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RESEARCH ARTICLE

Model Reference Based Sliding Mode Control for an Inventory System With a Novel Demand Profile

KATARZYNA ADAMIAK AND TYMOTEUSZ ZWIERZCHOWSKI
Institute of Automatic Control, Łódź University of Technology, 90-537 Łódź, Poland
Corresponding author: Katarzyna Adamiak (katarzyna.adamiak@p.lodz.pl)

ABSTRACT This paper presents a new approach to the inventory management problem. The work considers control design for a single product warehouse with multiple suppliers with different delivery times. The demand is divided into two parts: the a priori known time-variant contractual demand and the unknown but bounded random demand. The controller's task is to generate an order sequence, such that the stock always satisfies the contractual demand and some spare products for the random buyers are stored. Therefore, we design a reference system with one supplier and one lead time only. To such a reference model we apply a sliding mode controller, which generates the appropriate order values to satisfy the contractual demand at each time instant. The model is designed so that at each time instant the warehouse receives the exact amount of goods to match the demand, consequently minimizing the required warehouse capacity. As a result, the warehouse is emptied by the end of each day. Next, we treat these values as a reference trajectory for the multiple supplier system and design a trajectory following control law. The paper proves that with appropriate compensation of the random sales, such a control strategy ensures satisfaction of the contractual demand at all time instants.

INDEX TERMS Control design, discrete-time systems, inventory control, model reference control, sliding mode control.

I. INTRODUCTION

The invention of Internet has recently led to globalization of markets and rapid changes in the consumers' demand profiles. As the time of introducing products to the market has been reduced, customer needs are prone to change faster than a few decades ago. Production and distribution centers are consequently required to become more flexible and be able to adjust their resupply chain according to the customer needs. Therefore, the problem of supply chain management for inventory systems has recently become more and more commonly discussed by the control engineering community. Nowadays, control strategies for supply and distribution chains must not only provide a stable performance, but also need to be able to quickly respond to changing customers' demand. A control theory based approach to logistics problems can be widely found in the literature [1], [2], [3], [4], [5].

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The first attempts date back to the 1950s, when a continuous time single product inventory system was considered and servo mechanism control algorithm was applied to ensure efficient goods replenishment [6]. Later on a similar approach was taken but a discrete time servo mechanism was proposed [7]. Ever since various control theory based tools have been applied to solve the supply chain management problem. Most commonly it is assumed that a control action should be designed either to always fulfill the warehouse maximum capacity or prepare for the maximum market demand. However, with changing demand, delays in the supply chain and possibility of goods degradation handling the disturbances has become the greatest challenge. Popular inventory control methods include PID controllers [1] and H∞ norm minimizing strategies [8], [9]. Recently, great attention is given to optimal control, where an objective function is defined in order to maximize the sales, reduce the cost or degradation or eliminate unsatisfied demand. Such control strategies for supply chain management or production control

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have been considered in [10], [11], [12], [13], [14], and [15]. Another intensively studied branch is model predictive control, which deals with the stochastic nature of disturbance and information exchange difficulties in the supply chain [16], [17], [18]. Most recently the learning capabilities of neural networks have also been applied in perishable inventory control [19]. However effective, all the aforementioned control methods are of great computational complexity, which may turn out inefficient in practice. Therefore, another flow of research considers application of sliding mode control strategies [20], [21], [22], [23], which are relatively less complex and may turn out more practical to apply.

Sliding mode control, originally proposed for continuous time systems [24], [25], is nowadays the most exploited branch of variable structure control. The concept of sliding mode is based on selecting a sliding hyperplane, which ensures stable steady state performance of the system and determining a control law, which drives the representative point onto it. The structure of the controller is therefore changed according to the current position of the system's representative point relative to the sliding surface. The most common sliding mode design method is through the reaching law approach. This technique, proposed by Gao and Hung [29] for continuous time systems and then extended to the discrete time case [46], assumes explicitly specifying the evolution of the sliding variable so that it reaches the sliding hyperplane and deriving a control law, which results in such motion of the system. The qualities of sliding mode control systems, such as stability, finite time convergence and insensitivity to matched external disturbances and modelling uncertainties, thoroughly studied in the literature [26], [27], [28], [29], [30], [31], have led to quick popularization of the method and development of more complex design techniques, such as adaptive sliding mode [32], [33] or integral sliding mode [34], [35], [36], [37], [38], [39]. Thanks to the advantages of sliding more control the approach is nowadays popular in heavily disturbed systems, prone to rapid parameter changes, such as unmanned vehicles [35], [36], [37], robotic manipulators [38], [39] or maglev trains [40], [41]. The applications of sliding mode control have also been extended to the discrete time domain [42], [43], [44], [45], [46], [47], [48], where the reaching law approach has been exploited. This research led to application of exponential functions [49], [50], formation of dead zone reaching law [51] and power-rate reaching laws [52], [53]. The study of sliding mode control for discrete time systems thrives in the field of multi-rate output feedback algorithms [54], [55], higher relative degree outputs [56], [57] and event-triggered control [58], [59], [60].

In this paper, a model based sliding mode control will be applied to a single product inventory, with multiple suppliers and different delivery times. This system is a classic example of a discrete time integrating plant with multiple delays. In such a case it is common to apply a simple ordering policy where the controller always tries to fulfill maximum warehouse capacity or the maximum market demand. However, the storage space is nowadays of great importance from the economical point of view. Larger amount of stored products not only occupies space, which is costly, but also requires more energy for maintenance, i.e. temperature control, electricity consumption, transportation and utilization costs. Therefore, we aim to reduce the necessary amount of stored products in order to improve the system's efficiency. In the considered case, it is assumed that the demand is partially known and represented by contractual sales. This is to mirror real-life conditions. Therefore, the aim is to satisfy the contractual demand and store a certain number of additional products reserved for random buyers. For the control design purpose, a simplified model of the warehouse with one supplier, and a priori known contractual demand only, will be generated. For such a model a discrete time sliding mode control law will be applied in order to obtain the required state trajectory. The considered quasi-sliding motion will be designed in order to drive the model's representative point onto the sliding surface in one step. In the absence of any external disturbances, the model's trajectory will remain on the sliding surface ever after, eliminating any chattering risk. Such predefined trajectory will be consecutively applied as a reference to control the real inventory system with a model following control law. It will be shown that in spite of the simplicity of the proposed method, the developed control ensures that both contractual and random demands are always satisfied within their bounds. Moreover, the knowledge of the supply structure of the real inventory system allows to further reduce the stock level without compromising the contract.

The novelty of the paper lies mainly in the unconventional demand definition consisting of two parts (a priori known contractual component and random term) and in the application of a simplified, one supplier based model in order to control a more complex inventory system. Thanks to the application of discrete time sliding mode control scheme the system's stability is ensured and the computational effort is minimal. Moreover, we develop a further compensation term allowing to decrease the initial stock value. Therefore, our control method allows to satisfy both contractual and random customer demand without the need to accumulate unnecessary amount of goods, which will further benefit the warehouse maintenance costs.

The paper is organized as follows. Section II presents the considered inventory system. Further, subsection A of section III describes the control for simplified warehouse model and subsection B proposes the model reference based control for the actual system. Finally, section IV provides simulation results and section V contains the conclusions.

II. SYSTEM PRESENTATION

In this paper we consider an inventory management system with one product, multiple suppliers and a single warehouse. These suppliers require a certain lead time to deliver the product to the warehouse. The product is then sold according to a demand separated into two parts – an a priori known contractual demand and unknown, opportunistic demand. The main objective of the system is to satisfy the contractual part of the demand. The random demand may be satisfied with the leftover product. Therefore, the warehouse capacity and the transportation costs may be reduced. The objective will be achieved with a desired trajectory following control scheme. In this section we present the notation related to the system, and describe it in the state space.

In the paper we use the following notation. Let q be a positive integer denoting the number of suppliers delivering the product to the warehouse. Furthermore, let n define the longest lead time among those suppliers. Therefore, the suppliers in the system have lead times in the range of

$$i = 1, 2, \ldots, n. \tag{1}$$

We consider all suppliers with the same lead time as a singular supplier. For every i we define a_i as the part of the controller's order allocated to the supplier with lead time equal to i. For example, the supplier with lead time equal to 1 is allocated an a_1 part of the order, and the supplier with lead time equal to n, an a_n part of the order. When a supplier with a certain lead time i does not exist, the a_i equals 0. Every a_i satisfies

$$0 \leq a_i \leq 1 \tag{2}$$

and the sum of all the a_i parameters equals 1,

$$\sum_{i=1}^{n} a_i = 1. \tag{3}$$

The order of the control system depends on the maximum lead time and is denoted with $n + 1$. The goal is to fulfill the customers' demand $d(k)$

$$d(k) = d_c(k) + d_r(k). \tag{4}$$

The contractual part of the demand $d_c(k)$ changes in time. Its expected, a priori known, values are denoted with $\tilde{d}_c(k)$. However, a situation when not all of the contracted goods are purchased at some instant k may exist. Therefore,

$$0 \leq d_c(k) \leq \tilde{d}_c(k). \tag{5}$$

The contractual sales have higher priority, and the unknown demand part $d_r(k)$ can be fulfilled with leftover product. The unknown demand is bounded as follows

$$0 \leq d_r(k) \leq d_{\max}. \tag{6}$$

We continue with the system's description in the state space. The dynamics of the inventory system are defined as

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}u(k) - \boldsymbol{f}h(k) \\ y(k) &= x_1(k), \end{aligned} \tag{7}$$

with the state vector $\boldsymbol{x}(k)$, of $n + 1$ order, containing the information about the amount of goods in the warehouse and the orders on their way to it. Matrix $\boldsymbol{A}$ is the state matrix of the system, and $\boldsymbol{b}$ is its input vector. The product stored currently in the warehouse is defined as the scalar output $y(k)$. Control signal $u(k)$ represents the amount of product ordered by the controller. The wares sold at the given instant are defined as $h(k)$. We will now define all of those vectors and matrices.

The state vector $\boldsymbol{x}(k)$ of the system has $n + 1$ elements. It can be defined as

$$\boldsymbol{x}(k) = \begin{bmatrix} x_1(k) & x_2(k) & \ldots & x_n(k) & x_{n+1}(k) \end{bmatrix}^T. \tag{8}$$

The vector's first element $x_1(k)$ holds the information about the amount of product stored inside the warehouse before any of the demand is fulfilled at any time instant k. All the other elements represent the wares on their way to the warehouse, and are delayed orders generated by the controller. The $x_{n+1}(k)$ is the amount of product ordered at the $k-1$ instant, the $x_n(k)$ – at the $k-2$ instant etc.

The state matrix $\boldsymbol{A}$ has to be created after considering the nature of the inventory system. The suppliers that fulfill the orders placed by the controller need a certain lead time before their deliveries can reach the warehouse. Moreover, we need to consider the fact that those lead times can differ. Therefore, the state matrix of the system takes the following form

$$\boldsymbol{A} = \begin{bmatrix} 1 & a_n & a_{n-1} & \cdots & a_2 & a_1 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n+1 \times n+1}. \tag{9}$$

This reflects the delayed orders from the controller. Those orders are then delivered to the warehouse in parts, according to the a_i parameters in the top row of the matrix. The input vector $\boldsymbol{b}$ of the system has $n + 1$ elements and is described as

$$\boldsymbol{b} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix}_{n+1}^T. \tag{10}$$

This ensures that the value of the control signal at the instant k becomes the last state variable of the system x_{n+1} at the instant $k + 1$.

Finally, we define the vector $\boldsymbol{f}$. It represents the sales at any given time instant. It has $n + 1$ elements and

$$\boldsymbol{f} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \end{bmatrix}_{n+1}^T. \tag{11}$$

The amount of product sold is $h(k)$. It's existence is justified by the fact that we cannot sell more product than what is stored in the warehouse. Moreover, we want to prioritize the known contractual part of the demand, so we need to distinguish both parts

$$h(k) = h_c(k) + h_r(k), \tag{12}$$

where $h_c(k)$ is the amount of fulfilled contract demand, and the $h_r(k)$ is the amount of opportunistic sales. Those two values cannot be greater than their respective demands.

$$h_c(k) \leq d_c(k) \tag{13}$$

$$h_r(k) \leq d_r(k). \tag{14}$$

As mentioned before, we prioritize the contractual part of the demand, so it is easy to see that if

$$h_c(k) < d_c(k), \tag{15}$$

then $h_r(k) = 0$.

To calculate the value of $h_c(k)$ we need to consider the current stock level $y(k)$, which yields

$$h_c(k) = \min[y(k), d_c(k)]. \tag{16}$$

If there is any leftover product in the warehouse after fulfilling the contractual part of the demand, i.e.

$$y(k) - h_c(k) > 0, \tag{17}$$

then we can calculate the value of $h_r(k)$. If we define the amount of leftover product as

$$y_r(k) = y(k) - h_c(k), \tag{18}$$

then

$$h_r(k) = \min[y_r(k), d_r(k)]. \tag{19}$$

Considering the sales, the current amount of goods stored in the warehouse for any $k \le n$ may be calculated as

$$y(k) = y_0 + \sum_{i=1}^{n-1} a_i \sum_{j=0}^{k-i-1} u(j) - \sum_{j=0}^{k-1} h(j), \tag{20}$$

and for any $k \ge n+1$ as

$$y(k) = y_0 + \sum_{j=0}^{k-n-1} u(j) + \sum_{i=1}^{n-1} a_i \sum_{j=k-n}^{k-i-1} u(j) - \sum_{j=0}^{k-1} h(j), \tag{21}$$

where $y_0 = x_1(0)$ represents the initial stock. In the end, we can also calculate the amount of product left in the warehouse at instant k after all the sales from instant k conclude. Let us denote it with $y_s(k)$, where

$$y_s(k) = y(k) - h(k). \tag{22}$$

In the next chapter we propose a model reference based order strategy for the presented system. The idea of the designed controller is to ensure that the contractual demand is always fulfilled, i.e.

$$h_c(k) = d_c(k) \tag{23}$$

and for any $k \ge 0$ and the leftover sales satisfy

$$0 \le h_r(k) \le d_r(k). \tag{24}$$

III. CONTROL STRATEGY

A. REFERENCE MODEL

In this work we propose an innovative approach to the popular problem of inventory management. We design the ordering strategy to ensure the contractual sales based on a simplified reference model. Next, we implement a model following sliding mode control law to the actual inventory system so that it follows the reference ordering policy. The control law we use is non-switching and therefore does not cause the unwelcome chattering effect.

Firstly, we define the reference model of an inventory system. As the inventory systems' modelling is usually a complex and time consuming process, this work aims to simplify it by using a one supplier based model. Therefore, we introduce a basic inventory system's model, with $q = 1$, which represents one supplier. The supplier's lead equals the maximum lead time of the real system described in the previous chapter, i.e. it is equal to n. As the system has one supplier only with the lead time n, the whole order is allocated to this supplier, which is denoted with $a_n = 1$ and $a_1, a_2, \ldots, a_{n-1} = 0$. Moreover, for the model we only consider the contractual sales $h_c(k)$, further denoted as $h_{cm}(k)$ in order to differentiate between the model and the actual control system. The way to obtain $h_{cm}(k)$ will be shown further in this chapter. It is worth pointing out that the $y(k)$ from (16) becomes $y_m(k)$ for the reference model. Such an inventory model is described by

$$\boldsymbol{x_m}(k+1) = \boldsymbol{A_m}\boldsymbol{x_m}(k) + \boldsymbol{b_m}u_m(k) - \boldsymbol{f}h_{cm}(k). \tag{25}$$

The state vector $\boldsymbol{x_m}(k)$ contains the current number of goods in the warehouse model as the first state variable $x_{m1}(k)$ and the number of goods that have already been ordered from the supplier and are on its way to the warehouse in the next n state variables. Therefore, it is of $n+1$ order, as n is the lead time of the system.

$$\boldsymbol{x_m}(k) = \begin{bmatrix} x_{m1}(k) & x_{m2}(k) & \ldots & x_{mn}(k) & x_{mn+1}(k) \end{bmatrix}^T. \tag{26}$$

The model's output signal is the current number of goods stored in the warehouse at instant k before any sales take place, represented by the first state variable

$$y_m(k) = x_{m1}(k). \tag{27}$$

The state matrix of the model is constructed in the same way as for the original inventory system. As all orders are allocated to one supplier with lead time n, $\boldsymbol{A_m}$ becomes

$$\boldsymbol{A_m} = \begin{bmatrix} 1 & a_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n+1 \times n+1}, \tag{28}$$

where $a_n = 1$. This means that the whole order generated by the controller reaches the reference warehouse after $n+1$ discrete instants. The control distribution vector is

$$\boldsymbol{b_m} = \boldsymbol{b} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T_{n+1}. \tag{29}$$

The scalar control signal $u_m(k)$ represents the number of goods to be ordered at current instant k. Vector $\boldsymbol{f}$ represents the sales, as in the actual plant. As for the reference system, only the contractual demand is considered the amount of sold goods is represented with $h_{cm}(k)$. The volume of these sales is equal to the contractual demand $d_c(k)$ as long as the number of stock is sufficient, which yields:

$$h_{cm}(k) = \min\left[d_c(k), y_m(k)\right]. \tag{30}$$

and the contractual demand $d_c(k)$ always matches the a priori known expected values, i.e.

$$d_c(k) = \tilde{d}_c(k). \tag{31}$$

Considering the sales and the structure of matrix $\boldsymbol{A_m}$ (28) the current stock may be calculated as

$$y_m(k) = y_{m0} + \sum_{j=0}^{k-n-1} u_m(j) - \sum_{j=0}^{k-1} h_{cm}(j), \tag{32}$$

where y_{m0} is the initial stock at $k = 0$. Naturally, the presented model shall be controlled so that the sales will always satisfy the contractual demand. Moreover, we denote the number of goods remaining in the warehouse model after the sales take place with:

$$y_{sm}(k) = y_m(k) - h_{cm}(k). \tag{33}$$

Considering that the contract is a priori known, we aim to design a control strategy that provides the exact number of goods planned for sale on the certain day. For that purpose we propose to apply a discrete time sliding mode control scheme with a time-variant sliding surface.

In the first step of control design, we define the model's demand state vector $\boldsymbol{x_{dm}}(k)$. This vector is of size $n+1$ and contains the contractual demand for the next $n+1$ time instants, so that

$$\boldsymbol{x_{dm}}(k) = \left[d_c(k) \quad d_c(k+1) \quad \ldots \quad d_c(k+n-1) \quad d_c(k+n)\right]_{n+1}^T, \tag{34}$$

where $d_c(k)$ denotes the contractual demand for the step k. Next, we proceed to define the sliding surface and the sliding variable $s_m(k)$. As the demand state vector changes in time, we define time-variant sliding plane as:

$$s_m(k) = \boldsymbol{c_m x_{dm}}(k) - \boldsymbol{c_m x_m}(k) = 0, \tag{35}$$

where $\boldsymbol{c_m}$ is a row vector such that $(\boldsymbol{c_m b_m})^{-1} \neq 0$. The model's sliding variable $s_m(k)$ describes the current position of the representative point of the system relative to the demand position $\boldsymbol{c_m x_{dm}}$. We assume that at the initial time instant $k = 0$, the model's representative point is on the sliding surface, so

$$\boldsymbol{c_m x_m}(0) = \boldsymbol{c_m x_{dm}}(0) \tag{36}$$

and

$$y_{m0} = x_{m1}(0) = \sum_{j=0}^{n} d_c(j). \tag{37}$$

This may be understood as a situation when at the beginning of the control process the stock of the modelled warehouse is sufficient to satisfy the contractual demand for the period $0 \leq k \leq n$. In other words, there is enough products to satisfy the sales up to the instant the highest lead time delivery is received, as the products ordered at $k = 0$ are received in the warehouse at $k = n + 1$.

The definition of the sliding variable (35) contains model's trajectory $\boldsymbol{c_m x_m}(k)$, which shall follow the contractual demand. For that purpose we apply a simple reaching law of Utkin and Drakunov [44], which keeps the sliding variable on the sliding surface for any $k > 0$, *i.e.*

$$s_m(k+1) = \boldsymbol{c_m}\left[\boldsymbol{x_{dm}}(k+1) - \boldsymbol{x_m}(k+1)\right] = 0. \tag{38}$$

By inserting the model's state equation (25) into the reaching law (38) we obtain the following control law for the model

$$u_m(k) = (\boldsymbol{c_m b_m})^{-1}\left[\boldsymbol{c_m x_{dm}}(k+1) - \boldsymbol{c_m A_m x_m}(k) + \boldsymbol{c_m f} h_{cm}(k)\right]. \tag{39}$$

At the instant k, when $u_m(k)$ is calculated, the sales value $h_{cm}(k)$ is still unknown. However, the control must ensure that $h_{cm}(k) = d_c(k)$. Therefore, we substitute this value in the control law, obtaining

$$u_m(k) = (\boldsymbol{c_m b_m})^{-1}\left[\boldsymbol{c_m x_{dm}}(k+1) - \boldsymbol{c_m A_m x_m}(k) + \boldsymbol{c_m f} d_c(k)\right]. \tag{40}$$

The parameters used in (40) originate from the model's state equation (25). $d_c(k)$ is a priori known, as it denotes the contractual demand. $\boldsymbol{x_{dm}}(k)$ is the desired state vector consisting of the contractual demand in step k and the n consecutive time instants, as shown in (34). Finally, the choice of control vector $\boldsymbol{c_m}$ is arbitrary. However, it must ensure stability of the sliding mode. A way to select $\boldsymbol{c_m}$ is shown further in the section.

The control (40) forces the representative point of the model to reach and remain on the sliding surface. As no external disturbances act on the model, the risk of chattering is hereby eliminated. From the state equation (25) with the control (40) one may obtain the closed-loop system's state matrix $\boldsymbol{A_{cl}}$ and the characteristic polynomial $M(z)$:

$$\begin{aligned} M(z) &= \det\left(z\mathbf{1}_{n+1} - \boldsymbol{A_{cl}}\right) \\ &= \det\left\{z\mathbf{1}_{n+1} - \left[\mathbf{1}_{n+1} - \boldsymbol{b_m}(\boldsymbol{c_m b_m})^{-1}\boldsymbol{c_m}\right]\boldsymbol{A_m}\right\} \end{aligned} \tag{41}$$

The control vector $\boldsymbol{c_m}$ is chosen in a deadbeat manner, so that $M(z) = z^{n+1}$ and in order to ensure the fastest possible convergence to the demand state. Therefore, $\boldsymbol{c_m}$ becomes:

$$\boldsymbol{c_m} = \left[1 \quad a_n \quad \sum_{i=n-1}^{n} a_i \quad \ldots \quad \sum_{i=2}^{n} a_i \quad \sum_{i=1}^{n} a_i\right]_{n+1}. \tag{42}$$

In order to simply further calculations we introduce the following Lemma.

Lemma 1: The choice of vector $\boldsymbol{c}$ according to (42) for the system defined as in (7) with matrix $\boldsymbol{A}$, vectors $\boldsymbol{b}$ and $\boldsymbol{f}$ defined with (9), (10), (11) guarantees the following

$$\boldsymbol{cA} = \boldsymbol{c} \tag{43}$$

$$\boldsymbol{cb} = 1 \tag{44}$$
$$\boldsymbol{cf} = 1. \tag{45}$$

Proof of Lemma 1: The above equalities may be obtained by simple mathematical multiplications. As vector $\boldsymbol{c}$ holds 1 in the first position (45) holds. Furthermore, it holds that $\sum_{i=1}^{n} a_i = 1$.Therefore, vector $\boldsymbol{c}$ always holds 1 in the last position as well. Therefore (44) holds. Finally, as matrix $\boldsymbol{A}$ has ones above the diagonal $\boldsymbol{cA}$ may be easily calculated

$$\boldsymbol{cA} = \begin{bmatrix} 1 & a_n & \sum_{i=n-1}^{n} a_i & \ldots & \sum_{i=2}^{n} a_i & \sum_{i=1}^{n} a_i \end{bmatrix}_{n+1}, \tag{46}$$

which ends the proof. ■

Considering the state matrix of the model $\boldsymbol{A_m}$, $\boldsymbol{c_m} = [1\ 1 \ldots 1\ 1]_{n+1}$. The calculated control law (40) ensures that the model's sliding variable reaches the sliding surface in one control step. As follows from (35), when $s_m(k) = 0$

$$\boldsymbol{c_m x_{dm}}(k) = \boldsymbol{c_m x_m}(k). \tag{47}$$

Theorem 1: The control law (40) applied to the inventory model (25) with the initial condition (37) ensures that for any $k \geq 0$ the control signal is

$$u_m(k) = d_c(k + n + 1). \tag{48}$$

Moreover, the model's state vector for $0 \leq k < n$ is

$$\boldsymbol{x_m}(k) = \begin{bmatrix} \sum_{j=k}^{n} d_c(j) & \underbrace{0 \ \ldots \ 0}_{n-k} & d_c(n+1) & \ldots & d_c(n+k) \end{bmatrix}_{n+1}^{T} \tag{49}$$

and for any $k \geq n$

$$\boldsymbol{x_m}(k) == \boldsymbol{x_{dm}}(k) = \begin{bmatrix} d_c(k) & d_c(k+1) & \ldots & d_c(k+n-1) & d_c(k+n) \end{bmatrix}_{n+1}^{T}. \tag{50}$$

Therefore, the model satisfies the contractual demand $d_c(k)$ for any $k \geq 0$. Furthermore, for any $k \geq n$, $s_m(k) = 0$, so the quasi-sliding mode is achieved in finite time.

Proof of Theorem 1: Let us begin the proof by calculating the control (40) at $k = 0$ with the initial condition (37)

$$u_m(0) = (\boldsymbol{c_m b_m})^{-1}\left[\boldsymbol{c_m x_{dm}}(1) - \boldsymbol{c_m A_m x_m}(0) + \boldsymbol{c_m f} d_c(0)\right]. \tag{51}$$

Using *Lemma 1*, this simplifies to

$$u_m(0) = \sum_{j=1}^{n+1} d_c(j) - \sum_{j=0}^{n} d_c(j) + d_c(0) = d_c(n+1). \tag{52}$$

Substituting (52) into the state equation (25) we obtain the model's state vector

$$\boldsymbol{x_m}(1) = \begin{bmatrix} \sum_{j=1}^{n} d_c(j) & \underbrace{0 \ \ldots \ 0}_{n-1} & d_c(n+1) \end{bmatrix}_{n+1}^{T}. \tag{53}$$

In the same manner we obtain $u_m(1)$

$$u_m(1) = \sum_{j=2}^{n+2} d_c(j) - \sum_{j=1}^{n+1} d_c(j) + d_c(1) = d_c(n+2) \tag{54}$$

and $\boldsymbol{x_m}(2)$

$$\boldsymbol{x_m}(2) = \begin{bmatrix} \sum_{j=2}^{n} d_c(j) & \underbrace{0 \ \ldots \ 0}_{n-2} & d_c(n+1) & d_c(n+2) \end{bmatrix}_{n+1}^{T}. \tag{55}$$

By continuing the same reasoning for $k = n$ we get

$$\boldsymbol{x_m}(n) = \begin{bmatrix} d_c(n) & d_c(n+1) & \ldots & d_c(n+n) \end{bmatrix}_{n+1}^{T}, \tag{56}$$

which proves that for $0 \leq k < n$ (48) and (49) hold. The control at $k = n$ is

$$u_m(n) = d_c(n + n + 1). \tag{57}$$

By substituting (57) into (25) we get

$$\boldsymbol{x_m}(n+1) = \begin{bmatrix} d_c(n+1) & d_c(n+2) & \ldots & d_c(n+n+1) \end{bmatrix}_{n+1}^{T}. \tag{58}$$

We conclude that for any $k \geq n$ the model's state vector becomes

$$\boldsymbol{x_m}(k) = \begin{bmatrix} d_c(k) & d_c(k+1) & \ldots & d_c(k+n) \end{bmatrix}_{n+1}^{T}. \tag{59}$$

and the control equals (48). Considering (59) and (34), one may see that (50) holds. Furthermore, taking into account the definition of the sliding surface (35), we conclude that $s_m(k) = 0$, for any $k \geq n$, which ends the proof. ■

Theorem 1 shows that at any $k > 0$ the current warehouse stock, represented by the first state variable is greater than or equal to the contractual demand, i.e.

$$y_m(k) = x_{m1}(k) \geq d_c(k). \tag{60}$$

Therefore, the sales are expressed as

$$h_{cm}(k) = d_c(k). \tag{61}$$

As follows, the remaining number of goods in the warehouse becomes

$$y_{sm}(k) = y_m(k) - h_{cm}(k) = d_c(k) - d_c(k) = 0, \tag{62}$$

for any $k \geq n$. In other words, at each time instant the delivery exactly matches the contractual demand and at the end of the day the warehouse is emptied. Therefore, the volume of stored products is reduced to minimum, which allows to reduce the storage costs.

B. MODEL FOLLOWING CONTROL

In this section we design the control law for a real warehouse with multiple suppliers and contractual plus random sales, introduced in Section II, described with (7). The warehouse has q suppliers, whose highest lead time is denoted with n. Therefore, the ordered goods appear in stock latest after $n+1$ time instants, so the system is of $n+1$ order. In this case both contractual and random demand and sales are considered. Therefore, $h(k)$ represents the actual sales in the system at the time instant k. We also define $\tilde{h}_c(k)$, which represents the expected number of sold goods, according to the contract and is calculated as

$$\tilde{h}_c(k) = \min\left[y(k),\ \tilde{d}_c(k)\right]. \tag{63}$$

There might exist instants such that $\tilde{h}_c(k) \neq h_c(k)$. This is to enable a situation when the contracting entities do not purchase some of the contracted goods.

Having introduced the necessary notation, we proceed to design a reference trajectory following control scheme for the system. We would like to control the system according to the reference order sequence generated by the simplified single supplier based model. The ordering sequence is expressed with the model's trajectory $\boldsymbol{c_m}x_{\boldsymbol{m}}(k)$. This trajectory may be generated in advance, as the contractual demand is known a priori, and transferred to the system's controller. We denote the reference trajectory with $s_d(k)$

$$s_d(k) = \boldsymbol{c_m x_m}(k). \tag{64}$$

Next, we describe the position of the representative point of the system relative to the reference position with the sliding variable

$$s(k) = s_d(k) - \boldsymbol{cx}(k), \tag{65}$$

where $\boldsymbol{c}$ is a vector chosen according to (42). We define the sliding plane as $s(k) = 0$.

As the reference trajectory varies in time, such definition may be understood as a time-varying sliding surface. The current value of the sliding variable $s(k)$ represents the error between the position of the representative point of the system $\boldsymbol{cx}(k)$ and the reference position $s_d(k)$. It is conventionally considered that at the initial time $k=0$, the sliding variable is on the sliding surface, so the initial condition of the system satisfies

$$y_0 = x_1(0) = x_{m1}(0) \tag{66}$$

and $s(0) = 0$. In other words, we assume that the initial stock of the warehouse is sufficient to satisfy the contractual obligations for the first n time instants, up to the instant $n+1$, when the delivery of longest lead time is received. However, in the discussed case both contractual and random demands occur from the very beginning of the control process. Consequently, the initial stock shall also provide a reserve of products for random buyers. Therefore, the initial condition will be further modified with a designed compensation term.

The control's objective is to drive the representative point of the system to the demand position $s_d(k)$ at each discrete time instant. This is expressed with

$$s(k+1) = s_d(k+1) - \boldsymbol{cx}(k+1) = 0 \tag{67}$$

Considering (67) and (7) the following control law may be derived

$$u(k) = (\boldsymbol{cb})^{-1}\left\{\boldsymbol{c_m x_m}(k+1) - \boldsymbol{cAx}(k) + \boldsymbol{cf}h(k)\right\}. \tag{68}$$

However, at the control design stage, the sales $h(k)$ are unknown, as it contains both contractual and random sales, upper bounded by (13) and (14), respectively. Considering the expected sales volume (5) and the upper bounds of random sales in a single step k, described by (6), we get

$$h(k) \le d_c(k) + d_r(k) \le \tilde{d}_c(k) + d_{\max}. \tag{69}$$

In order to make the control feasible we substitute vector $h(k)$ with its maximum bounds

$$h(k) = \tilde{d}_c(k) + d_{\max}. \tag{70}$$

Finally, one must notice that the compensation term d_{max} only compensates for a single step random sale appearing at instant k. The goods ordered according to this term will arrive at the warehouse at instant $k+n+1$ and then will be available for sale. However, at steps $k+1$, $k+2$, ..., $k+n+1$ there will be no reserve product for random buyers. Consequently, an additional compensation term is required to provide a reserve for randomized buyers for the whole period of $n+1$ time instants. Therefore, we introduce the compensation vector $\boldsymbol{D_{max}}$ such that

$$\boldsymbol{D_{max}} = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \end{bmatrix}_{n+1}^{T} d_{\max}, \tag{71}$$

which provides a reserve of products for the whole reference trajectory. With this compensation term, the control law becomes

$$u(k) = (\boldsymbol{cb})^{-1}\Big\{\boldsymbol{c_m x_m}(k+1) - \boldsymbol{cAx}(k) + \boldsymbol{cf}\left[\tilde{d}_c(k) + d_{\max}\right] + \boldsymbol{cD_{max}}\Big\}, \tag{72}$$

where all the variables are known, so the control is realizable. Furthermore, the initial condition of the system must be modified, in order to provide spare product for random sales for steps $k = 0, 1, 2, \dots, n$. The initial stock satisfying these demands is described as

$$y_0 = x_1(0) = \sum_{j=0}^{n} d_c(j) + \boldsymbol{cD_{max}}. \tag{73}$$

The control law (72), presented above, ensures that the representative point of the system remains in the vicinity of sliding surface for any $k \ge 0$. The width of this vicinity is represented by

$$|s(k)| = |s_d(k) - \boldsymbol{cx}(k)| \le \boldsymbol{cf}d_{\max} + \boldsymbol{cD_{max}}. \tag{74}$$

Next, it will be proved that the stock level satisfies the contractual demand in the presence of random buyers.

Theorem 2: The control law (72) applied to the inventory system (7) with the initial condition (73) ensures that for any $k \geq 0$ the control signal $u(k)$ satisfies

$$u(k) \leq d_c(k+n+1) + d_{\max}. \tag{75}$$

For any $k \geq 0$ the stock level satisfies

$$y(k) \geq d_c(k) + d_{\max}. \tag{76}$$

Therefore, it holds that

$$h_c(k) = d_c(k) \tag{77}$$

and

$$y_r(k) \geq d_{\max}. \tag{78}$$

Consequently, both contractual and random demands are always satisfied within their bounds.

Proof of Theorem 2: To prove the above Theorem we will consider the worst possible case, when the random demand always takes its maximum value, so $d_r(k) = d_{\max}$. With this assumption, we will show that the proposed control provides enough product in stock to satisfy both contractual and random demand at any $k \geq 0$. Considering Lemma 1, the control (72) simplifies to

$$u(k) = \boldsymbol{c_m x_m}(k+1) - \boldsymbol{cx}(k) + \tilde{d}_c(k) + d_{\max} + \boldsymbol{cD_{max}}. \tag{79}$$

Moreover, let us notice that the term $\boldsymbol{cD_{max}}$ has the following structure

$$\boldsymbol{cD_{max}} = \begin{bmatrix} 1 & a_n & \sum_{i=n-1}^{n} a_i & \dots & \sum_{i=2}^{n} a_i & \underbrace{\sum_{i=1}^{n} a_i}_{1} \end{bmatrix}_{n+1} \times \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \end{bmatrix}_{n+1}^{T} d_{\max}. \tag{80}$$

After multiplication of the right hand side of (80) we obtain

$$\boldsymbol{cD_{max}} = d_{\max} + a_n d_{\max} + \sum_{i=n-1}^{n} a_i d_{\max} + \dots + \sum_{i=2}^{n} a_i d_{\max} + d_{\max} \geq 2d_{\max}. \tag{81}$$

Considering the system's initial condition (73) let us define $u(0)$

$$\begin{aligned} u(0) &= \sum_{j=1}^{n+1} d_c(j) - \sum_{j=0}^{n} d_c(j) - \boldsymbol{cD_{max}} \\ &\quad + \tilde{d}_c(0) + d_{\max} + \boldsymbol{cD_{max}} \\ &= \tilde{d}_c(0) - d_c(0) + d_c(n+1) + d_{\max}. \end{aligned} \tag{82}$$

We consider the worst case, i.e. when the maximum number of products is demanded and sold at each time instant. Therefore:

$$\Delta d_c(k) = \tilde{d}_c(k) - d_c(k) = 0. \tag{83}$$

For such a case the control signal becomes

$$u(0) = d_c(n+1) + d_{\max}. \tag{84}$$

As the initial stock provides enough product, the contractual sales at $k = 0$ are $h_c(0) = d_c(0)$ and the random sales are assumed to be maximum. Therefore the state vector at $k = 1$, according to (7), becomes

$$\boldsymbol{x}(1) = \begin{bmatrix} \sum_{j=1}^{n} d_c(j) + \boldsymbol{cD_{max}} - d_{\max} \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} n-1 \\ d_c(n+1) + d_{\max} \end{bmatrix}_{n+1} \tag{85}$$

and it is satisfied that

$$\boldsymbol{cD_{max}} - d_{\max} = a_n d_{\max} + \sum_{i=n-1}^{n} a_i d_{\max} + \dots + \sum_{i=2}^{n} a_i d_{\max} + d_{\max} \geq d_{\max}. \tag{86}$$

Next, let us calculate $u(1)$

$$\begin{aligned} u(1) &= \sum_{j=2}^{n+2} d_c(j) - \sum_{j=1}^{n} d_c(j) - \boldsymbol{cD_{max}} \\ &\quad + d_{\max} - d_c(n+1) - d_{\max} \\ &\quad + \tilde{d}_c(1) + d_{\max} + \boldsymbol{cD_{max}} \\ &= \tilde{d}_c(1) - d_c(1) + d_c(n+2) + d_{\max}. \end{aligned} \tag{87}$$

Further assuming that maximum demand occurs, so (83) holds, the control becomes

$$u(1) = d_c(n+2) + d_{\max}. \tag{88}$$

Considering the stock, sales at $k = 1$ are $h_c(1) = d_c(1)$ and $h_r(1) = d_{\max}$. Further, $\boldsymbol{x}(2)$ becomes

$$\boldsymbol{x}(2) = \begin{bmatrix} y(2) \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} n-2 \\ d_c(n+1) + d_{\max} \\ d_c(n+2) + d_{\max} \end{bmatrix}_{n+1}, \tag{89}$$

where

$$y(2) = \sum_{j=2}^{n} d_c(j) + \boldsymbol{cD_{max}} - 2d_{\max} + a_1 d_c(n+1) + a_1 d_{\max}. \tag{90}$$

For the sake of clarity we denote the terms compensating for random sales with $g(k)$. Considering the stock $y(2)$ and (81) one may notice that

$$\begin{aligned} g(2) &= \boldsymbol{cD_{max}} - 2d_{\max} + a_1 d_{\max} \\ &= a_n d_{\max} + \sum_{i=n-1}^{n} a_i d_{\max} + \dots + \sum_{i=2}^{n} a_i d_{\max} + a_1 d_{\max}. \end{aligned} \tag{91}$$

As (3) holds, (91) becomes

$$g\left(2\right)=a_n d_{\max}+\sum_{i=n-1}^{n}a_i d_{\max}+\ldots+\sum_{i=3}^{n}a_i d_{\max}+\underbrace{\sum_{i=2}^{n}a_i d_{\max}+a_1 d_{\max}}_{d_{\max}}\geq d_{\max}. \tag{92}$$

Therefore, sales at $k=2$ are $h_c(2)=d_c(2)$ and $h_r(2)=d_{\max}$. Following the same reasoning we calculate $u(2)$

$$\begin{aligned}u\left(2\right)&=\sum_{j=3}^{n+3}d_c\left(j\right)-\sum_{j=2}^{n}d_c\left(j\right)-\boldsymbol{cD_{max}}+2d_{\max}\\&-\underbrace{\left\{a_1\left[d_c\left(n+1\right)+d_{\max}\right]+\sum_{i=2}^{n}a_i\left[d_c\left(n+1\right)+d_{\max}\right]\right\}}_{d_c(n+1)+d_{\max}}\\&-d_c\left(n+2\right)-d_{\max}+\tilde{d}_c\left(2\right)+d_{\max}+\boldsymbol{cD_{max}}\\&=\tilde{d}_c\left(2\right)-d_c\left(2\right)+d_c\left(n+3\right)+d_{\max}.\end{aligned} \tag{93}$$

As follows from (83)

$$u\left(2\right)=d_c\left(n+3\right)+d_{\max}. \tag{94}$$

The state vector $\boldsymbol{x}(3)$ is

$$\boldsymbol{x}\left(3\right)=\begin{bmatrix}y\left(3\right)\\ \left.\begin{matrix}0\\ \vdots\\ 0\end{matrix}\right\}n-3\\ d_c\left(n+1\right)+d_{\max}\\ d_c\left(n+2\right)+d_{\max}\\ d_c\left(n+3\right)+d_{\max}\end{bmatrix}_{n+1}, \tag{95}$$

where

$$y\left(3\right)=\sum_{j=3}^{n}d_c\left(j\right)+\sum_{i=1}^{2}a_i d_c\left(n+1\right)+a_1 d_c\left(n+2\right)+g\left(3\right), \tag{96}$$

and, considering that the sum of elements a_1, a_2, …, a_n equals 1,

$$\begin{aligned}g\left(3\right)&=cD_{\max}-3d_{\max}+\sum_{i=1}^{2}a_i d_{\max}=\\&=a_n d_{\max}+\sum_{i=n-1}^{n}a_i d_{\max}+\ldots+\sum_{i=4}^{n}a_i d_{\max}\\&+\underbrace{\sum_{i=3}^{n}a_i d_{\max}+\sum_{i=1}^{2}a_i d_{\max}}_{d_{\max}}\geq d_{\max}.\end{aligned} \tag{97}$$

Therefore, sales at $k=3$ are $h_c(3)=d_c(3)$ and $h_r(3)=d_{\max}$. At this point one may notice that for any $0\leq k\leq n$ the state vector becomes

$$\boldsymbol{x}\left(k\right)=\begin{bmatrix}y\left(k\right)\\ \left.\begin{matrix}0\\ \vdots\\ 0\end{matrix}\right\}n-k\\ d_c\left(n+1\right)+d_{\max}\\ \vdots\\ d_c\left(n+k\right)+d_{\max}\end{bmatrix}_{n+1}. \tag{98}$$

Following the same reasoning

$$u\left(3\right)=d_c\left(n+4\right)+d_{\max}. \tag{99}$$

And $y(4)$ is

$$y\left(4\right)=\sum_{j=4}^{n}d_c\left(j\right)+\sum_{i=1}^{3}a_i d_c\left(n+1\right)+\sum_{i=1}^{2}a_i d_c\left(n+2\right)+a_1 d_c\left(n+3\right)+g\left(4\right), \tag{100}$$

where

$$g\left(4\right)=a_n d_{\max}+\sum_{i=n-1}^{n}a_i d_{\max}+\ldots+\underbrace{\sum_{i=4}^{n}a_i d_{\max}+\sum_{1}^{3}a_i d_{\max}}_{d_{\max}}\geq d_{\max}. \tag{101}$$

As follows the sales again satisfy $h_c(4)=d_c(4)$ and $h_r(4)=d_{\max}$. With the same reasoning for any $k\geq 0$ we obtain

$$u\left(k\right)=d_c\left(k+n+1\right)+d_{\max}. \tag{102}$$

As the maximum demand values are considered, for $k=n$ we get

$$\boldsymbol{x}\left(n\right)=\left[y\left(n\right)\ \ d_c\left(n+1\right)+d_{\max}\ \ \ldots\ \ d_c\left(n+n\right)+d_{\max}\right]_{n+1}^{T}, \tag{103}$$

where

$$y\left(n\right)=d_c\left(n\right)+\sum_{i=1}^{n-1}a_i\sum_{j=n+1}^{n+n-i}d_c\left(j\right)+g\left(n\right) \tag{104}$$

and

$$g\left(n\right)=a_n d_{\max}+\sum_{i=1}^{n-1}a_i d_{\max}=d_{\max}. \tag{105}$$

We conclude that, for any $0\leq k\leq n$, with the maximum demands considered, the stock level may be expressed as

$$y\left(k\right)=\sum_{j=k}^{n}d_c\left(j\right)+\sum_{i=1}^{n-1}a_i\sum_{j=n+1}^{k+n-i}d_c\left(j\right)+g\left(k\right), \tag{106}$$

where

$$g(k)=\underbrace{\sum_{i=k}^{n}a_i d_{\max}+\sum_{i=1}^{k-1}a_i d_{\max}}_{d_{\max}}+\sum_{i=k+1}^{n}a_i d_{\max}+\sum_{i=k+2}^{n}a_i d_{\max}+\ldots++\sum_{i=n-1}^{n}a_i d_{\max}+a_n d_{\max}\geq d_{\max}. \quad (107)$$

From (106) and (107) it is clear that (76) holds for steps $0\leq k\leq n$. Therefore, both contractual and random sales are satisfied.

Next let us consider step $k=n+1$. According to (102)

$$u(n+1)=d_c(n+n+1)+d_{\max}. \quad (108)$$

Considering the control and the state vector (98), the state vector at $k=n+1$ becomes

$$\boldsymbol{x}(n+1)=\left[y(n+1)\ \ d_c(n+2)+d_{\max}\ \ \ldots\ \ d_c(n+n+1)+d_{\max}\right]_{n+1}^{T}, \quad (109)$$

where

$$y(n+1)=\sum_{i=1}^{n}a_i d_c(n+1)+\sum_{i=1}^{n-1}a_i\sum_{j=n+2}^{n+n+1-i}d_c(j)+g(n+1) \quad (110)$$

and

$$g(n+1)=\sum_{i=1}^{n}a_i d_{\max}. \quad (111)$$

Considering that the sum of elements $a_1, a_2, \ldots, a_n$ equals 1, as stated in (3), for $k=n+1$ the stock level equals

$$y(n+1)=d_c(n+1)+\sum_{i=1}^{n-1}a_i\sum_{j=n+2}^{n+n+1-i}d_c(j)+d_{\max}. \quad (112)$$

Finally, for any $k\geq n+1$, the stock level is expressed as

$$y(k)=d_c(k)+\sum_{i=1}^{n-1}a_i\sum_{j=k+1}^{k+n-i}d_c(j)+d_{\max}=u(k-n-1)+\sum_{i=1}^{n-1}a_i\sum_{j=k-n}^{k-1-i}u(j)+d_{\max}. \quad (113)$$

Taking into account that the maximum demand values have been considered in the above reasoning, from (113) we conclude that

$$y(k)\geq d_c(k)+d_{\max} \quad (114)$$

for any $k\geq 0$, which ends the proof. ■

Above, we have shown that the presented simplified model based control strategy ensures satisfaction of both contractual and random demand for any $k\geq 0$. However, considering the stock level expressed with (112), it may be seen that some overload may exist in the warehouse. The orders are generated $n+1$ steps in advance and some of them are received earlier than others. Consequently, the current stock level may be further reduced without compromising the contract. This is due to the fact that orders from step k, meant to satisfy the contract at step $k+n+1$ arrive before the expected time, and can be used to fulfil the contract sooner. Consequently, the number of goods stored in the warehouse may be permanently reduced by the minimum number of goods arriving to the warehouse before their expected time. This helps increase the energy efficiency of the system, both when it comes to the energy required for the operations of the plant itself and the transportation of incoming goods. Such a term may be calculated as

$$\alpha=\min_{k\to\infty}\left[(\boldsymbol{c_m}-\boldsymbol{c})\boldsymbol{x_d}(k)\right]=\min_{k\to\infty}\left[\left(\mathbf{1}_{1\times(n+1)}-\boldsymbol{c}\right)\boldsymbol{x_d}(k)\right]. \quad (115)$$

Considering (115) the control signal (72) is further modified to

$$u(k)=(\boldsymbol{cb})^{-1}\left\{\boldsymbol{c_m x_m}(k+1)-\boldsymbol{cAx}(k)+\boldsymbol{c}\tilde{\boldsymbol{f}}_{\boldsymbol{c}}(k)+\boldsymbol{cd_{max}}+\boldsymbol{cD_{max}}-\alpha\right\}. \quad (116)$$

The same compensation term may be applied to the initial condition of the system, expressed with (73). As some of the goods ordered at steps $k=0, 1, 2, \ldots, n+1$ will arrive earlier than their expected sales time, the initial stock might be reduced, by the same compensation term, which gives

$$y_0=x_1(0)=\sum_{j=0}^{n}d_c(j)+\boldsymbol{cD_{max}}-\alpha. \quad (117)$$

The above modification ensures a reduction of warehouse space required to always satisfy both contractual obligations and random demand.

IV. SIMULATION EXAMPLE

In this section, a simulation of the system presented in Section II will be used to verify the control properties demonstrated in Section III. The goal is to ensure full consumers' demand satisfaction, both contractual and random, with a 3-supplier warehouse following a single supplier model. The considered warehouse stores a single product. In order to assign a universal measuring system, the amount of stored products is measured in pieces [pcs.]. The contractual demand vector is comprised of a repeated sequence of five values

$$d_c(k)\in\{10,\quad 30,\quad 50,\quad 40,\quad 10\}. \quad (118)$$

The demand vector shall be interpreted as follows. At the discrete time instant $k=1$ the contractual buyers require 10 pieces of the product, at $k=2$ 30 pieces, at $k=3$ 50 pieces, at $k=4$ 40 pieces and at $k=5$ 10 pieces. Afterwards, at $k=6$ the contractual demand sequence repeats, so at

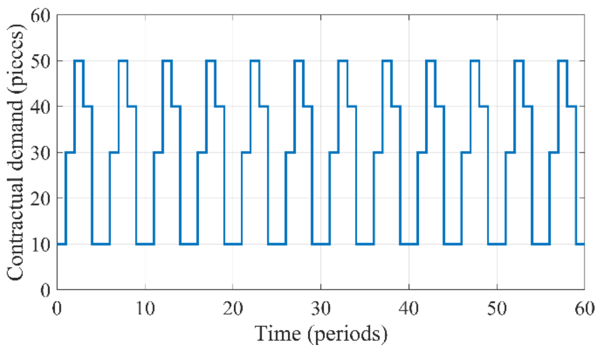

FIGURE 1. Contractual demand sequence in number of ordered pieces [pcs.].

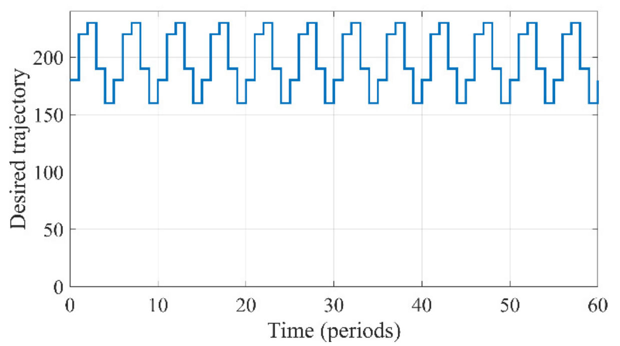

FIGURE 2. Desired trajectory $s_d(k)$ generated by the model.

$k = 6$ 10 pieces are required, at $k = 7$ 30 pieces are required and so on. Figure 1 shows the contractual demand. The random consumers' demand is defined as

$$d_r(k) = 3 + 2 * (-1)^{[k/20]} \tag{119}$$

with $d_{max} = 5$. In other words, the random demand changes between its maximum and minimum value every 20 discrete time instants. Therefore, we present the system's behaviors in two extreme cases: with the smallest disturbance impact, i.e. $d_r(k) = 1$, and under the largest disturbance influence, i.e. when $d_r(k) = 5$.

The simulations will prove, that even with largest admissible demand from random customers, the proposed control strategy still provides satisfactory number of products in stock.

The plant considered for the simulations is a 7-th order inventory system with the following three ($q = 3$) suppliers:

- supplier 1 with a lead time equal to 6, delivering 50% of the ordered product,
- supplier 2 with a lead time equal to 5, delivering 30%,
- supplier 3 with a lead time equal to 3, delivering 20%.

With that considered, the maximum lead time $n = 6$, and the system's state equation is

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 1 & 0.5 & 0.3 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k) - \boldsymbol{f}(k)$$

$$y(k) = x_1(k). \tag{120}$$

This means that 20% of the orders generated by the controller will reach the warehouse after 3 instants, 30% after 5 instants, and 50% after 6 time instants. This can be seen in the values of the a parameters – $a_3 = 0.2$, $a_5 = 0.3$ and $a_6 = 0.5$. The plant's vector $\boldsymbol{c}$ is chosen according to the plant's state matrix $\boldsymbol{A}$ and (42), which gives

$$\boldsymbol{c} = \begin{bmatrix} 1 & 0.5 & 0.8 & 0.8 & 1 & 1 & 1 \end{bmatrix}. \tag{121}$$

For such a system we introduce a simplified model. The model used in the simulation is a 7-th order inventory system as well, with a single supplier ($q = 1$) with the lead time equal to $n = 6$. Therefore

$$\boldsymbol{A_m} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{7\times 7}. \tag{122}$$

In this system, the $a_6 = 1$ and $a_1, a_2, \ldots, a_{n-1} = 0$, as there are no other suppliers. The model's vector $\boldsymbol{c_m}$ is chosen according to (42):

$$\boldsymbol{c_m} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{123}$$

This allows us to calculate the desired trajectory $s_d(k)$, as generated by the model. Figure 2 depicts this desired trajectory. It is defined in (64) and is the linear combination of the reference model's state variables. The initial amount of stock required is

$$x_{m1}(0) = x_1(0) = 180, \tag{124}$$

which fulfills the contractual demand for $k \in [0, 6]$.

Having generated the desired state trajectory $s_d(k)$, we proceed to design the control for the actual inventory system, described in (120). The system shall follow the demand trajectory $s_d(k)$ according to (65). However, as the disturbance, in the form of random sales, appears, it must be compensated for. Figure 3 shows the random consumers' demand. As explained before, it presents the cases with the highest disturbance (with $d_r(k) = 5$) and the lowest disturbance (with

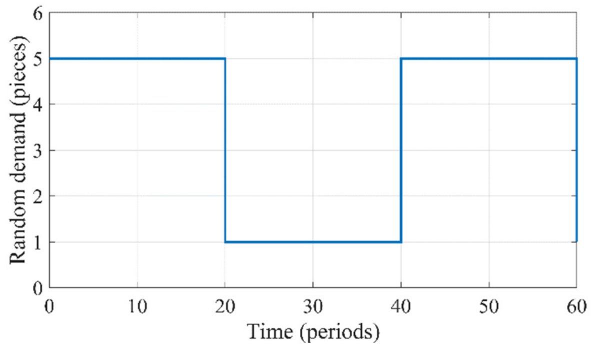

FIGURE 3. Random demand in the plant [pcs.].

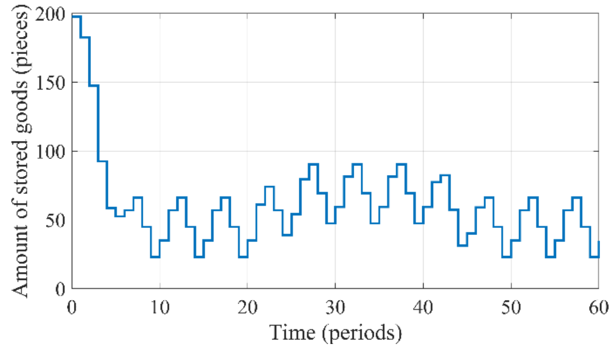

FIGURE 4. Amount of goods stored inside the warehouse $y(k)$ [pcs.].

$d_r(k) = 1$). Its appearance creates the need to compensate it, and to do so, we calculate the required compensation vector according to (71)

$$D_{max} = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 & 5 & 5 \end{bmatrix}^T . \tag{125}$$

Next, we compare the desired trajectory $\boldsymbol{c_m x_d}(k)$ with $\boldsymbol{c x_d}(k)$ and lower the desired trajectory according to (115)

$$\alpha = \min \left[\boldsymbol{c_m x_d}(k) - \boldsymbol{c x_d}(k) \right] = 13. \tag{126}$$

This reduces the amount of goods stored in the warehouse to the minimum, as there are instants where the stock falls to zero. With the considered control strategy this is the ideal situation, minimizing the usage of storage space and increasing the efficiency of transportation. We also calculate the amount of initial stock needed

$$x_1(0) = 180 + \boldsymbol{c D_{max}} - \alpha = 197.5, \tag{127}$$

which satisfies both contractual and random demands for $k \in [0, 6]$, taking into account that the resupply order generated at $k = 0$ will be partially received at $k = 4$, $k = 6$ and $k = 7$. After consideration of both compensation terms, the control law (116) may be applied. Consequently, we achieve the system with the smallest amount of product stored in the warehouse that can still satisfy the customers' demand. Figure 4 illustrates the amount of goods in the warehouse $y(k)$. We can clearly see the effect of the random demand on the inventory level – with $d_r(k) = 5$ the amount of goods in the warehouse decreases in order to satisfy the customers' needs.

Figure 5 demonstrates the customers' demand unfulfilled due to insufficient amount of product in the warehouse. It is clear that the system is capable of fulfilling all of the customers' demand. Figure 6 depicts the control signal in the system. It may be noticed that the desired trajectory forces the controller into ordering the contracted amount of goods, increased by the additional need to satisfy the random demand. The controller reacts to the decreased random demand by lowering the amount of ordered goods in the very next moment.

Additionally, Figure 7 shows the amount of goods in the warehouse after the contractual sale and after the random sale.

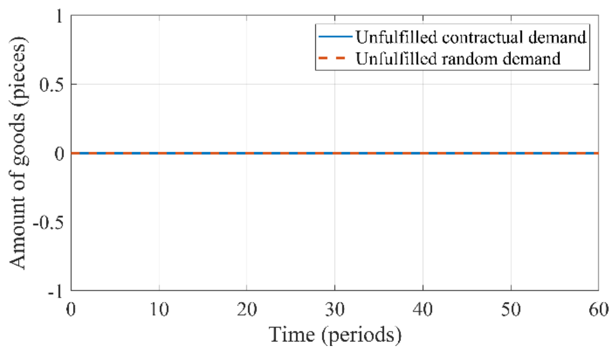

FIGURE 5. Unfulfilled demand in the system [pcs.].

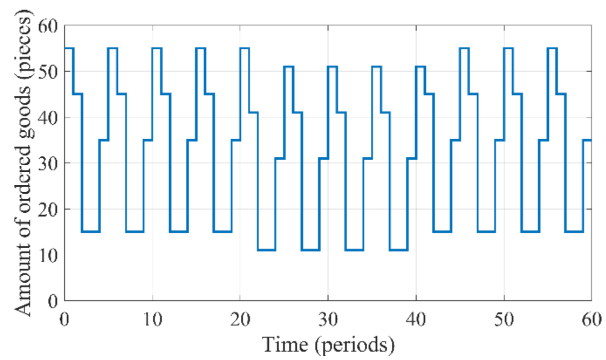

FIGURE 6. Amount of goods ordered by the controller $u(k)$ [pcs.].

Here, one can observe the instants where constant high random demand clears out the warehouse's stock. This happens after each contractual demand sequence ends. However, if the contractual demand wasn't a repeated sequence, such a case might not occur.

Considering all the above, we have successfully designed an inventory system with multiple suppliers that can fully satisfy its customers by following a reference trajectory generated by a single supplier model. The effect of the random demand (Fig.3) on the inventory levels (Fig.4) and the orders generated by the controller (Fig.6) can be clearly seen. The increase in demand results in larger orders and lowers the amount of product stored in the warehouse, but we never

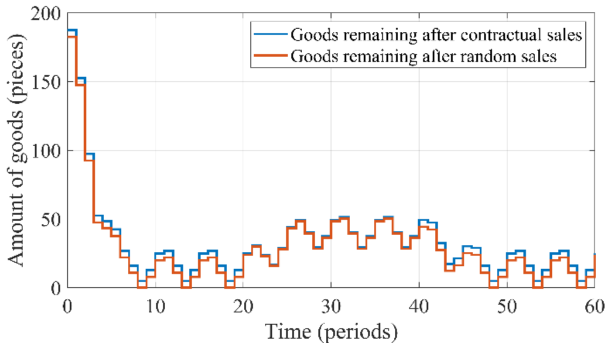

FIGURE 7. The $y_S(k)$ [pcs.], goods remaining in the warehouse after satisfying the demand.

run into a situation when the warehouse fails to fulfill the customers' needs (Fig.5). From Fig. 7, one may see that with the maximum disturbance (maximum random demand) the warehouse may be emptied. Therefore, neither the initial condition nor the order values may be further lowered, as it would result in some unsatisfied customers. We conclude that the obtained results present an optimal resupply sequence, where customer needs are fulfilled, no sales losses are encountered and at the same time no unnecessary stock is stored.

V. CONCLUSION

This study tackled the problem of inventory control. The paper considered a single product warehouse with multiple suppliers. Each of the suppliers required a non-negligible amount of time to deliver the ordered goods. Moreover, in the considered case, we assumed that a part of the customer demand is known a priori, denoted as contractual demand. Conventionally, in such a system a resupply order sequence is designed in order to fulfill the maximum warehouse capacity at any step. However, this may lead to storage of too large number of goods, cause degradation of products and generate unnecessary costs.

We aimed to prevent these effects by introduction of a novel model reference based control strategy for the inventory management problem. Having the knowledge of contractual demand, we have developed a hypothetical inventory model, with one supplier and contractual demand only. For such a system we applied a discrete time sliding mode control law, which allowed us to generate the appropriate order volume to satisfy the contractual buyers. Afterwards, the generated order profile was used as a reference for the actual system, with multiple suppliers and contractual plus random demand. For the system we proposed a model following quasi-sliding mode control strategy. It is proved that, with proper compensation of random sales, such a control law provides enough stock product to fulfill both contractual and random demand. Moreover, the knowledge of contractual sale allows to reduce the number of spare products, stored in the warehouse for random buyers. The benefits of the proposed control strategy have been finally demonstrated with a simulation example. It is explicitly shown that when the random demand is maximized the warehouse may be emptied. However, no unsatisfied demand appears.

We believe that the proposed control strategy significantly simplifies the inventory control problem. In the proposed scenario, we base our control on a single supplier model instead of the real multi supplier system. Therefore, the control design is less complicated. However, the proposed control strategy requires precise knowledge of the initial stock and demand for the n following discrete time instants which may cause some design difficulties.

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KATARZYNA ADAMIAK received the B.Sc. degree in mechatronics and the M.Sc. degree in control engineering from the Łódź University of Technology, Łódź, Poland, in 2017 and 2018, respectively, where she is currently pursuing the Ph.D. degree. Her main area of interest is discrete time sliding mode control.

TYMOTEUSZ ZWIERZCHOWSKI received the B.Sc. and M.Sc. degrees in control engineering and robotics from the Łódź University of Technology, Łódź, Poland, in 2020 and 2021, respectively. His current research interests include discrete-time sliding mode control and supply chain systems.