ +4\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)|, \\ |a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\end{array}\right)\end{array}\right]$$
$$= 0.$$

Conversely, if $\delta(\mathfrak{D}, \mathfrak{D}^c) = 0$, then it follows that

$$\left[\begin{array}{c}\left(2|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)|+2|a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\right) \\ +4\max\left(|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)|, |a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\right)\end{array}\right] = 0$$

which implies that $|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)| = 0$ and $|a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)| = 0$. Thus, we get $\varphi_{\mathfrak{D}}(\varrho_i) = \upsilon_{\mathfrak{D}}(\varrho_i)$ and $a_{\mathfrak{D}}(\varrho_i) = b_{\mathfrak{D}}(\varrho_i)$ for all $\varrho \in \mathfrak{R}$. This completes the proof.

Proposition 4: For any two LDFSs $\mathfrak{D}$ and $\mathfrak{E}$, the following equalities hold:

1. $\delta(\mathfrak{D}, \mathfrak{E}) = \delta(\mathfrak{D}\cap\mathfrak{E}, \mathfrak{D}\cup\mathfrak{E})$,
2. $\delta(\mathfrak{D}, \mathfrak{D}\cap\mathfrak{E}) = \delta(\mathfrak{E}, \mathfrak{D}\cup\mathfrak{E})$,
3. $\delta(\mathfrak{D}, \mathfrak{D}\cup\mathfrak{E}) = \delta(\mathfrak{E}, \mathfrak{D}\cap\mathfrak{E})$.

B. SIMILARITY MEASURE FOR LDFS

Definition 8: A similarity measure between LDFSs is a mapping $\sigma : LDFS(\mathfrak{R}) \times LDFS(\mathfrak{R}) \to [0, 1]$ satisfies the following properties, for every $\mathfrak{D}, \mathfrak{E}, \mathfrak{F} \in LDFS(\mathfrak{R})$,

S1) $0 \leq \sigma(\mathfrak{D}, \mathfrak{E}) \leq 1$,
S2) $\sigma(\mathfrak{D}, \mathfrak{E}) = 1$ iff $\mathfrak{D} = \mathfrak{E}$,
S3) $\sigma(\mathfrak{D}, \mathfrak{E}) = \sigma(\mathfrak{E}, \mathfrak{D})$,
S4) $\sigma(\mathfrak{D}, \mathfrak{D}^c) = 0$ iff $\mathfrak{D}$ is crisp set,
S5) If $\mathfrak{D} \subseteq \mathfrak{E} \subseteq \mathfrak{F}$, then $\sigma(\mathfrak{D}, \mathfrak{F}) \leq \sigma(\mathfrak{D}, \mathfrak{E})$ and $\sigma(\mathfrak{D}, \mathfrak{F}) \leq \sigma(\mathfrak{E}, \mathfrak{F})$.

Let $\mathfrak{D}$ and $\mathfrak{E}$ be two LDFSs on reference set $\mathfrak{R}$. We define a mapping on $LDF(\mathfrak{R})$ as follow:

$$\sigma(\mathfrak{D}, \mathfrak{E}) = \frac{1}{n}\sum_{i=1}^{n} \frac{2 - \left[\begin{array}{l}\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\\ +\max\left(\begin{array}{c}|a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right]}{2 + \left[\begin{array}{l}\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\\ +\max\left(\begin{array}{c}|a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right]}.$$

Theorem 2: Let $\mathfrak{D}$ and $\mathfrak{E}$ be two LDFSs, then $\sigma(\mathfrak{D}, \mathfrak{E})$ is a similarity measure.

Proof: In order for $\sigma(\mathfrak{D}, \mathfrak{E})$ to be described as similarity measure for LDFS, it must satisfy S1–S5 of axiomatic requirements.

S1) By the definition of LDFS, the following inequalities hold:

$$\begin{aligned}
&0 \leq \varphi_{\mathfrak{D}}(\varrho) \leq 1, \quad 0 \leq \varphi_{\mathfrak{E}}(\varrho) \leq 1 \Rightarrow 0 \\
&\qquad\qquad \leq |\varphi_{\mathfrak{D}}(\varrho) - \varphi_{\mathfrak{E}}(\varrho)| \leq 1, \\
&0 \leq \upsilon_{\mathfrak{D}}(\varrho) \leq 1, \quad 0 \leq \upsilon_{\mathfrak{E}}(\varrho) \leq 1 \Rightarrow 0 \\
&\qquad\qquad \leq |\upsilon_{\mathfrak{D}}(\varrho) - \upsilon_{\mathfrak{E}}(\varrho)| \leq 1, \\
&0 \leq a_{\mathfrak{D}}(\varrho) \leq 1, \quad 0 \leq a_{\mathfrak{E}}(\varrho) \leq 1 \Rightarrow 0 \\
&\qquad\qquad \leq |a_{\mathfrak{D}}(\varrho) - a_{\mathfrak{E}}(\varrho)| \leq 1, \\
&0 \leq b_{\mathfrak{D}}(\varrho) \leq 1, \quad 0 \leq b_{\mathfrak{E}}(\varrho) \leq 1 \Rightarrow 0 \\
&\qquad\qquad \leq |b_{\mathfrak{D}}(\varrho) - b_{\mathfrak{E}}(\varrho)| \leq 1.
\end{aligned}$$

Then,

$$\begin{aligned}
&\sigma(\mathfrak{D}, \mathfrak{E}) \\
&= \frac{1}{n}\sum_{i=1}^{n} \frac{2 - \left[\begin{array}{l}\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\\ +\max\left(\begin{array}{c}|a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right]}{2 + \left[\begin{array}{l}\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\\ +\max\left(\begin{array}{c}|a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right]} \\
&\geq \frac{1}{n}\sum_{i=1}^{n} \frac{2 - [1 + 1]}{2 + [1 + 1]} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\sigma(\mathfrak{D}, \mathfrak{E}) \\
&= \frac{1}{n}\sum_{i=1}^{n} \frac{2 - \left[\begin{array}{l}\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\\ +\max\left(\begin{array}{c}|a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right]}{2 + \left[\begin{array}{l}\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\\ +\max\left(\begin{array}{c}|a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right]} \\
&\leq \frac{1}{n}\sum_{i=1}^{n} \frac{2 - [0 + 0]}{2 + [0 + 0]} \\
&= 1.
\end{aligned}$$

S2) If $\mathfrak{D} = \mathfrak{E}$, then $\varphi_{\mathfrak{D}}(\varrho) = \varphi_{\mathfrak{E}}(\varrho)$, $\upsilon_{\mathfrak{D}}(\varrho) = \upsilon_{\mathfrak{E}}(\varrho)$, $a_{\mathfrak{D}}(\varrho) = a_{\mathfrak{E}}(\varrho)$ and $b_{\mathfrak{D}}(\varrho) = b_{\mathfrak{E}}(\varrho)$ for all $\varrho \in \mathfrak{R}$. So

$$\begin{aligned}
&\sigma(\mathfrak{D}, \mathfrak{E}) \\
&= \frac{1}{n}\sum_{i=1}^{n} \frac{2 - \left[\begin{array}{l}\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\\ +\max\left(\begin{array}{c}|a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right]}{2 + \left[\begin{array}{l}\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\\ +\max\left(\begin{array}{c}|a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right]} \\
&= \frac{1}{n}\sum_{i=1}^{n} \frac{2 - [0 + 0]}{2 + [0 + 0]} \\
&= 1.
\end{aligned}$$

Otherwise, if $\sigma(\mathfrak{D}, \mathfrak{E}) = 1$, then

$$\begin{aligned}
&2 - \left[\begin{array}{l}\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\\ +\max\left(\begin{array}{c}|a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right] \\
&\quad = 2 + \left[\begin{array}{l}\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\\ +\max\left(\begin{array}{c}|a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right] \\
&\quad \Rightarrow - \left[\begin{array}{l}\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\\ +\max\left(\begin{array}{c}|a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right] \\
&\quad = \left[\begin{array}{l}\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\\ +\max\left(\begin{array}{c}|a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right] \\
&\quad \Rightarrow \left[\begin{array}{l}\max\left(\begin{array}{c}|\varphi_{\mathfrak{D}}(\varrho_i) - \varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i) - \upsilon_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\\ +\max\left(\begin{array}{c}|a_{\mathfrak{D}}(\varrho_i) - a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i) - b_{\mathfrak{E}}(\varrho_i)|\end{array}\right)\end{array}\right] = 0.
\end{aligned}$$

Thus, we have $|\varphi_{\mathfrak{D}}(\varrho)-\varphi_{\mathfrak{E}}(\varrho)|=0$, $|\upsilon_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{E}}(\varrho)|=0$, $|a_{\mathfrak{D}}(\varrho)-a_{\mathfrak{E}}(\varrho)|=0$ and $|b_{\mathfrak{D}}(\varrho)-b_{\mathfrak{E}}(\varrho)|=0$. Hence, we get $\mathfrak{D}=\mathfrak{E}$.

S3)

$$\begin{aligned}
&\sigma(\mathfrak{D},\mathfrak{E})\\
&=\frac{1}{n}\sum_{i=1}^{n}\frac{2-\left[\begin{array}{l}\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|,\\|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\\+\max\begin{pmatrix}|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|,\\|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\end{array}\right]}{2+\left[\begin{array}{l}\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|,\\|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\\+\max\begin{pmatrix}|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|,\\|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\end{array}\right]}\\
&=\frac{1}{n}\sum_{i=1}^{n}\frac{2-\left[\begin{array}{l}\max\begin{pmatrix}|\varphi_{\mathfrak{E}}(\varrho_i)-\varphi_{\mathfrak{D}}(\varrho_i)|,\\|\upsilon_{\mathfrak{E}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)|\end{pmatrix}\\+\max\begin{pmatrix}|a_{\mathfrak{E}}(\varrho_i)-a_{\mathfrak{D}}(\varrho_i)|,\\|b_{\mathfrak{E}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\end{pmatrix}\end{array}\right]}{2+\left[\begin{array}{l}\max\begin{pmatrix}|\varphi_{\mathfrak{E}}(\varrho_i)-\varphi_{\mathfrak{D}}(\varrho_i)|,\\|\upsilon_{\mathfrak{E}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)|\end{pmatrix}\\+\max\begin{pmatrix}|a_{\mathfrak{E}}(\varrho_i)-a_{\mathfrak{D}}(\varrho_i)|,\\|b_{\mathfrak{E}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\end{pmatrix}\end{array}\right]}\\
&=\sigma(\mathfrak{E},\mathfrak{D}).
\end{aligned}$$

S4) If $\sigma(\mathfrak{D},\mathfrak{D}^c)=0$, then

$$2-\left[\begin{array}{l}\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{D}^c}(\varrho_i)|,\\|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}^c}(\varrho_i)|\end{pmatrix}\\+\max\begin{pmatrix}|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{D}^c}(\varrho_i)|,\\|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}^c}(\varrho_i)|\end{pmatrix}\end{array}\right]=0$$

$$\left[\begin{array}{l}\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)|,\\|\upsilon_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{D}}(\varrho_i)|\end{pmatrix}\\+\max\begin{pmatrix}|a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|,\\|b_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{D}}(\varrho_i)|\end{pmatrix}\end{array}\right]=2$$

$$|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)|,|\upsilon_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{D}}(\varrho_i)|=2.$$

Since $0\le\varphi_{\mathfrak{D}}(\varrho)\le1$, $0\le\upsilon_{\mathfrak{D}}(\varrho)\le1$, $0\le a_{\mathfrak{D}}(\varrho)\le1$ and $0\le b_{\mathfrak{D}}(\varrho)\le1$ for all $\varrho\in\mathfrak{R}$, $0\le|\varphi_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{D}}(\varrho)|\le1$ and $0\le|a_{\mathfrak{D}}(\varrho)-b_{\mathfrak{D}}(\varrho)|\le1$. The latest equation is provided when $|\varphi_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{D}}(\varrho)|=1$ and $|a_{\mathfrak{D}}(\varrho)-b_{\mathfrak{D}}(\varrho)|=1$. Hence, $\mathfrak{D}$ is crisp set.

If $\mathfrak{D}$ is crisp set, then

$$\begin{aligned}
&\sigma\left(\mathfrak{D},\mathfrak{D}^c\right)\\
&=\frac{1}{n}\sum_{i=1}^{n}\frac{2-\left[\begin{array}{l}\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{D}^c}(\varrho_i)|,\\|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}^c}(\varrho_i)|\end{pmatrix}\\+\max\begin{pmatrix}|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{D}^c}(\varrho_i)|,\\|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}^c}(\varrho_i)|\end{pmatrix}\end{array}\right]}{2+\left[\begin{array}{l}\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{D}^c}(\varrho_i)|,\\|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}^c}(\varrho_i)|\end{pmatrix}\\+\max\begin{pmatrix}|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{D}^c}(\varrho_i)|,\\|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}^c}(\varrho_i)|\end{pmatrix}\end{array}\right]}\\
&=\frac{1}{n}\sum_{i=1}^{n}\frac{2-[1+1]}{2+[1+1]}\\
&=0.
\end{aligned}$$

S5) For any LDFS $\mathfrak{F}=\{(\varrho,\langle\varphi_{\mathfrak{F}}(\varrho),\upsilon_{\mathfrak{F}}(\varrho)\rangle,\langle a_{\mathfrak{F}}(\varrho)),b_{\mathfrak{F}}(\varrho)\rangle):\varrho\in\mathfrak{R}\}$, if $\mathfrak{D}\subseteq\mathfrak{E}\subseteq\mathfrak{F}$, then we have $|\varphi_{\mathfrak{D}}(\varrho)-\varphi_{\mathfrak{E}}(\varrho)|\le|\varphi_{\mathfrak{D}}(\varrho)-\varphi_{\mathfrak{F}}(\varrho)|$, $|\upsilon_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{E}}(\varrho)|\le|\upsilon_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{F}}(\varrho)|$, $|a_{\mathfrak{D}}(\varrho)-a_{\mathfrak{E}}(\varrho)|\le|a_{\mathfrak{D}}(\varrho)-a_{\mathfrak{F}}(\varrho)|$ and $|b_{\mathfrak{D}}(\varrho)-b_{\mathfrak{E}}(\varrho)|\le|b_{\mathfrak{D}}(\varrho)-b_{\mathfrak{F}}(\varrho)|$. Hence, we have

$$\left[\begin{array}{l}\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho)-\varphi_{\mathfrak{F}}(\varrho)|,\\|\upsilon_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{F}}(\varrho)|\end{pmatrix}\\+\max\begin{pmatrix}|a_{\mathfrak{D}}(\varrho)-a_{\mathfrak{F}}(\varrho)|,\\|b_{\mathfrak{D}}(\varrho)-b_{\mathfrak{F}}(\varrho)|\end{pmatrix}\end{array}\right]\ge\left[\begin{array}{l}\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho)-\varphi_{\mathfrak{E}}(\varrho)|,\\|\upsilon_{\mathfrak{D}}(\varrho)-\upsilon_{\mathfrak{E}}(\varrho)|\end{pmatrix}\\+\max\begin{pmatrix}|a_{\mathfrak{D}}(\varrho)-a_{\mathfrak{E}}(\varrho)|,\\|b_{\mathfrak{D}}(\varrho)-b_{\mathfrak{E}}(\varrho)|\end{pmatrix}\end{array}\right]$$

and then we get $\sigma(\mathfrak{D},\mathfrak{F})\le\sigma(\mathfrak{D},\mathfrak{E})$. Similarly, $\sigma(\mathfrak{D},\mathfrak{F})\le\sigma(\mathfrak{E},\mathfrak{F})$.

This completes the proof.

Proposition 5: For any two LDFSs $\mathfrak{D}$ and $\mathfrak{E}$, we have

1. $\sigma(\mathfrak{D},\mathfrak{E}^c)=\sigma(\mathfrak{D}^c,\mathfrak{E})$,
2. $\sigma(\mathfrak{D},\mathfrak{E})=\sigma(\mathfrak{D}\cap\mathfrak{E},\mathfrak{D}\cup\mathfrak{E})$,
3. $\sigma(\mathfrak{D},\mathfrak{D}\cap\mathfrak{E})=\sigma(\mathfrak{E},\mathfrak{D}\cup\mathfrak{E})$,
4. $\sigma(\mathfrak{D},\mathfrak{D}\cup\mathfrak{E})=\sigma(\mathfrak{E},\mathfrak{D}\cap\mathfrak{E})$,
5. $\sigma(\mathfrak{D},\mathfrak{D}\otimes\mathfrak{E})=\sigma(\mathfrak{E},\mathfrak{D}\oplus\mathfrak{E})$,
6. $\sigma(\mathfrak{D},\mathfrak{D}\oplus\mathfrak{E})=\sigma(\mathfrak{E},\mathfrak{D}\otimes\mathfrak{E})$.

Theorem 3: Let δ be a distance measure for LDFSs $\mathfrak{D}$ and $\mathfrak{E}$, then $\sigma(\mathfrak{D},\mathfrak{E})=1-\delta(\mathfrak{D},\mathfrak{E})$ is a similarity measure between $\mathfrak{D}$ and $\mathfrak{E}$.

Proof:

S1) By the Definition of distance $0\le\delta(\mathfrak{D},\mathfrak{E})\le1$. Thus, we get $0\le\sigma(\mathfrak{D},\mathfrak{E})\le1$.

S2) If $\mathfrak{D}=\mathfrak{E}$, then $\delta(\mathfrak{D},\mathfrak{E})=0$. Hence $\sigma(\mathfrak{D},\mathfrak{E})=1-\delta(\mathfrak{D},\mathfrak{E})=1$. Otherwise, $\sigma(\mathfrak{D},\mathfrak{E})=1\Rightarrow1-\delta(\mathfrak{D},\mathfrak{E})=1\Rightarrow\delta(\mathfrak{D},\mathfrak{E})=0\Rightarrow\mathfrak{D}=\mathfrak{E}$.

S3) Since $\delta(\mathfrak{D},\mathfrak{E})=\delta(\mathfrak{E},\mathfrak{D})$, we get $\sigma(\mathfrak{D},\mathfrak{E})=1-\delta(\mathfrak{D},\mathfrak{E})=1-\delta(\mathfrak{E},\mathfrak{D})=\sigma(\mathfrak{E},\mathfrak{D})$.

S4) If $\mathfrak{D}$ is crisp set, then $\sigma(\mathfrak{D},\mathfrak{D}^c)=1-\delta(\mathfrak{D},\mathfrak{D}^c)=0$, by Proposition 2. Conversely, $\sigma(\mathfrak{D},\mathfrak{D}^c)=0\Rightarrow1-\delta(\mathfrak{D},\mathfrak{D}^c)=0\Rightarrow\delta(\mathfrak{D},\mathfrak{D}^c)=1$, so $\mathfrak{D}$ is crisp set.

S5) If $\mathfrak{D}\subseteq\mathfrak{E}\subseteq\mathfrak{F}$, then $\delta(\mathfrak{D},\mathfrak{F})\ge\delta(\mathfrak{D},\mathfrak{E})$ and $\delta(\mathfrak{D},\mathfrak{F})\ge\delta(\mathfrak{E},\mathfrak{F})$ for all $\mathfrak{D},\mathfrak{E},\mathfrak{F}\in LDFS(\mathfrak{R})$, by the definition of distance measure. Hence $\sigma(\mathfrak{D},\mathfrak{F})=1-\delta(\mathfrak{D},\mathfrak{F})\le1-\delta(\mathfrak{D},\mathfrak{E})=\sigma(\mathfrak{D},\mathfrak{E})$ and $\sigma(\mathfrak{D},\mathfrak{F})=1-\delta(\mathfrak{D},\mathfrak{F})\le1-\delta(\mathfrak{E},\mathfrak{F})=\sigma(\mathfrak{E},\mathfrak{F})$ for all $\mathfrak{D},\mathfrak{E},\mathfrak{F}\in LDFS(\mathfrak{R})$.

For $\mathfrak{D},\mathfrak{E}\in LDFS(\mathfrak{R})$, we have

$$\begin{aligned}
&\sigma_{\delta}(\mathfrak{D},\mathfrak{E})\\
&=1-\frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|\\+|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|\\+|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|\\+|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\\+4\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|,\\|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|,\\|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|,\\|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\end{array}\right].
\end{aligned}$$

Now, we propose weighted similarity measures between the LDFSs $\mathfrak{D}$ and $\mathfrak{E}$ as follow:

$$\sigma^{\omega}(\mathfrak{D},\mathfrak{E}) = \frac{1}{n}\sum_{i=1}^{n}\omega_i \frac{2-\left[\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|\end{pmatrix} + \max\begin{pmatrix}|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\right]}{2+\left[\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|\end{pmatrix} + \max\begin{pmatrix}|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\right]},$$

$$\sigma_{\delta}^{\omega}(\mathfrak{D},\mathfrak{E}) = 1-\frac{1}{8n}\sum_{i=1}^{n}\omega_i\left[\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|\\ +|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|\\ +|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|\\ +|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{pmatrix} + 4\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|,\\ |a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\right].$$

C. COMPARISON OF SIMILARITY MEASURES

In this subsection, we present a comparative study of the proposed similarity measures and existing similarity measures in the literature. The existing similarity measures defined by Mohammed et al. [31] are given below.

- Jaccard Similarity Measure

$$\sigma_{JSM}(\mathfrak{D},\mathfrak{E}) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\varphi_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{E}}(\varrho_i)+\upsilon_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{E}}(\varrho_i) + a_{\mathfrak{D}}(\varrho_i)a_{\mathfrak{E}}(\varrho_i)+b_{\mathfrak{D}}(\varrho_i)b_{\mathfrak{E}}(\varrho_i)}{\begin{matrix}\left(\varphi_{\mathfrak{D}}^2(\varrho_i)+\upsilon_{\mathfrak{D}}^2(\varrho_i)+a_{\mathfrak{D}}^2(\varrho_i)+b_{\mathfrak{D}}^2(\varrho_i)\right)\\ +\left(\varphi_{\mathfrak{E}}^2(\varrho_i)+\upsilon_{\mathfrak{E}}^2(\varrho_i)+a_{\mathfrak{E}}^2(\varrho_i)+b_{\mathfrak{E}}^2(\varrho_i)\right)\\ -\begin{pmatrix}\varphi_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{E}}(\varrho_i)+\upsilon_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{E}}(\varrho_i)\\ +a_{\mathfrak{D}}(\varrho_i)a_{\mathfrak{E}}(\varrho_i)+b_{\mathfrak{D}}(\varrho_i)b_{\mathfrak{E}}(\varrho_i)\end{pmatrix}\end{matrix}}\right]$$

- Exponential Similarity Measure

$$\sigma_{ESM}(\mathfrak{D},\mathfrak{E}) = e^{-\left\{\frac{1}{4n}\sum_{i=1}^{n}\left[\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|+|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|\\ +|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|+|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\right]\right\}}.$$

- Cosine Similarity Measures

$$\sigma_{Cos_1}(\mathfrak{D},\mathfrak{E}) = \frac{1}{n}\sum_{i=1}^{n}\cos\left[\frac{\pi}{2}\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|,\\ |a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\right]$$

$$\sigma_{Cos_2}(\mathfrak{D},\mathfrak{E}) = \frac{1}{n}\sum_{i=1}^{n}\cos\left[\frac{\pi}{4}\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|\\ +|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|\\ +|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|\\ +|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\right]$$

- Cotangent Similarity Measures

$$\sigma_{Cot_1}(\mathfrak{D},\mathfrak{E}) = \frac{1}{n}\sum_{i=1}^{n}\cot\left[\frac{\pi}{4}+\frac{\pi}{4}\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|,\\ |\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|,\\ |a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|,\\ |b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\right]$$

$$\sigma_{Cot_2}(\mathfrak{D},\mathfrak{E}) = \frac{1}{n}\sum_{i=1}^{n}\cot\left[\frac{\pi}{4}+\frac{\pi}{4}\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\varphi_{\mathfrak{E}}(\varrho_i)|\\ +|\upsilon_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{E}}(\varrho_i)|\\ +|a_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{E}}(\varrho_i)|\\ +|b_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)|\end{pmatrix}\right]$$

The comparison results of the existing similarity measures and the proposed similarity measures are given in Table 2, where all values in bold demonstrate unreasonable results. From Table 2, it can be seen that the similarity measures σ_{JSM}, σ_{ESM}, σ_{Cos_1}, σ_{Cos_2}, σ_{Cot_1} and σ_{Cot_2} get unreasonable results in some situations.

- The Jaccard similarity measure σ_{JSM} has found the same result in Case 3 and Case 4. The exponential similarity measure σ_{ESM} produces unreasonable result in Case 1 and has found the same result in Case 5 and Case 6.
- The cosine similarity measure σ_{Cos_1} has found same result in Case 1 and Case 2.
- The cosine similarity measure σ_{Cos_2} is not met condition S1 in Definition 8 in Case 1and has obtained an unreasonable result in Case 2 and has found the same result in Cases 5 and 6.
- The cotangent similarity measure σ_{Cot_1} has found same result in Case 1 and Case 2.
- The cotangent similarity measure σ_{Cot_2} has obtained an unreasonable result in Case 1 and is not met condition S1 in Definition 8 in Cases 2-6.

From Table 2, it can be seen that the proposed similarity measures σ, σ_{δ} have produced reasonable results for all cases.

D. ENTROPY MEASURE FOR LDFS

Definition 9: An entropy measure ε is a mapping $\varepsilon : LDFS(\mathfrak{R}) \to [0,1]$, provides the following features, for every $\mathfrak{D},\mathfrak{E} \in LDFS(\mathfrak{R})$,

E1) $\varepsilon(\mathfrak{D}) = 0$ if and only if $\mathfrak{D}$ is crisp set,

E2) $\varepsilon(\mathfrak{D}) = 1$ if $\varphi_{\mathfrak{D}}(\varrho) = \upsilon_{\mathfrak{D}}(\varrho)$ and $a_{\mathfrak{D}}(\varrho) = \beta_{\mathcal{L}}(x)$ for all $\varrho \in \mathfrak{R}$,

E3) $\varepsilon(\mathfrak{D}) = \varepsilon(\mathfrak{D}^c)$,

E4) $\varepsilon(\mathfrak{D}) \leq \varepsilon(\mathfrak{E})$ if $\mathfrak{D}$ is less fuzzy than $\mathfrak{E}$, i.e. $\mathfrak{D} \subseteq \mathfrak{E}$ for $\varphi_{\mathfrak{E}}(\varrho) \leq \upsilon_{\mathfrak{E}}(\varrho)$ and $a_{\mathfrak{E}}(\varrho) \leq b_{\mathfrak{E}}(\varrho)$ or $\mathfrak{E} \subseteq \mathfrak{D}$ for $\upsilon_{\mathfrak{E}}(\varrho) \leq \varphi_{\mathfrak{E}}(\varrho)$ and $b_{\mathfrak{E}}(\varrho) \leq a_{\mathfrak{E}}(\varrho)$

Let $\mathfrak{D} = \{(\varrho, \langle\varphi_{\mathfrak{D}}(\varrho), \upsilon_{\mathfrak{D}}(\varrho)\rangle, \langle a_{\mathfrak{D}}(\varrho), b_{\mathfrak{D}}(\varrho)\rangle) : \varrho \in \mathfrak{R}\}$ be an LDFS on reference set $\mathfrak{R}$. We define a mapping on $LDFS(\mathfrak{R})$ as follows

$$\varepsilon(\mathfrak{D}) = \frac{1}{n}\sum_{i=1}^{n}\frac{1-|a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)|}{1+|a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)|}.$$

TABLE 2. Comparison similarity measures.

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
	$\mathfrak{D}=\begin{Bmatrix}\langle 1.00,0.00\rangle,\\ \langle 1.00,0.00\rangle\end{Bmatrix}$	$\mathfrak{D}=\begin{Bmatrix}\langle 1.00,0.00\rangle,\\ \langle 0.50,0.50\rangle\end{Bmatrix}$	$\mathfrak{D}=\begin{Bmatrix}\langle 0.85,0.30\rangle,\\ \langle 0.10,0.50\rangle\end{Bmatrix}$	$\mathfrak{D}=\begin{Bmatrix}\langle 0.85,0.30\rangle,\\ \langle 0.10,0.50\rangle\end{Bmatrix}$	$\mathfrak{D}=\begin{Bmatrix}\langle 1.00,0.40\rangle,\\ \langle 0.30,0.50\rangle\end{Bmatrix}$	$\mathfrak{D}=\begin{Bmatrix}\langle 1.00,0.40\rangle,\\ \langle 0.30,0.50\rangle\end{Bmatrix}$
	$\mathfrak{E}=\begin{Bmatrix}\langle 0.00,1.00\rangle,\\ \langle 0.00,1.00\rangle\end{Bmatrix}$	$\mathfrak{E}=\begin{Bmatrix}\langle 0.00,1.00\rangle,\\ \langle 0.50,0.50\rangle\end{Bmatrix}$	$\mathfrak{E}=\begin{Bmatrix}\langle 0.35,0.45\rangle,\\ \langle 0.70,0.25\rangle\end{Bmatrix}$	$\mathfrak{E}=\begin{Bmatrix}\langle 0.55,1.00\rangle,\\ \langle 0.65,0.15\rangle\end{Bmatrix}$	$\mathfrak{E}=\begin{Bmatrix}\langle 0.60,0.85\rangle,\\ \langle 0.55,0.20\rangle\end{Bmatrix}$	$\mathfrak{E}=\begin{Bmatrix}\langle 0.50,0.70\rangle,\\ \langle 0.55,0.15\rangle\end{Bmatrix}$
σ	0.0000	0.3333	0.2903	0.2308	0.4545	0.4035
σ_δ	0.0000	0.2500	0.5125	0.4125	0.6000	0.5750
σ_{JSM}	0.0000	0.2000	**0.4745**	**0.4745**	0.7006	0.6602
σ_{ESM}	**0.3679**	0.6065	0.6873	0.6219	**0.7047**	**0.7047**
σ_{Cos_1}	**0.0000**	**0.0000**	0.5878	0.4540	0.7604	0.7071
σ_{Cos_2}	**-1.0000**	**0.0000**	0.3827	0.0785	**0.4540**	**0.4540**
σ_{Cot_1}	**0.0000**	**0.0000**	0.3249	0.2401	0.4610	0.4142
σ_{Cot_2}	**1.0000**	**-1.0000**	**-0.4142**	**-0.8541**	**-0.3249**	**-0.3249**

Theorem 4: Let $\mathfrak{D}$ be an LDFS, then $\varepsilon(\mathfrak{D})$ is an entropy measure.

Proof: In order for $\varepsilon(\mathfrak{D})$ to be described as entropy measure for LDFS, it must satisfy properties of Definition 9.

E1) If $\mathfrak{D}$ is crisp set, then $\varphi_{\mathfrak{D}}(\varrho)=1$, $\upsilon_{\mathfrak{D}}(\varrho)=0$, $a_{\mathfrak{D}}(\varrho)=1$ and $b_{\mathfrak{D}}(\varrho)=0$ or $\varphi_{\mathfrak{D}}(\varrho)=0$, $\upsilon_{\mathfrak{D}}(\varrho)=1$, $a_{\mathfrak{D}}(\varrho)=0$ and $b_{\mathfrak{D}}(\varrho)=1$. In both case, $\varepsilon(\mathfrak{D})=0$. Conversely, if $\varepsilon(\mathfrak{D})=0$, then

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1-|a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)|}{1+|a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)|}=0$$
$$1-|a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)|=0$$
$$|a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)|=1.$$

Since $a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)+b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)\leq 1$ for all $\varrho\in\mathfrak{R}$, we obtain $\mathfrak{D}$ is a crisp set.

E2) If $\varphi_{\mathfrak{D}}(\varrho)=\upsilon_{\mathfrak{D}}(\varrho)$ and $a_{\mathfrak{D}}(\varrho)=b_{\mathfrak{D}}(\varrho)$ for all $\varrho\in\mathfrak{R}$, then

$$\begin{aligned}\varepsilon(\mathfrak{D})&=\frac{1}{n}\sum_{i=1}^{n}\frac{1-|a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)|}{1+|a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)|}\\&=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1}\\&=1.\end{aligned}$$

E3)

$$\begin{aligned}&\varepsilon(\mathfrak{D})\\&=\frac{1}{n}\sum_{i=1}^{n}\frac{1-|a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)|}{1+|a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)|}\\&=\frac{1}{n}\sum_{i=1}^{n}\frac{1-|b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)|}{1+|b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)-a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)|}\\&=\frac{1}{n}\sum_{i=1}^{n}\frac{1-|a_{\mathfrak{D}^c}(\varrho_i)\varphi_{\mathfrak{D}^c}(\varrho_i)-b_{\mathfrak{D}^c}(\varrho_i)\upsilon_{\mathfrak{D}^c}(\varrho_i)|}{1+|a_{\mathfrak{D}^c}(\varrho_i)\varphi_{\mathfrak{D}^c}(\varrho_i)-b_{\mathfrak{D}^c}(\varrho_i)\upsilon_{\mathfrak{D}^c}(\varrho_i)|}\\&=\varepsilon\left(\mathfrak{D}^c\right).\end{aligned}$$

E4) If $\mathfrak{D}\subseteq\mathfrak{E}$, for $\varphi_{\mathfrak{E}}(\varrho)\leq\upsilon_{\mathfrak{E}}(\varrho)$ and $a_{\mathfrak{E}}(\varrho)\leq b_{\mathfrak{E}}(\varrho)$, we get $\varphi_{\mathfrak{D}}(\varrho)\leq\varphi_{\mathfrak{E}}(\varrho)\leq\upsilon_{\mathfrak{E}}(\varrho)\leq\upsilon_{\mathfrak{D}}(\varrho)$, $a_{\mathfrak{D}}(\varrho)\leq a_{\mathfrak{E}}(\varrho)\leq b_{\mathfrak{E}}(\varrho)\leq b_{\mathfrak{D}}(\varrho)$. If $\mathfrak{E}\subseteq\mathfrak{D}$, for $\varphi_{\mathfrak{E}}(\varrho)\geq\upsilon_{\mathfrak{E}}(\varrho)$ and $a_{\mathfrak{E}}(\varrho)\geq b_{\mathfrak{E}}(\varrho)$, we get $\varphi_{\mathfrak{D}}(\varrho)\geq\varphi_{\mathfrak{E}}(\varrho)\geq\upsilon_{\mathfrak{E}}(\varrho)\geq\upsilon_{\mathfrak{D}}(\varrho)$, $a_{\mathfrak{D}}(\varrho)\geq a_{\mathfrak{E}}(\varrho)\geq b_{\mathfrak{E}}(\varrho)\geq b_{\mathfrak{D}}(\varrho)$. In both cases, we have

$$\begin{aligned}\varepsilon(\mathfrak{D})&=\frac{1-|a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)|}{1+|a_{\mathfrak{D}}(\varrho_i)\varphi_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)\upsilon_{\mathfrak{D}}(\varrho_i)|}\\&\leq\frac{1}{n}\sum_{i=1}^{n}\frac{1-|a_{\mathfrak{E}}(\varrho_i)\varphi_{\mathfrak{E}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)\upsilon_{\mathfrak{E}}(\varrho_i)|}{1+|a_{\mathfrak{E}}(\varrho_i)\varphi_{\mathfrak{E}}(\varrho_i)-b_{\mathfrak{E}}(\varrho_i)\upsilon_{\mathfrak{E}}(\varrho_i)|}\\&=\varepsilon(\mathfrak{E}).\end{aligned}$$

Theorem 5: Suppose δ is a distance measure of LDFSs and σ is a similarity measure of LDFSs, for $\mathfrak{D}\in LDFS(\mathfrak{R})$, then $\varepsilon(\mathfrak{D})=\sigma(\mathfrak{D},\mathfrak{D}^c)=1-\delta(\mathfrak{D},\mathfrak{D}^c)$ is the entropy measure of LDFS $\mathfrak{D}$.

Proof: The proof is obvious.

For $\mathfrak{D}\in LDFS(\mathfrak{R})$, we have

$$\begin{aligned}\varepsilon_\sigma(\mathfrak{D})&=\sigma\left(\mathfrak{D},\mathfrak{D}^c\right)\\&=\frac{1}{n}\sum_{i=1}^{n}\frac{2-\begin{bmatrix}\max\left(|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)|\right)\\+\max\left(|a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\right)\end{bmatrix}}{2+\begin{bmatrix}\max\left(|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)|\right)\\+\max\left(|a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\right)\end{bmatrix}},\\\varepsilon_\delta(\mathfrak{D})&=\sigma_\delta\left(\mathfrak{D},\mathfrak{D}^c\right)\\&=1-\delta\left(\mathcal{L},\mathcal{L}^c\right)=1\\&\quad-\frac{1}{4n}\sum_{i=1}^{n}\begin{bmatrix}\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)|\\+|a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\end{pmatrix}\\+2\max\begin{pmatrix}|\varphi_{\mathfrak{D}}(\varrho_i)-\upsilon_{\mathfrak{D}}(\varrho_i)|,\\|a_{\mathfrak{D}}(\varrho_i)-b_{\mathfrak{D}}(\varrho_i)|\end{pmatrix}\end{bmatrix}.\end{aligned}$$

IV. THE PROPOSED ENTROPY-BASED VIKOR METHOD FOR LDFSS

The VIKOR method is utilized for obtaining the best alternative based on measuring closeness to ideal solutions. The significant advantage of using the method of VIKOR is its ability to provide a compromise solution. The proposed VIKOR method procedure is presented as follows:

V1: Construct the decision matrix $\mathrm{M}=\left[m_{ij}\right]_{m\times n}$ with the help of LDFN $\mathfrak{N}=(\langle\varphi_{\mathfrak{N}},\upsilon_{\mathfrak{N}}\rangle,\langle a_{\mathfrak{N}},b_{\mathfrak{N}}\rangle)$ as:

V2: Establish the normalized decision matrix $\mathrm{M}^{*}=[m_{ij}^{*}]_{m\times n}$ as follows:

$$m_{ij}^{*}=\begin{cases} m_{ij}, & \text{for Benefit criteria (BC)}\\ \left(m_{ij}\right)^{c}, & \text{for Cost Criteria (CC)}\end{cases}$$

where $\left(m_{ij}\right)^{c}=\langle\upsilon_{\mathfrak{N}_{ij}},\varphi_{\mathfrak{N}_{ij}}\rangle,\langle b_{\mathfrak{N}_{ij}},a_{\mathfrak{N}_{ij}}\rangle$ denotes the complement of m_{ij}.

V3: Calculate the weight of criteria by using entropy measure as follow:

$$\omega_{j}=\frac{1-\varepsilon_{j}}{\sum_{j=1}^{n}\left(1-\varepsilon_{j}\right)}$$

where ε_{j} is entropy of j^{th} criteria. Here, we use the following entropy measure ε:

$$\varepsilon\left(\mathfrak{D}\right)=\frac{1}{n}\sum_{i=1}^{n}\frac{1-\left|a_{\mathfrak{D}}\left(\varrho_{i}\right)\varphi_{\mathfrak{D}}\left(\varrho_{i}\right)-b_{\mathfrak{D}}\left(\varrho_{i}\right)\upsilon_{\mathfrak{D}}\left(\varrho_{i}\right)\right|}{1+\left|a_{\mathfrak{D}}\left(\varrho_{i}\right)\varphi_{\mathfrak{D}}\left(\varrho_{i}\right)-b_{\mathfrak{D}}\left(\varrho_{i}\right)\upsilon_{\mathfrak{D}}\left(\varrho_{i}\right)\right|}.$$

V4: Determine the Linear Diophantine Fuzzy Positive Ideal Solution (P_j) and Linear Diophantine Fuzzy Negative Ideal Solution (N_j) as follow:

$$\varphi_{\mathfrak{N}_{j}}^{+}=\begin{cases}\max\limits_{i}\varphi_{\mathfrak{N}_{ij}}, & \text{for BC}\\ \min\limits_{i}\varphi_{\mathfrak{N}_{ij}}, & \text{for CC}\end{cases}$$

$$\upsilon_{\mathfrak{N}_{j}}^{+}=\begin{cases}\min\limits_{i}\upsilon_{\mathfrak{N}_{ij}}, & \text{for BC}\\ \max\limits_{i}\upsilon_{\mathfrak{N}_{ij}}, & \text{for CC}\end{cases}$$

$$a_{\mathfrak{N}_{j}}^{+}=\begin{cases}\max\limits_{i}a_{\mathfrak{N}_{ij}}, & \text{for BC}\\ \min\limits_{i}a_{\mathfrak{N}_{ij}}, & \text{for CC}\end{cases}$$

$$b_{\mathfrak{N}_{j}}^{+}=\begin{cases}\min\limits_{i}b_{\mathfrak{N}_{ij}}, & \text{for BC}\\ \max\limits_{i}b_{\mathfrak{N}_{ij}}, & \text{for CC}\end{cases}$$

$$\varphi_{\mathfrak{N}_{j}}^{-}=\begin{cases}\min\limits_{i}\varphi_{\mathfrak{N}_{ij}}, & \text{for BC}\\ \max\limits_{i}\varphi_{\mathfrak{N}_{ij}}, & \text{for CC}\end{cases}$$

$$\upsilon_{\mathfrak{N}_{j}}^{-}=\begin{cases}\max\limits_{i}\upsilon_{\mathfrak{N}_{ij}}, & \text{for BC}\\ \min\limits_{i}\upsilon_{\mathfrak{N}_{ij}}, & \text{for CC}\end{cases}$$

$$a_{\mathfrak{N}_{j}}^{-}=\begin{cases}\min\limits_{i}a_{\mathfrak{N}_{ij}}, & \text{for BC}\\ \max\limits_{i}a_{\mathfrak{N}_{ij}}, & \text{for CC}\end{cases}$$

$$b_{\mathfrak{N}_{j}}^{-}=\begin{cases}\max\limits_{i}b_{\mathfrak{N}_{ij}}, & \text{for BC}\\ \min\limits_{i}b_{\mathfrak{N}_{ij}}, & \text{for CC}\end{cases}$$

and then define $P_{j}=\left(\langle\varphi_{\mathfrak{N}_{j}}^{+},\upsilon_{\mathfrak{N}_{j}}^{+}\rangle,\langle a_{\mathfrak{N}_{j}}^{+},b_{\mathfrak{N}_{j}}^{+}\rangle\right)$, $N_{j}=\left(\langle\varphi_{\mathfrak{N}_{j}}^{-},\upsilon_{\mathfrak{N}_{j}}^{-}\rangle,\langle a_{\mathfrak{N}_{j}}^{-},b_{\mathfrak{N}_{j}}^{-}\rangle\right),j=1,2,\ldots,n$.

V5: Compute the utility measure U_i and regret measure R_i as follow:

$$U_{i}=\sum_{j=1}^{n}\omega_{j}\frac{\delta\left(P_{j},m_{ij}\right)}{\delta\left(P_{j},I_{j}\right)},$$

$$R_{i}=\left(\omega_{j}\frac{\delta\left(P_{j},m_{ij}\right)}{\delta\left(P_{j},I_{j}\right)}\right)$$

for $i=1,2,\ldots,m$. In this step, we use proposed distance (δ) measure formula as follow.

$$\delta\left(\mathfrak{D},\mathfrak{E}\right)=\frac{1}{8n}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\begin{array}{c}\left|\varphi_{\mathfrak{D}}\left(\varrho_{i}\right)-\varphi_{\mathfrak{E}}\left(\varrho_{i}\right)\right|\\+\left|\upsilon_{\mathfrak{D}}\left(\varrho_{i}\right)-\upsilon_{\mathfrak{E}}(\varrho_{i})\right|\\+\left|a_{\mathfrak{D}}\left(\varrho_{i}\right)-a_{\mathfrak{E}}\left(\varrho_{i}\right)\right|\\+\left|b_{\mathfrak{D}}\left(\varrho_{i}\right)-b_{\mathfrak{E}}\left(\varrho_{i}\right)\right|\end{array}\right)\\+4\max\left(\begin{array}{c}\left|\varphi_{\mathfrak{D}}\left(\varrho_{i}\right)-\varphi_{\mathfrak{E}}\left(\varrho_{i}\right)\right|,\\\left|\upsilon_{\mathfrak{D}}\left(\varrho_{i}\right)-\upsilon_{\mathfrak{E}}(\varrho_{i})\right|,\\\left|a_{\mathfrak{D}}\left(\varrho_{i}\right)-a_{\mathfrak{E}}\left(\varrho_{i}\right)\right|,\\\left|b_{\mathfrak{D}}\left(\varrho_{i}\right)-b_{\mathfrak{E}}\left(\varrho_{i}\right)\right|\end{array}\right)\end{array}\right].$$

V6: Determine the index value Q_i as follow:

$$Q_{i}=\rho\left[\frac{U_{i}-U^{+}}{U^{-}-U^{+}}\right]+(1-\rho)\left[\frac{R_{i}-R^{+}}{R^{-}-R^{+}}\right]$$

for $i=1,2,\ldots,m$. Here $U^{+}=\min\limits_{i}U_{i}$, $U^{-}=\max\limits_{i}U_{i}$, $R^{+}=\min\limits_{i}R_{i}$ and $R^{-}=\max\limits_{i}R_{i}$. The coefficients ρ and $1-\rho$ is identified as a weight for maximum group utility U_i and individual regret R_i. In this study, ρ is assumed as 0.5.

V7: Establish the rank of the alternatives $\mathcal{X}_1,\mathcal{X}_2,\ldots,\mathcal{X}_m$ according to Q_i, S_i, and R_i $(i=1,2,\ldots,m)$ in decreasing order.

V8: Determining the best or comparison solution.

(a) If $Q\left(\mathcal{X}^{(2)}\right)-Q\left(\mathcal{X}^{(1)}\right)\geq\frac{1}{m-1}$ where the alternative $\mathcal{X}^{(1)}$ is first position and the alternative $\mathcal{X}^{(2)}$ is second position in the ranking list of Q, and m represent the number of alternatives.

(b) The alternative $\mathcal{X}^{(1)}$ should also put in order first in the ranking lists U or/and R.

If one of the conditions (a) and (b) is not satisfied, then a compromise solution set can be proposed.

(i) If the condition (b) is not satisfied, $\mathcal{X}^{(1)}$, $\mathcal{X}^{(2)}$ are the compromise solutions.

(ii) If the condition (a) is not satisfied, $\mathcal{X}^{(1)}$, $\mathcal{X}^{(2)}$, $\mathcal{X}^{(3)}$,..., $\mathcal{X}^{(t)}$ will be a set of compromise solutions where $\mathcal{X}^{(t)}$ is determined by the following equation

$$Q\left(\mathcal{X}^{(t)}\right)-Q\left(\mathcal{X}^{(1)}\right)<\frac{1}{m-1}$$

for maximum t.

In the next Section, we give an application of the proposed entropy based LDF-VIKOR with the help of an illustrative example. We compared our results using all proposed entropies and different distance measures.

TABLE 3. Original decision matrix ($\mathrm{M} = [m_{ij}]$).

	C_1				C_2				C_3			
	φ_{i1}	υ_{i1}	a_{i1}	b_{i1}	φ_{i2}	υ_{i2}	a_{i2}	b_{i2}	φ_{i3}	υ_{i3}	a_{i3}	b_{i3}
$\mathcal{X}_1$	0.81	0.47	0.52	0.39	0.47	0.81	0.32	0.51	0.56	0.71	0.79	0.15
$\mathcal{X}_2$	0.56	0.27	0.37	0.41	0.56	0.31	0.25	0.61	0.16	0.81	0.71	0.31
$\mathcal{X}_3$	0.32	0.56	0.11	0.81	0.23	0.73	0.31	0.75	0.79	0.33	0.11	0.21
$\mathcal{X}_4$	0.16	0.79	0.14	0.76	0.45	0.37	0.56	0.37	0.52	0.71	0.21	0.56
$\mathcal{X}_5$	0.81	0.31	0.81	0.13	0.71	0.22	0.67	0.29	0.32	0.89	0.33	0.56
	C_4				C_5				C_6			
	φ_{i4}	υ_{i4}	a_{i4}	b_{i4}	φ_{i5}	υ_{i5}	a_{i5}	b_{i5}	φ_{i6}	υ_{i6}	a_{i6}	b_{i6}
$\mathcal{X}_1$	0.36	0.73	0.31	0.63	0.79	0.36	0.16	0.56	0.36	0.43	0.57	0.31
$\mathcal{X}_2$	0.27	0.67	0.11	0.32	0.37	0.56	0.36	0.67	0.37	0.71	0.37	0.21
$\mathcal{X}_3$	0.19	0.39	0.32	0.56	0.71	0.81	0.26	0.76	0.56	0.63	0.27	0.32
$\mathcal{X}_4$	0.27	0.81	0.45	0.35	0.56	0.79	0.23	0.51	0.71	0.53	0.37	0.21
$\mathcal{X}_5$	0.69	0.23	0.67	0.13	0.93	0.63	0.53	0.21	0.83	0.21	0.76	0.13

V. AN APPLICATION ON HOSPITAL-BASED PAC-CVD

To illustrate the applicability of the proposed LDF-VIKOR method, we apply it to an MADM problem. Iampan et al. [63] generated the original dataset and accomplished LDF Einstein aggregation operators on it. In this work, we conducted a more inclusive MADM application. We analyzed the results obtained by using distinct entropies and distance measures to demonstrate more suitable information measures.

V1: The accumulated views of decision makers are represented by LDF decision matrix $\mathrm{M} = \left[m_{ij}\right]_{5\times 6}$ given Table 3.

V2: The normalized decision matrix $\mathrm{M}^* = \left[m^*_{ij}\right]_{5\times 6}$ is determined and it is shown in Table 4.

V3: We denote by $C_j (1 \leq j \leq 6)$ the criteria. The weights of criteria are obtained by using entropy measure ε. The entropy value of the criteria C_3 is calculated as follows:

$$\varepsilon\left(C_3\right) = \frac{1}{5}\left[\begin{array}{l} \dfrac{1-|0.79*0.56-0.15*0.71|}{1+|0.79*0.56-0.15*0.71|} \\ +\dfrac{1-|0.71*0.16-0.31*0.81|}{1+|0.71*0.16-0.31*0.81|} \\ +\dfrac{1-|0.11*0.79-0.21*0.33|}{1+|0.11*0.79-0.21*0.33|} \\ +\dfrac{1-|0.21*0.52-0.56*0.71|}{1+|0.21*0.52-0.56*0.71|} \\ +\dfrac{1-|0.33*0.32-0.56*0.89|}{1+|0.33*0.32-0.56*0.89|} \end{array}\right] = 0.6418.$$

Similarly, the entropy values of other criteria are $\varepsilon\left(C_1\right) = 0.4710$, $\varepsilon\left(C_2\right) = 0.6110$, $\varepsilon\left(C_4\right) = 0.6033$, $\varepsilon\left(C_5\right) = 0.5815$ and $\varepsilon\left(C_6\right) = 0.7460$. Then the weight of C_3 is computed as follows:

$$\omega_3 = \frac{1-0.6418}{6-\left(\begin{array}{c} 0.4710+0.6110+0.6418 \\ +0.6033+0.5815+0.7460 \end{array}\right)} = 0.1527.$$

The weights of criteria are $\omega_1 = 0.2256$, $\omega_2 = 0.1658$, $\omega_3 = 0.1527$, $\omega_4 = 0.1691$, $\omega_5 = 0.1784$, and $\omega_6 = 0.1083$.

V4: Positive and Negative Ideal Solutions are depicted in Table 5.

Step 5-6: The utility measure U_i, regret measure R_i and index value Q_i are calculated based on proposed distance is shown in Table 6.

Step 7: The ranking of all alternatives according to U_i, R_i and Q_i in increasing order based on proposed distance is displayed in Table 7. According to the proposed LDF-VIKOR method, $\mathcal{X}_5$ and $\mathcal{X}_1$ alternatives with first and second position in the ranking list of U, R and Q, respectively. Also $\mathcal{X}_1 - \mathcal{X}_5 = 0.4236 - 0.000 = 0.4236 \geq \frac{1}{4} = 0.25$. This implies that conditions (a) and (b) is satisfied. Hence the alternative $\mathcal{X}_5$ is the best solution. The ranking of all alternatives is $\mathcal{X}_5 \succ \mathcal{X}_1 \succ \mathcal{X}_2 \succ \mathcal{X}_4 \succ \mathcal{X}_3$ according to the U and Q.

VI. COMPARISON ANALYSIS

To show the accuracy of the proposed approach, the rankings of the alternatives were compared by applying two different distance measures. For this analysis, the LDF-VIKOR

$$\mathrm{M} = \left[m_{ij}\right]_{m\times n} = \begin{bmatrix} \left(\langle\varphi_{\Re_{11}}, \upsilon_{\Re_{11}}\rangle, \langle a_{\Re_{11}}, b_{\Re_{11}}\rangle\right) & \cdots & \left(\langle\varphi_{\Re_{1n}}, \upsilon_{\Re_{1n}}\rangle, \langle a_{\Re_{1n}}, b_{\Re_{1n}}\rangle\right) \\ \vdots & \vdots & \vdots \\ \left(\langle\varphi_{\Re_{m1}}, \upsilon_{\Re_{m1}}\rangle, \langle a_{\Re_{m1}}, b_{\Re_{m1}}\rangle\right) & \cdots & \left(\langle\varphi_{\Re_{mn}}, \upsilon_{\Re_{mn}}\rangle, \langle a_{\Re_{mn}}, b_{\Re_{mn}}\rangle\right) \end{bmatrix}$$

TABLE 4. Normalized decision matrix ($M^* = [m^*_{ij}]$).

	C_1				C_2				C_3			
	φ_{i1}	υ_{i1}	a_{i1}	b_{i1}	φ_{i2}	υ_{i2}	a_{i2}	b_{i2}	φ_{i3}	υ_{i3}	a_{i3}	b_{i3}
$\mathcal{X}_1$	0.81	0.47	0.52	0.39	0.47	0.81	0.32	0.51	0.56	0.71	0.79	0.15
$\mathcal{X}_2$	0.56	0.27	0.37	0.41	0.56	0.31	0.25	0.61	0.16	0.81	0.71	0.31
$\mathcal{X}_3$	0.32	0.56	0.11	0.81	0.23	0.73	0.31	0.75	0.79	0.33	0.11	0.21
$\mathcal{X}_4$	0.16	0.79	0.14	0.76	0.45	0.37	0.56	0.37	0.52	0.71	0.21	0.56
$\mathcal{X}_5$	0.81	0.31	0.81	0.13	0.71	0.22	0.67	0.29	0.32	0.89	0.33	0.56
	C_4				C_5				C_6			
	φ_{i4}	υ_{i4}	a_{i4}	b_{i4}	φ_{i5}	υ_{i5}	a_{i5}	b_{i5}	φ_{i6}	υ_{i6}	a_{i6}	b_{i6}
$\mathcal{X}_1$	0.36	0.73	0.31	0.63	0.79	0.36	0.16	0.56	0.43	0.36	0.31	0.57
$\mathcal{X}_2$	0.27	0.67	0.11	0.32	0.37	0.56	0.36	0.67	0.71	0.37	0.21	0.37
$\mathcal{X}_3$	0.19	0.39	0.32	0.56	0.71	0.81	0.26	0.76	0.63	0.56	0.32	0.27
$\mathcal{X}_4$	0.27	0.81	0.45	0.35	0.56	0.79	0.23	0.51	0.53	0.71	0.21	0.37
$\mathcal{X}_5$	0.69	0.23	0.67	0.13	0.93	0.63	0.53	0.21	0.21	0.83	0.13	0.76

TABLE 5. Positive ideal solutions (P_j) and negative ideal solutions (N_j).

	C_1				C_2				C_3			
	φ_{i1}	υ_{i1}	a_{i1}	b_{i1}	φ_{i2}	υ_{i2}	a_{i2}	b_{i2}	φ_{i3}	υ_{i3}	a_{i3}	b_{i3}
P	0.81	0.27	0.81	0.13	0.71	0.22	0.67	0.29	0.79	0.33	0.79	0.15
N	0.16	0.79	0.11	0.81	0.23	0.81	0.25	0.75	0.16	0.89	0.11	0.56
	C_4				C_5				C_6			
	φ_{i4}	υ_{i4}	a_{i4}	b_{i4}	φ_{i5}	υ_{i5}	a_{i5}	b_{i5}	φ_{i6}	υ_{i6}	a_{i6}	b_{i6}
P	0.69	0.23	0.67	0.13	0.93	0.36	0.53	0.21	0.21	0.83	0.13	0.76
N	0.19	0.81	0.11	0.63	0.37	0.81	0.16	0.76	0.71	0.36	0.32	0.27

TABLE 6. The values of U_i, R_i and Q_i based on proposed distance.

U_i		R_i		Q_i	
$\mathcal{X}_1$	0.6176	$\mathcal{X}_1$	0.1447	$\mathcal{X}_1$	0.4236
$\mathcal{X}_2$	0.7388	$\mathcal{X}_2$	0.1553	$\mathcal{X}_2$	0.5745
$\mathcal{X}_3$	0.8501	$\mathcal{X}_3$	0.2091	$\mathcal{X}_3$	0.9558
$\mathcal{X}_4$	0.7506	$\mathcal{X}_4$	0.2171	$\mathcal{X}_4$	0.9243
$\mathcal{X}_5$	0.1927	$\mathcal{X}_5$	0.1265	$\mathcal{X}_5$	0.0000

approach was performed with the Euclidean and Hamming distance measures introduced by Mohammad et al. [31]. The utility measure U_i, regret measure R_i and index value Q_i are calculated based on Euclidean and Hamming distance measures and the rankings of all alternatives according to U_i, R_i and Q_i in increasing order are displayed in Table 8-9.

In accordance with the LDF-VIKOR method based on Euclidean distance measure, $\mathcal{X}_5$ and $\mathcal{X}_1$ alternatives with first and second position in the ranking list of U, R and Q, respectively. Also $\mathcal{X}_1 - \mathcal{X}_5 = 0.3646 - 0.000 = 0.3646 \geq \frac{1}{4} = 0.25$. This implies that conditions (a) and (b) is satisfied. Thus, the alternative $\mathcal{X}_5$ is the best solution. The ranking of all alternatives is $\mathcal{X}_5 \succ \mathcal{X}_1 \succ \mathcal{X}_2 \succ \mathcal{X}_3 \succ \mathcal{X}_4$ regarding to the Q.

According to the LDF-VIKOR method based on Hamming distance measure, the condition (b) is not satisfied. Hence the alternative $\mathcal{X}_5$ and $\mathcal{X}_1$ are the compromise solution. The ranking of all alternatives is $\mathcal{X}_5 \succ \mathcal{X}_1 \succ \mathcal{X}_2 \succ \mathcal{X}_3 \succ \mathcal{X}_4$ according to the Q.

As seen, the best solution is obtained by using the proposed approach and the LDF-VIKOR method based on Euclidean distance measure. The LDF-VIKOR method based on Hamming distance measure determined a compromise solution. Also, the first three of alternatives are in the same order in the rankings of all alternatives according to Q, is displayed in Figure 1. In proposed method, only the alternatives $\mathcal{X}_4$ and $\mathcal{X}_5$ has different order from LDF-VIKOR method based on Euclidean and Hamming distance measures.

The same problem is solved with the LDF extension of TOPSIS based on Euclidean and Hamming distance measures was proposed by Gül and Aydoğdu [32]. All the values are

TABLE 7. The ranking of alternatives based on proposed distance.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$
U_i	2	3	5	4	1
R_i	2	3	4	5	1
Q_i	2	3	5	4	1

TABLE 8. The values of U_i, R_i and Q_i and ranking of alternatives based on Euclidean distance.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$	Ranking
U_i	0.5684	0.6758	0.7983	0.7040	0.1824	$\mathcal{X}_5 \succ \mathcal{X}_1 \succ \mathcal{X}_2 \succ \mathcal{X}_4 \succ \mathcal{X}_3$
R_i	0.1355	0.1407	0.1987	0.2181	0.1261	$\mathcal{X}_5 \succ \mathcal{X}_1 \succ \mathcal{X}_2 \succ \mathcal{X}_3 \succ \mathcal{X}_4$
Q_i	0.3646	0.4798	0.8944	0.9234	0.0000	$\mathcal{X}_5 \succ \mathcal{X}_1 \succ \mathcal{X}_2 \succ \mathcal{X}_3 \succ \mathcal{X}_4$

TABLE 9. The values of U_i, R_i and Q_i and ranking of alternatives based on hamming distance.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$	Ranking
U_i	0.5089	0.6092	0.7361	0.6823	0.1558	$\mathcal{X}_5 \succ \mathcal{X}_1 \succ \mathcal{X}_2 \succ \mathcal{X}_4 \succ \mathcal{X}_3$
R_i	0.1336	0.1285	0.1911	0.2185	0.1273	$\mathcal{X}_5 \succ \mathcal{X}_2 \succ \mathcal{X}_1 \succ \mathcal{X}_3 \succ \mathcal{X}_4$
Q_i	0.3388	0.3974	0.8497	0.9537	0.0000	$\mathcal{X}_5 \succ \mathcal{X}_1 \succ \mathcal{X}_2 \succ \mathcal{X}_3 \succ \mathcal{X}_4$

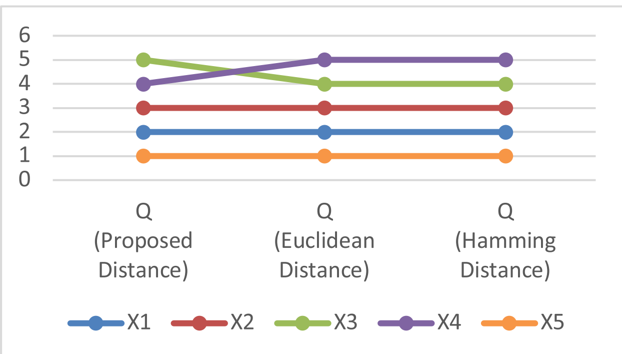

FIGURE 1. The rankings of all alternatives according to *Q*.

TABLE 10. Comparison rankings of all alternatives based on proposed distance (δ), euclidean distance (δ_M^E) and hamming distance (δ_M^H).

	LDF-VIKOR			LDF-TOPSIS [32]	
	d	δ_M^E	δ_M^H	δ_M^E	δ_M^H
$\mathcal{X}_1$	2	2	2	2	2
$\mathcal{X}_2$	3	3	3	3	3
$\mathcal{X}_3$	5	4	4	4	4
$\mathcal{X}_4$	4	5	5	5	5
$\mathcal{X}_5$	1	1	1	1	1

TABLE 11. The weights of the criteria according to different entropies.

	ε	ε_σ	ε_δ	ε_{GA_1}	ε_{GA_2}
ω_1	0.2256	0.1934	0.2086	0.2074	0.2100
ω_2	0.1658	0.1629	0.1540	0.1601	0.1540
ω_3	0.1527	0.1848	0.2004	0.1860	0.1860
ω_4	0.1691	0.1713	0.1669	0.1681	0.1621
ω_5	0.1784	0.1600	0.1488	0.1522	0.1439
ω_6	0.1083	0.1276	0.1214	0.1263	0.1439

summarized in Table 10. It seen that the alternative $\mathcal{X}_5$ is the best solution according to the LDF-TOPSIS and LDF-VIKOR except the LDF-VIKOR method based on Hamming distance measure. The compromise solution set acquired via the Hamming distance is included the alternative $\mathcal{X}_5$. Besides the ranking of alternatives according to index values remained the same except only one.

VII. SENSITIVITY ANALYSIS

To illustrate the accuracy, stability, and validity of our proposed model, we applied a sensitivity analysis. In the sensitivity analysis, the weights of the criteria were re-determined by using different entropies. Then, the effect on the final ranking was analyzed. In Table 11, we gave the weights of criteria by using the entropies ε, ε_σ, ε_δ, ε_{GA_1} and ε_{GA_2}.

The change in the weights of criteria is shown in Figure 2.

The ranking of all alternatives according to U_i, R_i and Q_i in increasing order based on proposed, Euclidean and Hamming distance measures with entropy ε_σ is displayed in Table 12-14. According to the LDF-VIKOR methods,

TABLE 12. The values of U_i, R_i and Q_i based on proposed distance with entropy measure ε_σ.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$	Ranking				
U_i	0.6242	0.7493	0.8428	0.7450	0.2120	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_4 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_3$
R_i	0.1421	0.1479	0.1793	0.1862	0.1530	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_5 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$
Q_i	0.3267	0.4914	0.9221	0.9225	0.1238	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$

TABLE 13. The values of U_i, R_i and Q_i based on euclidean distance with entropy measure ε_σ.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$	Ranking				
U_i	0.5722	0.6869	0.7889	0.6954	0.2028	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_4 \succ$	$\mathcal{X}_3$
R_i	0.1373	0.1355	0.1704	0.1871	0.1526	$\mathcal{X}_2 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_5 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$
Q_i	0.3322	0.4129	0.8383	0.9202	0.1655	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$

TABLE 14. The values of U_i, R_i and Q_i based on hamming distance with entropy measure ε_σ.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$	Ranking				
U_i	0.5118	0.6195	0.7197	0.6721	0.1794	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_4 \succ$	$\mathcal{X}_3$
R_i	0.1353	0.1289	0.1639	0.1874	0.1540	$\mathcal{X}_2 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_5 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$
Q_i	0.3623	0.4073	0.7990	0.9559	0.2147	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$

TABLE 15. The values of U_i, R_i and Q_i based on proposed distance with entropy measure ε_δ.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$	Ranking				
U_i	0.6135	0.7442	0.8407	0.7522	0.2219	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_4 \succ$	$\mathcal{X}_3$
R_i	0.1380	0.1551	0.1934	0.2008	0.1659	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_5 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$
Q_i	0.3164	0.5580	0.9410	0.9285	0.2223	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_4 \succ$	$\mathcal{X}_3$

TABLE 16. The values of U_i, R_i and Q_i based on euclidean distance with entropy measure ε_δ.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$	Ranking				
U_i	0.5628	0.6816	0.7858	0.7033	0.2131	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_4 \succ$	$\mathcal{X}_3$
R_i	0.1337	0.1405	0.1837	0.2017	0.1655	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_5 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$
Q_i	0.3053	0.4591	0.8678	0.9280	0.2335	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$

the condition (a) is not satisfied. Consequently, the alternatives $(\mathcal{X}_5, \mathcal{X}_1)$ are the compromise solutions according to the LDF-VIKOR method based on proposed distance measure. Similarly, it seen that the alternatives $(\mathcal{X}_5, \mathcal{X}_1, \mathcal{X}_2)$ are the compromise solutions according to the LDF-VIKOR methods based on Euclidean and Hamming distance measures.

Considering the values of U_i, R_i and Q_i that are shown in Table 15-17, the condition (a) is not satisfied according to the LDF-VIKOR method based on proposed, Euclidean and Hamming distance measures with entropy ε_δ. Hence, the alternatives $(\mathcal{X}_5, \mathcal{X}_1)$ are the compromise solutions according to the LDF-VIKOR method based on proposed distance measure and the alternatives $(\mathcal{X}_5, \mathcal{X}_1, \mathcal{X}_2)$ are the compromise solutions according to the LDF-VIKOR methods based on Euclidean and Hamming distance measures.

The values of U_i, R_i and Q_i, which are obtained the LDF-VIKOR method based on proposed, Euclidean and Hamming distance measures with entropy ε_{GA_1}, are shown in Table 18-20. It seen that the condition (a) is not satisfied so the alternatives $(\mathcal{X}_5, \mathcal{X}_1)$ are the compromise solution according to the LDF-VIKOR methods based on proposed and Euclidean distance measures, and the alternatives $(\mathcal{X}_5, \mathcal{X}_1, \mathcal{X}_2)$ are the compromise solutions according to the LDF-VIKOR method based on Hamming distance measure.

Analyzing the values of U_i, R_i and Q_i that are shown in Table 21-23, the condition (a) is not satisfied. Thus, the

TABLE 17. The values of U_i, R_i and Q_i based on hamming distance with entropy measure ε_δ.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$			Ranking		
U_i	0.5016	0.6133	0.7126	0.6807	0.1911	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_4 \succ$	$\mathcal{X}_3$
R_i	0.1318	0.1255	0.1767	0.2020	0.1670	$\mathcal{X}_2 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_5 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$
Q_i	0.3385	0.4048	0.8343	0.9694	0.2711	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$

TABLE 18. The values of U_i, R_i and Q_i based on proposed distance with entropy measure ε_{GA_1}.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$			Ranking		
U_i	0.6191	0.7448	0.8435	0.7488	0.2110	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_4 \succ$	$\mathcal{X}_3$
R_i	0.1397	0.1451	0.1922	0.1996	0.1540	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_5 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$
Q_i	0.3226	0.4671	0.9386	0.9252	0.1193	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_4 \succ$	$\mathcal{X}_3$

TABLE 19. The values of U_i, R_i and Q_i based on euclidean distance with entropy measure ε_{GA_1}.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$			Ranking		
Ui	0.5676	0.6825	0.7896	0.7000	0.2021	$\mathcal{X}_5\succ$	$\mathcal{X}_1\succ$	$\mathcal{X}_2\succ$	$\mathcal{X}_4\succ$	$\mathcal{X}_3$
Ri	0.1347	0.1330	0.1827	0.2005	0.1536	$\mathcal{X}_2\succ$	$\mathcal{X}_1\succ$	$\mathcal{X}_5\succ$	$\mathcal{X}_3\succ$	$\mathcal{X}_4$
Qi	0.3239	0.4089	0.8677	0.9237	0.1524	$\mathcal{X}_5\succ$	$\mathcal{X}_1\succ$	$\mathcal{X}_2\succ$	$\mathcal{X}_3\succ$	$\mathcal{X}_4$

TABLE 20. The values of U_i, R_i and Q_i based on hamming distance with entropy measure ε_{GA_1}.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$			Ranking		
U_i	0.5074	0.6150	0.7201	0.6770	0.1795	$\mathcal{X}_5\succ$	$\mathcal{X}_1\succ$	$\mathcal{X}_2\succ$	$\mathcal{X}_4\succ$	$\mathcal{X}_3$
R_i	0.1327	0.1264	0.1756	0.2009	0.1550	$\mathcal{X}_2\succ$	$\mathcal{X}_1\succ$	$\mathcal{X}_5\succ$	$\mathcal{X}_3\succ$	$\mathcal{X}_4$
Q_i	0.3455	0.4028	0.8306	0.9602	0.1917	$\mathcal{X}_5\succ$	$\mathcal{X}_1\succ$	$\mathcal{X}_2\succ$	$\mathcal{X}_3\succ$	$\mathcal{X}_4$

TABLE 21. The values of U_i, R_i and Q_i based on proposed distance with entropy measure ε_{GA_2}.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$			Ranking		
U_i	0.6194	0.7467	0.8447	0.7499	0.2085	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_4 \succ$	$\mathcal{X}_3$
R_i	0.1343	0.1440	0.1947	0.2022	0.1540	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_5 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$
Q_i	0.3229	0.4941	0.9450	0.9255	0.1452	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_4 \succ$	$\mathcal{X}_3$

TABLE 22. The values of U_i, R_i and Q_i based on euclidean distance with entropy measure ε_{GA_2}.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$	$\mathcal{X}_5$			Ranking		
U_i	0.5664	0.6851	0.7902	0.7009	0.1999	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_4 \succ$	$\mathcal{X}_3$
R_i	0.1299	0.1311	0.1850	0.2031	0.1536	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_5 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$
Q_i	0.3105	0.4191	0.8764	0.9244	0.1619	$\mathcal{X}_5 \succ$	$\mathcal{X}_1 \succ$	$\mathcal{X}_2 \succ$	$\mathcal{X}_3 \succ$	$\mathcal{X}_4$

compromise solution obtained by LDF-VIKOR method based on proposed and Euclidean distance measures with entropy measure ε_{GA_2} is the alternatives $(\mathcal{X}_5, \mathcal{X}_1)$, and the alternatives $(\mathcal{X}_5, \mathcal{X}_1, \mathcal{X}_2)$ are the compromise solutions according to the LDF-VIKOR method based on Hamming distance measure.

TABLE 23. The values of U_i, R_i and Q_i based on hamming distance with entropy measure ε_{GA_2}.

	x_1	x_2	x_3	x_4	x_5	Ranking				
U_i	0.5067	0.6178	0.7209	0.6775	0.1785	$x_5 \succ$	$x_1 \succ$	$x_2 \succ$	$x_4 \succ$	x_3
R_i	0.1280	0.1247	0.1779	0.2035	0.1550	$x_2 \succ$	$x_1 \succ$	$x_5 \succ$	$x_3 \succ$	x_4
Q_i	0.3236	0.4049	0.8379	0.9600	0.1926	$x_5 \succ$	$x_1 \succ$	$x_2 \succ$	$x_3 \succ$	x_4

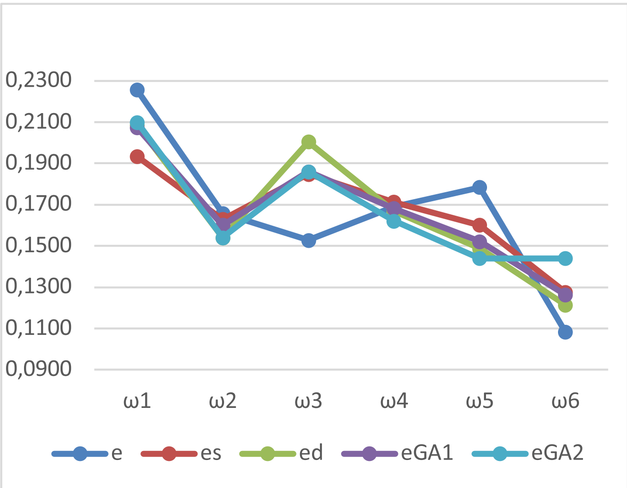

FIGURE 2. The criteria weights.

VIII. CONCLUSION

In this paper, the VIKOR method is extended for LDF sets. Firstly, we gave information measure for LDFS are introduced and studied some properties of these measures. In addition, the applicability of proposed measures in MCDM problems has been demonstrated. The LDF-VIKOR method is successfully applied to a healthcare decision-making problem. Firstly, the solutions of the LDF-VIKOR method based on the proposed distance measure were compared with the results obtained by using the Euclidean and Hamming distance measures. In extended VIKOR method with the proposed entropy measure ε, the best solution was obtained via the proposed distance measure δ and the Euclidean distance measure δ_M^E. When using the Hamming distance measure δ_M^H, a compromise solution was determined. Then, to show the validity of the extended VIKOR method, the solutions were compared with results of the LDF-TOPSIS method, which is the only extension of MCDM methods into the LDF environment. To demonstrate the effect of different entropy measures on the solutions of the LDF-VIKOR method, a sensitivity analysis was given. It has been observed that the entropy measures used in determining weights of the criteria have a significant effect on obtaining the best and compromise solutions in the LDF-VIKOR method. Therefore, the effect of the preferred entropy measure on the decision makers' determining of the solution closest to the ideal is undeniable in the objective weighting method. The model will provide mathematical model to solve for many decision-making problems including AI, robotics, machine learning, medical analysis, engineering, economics, etc. by finding more effective and accurate results in the LDF sets according to the decision makers' preference. As a future work, the presented measures can be adapted to different MCDM models such as TODIM, CODAS, MULTIMOORA, MABAC methods.

Although the LDF-VIKOR some unique characteristics, the method used in this paper need some advancements. The proposed model can be extended to multi-criteria group decision-making models in which the DMs' weights are determined by the objective weighting method. Models can be developed in which experts can assign membership and non-membership degrees and reference parameters in linguistic terms instead of specifying crisp numbers in the range of [0, 1].

DATA AVAILABILITY

No underlying data was collected or produced in this study.

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CONFLICT OF INTEREST

The author(s) declare that they have no conflict of interest.

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ALİ AYDOĞDU received the degree from the Kazım Karabekir Faculty of Education, Atatürk University, in 2008, and the Ph.D. degree in mathematics from the Institute of Natural and Applied Sciences, Atatürk University, in 2014. He is currently an Assistant Professor with Turkish National Defence University, İzmir. His research interests include fuzzy sets, fuzzy logic, number theory, math basics, and mathematical logic.