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RESEARCH ARTICLE

Properties and Applications of a Symmetric Toeplitz Matrix Generated by $C + 1/C$ Elements

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ABSTRACT Utilizing derivations for the properties of a symmetric Toeplitz matrix, we obtain analytical expressions for the performance evaluation of wireless communication systems using multiple antennas at the transmitter and/or the receiver, including those for keyhole channels, beamforming, and noncoherent detection. Our derivations of the analytical expressions are based upon closed form expressions we have obtained for the eigenvalues and eigenvectors of the $L \times L$ symmetric Toeplitz matrix whose element in the ith row and the jth column is given by $C^{i-j} + C^{j-i}$, where $C \in \mathbb{C} \setminus \{-1, 0, 1\}$, with $\mathbb{C}$ denoting the set of complex numbers. Each element of this matrix can be expressed as a polynomial in $C+1/C$. Furthermore, the special cases of real nonzero C and of complex C with magnitude one are discussed. Using these new results, analytical expressions for the performance of wireless communication systems using multiple antennas at the transmitter and/or the receiver can be obtained.

INDEX TERMS Applications in wireless communications, eigenvalues, eigenvectors, symmetric Toeplitz matrix.

I. INTRODUCTION

Toeplitz matrices play an important role in engineering applications, such as in the correlation structure of wide sense stationary colored noise, in the discrete-time modeling of linear time-invariant systems, and in the modeling and analysis of shift-invariant imaging systems. There are only a few analytical results available for Toeplitz matrices and therefore the study of these systems inevitably requires further investigations into the analysis of such matrices. An algorithm for inversion Toeplitz matrices has been presented in [1]. Properties of special types of Toeplitz matrices, such as matrices generated by rational functions [2], symmetric matrices [3], [4], tridiagonal and other banded matrices [5], [6], and real symmetric matrices with linearly increasing entries [7], have also been studied.

In this paper, we present analytical expressions in closed form for the eigenvalues and eigenvectors of the $L \times L$ complex symmetric Toeplitz matrix whose element in the ith row and the jth column is given by $C^{i-j} + C^{j-i}$, where $C \in \mathbb{C} \setminus \{-1, 0, 1\}$, with $\mathbb{C}$ denoting the set of complex numbers. Each element of this matrix can be expressed as a polynomial in $C+1/C$. For $L = 2$, the rank, eigenvalues, and eigenvectors, obtained in a simple manner, are presented. For $L \geq 3$, using a second order linear homogeneous recurrence followed by the matrix elements, the rank is found, and, from the structure of the characteristic polynomial, closed form expressions for the eigenvalues are obtained; furthermore, closed form expressions for the eigenvectors for the cases of even L and odd L are given by Propositions 1 and 2, respectively. Complex orthogonal symmetric and skew-symmetric eigenvectors corresponding to the eigenvalue zero for $L = 4$ and $L = 5$ are presented; and, by using the symmetric and skew-symmetric properties of the eigenvectors, a method of obtaining complex orthogonal eigenvectors corresponding to the eigenvalue zero for $L \geq 4$ is described. Furthermore, the special cases of real nonzero C and of complex C with magnitude one are discussed. The results are applied to the performance evaluation of wireless communication systems using multiple antennas at the transmitter and/or the receiver; this is an important application in the area of electrical communication engineering. The results may also be able to assist

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in the analysis of other important engineering applications and provide expressions for analytical bounds and tradeoffs on performance.

The organization of the paper is as follows. Some properties of the matrix elements and the rank are presented in Section II. The eigenvalues are obtained in Section III and the eigenvectors are presented in Section IV. In Section V, the special cases of real nonzero C and of complex C with magnitude one are discussed. Section VI shows applications of the results in the area of wireless communications. Some concluding remarks are given in Section VII.

II. THE MATRIX ELEMENTS AND RANK

Consider the $L \times L$ symmetric Toeplitz matrix $\mathbf{T}_L(C)$ whose element in the ith row and the jth column is given by

$$\begin{aligned}\left[\mathbf{T}_L(C)\right]_{i,j} &= C^{i-j} + C^{j-i} \\ &= C^{|i-j|} + C^{-|i-j|}, \\ & i = 1, \ldots, L, \ \ j = 1, \ldots, L, \\ & C \in \mathbb{C} \setminus \{-1, 0, 1\}. \end{aligned} \tag{1}$$

It is clear from (1) the matrix $\mathbf{T}_L(C)$ has twos as its diagonal elements. Furthermore, note that $\mathbf{T}_L(C)$ is a persymmetric (a Toeplitz matrix is persymmetric, that is, symmetric across its lower-left to upper-right diagonal) matrix, apart from being symmetric.

Define the function $g(C, \nu)$ as

$$g(C, \nu) \triangleq C^{\nu} + C^{-\nu}, \quad C \in \mathbb{C} \setminus \{-1, 0, 1\}, \quad \nu \in \mathbb{R}, \tag{2}$$

with $\mathbb{R}$ denoting the set of real numbers, which implies that (1) can be expressed as

$$\begin{aligned}\left[\mathbf{T}_L(C)\right]_{i,j} &= g(C, i-j) = g(C, j-i), \\ & i = 1, \ldots, L, \ \ j = 1, \ldots, L. \end{aligned} \tag{3}$$

We find from (2) that $g(C, \nu)$ has the properties

$$g(C, -\nu) = g(C, \nu), \tag{4a}$$
$$g(-C, \nu) = (-1)^{\nu} g(C, \nu), \tag{4b}$$
$$g\left(C^{-1}, \nu\right) = g(C, \nu), \tag{4c}$$
$$g(C, \nu_1) g(C, \nu_2) = g(C, \nu_1 - \nu_2) + g(C, \nu_1 + \nu_2), \tag{4d}$$
$$|g(C, \nu)| \geq 2, \tag{4e}$$

and that $g(C, \nu)$ follows the second order linear homogeneous recurrence (in terms of ν) [8]

$$g(C, \nu) = \left(C + C^{-1}\right) g(C, \nu - 1) - g(C, \nu - 2). \tag{5}$$

We find from (5) that *each element of $\mathbf{T}_L(C)$ can be expressed as a polynomial in $C + C^{-1}$*, that is, $C + 1/C$; for example,

$$\left[\mathbf{T}_L(C)\right]_{j,j} = g(C, 0) = 2,$$

$$\left[\mathbf{T}_L(C)\right]_{j+1,j} = g(C, 1) = C + \frac{1}{C},$$

$$\left[\mathbf{T}_L(C)\right]_{j+2,j} = g(C, 2) = \left(C + \frac{1}{C}\right)^2 - 2,$$

$$\left[\mathbf{T}_L(C)\right]_{j+3,j} = g(C, 3) = \left(C + \frac{1}{C}\right)^3 - 3\left(C + \frac{1}{C}\right),$$

and so on. Using [8, Proposition 9], we can express the element in the $(j+k)$th row and the jth column of the $L \times L$ symmetric Toeplitz matrix $\mathbf{T}_L(C)$ as

$$\begin{aligned}\left[\mathbf{T}_L(C)\right]_{j+k,j} &= g(C, k) \\ &= \left(C + \frac{1}{C}\right)^k \\ &\quad + \sum_{m=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^m \left[\binom{k-m}{m} + \binom{k-m-1}{m-1}\right] \\ &\quad \times \left(C + \frac{1}{C}\right)^{k-2m}, \\ & j = 1, \ldots, L-k, \quad k = 0, \ldots, L-1. \end{aligned} \tag{6}$$

Thus each element of $\mathbf{T}_L(C)$ is generated by $C + 1/C$.

Since $C \in \mathbb{C} \setminus \{-1, 0, 1\}$, $\mathbf{T}_L(C)$ is a complex symmetric Toeplitz matrix. Furthermore, since the elements of $\mathbf{T}_L(C)$ are polynomials (in $C + 1/C$) given by (6), $\mathbf{T}_L(C)$ is a real symmetric matrix if and only if $C + 1/C$ is real, and this happens under either one of the following two conditions on C:

1) $\Im(C) = 0$ and $\Re(C) \neq 0$, where $\Im(\cdot)$ and $\Re(\cdot)$ denote the imaginary part and real part operators, respectively, that is, $C \in \mathbb{R} \setminus \{-1, 0, 1\}$;
2) $|C| = 1$ and $\Im(C) \neq 0$, that is, C is on the unit circle centered at the origin on the complex plane, excluding the points -1 and 1.

A. RANK

The element is the ith row and the jth column of $\mathbf{T}_L(C)$ is $g(C, i-j)$. For $L = 2$,

$$\begin{aligned}\mathbf{T}_2(C) &= \begin{bmatrix} 2 & g(C,1) \\ g(C,1) & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & C + C^{-1} \\ C + C^{-1} & 2 \end{bmatrix}, \end{aligned} \tag{7}$$

and it is clear that its rank is 2.

To obtain the rank of $\mathbf{T}_L(C)$ for $L \geq 3$, for each $m = 1, \ldots, L-2$, we apply the following row operation on the $(L+1-m)$th row of $\mathbf{T}_L(C)$, starting with the Lth row and ending with the 3rd row:

$$\begin{aligned} &\text{row}_{L+1-m} - \left(C + C^{-1}\right) \text{row}_{L-m} + \text{row}_{L-1-m} \\ &\quad \longrightarrow \text{row}_{L+1-m}. \end{aligned}$$

The element in the $(L+1-m)$th row and jth column is replaced by

$$\begin{aligned} & g(C, L+1-m-j) \\ & - \left(C + C^{-1}\right) g(C, L-m-j) + g(C, L-1-m-j), \end{aligned}$$

which is zero from recurrence (5), for $m = 1, \ldots, L-2$, implying that the rank of $\mathbf{T}_L(C)$ is at most 2. Since the first two rows of $\mathbf{T}_L(C)$ are linearly independent, we conclude that its rank is 2.

Thus the rank of $\mathbf{T}_L(C)$ is 2 for $L \geq 2$.

Alternatively, if we express the $L \times 1$ vectors $\mathbf{u}_L(C)$ and $\mathbf{u}_L(1/C)$ as

$$\mathbf{u}_L(C) = \begin{bmatrix} C \\ C^2 \\ \vdots \\ C^L \end{bmatrix}, \quad \mathbf{u}_L(1/C) = \begin{bmatrix} C^{-1} \\ C^{-2} \\ \vdots \\ C^{-L} \end{bmatrix}, \tag{8}$$

then $\mathbf{T}_L(C)$ can be written as

$$\begin{aligned}\mathbf{T}_L(C) &= \mathbf{u}_L(C)\mathbf{u}_L^T(1/C) + \mathbf{u}_L(1/C)\mathbf{u}_L^T(C) \\ &= \left[\mathbf{u}_L(C)\ \mathbf{u}_L(1/C)\right]\begin{bmatrix}\mathbf{u}_L^T(1/C) \\ \mathbf{u}_L^T(C)\end{bmatrix},\end{aligned} \tag{9}$$

where $(\cdot)^T$ denotes the transpose operator, which implies that the rank of $\mathbf{T}_L(C)$ is 2.

III. EIGENVALUES

Consider the matrix $\lambda\mathbf{I}_L - \mathbf{T}_L(C)$, where λ is an eigenvalue of $\mathbf{T}_L(C)$ and $\mathbf{I}_L$ is the $L \times L$ identity matrix.

For $L = 2$, we find from (7) that

$$\det\left(\lambda\mathbf{I}_2 - \mathbf{T}_2(C)\right) = (\lambda - 2)^2 - \left(C + C^{-1}\right)^2,$$

implying the characteristic equation

$$(\lambda - 2)^2 - \left(C + C^{-1}\right)^2 = 0, \tag{10}$$

which can be rewritten as

$$\lambda^2 - 4\lambda + \left(2 - \left(C^2 + C^{-2}\right)\right) = 0. \tag{11}$$

Solving (10) for λ, we obtain the *eigenvalues of* $\boldsymbol{T}_2(C)$ as

$$\lambda = 2 \pm \left(C + C^{-1}\right). \tag{12}$$

Denote the two-sided sequence r_k as

$$r_k = g(C, k) = C^k + C^{-k}, \quad k \in \mathbb{Z}, \tag{13}$$

where $\mathbb{Z}$ denotes the set of integers. Note from (5) that r_k follows the recurrence

$$r_k = \left(C + C^{-1}\right) r_{k-1} - r_{k-2}, \quad k \in \mathbb{Z}, \tag{14}$$

with conditions

$$r_0 = 2, \quad r_1 = g(C, 1) = C + C^{-1}. \tag{15}$$

Furthermore, (4a) implies

$$r_k = r_{-k}, \quad k \in \mathbb{Z}. \tag{16}$$

For $L \geq 3$, we apply on $\mathbf{T}_L(C) - \lambda\mathbf{I}_L$ the row operations

$$\text{row}_{L+1-m} - \left(C + C^{-1}\right)\text{row}_{L-m} + \text{row}_{L-1-m} \longrightarrow \text{row}_{L+1-m}, \quad m = 1, \ldots, L-2,$$

starting with the Lth row and ending with the 3rd row. This results in the characteristic polynomial

$$\begin{aligned}\det\left(\lambda\mathbf{I}_L - \mathbf{T}_L(C)\right) &= (-1)^L\det\left(\mathbf{T}_L(C) - \lambda\mathbf{I}_L\right) \\ &= \lambda^{L-2}\begin{vmatrix}\mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22}\end{vmatrix},\end{aligned} \tag{17}$$

where

$$\mathbf{M}_{11} = \begin{bmatrix} 2-\lambda & r_1 & r_2 & \cdots & r_{L-3} \\ r_1 & 2-\lambda & r_1 & \cdots & r_{L-4}\end{bmatrix}, \tag{18}$$

$$\mathbf{M}_{12} = \begin{bmatrix} r_{L-2} & r_{L-1} \\ r_{L-3} & r_{L-2}\end{bmatrix}, \tag{19}$$

$$\mathbf{M}_{22} = \begin{bmatrix} \mathbf{0}_{L-4} & \mathbf{0}_{L-4} \\ 1 & 0 \\ -r_1 & 1 \end{bmatrix}, \tag{20}$$

with $\mathbf{0}_{L-4}$ being the $(L-4) \times 1$ vector of zeros, and $\mathbf{M}_{21}$ is an $(L-2) \times (L-2)$ upper triangular triple-band Toeplitz matrix whose element in the ith row and the jth column is given by

$$\left[\mathbf{M}_{21}\right]_{i,j} = \begin{cases} 1 & \text{if } j = i, \\ -r_1 & \text{if } j = i+1, \\ 1 & \text{if } j = i+2, \\ 0 & \text{otherwise,}\end{cases} \quad i, j = 1, \ldots, L-2. \tag{21}$$

Now

$$\begin{aligned}\begin{vmatrix}\mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22}\end{vmatrix} &= \det(\mathbf{M}_{21})\det\left(\mathbf{M}_{12} - \mathbf{M}_{11}\mathbf{M}_{21}^{-1}\mathbf{M}_{22}\right) \\ &= \det\left(\mathbf{M}_{12} - \mathbf{M}_{11}\mathbf{M}_{21}^{-1}\mathbf{M}_{22}\right),\end{aligned} \tag{22}$$

since, from (21), we get $\det(\mathbf{M}_{21}) = 1$.

It is clear from (17)-(21) that $\mathbf{T}_L(C)$ has eigenvalue zero with multiplicity $L-2$ and 2 nonzero eigenvalues; therefore $\det\left(\mathbf{M}_{12} - \mathbf{M}_{11}\mathbf{M}_{21}^{-1}\mathbf{M}_{22}\right)$ is a polynomial of degree 2 in λ.

From (3) and (13), we can express the element in the ith row and the jth column of $\mathbf{T}_L(C)$ as

$$\left[\mathbf{T}_L(C)\right]_{i,j} = r_{i-j} = r_{j-i}, \quad i = 1, \ldots, L, \quad j = 1, \ldots, L. \tag{23}$$

Let $\lambda_{(+)}$ and $\lambda_{(-)}$ denote the two nonzero eigenvalues of $\mathbf{T}_L(C)$, where $\lambda_{(+)}$ is the eigenvalue associated with the positive square root of the discriminant and $\lambda_{(-)}$ is the eigenvalue associated with the negative square root of the discriminant of the quadratic equation (in λ)

$$\det\left(\mathbf{M}_{12} - \mathbf{M}_{11}\mathbf{M}_{21}^{-1}\mathbf{M}_{22}\right) = 0, \tag{24}$$

where $\mathbf{M}_{11}, \mathbf{M}_{12}, \mathbf{M}_{22}$, and $\mathbf{M}_{21}$ are given by (18), (19), (20), and (21), respectively. Now

$$\begin{aligned}\lambda_{(+)} + \lambda_{(-)} &= \text{trace}\left(\mathbf{T}_L(C)\right), \\ \lambda_{(+)}^2 + \lambda_{(-)}^2 &= \text{trace}\left(\mathbf{T}_L^2(C)\right),\end{aligned} \tag{25}$$

which results in

$$\begin{aligned}\lambda_{(+)} &= \frac{1}{2}\text{trace}\left(\mathbf{T}_L(C)\right) + \sqrt{\frac{1}{2}\text{trace}\left(\mathbf{T}_L^2(C)\right) - \frac{1}{4}\left(\text{trace}\left(\mathbf{T}_L(C)\right)\right)^2}, \\ \lambda_{(-)} &= \frac{1}{2}\text{trace}\left(\mathbf{T}_L(C)\right) - \sqrt{\frac{1}{2}\text{trace}\left(\mathbf{T}_L^2(C)\right) - \frac{1}{4}\left(\text{trace}\left(\mathbf{T}_L(C)\right)\right)^2}.\end{aligned} \tag{26}$$

From (23), we get

$$\text{trace}\left(\mathbf{T}_L(C)\right) = 2L \tag{27}$$

and

$$\text{trace}\left(\mathbf{T}_L^2(C)\right) = \sum_{i=1}^{L}\sum_{j=1}^{L} r_{i-j}^2. \tag{28}$$

Using (13), (28) can be expressed as

$$\text{trace}\left(\mathbf{T}_L^2(C)\right) = \sum_{i=1}^{L}\sum_{j=1}^{L}\left[2 + C^{2i-2j} + C^{2j-2i}\right]$$

$$= 2L^2 + 2\left[\sum_{i=1}^{L} C^{2i}\right]\left[\sum_{j=1}^{L} C^{-2j}\right]$$
$$= 2L^2 + 2\left[\frac{C^2\left(C^{2L}-1\right)}{\left(C^2-1\right)}\right] \times \left[\frac{C^{-2}\left(1-C^{-2L}\right)}{\left(1-C^{-2}\right)}\right]$$
$$= 2L^2 + \frac{2\left(C^{2L}+C^{-2L}-2\right)}{\left(C^2+C^{-2}-2\right)},$$

and this can be rewritten as

$$\mathrm{trace}\left(\mathbf{T}_L^2(C)\right) = 2L^2 + \frac{2\left(C^L-C^{-L}\right)^2}{\left(C-C^{-1}\right)^2}. \quad (29)$$

Substituting (27) and (29) in (26), we get

$$\lambda_{(+)} = L + \frac{\left(C^L-C^{-L}\right)}{\left(C-C^{-1}\right)},$$
$$\lambda_{(-)} = L - \frac{\left(C^L-C^{-L}\right)}{\left(C-C^{-1}\right)}. \quad (30)$$

Note that (30) for $L = 2$ results in (12).

From (30), we obtain the *eigenvalues of* $\mathbf{T}_L(C)$ *for* $L \geq 3$ *in closed form* as

$$\lambda{=}0 \text{ (with multiplicity } L-2\text{)},\ L \pm \frac{\left(C^L-C^{-L}\right)}{\left(C-C^{-1}\right)}$$
$$=0 \text{ (with multiplicity } L-2\text{)},\ L \pm \sum_{k=0}^{L-1} C^{L-1-2k}. \quad (31)$$

Denote the two-sided sequence v_k as

$$v_k = \frac{\left(C^k-C^{-k}\right)}{\left(C-C^{-1}\right)}, \quad k \in \mathbb{Z}. \quad (32)$$

It can be easily from (32) shown that v_k follows the recurrence

$$v_k = \left(C+C^{-1}\right)v_{k-1} - v_{k-2}, \quad k \in \mathbb{Z}, \quad (33)$$

with conditions

$$v_0 = 0, \quad v_1 = 1. \quad (34)$$

Note that from (32) we also get

$$v_k = -v_{-k}, \quad k \in \mathbb{Z}, \quad (35)$$

and

$$v_2 = r_1 = C + C^{-1}. \quad (36)$$

Furthermore, it can be easily shown from (13) and (32) that

$$r_k = v_{k+1} - v_{k-1}, \quad k \in \mathbb{Z}. \quad (37)$$

Applying (32) and (13) to (31), the eigenvalues can alternatively be written as

$$\lambda = 0 \text{ (with multiplicity } L-2\text{)},\ L \pm v_L$$
$$= 0 \text{ (with multiplicity } L-2\text{)},$$
$$L \pm \left[\frac{\left((-1)^L-1\right)}{2} + \sum_{k=0}^{\left\lfloor\frac{L-1}{2}\right\rfloor} r_{L-1-2k}\right]. \quad (38)$$

Note from (9) that $\lambda_{(+)}$ and $\lambda_{(-)}$, given by (30), are the two eigenvalues of the 2×2 matrix

$$\begin{bmatrix} \mathbf{u}_L^T(1/C) \\ \mathbf{u}_L^T(C) \end{bmatrix}\left[\mathbf{u}_L(C)\ \mathbf{u}_L(1/C)\right] = \begin{bmatrix} L & \mathbf{u}_L^T(1/C)\mathbf{u}_L(1/C) \\ \mathbf{u}_L^T(C)\mathbf{u}_L(C) & L \end{bmatrix}.$$

IV. EIGENVECTORS

For $L = 2$, the two eigenvalues of $\mathbf{T}_2(C)$, denoted as $\lambda_{(-)}$ and $\lambda_{(+)}$, are given by (see (12) and (30))

$$\lambda_{(-)} = 2 - \left(C+C^{-1}\right), \quad \lambda_{(+)} = 2 + \left(C+C^{-1}\right). \quad (39)$$

We find from (7) and (39) that the eigenvector $\mathbf{e}_{(-)}$ of $\mathbf{T}_2(C)$ corresponding to eigenvalue $\lambda_{(-)}$ and the eigenvector $\mathbf{e}_{(+)}$ of $\mathbf{T}_2(C)$ corresponding to eigenvalue $\lambda_{(+)}$ are given by

$$\mathbf{e}_{(-)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{e}_{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad (40)$$

and these are complex orthogonal ($\mathbf{e}_{(-)}^T\mathbf{e}_{(+)} = 0$), as well as orthogonal ($\mathbf{e}_{(-)}^H\mathbf{e}_{(+)} = 0$, where $(\cdot)^H$ denotes the Hermitian (complex conjugate transpose) operator).

Closed form expressions for the eigenvectors of $\mathbf{T}_L(C)$ when $L \geq 3$ for the cases of even L and odd L are given by the following two propositions.

Proposition 1: When L is even, such that

$$L = 2M, \quad M \in \mathbb{N} - \{1\}, \quad (41)$$

where $\mathbb{N}$ denotes the set of natural numbers, the eigenvectors $\mathbf{e}_{(-)}$, $\mathbf{e}_{(+)}$ corresponding to the eigenvalues

$$\lambda_{(-)} = 2M - v_{2M} = 2M - \frac{\left(C^{2M}-C^{-2M}\right)}{\left(C-C^{-1}\right)}, \quad (42a)$$
$$\lambda_{(+)} = 2M + v_{2M} = 2M + \frac{\left(C^{2M}-C^{-2M}\right)}{\left(C-C^{-1}\right)}, \quad (42b)$$

respectively, and the $2M - 2$ eigenvectors $\mathbf{e}_0(1), \ldots, \mathbf{e}_0(2M-2)$ corresponding to the eigenvalue zero of multiplicity $2M-2$, are given by

$$\mathbf{e}_{(-)} = \begin{bmatrix} e_{1,-} \\ \vdots \\ e_{2M,-} \end{bmatrix}, \quad \mathbf{e}_{(+)} = \begin{bmatrix} e_{1,+} \\ \vdots \\ e_{2M,+} \end{bmatrix}, \quad (43a)$$

where

$$e_{k,-} = v_{M+1-k} + v_{M-k} = \frac{\left(\left[C^{M+1-k}-C^{-(M+1-k)}\right]+\left[C^{M-k}-C^{-(M-k)}\right]\right)}{\left(C-C^{-1}\right)},$$
$$k = 1, \ldots, 2M, \quad (43b)$$
$$e_{k,+} = v_{M+1-k} - v_{M-k} = \frac{\left(\left[C^{M+1-k}-C^{-(M+1-k)}\right]-\left[C^{M-k}-C^{-(M-k)}\right]\right)}{\left(C-C^{-1}\right)},$$
$$k = 1, \ldots, 2M, \quad (43c)$$

and

$$\mathbf{e}_0(j) = \begin{bmatrix} e_{1,0}(j) \\ \vdots \\ e_{2M,0}(j) \end{bmatrix}, \quad j = 1, \ldots, 2M-2, \tag{44a}$$

where

$$e_{k,0}(j) = \begin{cases} 1 & \text{if } k = j, \\ -\left(C + C^{-1}\right) & \text{if } k = j+1, \\ 1 & \text{if } k = j+2, \\ 0 & \text{otherwise}, \end{cases}$$
$$k = 1, \ldots, 2M, \quad j = 1, \ldots, 2M-2. \tag{44b}$$

Proof: The proof is presented in Appendix A. ■

Proposition 2: When L is odd, such that

$$L = 2M + 1, \quad M \in \mathbb{N}, \tag{45}$$

the eigenvectors $\mathbf{e}_{(-)}$, $\mathbf{e}_{(+)}$ corresponding to the eigenvalues

$$\lambda_{(-)} = 2M + 1 - v_{2M+1} = 2M + 1 - \frac{\left(C^{2M+1} - C^{-(2M+1)}\right)}{\left(C - C^{-1}\right)}, \tag{46a}$$

$$\lambda_{(+)} = 2M + 1 + v_{2M+1} = 2M + 1 + \frac{\left(C^{2M+1} - C^{-(2M+1)}\right)}{\left(C - C^{-1}\right)}, \tag{46b}$$

respectively, and the $2M - 1$ eigenvectors $\mathbf{e}_0(1), \ldots, \mathbf{e}_0(2M-1)$ corresponding to the eigenvalue zero of multiplicity $2M - 1$, are given by

$$\mathbf{e}_{(-)} = \begin{bmatrix} e_{1,-} \\ \vdots \\ e_{2M+1,-} \end{bmatrix}, \quad \mathbf{e}_{(+)} = \begin{bmatrix} e_{1,+} \\ \vdots \\ e_{2M+1,+} \end{bmatrix}, \tag{47a}$$

where

$$e_{k,-} = v_{M+1-k} = \frac{\left(C^{M+1-k} - C^{-(M+1-k)}\right)}{\left(C - C^{-1}\right)}, \quad k = 1, \ldots, 2M+1, \tag{47b}$$

$$e_{k,+} = r_{M+1-k} = v_{M+2-k} - v_{M-k} = C^{M+1-k} + C^{-(M+1-k)}, \quad k = 1, \ldots, 2M+1, \tag{47c}$$

and

$$\mathbf{e}_0(j) = \begin{bmatrix} e_{1,0}(j) \\ \vdots \\ e_{2M+1,0}(j) \end{bmatrix}, \quad j = 1, \ldots, 2M-1, \tag{48a}$$

where

$$e_{k,0}(j) = \begin{cases} 1 & \text{if } k = j, \\ -\left(C + C^{-1}\right) & \text{if } k = j+1, \\ 1 & \text{if } k = j+2, \\ 0 & \text{otherwise}, \end{cases}$$
$$k = 1, \ldots, 2M+1, \quad j = 1, \ldots, 2M-1. \tag{48b}$$

Proof: The proof is presented in Appendix B. ■

A. COMPLEX ORTHOGONALITY AMONG EIGENVECTORS FOR $L \geq 3$

Using Propositions 1 and 2, it can be shown that for $L \geq 3$,

$$\begin{aligned} &\mathbf{e}_{(+)}^T \mathbf{e}_{(-)} = 0, \\ &\mathbf{e}_{(-)}^T \mathbf{e}_0(j) = 0, \quad j = 1, \ldots, L-2, \\ &\mathbf{e}_{(+)}^T \mathbf{e}_0(j) = 0, \quad j = 1, \ldots, L-2. \end{aligned} \tag{49}$$

Thus, for $L \geq 3$, $\{\mathbf{e}_{(-)}, \mathbf{e}_{(+)}, \mathbf{e}_0(j)\}$ is a set of complex orthogonal eigenvectors for each $j \in \{1, \ldots, L-2\}$. However, $\{\mathbf{e}_0(1), \ldots, \mathbf{e}_0(L-2)\}$ for $L \geq 4$ is a set of linearly independent eigenvectors which are not complex orthogonal.

B. SYMMETRIC AND SKEW-SYMMETRIC EIGENVECTORS

Let $\mathbf{x} = [x_1, \cdots, x_L]^T$ be an eigenvector of $\mathbf{T}_L(C)$; it is called symmetric if it satisfies

$$x_k = x_{L+1-k}, \quad k = 1, \ldots, L,$$

and it is called skew-symmetric if it satisfies

$$x_k = -x_{L+1-k}, \quad k = 1, \ldots, L.$$

Since $\mathbf{T}_L(C)$ is a complex symmetric Toeplitz matrix, its eigenvectors form a complex orthogonal basis consisting of $\lfloor L/2 \rfloor$ symmetric and $L - \lfloor L/2 \rfloor$ skew-symmetric eigenvectors.

From (40), (43), and (47), we find that for $L \geq 2$, $\mathbf{e}_{(-)}$ is skew-symmetric and $\mathbf{e}_{(+)}$ is symmetric.

For $L = 3$, we find from (48) that $\mathbf{e}_0(1)$ is symmetric, and we find from (49) that $\{\mathbf{e}_{(-)}, \mathbf{e}_{(+)}, \mathbf{e}_0(1)\}$ is a set of three orthogonal eigenvectors: of these, one is skew-symmetric and two are symmetric, and, from Proposition 2, they can be expressed as

$$\mathbf{e}_{(-)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{e}_{(+)} = \begin{bmatrix} r_1 \\ 2 \\ r_1 \end{bmatrix}, \quad \mathbf{e}_0(1) = \begin{bmatrix} 1 \\ -r_1 \\ 1 \end{bmatrix}, \tag{50}$$

where $r_1 = C + C^{-1}$.

For $L = 4$, we find from (44) that $\mathbf{e}_0(1)$ and $\mathbf{e}_0(2)$ are linearly independent but not complex orthogonal. However, $\mathbf{e}_0(1) - \mathbf{e}_0(2)$ and $\mathbf{e}_0(1) + \mathbf{e}_0(2)$ are complex orthogonal, and therefore $\{\mathbf{e}_{(-)}, \mathbf{e}_{(+)}, \mathbf{w}_0(1), \mathbf{w}_0(2)\}$, where $\mathbf{w}_0(1) = \mathbf{e}_0(1) - \mathbf{e}_0(2)$ and $\mathbf{w}_0(2) = \mathbf{e}_0(1) + \mathbf{e}_0(2)$, is a set of four complex orthogonal eigenvectors: of these, two ($\mathbf{e}_{(-)}$ and $\mathbf{w}_0(1)$) are skew-symmetric and two ($\mathbf{e}_{(+)}$ and $\mathbf{w}_0(2)$) are symmetric, and, from Proposition 1, they can be expressed as

$$\mathbf{e}_{(-)} = \begin{bmatrix} r_1 + 1 \\ 1 \\ -1 \\ -(r_1 + 1) \end{bmatrix}, \quad \mathbf{e}_{(+)} = \begin{bmatrix} r_1 - 1 \\ 1 \\ 1 \\ r_1 - 1 \end{bmatrix},$$

$$\mathbf{w}_0(1) = \begin{bmatrix} 1 \\ -(r_1 + 1) \\ r_1 + 1 \\ -1 \end{bmatrix},$$

$$\mathbf{w}_0(2) = \begin{bmatrix} 1 \\ -(r_1-1) \\ -(r_1-1) \\ 1 \end{bmatrix}. \quad (51)$$

For $L = 5$, we find from (48) that $\mathbf{e}_0(1)$, $\mathbf{e}_0(2)$, and $\mathbf{e}_0(3)$ are linearly independent but not complex orthogonal. However, $\mathbf{e}_0(1) - \mathbf{e}_0(3)$ and $\mathbf{e}_0(1) + \mathbf{e}_0(3)$, and also $\mathbf{e}_0(1) - \mathbf{e}_0(3)$ and $\mathbf{e}_0(2)$, are complex orthogonal pairs. We can get complex orthogonal eigenvectors $\mathbf{w}_0(1), \mathbf{w}_0(2), \mathbf{w}_0(3)$ from $\mathbf{e}_0(1), \mathbf{e}_0(2), \mathbf{e}_0(3)$ as

$$\begin{aligned} \mathbf{w}_0(1) &= \mathbf{e}_0(1) - \mathbf{e}_0(3), \\ \mathbf{w}_0(2) &= \mathbf{e}_0(1) + \mathbf{e}_0(3), \\ \mathbf{w}_0(3) &= \frac{\left(\mathbf{e}_0(1)+\mathbf{e}_0(3)\right)^T\left(\mathbf{e}_0(1)+\mathbf{e}_0(3)\right)}{2} \\ &\quad \times \left[\mathbf{e}_0(2) - \frac{\left(\mathbf{e}_0(1)+\mathbf{e}_0(3)\right)^T \mathbf{e}_0(2)}{\left(\mathbf{e}_0(1)+\mathbf{e}_0(3)\right)^T\left(\mathbf{e}_0(1)+\mathbf{e}_0(3)\right)} \right. \\ &\quad \left. \times \left(\mathbf{e}_0(1)+\mathbf{e}_0(3)\right)\right]. \end{aligned}$$

Note that $\mathbf{w}_0(3)$ is obtained by a method similar to Gram-Schmidt orthogonalization. Therefore $\{\mathbf{e}_{(-)}, \mathbf{e}_{(+)}, \mathbf{w}_0(1), \mathbf{w}_0(2), \mathbf{w}_0(3)\}$ is a set of five complex orthogonal eigenvectors: of these, two ($\mathbf{e}_{(-)}$ and $\mathbf{w}_0(1)$) are skew-symmetric and three ($\mathbf{e}_{(+)}$, $\mathbf{w}_0(2)$, and $\mathbf{w}_0(3)$) are symmetric, and, from Proposition 2, they can be expressed as

$$\mathbf{e}_{(-)} = \begin{bmatrix} r_1 \\ 1 \\ 0 \\ -1 \\ -r_1 \end{bmatrix}, \quad \mathbf{e}_{(+)} = \begin{bmatrix} r_1^2-2 \\ r_1 \\ 2 \\ r_1 \\ r_1^2-2 \end{bmatrix},$$
$$\mathbf{w}_0(1) = \begin{bmatrix} 1 \\ -r_1 \\ 0 \\ r_1 \\ -1 \end{bmatrix}, \quad \mathbf{w}_0(2) = \begin{bmatrix} 1 \\ -r_1 \\ 2 \\ -r_1 \\ 1 \end{bmatrix},$$
$$\mathbf{w}_0(3) = \begin{bmatrix} 2r_1 \\ -\left(r_1^2-3\right) \\ -\left(r_1^3-r_1\right) \\ -\left(r_1^2-3\right) \\ 2r_1 \end{bmatrix}. \quad (52)$$

Now consider the general case of even L, such that $L = 2M$, $M \in \mathbb{N} - \{1\}$. Let the skew-symmetric eigenvectors $\mathbf{h}_{0,skew-symm}(1), \ldots, \mathbf{h}_{0,skew-symm}(M-1)$ and symmetric eigenvectors $\mathbf{h}_{0,symm}(1), \ldots, \mathbf{h}_{0,symm}(M-1)$ of $\mathbf{T}_{2M}(C)$ corresponding to the eigenvalue zero be given by

$$\mathbf{h}_{0,skew-symm}(j) = \begin{bmatrix} \mathbf{a}(j) \\ -\mathbf{J}_M \mathbf{a}(j) \end{bmatrix},$$
$$\mathbf{h}_{0,symm}(j) = \begin{bmatrix} \mathbf{b}(j) \\ \mathbf{J}_M \mathbf{b}(j) \end{bmatrix}, \quad j = 1, \ldots, M-1, \quad (53)$$

where $\mathbf{J}_M$ is the $M \times M$ exchange matrix, and

$$\mathbf{a}(j) = \begin{bmatrix} a_1(j) \\ \vdots \\ a_M(j) \end{bmatrix}, \quad \mathbf{b}(j) = \begin{bmatrix} b_1(j) \\ \vdots \\ b_M(j) \end{bmatrix}. \quad (54)$$

Note that each of the $M-1$ skew-symmetric eigenvectors is complex orthogonal to each of the $M-1$ symmetric eigenvectors. Multiplying the first row of $\mathbf{T}_{2M}(C)$ with $\mathbf{h}_{0,skew-symm}(j)$, we get

$$\sum_{k=1}^{M} (r_{k-1} - r_{2M-k}) a_k(j) = 0, \quad (55)$$

while multiplying the first row of $\mathbf{T}_{2M}(C)$ with $\mathbf{h}_{0,symm}(j)$, we get

$$\sum_{k=1}^{M} (r_{k-1} + r_{2M-k}) b_k(j) = 0. \quad (56)$$

From (13), we obtain

$$\begin{aligned} r_{k-1} - r_{2M-k} &= -\left(C^{M-\frac{1}{2}} - C^{-\left(M-\frac{1}{2}\right)}\right) \\ &\quad \times \left(C^{M-k+\frac{1}{2}} - C^{-\left(M-k+\frac{1}{2}\right)}\right), \quad (57a) \\ r_{k-1} + r_{2M-k} &= \left(C^{M-\frac{1}{2}} + C^{-\left(M-\frac{1}{2}\right)}\right) \\ &\quad \times \left(C^{M-k+\frac{1}{2}} + C^{-\left(M-k+\frac{1}{2}\right)}\right). \quad (57b) \end{aligned}$$

Substitution of (57a) in (55) and (57b) in (56) results in

$$\sum_{k=1}^{M} \left(C^{M-k+\frac{1}{2}} - C^{-\left(M-k+\frac{1}{2}\right)}\right) a_k(j) = 0, \quad (58a)$$
$$\sum_{k=1}^{M} \left(C^{M-k+\frac{1}{2}} + C^{-\left(M-k+\frac{1}{2}\right)}\right) b_k(j) = 0. \quad (58b)$$

We can choose linearly independent $\mathbf{a}(1), \ldots, \mathbf{a}(M-1)$ and linearly independent $\mathbf{b}(1), \ldots, \mathbf{b}(M-1)$ from (58a) and (58b), respectively, as

$$a_k(j) = \begin{cases} 1 & \text{if } k = j, \\ -\dfrac{\left(C^{M-j+\frac{1}{2}} - C^{-\left(M-j+\frac{1}{2}\right)}\right)}{\left(C^{\frac{1}{2}} - C^{-\frac{1}{2}}\right)} & \text{if } k = M, \\ 0 & \text{otherwise,} \end{cases}$$
$$k = 1, \ldots, M, \quad j = 1, \ldots, M-1, \quad (59a)$$

$$b_k(j) = \begin{cases} 1 & \text{if } k = j, \\ -\dfrac{\left(C^{M-j+\frac{1}{2}} + C^{-\left(M-j+\frac{1}{2}\right)}\right)}{\left(C^{\frac{1}{2}} + C^{-\frac{1}{2}}\right)} & \text{if } k = M, \\ 0 & \text{otherwise,} \end{cases}$$
$$k = 1, \ldots, M, \quad j = 1, \ldots, M-1. \quad (59b)$$

By complex orthogonalizing $\mathbf{a}(1), \ldots, \mathbf{a}(M-1)$ whose elements are given by (59a), we obtain a complex orthogonal set $\{\mathbf{a}_{cporth}(1), \ldots, \mathbf{a}_{cporth}(M-1)\}$, and by complex orthogonalizing $\mathbf{b}(1), \ldots, \mathbf{b}(M-1)$ whose elements are given by (59b), we obtain another complex orthogonal set $\{\mathbf{b}_{cporth}(1), \ldots, \mathbf{b}_{cporth}(M-1)\}$. A set $\{\mathbf{w}_0(1), \ldots, \mathbf{w}(2M-2)\}$ of complex orthogonal eigenvectors corresponding to the eigenvalue zero can now be expressed as

$$\mathbf{w}_0(j) = \begin{bmatrix} \mathbf{a}_{cporth}(j) \\ -\mathbf{J}_M \mathbf{a}_{cporth}(j) \end{bmatrix}, \quad \mathbf{w}_0(M-1+j) = \begin{bmatrix} \mathbf{b}_{cporth}(j) \\ \mathbf{J}_M \mathbf{b}_{cporth}(j) \end{bmatrix}, \quad j = 1, \ldots, M-1, \tag{60}$$

where $\mathbf{w}_0(1), \ldots, \mathbf{w}_0(M-1)$ are skew-symmetric and $\mathbf{w}_0(M), \ldots, \mathbf{w}_0(2M-2)$ are symmetric.

Next consider the general case of odd L, such that $L = 2M+1, M \in \mathbb{N} - \{1\}$. Let the skew-symmetric eigenvectors $\mathbf{h}_{0,skew-symm}(1), \ldots, \mathbf{h}_{0,skew-symm}(M-1)$ and symmetric eigenvectors $\mathbf{h}_{0,symm}(1), \ldots, \mathbf{h}_{0,symm}(M)$ of $\mathbf{T}_{2M+1}(C)$ corresponding to the eigenvalue zero be given by

$$\mathbf{h}_{0,skew-symm}(j) = \begin{bmatrix} \mathbf{a}(j) \\ 0 \\ -\mathbf{J}_M \mathbf{a}(j) \end{bmatrix}, \quad j = 1, \ldots, M-1,$$
$$\mathbf{h}_{0,symm}(\ell) = \begin{bmatrix} \mathbf{b}(\ell) \\ b_{M+1}(\ell) \\ \mathbf{J}_M \mathbf{b}(\ell) \end{bmatrix}, \quad \ell = 1, \ldots, M, \tag{61}$$

where

$$\mathbf{a}(j) = \begin{bmatrix} a_1(j) \\ \vdots \\ a_M(j) \end{bmatrix}, \quad \mathbf{b}(\ell) = \begin{bmatrix} b_1(\ell) \\ \vdots \\ b_M(\ell) \end{bmatrix}. \tag{62}$$

Note that each of the $M-1$ skew-symmetric eigenvectors is complex orthogonal to each of the M symmetric eigenvectors. Multiplying the first row of $\mathbf{T}_{2M+1}(C)$ with $\mathbf{h}_{0,skew-symm}(j)$, we get

$$\sum_{k=1}^{M} (r_{k-1} - r_{2M+1-k}) a_k(j) = 0, \tag{63}$$

while multiplying the first row of $\mathbf{T}_{2M+1}(C)$ with $\mathbf{h}_{0,symm}(\ell)$, we get

$$r_M b_{M+1}(\ell) + \sum_{k=1}^{M} (r_{k-1} + r_{2M+1-k}) b_k(\ell) = 0. \tag{64}$$

From (13), we obtain

$$r_{k-1} - r_{2M+1-k} = -\left(C^M - C^{-M}\right) \times \left(C^{M-k+1} - C^{-(M-k+1)}\right), \tag{65a}$$
$$r_{k-1} + r_{2M+1-k} = \left(C^M + C^{-M}\right) \times \left(C^{M-k+1} + C^{-(M-k+1)}\right). \tag{65b}$$

Substitution of (65a) in (63) and (65b) in (64) results in

$$\sum_{k=1}^{M} \left(C^{M-k+1} - C^{-(M-k+1)}\right) a_k(j) = 0, \tag{66a}$$

$$b_{M+1}(\ell) + \sum_{k=1}^{M} \left(C^{M-k+1} + C^{-(M-k+1)}\right) b_k(\ell) = 0. \tag{66b}$$

We can choose linearly independent $\mathbf{a}(1), \ldots, \mathbf{a}(M-1)$ and linearly independent $\mathbf{b}(1), \ldots, \mathbf{b}(M)$ from (66a) and (66b), respectively, as

$$a_k(j) = \begin{cases} 1 & \text{if } k = j, \\ -\dfrac{\left(C^{M-j+1} - C^{-(M-j+1)}\right)}{\left(C - C^{-1}\right)} & \text{if } k = M, \\ 0 & \text{otherwise,} \end{cases} \quad k = 1, \ldots, M, \quad j = 1, \ldots, M-1, \tag{67a}$$

$$b_k(\ell) = \begin{cases} 1 & \text{if } k = \ell, \\ -\left(C^{M-\ell+1} + C^{-(M-\ell+1)}\right) & \text{if } k = M+1, \\ 0 & \text{otherwise,} \end{cases} \quad k = 1, \ldots, M+1, \quad \ell = 1, \ldots, M. \tag{67b}$$

By complex orthogonalizing $\mathbf{a}(1), \ldots, \mathbf{a}(M-1)$ whose elements are given by (67a), we obtain a complex orthogonal set $\{\mathbf{a}_{cporth}(1), \ldots, \mathbf{a}_{cporth}(M-1)\}$, and by complex orthogonalizing appropriately

$$\begin{bmatrix} \mathbf{b}(1) \\ b_{M+1}(1) \\ \mathbf{J}_M \mathbf{b}(1) \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{b}(M) \\ b_{M+1}(M) \\ \mathbf{J}_M \mathbf{b}(M) \end{bmatrix}$$

whose elements are given by (67b), we obtain another complex orthogonal set

$$\left\{ \begin{bmatrix} \mathbf{b}_{cporth}(1) \\ b_{M+1,cporth}(1) \\ \mathbf{J}_M \mathbf{b}_{cporth}(1) \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{b}_{cporth}(M) \\ b_{M+1,cporth}(M) \\ \mathbf{J}_M \mathbf{b}_{cporth}(M) \end{bmatrix} \right\}.$$

A set $\{\mathbf{w}_0(1), \ldots, \mathbf{w}(2M-1)\}$ of complex orthogonal eigenvectors corresponding to eigenvalue zero can now be expressed as

$$\mathbf{w}_0(j) = \begin{bmatrix} \mathbf{a}_{cporth}(j) \\ 0 \\ -\mathbf{J}_M \mathbf{a}_{cporth}(j) \end{bmatrix}, \quad j = 1, \ldots, M-1,$$
$$\mathbf{w}_0(M-1+\ell) = \begin{bmatrix} \mathbf{b}_{cporth}(\ell) \\ b_{M+1,cporth}(\ell) \\ \mathbf{J}_M \mathbf{b}_{cporth}(\ell) \end{bmatrix}, \quad \ell = 1, \ldots, M, \tag{68}$$

where $\mathbf{w}_0(1), \ldots, \mathbf{w}_0(M-1)$ are skew-symmetric and $\mathbf{w}_0(M), \ldots, \mathbf{w}_0(2M-1)$ are symmetric.

V. SPECIAL CASES OF REAL NONZERO C AND COMPLEX C WITH MAGNITUDE ONE

We consider two special cases of the results obtained for $C \in \mathbb{C} \setminus \{-1, 0, 1\}$, which are $C \in \mathbb{R} \setminus \{-1, 0, 1\}$ and $|C| = 1, \Im(C) \neq 0$. In both of these cases, $C + 1/C$ is real, which implies $\mathbf{T}_L(C)$ is real.

In the case of $C \in \mathbb{R} \setminus \{-1, 0, 1\}$, observing from (1) that

$$\left[\mathbf{T}_L(C)\right]_{i,j} = \left(\text{sgn}(C)\right)^{i-j} \left[\exp\left((i-j)\ln|C|\right)\right.$$

$$+\exp\Big(-(i-j)\ln|C|\Big)\Big]$$
$$= 2\big(\mathrm{sgn}(C)\big)^{i-j}\cosh\Big((i-j)\ln|C|\Big),$$
$$i=1,\ldots,L,\quad j=1,\ldots,L, \tag{69}$$

where $\mathrm{sgn}(\cdot)$ denotes the signum or sign function which takes the value 1 for positive argument, 0 for zero argument, and -1 for negative argument, we can express (38) as

$$\lambda=0 \text{ (with multiplicity } L-2),\quad L\pm\big(\mathrm{sgn}(C)\big)^{L-1}\left[\frac{\sinh\big(L\ln|C|\big)}{\sinh\big(\ln|C|\big)}\right]$$
$$=0 \text{ (with multiplicity } L-2),\quad L\pm\left[\begin{array}{l}\dfrac{\Big((-1)^L-1\Big)}{2}\\ +2\big(\mathrm{sgn}(C)\big)^{L-1}\\ \times\displaystyle\sum_{k=0}^{\left\lfloor\frac{L-1}{2}\right\rfloor}\cosh\Big((L-1-2k)\ln|C|\Big)\end{array}\right]. \tag{70}$$

Furthermore, we obtain from (13)

$$r_k = 2\big(\mathrm{sgn}(C)\big)^k\cosh\big(k\ln|C|\big),\quad k\in\mathbb{Z}, \tag{71}$$

and from (32)

$$v_k = \big(\mathrm{sgn}(C)\big)^{k-1}\left[\frac{\sinh\big(k\ln|C|\big)}{\sinh\big(\ln|C|\big)}\right],\quad k\in\mathbb{Z}, \tag{72}$$

and the eigenvectors are obtained from Propositions 1 and 2 with appropriate substitutions using (71) and (72).

In the case of complex C with magnitude one, $\Im(C)\neq 0$, putting

$$C=\exp\{\jmath\Theta\},\quad \Theta\in(0,2\pi)\setminus\{\pi\},$$

where $\jmath=\sqrt{-1}$, gives

$$\Big[\mathbf{T}_L\left(\exp\{\jmath\Theta\}\right)\Big]_{i,j}+2\cos\Big((i-j)\Theta\Big)$$
$$i+1,\ldots,L,\quad j=1,\ldots,L,$$
$$\Theta\in(0,2\pi)\setminus\{\pi\}. \tag{73}$$

The eigenvalues of $\mathbf{T}_L\left(\exp\{\jmath\Theta\}\right)$ are obtained from (70) as

$$\lambda=0 \text{ (with multiplicity } L-2),\ L\pm\frac{\sin\big(L\Theta\big)}{\sin\Theta}$$
$$=0 \text{ (with multiplicity } L-2),\quad L\pm\left[\begin{array}{l}\dfrac{\Big((-1)^L-1\Big)}{2}\\ +2\displaystyle\sum_{k=0}^{\left\lfloor\frac{L-1}{2}\right\rfloor}\cos\Big((L-1-2k)\Theta\Big)\end{array}\right]. \tag{74}$$

Furthermore, we obtain from (13)

$$r_k = 2\cos\big(k\Theta\big),\quad k\in\mathbb{Z}, \tag{75}$$

and from (32)

$$v_k=\frac{\sin\big(k\Theta\big)}{\sin\Theta},\quad k\in\mathbb{Z}, \tag{76}$$

and the eigenvectors are obtained from Propositions 1 and 2 with appropriate substitutions using (75) and (76).

VI. APPLICATIONS IN WIRELESS COMMUNICATIONS

We present here some applications of the results obtained to the performance evaluation of wireless communication systems using multiple antennas at the transmitter and/or the receiver.

A. CHANNEL CAPACITY AND TRANSMIT WEIGHT VECTOR OF OPTICAL WIRELESS COMMUNICATION SYSTEM

Consider a multiple-input multiple-output (MIMO) optical wireless communication system using intensity modulation with symbol-by-symbol transmission, L transmit antennas, and L receive antennas [9], [10], [11]. The L transmit antennas are separated from the L receive antennas by an opaque wall having two keyholes [12], [13] through which transmission occurs, resulting in a *double-keyhole channel*, with a *deterministic (not random or stochastic)* $L\times L$ *real-valued channel matrix* $\mathbf{H}$. If the transmitted information-bearing symbol (a non-negative real number) over a symbol time interval is s and the $L\times 1$ real-valued *transmit weight vector* is $\mathbf{t}$ ($\mathbf{t}$ is a unit vector, that is, it has Euclidean norm of 1), then the $L\times 1$ received signal vector $\mathbf{r}$ is given by

$$\mathbf{r}=\mathbf{H}\mathbf{t}s+\mathbf{n}, \tag{77}$$

where $\mathbf{n}$ is the $L\times 1$ real-valued additive white Gaussian noise (AWGN) vector which has L independent and identically distributed (i.i.d.) random elements, each having a Gaussian or normal distribution with mean zero and variance σ_n^2, that is, the $\mathcal{N}\big(0,\sigma_n^2\big)$ distribution.

The $L\times L$ channel matrix $\mathbf{H}$ of this MIMO system is expressed as

$$\mathbf{H}=\frac{1}{2}\left(\mathbf{x}\mathbf{y}^T+\mathbf{y}\mathbf{x}^T\right), \tag{78a}$$

where $\mathbf{x}$ and $\mathbf{y}$ are real-valued $L\times 1$ vectors which are given by

$$\mathbf{x}=\begin{bmatrix}1\\h\\h^2\\ \vdots\\h^{L-1}\end{bmatrix},\quad \mathbf{y}=\begin{bmatrix}1\\h^{-1}\\h^{-2}\\ \vdots\\h^{-(L-1)}\end{bmatrix},\quad h>1; \tag{78b}$$

this implies $\mathbf{H}=(1/2)\mathbf{T}_L(h)$ with $h>1$, where $\mathbf{T}_L(\cdot)$ is given by (1). The channel state information, which is the value of $\mathbf{H}$, is known to both the transmitter and the receiver.

The information-bearing symbol s takes one of a finite number $\mathcal{M}$ of distinct real non-negative values $S_1,\ldots,S_{\mathcal{M}}$, therefore the intensity modulation is $\mathcal{M}$-ary and it is a digital communication system. One of these $\mathcal{M}$ values is transmitted with probability $1/\mathcal{M}$ in a symbol interval, resulting in an average symbol energy E_{av} given by

$$E_{av}=\frac{1}{\mathcal{M}}\sum_{i=1}^{\mathcal{M}}|S_i|^2. \tag{79}$$

One of the important performance measures of this system is the channel capacity (in bits/channel use), which, as a function of the transmit weight vector $\mathbf{t}$, is given by

$$\mathcal{C}(\mathbf{t}) = \log_2\left(1 + \frac{E_{av}}{\sigma_n^2}\mathbf{t}^T\mathbf{H}^T\mathbf{H}\mathbf{t}\right). \tag{80}$$

Since $\mathbf{H}$ is a symmetric matrix, (80) can be rewritten as

$$\mathcal{C}(\mathbf{t}) = \log_2\left(1 + \frac{E_{av}}{\sigma_n^2}\mathbf{t}^T\mathbf{H}^2\mathbf{t}\right). \tag{81}$$

We choose $\mathbf{t}$ such that the capacity is maximized. Let λ_{max} denote the maximum eigenvalue of $\mathbf{H}$ and $\mathbf{t}_{max}$ the normalized eigenvector corresponding to this eigenvalue. The capacity (81) attains a maximum $\mathcal{C}_{max}$ when $\mathbf{t} = \mathbf{t}_{max}$, which results in

$$\begin{aligned}\mathcal{C}_{max} &= \mathcal{C}(\mathbf{t}_{max}) \\ &= \log_2\left(1 + \frac{E_{av}}{\sigma_n^2}\lambda_{max}^2\right). \end{aligned}\tag{82}$$

From (70) and (78), we get

$$\lambda_{max} = \frac{1}{2}\left(L + \left[\frac{\sinh\big(L\ln h\big)}{\sinh\big(\ln h\big)}\right]\right). \tag{83}$$

Substitution of (83) in (82) gives the *maximum capacity* as

$$\mathcal{C}_{max} = \log_2\left(1 + \frac{E_{av}}{4\sigma_n^2}\left(L + \left[\frac{\sinh\big(L\ln h\big)}{\sinh\big(\ln h\big)}\right]\right)^2\right). \tag{84}$$

Furthermore, from Proposition 1, Proposition 2, (71), (72), and (78), we get the *capacity maximizing normalized transmit weight vector* as

$$\mathbf{t}_{max} = \frac{1}{\sqrt{\sum_{\ell=1}^{L} e_\ell^2}}\begin{bmatrix} e_1 \\ \vdots \\ e_L\end{bmatrix}, \tag{85a}$$

where

$$e_k = \begin{cases} \sinh\Big(\big[\frac{L}{2} + 1 - k\big]\ln h\Big) \\ -\sinh\Big(\big[\frac{L}{2} - k\big]\ln h\Big) \\ \text{if } L \text{ is even,} \\ \\ \cosh\Big(\Big[\frac{(L+1)}{2} - k\Big]\ln h\Big) \\ \text{if } L \text{ is odd,}\end{cases}$$
$$k = 1, \ldots, L. \tag{85b}$$

B. TRANSMIT BEAMFORMING AND RECEIVE COMBINING IN OPTICAL WIRELESS COMMUNICATION SYSTEM WITH INTERFERER

Consider a MIMO optical wireless communication system using intensity modulation with symbol-by-symbol transmission, with K transmit antennas and L ($L \geq 3$) receive antennas [9], [10], [11] in the presence of an interferer. The channel between the transmitter and the receiver is characterized by a deterministic $L \times K$ real-valued *transmitter-receiver channel matrix* $\mathbf{G}$ (with non-negative elements), while the channel between the interferer and the receiver is characterized by a deterministic $L \times L$ real-valued double-keyhole *interferer-receiver channel matrix* $\mathbf{H}$ as in (78) of Subsection VI-A. If the transmitted information-bearing symbol (a non-negative real number) over a symbol time interval is s_t, the $K \times 1$ real-valued transmit weight vector or *transmit beamforming vector* is $\mathbf{v}$, and the $L \times 1$ symbol vector of the interferer is $\mathbf{s}_{int}$, then the $L \times 1$ received signal vector $\mathbf{r}$ is given by

$$\mathbf{r} = \mathbf{G}\mathbf{v}s_t + \mathbf{H}\mathbf{s}_{int} + \mathbf{n}, \tag{86}$$

where $\mathbf{n}$ is the AWGN vector with L i.i.d. random elements, each distributed as $\mathcal{N}(0, \sigma_n^2)$. It is assumed that the transmitter and receiver have knowledge of the channel matrices $\mathbf{G}$ and $\mathbf{H}$. From (78), the element in the ith row and the jth column of the interferer-receiver channel matrix $\mathbf{H}$ is given by

$$\big[\mathbf{H}\big]_{i,j} = \frac{1}{2}\left(h^{|i-j|} + h^{-|i-j|}\right),$$
$$i = 1, \ldots, L,\quad j = 1, \ldots, L, \quad h > 1. \tag{87}$$

It is clear from (44) and (48) that the $L-2$ eigenvectors $\mathbf{f}_0(1), \ldots, \mathbf{f}_0(L-2)$ of $\mathbf{H}$ corresponding to the eigenvalue zero are given by

$$\mathbf{f}_0(j) = \begin{bmatrix} f_{1,0}(j) \\ \vdots \\ f_{L,0}(j)\end{bmatrix}, \quad j = 1, \ldots, L-2, \tag{88a}$$

where

$$\begin{aligned} &f_{k,0}(j) = 0, \quad k \in \{1, \ldots, L\} \setminus \{j, j+1, j+2\}, \\ &f_{j,0}(j) = f_{j+2,0}(j) = 1, \quad f_{j+1,0} = -\left(h + h^{-1}\right), \\ &\quad j = 1, \ldots, L-2. \end{aligned}\tag{88b}$$

The information-bearing symbol s_t belongs to a one-sided $\mathcal{M}$-ary amplitude-shift keying (ASK) constellation $\mathcal{S}_\mathcal{M}$ given by

$$\mathcal{S}_\mathcal{M} = \left\{\sqrt{E_1}, \ldots, \sqrt{E_\mathcal{M}}\right\}. \tag{89}$$

Note that E_i is the energy of the ith symbol, $i = 1, \ldots, \mathcal{M}$. The symbol energies $E_1, \ldots, E_M$ are in ascending order, that is,

$$0 \leq E_1 < E_2 < \cdots < E_\mathcal{M}. \tag{90}$$

In each symbol interval, one of the one-sided $\mathcal{M}$-ASK symbols is transmitted with probability $1/\mathcal{M}$.

One of the objectives of the receiver is to cancel the interference and create a situation when the signal-to-noise ratio (SNR) is the maximum, so that the best error performance of symbol-by-symbol detection of the information-bearing symbol s_t can be attained. This objective is achieved by the following:

1) The receiver linearly combines the elements of $\mathbf{r}$ using a receive weight vector or *receive combining vector* $\mathbf{w}$ in such a way that the interference term $\mathbf{H}\mathbf{s}_{int}$ is nullified through the operation $\mathbf{w}^T\mathbf{r}$. This can be done by choosing $\mathbf{w}$ as any one of $\mathbf{f}_0(1), \ldots, \mathbf{f}_0(L-2)$, since $\mathbf{f}_0^T(j)\mathbf{H}\mathbf{s}_{int} = 0$ for all $j \in \{1, \ldots, L-2\}$.

2) Now
$$\mathbf{f}_0^T(j)\mathbf{r} = \mathbf{f}_0^T(j)\mathbf{G}\mathbf{v}s_t + \mathbf{f}_0^T(j)\mathbf{n}, \quad j = 1, \ldots, L-2, \tag{91}$$
and the SNR as a function of j, which we denote as $\gamma(j)$, is given by
$$\gamma(j) = \frac{\left(\mathbf{f}_0^T(j)\mathbf{G}\mathbf{v}\right)^2 s_t^2}{\mathbb{E}\left[\left|\mathbf{f}_0^T(j)\mathbf{n}\right|^2\right]}, \quad j = 1, \ldots, L-2, \tag{92}$$
where $\mathbb{E}[\cdot]$ denotes the expectation operator. For a given j, the numerator on the right-hand side of (92) is maximized by choosing the transmit beamforming vector $\mathbf{v}$ as
$$\mathbf{v} = \mathbf{G}^T\mathbf{f}_0(j). \tag{93}$$
Moreover, we have
$$\begin{aligned}\mathbb{E}\left[\left|\mathbf{f}_0^T(j)\mathbf{n}\right|^2\right] &= \mathbf{f}_0^T(j)\mathbb{E}\left[\mathbf{n}\mathbf{n}^T\right]\mathbf{f}_0(j) \\ &= \sigma_n^2\left[2 + \left(h + h^{-1}\right)^2\right] \\ &= \sigma_n^2\left(4 + h^2 + h^{-2}\right).\end{aligned} \tag{94}$$
Substituting (93) and (94) in (92), we get
$$\gamma(j) = \frac{\left(\mathbf{f}_0^T(j)\mathbf{G}\mathbf{G}^T\mathbf{f}_0(j)\right)^2 s_t^2}{\sigma_n^2\left(4 + h^2 + h^{-2}\right)}, \quad j = 1, \ldots, L-2. \tag{95}$$

3) Let the $L \times K$ transmitter-receiver channel matrix $\mathbf{G}$ be expressed in terms of its rows as
$$\mathbf{G} = \begin{bmatrix}\mathbf{g}_1^T \\ \vdots \\ \mathbf{g}_L^T\end{bmatrix}, \tag{96}$$
where $\mathbf{g}_k$ is $K \times 1$ and $\mathbf{g}_k^T$ denotes the kth row of $\mathbf{G}$ for $k = 1, \ldots, L$. Substituting (88) and (96) in (95), we get
$$\gamma(j) = \frac{\left\|\mathbf{g}_j - \left(h + h^{-1}\right)\mathbf{g}_{j+1} + \mathbf{g}_{j+2}\right\|^4 s_t^2}{\sigma_n^2\left(4 + h^2 + h^{-2}\right)}, \quad j = 1, \ldots, L-2, \tag{97}$$
where $\|\cdot\|$ denotes the Euclidean norm. The SNR $\gamma(j)$ in (97) is maximized when $j = j_{\max}$, where
$$j_{\max} = \arg\max_{j\in\{1,\ldots,L-2\}} \left\|\mathbf{g}_j - \left(h + h^{-1}\right)\mathbf{g}_{j+1} + \mathbf{g}_{j+2}\right\|^2. \tag{98}$$
We therefore choose the transmit beamforming vector as $\mathbf{v} = \mathbf{v}_{\max}$, where
$$\mathbf{v}_{\max} = \mathbf{g}_{j_{\max}} - \left(h + h^{-1}\right)\mathbf{g}_{j_{\max}+1} + \mathbf{g}_{j_{\max}+2}, \tag{99}$$
and the receive combining vector as $\mathbf{w} = \mathbf{w}_{\max}$, where
$$\mathbf{w}_{\max} = \mathbf{f}_0(j_{\max}). \tag{100}$$

From (88), (91), (96), (99), and (100), we get
$$\mathbf{w}_{\max}^T\mathbf{r} = \|\mathbf{v}_{\max}\|^2 s_t + \mathbf{w}_{\max}^T\mathbf{n}, \tag{101}$$
where $\mathbf{w}_{\max}^T\mathbf{n} \sim \mathcal{N}\left(0, \sigma_n^2\left(4 + h^2 + h^{-2}\right)\right)$. Using the conditional probability density function of $\left(\mathbf{w}_{\max}^T\mathbf{n}\right)^2$, conditioned on s_t, the decision rule for the optimum noncoherent one-sided $\mathcal{M}$-ASK symbol detector is obtained as
$$\hat{s}_t = \arg\max_{s_t\in\{\sqrt{E_1},\ldots,\sqrt{E_\mathcal{M}}\}} \exp\left\{-\frac{s_t^2\|\mathbf{v}_{\max}\|^4}{2\sigma_n^2\left(4 + h^2 + h^{-2}\right)}\right\} \times \cosh\left(\frac{s_t\|\mathbf{v}_{\max}\|^2\left|\mathbf{w}_{\max}^T\mathbf{r}\right|}{\sigma_n^2\left(4 + h^2 + h^{-2}\right)}\right). \tag{102}$$

C. LOW SNR APPROXIMATION OF MUTUAL INFORMATION OF ONE-SIDED ASK IN RAYLEIGH FADING WITH NONCOHERENT ENERGY DETECTION

Consider a single-input multiple-output (SIMO) digital wireless communication system in flat Rayleigh fading, with one transmit antenna and L receive antennas, that performs noncoherent energy detection of one-sided $\mathcal{M}$-ASK symbols from the $\mathcal{M}$-ary constellation $\mathcal{S}_\mathcal{M}$ given by (89) and having symbol energies $E_1, \ldots, E_\mathcal{M}$ in ascending order as in (90). If the transmitted information-bearing one-sided $\mathcal{M}$-ASK symbol (a non-negative real number) over a symbol time interval is s, then the $L \times 1$ complex baseband received signal vector $\mathbf{r}$ for this SIMO system is given by [14]
$$\mathbf{r} = \mathbf{h}s + \mathbf{n}, \tag{103}$$
where $\mathbf{h}$ is the random complex normal fading gain vector distributed as $\mathcal{CN}\left(\mathbf{0}_L, \sigma_h^2\mathbf{I}_L\right)$ and $\mathbf{n}$ is the complex normal AWGN vector distributed as $\mathcal{CN}\left(\mathbf{0}_L, \sigma_n^2\mathbf{I}_L\right)$, with $\mathbf{0}_L$ denoting the $L \times 1$ vector of zeros and $\mathbf{I}_L$ denoting the $L \times L$ identity matrix. The signal-plus-noise ratios over consecutive symbols are assumed to have a common ratio R given by
$$R = \frac{\left(E_{i+1}\sigma_h^2 + \sigma_n^2\right)}{\left(E_i\sigma_h^2 + \sigma_n^2\right)}, \quad i = 1, \ldots, \mathcal{M}-1. \tag{104}$$
It is clear from (90) and (104) that $R > 1$. Let the $\mathcal{M} \times 1$ *input probability vector* $\mathbf{p}$ be expressed as
$$\underline{p} = \begin{bmatrix}\Pr\left[s = \sqrt{E_1}\right] \\ \vdots \\ \Pr\left[s = \sqrt{E_\mathcal{M}}\right]\end{bmatrix} = \begin{bmatrix}p_1 \\ \vdots \\ p_\mathcal{M}\end{bmatrix}, \tag{105}$$
where
$$\sum_{i=1}^{\mathcal{M}} p_i = 1, \quad 0 \le p_i \le 1, \quad i = 1, \ldots, \mathcal{M}. \tag{106}$$

The *mutual information* between the *input* s and the *output* $\mathbf{r}$ under the low SNR condition $(R-1) \ll 1$, which is denoted as $\mathcal{I}_{loSNR}(s; \mathbf{r})$, is expressed as [14, eq. (50)]
$$\mathcal{I}_{loSNR}(s; \mathbf{r}) = -\frac{L}{2\ln 2} + \frac{L}{4\ln 2}\mathbf{p}^T\mathbf{T}_L(R)\mathbf{p}, \tag{107}$$
where $\mathbf{T}_L(\cdot)$ is given by (1). The mutual information in (107) can be maximized over $\mathbf{p}$ subject to the constraints (106) to obtain the channel capacity.

VII. CONCLUSION

For the $L \times L$ rank-two matrix $\mathbf{T}_L(C)$, where $C \in \mathbb{C}\setminus\{-1, 0, 1\}$, closed form expressions for the eigenvalues and eigenvectors are presented. For $L \geq 4$, a method of complex orthogonalization of the eigenvectors corresponding to the eigenvalue zero by using the symmetric and skew-symmetric properties of the eigenvectors is described. The special cases of $C \in \mathbb{R}\setminus\{-1, 0, 1\}$ and $|C| = 1, \Im(C) \neq 0$ are discussed, and applications of the results to the performance evaluation of wireless communication systems using multiple antennas at the transmitter and/or the receiver are shown.

APPENDIX A
PROOF OF PROPOSITION 1

Proof: We have $L = 2M$, where $M \in \mathbb{N} - \{1\}$. Consider the eigenvector $\mathbf{e}_{(-)}$. The ith row element of $\mathbf{T}_{2M}(C)\mathbf{e}_{(-)}$, where $i = 1, \ldots, 2M$, is $\sum_{k=1}^{2M} r_{i-k} e_{k,-}$, which can be expressed using (13), (32), and (43b) as

$$\begin{aligned}\sum_{k=1}^{2M} r_{i-k} e_{k,-} &= \frac{1}{(C - C^{-1})} \\ &\quad \times \sum_{k=1}^{2M} \left(C^{i-k} + C^{k-i}\right) \\ &\quad \times \begin{pmatrix} C^{M+1-k} - C^{k-M-1} \\ +C^{M-k} - C^{k-M} \end{pmatrix} \\ &= 2M(v_{M+1-i} + v_{M-i}) \\ &\quad + \frac{C^{M+i+1}}{(C - C^{-1})} \sum_{k=1}^{2M} C^{-2k} \\ &\quad + \frac{C^{M+i}}{(C - C^{-1})} \sum_{k=1}^{2M} C^{-2k} \\ &\quad - \frac{C^{-M-i-1}}{(C - C^{-1})} \sum_{k=1}^{2M} C^{2k} \\ &\quad - \frac{C^{-M-i}}{(C - C^{-1})} \sum_{k=1}^{2M} C^{2k}. \end{aligned} \tag{108}$$

Now

$$\begin{aligned}\sum_{k=1}^{2M} C^{-2k} &= \frac{C^{-2}\left(1 - C^{-4M}\right)}{\left(1 - C^{-2}\right)} \\ &= C^{-2M-1} v_{2M}, \end{aligned} \tag{109a}$$

$$\begin{aligned}\sum_{k=1}^{2M} C^{2k} &= \frac{C^{2}\left(C^{4M} - 1\right)}{\left(C^{2} - 1\right)} \\ &= C^{2M+1} v_{2M}. \end{aligned} \tag{109b}$$

Substitution of (109) in (108) and subsequent simplification results in

$$\begin{aligned}\sum_{k=1}^{2M} r_{i-k} e_{k,-} &= (2M - v_{2M})(v_{M+1-i} + v_{M-i}) \\ &= \lambda_{(-)} e_{i,-}, \quad i = 1, \ldots, 2M. \end{aligned} \tag{110}$$

Now consider the eigenvector $\mathbf{e}_{(+)}$. The ith row element of $\mathbf{T}_{2M}(C)\mathbf{e}_{(+)}$, where $i = 1, \ldots, 2M$, is $\sum_{k=1}^{2M} r_{i-k} e_{k,+}$, which can be expressed using (13), (32), and (43c) as

$$\begin{aligned}\sum_{k=1}^{2M} r_{i-k} e_{k,+} &= \frac{1}{(C - C^{-1})} \\ &\quad \times \sum_{k=1}^{2M} \left(C^{i-k} + C^{k-i}\right) \\ &\quad \times \begin{pmatrix} C^{M+1-k} - C^{k-M-1} \\ -C^{M-k} + C^{k-M} \end{pmatrix} \\ &= 2M(v_{M+1-i} - v_{M-i}) \\ &\quad + \frac{C^{M+i+1}}{(C - C^{-1})} \sum_{k=1}^{2M} C^{-2k} \\ &\quad - \frac{C^{M+i}}{(C - C^{-1})} \sum_{k=1}^{2M} C^{-2k} \\ &\quad - \frac{C^{-M-i-1}}{(C - C^{-1})} \sum_{k=1}^{2M} C^{2k} \\ &\quad + \frac{C^{-M-i}}{(C - C^{-1})} \sum_{k=1}^{2M} C^{2k}. \end{aligned} \tag{111}$$

Substitution of (109) in (111) and subsequent simplification results in

$$\begin{aligned}\sum_{k=1}^{2M} r_{i-k} e_{k,+} &= (2M + v_{2M})(v_{M+1-i} - v_{M-i}) \\ &= \lambda_{(+)} e_{i,+}, \quad i = 1, \ldots, 2M. \end{aligned} \tag{112}$$

Consider next the eigenvector $\mathbf{e}_0(j)$, where $j = 1, \ldots, 2M - 2$. The ith row element of $\mathbf{T}_{2M}(C)\mathbf{e}_0(j)$, where $i = 1, \ldots, 2M$, is $\sum_{k=1}^{2M} r_{i-k} e_0(j)$; this can be expressed using (44b) as

$$\sum_{k=1}^{2M} r_{i-k} e_{k,0}(j) = r_{i-j} - \left(C + C^{-1}\right) r_{i-j-1} + r_{i-j-2}. \tag{113}$$

Applying recurrence (14) to (113), we get

$$\sum_{k=1}^{2M} r_{i-k} e_{k,0}(j) = 0, \quad i = 1, \ldots, 2M, \quad j = 1, \ldots, 2M - 2. \tag{114}$$

From (110), we conclude that $\mathbf{e}_{(-)}$, given by (43a) and (43b), is the eigenvector corresponding to the eigenvalue $\lambda_{(-)}$, given by (42a); from (112), we conclude that $\mathbf{e}_{(+)}$, given by (43a) and (43c), is the eigenvector corresponding to the eigenvalue $\lambda_{(+)}$, given by (42b); and from (114), we conclude that $\mathbf{e}_0(1), \ldots, \mathbf{e}_0(2M - 2)$, given by (44), are the $2M - 2$ eigenvectors corresponding to the zero eigenvalue of multiplicity $2M - 2$. This proves Proposition 1. ■

APPENDIX B
PROOF OF PROPOSITION 2

Proof: We have $L = 2M + 1$, where $M \in \mathbb{N}$. Consider the eigenvector $\mathbf{e}_{(-)}$. The ith row element of $\mathbf{T}_{2M+1}(C)\mathbf{e}_{(-)}$, where $i = 1, \ldots, 2M+1$, is $\sum_{k=1}^{2M+1} r_{i-k} e_{k,-}$, which can be expressed using (13), (32), and (47b) as

$$\sum_{k=1}^{2M+1} r_{i-k} e_{k,-} = \frac{1}{\left(C - C^{-1}\right)} \sum_{k=1}^{2M+1} \left(C^{i-k} + C^{k-i}\right) \times \left(C^{M+1-k} - C^{k-M-1}\right)$$
$$= (2M+1) v_{M+1-i} + \frac{C^{M+i+1}}{\left(C - C^{-1}\right)} \sum_{k=1}^{2M+1} C^{-2k} - \frac{C^{-M-i-1}}{\left(C - C^{-1}\right)} \sum_{k=1}^{2M+1} C^{2k}. \tag{115}$$

Now

$$\sum_{k=1}^{2M+1} C^{-2k} = \frac{C^{-2}\left(1 - C^{-4M-2}\right)}{\left(1 - C^{-2}\right)} = C^{-2M-2} v_{2M+1}, \tag{116a}$$

$$\sum_{k=1}^{2M+1} C^{2k} = \frac{C^{2}\left(C^{4M+2} - 1\right)}{\left(C^{2} - 1\right)} = C^{2M+2} v_{2M+1}. \tag{116b}$$

Substitution of (116) in (115) and subsequent simplification results in

$$\sum_{k=1}^{2M+1} r_{i-k} e_{k,-} = (2M+1 - v_{2M+1}) v_{M+1-i} = \lambda_{(-)} e_{i,-}, \quad i = 1, \ldots, 2M+1. \tag{117}$$

Now consider the eigenvector $\mathbf{e}_{(+)}$. The ith row element of $\mathbf{T}_{2M+1}(C)\mathbf{e}_{(+)}$, where $i = 1, \ldots, 2M+1$, is $\sum_{k=1}^{2M+1} r_{i-k} e_{k,+}$, which can be expressed using (13) and (47c) as

$$\sum_{k=1}^{2M+1} r_{i-k} e_{k,+} = \sum_{k=1}^{2M+1} \left(C^{i-k} + C^{k-i}\right) \times \left(C^{M+1-k} + C^{k-M-1}\right)$$
$$= (2M+1) r_{M+1-i} + C^{M+i+1} \sum_{k=1}^{2M+1} C^{-2k} + C^{-M-i-1} \sum_{k=1}^{2M+1} C^{2k}. \tag{118}$$

Substitution of (116) in (118) and subsequent simplification results in

$$\sum_{k=1}^{2M+1} r_{i-k} e_{k,+} = (2M+1 + v_{2M+1}) r_{M+1-i} = \lambda_{(+)} e_{i,+}, \quad i = 1, \ldots, 2M+1. \tag{119}$$

Consider next the eigenvector $\mathbf{e}_0(j)$, where $j = 1, \ldots, 2M-1$. The ith row element of $\mathbf{T}_{2M+1}(C)\mathbf{e}_0(j)$, where $i = 1, \ldots, 2M+1$, is $\sum_{k=1}^{2M+1} r_{i-k} e_0(j)$; this can be expressed using (48b) as

$$\sum_{k=1}^{2M+1} r_{i-k} e_{k,0}(j) = r_{i-j} - \left(C + C^{-1}\right) r_{i-j-1} + r_{i-j-2}. \tag{120}$$

Applying recurrence (14) to (120), we get

$$\sum_{k=1}^{2M+1} r_{i-k} e_{k,0}(j) = 0, \quad i = 1, \ldots, 2M+1, \quad j = 1, \ldots, 2M-1. \tag{121}$$

From (117), we conclude that $\mathbf{e}_{(-)}$, given by (47a) and (47b), is the eigenvector corresponding to the eigenvalue $\lambda_{(-)}$, given by (46a); from (119), we conclude that $\mathbf{e}_{(+)}$, given by (47a) and (47c), is the eigenvector corresponding to the eigenvalue $\lambda_{(+)}$, given by (46b); and from (121), we conclude that $\mathbf{e}_0(1), \ldots, \mathbf{e}_0(2M-1)$, given by (48), are the $2M-1$ eigenvectors corresponding to the zero eigenvalue of multiplicity $2M-1$. This proves Proposition 2. ■

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