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RESEARCH ARTICLE

Rough Substructures Based on Overlaps of Successor in Quantales Under Serial Fuzzy Relations

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ABSTRACT In this research article, a new connection between serial fuzzy relations and an extended version of rough sets in an algebraic structure quantale is established. The extended notion of rough sets consists of successor class and an overlap of the successor class of an element of a quantale. Thus a new approximation space based on serial fuzzy relations via the overlaps of successor in quantales, are introduced. The main purpose of this study is to provide basic algebraic structures based on serial fuzzy-relations. In this way, the new approximation space acquires certain appealing algebraic properties. Compatible fuzzy relations in quantale are being applied to introduce the notions of rough multiplicative set, rough m-system and further rough substructures of quantales. Following that, various quantale substructures are described in terms of successor overlaps under serial fuzzy relations, leading to the development of some key theorems. Moreover, several results including quantale homomorphism between rough substructures and their homomorphic images are provided. It is concluded that this new study is significantly easy and superior to various types of approximations in various types of algebraic structures. Furthermore, different examples are given to show the effectiveness of the developed approach and a comparative study of the investigated approach with some existing methods are expressed broadly which show that the investigated approach are more effective and easy than the existing approaches.

INDEX TERMS Quantale, rough ideal, SFrelations, TCFR, rough multiplicative set, rough m-system.

LIST OF ACRONYMS/ABBREVIATIONS

Acronyms	Representation
$\circledcirc$	Binary operation on quantale
$Sub_{\mathbb{G}}$	Subquantale
UP. appr.	Upper approximation
LW. appr.	Lower approximation
FZ.Subset	Fuzzy Subset
FZ-Relation	Fuzzy Relation
TCFR	Transitive compatible fuzzy relation
CFZR	Complete fuzzy relation
AP.SP.	Approximation space
SFrelations	Serial fuzzy relation

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I. INTRODUCTION

Managing ambiguous and vague information has always been difficult. Many theories, like theory of rough sets [1] and theory of fuzzy sets (FSs) [2], have been proposed to address the imprecision and uncertainty found in practically all real-world problems. Zadeh's FS is a remarkable idea and is heavily used in many situations of uncertainty including decision-making problems, pattern recognition, clustering, networking, and many other fields of computer and engineering. Each of these theories has unique qualities and benefits. Another characteristics and advantages of FS are used to characterize different properties of algebraic structures in terms of fuzzy substructures. For example fuzzy substructures in semigroups and quantale were proposed by Kuroki [3] and Farooq et al. [4] respectively. t-intuitionistic subgroups with Fuzzy set was characterized by Gulzar et al., [5], [6]. A Multi-attribute decision-making method in terms of complex q-rung orthopair via Einstein geometric aggregation operators were studied by Wu et al., [7].

Rough set theory, which has several applications, was developed by Pawalk [8]. It is becoming a highly helpful method for addressing uncertainty among the elements of a set. Consequently, various more general rough set models were presented in [9] and [10] to avoid equivalence relation, a necessary part in Pawalk rough set theory. By Dubois and Prade, the idea of roughness in fuzzy sets was introduced [11]. Recently, Fuzzy convexities was investigated via overlap functions by Pang [12]. Important Hamacher aggregation operators dependent on the interval-valued intuitionistic fuzzy numbers related to decision making was proposed by Liu [13]. Fuzzy formal contexts and fuzzy relations between objects of different types in the form of fuzzy relational context families was investigated by Boffa [14].

In the literature, there are many examples of how various algebraic structures are combined with rough and fuzzy sets, and different applications can be seen. Roughness in different algebraic structures like quantale and quantale modules through congruence relations were investigated by Yang and Xu [15], Qurashi and Shabir [16], respectively. Many authors studied roughness in different other algebraic structures; for more information, see [17], [18], [19], and [20]. Different characterizations of important residual implications in terms of Copulas was presented by Ji and Xie [21], [22]. The character and applications of aggregating intuitionistic uncertain linguistic variables to group decision making were proposed by Liu and Jin [23].

To our knowledge, there has never been a study of roughness for algebraic structures of quantale based on serial fuzzy relations via the overlaps of successor in quantale. From two perspectives, we attempt to generalize Mareay's work [24] in this research article. First, to weaken Rosenthal's conditions for quantale congruence [25], we shall first establish the concept of compatibility in newly rough model connected with quantale compatibility. Secondly, we will introduce roughness of substructures through these transitive and compatible fuzzy relations.

A. SOME BACKGROUND STUDIES AND IMPORTANCE OF QUANTALE IN THE LITERATURE

The notion of quantale, which designates a complete lattice equipped with associative binary multiplication distributing over arbitrary joins, was used for the first time by Mulvey [26]. However, multiplicative ordered structures were studied already in 1930s in the form of lattices of ring ideals. Frames, various ideal lattices of rings and C^*-algebras, and the power set of a semigroup are just a few of the many examples of quantales. The study of such partially ordered algebraic structures dates back to the late 1930s work on residual lattices by Ward and Dilworth [27], [28] which was driven by ring-theoretic problems. Derivation is helpful to the research of structure and property in algebraic system. Derivations in quantale was studied by Xiao and Liu [29]. Quantale module developed on quantale as a structure was studied by Abramsky and Vickers [30]. Quantales can be seen as a framework for a non-commutative topology. Further, regular and normal quantales were defined by Paseka [31]. He further studied the notion of w-quantale and conjunctivity in quantale. Moreover, simple and semisimple quantales and quantale that classify C^*-algebras were presented by Kruml, and Paseka and Kruml, Resende [32], [33], respectively. Morphisms, theory of locales and the presheaves and sheaves on a quantale were studied by Borceux and Van den Bossche [34].

B. LITERATURE REVIEW IN DETAIL

For the purpose of studying the spectrum of C*-algebras and the foundations of quantum mechanics, quantales were introduced. From this last point of view, a quantale is a semigroup whose multiplication $a \circledcirc b$ can be temporally interpreted as "a and then b". This idea has also appeared in [35], when studying non-commutative versions of the linear logic of Girard [36], and later in [31], where a quantale can be understood as an algebra of observations on concurrent systems.

Roughness to the substructures of quantales including ideals, prime, semiprime and primary ideals were studied by Wang and Zhao [37] in 2013. They actually used congruence relations to develop different rough structures. Further in 2014, rough set theory applied to quantale in a different way but this is done again congruence relations by Luo and Wang [38]. They also discussed rough fuzzy substructures in quantale. Generalized or T-roughness by set-valued homomorphism in quantalewas applied by Xiao and Li [39]. Further, rough set theory to quantale was applied by Qurashi and Shabir with the help of soft relations under aftersets and foresets [40]. More generalized forms of rough fuzzy substructures via $(\epsilon.\epsilon\bigvee q)$-fuzzy type were also studied by Qurashi and Shabir [41].

C. THE MOTIVATION OF THE STUDY AND THE RESEARCH GAP IN THE LITERATURE CURRENTLY AVAILABLE

In the above literature review, some advancements in both classical theory and rough set theory are highlighted. Also, despite the fact that several findings about rough subquantale, rough ideals of quantale, rough fuzzy substructures and rough substructures based on set-valued homomorphism of quantales have been demonstrated, certain problems remain unresolved and should be answered.

1. There have been numerous contributions to classical quantale theory, but its generalization has received little attention. We point out some of them, for example soft substructures in quantale and its characterization by different means like $(\epsilon, \epsilon\bigvee q)$-fuzzy soft types substructures, $(\epsilon, \epsilon\bigvee q_k)$-fuzzy soft type's substructures of quantale are included. Moreover, less attention is being paid to $(\epsilon_\gamma, \epsilon_\gamma\bigvee q_\delta)$-fuzzy soft types substructures of quantale. Further, rough neutrosophic soft ideals and fuzzy bipolar soft ideals in quantale have received less attention.

2. Roughness associated with soft relations under aftersets and forests have been studied by Kanwal and Shabir [42], [43] and Kanwal et al., [44]. This type of roughness was being applied to substructures and fuzzy substructures of semigroups and quantales. Can we extend the concept of roughness and its results under serial fuzzy relations via the overlaps of successor and obtain the similar results easily. So, the study proposed is important.

3. Roughness techniques applied to substructures and fuzzy substructures of quantale with the help of congruence and set-valued homomorphism are in the literature discussed above. What will be the behavior of rough substructures when subjected to serial fuzzy relations via the overlaps of successor is a logical question that naturally arises.

4. Some important theorems related to quantale homomorphism have been discussed in the references [40] and [44] above. These remarkable theorems must therefore be discussed in the context of quantale homomorphism under serial fuzzy relations and compatible fuzzy relations based on overlaps of successor.

5. The literature has examined many algebraic aspects of rough and fuzzy substructures of quantale and others structures with congruence, set-valued homomorphism, and through soft relations. These works have not yet been thoroughly analyzed in the context of rough ideals in quantales under serial fuzzy relations and compatible fuzzy relations based on overlaps of successor classes. As the suggested approach in this paper is easier to develop rough substructures and discuss different properties. So it is concluded that this new study is much better.

Addressing the aforementioned open questions and filling the knowledge gap in the available literature are the ultimate goals of this research.

D. COMPARATIVE RESEARCH AND THE DEFICIENCIES OF THE EXISTING FIELD OF RESEARCH

The results reported in this research hold true for rough substructures in quantale module based on fuzzy relations. Moreover the current analysis is also applicable to fuzzy substructures and intuitionistic fuzzy (IFS) substructures in quantale through serial fuzzy relations and compatible fuzzy relations based on overlaps of successor classes because every fuzzy set is an IFS. Bilal and Shabir provided rough Pythagorean fuzzy sets using soft binary relations [45]. As a result, we can define Rough Pythagorean substructure in quantale dependent on overlaps of successor classes under serial fuzzy relations. However, there are some restrictions on how far we can pursue our work. For example, we cannot take q-rung orthopair fuzzy sets, picture fuzzy sets and fuzzy soft hyper to establish overlaps of successor classes under serial fuzzy relations. So distinct research are advised for these generalized structures. Our research is primarily constrained by this.

Following is a description of how the paper is organized. In section-II, first of all substructures of quantales, fuzzy relations and its types and successor class of an element of quantale, overlap of the successor class, are discussed. In section-III, roughness of substructures of quantale dependent on compatible fuzzy relation and transitive compatible fuzzy relation are defined. Moreover, complete fuzzy relations are defined and different important results are developed. These rough substructures based on overlaps of successor and their homomorphic images under quantale homomorphism are discussed in section-IV. At the end, the conclusion is given in section V.

II. PRELIMINARIES

In this section, we will discuss some important definitions like quantale and its substructures, generalized rough set and related results. We will use symbols $\mathbb{G}$ and $\mathbb{H}$ for quantales throughout the paper where $\mathbb{G}$ and $\mathbb{H}$ are nonempty universal sets.

Definition 1 [26]: Let a nonempty set $\mathbb{G}$ be a complete lattice associate with a binary operation $\circledcirc$ satisfying the following conditions $\forall g, g_j \in G$

1. $g \circledcirc \left(\bigvee_{\jmath\in\mathcal{J}} g_\jmath\right) = \bigvee_{\jmath\in\mathcal{J}}(g \circledcirc g_\jmath)$
2. $\left(\bigvee_{\jmath\in\mathcal{J}} g_\jmath\right) \circledcirc g = \bigvee_{\jmath\in\mathcal{J}}(g_\jmath \circledcirc g)$

Then this $\mathbb{G}$ is called quantale. Let $\mathcal{W}_j\mathcal{W}, \mathcal{Z} \subseteq \mathbb{G}$ Then we define arbitrary join and binary operation as

$$\mathcal{W} \vee \mathcal{Z} = \{g_1 \vee g_2 | g_1 \in \mathcal{W}, g_2 \in \mathcal{Z}\},$$
$$\mathcal{W} \circledcirc \mathcal{Z} = \{g_1 \circledcirc g_2 | g_1 \in \mathcal{W}, g_2 \in \mathcal{Z}\},$$
$$\bigvee_{j\in J} \mathcal{W}_j = \{\bigvee_{\jmath\in|} | g_\jmath \in \mathcal{W}_\jmath\}.$$

Definition 2 [25]: A nonempty subset $\mathbb{E}$ of a quantale $\mathbb{G}$ is called a subquantale($Sub_{\mathbb{G}}$) of a quantale $\mathbb{G}$ if following properties hold, for all $e_1, e_2, e_\jmath \in \mathbb{E}$

i. $\bigvee_{\jmath\in\mathcal{J}} e_j \in \mathbb{E}$ ii. $e_1 \circledcirc e_2 \in \mathbb{E}$.

A nonempty subset $\mathbb{E} \subseteq \mathbb{G}$ is called an m-system of $\mathbb{G}$, if for all $p, q \in \mathbb{E}$, $\downarrow (p \circledcirc 1 \circledcirc q) \cap \mathbb{E} \neq \emptyset$.

A nonempty subset $\mathbb{E} \subseteq \mathbb{G}$ is called a multiplicative set of $\mathbb{G}$, if $p \circledcirc q \in \mathbb{E}$ for all $p, q \in \mathbb{E}$.

Definition 3 [37]: A subset $\emptyset \neq \mathbb{E}$ of a quantale $\mathbb{G}$ is called an ideal of $\mathbb{G}$ if

i. $e_1 \vee e_2 \in \mathbb{E} \forall e_1, e_2 \in \mathbb{E}$

ii. $\forall e_1, e_2 \in \mathbb{G}$ and $e_2 \in \mathbb{E}$ such that $e_1 \leq e_2 \in \mathbb{E} \Longrightarrow e_1 \in \mathbb{E}$

iii. $\forall g \in \mathbb{G}$ and $e \in \mathbb{E} \Longrightarrow g \circledcirc e \in \mathbb{E}$ and $e \circledcirc g \in \mathbb{E}$.

Definition 4 [37]: An ideal $\mathbb{E}$ of a quantale $\mathbb{G}$ is called

i. prime ideal if $e_1 \circledcirc e_2 \in \mathbb{E} \Longrightarrow e_1 \in \mathbb{E}$ or $e_2 \in \mathbb{E} \forall e_1, e_2 \in \mathbb{G}$.

ii. semi prime ideal if $e \circledcirc e \in \mathbb{E} \Longrightarrow e \in \mathbb{E} \forall e \in \mathbb{G}$.

iii.primary ideal if $\mathbb{E} \neq \mathbb{G}$ and $\forall e_1, e_2 \in \mathbb{G}$

$e_1 \circledcirc e_2 \in \mathbb{E}$ *and* $e_1 \notin \mathbb{E} \Longrightarrow e_2^n \in \mathbb{E}$ for some $n > 0$, where $e_2^n = e_2 \circledcirc e_2 \circledcirc \cdots \circledcirc e_2$.

Definition 5 [2]: Let $\mathbb{G}$ be a nonempty universal set then the function $\mathcal{L}$ of $\mathbb{G}$ into the closed interval [0, 1] is called FZ.Subsetof $\mathbb{G}$.

Definition 6 [2]: Let $\mathbb{G}$ and $\mathbb{H}$ be two nonempty universal sets then mapping $\sigma : \mathbb{G} \times \mathbb{H} \to [0, 1]$ is said to be FZ-Relation from $\mathbb{G}$ to $\mathbb{H}$. A mapping $\sigma : \mathbb{G} \times \mathbb{G} \to [0, 1]$ is called FZ-Relation on $\mathbb{G}$. A FZ-Relation in the form of matrix is denoted by

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \dots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} \dots & \sigma_{2m} \\ \vdots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{nm} \end{pmatrix}$$

Definition 7 [46]: If there exists g $\in \mathbb{G}$ and for all $h \in \mathbb{H}$ such that $\sigma(g, h) = 1$, where σ is a FZ-Relation of $\mathbb{G}$ into $\mathbb{H}$. Then σ is called serial FZ-Relation.

Definition 8 [46]: Assume that σ is a FZ-Relation on $\mathbb{G}$.

i. If $\sigma(g, g) = 1$ for all g $\in \mathbb{G}$ then σ is called a reflexive FZ-Relation.

ii. If $\sigma(g_1, g_2) = \sigma(g_2, g_1)$ for all $g_1, g_2 \in \mathbb{Q}$ then σ is called symmetric FZ-Relation.

iii. If $\sigma(g_1, g_2) \geq \vee_{g_3 \in \mathbb{G}}(\sigma(g_1, g_3) \wedge \sigma(g_3, g_2))$ for all $g_1, g_2 \in \mathbb{G}$ then σ is called Transitive.

Definition 9 [46]: If σ is reflexive, symmetric and transitive then σ is called similarity FZ-Relation.

Definition 10 [47]: Assume that σ is a FZ-Relation from $\mathbb{G}$ to $\mathbb{H}$ and $t \in [0, 1]$. For $g \in \mathbb{G}$,

$$SC_\sigma(g; t) := \{h \in \mathbb{H} : \sigma(g, h) \geq t\}$$

is called a successor class of g related to t-level under σ.

Definition 11 [47]: Assume that σ is a serial FZ-Relation from $\mathbb{G}$ to $\mathbb{H}$ and $t \in [0, 1]$. For $g_1 \in \mathbb{G}$,

$$OSC_\sigma(g_1; t) := \{g_2 \in \mathbb{G} : SC_\sigma(g_1; t) \cap SC_\sigma(g_2, t) \neq \emptyset\}$$

is called an overlap of the successor class of g_1 related to t-level under σ.

The collection of $OSC_\sigma(g; t)$ for all g $\in \mathbb{G}$ is denoted by $\mathcal{OSC}_\sigma(\mathbb{G}; t)$.

Remark 1 [47]: Assume that σ is a serial FZ-Relation from $\mathbb{G}$ to $\mathbb{H}$ and $t \in [0, 1]$. Then $\forall g \in \mathbb{G} SC_\sigma(g; t) \neq \emptyset$.

Proposition 1 [47]: Let σ be a serial FZ-Relation from $\mathbb{G}$ to $\mathbb{H}$ and $t \in [0, 1]$. Then

1. $g \in OSC_\sigma(g; t)$ for all $g \in \mathbb{G}$.

2.$q \in OSC_\sigma(p; t)$ iff $OSC_\sigma(p; t) = OSC_\sigma(q; t)$.

Definition 12 [47]: Let σ be a serial FZ-Relation from $\mathbb{G}$ to $\mathbb{H}$ and $t \in [0, 1]$. A triple $(\mathbb{G}, \mathbb{H}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ is called an approximation space (AP.SP.) based on $\mathcal{OSC}_\sigma(\mathbb{G}; t)$. If $\mathbb{G} = \mathbb{H}$, Then $(\mathbb{G}, \mathbb{H}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ is replaced by $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$.

Definition 13 [47]: Let $(\mathbb{G}, \mathbb{H}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. and $\emptyset \neq \mathbb{E} \subseteq \mathbb{G}$, Then we define UP.appr. of $\mathbb{E}$ in $(\mathbb{G}, \mathbb{H}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ and LW.appr. of $\mathbb{E}$ in $(\mathbb{G}, \mathbb{H}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ as $\overline{\sigma}(\mathbb{E}; t) := \{g \in \mathbb{G} : OSC_\sigma(g; t) \cap \mathbb{E} \neq \emptyset\}$ and $\underline{\sigma}(\mathbb{S}; u) := \{g \in \mathbb{G} : OSC_\sigma(g; t) \subseteq \mathbb{E}\}$.

The $\sigma R(\mathbb{E}; t) := (\overline{\sigma}(\mathbb{E}; t), \underline{\sigma}(\mathbb{E}; t))$ is called a rough set of $\mathbb{E}$ in $(\mathbb{G}, \mathbb{H}, \mathcal{OSC}_\Omega(\mathbb{G}; t))$ if $\overline{\sigma}(\mathbb{E}; t) \neq \underline{\sigma}(\mathbb{E}; t)$.

Proposition 2 [47]: Let $(\mathbb{G}, \mathbb{H}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. If $\emptyset \neq \mathbb{E}, \mathbb{F} \subseteq \mathbb{G}$. Then

1. $\overline{\sigma}(\emptyset; t) = \emptyset$ and $\underline{\sigma}(\emptyset; t) = \emptyset$
2. $\overline{\sigma}(\mathbb{G}; t) = \mathbb{G}$ and $\underline{\sigma}(\mathbb{G}; t) = \mathbb{G}$
3. $\mathbb{E} \subseteq \overline{\sigma}(\mathbb{E}; t)$ and $\underline{\sigma}(\mathbb{E}; t) \subseteq \mathbb{E}$
4. $\underline{\sigma}(\mathbb{E} \cup \mathbb{F}; t) \supseteq \underline{\sigma}(\mathbb{E}; t) \cup \underline{\sigma}(\mathbb{F}; t)$ and $\overline{\sigma}(\mathbb{E} \cap \mathbb{F}; t) \subseteq \overline{\sigma}(\mathbb{E}; t) \cap \overline{\sigma}(\mathbb{F}; t)$
5. $\overline{\sigma}(\mathbb{E} \cup \mathbb{F}; t) = \overline{\sigma}(\mathbb{E}; t) \cup \overline{\sigma}(\mathbb{F}; t)$ and $\underline{\sigma}(\mathbb{E} \cap \mathbb{F}; t) = \underline{\sigma}(\mathbb{E}; t) \cap \underline{\sigma}(\mathbb{F}; t)$.
6. If $\mathbb{E} \subseteq \mathbb{F}$, then $\overline{\sigma}(\mathbb{E}; t) \subseteq \overline{\sigma}(\mathbb{F}; t)$ and $\underline{\sigma}(\mathbb{E}; t) \subseteq \underline{\sigma}(\mathbb{F}; t)$.
7. $\underline{\sigma}(\mathbb{E}^c; t) = (\overline{\sigma}(\mathbb{E}; t))^c$, where $\mathbb{E}^c$ and $(\overline{\sigma}(\mathbb{E}; t))^c$ are complements of $\mathbb{E}$ and $\overline{\sigma}(\mathbb{E}; t)$, respectively.

Theorem 1: Let $(\mathbb{G}, \mathbb{H}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. If $\mathbb{E}, \mathbb{F} \subseteq \mathbb{G}$. Then

1. $\underline{\sigma}(\mathbb{E}; t) \cap \underline{\sigma}(\mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; t)$,
   if $\mathbb{G}$ is idempotent quantale.
2. $\underline{\sigma}(\mathbb{E}; t) \cup \underline{\sigma}(\mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \vee \mathbb{F}; t)$, if $0 \in \mathbb{E} \cap \mathbb{F}$.
3. $\underline{\sigma}(\mathbb{E}; t) \cup \underline{\sigma}(\mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; t)$, if $e \in \mathbb{E} \cap \mathbb{F}$.
4. $\overline{\sigma}(\mathbb{E}; t) \cup \overline{\sigma}(\mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \vee \mathbb{F}; t)$, if $0 \in \mathbb{E} \cap \mathbb{F}$.
5. $\overline{\sigma}(\mathbb{E}; t) \cup \overline{\sigma}(\mathbb{F}; t) \subseteq \overline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; t)$, if $e \in \mathbb{S} \cap \mathbb{T}$.
6. $\underline{\sigma}(\mathbb{E}; t) \cap \underline{\sigma}(\mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \vee \mathbb{F}; t)$.

Proof: **1.** Since $\mathbb{G}$ is an idempotent quantale. Therefore we have $\mathbb{E} \cap \mathbb{F} \subseteq \mathbb{E} \circledcirc \mathbb{F}$. From Proposition 2 we have

$$\underline{\sigma}(\mathbb{E} \cap \mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; t)$$

From Proposition 2 we have

$$\underline{\sigma}(\mathbb{E}; t) \cap \underline{\sigma}(\mathbb{F}; t) = \underline{\sigma}(\mathbb{E} \cap \mathbb{F}; t)$$

Therefore, $\underline{\sigma}(\mathbb{E}; t) \cap \underline{\sigma}(\mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; t)$

2. Let $e \in \mathbb{E}$, we have for $0 \in \mathbb{F} e = e \vee 0 \in \mathbb{E} \vee \mathbb{F}$. This implies that $e \in \mathbb{E} \vee \mathbb{F}$. Hence $\mathbb{E} \subseteq \mathbb{E} \vee \mathbb{F}$. Similarly, we have $\mathbb{F} \subseteq \mathbb{E} \vee \mathbb{F}$. Thus, $\mathbb{E} \cup \mathbb{F} \subseteq \mathbb{E} \vee \mathbb{F}$. From Proposition 2 we have,

$$\underline{\sigma}(\mathbb{E} \cup \mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \vee \mathbb{F}; t)$$

Again from Proposition 2, we have

$$\underline{\sigma}(\mathbb{E}; t) \cup \underline{\sigma}(\mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \vee \mathbb{F}; t).$$

3. Let $e \in \mathbb{E}$ then for $0 \in \mathbb{F}$, we have $e = e \circledcirc 0 \in \mathbb{E} \circledcirc \mathbb{F}$. This implies that $e \in \mathbb{E} \circledcirc \mathbb{F}$, hence $\mathbb{E} \subseteq \mathbb{E} \circledcirc \mathbb{F}$. Similarly, we have $\mathbb{E} \subseteq \mathbb{E} \circledcirc \mathbb{F}$. Thus, $\mathbb{E} \cup \mathbb{F} \subseteq \mathbb{E} \circledcirc \mathbb{F}$. From Proposition 2 we have, $\underline{\sigma}(\mathbb{E} \cup \mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; t)$

Again from Proposition 2, we have

$$\underline{\sigma}(\mathbb{E}; t) \cup \underline{\sigma}(\mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; t).$$

4. Let $e \in \mathbb{E}$. Then for $0 \in \mathbb{F}$, we have $e = e \vee 0 \in \mathbb{E} \vee \mathbb{F}$. This implies that $e \in \mathbb{S} \vee \mathbb{T}$, hence $\mathbb{E} \subseteq \mathbb{E} \vee \mathbb{F}$. Similarly, we have $\mathbb{F} \subseteq \mathbb{E} \vee \mathbb{F}$. Therefore $\mathbb{E} \cup \mathbb{F} \subseteq \mathbb{E} \vee \mathbb{F}$. From Proposition 2 we have,

$$\overline{\sigma}\,(\mathbb{E} \cup \mathbb{F}; t) \subseteq \overline{\sigma}(\mathbb{E} \vee \mathbb{F}; t)$$

Again from Proposition 2, we have

$$\overline{\sigma}\,(\mathbb{E}; t) \cup \overline{\sigma}\,(\mathbb{F}; t) \subseteq \overline{\sigma}(\mathbb{E} \vee \mathbb{F}; t).$$

5. Let $e \in \mathbb{E}$, then for $e \in \mathbb{F}$ we have $e = e \circledcirc 0 \in \mathbb{E} \circledcirc \mathbb{F}$. This implies that $e \in \mathbb{E} \circledcirc \mathbb{F}$. Hence, $\mathbb{E} \subseteq \mathbb{E} \circledcirc \mathbb{F}$. Similarly, we have $\mathbb{F} \subseteq \mathbb{E} \circledcirc \mathbb{F}$. Therefore, $\mathbb{E} \cup \mathbb{F} \subseteq \mathbb{E} \circledcirc \mathbb{F}$.From Proposition 2, we have, $\overline{\sigma}\,(\mathbb{E} \cup \mathbb{F}; t) \subseteq \overline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; t)$.

Again from Proposition 2 we have

$$\overline{\sigma}\,(\mathbb{E}; t) \cup \overline{\sigma}\,(\mathbb{F}; t) \subseteq \overline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; t).$$

6. Clearly $\mathbb{E} \cap \mathbb{F} \subseteq \mathbb{E} \vee \mathbb{F}$. From Proposition 2 we have $\underline{\sigma}\,(\mathbb{E} \cap \mathbb{F}; t) \subseteq \underline{\sigma}\,(\mathbb{E} \vee \mathbb{F}; t)$. From Proposition 2 we have

$$\underline{\sigma}\,(\mathbb{E}; t) \cap \underline{\sigma}\,(\mathbb{F}; t) = \underline{\sigma}\,(\mathbb{E} \cap \mathbb{F}; t)$$

Therefore, $\underline{\sigma}\,(\mathbb{E}; t) \cap \underline{\sigma}\,(\mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \vee \mathbb{F}; t)$.

Preliminaries section contains some importatnt definitions including quantale and its substructures. In substructures of quantale, idelas, subquantale, m-system and multiplicative set are presented. Further, successor class of g related to t-level underFZ-Relation and overlap of the successor class of an element of quantale are being discussed. The above all are very important because rough m-system, rough multiplicative set and rough ideals are defined which are dependent on overlap of the successor classes of quantale. In fact, Proposition 2 shows the usefulness of Definition 12 and Definition 13. However, we have generalized Proposition 2 in Theorem 1 which shows the validity of definition 13 and 14 in quantale.

III. ROUGH SUBSTRUCTURES IN QUANTALES INDUCED BY SERIAL FUZZY RELATIONS

Compatible fuzzy relations and transitive compatible fuzzy relations in quantale are defined in this section. Further, more generalized results dependent on transitive compatible fuzzy relation and complete fuzzy relation are discussed.

Definition 14: Let σ be a FZ-Relation on $\mathbb{G}$.

if $\forall g_1, g_2, g_3, g_4, e_j, f_j \in \mathbb{G}$

1. $\sigma(g_1 \circledcirc g_3, g_2 \circledcirc g_4) \geq \sigma(g_1, g_2) \wedge \sigma(g_3, g_4)$
2. $\sigma\left(\bigvee_{j \in \mathcal{J}} e_j, \bigvee_{j \in \mathcal{J}} f_j\right) \geq \bigwedge_{j \in \mathcal{J}} \sigma(e_j, f_j)$

Then this is called compatible FZ-Relation.

Definition 15: Let $(\mathbb{G}, \mathcal{OSC}(\mathbb{G}; t))$ be an $\mathcal{OSC}(\mathbb{G}; t)$-AP.SP. If σ is a transitive and Compatible FZ-Relation then $(\mathbb{G}, \mathcal{OSC}(\mathbb{G}; t))$ is called an $\mathcal{OSC}(\mathbb{G}; t)$-AP.SP. of TCFR.

Proposition 3: Assume that $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ is a $\mathcal{OSC}_\sigma$ $(\mathbb{G}; t)$-AP.SP. of TCFR. Then for all $g_1, g_2 \in \mathbb{G}$

$$(OSC_\sigma\,(g_1; t)) \circledcirc (OSC_\sigma\,(g_2; t)) \subseteq OSC_\sigma\,(g_1 \circledcirc g_2; t)\,.$$

Proof: Let $g_3 \in (OSC_\sigma(g_1; t)) \circledcirc (OSC_\sigma(g_2; t))$. Then there exists $g_4 \in OSC_\sigma(g_1; t)$ and $g_5 \in OSC_\sigma(g_2; u)$ such that $g_3 = g_4 \circledcirc g_5$. Thus,

$$SC_\sigma\,(g_1; t) \cap SC_\sigma\,(g_4; t) \neq \emptyset \text{ and}$$

$$SC_\sigma\,(g_2; t) \cap SC_\sigma\,(g_5; t) \neq \emptyset.$$

Let $g_6 \in SC_\sigma\,(g_1; t) \cap SC_\sigma\,(g_4; t)$ and $g_7 \in SC_\sigma\,(g_2; t) \cap SC_\sigma\,(g_5; t)$. Then we have $\sigma\,(g_1, g_6) \geq t, \sigma\,(g_4, g_6) \geq t, \sigma\,(g_2, g_7) \geq t and \sigma\,(g_5, g_7) \geq t$. Since σ is a serial FZ-Relation, we have $(g_1, g_1) = 1 \geq t, \sigma\,(g_7, g_7) = 1 \geq t, \sigma\,(g_5, g_5) = 1 \geq t$ and $\sigma\,(g_6, g_6) = 1 \geq t$. Since σ is transitive and compatible, we have

$$\begin{aligned}
&\sigma\,(g_1 \circledcirc g_2, g_6 \circledcirc g_7) \\
&\quad \geq \bigvee_{g_8 \in \mathbb{G}} (\sigma\,(g_1 \circledcirc g_2, g_8) \\
&\quad\quad \wedge \sigma\,(g_8, g_6 \circledcirc g_7)) \\
&\quad \geq \sigma\,(g_1 \circledcirc g_2, g_1 \circledcirc g_7) \wedge \sigma\,(g_1 \circledcirc g_7, g_6 \circledcirc g_7) \\
&\quad \geq \sigma\,(g_1, g_1) \wedge \sigma\,(g_2, g_7) \wedge \sigma\,(g_1, g_6) \wedge \sigma\,(g_7, g_7) \\
&\quad \geq t \wedge t \wedge t \wedge t = t.
\end{aligned}$$

Hence, $\sigma\,(g_1 \circledcirc g_2, g_6 \circledcirc g_7) \geq t$ and so $g_6 \circledcirc g_7 \in SC_\sigma(g_1 \circledcirc g_2; t)$. Since σ is transitive and compatible, we have

$$\begin{aligned}
&\sigma\,(g_4 \circledcirc g_5, g_6 \circledcirc g_7) \\
&\quad \geq \textstyle\bigvee_{g_9 \in \mathbb{G}} (\sigma\,(g_4 \circledcirc g_5, g_9) \wedge \sigma\,(g_9, g_6 \circledcirc g_7)) \\
&\quad \geq \sigma\,(g_4 \circledcirc g_5, g_6 \circledcirc g_5) \wedge \sigma\,(g_6 \circledcirc g_5, g_6 \circledcirc g_7) \\
&\quad \geq \sigma\,(g_4, g_6) \wedge \sigma\,(g_5, g_5) \wedge \sigma\,(g_6, g_6) \wedge \sigma\,(g_5, g_7) \\
&\quad \geq t \wedge t \wedge t \wedge t = t.
\end{aligned}$$

Hence, $\sigma\,(g_4 \circledcirc g_5, g_6 \circledcirc g_7) \geq t$ and so $g_6 \circledcirc g_7 \in SC_\sigma(g_4 \circledcirc g_5; t)$. Thus, $SC_\sigma\,(g_1 \circledcirc g_2; t) \cap SC_\sigma\,(g_4 \circledcirc g_5; t) \neq \emptyset$. Therefore, $g_3 = g_4 \circledcirc g_5 \in OSC_\sigma\,(g_1 \circledcirc g_2; t)$.

Hence, $(OSC_\sigma(g_1; t)) \circledcirc (OSC_\sigma(g_2; t)) \subseteq OSC_\sigma(g_1 \circledcirc g_2; t)$.

Proposition 4: Assume that $(\mathbb{G}, \mathcal{OSC}_\sigma\ (\mathbb{G}; t))$ is a $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of TCFR. Then for all $g_1, g_2 \in \mathbb{G}$

$$(OSC_\sigma\,(g_1; t)) \vee (OSC_\sigma\,(g_2; t)) \subseteq OSC_\sigma\,(g_1 \vee g_2; t)\,.$$

Proof: Let $g_3 \in (OSC_\sigma(g_1; t)) \vee (OSC_\sigma(g_2; t))$. Then there exists $g_4 \in OSC_\Omega(g_1; t)$ and $g_5 \in OSC_\Omega(g_2; u)$ such that $g_3 = g_4 \vee g_5$. Thus,

$$SC_\sigma\,(g_1; t) \cap SC_\sigma\,(g_4; t) \neq \emptyset \text{ and}$$
$$SC_\sigma\,(g_2; t) \cap SC_\sigma\,(g_5; t) \neq \emptyset.$$

Let $g_6 \in SC_\sigma\,(g_1; t) \cap SC_\sigma\,(g_4; t)$ and $g_7 \in SC_\sigma\,(g_2; t) \cap SC_\sigma\,(g_5; t)$. Then we have $\sigma\,(g_1, g_6) \geq t, \sigma\,(g_4, g_6) \geq t, \sigma\,(g_2, g_7) \geq t$ *and* $\sigma\,(g_5, g_7) \geq t$. Since σ is a serial FZ-Relation, we have $(g_1, g_1) = 1 \geq t, \sigma\,(g_7, g_7) = 1 \geq t, \sigma\,(g_5, g_5) = 1 \geq t$ and $\sigma\,(g_6, g_6) = 1 \geq t$. Since σ is transitive and compatible, we have

$$\begin{aligned}
&\sigma\,(g_1 \vee g_2, g_6 \vee g_7) \\
&\quad \geq \bigvee_{g_8 \in \mathbb{G}} (\sigma\,(g_1 \vee g_2, g_8) \\
&\quad\quad \wedge \sigma\,(g_8, g_6 \vee g_7)) \\
&\quad \geq \sigma\,(g_1 \vee g_2, g_1 \vee g_7) \wedge \sigma\,(g_1 \vee g_7, g_6 \vee g_7) \\
&\quad \geq \sigma\,(g_1, g_1) \wedge \sigma\,(g_2, g_7) \wedge \sigma\,(g_1, g_6) \wedge \sigma\,(g_7, g_7) \\
&\quad \geq t \wedge t \wedge t \wedge t = t.
\end{aligned}$$

TABLE 1. Binary operation $\circledcirc$ on quantale $\mathbb{G}$.

$\circledcirc$	$0'$	p'	q'	r'	s'	$1'$
$0'$	$0'$	p'	q'	q'	s'	$1'$
p'	p'	p'	p'	p'	p'	$1'$
q'	q'	p'	q'	q'	s'	$1'$
r'	q'	p'	q'	q'	s'	$1'$
s'	s'	p'	s'	s'	s'	$1'$
$1'$	$1'$	$1'$	$1'$	$1'$	$1'$	$1'$

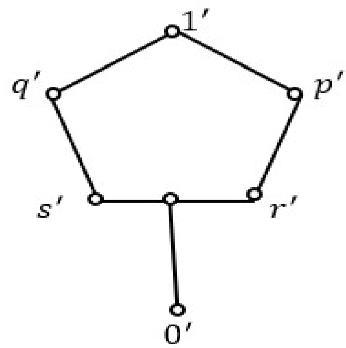

FIGURE 1. Complete lattice in quantale $\mathbb{G}$.

Hence, $\sigma(g_1 \vee g_2, g_6 \vee g_7) \geq t$ and so $g_6 \vee g_7 \in SC_\sigma(g_1 \vee g_2; t)$. Since σ is transitive and compatible, we have

$$\begin{aligned}
&\sigma(g_4 \vee g_5, g_6 \vee g_7)\\
&\quad \geq \bigvee_{g_9 \in \mathbb{G}}(\sigma(g_4 \vee g_5, g_9) \wedge \sigma(g_9, g_6 \vee g_7))\\
&\quad \geq \sigma(g_4 \vee g_5, g_6 \vee g_5) \wedge \sigma(g_6 \vee g_5, g_6 \vee g_7)\\
&\quad \geq \sigma(g_4, g_6) \wedge \sigma(g_5, g_5) \wedge \sigma(g_6, g_6) \wedge \sigma(g_5, g_7)\\
&\quad \geq t \wedge t \wedge t \wedge t = t.
\end{aligned}$$

Hence, $\sigma(g_4 \vee g_5, g_6 \vee g_7) \geq t$ and so $g_6 \vee g_7 \in SC_\sigma(g_4 \vee g_5; t)$. Thus, $SC_\sigma(g_1 \vee g_2; t) \cap SC_\sigma(g_4 \vee g_5; t) \neq \emptyset$. Therefore, $g_3 = g_4 \vee g_5 \in OSC_\sigma(g_1 \vee g_2; t)$.

Hence, $(OSC_\sigma(g_1; t)) \vee (OSC_\sigma(g_2; t)) \subseteq OSC_\sigma(g_1 \vee g_2; t)$.

Example 1: Let $\mathbb{G} = \{0', p', q', r', s', 1'\}$ be a quantale with binary operation $\circledcirc$ defined in Table 1 and Shown in Figure 1.

Define the membership grades of relationship between any two elements in $\mathbb{G}$ under FZ-Relations σ on $\mathbb{G}$ as follows

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 1 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Clearly, σ is transitive and compatible. For $t = 0.6$, the successor class of each element in $\mathbb{G}$ related to 0.6 under σ are

$$\begin{aligned}
SC_\sigma(0'; 0.6) &= \{0'\},\\
SC_\sigma(p'; 0.6) &= \{q', s'\},\\
SC_\sigma(q'; 0.6) &= \{q', s'\},\\
SC_\sigma(r'; 0.6) &= \{r'\}.\\
SC_\sigma(s'; 0.6) &= \{q', s'\} \quad \text{and}\\
SC_\sigma(1'; 0.6) &= \{1'\}.
\end{aligned}$$

Hence, the core of successor class of each element in $\mathbb{G}$ related to 0.6 level under σ are

$$\begin{aligned}
OSC_\sigma(0'; 0.6) &= \{0'\},\\
OSC_\sigma(p'; 0.6) &= \{p', q', s'\},\\
OSC_\sigma(q'; 0.6) &= \{p', q', s'\},\\
OSC_\sigma(r'; 0.6) &= \{r'\},\\
OSC_\sigma(s'; 0.6) &= \{p', q', s'\},\\
OSC_\sigma(1'; 0.6) &= \{1'\},
\end{aligned}$$

Here it is easy to verify that for all $g_1, g_2 \in \mathbb{G}$

$$\begin{aligned}
&(OSC_\sigma(g_1; 0.6)) \circledcirc (OSC_\sigma(g_2; 0.6))\\
&\quad \subseteq OSC_\sigma(g_1 \circledcirc g_2; 0.6) \quad \textit{and}\\
&(OSC_\sigma(g_1; 0.6)) \vee (OSC_\sigma(g_2; 0.6))\\
&\quad \subseteq OSC_\sigma(g_1 \vee g_2; 0.6)
\end{aligned}$$

Observe that in this example equality in general does not hold. Let us consider the following Example.

Example 2: Let $\mathbb{G} = \{0', e', f', g', h', 1'\}$ be a quantale with binary operation $\circledcirc$ defined in Table 2 and Shown in Figure 2.

TABLE 2. Binary operation $\circledcirc$ on quantale $\mathbb{G}$.

$\circledcirc$	$0'$	e'	f'	g'	h'	$1'$
$0'$	$0'$	$0'$	$0'$	$0'$	$0'$	$1'$
e'	$0'$	e'	e'	e'	h'	$1'$
f'	$0'$	e'	f'	e'	h'	$1'$
g'	$0'$	e'	e'	g'	h'	$1'$
h'	$0'$	h'	h'	h'	h'	$1'$
$1'$	$1'$	$1'$	$1'$	$1'$	$1'$	$1'$

Define the membership grades of relationship between any two elements in $\mathbb{G}$ under FZ-Relations σ on $\mathbb{G}$ as follows

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0\\
0 & 1 & 1 & 1 & 0 & 0\\
0 & 1 & 1 & 1 & 0 & 0\\
0 & 1 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

It easy to verify to σ is transitive and compatible.

For $\ell = 0.6$, the successor class of each element in $\mathbb{G}$ related to 0.6 under σ are

$$SC_\sigma(0'; 0.6) = \{h'\},$$
$$SC_\sigma(e'; 0.6) = \{e', f', g'\},$$
$$SC_\sigma(f'; 0.6) = \{e', f', g'\},$$
$$SC_\sigma(g'; 0.6) = \{e', f', g'\},$$
$$SC_\sigma(h'; 0.6) = \{h'\} \quad \text{and}$$
$$SC_\sigma(1'; 0.6) = \{1'\}.$$

Hence, the core of successor class of each element in $\mathbb{G}$ related to 0.6 level under σ are

$$OSC_\sigma(0'; 0.6) = \{0', h'\},$$
$$OSC_\sigma(e'; 0.6) = \{e', f', g'\},$$
$$OSC_\sigma(f'; 0.6) = \{e', f', g'\},$$
$$OSC_\sigma(g'; 0.6) = \{e', f', g'\},$$
$$OSC_\sigma(h'; 0.6) = \{0', h'\} \quad \text{and}$$
$$OSC_\sigma(1'; 0.6) = \{1'\}.$$

Here it is easy to verify that for all $g_1, g_2 \in \mathbb{G}$

$$(OSC_\sigma(g_1; 0.6)) \circledcirc (OSC_\sigma(g_2; 0.6)) = OSC_\sigma(g_1 \circledcirc g_2; 0.6) \quad \textit{and}$$
$$(OSC_\sigma(g_1; 0.6)) \vee (OSC_\sigma(g_2; 0.6)) = OSC_\sigma(g_1 \vee g_2; 0.6).$$

Definition 16: Let $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; \ell))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; \ell)$-AP.SP. of TCFR. Then for all $g_1, g_2 \in \mathbb{G}$,

$$(OSC_\sigma(g_1; \ell)) \circledcirc (OSC_\sigma(g_2; \ell)) = OSC_\sigma(g_1 \circledcirc g_2; \ell)$$

Then the collection $\mathcal{OSC}_\sigma(\mathbb{G}; \ell)$ is called $\circledcirc$-Complete.

If for all $g_1, g_2 \in \mathbb{G}$,

$$(OSC_\sigma(g_1; \ell)) \vee (OSC_\sigma(g_2; \ell)) = OSC_\sigma(g_1 \vee g_2; \ell)$$

Then the collection $\mathcal{OSC}_\sigma(\mathbb{G}; \ell)$ is called $\vee$-Complete.

Although approximation through overlaps of successor inquantales is totally dependent on overlap of the successor class $(OSC_\sigma(g; \ell))$ of an element of quantale yetwe have observed some interesting properties of these classes under serial fuzzy relations. It is noticed that these classes under $\bigvee$ and $\circledcirc$ always show always containment. That is $(OSC_\sigma(g_1; t)) \vee (OSC_\sigma(g_2; t)) \subseteq OSC_\sigma(g_1 \vee g_2; t)$ and $(OSC_\sigma(g_1; t)) \circledcirc (OSC_\sigma(g_2; t)) \subseteq OSC_\sigma(g_1 \circledcirc g_2; t)$. Further, it is observed that equality does not hold in general. So in next results, we have applied the conditions of transitive compatible fuzzy relation(TCFR) and complete fuzzy relation (CFZR) to fulfil the condition of equality.

Definition 17: Let $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; \ell))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; \ell)$-AP.SP. of TCFR. Then the collection $\mathcal{OSC}_\sigma(\mathbb{G}; \ell)$ is called σ-Complete if it is both $\circledcirc$-Complete and $\vee$-Complete.

Definition 18: Let $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; \ell))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; \ell)$-AP.SP. of TCFR. Then σ is called complete FZ-Relation(CFZR) if $\mathcal{OSC}_\sigma(\mathbb{G}; \ell)$ is complete induced by σ.

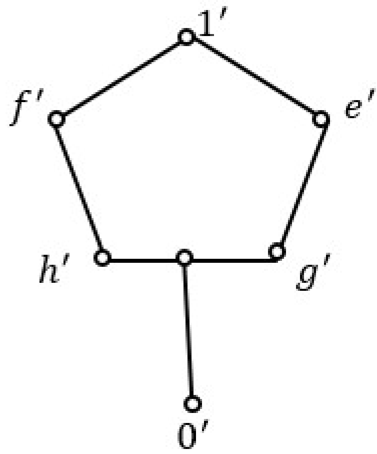

FIGURE 2. Complete lattice in quantale $\mathbb{G}$.

$(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; \ell))$ is called an $\mathcal{OSC}_\sigma(\mathbb{G}; \ell)$-AP.SP. of CFZR if σ is complete.

Theorem 2: Let $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; \ell))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; \ell)$-AP.SP. of TCFR and $\emptyset \neq \mathbb{E}, \mathbb{F} \subseteq \mathbb{G}$. Then

1. $\overline{\sigma}(\mathbb{E}; \ell) \circledcirc \overline{\sigma}(\mathbb{F}; \ell) \subseteq \overline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; \ell)$
2. $\overline{\sigma}(\mathbb{E}; \ell) \vee \overline{\sigma}(\mathbb{F}; \ell) \subseteq \overline{\sigma}(\mathbb{E} \vee \mathbb{F}; \ell)$.

Proof: **1.** Let $g_1 \in \overline{\sigma}(\mathbb{E}; \ell) \circledcirc \overline{\sigma}(\mathbb{F}; \ell)$ Then there exists $g_2 \in \overline{\sigma}(\mathbb{E}; \ell)$ and $g_3 \in \overline{\sigma}(\mathbb{F}; \ell)$ be such that $g_1 = g_2 \circledcirc g_3$. Then $OSC_\sigma(g_2; \ell) \cap \mathbb{E} \neq \emptyset$ and $OSC_\sigma(g_3; \ell) \cap \mathbb{F} \neq \emptyset$. There exists $g_4, g_5 \in \mathbb{G}$ be such that $g_4 \in OSC_\sigma(g_2; \ell) \cap \mathbb{E}$ and $g_5 \in OSC_\sigma(g_3; \ell) \cap \mathbb{F}$. This means that $g_4 \in OSC_\sigma(g_2; \ell)$, $g_4 \in \mathbb{E}$ and $g_5 \in OSC_\sigma(g_3; \ell)$, $g_5 \in \mathbb{F}$. This implies that $g_4 \circledcirc g_5 \in \mathbb{E} \circledcirc \mathbb{F}$ and $g_4 \circledcirc g_5 \in OSC_\sigma(g_2; \ell) \circledcirc OSC_\sigma(g_3; \ell)$. From Proposition 3, we get

$$\begin{aligned} g_4 \circledcirc g_5 &\in (OSC_\sigma(g_2; \ell)) \circledcirc (OSC_\sigma(g_3; \ell)) \\ &\subseteq OSC_\sigma(g_2 \circledcirc g_3; \ell) \\ &\Rightarrow g_4 \circledcirc g_5 \in OSC_\sigma(g_2 \circledcirc g_3; \ell) \end{aligned}$$

So, we have $OSC_\sigma(g_2 \circledcirc g_3; \ell) \cap \mathbb{E} \circledcirc \mathbb{F} \neq \emptyset$. This implies that $g_1 = g_2 \circledcirc g_3 \in \overline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; \ell)$.

Hence, $\overline{\sigma}(\mathbb{E}; \ell) \circledcirc \overline{\sigma}(\mathbb{F}; \ell) \subseteq \overline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; \ell)$.

2. Let $g_1 \in \overline{\sigma}(\mathbb{E}; \ell) \vee \overline{\sigma}(\mathbb{F}; \ell)$ Then there exists $g_2 \in \overline{\sigma}(\mathbb{E}; \ell)$ and $g_3 \in \overline{\sigma}(\mathbb{F}; \ell)$ such that $g_1 = g_2 \vee g_3$. Then $OSC_\sigma(g_2; \ell) \cap \mathbb{E} \neq \emptyset$ and $OSC_\sigma(g_3; \ell) \cap \mathbb{F} \neq \emptyset$. There exists $g_4, g_5 \in \mathbb{G}$ such that $g_4 \in OSC_\sigma(g_2; \ell) \cap \mathbb{E}$ and $g_5 \in OSC_\sigma(g_3; \ell) \cap \mathbb{F}$. This means that $g_4 \in OSC_\sigma(g_2; \ell)$, $g_4 \in \mathbb{E}$ and $g_5 \in OSC_\sigma(g_3; \ell)$, $g_5 \in \mathbb{F}$. This implies that $g_4 \vee g_5 \in \mathbb{E} \vee \mathbb{F}$ and $g_4 \vee g_5 \in OSC_\sigma(g_2; \ell) \vee OSC_\sigma(g_3; \ell)$.

From Proposition 4, we get

$$\begin{aligned} g_4 \vee g_5 &\in (OSC_\sigma(g_2; \ell)) \vee (OSC_\sigma(g_3; \ell)) \\ &\subseteq OSC_\sigma(g_2 \vee g_3; \ell) \\ &\Rightarrow g_4 \vee g_5 \in OSC_\sigma(g_2 \vee g_3; \ell) \end{aligned}$$

So, we have, $OSC_\sigma(g_2 \vee g_3; t) \cap \mathbb{E} \vee \mathbb{F} \neq \emptyset$. This implies that $g_1 = g_2 \vee g_3 \in \overline{\sigma}(\mathbb{E} \vee \mathbb{F}; t)$.

Hence, $\overline{\sigma}(\mathbb{E}; t) \vee \overline{\sigma}(\mathbb{F}; t) \subseteq \overline{\sigma}(\mathbb{E} \vee \mathbb{F}; t)$.

Theorem 3: Let $(\mathbb{G}, \mathbb{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of CFZR and $\emptyset \neq \mathbb{E}, \mathbb{F} \subseteq \mathbb{G}$ Then

1. $\underline{\sigma}(\mathbb{E}; t) \circledcirc \underline{\sigma}(\mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; t)$
2. $\underline{\sigma}(\mathbb{E}; t) \vee \underline{\sigma}(\mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \vee \mathbb{F}; t)$.

Proof: **1.** Suppose that $g_3 \in \underline{\sigma}(\mathbb{E}; t) \circledcirc \underline{\sigma}(\mathbb{F}; t)$ then there exists $g_1 \in \underline{\sigma}(\mathbb{E}; t)$ and $g_2 \in \underline{\sigma}(\mathbb{F}; t)$ such that $g_3 = g_1 \circledcirc g_2$. This implies that $OSC_\sigma(g_1; t) \subseteq \mathbb{E}$ and $OSC_\sigma(g_2; t) \subseteq \mathbb{F}$. This shows that $OSC_\sigma(g_1; t) \circledcirc OSC_\sigma(g_2; t) \subseteq \mathbb{E} \circledcirc \mathbb{F}$. Since σ is $\circledcirc$-Complete, we have $OSC_\sigma(g_1; t) \circledcirc OSC_\sigma(g_2; t) = OSC_\sigma(g_1 \circledcirc g_2; t) \subseteq \mathbb{S} \circledcirc \mathbb{T}$. This implies that

$$OSC_\sigma(g_1 \circledcirc g_2; t) \subseteq \mathbb{E} \circledcirc \mathbb{F}.$$

$\Rightarrow g_3 = g_1 \circledcirc g_2 \in \underline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; t)$. Hence, $\underline{\sigma}(\mathbb{E}; t) \circledcirc \underline{\sigma}(\mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \circledcirc \mathbb{F}; t)$.

2. Suppose that $g_3 \in \underline{\sigma}(\mathbb{E}; t) \vee \underline{\sigma}(\mathbb{F}; t)$ then there exists $g_1 \in \underline{\sigma}(\mathbb{E}; t)$ and $g_2 \in \underline{\sigma}(\mathbb{F}; t)$ such that $g_3 = g_1 \vee g_2$. This implies that $OSC_\sigma(g_1; t) \subseteq \mathbb{E}$ and $OSC_\sigma(g_2; t) \subseteq \mathbb{F}$. Implies that $OSC_\sigma(g_1; t) \vee OSC_\sigma(g_2; t) \subseteq \mathbb{E} \vee \mathbb{F}$.

Since σ is $\vee$-Complete So, we have $OSC_\sigma(g_1; t) \vee OSC_\sigma(g_2; t) = OSC_\sigma(g_1 \vee g_2; t) \subseteq \mathbb{S} \vee \mathbb{T}$. This implies that

$$OSC_\sigma(g_1 \vee g_2; t) \subseteq \mathbb{E} \vee \mathbb{F}.$$

$\Rightarrow g_3 = g_1 \vee g_2 \in \underline{\sigma}(\mathbb{E} \vee \mathbb{F}; t)$. Hence, $\underline{\sigma}(\mathbb{E}; t) \vee \underline{\sigma}(\mathbb{F}; t) \subseteq \underline{\sigma}(\mathbb{E} \vee \mathbb{F}; t)$.

Definition 19: Let $\emptyset \neq \mathbb{E} \subseteq \mathbb{G}$ and $(\mathbb{G}, \mathbb{OSC}_\Omega(\mathbb{G}; t))$ be an $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of TCFR. Then a nonempty $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. $\overline{\sigma}(\mathbb{E}; t)$ of $\mathbb{E}$ in $(\mathbb{G}, \mathbb{OSC}_\Omega(\mathbb{G}; t))$ is a $Sub_\mathbb{G}$ of $\mathbb{G}$ then this is called an $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. Subquantale.

A nonempty $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-LW.appr. $\underline{\sigma}(\mathbb{E}; t)$ of $\mathbb{S}$ in $(\mathbb{G}, \mathbb{OSC}_\sigma(\mathbb{G}; t))$ is a $Sub_\mathbb{G}$ of $\mathbb{G}$ then this is called an $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-LW.appr. Subquantale.

Similarly, we can define ideals(prime, semi-prime, primary, multiplicative set, m-system).

Theorem 4: Assume that $\emptyset \neq \mathbb{E} \subseteq \mathbb{G}$ and $(\mathbb{G}, \mathbb{OSC}_\sigma(\mathbb{G}; t))$ is a $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of TCFR. If $\mathbb{E}$ is a closed under arbitrary joins, then $\overline{\sigma}(\mathbb{E}; t)$ is closed under arbitrary joins.

Proof: Let $\mathbb{F} \subseteq \overline{\sigma}(\mathbb{E}; t)$ then for each $f \in \mathbb{F}$, we obtain $f \in \overline{\sigma}(\mathbb{E}; t)$, then $OSC_\sigma(f; t) \cap \mathbb{E} \neq \emptyset$. There exists $x_f \in OSC_\sigma(f; t) \cap \mathbb{E}$. Therefore, we get $x_f \in OSC_\sigma(f; t)$ and $x_f \in \mathbb{E}$. Now,

$$\bigvee_{f \in \mathbb{F}} x_f \in OSC_\sigma(f; t) \vee OSC_\sigma(f; t) \vee \cdots \vee OSC_\sigma(f; t)$$

From Proposition 4, we get

$$\subseteq OSC_\sigma(f \vee f \vee f \vee \cdots \vee f; t) = OSC_\sigma(\vee\mathbb{F}; t)$$
$$\bigvee_{f \in \mathbb{F}} x_f \in OSC_\sigma(\vee\mathbb{F}; t)$$

Since $\mathbb{E}$ is a closed under arbitrary joins, we have $\bigvee_{f \in \mathbb{F}} x_f \in \mathbb{E}$. Therefore, we have

$$\begin{aligned} \bigvee_{f \in \mathbb{F}} x_f &\in OSC_\sigma(\vee\mathbb{F}; t) \cap \mathbb{E}, \\ &\Rightarrow OSC_\sigma(\vee\mathbb{F}; t) \cap \mathbb{E} \neq \emptyset \\ &\Rightarrow \vee\mathbb{F} \in \overline{\sigma}(\mathbb{E}; t) \end{aligned}$$

Thus, $\overline{\sigma}(\mathbb{E}; t)$ is closed under arbitrary joins.

Theorem 5: Assume that $\emptyset \neq \mathbb{E} \subseteq \mathbb{G}$ and $(\mathbb{G}, \mathbb{OSC}_\sigma(\mathbb{G}; t))$ is a $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of $\vee$-Complete FZ-Relations. Let $\mathbb{E}$ be a closed under arbitrary Joins, then $\underline{\sigma}(\mathbb{E}; t)$ is closed under arbitrary joins.

Proof: Let $\mathbb{F} \subseteq \underline{\sigma}(\mathbb{E}; t)$ then for each $f \in \mathbb{F}$, we have $f \in \underline{\sigma}(\mathbb{E}; t)$. Then $OSC_\sigma(f; t) \subseteq \mathbb{E}$.

Since σ is $\vee$-Complete FZ-Relation. Therefore,

$$\begin{aligned} &OSC_\sigma(f \vee f \vee f \vee \cdots \vee f; t) \\ &\quad = OSC_\sigma(f; t) \vee CC_\sigma(f; t) \vee \cdots \vee CC_\sigma(f; t) \\ &\quad \Rightarrow OSC_\sigma(\vee\mathbb{F}; t) = \vee OSC_\sigma(\mathbb{F}; t) \end{aligned}$$

Assume that $w \in OSC_\sigma(\vee\mathbb{F}; t) = \vee OSC_\sigma(\mathbb{F}; t)$. There exists $x_f \in OSC_\sigma(f; t) \subseteq \mathbb{E}(f \in \mathbb{F})$ such that $w = \bigvee_{f \in \mathbb{F}} x_f$. Since $\mathbb{E}$ is a closed under arbitrary joins, we obtain w $= \bigvee_{f \in \mathbb{F}} x_f \in \mathbb{E}$. Therefore, we get $OSC_\sigma(\vee\mathbb{F}; t) \subseteq \mathbb{E}$,

$$\Rightarrow \vee\mathbb{F} \in \underline{\sigma}(\mathbb{E}; t)$$

Hence, $\underline{\sigma}(\mathbb{E}; t)$ is closed under arbitrary join.

Theorem 6: Let $(\mathbb{G}, \mathbb{OSC}_\Omega(\mathbb{G}; t))$ be an $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of TCFR and $\mathbb{E}$ is a $Sub_\mathbb{G}$ of $\mathbb{G}$.Then $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathbb{CC}_\sigma(\mathbb{G}; t)$-UP.appr. Subquantale.

Proof: Let $\mathbb{E}$ be a $Sub_\mathbb{G}$ of $\mathbb{G}$, thenby definition of $Sub_\mathbb{G}$, we have $\mathbb{E} \circledcirc \mathbb{E} \subseteq \mathbb{E}$ and $\mathbb{E} \vee \mathbb{E} \subseteq \mathbb{E}$. By Proposition 2, we obtain $\emptyset \neq \mathbb{E} \subseteq \overline{\sigma}(\mathbb{E}; t)$. Hence $\overline{\sigma}(\mathbb{E}; t)$ is a nonempty $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. As $\circledcirc\mathbb{E} \subseteq \mathbb{E}$. By Proposition 2 we get $\overline{\sigma}(\mathbb{E} \circledcirc \mathbb{E}; t) \subseteq \overline{\sigma}(\mathbb{E}; t)$. By Theorem 2 we obtain $\overline{\sigma}(\mathbb{E}; t) \circledcirc \overline{\sigma}(\mathbb{E}; t) \subseteq \overline{\sigma}(\mathbb{E} \circledcirc \mathbb{E}; t) \subseteq \overline{\sigma}(\mathbb{E}; t)$.

Thus, $\overline{\sigma}(\mathbb{E}; t) \circledcirc \overline{\sigma}(\mathbb{E}; t) \subseteq \overline{\sigma}(\mathbb{E}; t)$.

Also, As $\vee\mathbb{E} \subseteq \mathbb{E}$. By Proposition 2 we obtain $\overline{\sigma}(\mathbb{E} \vee \mathbb{E}; t) \subseteq \overline{\sigma}(\mathbb{E}; t)$. By Theorem 2 we have $\overline{\sigma}(\mathbb{E}; t) \vee \overline{\sigma}(\mathbb{E}; t) \subseteq \overline{\sigma}(\mathbb{E} \vee \mathbb{E}; t) \subseteq \overline{\sigma}(\mathbb{E}; t)$.

Thus, $\overline{\sigma}(\mathbb{E}; t) \vee \overline{\sigma}(\mathbb{E}; t) \subseteq \overline{\sigma}(\mathbb{E}; t)$.

Hence, $\overline{\sigma}(\mathbb{E}; t)$ is a $Sub_\mathbb{G}$ of $\mathbb{G}$. Therefore, $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. quantale.

Theorem 7: Let $(\mathbb{G}, \mathbb{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-AP.SP of CFZR and $\mathbb{E}$ is a $Sub_\mathbb{G}$ of $\mathbb{G}$ with $\underline{\sigma}(\mathbb{E}; t) \neq \emptyset$ Then $\underline{\sigma}(\mathbb{E}; t)$ is an $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-LW.appr. quantale.

Proof: Let $\mathbb{E}$ be a $Sub_\mathbb{G}$ of $\mathbb{G}$, then by definition of $Sub_\mathbb{G}$, we have $E \circledcirc E \subseteq E$ and $\mathbb{E} \vee \mathbb{E} \subseteq \mathbb{E}$. Also, $\underline{\sigma}(\mathbb{E}; t)$ is a nonempty $\mathbb{OSC}_\sigma(\mathbb{G}; t)$-LW.appr.

As $\circledcirc\mathbb{E} \subseteq \mathbb{E}$. By Proposition 1 we have $\underline{\sigma}(\mathbb{E} \circledcirc \mathbb{E}; t) \subseteq \underline{\sigma}(\mathbb{E}; t)$. By Theorem 3 we get $\underline{\sigma}(\mathbb{E}; t) \circledcirc \underline{\sigma}(\mathbb{E}; t) \subseteq \underline{\sigma}(\mathbb{E} \circledcirc \mathbb{E}; t) \subseteq \underline{\sigma}(\mathbb{E}; t)$.

Thus, $\underline{\sigma}(\mathbb{E}; t) \circledcirc \underline{\sigma}(\mathbb{E}; t) \subseteq \underline{\sigma}(\mathbb{E}; t)$.

Also, as $E \vee E \subseteq E$. By Proposition 2 we have $\underline{\sigma}(\mathbb{E} \vee \mathbb{E}; t) \subseteq \underline{\sigma}(\mathbb{E}; t)$. By Theorem 3 we get $\underline{\sigma}(\mathbb{E}; t) \vee \underline{\sigma}(\mathbb{E}; t) \subseteq \underline{\sigma}(\mathbb{E} \vee \mathbb{E}; t) \subseteq \underline{\sigma}(\mathbb{E}; t)$.

Thus, $\underline{\sigma}(\mathbb{E};t) \vee \underline{\sigma}(\mathbb{E};t) \subseteq \underline{\sigma}(\mathbb{E};t)$.

Hence, $\underline{\sigma}(\mathbb{E};t)$ is a $Sub_{\mathbb{G}}$ of $\mathbb{G}$. Therefore, $\underline{\sigma}(\mathbb{E};t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G};t)$-LW.appr. quantale.

Example 3: Assume that $\mathbb{E} := \{0', f', g', h'\} \subseteq \mathbb{G}$ from Example 2, then we have $\overline{\sigma}(\mathbb{E};0.6) = \{0'e', f', g', h'\}$,and $\underline{\sigma}(\mathbb{E};0.6) = 0', h'$

Note that $\overline{\sigma}(\mathbb{E};0.6) \neq \underline{\sigma}(\mathbb{E};0.6)$. Hence, it is simple to check that $\overline{\sigma}(\mathbb{E};0.6)$ is an $\mathcal{OSC}_\sigma(\mathbb{G};0.6)$-UP.appr. quantaleand $\underline{\sigma}(\mathbb{E};0.6)$ is an $\mathcal{OSC}_\sigma(\mathbb{G};0.6)$-LW.appr. quantale. However, $\mathbb{E}$ is not a $Sub_{\mathbb{G}}$ of $\mathbb{G}$.

Theorem 8: Let $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G};t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G};t)$-AP.SP. of $\vee$-Complete FZ-Relations and $\mathbb{E}$ is an ideal of $\mathbb{G}$.Then $\overline{\sigma}(\mathbb{E};t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G};t)$-UP.appr. ideal.

Proof: **1.** Suppose $p, q \in \overline{\sigma}(\mathbb{E};t)$ then $OSC_\sigma(p;t) \cap \mathbb{E} \neq \emptyset$ and $OSC_\sigma(q;t) \cap \mathbb{E} \neq \emptyset$

There exists $r \in OSC_\sigma(p;t) \cap \mathbb{E}$ and $s \in OSC_\sigma(q;t) \cap \mathbb{E}$, we have $r \in OSC_\sigma(p;t)$, $r \in \mathbb{E}$ and $s \in OSC_\sigma(q;t)$, $s \in \mathbb{E}$. As $\mathbb{E}$ is an ideal so we get $r \vee s \in \mathbb{E}$ and $r \vee s \in OSC_\sigma(p;t) \vee OSC_\sigma(q;t)$. From Proposition 3 we obtain,

$$\begin{aligned} r \vee s &\in OSC_\sigma(p;t) \vee OSC_\sigma(q;t) \subseteq OSC_\sigma(p \vee q;t) \\ &\Rightarrow r \vee s \in OSC_\sigma(p \vee q;t) \\ &\Rightarrow r \vee s \in OSC_\sigma(p \vee q;t) \cap \mathbb{E} \end{aligned}$$

$OSC_\sigma(p \vee q;t) \cap \mathbb{E} \neq \emptyset$, Thus, $p \vee q \in \overline{\sigma}(\mathbb{E};t)$.

2. Let $p \leq q \in \overline{\sigma}(\mathbb{E};t)$. Then there exists $w \in OSC_\sigma(q;t) \cap \mathbb{E}$. From this we obtain $w \in OSC_\sigma(q;t)$ and $w \in \mathbb{E}$. Now,

$$OSC_\sigma(q;t) = OSC_\sigma(p \vee q;t) \quad \because p \vee q = q$$

Since σ is $\vee$-Complete FZ-Relations, Therefore, $OSC_\sigma(p \vee q;t) = OSC_\sigma(p;t) \vee OSC_\sigma(q;t)$. There exists $\in OSC_\sigma(p;t)$, $s \in OSC_\sigma(q;t)$ such that $w = r \vee s$. Since $\mathbb{E}$ is an ideal.

Therefore, $r \leq r \vee s = w \in \mathbb{E}$ implies that $r \in \mathbb{E}$. Thus, $r \in OSC_\sigma(p;t) \cap \mathbb{E}$ implies that $OSC_\sigma(p;t) \cap \mathbb{E} \neq \emptyset$. Thus $p \in \overline{\sigma}(\mathbb{E};t)$.

3. Let $r \in \mathbb{G}, p \in \overline{\sigma}(\mathbb{E};t)$ Then there exists $q \in OSC_\sigma(p;t) \cap \mathbb{E}$ such that $q \in OSC_\sigma(p;t)$ and $q \in \mathbb{E}$. As $\mathbb{E}$ is an ideal of $\mathbb{G}$, so we get, $q \circledcirc s \in \mathbb{E}$, $s \circledcirc q \in \mathbb{E}$ for each $s \in OSC_\sigma(r;t) \subseteq \mathbb{G}$. Therefore, we have

$$q \circledcirc s \in OSC_\sigma(p;t) \circledcirc OSC_\sigma(r;t)$$

From Proposition 2 we have

$$\begin{aligned} q \circledcirc s &\in OSC_\sigma(p;t) \circledcirc OSC_\sigma(r;t) \subseteq OSC_\sigma(p \circledcirc r;t) \\ &\Rightarrow q \circledcirc s \in OSC_\sigma(p \circledcirc r;t) \cap \mathbb{E} \\ &\Rightarrow OSC_\sigma(p \circledcirc r;t) \cap \mathbb{E} \neq \emptyset \Rightarrow p \circledcirc r \in \overline{\sigma}(\mathbb{E};t) \end{aligned}$$

Similarly, we have $r \circledcirc p \in \overline{\sigma}(\mathbb{E};t)$. Therefore, $\overline{\sigma}(\mathbb{E};t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G};t)$-UP.appr. ideal.

We give an example to illustrate that the condition for all $g_1, g_2 \in \mathbb{G}$,

$$(OSC_\sigma(g_1;0.6)) \vee (OSC_\sigma(g_2;0.6)) = OSC_\sigma(g_1 \vee g_2;0.6)$$

In Theorem 8 is indispensable.

Example 4: From Example 1we havefor all $g_1, g_2 \in \mathbb{G}$

$$(OSC_\sigma(g_1;0.6)) \vee (OSC_\sigma(g_2;0.6)) \neq OSC_\sigma(g_1 \vee g_2;0.6)$$

Because,

$$\left(OSC_\sigma(0';0.6)\right) \vee \left(OSC_\sigma(r';0.6)\right) = \{q'\}$$

and

$$OSC_\sigma(0' \vee r';0.6) = \{p', q', s'\}$$

This implies that

$$\left(OSC_\sigma(0';0.6)\right) \vee \left(OSC_\sigma(r';0.6)\right) \neq OSC_\sigma(0' \vee r';0.6)$$

Set $\mathbb{E} = \{0', p'\}$ then $\mathbb{E}$ is an ideal but $\overline{\sigma}(\mathbb{E};0.6) = \{0', p;, q', s'\}$ is not ideal because $\overline{\sigma}(\mathbb{E};0.6)$ is not a lower set.

Theorem 9: Let $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G};t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G};t)$-AP.SP. of CFZR and $\mathbb{E}$ be an ideal of $\mathbb{G}$ with $\underline{\sigma}(\mathbb{E};t) \neq \emptyset$. Then $\underline{\sigma}(\mathbb{E};t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G};t)$-LW.appr. ideal.

Proof: **1.** Let $p, q \in \underline{\sigma}(\mathbb{E};t)$. Then $OSC_\sigma(p;t) \subseteq \mathbb{E}$ and $OSC_\sigma(q;t) \subseteq \mathbb{E}$

Since σ is $\vee$-Complete FZ-Relations, we have

$$\begin{aligned} OSC_\sigma(p;t) \vee OSC_\sigma(q;t) &= OSC_\sigma(p \vee q;t) \subseteq \mathbb{E} \\ &\Rightarrow OSC_\sigma(p \vee q;t) \subseteq \mathbb{E}. \end{aligned}$$

Hence, $p \vee q \in \underline{\sigma}(\mathbb{E};t)$.

2. Let $p \leq q \in \underline{\sigma}(\mathbb{E};t)$. Then there exists $OSC_\sigma(q;t) \subseteq \mathbb{E}$. Let $w \in OSC_\sigma(p;t)$ and $r \in OSC_\sigma(q;t)$, we have

$$w \vee r \in OSC_\sigma(p;t) \vee OSC_\sigma(q;t)$$

From Proposition 4 we have

$$\begin{aligned} w \vee r &\in OSC_\sigma(p;t) \vee OSC_\sigma(q;t) \subseteq OSC_\sigma(p \vee q;t) \\ w \vee r &\in OSC_\sigma(p \vee q;t) = OSC_\sigma(q;t) \subseteq \mathbb{E} \\ &\because p \vee q = q \end{aligned}$$

As $\mathbb{E}$ is an ideal. Therefore, $w \leq w \vee r \in \mathbb{E} \Rightarrow w \in \mathbb{E}$. Thus, $OSC_\sigma(p;t) \subseteq \mathbb{E}$. Hence, $p \in \underline{\sigma}(\mathbb{E};t)$.

3. Let $r \in \mathbb{G}, p \in \underline{\sigma}(\mathbb{E};t)$. Then we have $OSC_\sigma(p;t) \subseteq \mathbb{E}$. Let $q \in OSC_\sigma(p \circledcirc r;t)$. Since σ is $\circledcirc$-Complete FZ-Relations so we have

$$q \in OSC_\sigma(p \circledcirc r;t) = OSC_\sigma(p;t) \circledcirc OSC_\sigma(r;t)$$

Then there exists $q_1 \in OSC_\sigma(p;t) \subseteq \mathbb{E}$ and $q_2 \in OSC_\sigma(r;t)$ such that $q = q_1 \circledcirc q_2$.

As $\mathbb{E}$ is an ideal of $\mathbb{G}$, we get $q = q_1 \circledcirc q_2 \in \mathbb{E}$. Therefore, we have $OSC_\sigma(p \circledcirc r;t) \subseteq \mathbb{E}$.

Hence, $p \circledcirc r \in \underline{\sigma}(\mathbb{E};t)$. Similarly, we have $r \circledcirc p \in \underline{\sigma}(\mathbb{E};t)$. Thus, $\underline{\sigma}(\mathbb{E};t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G};t)$-LP.appr. ideal.

Example 5: Assume that $\mathbb{E} := \{0', g', h', 1'\} \subseteq \mathbb{G}$ from Example 2, then we have $\overline{\sigma}(\mathbb{E};0.6) = \mathbb{G}$ and $\underline{\sigma}(\mathbb{E};0.6) = 0', h', 1'$. Note that $\overline{\sigma}(\mathbb{E};0.6) \neq \underline{\sigma}(\mathbb{E};0.6)$. Hence, it is simple to verify that $\overline{\sigma}(\mathbb{E};0.6)$ is an $\mathcal{OSC}_\sigma(\mathbb{G};0.6)$-UP.appr. ideal and $\underline{\sigma}(\mathbb{E};0.6)$ is an $\mathcal{OSC}_\sigma(\mathbb{G};0.7)$-LW.appr. ideal. But $\mathbb{E}$ is not a ideal of $\mathbb{G}$.

Theorem 10: Let $\mathbb{E}$ be a prime ideal of $\mathbb{G}$ and $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of CFZR. Then $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. prime ideal.

Proof: As $\mathbb{E}$ is an ideal of $\mathbb{G}$ so by Theorem 8 $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. ideal. Now we have to show that $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. prime ideal.

Let $g_1, g_2 \in \mathbb{G}$ such that $g_1 \circledcirc g_2 \in \overline{\sigma}(\mathbb{E}; t)$. Then $OSC_\sigma(g_1 \circledcirc g_2; t) \cap \mathbb{E} \neq \emptyset$.

As σ is $\circledcirc$-Complete FZ-Relation. Therefore, $OSC_\sigma(g_1; t) \circledcirc OSC_\sigma(g_2; t) \cap \mathbb{E} = OSC_\sigma(g_1 \circledcirc g_2; t) \cap \mathbb{E} \neq \emptyset$. There exists $g_3 \in OSC_\sigma(g_1; t)$ and $g_4 \in OSC_\sigma(g_2; t)$ such that $g_3 \circledcirc g_4 \in \mathbb{E}$. Since $\mathbb{E}$ is a prime ideal of $\mathbb{G}$, so we have $g_3 \in \mathbb{E}$ or $g_4 \in \mathbb{E}$. Therefore, we have $g_3 \in OSC_\sigma(g_1; t) \cap \mathbb{E}$ or $g_4 \in OSC_\sigma(g_2; t) \cap \mathbb{E}$. This implies that $OSC_\sigma(g_1; t) \cap \mathbb{E} \neq \emptyset$ or $OSC_\sigma(g_2; t) \cap \mathbb{E} \neq \emptyset$.

$g_1 \in \overline{\sigma}(\mathbb{E}; t)$ or $g_2 \in \overline{\sigma}(\mathbb{E}; t)$. Hence, $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. prime ideal.

Theorem 11: Let $\mathbb{E}$ be a prime ideal of $\mathbb{G}$ and $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of CFZR. Then $\underline{\sigma}(\mathbb{E}; t) \neq \emptyset$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-LW.appr. prime ideal.

Proof: Proof is similar to above.

Theorem 12: Let $\mathbb{E}$ be a semi-prime ideal of $\mathbb{G}$ and $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of CFZR. Then $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. semi-prime ideal.

Proof: As $\mathbb{E}$ is an ideal of $\mathbb{G}$, so by Theorem 8, $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. ideal. Now we have to show that $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. semi-prime ideal.

Let $g_1, \in \mathbb{G}$ such that$g_1 \circledcirc g_1 \in \overline{\sigma}(\mathbb{E}; t)$, then $OSC_\sigma(g_1 \circledcirc g_1; t) \cap \mathbb{E} \neq \emptyset$.

There exists $g_2 \in \mathrm{OSC}_\sigma(g_1 \circledcirc g_1; t) \cap \mathbb{E}$. This implies that $g_2 \in \mathrm{OSC}_\sigma(g_1 \circledcirc g_1; t)$ and $g_2 \in \mathbb{E}$.

Since σ is $\circledcirc$-Complete FZ-Relation. so we have $OSC_\sigma(g_1; t) \circledcirc OSC_\sigma(g_1; t) = OSC_\sigma(g_1 \circledcirc g_1; t)$. Then there exists $g_3 \in OSC_\sigma(g_1; t)$ such that $g_2 = g_3 \circledcirc g_3 \in \mathbb{E}$. Since $\mathbb{E}$ is a semi-prime ideal of $\mathbb{G}$, so we have $g_3 \in \mathbb{E}$. This implies that $g_3 \in OSC_\sigma(g_1; t) \cap \mathbb{E}$. This implies that $OSC_\sigma(g_1; t) \cap \mathbb{E} \neq \emptyset$. Thus $g_1 \in \overline{\sigma}(\mathbb{E}; t)$. Hence, $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. semi-prime ideal.

Theorem 13: Let $\mathbb{E}$ be a semi-prime ideal of $\mathbb{G}$ and $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of CFZR. Then $\underline{\sigma}(\mathbb{E}; t) \neq \emptyset$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-LW.appr. semi-prime ideal.

Proof: Proof is similar to above.

Theorem 14: Let $\mathbb{E}$ be a primary ideal of $\mathbb{G}$ and $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of CFZR. If $\overline{\sigma}(\mathbb{E}; t) \neq \emptyset$ and $\overline{\sigma}(\mathbb{E}; t) \neq \mathbb{Q}$. Then $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. primary ideal.

Proof: As $\mathbb{E}$ is an ideal of $\mathbb{G}$, so by Theorem 8, $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. ideal. Now we have to show that $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. primary ideal. Let $g_1, g_2 \in \mathbb{G}$ such that$g_1 \circledcirc g_2 \in \overline{\sigma}(\mathbb{E}; t)$ and $g_1 \notin \overline{\sigma}(\mathbb{E}; t)$ then there exists $p \in OSC_\sigma(g_1 \circledcirc g_2; t) \cap \mathbb{E}$. From this we have, $p \in OSC_\sigma(g_1 \circledcirc g_2; t)$ and $p \in \mathbb{E}$.

As σ is $\circledcirc$-Complete FZ-Relation so we have $p \in OSC_\sigma(g_1 \circledcirc g_2; t) = OSC_\sigma(g_1; t) \circledcirc OSC_\sigma(g_2; t) \Rightarrow p \in \mathrm{OSC}_\sigma(g_1; t) \circledcirc \mathrm{OSC}_\sigma(g_2; t)$. There exists $q \in OSC_\sigma(g_1; t)$ and $w \in OSC_\sigma(g_2; t)$ be such that $p = q \circledcirc w \in \mathbb{E}$. Since $g_1 \notin \overline{\sigma}(\mathbb{E}; t)$, we get $q \notin \mathbb{E}$. Since $\mathbb{E}$ is a primary ideal we have $w^n \in \mathbb{E}$ for some $n > 0$. Now, $w \circledcirc w \circledcirc w \circledcirc \ldots \circledcirc w \in OSC_\sigma(g_2; t) \circledcirc \mathrm{OSC}_\sigma(g_2; t) \circledcirc OSC_\sigma(g_2; t) \ldots \circledcirc OSC_\sigma(g_2; t)$.

Since σ is $\circledcirc$-Complete FZ-Relation. Thus, we have

$$w^n \in OSC_\sigma(g_2 \circledcirc g_2 \circledcirc \ldots \circledcirc g_2; t)$$
$$\Rightarrow w^n \in OSC_\sigma\left(g_2^n; t\right), \quad \Rightarrow w^n \in OSC_\sigma\left(g_2^n; t\right) \cap \mathbb{E}$$
$$\Rightarrow OSC_\sigma\left(g_2^n; t\right) \cap \mathbb{E} \neq \emptyset, \quad \Rightarrow g_2^n \in \overline{\sigma}(\mathbb{E}; t)$$

Hence, $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. primary ideal.

Theorem 15: Let $\mathbb{E}$ be a primary ideal of $\mathbb{G}$ and $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of CFZR. Then $\underline{\sigma}(\mathbb{E}; t) \neq \emptyset$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-LW.appr. primary ideal.

Proof: Proof is similar to above.

Theorem 16: Let $\mathbb{E}$ is a multiplicative set of $\mathbb{G}$ and $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of TCFR. Then $\overline{\sigma}(\mathbb{E}; t) \neq \emptyset$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. multiplicative set.

Proof: Assume that $g_1, g_2 \in \overline{\sigma}(\mathbb{E}; t)$, there exists $p \in OSC_\sigma(g_1; t) \cap \mathbb{E}$, $q \in OSC_\sigma(g_2; t) \cap \mathbb{E}$.

$\Rightarrow p \in OSC_\sigma(g_1; t), p \in \mathbb{E}$ and $q \in OSC_\sigma(g_2; t), q \in \mathbb{E}$, $\Rightarrow p \circledcirc q \in \mathrm{OSC}_\sigma(g_1; t) \circledcirc \mathrm{OSC}_\sigma(g_2; t)$.

From Proposition 3, we have $p \circledcirc q \in \mathrm{OSC}_\sigma(g_1; t) \circledcirc \mathrm{OSC}_\sigma(g_2; t) \subseteq \mathrm{OSC}_\sigma(g_1 \circledcirc g_2; t) \Rightarrow p \circledcirc q \in \mathrm{OSC}_\sigma(g_1 \circledcirc g_2; t)$.

Since $\mathbb{E}$ is a multiplicative set, so we have $p \circledcirc q \in \mathbb{E}$. Therefore, $p \circledcirc q \in \mathrm{OSC}_\sigma(g_1 \circledcirc g_2; t) \cap \mathbb{E}$.

$\Rightarrow \mathrm{OSC}_\sigma(g_1 \circledcirc g_2; t) \cap \mathbb{E} \neq \emptyset \Rightarrow g_1 \circledcirc g_2 \in \overline{\sigma}(\mathbb{E}; t)$.

Hence, $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. multiplicative set.

Theorem 17: Let $\mathbb{E}$ be a multiplicative set of $\mathbb{G}$ and $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of $\circledcirc$-Complete FZ-Relation. Then $\underline{\sigma}(\mathbb{E}; t) \neq \emptyset$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-LW.appr. multiplicative set.

Proof: Assume that $g_1, g_2 \in \underline{\sigma}(\mathbb{E}; t)$, then, $OSC_\sigma(g_1; t) \subseteq \mathbb{E}$ and $OSC_\sigma(g_2; t) \subseteq \mathbb{E}$.Let $p \in \mathrm{OSC}_\sigma(g_1 \circledcirc g_2; t)$.

As σ is $\circledcirc$-Complete FZ-Relation, we have $p \in \mathrm{OSC}_\sigma(g_1 \circledcirc g_2; t) = \mathrm{OSC}_\sigma(g_1; t) \circledcirc \mathrm{OSC}_\sigma(g_2; t)$.

Then there exists $q \in OSC_\sigma(g_1; t) \subseteq \mathbb{E}$ and $w \in OSC_\sigma(g_2; t) \subseteq \mathbb{E}$ such that $p = q \circledcirc w$.

Since, $\mathbb{E}$ is a multiplicative set, we have

$$p = q \circledcirc w \in \mathbb{E}, \Rightarrow p \in \mathbb{E}$$

So, $\mathrm{OSC}_\sigma(g_1 \circledcirc g_2; t) \subseteq \mathbb{E}, \Rightarrow g_1 \circledcirc g_2 \in \underline{\sigma}(\mathbb{E}; t)$

Hence, $\underline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-LW.appr. multiplicative set.

Theorem 18: Let $\mathbb{E} \subseteq \mathbb{G}$ is an m-system of $\mathbb{G}$ and $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t))$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-AP.SP. of $\vee$-Complete FZ-Relation. Then $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. m-system.

Proof: Assume that $g_1, g_2 \in \overline{\sigma}(\mathbb{E}; t)$, then $OSC_\sigma(g_1; t) \cap \mathbb{E} \neq \emptyset$ and $OSC_\sigma(g_2; t) \cap \mathbb{E} \neq \emptyset$.

Hence there exists $p_1 \in OSC_\sigma(g_1; t) \cap \mathbb{E}$, $p_2 \in OSC_\sigma(g_2; t) \cap \mathbb{E}$.

$\Rightarrow p_1 \in OSC_\sigma(g_1; t)$, $p_1 \in \mathbb{E}$ and $p_2 \in OSC_\sigma(q_2; t)$, $p_2 \in \mathbb{E}, \Rightarrow p_1, p_2 \in \mathbb{E}$. As $\mathbb{E}$ is an m-system, there is $p \in \mathbb{E}$ such that $p \leq p_1 \circledcirc 1 \circledcirc p_2$.

Hence, $p_1 \circledcirc 1 \circledcirc p_2 = p \vee (p_1 \circledcirc 1 \circledcirc p_2)$.
Now, $p_1 \circledcirc 1 \circledcirc p_2 \in OSC_\sigma(p_1 \circledcirc 1 \circledcirc p_2; t)$,

$$\Rightarrow p_1 \circledcirc 1 \circledcirc p_2 \in OSC_\sigma(p \vee (p_1 \circledcirc 1 \circledcirc p_2); t),$$

Since σ is $\vee$-Complete FZ-Relation so we have

$$p_1 \circledcirc 1 \circledcirc p_2 \in OSC_\sigma(p; t) \vee OSC_\sigma(p_1 \circledcirc 1 \circledcirc p_2; t)$$

There exists $q \in OSC_\sigma(p; t)$,
$w \in \mathrm{OSC}_\sigma(p_1 \circledcirc 1 \circledcirc p_2; t)$ such that $p_1 \circledcirc 1 \circledcirc p_2 = q \vee w$ and hence, $q \leq p_1 \circledcirc 1 \circledcirc p_2$. By Proposition 1, we have $q \in OSC_\sigma(p; t)$ if and only if

$$OSC_\sigma(p; t) = OSC_\sigma(q; t)$$

Since $p \in \mathbb{E}$, $p \in OSC_\sigma(p; t)$ this implies that

$$OSC_\sigma(p; t) \cap \mathbb{E} \neq \emptyset$$

$\Rightarrow \mathrm{OSC}_\sigma(\mathrm{q}; t) \cap \mathbb{E} \neq \emptyset$, $\Rightarrow y \in \overline{\sigma}(\mathbb{E}; t)$.
Hence, $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. m-system.

IV. HOMOMORPHIC IMAGES OF ROUGH SUBSTRUCTURES IN QUANTALE

This section is devoted to study the relations between rough substructures of quantale dependent on overlap of the successor classes and their homomorphic images. Moreover, some important theorems under quantale homomorphism are introduced.

Definition 20: Let $(\mathbb{G}, \circledcirc)$ and $(\mathbb{H}, \otimes)$ be two quantales. Then a mapping $\mathcal{F} : \mathbb{G} \to \mathbb{H}$ is known as a homomorphism in quantale if it satisfies the following Properties

1. $\mathcal{F}(g_1 \circledcirc g_2) = \mathcal{F}(g_1) \otimes \mathcal{F}(g_2)$
2. $\mathcal{F}(\vee_{j \in \mathcal{J}} g_j) = \vee_{j \in \mathcal{J}} \mathcal{F}(g_j) \forall g_1, g_2, g_j \in \mathbb{G}$,

A homomorphism $\mathcal{F}$ is said to be monomorphism if it is one-one and homomorphism $\mathcal{F}$ is said to be epimorphism if it is onto. A homomorphism $\mathcal{F}$ is said to be isomorphism if it is bijective. Note that σ is order preserving as $g_1 \leq g_2$ implies $\mathcal{F}(g_1) \leq \mathcal{F}(g_2)$.

Proposition 5 [25]: Let $\sigma(g_1, g_2) = \rho(\mathcal{F}(g_1), \mathcal{F}(g_2))$, $\forall g_1, g_2 \in \mathbb{G}$ where $\mathcal{F}$ is a surjective homomorphism from $\mathbb{G}$ in $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t)$ to $\mathbb{H}$ in $(\mathbb{H}; \mathcal{OSC}_\rho(\mathbb{H}; t))$ then following properties hold

1. $g_1 \in \mathcal{OSC}_\sigma(g_2; t)$ if and only if $\mathcal{F}(g_1) \in \mathcal{OSC}_\rho(\mathcal{F}(g_2); t)$, $\forall g_1, g_2 \in \mathbb{G}$.
2. For every nonempty subset $\mathbb{E}$ of $\mathbb{G}$ we have
$$\mathcal{F}(\overline{\sigma}(\mathbb{E}; t)) = \overline{\rho}(\mathcal{F}(\mathbb{E}); t).$$
3. For every nonempty subset $\mathbb{E}$ of $\mathbb{G}$ we have
$$\mathcal{F}\left(\underline{\sigma}(\mathbb{E}; t)\right) \subseteq \underline{\rho}(\mathcal{F}(\mathbb{E}); t).$$
4. For every nonempty subset $\mathbb{E}$ of $\mathbb{G}$ and if $\mathcal{F}$ is one-one then we have
$$\mathcal{F}\left(\underline{\sigma}(\mathbb{E}; t)\right) = \underline{\rho}(\mathcal{F}(\mathbb{E}); t)$$
5. If ρ is a transitive and compatible FZ-Relations, then σ is a transitive and compatible FZ-Relations.

Proof: **1.** Let $g_1 \in OSC_\sigma(g_2; t)$, where $g_1, g_2 \in \mathbb{G}$. Then $\mathcal{F}(g_1), \mathcal{F}(g_2) \in \mathbb{H}$ and $SC_\sigma(g_1; t) \cap SC_\sigma(g_2; t) \neq \emptyset$. Thus there exists $g_3 \in \mathbb{G}$ such that $g_3 \in SC_\sigma(g_1; t) \cap SC_\sigma(g_2; t)$.
Hence, $\sigma(g_1, g_3) \geq t$ and $\sigma(g_2, g_3) \geq t$. By the assumption, we get

$$\rho(\mathcal{F}(g_1), \mathcal{F}(g_3)) = \sigma(g_1, g_3) \geq t \quad \text{and}$$
$$\rho(\mathcal{F}(g_2), \mathcal{F}(g_3)) = \sigma(g_2, g_3) \geq t.$$

Thus, $\mathcal{F}(g_3) \in SC_\sigma(\mathcal{F}(g_1); t) \cap SC_\sigma(\mathcal{F}(g_2); t)$.
This implies that $SC_\sigma(\mathcal{F}(g_1); t) \cap SC_\sigma(\mathcal{F}(g_2); t) \neq \emptyset$.
Therefore, we have $\mathcal{F}(g_1) \in OSC_\sigma(\mathcal{F}(g_2); t)$.
Conversely, let $\mathcal{F}(g_1) \in OSC_\sigma(\mathcal{F}(g_2); t)$. Then $SC_\sigma(\mathcal{F}(g_1); t) \cap SC_\sigma(\mathcal{F}(g_2); t) \neq \emptyset$.
Then there exists $g_3 \in \mathbb{G}$ such that $\mathcal{F}(g_3) \in SC_\sigma(\mathcal{F}(g_1); t) \cap SC_\sigma(\mathcal{F}(g_2); t)$.
Then $\rho(\mathcal{F}(g_1), \mathcal{F}(g_3)) \geq t$ and $\rho(\mathcal{F}(g_2), \mathcal{F}(g_3)) \geq t$. This implies that

$$\rho(\mathcal{F}(g_1), \mathcal{F}(g_3)) = \sigma(g_1, g_3) \geq t \quad \text{and}$$
$$\rho(\mathcal{F}(g_2), \mathcal{F}(g_3)) = \sigma(g_2, g_3) \geq t.$$

This implies that $g_3 \in SC_\sigma(g_1; t) \cap SC_\sigma(g_2; t)$.
We get, $SC_\sigma(g_1; t) \cap SC_\sigma(g_2; t) \neq \emptyset$.
Hence, $g_1 \in OSC_\sigma(g_2; t)$.

2. Let $\mathbb{E} \neq \emptyset$ and $\mathbb{E} \subseteq \mathbb{G}$. Suppose that $h_1 \in \mathcal{F}(\overline{\sigma}(\mathbb{E}; t))$. Then there exists $g_1 \in \overline{\sigma}(\mathbb{E}; t)$ such that $\mathcal{F}(g_1) = h_1$, we have $OSC_\sigma(g_1; t) \cap \mathbb{E} \neq \emptyset$. There exists $g_2 \in \mathbb{G}$ such that $g_2 \in OSC_\sigma(g_1; t) \cap \mathbb{E}$ and $g_2 \in \mathbb{E}$. By property (1), we obtain that $\mathcal{F}(g_2) \in OSC_\rho(\mathcal{F}(g_1); t)$ and $\mathcal{F}(g_2) \in \mathcal{F}(\mathbb{E})$.
$OSC_\rho(\mathcal{F}(g_1); t) \cap \mathcal{F}(\mathbb{E}) \neq \phi$. So we have,

$$h_1 = \mathcal{F}(g_1) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$$

Thus, $\mathcal{F}\left(\overline{\Omega}(\mathbb{E}; t)\right) \subseteq \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$.
Now, let $h_2 \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ then there exists $g_3 \in \mathbb{G}$ such that $h_2 = \mathcal{F}(g_3)$ and so we have $OSC_\rho(\mathcal{F}(g_3); t) \cap \mathcal{F}(\mathbb{E}) \neq \phi$. There exists $g_4 \in \mathbb{E}$ be such that $\mathcal{F}(g_4) \in OSC_\rho(\mathcal{F}(g_3); t)$ and $\phi(g_4) \in \mathcal{F}(\mathbb{E})$. By property (1) we get $q_4 \in OSC_\sigma(g_3; t) \cap \mathbb{E}$ and $g_3 \in \mathbb{E}$, so we have $OSC_\sigma(g_3; t) \cap \mathbb{E} \neq \emptyset$. Hence, $g_3 \in \overline{\sigma}(\mathbb{E}; t)$ and therefore, $h_2 = \mathcal{F}(g_3) \in \mathcal{F}(\overline{\sigma}(\mathbb{E}; t))$. Thus,

$$\overline{\rho}(\mathcal{F}(\mathbb{E}); t) \subseteq \mathcal{F}(\overline{\sigma}(\mathbb{E}; t))$$

Hence, $\mathcal{F}(\overline{\sigma}(\mathbb{E}; t)) = \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$.

3. Let $\mathbb{E} \neq \emptyset$ and $\mathbb{E} \subseteq \mathbb{G}$. Suppose that $h_1 \in \mathcal{F}\left(\underline{\sigma}(\mathbb{E}; t)\right)$. Then there exists $g_1 \in \underline{\sigma}(\mathbb{E}; t)$ such that $\mathcal{F}(g_1) = h_1$, we have $OSC_\sigma(g_1; t) \subseteq \mathbb{E}$. We have to show that

$$OSC_\rho(h_1; t) \subseteq \mathcal{F}(\mathbb{E}).$$

Let $h_2 \in OSC_\rho(h_1; t)$. Then there exists $h_2 \in \mathbb{G}$ be such that $\mathcal{F}(g_2) = h_2$. $\Rightarrow \mathcal{F}(g_2) \in OSC_\rho(\mathcal{F}(g_1); t)$ By property (1) we obtain $g_2 \in OSC_\sigma(g_1; t)$ and $g_2 \in \mathbb{E}$. Hence, we have $h_2 = \mathcal{F}(g_2) \in \mathcal{F}(\mathbb{E})$. $\Rightarrow OSC_\rho(h_1; t) \subseteq \mathcal{F}(\mathbb{E})$, $\Rightarrow h_1 \in \underline{\rho}(\mathcal{F}(\mathbb{E}); t)$. Hence, $\mathcal{F}\left(\underline{\sigma}(\mathbb{E}; t)\right) \subseteq \underline{\rho}(\mathcal{F}(\mathbb{E}); t)$.

4. Let $\mathbb{E} \neq \emptyset$ and $\mathbb{E} \subseteq \mathbb{G}$. We have to show that $\underline{\rho}(\mathcal{F}(\mathbb{E}); t) \subseteq \mathcal{F}\left(\underline{\sigma}(\mathbb{E}; t)\right)$.

Suppose that $h_1 \in \underline{\rho}(\mathcal{F}(\mathbb{E});t)$ then there exists $g_1 \in \mathbb{G}$ such that $h_1 = \mathcal{F}(g_1)$ and so we have $OSC_\rho(\mathcal{F}(g_1);t) \subseteq \mathcal{F}(\mathbb{E})$. We have to show that

$$OSC_\sigma(g_1;t) \subseteq \mathbb{E}.$$

Let $g_2 \in OSC_\sigma(g_1;t)$ then by property (1) we have $\mathcal{F}(g_2) \in OSC_\rho(\mathcal{F}(g_1);t)$. This implies that $\mathcal{F}(g_2) \in \mathcal{F}(\mathbb{E})$. Then there exists $g_3 \in \mathbb{E}$ such that $\mathcal{F}(g_3) = \mathcal{F}(g_2)$. By assumption we have $g_2 \in \mathbb{E}$ and so $OSC_\sigma(g_1;t) \subseteq \mathbb{E}$. $\Rightarrow g_1 \in \underline{\sigma}(\mathbb{E};t)$.

Hence, $h_1 = \mathcal{F}(g_1) \in \mathcal{F}\left(\underline{\sigma}(\mathbb{E};t)\right)$, this implies

$$\underline{\rho}(\mathcal{F}(\mathbb{E});t) \subseteq \mathcal{F}\left(\underline{\sigma}(\mathbb{E};t)\right).$$

From this and property (3) we have

$$\underline{\rho}(\mathcal{F}(\mathbb{E});t) = \mathcal{F}\left(\underline{\sigma}(\mathbb{E};t)\right).$$

5. Transitive: Let ρ be a transitive then

$$\rho(\mathcal{F}(g_1),\mathcal{F}(g_2)) \geq \bigvee_{\mathcal{F}(g_3)\in\mathbb{H}} (\rho(\mathcal{F}(g_1),\mathcal{F}(g_3)) \wedge \rho(\mathcal{F}(g_3),\mathcal{F}(g_2)))$$
$$\Rightarrow \sigma(g_1,g_2) \geq \bigvee_{g_3\in\mathbb{G}} (\sigma(g_1,g_3) \wedge \sigma(g_3,g_2))$$

$\forall g_1, g_2 \in \mathbb{G}$ by definition.

This shows that σ is a transitive.

Compatibility: Suppose that ρ is compatible then $\forall g_1, g_2, g_3, e_j, f_j \in \mathbb{G}$, we have

$$\rho(\mathcal{F}(g_1 \circledcirc g_3),\mathcal{F}(g_2 \circledcirc g_4)) \geq \rho(\mathcal{F}(g_1),\mathcal{F}(g_2)) \wedge \rho(\mathcal{F}(g_3),\mathcal{F}(g_4)) \; and$$
$$\rho\left(\mathcal{F}\left(\vee_{j\in\mathcal{J}} e_j\right),\mathcal{F}\left(\vee_{j\in\mathcal{J}} f_j\right)\right) \geq \wedge_{j\in\mathcal{J}} \rho\left(\mathcal{F}\left(e_j\right),\mathcal{F}\left(f_j\right)\right).$$

This implies that $\sigma(g_1 \circledcirc g_3, g_2 \circledcirc g_4) \geq \sigma(g_1,g_2) \wedge \sigma(g_3,g_4)$ and $\sigma\left(\vee_{j\in\mathcal{J}} e_j, \vee_{j\in\mathcal{J}} f_j\right) \geq \wedge_{j\in\mathcal{J}} \sigma(e_j,f_j)$. This shows that σ is compatible.

Proposition 6: Let $\sigma(g_1,g_2) = \rho(\mathcal{F}(g_1),\mathcal{F}(g_2)) \forall g_1, g_2 \in \mathbb{G}$ where $\mathcal{F}$ is a surjective homomorphism from $\mathbb{G}$ in $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G};t)$ to $\mathbb{H}$ in $(\mathbb{H}; \mathcal{OSC}_\rho(\mathbb{H};t))$ then following statements holds

1. $g \in \overline{\sigma}(\mathbb{E};t) \Leftrightarrow \phi(g) \in \overline{\rho}(\mathcal{F}(\mathbb{E});t)$
2. $g \in \underline{\sigma}(\mathbb{E};t) \Leftrightarrow \phi(g) \in \underline{\rho}(\mathcal{F}(\mathbb{E});t$.

Proof: **1.** Let $g \in \overline{\sigma}(\mathbb{E};t)$ then $OSC_\sigma(g;t) \cap \mathbb{E} \neq \emptyset$. Then there exists $g_1 \in OSC_\sigma(g;t) \cap \mathbb{E} \Rightarrow g_1 \in OSC_\sigma(g;t), g_1 \in \mathbb{E}$.

By Proposition 5 we have $\mathcal{F}(g_1) \in OSC_\rho(\mathcal{F}(g);t)$, $\mathcal{F}(g_1) \in \mathcal{F}(\mathbb{E})$

This implies $\mathcal{F}(g_1) \in OSC_\rho(\mathcal{F}(g);t) \cap \mathcal{F}(\mathbb{E})$.

$$\Rightarrow OSC_\rho(\mathcal{F}(g);t) \cap \mathcal{F}(\mathbb{E}) \neq \emptyset$$
$$\Rightarrow \mathcal{F}(g) \in \overline{\rho}(\mathcal{F}(\mathbb{E});t).$$

Conversely, let $\mathcal{F}(g) \in \overline{\rho}(\mathcal{F}(\mathbb{E});t)$ then $OSC_\rho(\mathcal{F}(g);t) \cap \mathcal{F}(\mathbb{E}) \neq \emptyset$. Then there exists $\mathcal{F}(g_1) \in OSC_\rho(\mathcal{F}(g);t) \cap \mathcal{F}(\mathbb{E})$. This implies $\mathcal{F}(g_1) \in OSC_\rho(\mathcal{F}(g);t)$, $\mathcal{F}(g_1) \in \mathcal{F}(\mathbb{E})$.

By Proposition 5, we have

$$g_1 \in OSC_\sigma(g;t), q_1 \in \mathbb{E} \Rightarrow g_1 \in OSC_\sigma(g;t) \cap \mathbb{E}$$

$\Rightarrow OSC_\sigma(g;t) \cap \mathbb{E} \neq \emptyset, \Rightarrow g \in \overline{\sigma}(\mathbb{E};t)$.

Hence, $g \in \overline{\sigma}(\mathbb{E};t) \Leftrightarrow \phi(g) \in \overline{\rho}(\mathcal{F}(\mathbb{E});t)$

2. Let $g \in \underline{\sigma}(\mathbb{E};t)$ then $OSC_\sigma(g;t) \subseteq \mathbb{E}$. Then there exists $g_1 \in OSC_\sigma(g;t) \subseteq \mathbb{E} \Rightarrow g_1 \in OSC_\sigma(g;t), g_1 \in \mathbb{E}$. By Proposition 5 we have $\mathcal{F}(g_1) \in OSC_\rho(\mathcal{F}(g);t)$, $\mathcal{F}(g_1) \in \mathcal{F}(\mathbb{E})$.

This implies $\mathcal{F}(g_1) \in OSC_\rho(\mathcal{F}(g);t) \subseteq \mathcal{F}(\mathbb{E})$.

$$\Rightarrow OSC_\rho(\mathcal{F}(g);t) \subseteq \mathcal{F}(\mathbb{E})$$
$$\Rightarrow \mathcal{F}(g) \in \underline{\rho}(\mathcal{F}(\mathbb{E});t).$$

Conversely, let $\mathcal{F}(g) \in \underline{\rho}(\mathcal{F}(\mathbb{E});t)$ then $OSC_\rho(\mathcal{F}(g);t) \subseteq \mathcal{F}(\mathbb{E})$. Then there exists $\mathcal{F}(g_1) \in OSC_\rho(\mathcal{F}(g);t) \subseteq \mathcal{F}(\mathbb{E})$. This implies $\mathcal{F}(g_1) \in OSC_\rho(\mathcal{F}(g);t)$, $\mathcal{F}(g_1) \in \mathcal{F}(\mathbb{E})$.

By Proposition 5, we have

$$g_1 \in OSC_\sigma(g;t), g_1 \in \mathbb{E} \Rightarrow g_1 \in OSC_\sigma(g;t) \subseteq \mathbb{E}$$

$\Rightarrow OSC_\sigma(g;t) \subseteq \mathbb{E}, \Rightarrow g \in \underline{\sigma}(\mathbb{E};t)$.

Hence, $g \in \underline{\sigma}(\mathbb{E};t) \Leftrightarrow \mathcal{F}(g) \in \underline{\rho}(\mathcal{F}(\mathbb{E});t)$

Proposition 7: Let $\sigma(g_1,g_2) = \rho(\mathcal{F}(g_1),\mathcal{F}(g_2)) \forall g_1, g_2 \in \mathbb{G}$ where $\mathcal{F}$ is a bijective homomorphism from $\mathbb{G}$ in $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G};t)$ to $\mathbb{H}$ in $(\mathbb{H}; \mathcal{OSC}_\rho(\mathbb{H};t))$. If ρ is complete then σ is complete.

Proof: **1.** Let $g_3 \in OSC_\sigma(g_1 \circledcirc g_2;t)$. Then by Proposition 5 we get

$$\mathcal{F}(g_3) \in OSC_\rho(\mathcal{F}(g_1 \circledcirc g_2);t)$$

As ρ is complete and $\mathcal{F}$ is a homomorphism so,

$$\mathcal{F}(g_3) \in OSC_\rho(\mathcal{F}(g_1 \circledcirc g_2);t) = OSC_\rho(\mathcal{F}(g_1) \circledcirc \mathcal{F}(g_2);t) = OSC_\rho(\mathcal{F}(g_1);t) \circledcirc OSC_\rho(\mathcal{F}(g_2);t)$$

Then there exists $h_1 \in OSC_\rho(\mathcal{F}(g_1);t)$ and $h_2 \in OSC_\rho(\mathcal{F}(g_2);t)$ be such that $\mathcal{F}(g_3) = h_1 \circledcirc h_2$. As $\mathcal{F}$ is surjective, there exists $g_4, g_5 \in \mathbb{G}$ such that $\mathcal{F}(g_4) = h_1$ and $\mathcal{F}(g_5) = h_2$.

This implies that $\mathcal{F}(g_4) \circledcirc \mathcal{F}(g_5) = \mathcal{F}(g_3) \in OSC_\rho(\mathcal{F}(g_1);t) \circledcirc OSC_\rho(\mathcal{F}(g_2);t)$. It follows that $\mathcal{F}(g_4) \in OSC_\rho(\mathcal{F}(g_1);t)$, $\mathcal{F}(g_5) \in OSC_\rho(\mathcal{F}(g_2);t)$. By proposition 5(1), we get $g_4 \in OSC_\sigma(g_1;t)$ and $g_5 \in OSC_\sigma(g_2;t)$,since $\mathcal{F}$ is a homomorphism, we get $\mathcal{F}(g_4) \circledcirc \mathcal{F}(g_5) = \mathcal{F}(g_3) = \mathcal{F}(g_4 \circledcirc g_5)$.

Since $\mathcal{F}$ is one-one, we get that $g_3 = g_4 \circledcirc g_5$.

Therefore, we obtain $g_3 = g_4 \circledcirc g_5 \in OSC_\sigma(g_1;t) \circledcirc OSC_\sigma(g_2;t), \Rightarrow q_3 \in OSC_\sigma(q_1;t) \circledcirc OSC_\sigma(q_2;t)$. Hence,

$$OSC_\sigma((g_1 \circledcirc g_2);t) \subseteq OSC_\sigma(g_1;t) \circledcirc OSC_\sigma(g_2;t)$$

Now, by Proposition 3 and Proposition 5, we have

$$OSC_\sigma(g_1;t) \circledcirc OSC_\sigma(g_2;t) \subseteq OSC_\sigma((g_1 \circledcirc g_2);t).$$

Hence,

$$OSC_\sigma(g_1;t) \circledcirc OSC_\sigma(g_2;t) = OSC_\sigma((g_1 \circledcirc g_2);t).$$

2. Let $g_3 \in OSC_\sigma(g_1 \vee g_2; t)$ then by Proposition 5, we get

$$\mathcal{F}(g_3) \in OSC_\rho(\mathcal{F}(g_1 \vee g_2); t)$$

As ρ is complete and $\mathcal{F}$ is a homomorphism so,

$$\begin{aligned}\mathcal{F}(g_3) &\in OSC_\rho(\mathcal{F}(g_1 \vee g_2); t)\\ &= OSC_\rho(\mathcal{F}(g_1) \vee \mathcal{F}(g_2); t)\\ &= OSC_\rho(\mathcal{F}(g_1); t) \vee OSC_\rho(\mathcal{F}(g_2); t)\end{aligned}$$

Then there exists $h_1 \in OSC_\rho(\mathcal{F}(g_1); t)$ and $h_2 \in OSC_\rho(\mathcal{F}(g_2); t)$ be such that $\mathcal{F}(g_3) = h_1 \circledcirc h_2$. As $\mathcal{F}$ is surjective, there exists $g_4, g_5 \in \mathbb{G}$ such that $\mathcal{F}(g_4) = h_1$ and $\mathcal{F}(g_5) = h_2$.

This implies that $\mathcal{F}(g_4) \vee \mathcal{F}(g_5) = \mathcal{F}(g_3) \in OSC_\rho(\mathcal{F}(g_1); t) \vee OSC_\rho(\mathcal{F}(g_2); t)$. It follows that $\mathcal{F}(g_4) \in OSC_\rho(\mathcal{F}(g_1); t)$ and $\mathcal{F}(g_5) \in OSC_\rho(\mathcal{F}(g_2); t)$.

By Proposition 5 we get $g_4 \in OSC_\sigma(g_1; t)$ and $g_5 \in OSC_\sigma(g_2; t)$, since $\mathcal{F}$ is a homomorphism, we get

$$\mathcal{F}(g_4) \vee \mathcal{F}(g_5) = \mathcal{F}(g_3) = \mathcal{F}(g_4 \vee g_5)$$

Since $\mathcal{F}$ is one-one, we get that $g_3 = g_4 \vee g_5$.

Therefore, we obtain $g_3 = g_4 \vee g_5 \in OSC_\sigma(g_1; t) \vee OSC_\sigma(g_2; t), \Rightarrow q_3 \in OSC_\sigma(q_1; t) \vee OSC_\sigma(q_2; t)$. Hence,

$$OSC_\sigma((g_1 \vee g_2); t) \subseteq OSC_\sigma(g_1; t) \vee OSC_\sigma(g_2; t)$$

Now, by Proposition 4 and Proposition 5 we have

$$\begin{aligned}&OSC_\sigma(g_1; t) \vee OSC_\sigma(g_2; t)\\ &\quad \subseteq OSC_\sigma((g_1 \vee g_2); t).\end{aligned}$$

Hence,

$$\begin{aligned}&OSC_\sigma(g_1; t) \vee OSC_\sigma(g_2; t)\\ &\quad = OSC_\sigma((g_1 \vee g_2); t).\end{aligned}$$

Theorem 19: Let $\sigma(g_1, g_2) = \rho(\mathcal{F}(g_1), \mathcal{F}(g_2)) \forall g_1, g_2 \in \mathbb{G}$. where $\mathcal{F}$ is a surjective homomorphism from $\mathbb{G}$ in $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t)$ to $\mathbb{H}$ in $(\mathbb{H}; \mathcal{OSC}_\rho(\mathbb{H}; t))$ of TCFR and $\mathbb{E}$ be a nonempty subset of $\mathbb{G}$. Then $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an $\mathcal{OSC}_\rho(\mathbb{H}; t)$-UP.appr. quantale iff$\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. quantale.

Proof: Let $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ be an $\mathcal{OSC}_\rho(\mathbb{G}; t)$-UP.appr. quantale. We have to show that $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. quantale. For this we have to show that $\overline{\sigma}(\mathbb{E}; t)$ is a $Sub_{\mathbb{G}}$ of $\mathbb{G}$.

(1). Let $g_1 \in \overline{\sigma}(\mathbb{E}; t) \circledcirc \overline{\sigma}(\mathbb{E}; t)$, From Proposition 5 we have

$$\begin{aligned}\mathcal{F}(g_1) &\in \mathcal{F}(\overline{\sigma}(\mathbb{E}; t) \circledcirc \overline{\sigma}(\mathbb{E}; t))\\ &= \mathcal{F}(\overline{\sigma}(\mathbb{E}; t)) \circledcirc \phi(\overline{\sigma}(\mathbb{E}; t))\\ &= \overline{\rho}(\mathcal{F}(\mathbb{E}); t) \circledcirc \overline{\rho}(\mathcal{F}(\mathbb{E}); t) \subseteq \overline{\rho}(\mathcal{F}(\mathbb{E}); t)\\ &= \mathcal{F}(\overline{\sigma}(\mathbb{E}; t))\end{aligned}$$

Therefore, there exists $g_2 \in \overline{\sigma}(\mathbb{E}; t)$ such that $\mathcal{F}(g_1) = \mathcal{F}(g_2)$. We have $OSC_\sigma(g_2; t) \cap \mathbb{E} \neq \emptyset$.

From Proposition 1 we have $\mathcal{F}(g_1) \in OSC_\rho(\mathcal{F}(g_2); t)$. By Proposition 5, we obtain $g_1 \in OSC_\sigma(g_1; t)$ from Proposition 1 $OSC_\sigma(g_1; t) = OSC_\sigma(g_2; t)$ this implies that

$$\begin{aligned}&OSC_\sigma(g_1; t) \cap \mathbb{E} \neq \emptyset\\ &\Rightarrow g_1 \in \overline{\sigma}(\mathbb{E}; t)\end{aligned}$$

Hence, $\overline{\sigma}(\mathbb{E}; t) \circledcirc \overline{\sigma}(\mathbb{E}; t) \subseteq \overline{\sigma}(\mathbb{E}; t)$.

(2). Let $g_1 \in \overline{\sigma}(\mathbb{E}; t) \vee \overline{\sigma}(\mathbb{E}; t)$, From 2 Proposition 5 we have

$$\begin{aligned}\mathcal{F}(g_1) &\in \mathcal{F}(\overline{\sigma}(\mathbb{E}; t) \vee \overline{\sigma}(\mathbb{E}; t))\\ &= \mathcal{F}(\overline{\sigma}(\mathbb{E}; t)) \vee \phi(\overline{\sigma}(\mathbb{E}; t))\\ &= \overline{\rho}(\mathcal{F}(\mathbb{E}); t) \vee \overline{\rho}(\mathcal{F}(\mathbb{E}); t) \subseteq \overline{\rho}(\mathcal{F}(\mathbb{E}); t)\\ &= \mathcal{F}(\overline{\sigma}(\mathbb{E}; t))\end{aligned}$$

Therefore, there exists $g_2 \in \overline{\sigma}(\mathbb{E}; t)$ such that $\mathcal{F}(g_1) = \mathcal{F}(g_2)$, we have $OSC_\sigma(g_2; t) \cap \mathbb{E} \neq \emptyset$.

From Proposition 1 we have

$$\mathcal{F}(g_1) \in OSC_\rho(\mathcal{F}(g_2); t)$$

By Proposition 5, we obtain $g_1 \in OSC_\sigma(g_1; t)$ so from Proposition 1, we have $OSC_\sigma(g_1; t) = OSC_\sigma(g_2; t)$. This implies that

$$\begin{aligned}&OSC_\sigma(g_1; t) \cap \mathbb{E} \neq \emptyset\\ &\Rightarrow g_1 \in \overline{\sigma}(\mathbb{E}; t)\end{aligned}$$

Hence, $\overline{\sigma}(\mathbb{E}; t) \vee \overline{\sigma}(\mathbb{E}; t) \subseteq \overline{\sigma}(\mathbb{E}; t)$.

Thus, $\overline{\sigma}(\mathbb{E}; t)$ is a $Sub_{\mathbb{G}}$ of $\mathbb{G}$. Therefore, $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. quantale.

Conversely, let $\overline{\sigma}(\mathbb{E}; t)$ be an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. quantale. We have to show that $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an $\mathcal{OSC}_\rho(\mathbb{H}; t)$-UP.appr. quantale.

For this we have to show that $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is a $Sub_{\mathbb{H}}$ of $\mathbb{H}$.

(3). By Proposition 5 we have $\overline{\rho}(\mathcal{F}(\mathbb{E}); t) \circledcirc \overline{\rho}(\mathcal{F}(\mathbb{E}); t) = \mathcal{F}(\overline{\sigma}(\mathbb{E}; t)) \circledcirc \mathcal{F}(\overline{\sigma}(\mathbb{E}; t)) = \mathcal{F}(\overline{\sigma}(\mathbb{E}; t) \circledcirc \overline{\sigma}(\mathbb{E}; t))$, since $\mathcal{F}$ is homo. $\subseteq \mathcal{F}(\overline{\sigma}(\mathbb{E}; t))$ since $\overline{\sigma}(\mathbb{E}; t)$ is $Sub_{\mathbb{Q}} = \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ By Proposition 5

$$\overline{\rho}(\mathcal{F}(\mathbb{E}); t) \circledcirc \overline{\rho}(\mathcal{F}(\mathbb{E}); t) \subseteq \overline{\rho}(\mathcal{F}(\mathbb{E}); t).$$

(4). By Proposition 5 we have $\overline{\rho}(\mathcal{F}(\mathbb{E}); t) \vee \overline{\rho}(\mathcal{F}(\mathbb{E}); t) = \mathcal{F}(\overline{\sigma}(\mathbb{E}; t)) \vee \mathcal{F}(\overline{\sigma}(\mathbb{E}; t)) = \mathcal{F}(\overline{\sigma}(\mathbb{E}; t) \vee \overline{\sigma}(\mathbb{E}; t))$ since $\mathcal{F}$ is homo. $\subseteq \mathcal{F}(\overline{\sigma}(\mathbb{E}; t))$ since $\overline{\sigma}(\mathbb{E}; t)$ is $Sub_{\mathbb{Q}} = \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ by Proposition 5

$$\overline{\rho}(\mathcal{F}(\mathbb{E}); t) \vee \overline{\rho}(\mathcal{F}(\mathbb{E}); t) \subseteq \overline{\rho}(\mathcal{F}(\mathbb{E}); t).$$

This shows that $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is a $Sub_{\mathbb{H}}$ of $\mathbb{H}$. This shows that $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an $\mathcal{OSC}_\rho(\mathbb{H}; t)$-UP.appr. quantale.Hence proved.

Theorem 20: Let $\sigma(g_1, g_2) = \rho(\mathcal{F}(g_1), \mathcal{F}(g_2)) \forall g_1, g_2 \in \mathbb{G}$ where $\mathcal{F}$ is a bijective homomorphism from $\mathbb{G}$ in $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t)$ to $\mathbb{H}$ in $(\mathbb{H}; \mathcal{OSC}_\rho(\mathbb{H}; t))$ of TCFR and $\mathbb{E}$ be a nonempty subset of $\mathbb{G}$. Then $\underline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an $\mathcal{OSC}_\rho(\mathbb{H}; t)$-LW.appr. quantale iff $\underline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-LW.appr. quantale.

Proof: Proof of this theorem is similar to above Theorem 19.

Theorem 21: Let $\sigma(g_1, g_2) = \rho(\mathcal{F}(g_1), \mathcal{F}(g_2)) \forall g_1, g_2 \in \mathbb{G}$ where $\mathcal{F}$ is a surjective homomorphism from $\mathbb{G}$ in $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t)$ to $\mathbb{H}$ in $(\mathbb{H}; \mathcal{OSC}_\rho(\mathbb{H}; t))$ of TCFR and $\mathbb{E}$ be a nonempty subset of $\mathbb{G}$. Then $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an $\mathcal{OSC}_\rho(\mathbb{H}; t)$-UP.appr. ideal iff $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. ideal.

Proof: Let $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. ideal. We have to show that $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an $\mathcal{OSC}_\rho(\mathbb{H}; t)$-UP.appr. ideal.

(1). Let $h_1, h_2 \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$. Then there exists $g_1, g_2 \in \mathbb{G}$ such that $\mathcal{F}(g_1) = h_1, \mathcal{F}(g_2) = h_2$.

Since $\overline{\sigma}(\mathbb{E}; t)$ is an ideal. Therefore, $g_1 \vee g_2 \in \overline{\sigma}(\mathbb{E}; t)$. This implies that $\mathcal{F}(g_1 \vee g_2) \in \mathcal{F}(\overline{\sigma}(\mathbb{E}; t))$.

By Proposition 5 we have $\mathcal{F}(g_1 \vee g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$. This implies $\mathcal{F}(g_1) \vee \mathcal{F}(g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ since $\mathcal{F}$ is homo. $\Rightarrow h_1 \vee h_2 \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$.

(2). Assume that $h_1 \leq h_2 \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$. Then there exists $g_1 \in \mathbb{G}$ and $g_2 \in \overline{\sigma}(\mathbb{E}; t)$ such that $h_1 = \mathcal{F}(g_1)$ and $h_2 = \mathcal{F}(g_2)$. Since $\mathcal{F}(g_1) \leq \mathcal{F}(g_2)$ we have $\mathcal{F}(g_1) \vee \mathcal{F}(g_2) = \mathcal{F}(g_1 \vee g_2) = \mathcal{F}(g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$. This implies

$$\mathcal{F}(g_1 \vee g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$$

$\Rightarrow g_1 \vee g_2 \in \overline{\sigma}(\mathbb{E}; t)$ by Proposition 5.

Since $\overline{\sigma}(\mathbb{E}; t)$ is an ideal and

$$\begin{aligned} g_1 &\leq g_1 \vee g_2 \in \overline{\sigma}(\mathbb{E}; t) \\ &\Rightarrow g_1 \in \overline{\sigma}(\mathbb{E}; t), \quad \Rightarrow \mathcal{F}(g_1) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t). \\ &\Rightarrow h_1 \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t). \end{aligned}$$

(3). Assume that $h_1 \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$, $h_2 \in \mathbb{G}$ then there exists $g_1 \in \overline{\sigma}(\mathbb{E}; t)$, $g_2 \in \mathbb{G}$ such that $h_1 = \mathcal{F}(g_1)$ and $h_2 = \mathcal{F}(g_2)$. Since $\overline{\sigma}(\mathbb{E}; t)$ is an ideal, so we have $g_1 \circledcirc g_2 \in \overline{\sigma}(\mathbb{E}; t)$,

$$\Rightarrow \mathcal{F}(g_1 \circledcirc g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$$

$\Rightarrow \mathcal{F}(g_1) \circledcirc \mathcal{F}(g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$. Since $\mathcal{F}$ is homo. $\Rightarrow h_1 \circledcirc h_2 \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$. Similarly, $h_2 \circledcirc h_1 \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$. Hence, $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an $\mathcal{OSC}_\rho(\mathbb{H}; t)$-UP.appr. ideal.

Conversely, Assume that $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an $\mathcal{OSC}_\rho(\mathbb{H}; t)$-UP.appr. ideal. We have to show that $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. ideal.

(4). Let $g_1, g_2 \in \overline{\sigma}(\mathbb{E}; t)$ then $\mathcal{F}(g_1), \mathcal{F}(g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$. Since $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an ideal so we have $\mathcal{F}(g_1) \vee \mathcal{F}(g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$.

$\Rightarrow \mathcal{F}(g_1 \vee g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ since $\mathcal{F}$ is homo. $\Rightarrow g_1 \circledcirc g_2 \in \overline{\sigma}(\mathbb{E}; t)$.

(5). Let $g_1 \leq g_2 \in \overline{\sigma}(\mathbb{E}; t)$ then $\mathcal{F}(g_1) \leq \mathcal{F}(g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$. By Proposition 5.

Since $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an ideal so we have $\mathcal{F}(g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ this implies that $g_2 \in \overline{\sigma}(\mathbb{E}; t)$ by Proposition 5.

(6). Assume that $g_1 \in \overline{\sigma}(\mathbb{E}; t)$ and $g_2 \in \mathbb{G}$ then by Proposition 5 we have $\mathcal{F}(g_1) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$, $\phi(g_2) \in \mathbb{H}$. Since $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an ideal so we have

$$\mathcal{F}(g_1) \circledcirc \mathcal{F}(g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t),$$

$\Rightarrow \mathcal{F}(g_1 \circledcirc g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$, since $\mathcal{F}$ is homo.

$\Rightarrow g_1 \circledcirc g_2 \in \overline{\sigma}(\mathbb{E}; t)$ by Proposition 5 similarly, $g_2 \circledcirc g_1 \in \overline{\sigma}(\mathbb{E}; t)$. Hence, $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. ideal.

Theorem 22: Let $\sigma(g_1, g_2) = \rho(\mathcal{F}(g_1), \mathcal{F}(g_2)) \forall g_1, g_2 \in \mathbb{G}$ where $\mathcal{F}$ is a bijective homomorphism from $\mathbb{G}$ in $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t)$ to $\mathbb{H}$ in $(\mathbb{H}; \mathcal{OSC}_\rho(\mathbb{H}; t))$ of TCFR and $\mathbb{E}$ be a nonempty subset of $\mathbb{G}$. Then $\underline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an $\mathcal{OSC}_\rho(\mathbb{H}; t)$-LW.appr. ideal iff $\underline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-LW.appr. ideal.

Proof: The proof of this is similar to above theorem.

Theorem 23: Let $\sigma(g_1, g_2) = \rho(\mathcal{F}(g_1), \mathcal{F}(g_2)) \forall g_1, g_2 \in \mathbb{G}$ where $\mathcal{F}$ is a bijective homomorphism from $\mathbb{G}$ in $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t)$ to $\mathbb{H}$ in $(\mathbb{H}; \mathcal{OSC}_\rho(\mathbb{H}; t))$ of CFZR and $\mathbb{E}$ be a nonempty subset of $\mathbb{G}$. Then $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an $\mathcal{OSC}_\rho(\mathbb{H}; t)$-UP.appr. prime ideal iff $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. prime ideal.

Proof: Assume that $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{E}; t)$-UP.appr. prime ideal. Let $h_1, h_2 \in \mathbb{G}$ such that $h_1 \circledcirc h_2 \in \overline{\rho}(\mathcal{F}(\mathbb{E}), t)$. Then there exists $h_1, h_2 \in \mathbb{E}$ such that $\mathcal{F}(g_1) = h_1$ and $\mathcal{F}(g_2) = h_2$. Then

$$OSC_\rho(\mathcal{F}(g_1) \circledcirc \mathcal{F}(g_2); t) \cap \mathcal{F}(\mathbb{E}) \neq \emptyset.$$

Since ρ is complete, we have

$$\begin{aligned} (OSC_\rho(\mathcal{F}(g_1); t) \circledcirc OSC_\rho(\mathcal{F}(g_2); t)) \cap \mathcal{F}(\mathbb{E}) \\ = OSC_\rho(\mathcal{F}(g_1) \circledcirc \mathcal{F}(g_2); t) \cap \mathcal{F}(\mathbb{E}) \neq \emptyset. \end{aligned}$$

Then there exists $\mathcal{F}(q_3) \in OSC_\rho(\mathcal{F}(g_1); t)$ and $\mathcal{F}(g_4) \in OSC_\rho(\mathcal{F}(g_2); t)$ be such that $\mathcal{F}(g_3) \circledcirc \mathcal{F}(g_4) \in \mathcal{F}(\mathbb{E})$ and we have $\mathcal{F}(g_3 \circledcirc g_4) \in \mathcal{F}(\mathbb{E})$. Then there exists $g_5 \in \mathbb{E}$ such that $\mathcal{F}(g_3 \circledcirc g_4) = \mathcal{F}(g_5)$. By Proposition 5 we get $g_3 \in OSC_\sigma(g_1; t)$ and $g_4 \in OSC_\sigma(g_2; t)$. From Proposition 5 and Proposition 3, we obtain that $g_3 \circledcirc g_4 \in OSC_\sigma(g_1 \circledcirc g_2; t)$.

By Proposition 1 we have $OSC_\sigma(g_1 \circledcirc g_2; t) = OSC_\sigma(g_3 \circledcirc g_4; t)$. Note that $\mathcal{F}(g_3 \circledcirc g_4) \in OSC_\rho(\mathcal{F}(g_3 \circledcirc g_4); t)$.

Then $\mathcal{F}(g_5) \in OSC_\rho(\mathcal{F}(g_3 \circledcirc g_4); t)$. By Proposition 5, we have $g_5 \in OSC_\sigma(g_3 \circledcirc g_4; t) = OSC_\sigma(g_1 \circledcirc g_2; t)$. Thus, $OSC_\sigma(g_1 \circledcirc g_2; t) \cap \mathbb{E} \neq \emptyset$ and therefore we get, $g_1 \circledcirc g_2 \in \overline{\sigma}(\mathbb{E}; t)$. Since, $\overline{\sigma}(\mathbb{E}; t)$ is a prime ideal of $\mathbb{G}$, therefore, we have $g_1 \in \overline{\sigma}(\mathbb{E}; t)$ or $g_2 \in \overline{\sigma}(\mathbb{E}; t)$. We get that $\mathcal{F}(g_1) \in \mathcal{F}(\overline{\sigma}(\mathbb{E}; t))$ or $\mathcal{F}(g_2) \in \mathcal{F}(\overline{\sigma}(\mathbb{E}; t))$. From Proposition 5, we have $\mathcal{F}(g_1) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ or $\mathcal{F}(g_2) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$, this implies that $h_1 \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ or $h_2 \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$. This shows that $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is a prime ideal of $\mathbb{H}$. Hence, $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an $\mathcal{OSC}_\rho(\mathbb{H}; t)$-UP.appr. prime ideal.

Conversely, assume that $\overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an $\mathcal{OSC}_\rho(\mathbb{H}; t)$-UP.appr. prime ideal. Now, let $g_6, g_7 \in \mathbb{G}$ such that $g_6 \circledcirc g_7 \in \overline{\sigma}(\mathbb{E}; t)$. Then $\mathcal{F}(g_6 \circledcirc g_7) \in \mathcal{F}(\overline{\sigma}(\mathbb{E}; t))$. By Proposition 5 we get

$$\begin{aligned} \mathcal{F}(g_6) \circledcirc \mathcal{F}(g_7) = \mathcal{F}(g_6 \circledcirc g_7) \in \mathcal{F}(\overline{\sigma}(\mathbb{E}; t)) \\ = \overline{\rho}(\mathcal{F}(\mathbb{E}); t) \end{aligned}$$

Thus, $\mathcal{F}(g_6) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$ or $\mathcal{F}(g_7) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$. Now, we consider the following two cases.

Case 1. If $\mathcal{F}(g_6) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$, By Proposition 5 we have $\mathcal{F}(g_6) \in \mathcal{F}(\overline{\sigma}(\mathbb{E}; t))$. There exists $g_8 \in \overline{\sigma}(\mathbb{E}; t)$ such that $\mathcal{F}(g_6) = \mathcal{F}(g_8)$ then $OSC_\sigma(g_8; t) \cap \mathbb{E} \neq \emptyset$. By proposition 1, we get $\mathcal{F}(g_8) \in OSC_\rho(\mathcal{F}(g_8); t)$. Thus,

$\mathcal{F}(g_6) \in OSC_\rho(\mathcal{F}(g_8); t)$. By Proposition 5, we have $g_6 \in OSC_\sigma(g_8; t)$. From Proposition 1 we have $OSC_\sigma(g_6; t) = OSC_\sigma(g_8; t)$. Thus, we have $OSC_\sigma(g_6; t) \cap \mathbb{E} \neq \emptyset$ and so,

$$g_6 \in \overline{\sigma}(\mathbb{E}; t).$$

Case 2. If $\mathcal{F}(g_7) \in \overline{\rho}(\mathcal{F}(\mathbb{E}); t)$, By Proposition 5 we have $\mathcal{F}(g_7) \in \mathcal{F}(\overline{\sigma}(\mathbb{E}; t))$. There exists $g_9 \in \overline{\sigma}(\mathbb{E}; t)$ such that $\mathcal{F}(g_7) = \mathcal{F}(g_9)$ then $OSC_\sigma(g_9; t) \cap \mathbb{E} \neq \emptyset$. By Proposition 1 we get $\mathcal{F}(g_9) \in OSC_\rho(\mathcal{F}(g_9); t)$. Thus, $\mathcal{F}(g_7) \in OSC_\rho(\mathcal{F}(g_9); t)$. By Proposition 5 we have $g_7 \in OSC_\sigma(g_9; t)$. From Proposition 1 we have $OSC_\sigma(g_7; t) = OSC_\sigma(g_9; t)$. Thus, we have $OSC_\sigma(g_7; t) \cap \mathbb{E} \neq \emptyset$ and so, $g_7 \in \overline{\sigma}(\mathbb{E}; t)$. Hence, $\overline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-UP.appr. prime ideal.

Theorem 24: Let $\sigma(g_1, g_2) = \rho(\mathcal{F}(g_1), \mathcal{F}(g_2)) \forall g_1, g_2 \in \mathbb{G}$ where $\mathcal{F}$ is a bijective homomorphism from $\mathbb{G}$ in $(\mathbb{G}, \mathcal{OSC}_\sigma(\mathbb{G}; t)$ to $\mathbb{H}$ in $(\mathbb{H}; \mathcal{OSC}_\rho(\mathbb{H}; t))$ of CFZR and $\mathbb{E}$ be a nonempty subset of $\mathbb{G}$, then $\underline{\rho}(\mathcal{F}(\mathbb{E}); t)$ is an $\mathcal{OSC}_\rho(\mathbb{H}; t)$-LW.appr. prime ideal iff $\underline{\sigma}(\mathbb{E}; t)$ is an $\mathcal{OSC}_\sigma(\mathbb{G}; t)$-LW.appr. prime ideal.

Proof: The proof of this is similar to above Theorem 23.

In the following comparison Table 3, we are interested to express our approach how the proposed work is easy to previous work. Further we will show what the difficulties in the previous studies are and how the proposed work is free from all these difficulties.

V. CONCLUSION AND FUTURE WORK

This article identifies certain restrictions on the roughness specified by congruence and set-valued mappings and defines some benefits for rough structures built on serial fuzzy relations via successor overlaps. Then on the newly developed rough set model based on serial fuzzy relations, some new rough substructures are defined such as rough multiplicative set, rough m-system and further rough substructures of quantales.

The approaches used in the methods developed by Davvaz [10], Yang and Xu [15], Luo and Wang [38], Qurashi et al., [40], and Kanwal and Shabir [42], [43] are based on fundamental techniques such as roughness through set-valued mappings, with the aid of congruence relations, and roughness based on aftersets and foresets by soft relations, respectively. Although the aforementioned techniques are all well-developed and effective, they do have some limitations. We require numerous equivalence relations, for instance, in order to validate our results and examples while examining roughness through congruence relations. Finding equivalence relations and then congruence is never easy. In case of roughness through set-valued mappings, we need set-valued homomorphism to proceed our works. Sometimes it becomes difficult to find out set-valued homomorphism. Moreover, roughness through soft relations is yet more tedious due to the difficulty in determining compatible and complete relations with respect to aftersets and forests. Thus, we are not required such type of limitations in our paper.

In further work, we will broaden the applicability of the suggested approach to a variety of algebraic structures, including as quantale modules, ordered semigroups, rings, and near-rings. We will also focus on how the suggested approach may be applied to various real-life problems employing intuitionistic and Pythagorean fuzzy sets. Moreover, we will extend the developed method toothers

TABLE 3. Comparison table.

Sr. No	Previous Studies	Proposed Studies
1.	Roughness of ideals and roughness of fuzzy ideals in quantale by congruence relations were studied in[15], [38], respectively. Moreover, roughness of sub-module and submodule ideals in quantale module by congruence were studied by Qurashi and Shabir [16].	It is observed that equivalence relation and then congruence relations are not easy to find out while studying roughness in different algebraic structures. In this proposed study such type of equivalence relations and then congruence relations are not required.
2.	Moreover, rough substructures in quantales and quantale modules dependent on set valued homomorphism were presented in papers[16], [39], respectively. In these papers, set valued homomorphism are required.	It is difficult to find out set valued homomorphism and strong set valued homomorphism in different quantales. In our proposed study such types of mappings are not required.
3.	Rough approximations based on soft relations with respect to aftersets and foresets in substructures and fuzzy substructures in quantale and semigroups were studied by Qurashi *et al.*, [40], Kanwal and Shabir[42], [43], [44], respectively. Such type of approximations require compatible and complete relations respecting to aftersets and foresets.	In our proposed study, aftersets and foresets are not required. Soft relations are not needed. Since compatible and complete relations by using aftersets and foresets, based on soft relations are difficult to find out, so we have avoided them in our proposed research. All the important results discussed in [15], [39], [40], have obtained easily in the proposed method.

generalization of fuzzy sets as well and will be used to decision making techniques.

AUTHOR CONTRIBUTIONS

Saqib Mazher Qurashi: methodology, writing—original draft, writing—review and editing; Bander Almutairi: writing—review and editing; Rani Sumaira Kanwal: methodology, funding acquisition, writing—original draft, writing—review and editing; Mladen Krstić: writing—review and editing; and Muhammad Yousaf: conceptualization, methodology, writing—original draft, writing—review and editing.

DECLARATIONS

CONFLICT OF INTEREST

There is no conflict of interest.

Ethical approval and consent to participate: Not applicable.

Consent for publication: Not applicable.

DATA AVAILABILITY

The paper includes the information used to verify the study's findings.

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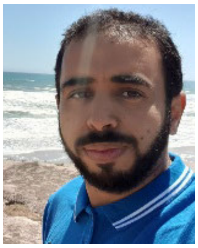

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