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RESEARCH ARTICLE

The Spectrum of Weighted Lexicographic Product on Self-Complementary Graphs

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ABSTRACT The lexicographic product, a powerful binary operation in graph theory, offers methods for creating a novel graph by establishing connections between each vertex of one graph and every vertex of another. Beyond its fundamental nature, this operation is found in various applications across computer science disciplines, including network analysis, data mining, and optimization. In this paper, we give a definition of the weight function to the lexicographic product graph $G[H]$, which enables us to capture the intricate interplay among the vertices of the constituent graphs and facilitate a deeper understanding of their relationships. We derive an expression for the spectrum of $G[H]$ by using the spectrums of G and H if the graph H is a self-complementary graph. Through a systematic analysis and careful computations, we derive a comprehensive expression for the spectrum of $G[H]$. Remarkably, we reveal an intriguing characteristic pertaining to self-complementarity within the weighted lexicographic product graph. Specifically, we show that the weighted lexicographic product graph can be self-complementary if this graph is a product of two connected weighted self-complementary graphs. Furthermore, we delve into the geometric properties of the lexicographic product, specifically examining the Ricci curvature for the product of two regular graphs. Through rigorous analysis, we have discovered that the lexicographic product of two regular graphs exhibits a lower bound on the Ricci curvature.

INDEX TERMS Lexicographic product, self-complementary, spectrum, Ricci curvature.

I. INTRODUCTION

A product graph is a mathematical structure which is used to represent the relationship between two or more graphs. It is created by taking the Cartesian product of the vertex sets and edge sets of the input graphs. The resulting product graph has vertices that are pairs of vertices from the input graphs, and edges that connect pairs of vertices according to a certain rule. The various types of graph products differ in terms of the specific mathematical condition that is used to create them. For example, the Cartesian product [13], [39], tensor product [22], [35], lexicographic product [6], [11], [16], [19], [29], and strong product [5], [34] each utilize a different set of rules to combine the input graphs into a new, composite graph. Depending on the particular requirements of a given problem, one type of graph product may be more suitable than another for modeling or analyzing relationships between graphs.

The lexicographic product, a powerful mathematical tool, has gained significant recognition and utilization in various fields including network analysis, data mining, and optimization. Furthermore, its versatile applications have contributed to a deeper understanding of complex systems, efficient data analysis, and problem-solving in diverse domains. Network analysis benefits greatly from the lexicographic product as it offers a robust framework for modeling systems characterized by strong interactions between nodes. For instance, social networks and communication networks heavily rely on the lexicographic product to capture the intricate dynamics and information flow within these interconnected structures. By referring to the works of Hu [23] and Mao [27] in

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their respective publications, researchers have successfully employed the lexicographic product to unravel the underlying mechanisms and identify crucial nodes that drive the overall network behavior. In the realm of data mining, the lexicographic product has emerged as a valuable tool for analyzing vast datasets and uncovering hidden patterns or relationships between different variables. The works of Chou [8] and Rute et al. [33], as cited in their publications, demonstrate how the lexicographic product facilitates the extraction of meaningful insights from complex data structures. By leveraging its capabilities, researchers can efficiently navigate through large volumes of information, identify relevant patterns, and make informed decisions based on the discovered knowledge. Moreover, the lexicographic product proves instrumental in optimization, enabling the resolution of intricate problems by breaking them down into smaller, more manageable sub-problems. This approach, as highlighted in the works of Bissoli et al. [7] and Guo et al. [20], allows for a systematic and efficient solution to complex optimization challenges. By decomposing the problem into smaller components, the lexicographic product simplifies the overall optimization process, leading to improved performance and enhanced decision-making.

The lexicographic product is a fundamental concept in the field of graph theory and has been extensively studied. While much of the research has focused on simple graphs, the lexicographic product was first introduced by Hausdorff [21]. Geller and Stahl [15] discovered that the independence number of a lexicographic product can be easily calculated from the independence numbers of its constituent graphs. Ravindra and Parthasarathy [32] established that a lexicographic product of two graphs is a perfect graph if and only if both factors are perfect. Additionally, Feigenbaum and Schäffer [14] demonstrated that determining whether a graph is a lexicographic product is equally complex as the graph isomorphism problem. These findings can be helpful for researchers to develop novel techniques for analyzing complex systems and solving challenging graph-related problems.

Recently, there has been an increased focus among researchers on understanding various properties of weighted graphs. Interested readers can refer to works [2], [3], [4], [37], [38]. Grigor'yan et al. have investigated the Kazdan-Warner equation on graphs in [18]. Additionally, Grigor'yan [17] focused on the Cartesian product of weighted graphs and demonstrated that all eigenvalues of the Laplace operator on the weighted Cartesian product graph are convex combinations of the eigenvalues of the Laplace operator of the original graphs.

Weighted graphs have important applications in electrical resistance. In circuit analysis, a weighted graph can be used to represent the connections and resistances between components in a circuit. By analyzing the weighted graph, one can calculate the current distribution, potential difference, and power dissipation in the circuit. This is very helpful for circuit design, optimization, and solving practical problems. In a weighted graph, nodes represent the endpoints or connection points in the circuit, while edges represent the wires or devices in the circuit. The weight of an edge represents the resistance value in the circuit. By using network analysis methods such as Kirchhoff's laws and Ohm's law, one can use the weighted graph to solve for voltage, current, power, and other parameters in the circuit. Furthermore, weighted lexicographic product graphs can help understand the topology and characteristics of a circuit. For example, for complex circuit boards or power grid systems, they can be abstracted into simple weighted graphs to simplify the analysis process. By changing the weights, one can simulate variations or faults in different components of the circuit. In summary, weighted graphs have important applications in electrical resistance, making circuit analysis more intuitive and efficient. They assist engineers in designing, optimizing circuits, and solving real-world problems. We refer the readers to the references [12] and [30] for more details.

Lin and Yau [26] established that the Ricci curvature, as defined by Bakry and Emery, of locally finite graphs is lower bounded by -1. Furthermore, they demonstrated that the Ricci curvature, as defined by Ollivier, for the simple random walk on graphs also possesses a lower bound. Chung and Yau's concept of a Ricci flat graph corresponds to a graph with Ricci curvature bounded below by zero. Münch and Wojciechowski [28] exhibited that a lower bound on the Ollivier curvature is equivalent to a particular Lipschitz decay pattern in solutions to the heat equation. Cushing et al. [10] proved that the curvature functions of the Cartesian product of two graphs, G_1 and G_2, equal an abstract product of curvature functions of G_1 and G_2. This paper focuses on analyzing the Ricci curvature of the lexicographic product graph and providing a lower bound for the same.

This paper takes inspiration from Grigor'yan's work on the Cartesian product of weighted graphs, as presented in [17]. In [36], we have already studied the spectral problem of the strong product weighted graphs. Here, we aim to extend our research by exploring the spectral properties of other kinds of weighted graphs, such as the weighted lexicographic product graphs. Conducting these studies will help us to gain a deeper understanding of the behavior and properties of weighted graphs, while also provide valuable insights that can inform future research in this field. The primary challenge in analyzing the connection between the spectrum of two original weighted graphs and their lexicographic product graph arises from the difficulty in assigning a suitable weight function to the weighted lexicographic product graph. Additionally, as Feigenbaum and Schäffer [14] showed, the lexicographic product is closely related to the isomorphism of graphs. To overcome this challenge, we adopt the weight function of the quasi-complement graph, which means that this weighted graph is isomorphic to its quasi-complement weighted graph. This is also a distinguishing factor from the spectral problem of strong product graphs.

The following structure is adopted in the remaining sections of this paper: Section II presents notations and definitions on weighted graphs along with our main results. Section II.*A* provides a formula for obtaining the spectrum of $G[H]$ using the spectrums of G and H, specifically in cases where graph H is a self-complementary graph. Additionally, when both graph G and H are self-complementary graphs, we establish that their weighted lexicographic product graph is also self-complementary. In Section II.*B*, we follow the method presented in [25] to determine the lower bound of Ricci curvature for the lexicographic product of two regular graphs. Finally, Sections III and IV offer the proofs of our results and the conclusion of this paper, respectively.

II. MAIN RESULTS

In this paper, we analyze connected, simple, undirected, and weighted graphs. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. We use $d_G(u, v)$ to represent the distance between two vertices u and v in G. Additionally, we denote the lexicographic product of graph G and H as $G[H]$. Here, we define a graph G as simple if it does not contain loops or multiple edges. We will now provide basic definitions for weighted graphs, which can be found in [3], [9], and [17]. Let V denote a finite discrete space that serves as the set of vertices for graph G, and E denote the set of edges in the graph. In addition, we define an edge weight function $\mu : V \times V \ni (x, y) \mapsto \mu_{xy} \in [0, \infty)$ that satisfies two conditions: (1) $\mu_{xy} = \mu_{yx}$ for every pair of vertices $x, y \in V$, and (2) $\sum_{y\in V} \mu_{xy} < \infty$ for every vertex $x \in V$. The edge weight function $\mu : V \times V \ni (x, y) \mapsto \mu_{xy} \in [0, \infty)$ induces a combinatorial (undirected) graph structure $G = (V, E)$ with a set of vertices V and a set of edges E. Specifically, for any pair of vertices $x, y \in V$, we have $\{x, y\} \in E$ if and only if $\mu_{xy} > 0$, which can be denoted by $x \sim y$. Alternatively, we can consider μ_{xy} as a positive function defined on the set of edges, which is extended to be 0 on non-edge pairs (x, y). Thus, a weighted graph can be represented as $G = (V, \mu)$. We say that a graph G has simple weights if the weight function μ satisfies either $\mu_{xy} = 1$ for all $x \sim y$ or $\mu_{xy} = 0$ for all $x \not\sim y$ in G. Given a weight function μ_{xy} on the edges of a graph $G = (V, E)$, we define a corresponding function on the vertices as follows:

$$\mu(x) = \sum_{y\sim x} \mu_{xy},$$

where $\mu(x)$ is called the weight of vertex x. For instance, if the weight function μ is simple, then $\mu(x)$ is equivalent to the degree of vertex x, denoted by $\deg(x)$.

A. SPECTRUM

Let us begin by recalling the definition of the lexicographic product of two unweighted graphs. Suppose we have two unweighted graphs (X, E_1) and (Y, E_2). The lexicographic product of these graphs is denoted by

$$(V, E) = (X, E_1)[(Y, E_2)],$$

where $V = X \times Y$ is the set of ordered pairs (x, y), and the set E of edges is defined as follows:

$$(x, y) \sim (x', y') \text{ if } \begin{cases} \text{either } x \sim x', \\ \text{or } x = x' \text{ and } y \sim y', \end{cases}$$

where $x, x' \in X$ and $y, y' \in Y$. Here, $|V|$ represents the total number of vertices in the lexicographic product graph, which is equal to the product of the number of vertices in each of the original graphs. The degree of a vertex (x, y) is the sum of the degrees of its corresponding vertices in the original graphs, taking into account the edges between pairs of vertices in X and Y. Finally, $|E|$ denotes the total number of edges, which is the sum of the edges in E_1 multiplied by the number of vertices in Y, plus the edges in E_2 multiplied by the number of vertices in X. When considering the lexicographic product of two weighted graphs, we define the vertex set and edge set of the product graph in a similar way as for unweighted graphs. However, defining an appropriate weight function for the product graph can be challenging. We will construct it through the weight function of the quasi-complement graph of the second graph. We say $\bar{G} = (Y, b)$ is the *quasi-complement* weighted graph of $G = (X, a)$, if $\bar{G} = (Y, b)$ satisfies: (1) $Y = X$; (2) For any $x, x' \in X$, there holds $x \sim x'$ in $G \Leftrightarrow x \not\sim x'$ in $\bar{G}$. We see that $\bar{G}$ is exactly the complement graph of G when both the weight functions are simple. And our construction to the weight function of product graph is as follows:

Definition 1: Let $G = (X, a)$ be a locally finite connected weighted graph and $H = (Y, b)$ be a finite connected weighted graph. Suppose that $\bar{H} = (Y, c)$ is the quasi-complement graph of $H = (Y, b)$. Fix four numbers $p_1, p_2, p_3, p_4 > 0$ and define the lexicographic product graph

$$(V, \mu) = G[H](p_1, p_2, p_3, p_4),$$

as follows: $V = X \times Y$ and the weight μ on V is defined by

$$\mu_{(x,y)(x',y')} = \begin{cases} p_1 a_{xx'} b_{yy'} c(y), & x \sim x' \text{ in } G, y \sim y' \text{ in } H; \\ p_2 a(x) b_{yy'} c(y), & x = x' \text{ in } G, y \sim y' \text{ in } H; \\ p_3 a_{xx'} b(y) c_{yy'}, & x \sim x' \text{ in } G, y \not\sim y' \text{ in } H; \\ p_4 a_{xx'} b(y) c(y), & x \sim x' \text{ in } G, y = y' \text{ in } H; \\ 0, & \textit{otherwise}. \end{cases} \tag{1}$$

Remark 1: For any two distinct vertices (x, y) and (x', y') of $\mathcal{G} = G[H](p_1, p_2, p_3, p_4)$, we observe that

$$d_{\mathcal{G}}((x, y), (x', y')) = \begin{cases} 1, & \text{if } x = x', y' \in N_H(y); \\ 2, & \text{if } x = x', y' \notin N_H(y); \\ d_G(x, x'), & \text{if } x \neq x'. \end{cases}$$

Let us revisit the definition of the Laplace and Markov operators on weighted graphs, as presented in [17]. Consider a locally finite weighted graph (V, μ) with no isolated points. For any function $f : V \to \mathbb{R}$, the function Δf is defined by

$$\Delta f(x) = \frac{1}{\mu(x)} \sum_{y} \mu_{xy}\big(f(y) - f(x)\big). \tag{2}$$

The operator Δ acting on functions on V, is called the weighted Laplace operator of (V, μ). For any function $f : V \to \mathbb{R}$, the function Pf is defined by

$$Pf(x) = \sum_{y} P(x, y)f(y), \tag{3}$$

where the Markov kernel $P(x, y) = \mu_{xy}/\mu(x)$ is the random walk with transition probability of moving from a vertex x to each of its neighbours y. This operator P is called the Markov operator. In this paper, we say a graph $G = (X, a)$ is *isomorphic* to another graph $H = (Y, b)$, if there exists a bijection σ between the vertex sets of G and H which satisfies: (1)$x \sim x'$ in G if and only if $\sigma(x) \sim \sigma(x')$ in H; (2) $a_{xx'} = b_{\sigma(x)\sigma(x')}$, for any $x, x' \in X$. A graph $G = (X, a)$ is a *self-complementary* means $G = (X, a)$ is isomorphic to its quasi-complement weighted graph $\bar{G} = (X, b)$. In view of the map $\sigma : X \to Y$ is a bijection, we know that if $X = Y$, the $\sigma(X)$ is exactly a permutation of X.

Theorem 1: Let $G = (X, a)$ be a finite connected weighted graph with m vertices and $H = (Y, b)$ be a finite connected self-complementary weighted graph with n vertices. Suppose that $\{\alpha_k\}_{k=1}^m$ and $\{\beta_l\}_{l=1}^n$ be the sequences of the eigenvalues of the Markov operators A on X and B on Y respectively, counted with multiplicities. Then all the eigenvalues of the Markov operator P on the lexicographic product $G[H](p_1, p_2, p_3, p_4)$ are given by the sequence

$$\left\{\frac{(p_1 + p_3)\alpha_k\beta_l + p_2\beta_l + p_4\alpha_k}{p_1 + p_2 + p_3 + p_4}\right\}$$

where $k = 1, 2, \ldots, m$ and $l = 1, 2, \ldots, n$.

According to (2) and (3), we know the Laplace operator Δ and the Markov operator P are related by a simple identity

$$\Delta = P - id,$$

where id is the identical operator in $\mathcal{F}$ which is the set of all real-valued functions on V. It is easy to see the same relation holds for the eigenvalues of Δ and P. Hence, by Theorem 1, we have

Corollary 1: Let $G = (X, a)$ be a finite connected weighted graph with m vertices and $H = (Y, b)$ be a finite connected self-complementary weighted graph with n vertices. Suppose that $\{\alpha_k\}_{k=1}^m$ and $\{\beta_l\}_{l=1}^n$ are the sequences of the eigenvalues of the Laplace operators A on X and B on Y respectively, counted with multiplicities. Then all the eigenvalues of the Laplace operator Δ on the lexicographic product $G[H](p_1, p_2, p_3, p_4)$ are given by the sequence

$$\left\{\frac{(p_1 + p_3)\alpha_k\beta_l + (p_1 + p_2 + p_3)\beta_l + (p_1 + p_3 + p_4)\alpha_k}{p_1 + p_2 + p_3 + p_4}\right\}$$

where $k = 1, 2, \ldots, m$ and $l = 1, 2, \ldots, n$.

Theorem 2: Let $G = (X, a)$ be a r-regular locally finite connected graph, $H = (Y, b)$ be a k-regular connected finite graph with n vertices. If G and H are two self-complementary graphs with simple weights, then their lexicographic product graph $G[H](\frac{1}{k}, \frac{1}{rk}, \frac{1}{k}, \frac{1}{k^2})$ is a $(r(2k + 1) + k)$-regular graph with simple weights. Further, if the quasi-complement graph of $G[H](\frac{1}{k}, \frac{1}{rk}, \frac{1}{k}, \frac{1}{k^2})$ also has a simple weight, then $G[H](\frac{1}{k}, \frac{1}{rk}, \frac{1}{k}, \frac{1}{k^2})$ is a self-complementary graph.

Theorem 3: Let $G = (X, a)$ be a locally finite connected self-complementary graph and $H = (Y, b)$ be a finite connected self-complementary graph. Then there exists a weight function on the quasi-complement graph of their lexicographic product graph $G[H](p_1, p_2, p_3, p_4)$ such that $G[H](p_1, p_2, p_3, p_4)$ is also a self-complementary graph.

B. RICCI CURVATURE

We will use similar notations as in [25] and [31]. A *probability distribution* over the vertex-set $V(G)$ is a mapping $m : V(G) \to [0, 1]$ satisfying $\sum_{x \in V(G)} m(x) = 1$. Let us assume that we have two probability distributions m_1 and m_2, both of them have finite support. A *coupling* between m_1 and m_2 is a mapping $A : V(G) \times V(G) \to [0, 1]$ with finite support so that

$$\sum_{y \in V(G)} A(x, y) = m_1(x) \text{ and } \sum_{x \in V(G)} A(x, y) = m_2(y).$$

Let $d(x, y)$ be the graph distance between two vertices x and y. The *transportation distance* between two probability distributions m_1 and m_2 is defined as follows:

$$W(m_1, m_2) = \inf_{A} \sum_{x, y \in V(G)} A(x, y)d(x, y),$$

where the infimum is taken over all coupling A between m_1 and m_2. For any vertex $x \in V(G)$, let $N(x)$ denote the set of neighborhood of x, i.e., $N(x) = \{y \in V(G) : y \sim x \text{ in } G\}$. For any $\alpha \in [0, 1]$ and any vertex x, the *probability measure* m_x^α is defined as

$$m_x^\alpha(v) = \begin{cases} \alpha, & \text{if } v = x; \\ \dfrac{1 - \alpha}{\deg(x)}, & \text{if } v \in N(x); \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

For any $x, y \in V$, we define α-Ricci-curvature k_α to be

$$k_\alpha(x, y) = 1 - \frac{W(m_x^\alpha, m_y^\alpha)}{d(x, y)},$$

and the Ricci curvature at (x, y) in the graph is

$$k(x, y) = \lim_{\alpha \to 1} \frac{k_\alpha(x, y)}{1 - \alpha}.$$

Theorem 4: Let G be a d_G-regular locally finite graph and H be a d_H-regular finite graph with n vertices. For $u_1 \sim u_2$ in G, $v_1, v_2 \in V(H)$, the Ricci curvature of $G[H]$ is bounded below, that is

$$k^{G[H]}((u_1, v_1), (u_2, v_2)) \geq \frac{-2nd_G}{nd_G + d_H}.$$

Theorem 5: Let G be a d_G-regular locally finite graph and H be a d_H-regular finite graph with n vertices. For $u \in V(G)$, $v_1 \sim v_2$ in H, the Ricci curvature of $G[H]$ is bounded below, that is

$$k^{G[H]}((u, v_1), (u, v_2)) \geq \frac{d_H k^H(v_1, v_2) - nd_G}{nd_G + d_H}.$$

From Lemma 2.3 in [25], we know if $k(u, v) \geq k_0$ for any $u \sim v$ in G, then $k(u, v) \geq k_0$ for any pair of vertices (u, v). Hence, we have

Corollary 2: Let G be a d_G-regular locally finite graph and H be a d_H-regular finite graph with n vertices. For $u_1, u_2 \in V(G)$, $v_1, v_2 \in V(H)$, the Ricci curvature of $G[H]$ is bounded below, that is

$$k^{G[H]}((u_1, v_1), (u_2, v_2)) \geq \min\left\{\frac{-2nd_G}{nd_G + d_H}, \frac{d_H k^H(v_1, v_2) - nd_G}{nd_G + d_H}\right\}.$$

III. PROOFS

Theorem 6: Let $G = (X, a)$ be a locally finite connected weighted graph and $H = (Y, b)$ be a finite connected weighted graph, and $\bar{H} = (Y, c)$ be the quasi-complement graph of $H = (Y, b)$. Suppose that A, B and C are the Markov kernels on X in G, Y in H and Y in $\bar{H}$, respectively. Then the Markov kernel P on the lexicographic product $G[H](p_1, p_2, p_3, p_4) = (V, \mu)$ is given by, as shown in the equation at the bottom of the page, where $x, x' \in X$, $y, y' \in Y$ and p_1, p_2, p_3, p_4 are four given positive numbers.

Proof: From the definition, the weight on the vertices of V is

$$\begin{aligned}\mu(x, y) &= \sum_{(x',y')\sim(x,y)} \mu_{(x,y)(x',y')} \\ &= \sum_{\substack{x'\sim x\\ y'\sim y}} \mu_{(x,y)(x',y')} + \sum_{\substack{x'=x\\ y'\sim y}} \mu_{(x,y)(x',y')} + \sum_{\substack{x'\sim x\\ y'\nsim y}} \mu_{(x,y)(x',y')} \\ &\quad + \sum_{\substack{x'\sim x\\ y'=y}} \mu_{(x,y)(x',y')} \\ &= p_1 c(y) \sum_{\substack{x'\sim x\\ y'\sim y}} a_{xx'} b_{yy'} + p_2 a(x) c(y) \sum_{\substack{x'=x\\ y'\sim y}} b_{yy'} \\ &\quad + p_3 b(y) \sum_{\substack{x'\sim x\\ y'\nsim y}} a_{xx'} c_{yy'} + p_4 b(y) c(y) \sum_{\substack{x'\sim x\\ y'=y}} a_{xx'} \\ &= (p_1 + p_2 + p_3 + p_4) a(x) b(y) c(y).\end{aligned}$$

In the case $x \sim x'$, $y \sim y'$, by (1), we have

$$\begin{aligned}P((x, y), (x', y')) &= \frac{\mu_{(x,y)(x',y')}}{\mu(x, y)} \\ &= \frac{p_1 a_{xx'} b_{yy'} c(y)}{(p_1 + p_2 + p_3 + p_4) a(x) b(y) c(y)} \\ &= \frac{p_1}{p_1 + p_2 + p_3 + p_4} A(x, x') B(y, y'),\end{aligned}$$

and other cases are treated similarly. ■

Lemma 1: If $G = (X, a)$, $H = (Y, b)$ and $\bar{H} = (Y, c)$ are r-regular, k-regular and $(|Y| - k - 1)$-regular graphs with simple weights, respectively. Then their lexicographic product

$$G[H]\left(\frac{1}{|Y| - k - 1}, \frac{1}{r(|Y| - k - 1)}, \frac{1}{k}, \frac{1}{k(|Y| - k - 1)}\right)$$

is a $(r|Y| + k)$-regular graph with a simple weight.

Proof: Since a, b and c are simple weights of G, H and $\bar{H}$ with the regularity of graphs G and H, we have

$$\begin{aligned}a(x) &= \deg(x) = r, \\ b(y) &= \deg(y) = k, \\ c(y) &= \deg(y) = |Y| - k - 1.\end{aligned}$$

Hence, by the definition of the weight function on lexicographic product, we get

$$\mu_{(x,y)(x',y')} = \begin{cases} p_1(|Y| - k - 1), & x \sim x' \text{ in } G, y \sim y' \text{ in } H; \\ p_2 r(|Y| - k - 1), & x = x' \text{ in } G, y \sim y' \text{ in } H; \\ p_3 k, & x \sim x' \text{ in } G, y \nsim y' \text{ in } H; \\ p_4 k(|Y| - k - 1), & x \sim x' \text{ in } G, y = y' \text{ in } H; \\ 0, & \textit{otherwise}. \end{cases} \quad (5)$$

Therefore, when taking the parameters p_1, p_2, p_3, p_4 as

$$\begin{aligned}p_1 &= \frac{1}{|Y| - k - 1}, \\ p_2 &= \frac{1}{r(|Y| - k - 1)}, \\ p_3 &= \frac{1}{k}, \\ p_4 &= \frac{1}{k(|Y| - k - 1)},\end{aligned}$$

we have $\mu_{(x,y)(x',y')} = 1$ for any $(x, y) \sim (x', y')$ and $\mu_{(x,y)(x',y')} = 0$ for any $(x, y) \nsim (x', y')$ in $G[H]$, that is the weight μ is also simple. ■

Now, we shall prove our main results for the spectrum of the Lexicographic Product.

$$P((x, y), (x', y')) = \begin{cases} \frac{p_1}{p_1 + p_2 + p_3 + p_4} A(x, x') B(y, y'), & x \sim x' \text{ in } G, y \sim y' \text{ in } H; \\ \frac{p_2}{p_1 + p_2 + p_3 + p_4} B(y, y'), & x = x' \text{ in } G, y \sim y' \text{ in } H; \\ \frac{p_3}{p_1 + p_2 + p_3 + p_4} A(x, x') C(y, y'), & x \sim x' \text{ in } G, y \nsim y' \text{ in } H; \\ \frac{p_4}{p_1 + p_2 + p_3 + p_4} A(x, x'), & x \sim x' \text{ in } G, y = y' \text{ in } H; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof of Theorem 1:

Proof: Assume that $\bar{H} = (Y, c)$ is a quasi-complement graph of H and C is the Markov operator on $\bar{H}$. Since H is a self-complementary graph, there exists a bijection $\sigma : Y \to Y$ such that H is isomorphic to $\bar{H}$. Suppose $Y = \{y_1, y_2, \ldots, y_n\}$, then $(\sigma(y_1)\sigma(y_2)\cdots\sigma(y_n))$ is a permutation of $(y_1y_2\cdots y_n)$ essentially. For convenience, we can denote $(\sigma(y_1), \sigma(y_2), \ldots, \sigma(y_n))$ by $(y_{s_1}, y_{s_2}, \ldots, y_{s_n})$ and $(s_1s_2\cdots s_n)$ is some permutation of $(12\cdots n)$. Let f be an eigenfunction of A with the eigenvalue α and g be an eigenfunction of B with the eigenvalue β. That is to say, for any $x \in X$ and $y \in Y$, there holds

$$Af(x) = \sum_{x' \in X} A(x, x')f(x') = \alpha f(x),$$
$$Bg(y) = \sum_{y' \in Y} B(y, y')g(y') = \beta g(y).$$

Now we claim that for operators B and C, there holds if the function g is the eigenfunction of B with the eigenvalue β, then g is also the eigenfunction of C with the eigenvalue β, i.e., if $Bg = \beta g$ then $Cg = \beta g$. Suppose that $H_B = (B(y_i, y_j))_{n\times n}$ and $H_C = (C(y_i, y_j))_{n\times n}$ are the matrix of operator B and C respectively, where $i, j \in \{1, 2, \ldots n\}$. For our goals, we will construct an invertible matrix $M_{n\times n}$ such that $M^{-1}H_CM = H_B$.

Step 1: If $s_1 = 1$, that is $\sigma(y_1) = y_1$, we take $P_1 = I_{n\times n}$, here $I_{n\times n}$ is the identity matrix. If $s_1 \neq 1$, we take P_1 as the elementary matrix from exchanging the positions of columns 1 and s_1 in the identity matrix $I_{n\times n}$. Now, we suppose that the label about columns of H_CP_1 be $(s_1, 2^{(1)}, 3^{(1)}, \ldots, n^{(1)})^T$.

Step 2: If $s_2 = 2^{(1)}$, we take $P_2 = I_{n\times n}$. If $s_2 \neq 2^{(1)}$, we take P_2 as the elementary matrix from exchanging the positions of columns $2^{(1)}$ and s_2 in the identity matrix $I_{n\times n}$. And suppose that the label about columns of $H_CP_1P_2$ be $(s_1, s_2, 3^{(2)}, \ldots, n^{(2)})^T$.

We continue this process for $n - 1$ times, then the label about columns of $H_CP_1P_2\cdots P_{n-1}$ is $(s_1, s_2, \ldots, s_n)^T$. Taking into account the symmetry of the matrix H_C, we know the label about rows of $P_{n-1}\cdots P_2P_1H_CP_1P_2\cdots P_{n-1}$ also is $(s_1, s_2, \ldots, s_n)$. Let $M = P_1P_2\cdots P_{n-1}$. It is easy to see that $M^T = P_{n-1}\cdots P_2P_1$ and $M^{-1} = M^T$. That is, $P_{n-1}\cdots P_2P_1H_CP_1P_2\cdots P_{n-1} = M^TH_CM$. In view of $c_{yy'} = b_{\sigma^{-1}(y)\sigma^{-1}(y')}$, we obtain

$$c(y) = \sum_{y'\sim y \text{ in } G_2} c_{yy'} = \sum_{\sigma(y')\sim\sigma(y) \text{ in } \bar{G}_2} b_{\sigma^{-1}(y)\sigma^{-1}(y')}$$
$$= b(\sigma^{-1}(y)),$$

then

$$C(y, y') = \frac{c_{yy'}}{c(y)} = \frac{b_{\sigma^{-1}(y)\sigma^{-1}(y')}}{b(\sigma^{-1}(y))} = B(\sigma^{-1}(y), \sigma^{-1}(y')).$$

Hence, the element of s_i-th row and s_j-th column in matrix M^TH_CM is $C(y_{s_i}, y_{s_j}) = C(\sigma(y_i), \sigma(y_j)) = B(y_i, y_j)$. Therefore, we have $M^TH_CM = H_B$, i.e., the matrix H_C is similar to the matrix H_B, then H_C and H_B have the same eigenvalues. As well as, if

$$H_B(g(y_1), g(y_2), \ldots, g(y_n))^T = \beta(g(y_1), g(y_2), \ldots, g(y_n))^T,$$

then

$$H_CM(g(y_1), g(y_2), \ldots, g(y_n))^T$$
$$= \beta M(g(y_1), g(y_2), \ldots, g(y_n))^T.$$

From the construction process of M, we can see that the action on some matrix by left multiplying these elementary matrices $P_1, P_2, \ldots, P_{n-1}$ is equivalent to transform the rows' positions of this matrix. So, we have

$$M(g(y_1), g(y_2), \ldots, g(y_n))^T$$
$$= P_1P_2\cdots P_{n-1}(g(y_1), g(y_2), \ldots, g(y_n))^T$$
$$= (g(y_{t_1}), g(y_{t_2}), \ldots, g(y_{t_n}))^T,$$

where $(t_1t_2\cdots t_n)$ is some permutation of $(12\cdots n)$.

Hence,

$$H_C(g(y_{t_1}), g(y_{t_2}), \ldots, g(y_{t_n}))^T = \beta(g(y_{t_1}), g(y_{t_2}), \ldots, g(y_{t_n}))^T.$$

Now, let us show that the function $h(x, y) = f(x)g(y)$ is the eigenfunction of P with the eigenvalue $\frac{(p_1+p_3)\alpha\beta+p_2\beta+p_4\alpha}{p_1+p_2+p_3+p_4}$. For any $(x, y) \in X \times Y$, by Theorem 6, we have

$$Ph(x, y)$$
$$= \sum_{\substack{x'\in X\\ y'\in Y}} P\left((x, y), (x', y')\right) h(x', y')$$
$$= \frac{p_1}{p_1+p_2+p_3+p_4} \sum_{\substack{x'\sim x\\ y'\sim y}} A(x, x')B(y, y')f(x')g(y')$$
$$+ \frac{p_2}{p_1+p_2+p_3+p_4} \sum_{\substack{x'=x\\ y'\sim y}} B(y, y')f(x')g(y')$$
$$+ \frac{p_3}{p_1+p_2+p_3+p_4} \sum_{\substack{x'\sim x\\ y'\nsim y}} A(x, x')C(y, y')f(x')g(y')$$
$$+ \frac{p_4}{p_1+p_2+p_3+p_4} \sum_{\substack{x'\sim x\\ y'=y}} A(x, x')f(x')g(y')$$
$$= \frac{p_1}{p_1+p_2+p_3+p_4}\alpha f(x)\beta g(y)$$
$$+ \frac{p_2}{p_1+p_2+p_3+p_4} f(x)\beta g(y)$$
$$+ \frac{p_3}{p_1+p_2+p_3+p_4}\alpha f(x)\beta g(y)$$
$$+ \frac{p_4}{p_1+p_2+p_3+p_4}\alpha f(x)g(y)$$
$$= \frac{(p_1+p_3)\alpha\beta+p_2\beta+p_4\alpha}{p_1+p_2+p_3+p_4} h(x, y),$$

which is to be proved.

Let $\{f_k\}$ be a basis in the space of functions on X such that $Af_k = \alpha_k f_k$, and $\{g_l\}$ be a basis in the space of functions on

Y such that $Bg_l = \beta_l g_l$. Then $\{h_{kl}(x,y) = f_k(x)g_l(y)\}$ is a linearly independent sequence of functions on $X \times Y$. Since the number of such functions is $mn = |X \times Y|$, we see that h_{kl} is a basis in the space of functions on $X \times Y$. Since h_{kl} is the eigenfunction with the eigenvalue $\frac{(p_1+p_3)\alpha_k\beta_l+p_2\beta_l+p_4\alpha_k}{p_1+p_2+p_3+p_4}$, we conclude that the sequence $\frac{(p_1+p_3)\alpha_k\beta_l+p_2\beta_l+p_4\alpha_k}{p_1+p_2+p_3+p_4}$ exhausts all the eigenvalues of P. ■

Proof of Theorem 2:

Proof: Assume that $\bar{G} = (X, d)$ and $\bar{H} = (Y, c)$ are the quasi-complement graphs to $G = (X, a)$ and $H = (Y, b)$ respectively. Since H is a k-regular self-complementary graph with n vertices and a simple weight, we know $\bar{H}$ is a k-regular graph and $n = 2k + 1$. By Lemma 1, the graph $G[H](\frac{1}{k}, \frac{1}{rk}, \frac{1}{k}, \frac{1}{k^2})$ is a $(r(2k+1)+k)$-regular graph with a simple weight.

In the following, we will prove $G[H](\frac{1}{k}, \frac{1}{rk}, \frac{1}{k}, \frac{1}{k^2})$ is also a self-complementary graph. For convenience, we denote $G[H](\frac{1}{k}, \frac{1}{rk}, \frac{1}{k}, \frac{1}{k^2})$ by $\mathcal{G}$. Since G is isomorphic to $\bar{G}$, there exists a bijection $\sigma_1 : X \to X$ satisfies for any $x_i, x_j \in X$, $x_i \sim x_j$ in G if and only if $\sigma_1(x_i) \sim \sigma_1(x_j)$ in $\bar{G}$. Similarly, there exists a bijection $\sigma_2 : Y \to Y$ satisfies for any $y_s, y_t \in Y$, $y_s \sim y_t$ in H if and only if $\sigma_2(y_s) \sim \sigma_2(y_t)$ in $\bar{H}$. Now we construct a mapping $\sigma : X \times Y \to X \times Y$ satisfies $\sigma((x,y)) = (\sigma_1(x), \sigma_2(y))$ for any $x \in X, y \in Y$. Clearly, the mapping σ is a bijection. We claim that for $x_i, x_j \in X$, $y_s, y_t \in Y$, there holds $(x_i, y_s) \sim (x_j, y_t)$ in $\mathcal{G}$ if and only if $(\sigma_1(x_i), \sigma_2(y_s)) \sim (\sigma_1(x_j), \sigma_2(y_t))$ in $\bar{\mathcal{G}}$.

Since $(x_i, y_s) \sim (x_j, y_t)$ in $\mathcal{G} \Leftrightarrow$ either $i \neq j$, $x_i \sim x_j$ in G or $i = j$, $y_s \sim y_t$ in H. Case 1. $i \neq j$, $x_i \sim x_j$ in $G \Leftrightarrow i \neq j$, $\sigma_1(x_i) \sim \sigma_1(x_j)$ in $\bar{G} \Leftrightarrow i \neq j$, $\sigma_1(x_i) \not\sim \sigma_1(x_j)$ in G. Case 2. $i = j$, $y_s \sim y_t$ in $H \Leftrightarrow i = j$, $\sigma_2(y_s) \sim \sigma_2(y_t)$ in $\bar{H} \Leftrightarrow i = j$, $\sigma_2(y_s) \not\sim \sigma_2(y_t)$ in H. Combining these two cases, we have $(x_i, y_s) \sim (x_j, y_t)$ in $\mathcal{G} \Leftrightarrow (\sigma_1(x_i), \sigma_2(y_s)) \not\sim (\sigma_1(x_j), \sigma_2(y_t))$ in $\mathcal{G} \Leftrightarrow (\sigma_1(x_i), \sigma_2(y_s)) \sim (\sigma_1(x_j), \sigma_2(y_t))$ in $\bar{\mathcal{G}}$. That is, if $(x_i, y_s) \sim (x_j, y_t)$ in $\mathcal{G}$, then $\sigma((x_i, y_s)) \sim \sigma((x_j, y_t))$ in $\bar{\mathcal{G}}$.

Since $\mathcal{G}$ and $\bar{\mathcal{G}}$ have simple weights, there exists a bijection σ such that $\mathcal{G}$ is isomorphic to $\bar{\mathcal{G}}$, which is equivalent to $\mathcal{G}$ is a self-complementary graph. ■

Proof of Theorem 3:

Proof: Suppose that $\mathcal{G} = (V, \mu) = G[H](p_1, p_2, p_3, p_4)$ and $\bar{\mathcal{G}} = (V, \omega)$ are the quasi-complement graph of $\mathcal{G}$. And assume $\bar{G} = (X, d)$ and $\bar{H} = (Y, c)$ are the quasi-complement graphs of $G = (X, a)$ and $H = (Y, b)$, respectively. Since G is a self-complementary graph, there exists a bijection $\sigma_1 : X \to X$ such that G is isomorphic to $\bar{G}$. Similarly, there exists a bijection $\sigma_2 : Y \to Y$ such that H is isomorphic to $\bar{H}$.

Now we construct a mapping $\sigma : X \times Y \to X \times Y$ such that $\sigma((x,y)) = (\sigma_1(x), \sigma_2(y))$ for any $x \in X, y \in Y$. Clearly, the mapping σ is a bijection. As the similar proof in Theorem 2, we know that for $x_i, x_j \in X$, $y_s, y_t \in Y$, there holds $(x_i, y_s) \sim (x_j, y_t)$ in $\mathcal{G}$ if and only if $(\sigma_1(x_i), \sigma_2(y_s)) \sim (\sigma_1(x_j), \sigma_2(y_t))$ in $\bar{\mathcal{G}}$.

If we take the the weight ω in $\bar{\mathcal{G}}$ as follows:

$$\omega_{(\sigma_1(x),\sigma_2(y))(\sigma_1(x'),\sigma_2(y'))} = \begin{cases} p_1 d_{\sigma_1(x)\sigma_1(x')} c_{\sigma_2(y)\sigma_2(y')} b(\sigma_2^{-1}(y)), & x \sim x', y \sim y'; \\ p_2 d(\sigma_1(x)) c_{\sigma_2(y)\sigma_2(y')} b(\sigma_2^{-1}(y)), & x = x', y \sim y'; \\ p_3 d_{\sigma_1(x)\sigma_1(x')} c(\sigma_2(y)) b_{\sigma_2^{-1}(y)\sigma_2^{-1}(y')}, & x \sim x', y \not\sim y'; \\ p_4 d_{\sigma_1(x)\sigma_1(x')} c(\sigma_2(y)) b(\sigma_2^{-1}(y)), & x \sim x', y = y'; \\ 0, & \textit{otherwise}. \end{cases}$$

Then $\mu_{(x,y)(x',y')} = \omega_{(\sigma_1(x),\sigma_2(y))(\sigma_1(x'),\sigma_2(y'))}$. Noting that $a_{xx'} = d_{\sigma_1(x)\sigma_1(x')}$, $b_{yy'} = c_{\sigma_2(y)\sigma_2(y')}$ we have

$$a(x) = \sum_{x' \sim x \text{ in } G} a_{xx'} = \sum_{\sigma_1(x') \sim \sigma_1(x) \text{ in } \bar{G}} d_{\sigma_1(x)\sigma_1(x')} = d(\sigma_1(x)).$$

Similarly, there hold $b(y) = c(\sigma_2(y))$ and $c(y) = b(\sigma_2^{-1}(y))$. Hence, by (1), we know

$$\mu_{(x,y)(x',y')} = \omega_{(\sigma_1(x),\sigma_2(y))(\sigma_1(x'),\sigma_2(y'))} = \omega_{(\sigma(x,y)\sigma(x',y'))},$$

for any $x, x' \in X, y, y' \in Y$. Therefore, there exists a bijection σ such that $\mathcal{G}$ is isomorphic to $\bar{\mathcal{G}}$, which is equivalent to $G[H](p_1, p_2, p_3, p_4)$ is a self-complementary graph. ■

Example 1: Let G be a path with four vertices. It is easy to see that G is a self-complementary graph when it has simple weight. And the lexicographic product of G and G with a simple weight is also a self-complementary graph. We can see this from figure 1.

In the following, we shall prove our main results on Ricci curvature of the lexicographic product of two regular graphs.

Proof of Theorem 4:

Proof: Assume that A is a coupling between $m_{v_1}^{\alpha}$ and $m_{v_2}^{\alpha}$ which defined as (4). Since $u_1 \sim u_2$ in G, $(u_1, v_1) \sim (u_2, v_2)$ in $G[H]$. We define a function $D : V(G[H]) \times V(G[H]) \to [0, 1]$, as shown in the equation at the bottom of the next page.

Now we claim that D is a coupling between $m_{(u_1,v_1)}^{\alpha}$ and $m_{(u_2,v_2)}^{\alpha}$. Set a characteristic function as follows:

$$\mathbf{1}_S(x) = \begin{cases} 1, & \text{if } x \in S; \\ 0, & \textit{otherwise}. \end{cases}$$

$$\begin{aligned} &\sum_{(x_1,y_1)\in V(G[H])} D((x_1, y_1), (x_2, y_2)) \\ &= \sum_{y_1 \in V(H)} D((u_1, y_1), (u_2, y_2))\delta_{u_2}(x_2) \\ &\quad + \sum_{\substack{x_1 \in N_G(u_1) \\ y_1 \in V(H)}} D((x_1, y_1), (x_2, y_2))\mathbf{1}_{N_G(u_2)}(x_2) \\ &= \frac{d_H}{nd_G + d_H}\delta_{u_2}(x_2) \sum_{y_1 \in V(H)} A(y_1, y_2) \\ &\quad + \alpha\frac{nd_G}{nd_G + d_H}\delta_{v_2}(y_2)\delta_{u_2}(x_2) + \frac{1-\alpha}{nd_G + d_H}\mathbf{1}_{N_G(u_2)}(x_2) \\ &= \frac{d_H}{nd_G + d_H}\delta_{u_2}(x_2) m_{v_2}^{\alpha}(y_2) + \alpha\frac{nd_G}{nd_G + d_H}\delta_{v_2}(y_2)\delta_{u_2}(x_2) \end{aligned}$$

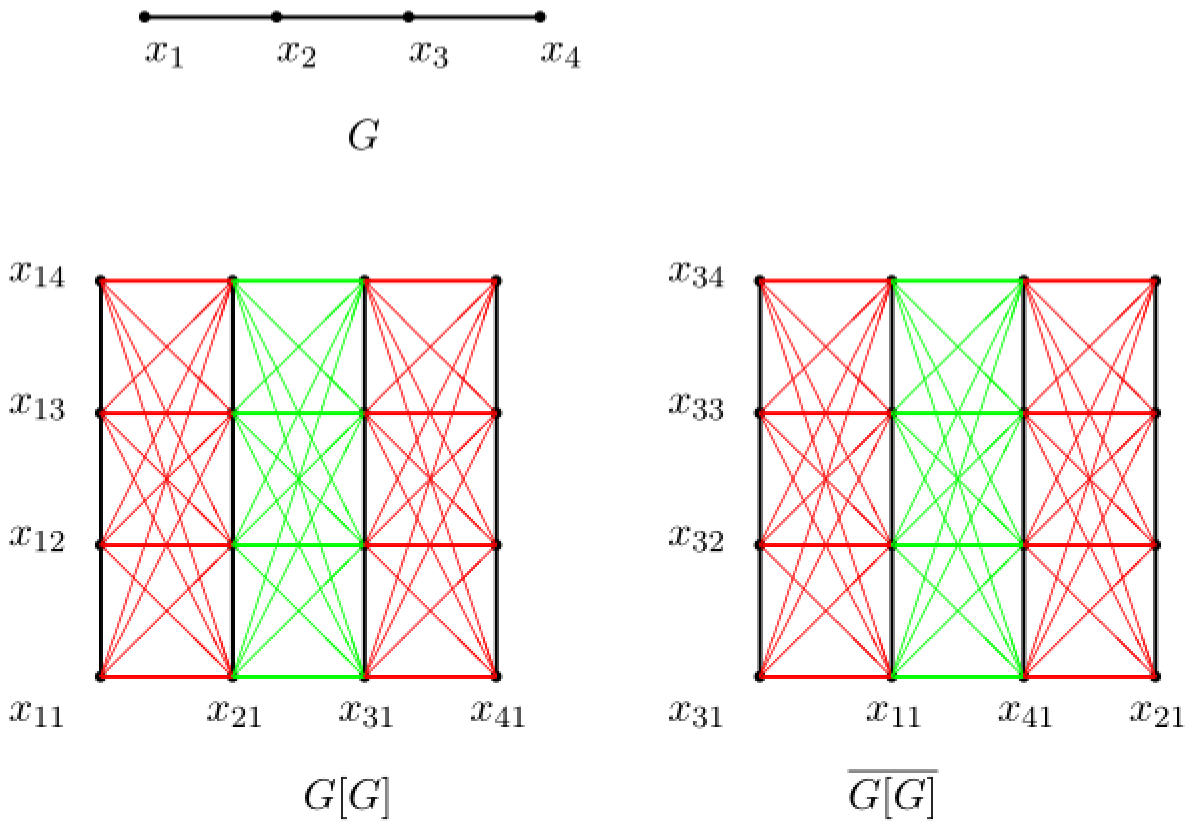

FIGURE 1. G[G] is a self-complementary graph.

$$+\frac{1-\alpha}{nd_G+d_H}\mathbf{1}_{N_G(u_2)}(x_2)$$
$$= m^{\alpha}_{(u_2,v_2)}(x_2,y_2)$$

Similarly, we have

$$\sum_{(x_2,y_2)\in V(G[H])} D((x_1,y_1),(x_2,y_2)) = m^{\alpha}_{(u_1,v_1)}(x_1,y_1).$$

Noting that $\sum_{y_1,y_2\in V(H)} A(x,y) = 1$ and $u_1 \sim u_2$ in G, for $x_1 \in N_G(u_1),\ x_2 \in N_G(u_2),\ y_1,y_2 \in V(H)$, we have

$$d((x_1,y_1),(x_2,y_2)) \le 3,$$

then we obtain

$$\begin{aligned} W(m^{\alpha}_{(u_1,v_1)}, m^{\alpha}_{(u_2,v_2)}) &\le \sum_{\substack{(x_1,y_1)\in V(G[H])\\(x_2,y_2)\in V(G[H])}} D((x_1,y_1),(x_2,y_2))d((x_1,y_1),(x_2,y_2)) \\ &= \sum_{y_1,y_2\in V(H)} D((u_1,y_1),(u_2,y_2))d((u_1,y_1),(u_2,y_2)) \\ &\quad + \frac{1-\alpha}{nd_G(nd_G+d_H)} \sum_{\substack{x_1\in N_G(u_1),x_2\in N_G(u_2)\\ y_1,\ y_2\in V(H)}} d((x_1,y_1),(x_2,y_2)) \end{aligned}$$

$$D((x_1,y_1),(x_2,y_2)) = \begin{cases} \frac{d_H}{nd_G+d_H}A(v_1,v_2) + \alpha\frac{nd_G}{nd_G+d_H}, & x_1=u_1, y_1=v_1, \\ & x_2=u_2, y_2=v_2; \\ \frac{d_H}{nd_G+d_H}A(y_1,y_2), & x_1=u_1, x_2=u_2, \\ & (y_1,y_2)\neq(v_1,v_2); \\ \frac{1-\alpha}{nd_G(nd_G+d_H)}, & x_1\in N_G(u_1), x_2\in N_G(u_2); \\ 0, & otherwise. \end{cases}$$

$$\leq \frac{d_H}{nd_G+d_H}\sum_{y_1,y_2\in V(H)} A(y_1,y_2)+\frac{\alpha nd_G}{nd_G+d_H} + \frac{3(1-\alpha)nd_G}{nd_G+d_H} = \frac{d_H+\alpha nd_G+3(1-\alpha)nd_G}{nd_G+d_H}.$$

Thus, for any $\alpha \in [0,1]$, we get

$$\begin{aligned} k^{G[H]}((u_1,v_1),(u_2,v_2)) &= \lim_{\alpha\to 1}\frac{k_\alpha^{G[H]}((u_1,v_1),(u_2,v_2))}{1-\alpha} \\ &= \lim_{\alpha\to 1}\frac{1-W(m^\alpha_{(u_1,v_1)},m^\alpha_{(u_2,v_2)})}{1-\alpha} \\ &\geq \frac{-2nd_G}{nd_G+d_H}. \end{aligned}$$

■

Proof of Theorem 5:

Proof: Suppose that A is a coupling between $m^\alpha_{v_1}$ and $m^\alpha_{v_2}$ which reaches the infimum in the definition of $W_H(m^\alpha_{v_1},m^\alpha_{v_2})$. So, we have

$$W_H(m^\alpha_{v_1},m^\alpha_{v_2}) = \sum_{y_1,y_2\in V(H)} A(y_1,y_2)d(y_1,y_2).$$

Define the function $D : V(G[H]) \times V(G[H]) \to [0,1]$ as follows:

$$D((x_1,y_1),(x_2,y_2)) = \begin{cases} \frac{d_H}{nd_G+d_H}A'(v_1,v_2)+\alpha\frac{nd_G}{nd_G+d_H}, & x_1=x_2=u,\ y_1=v_1,y_2=v_2; \\ \frac{d_H}{nd_G+d_H}A'(y_1,y_2), & x_1=x_2=u,\ (y_1,y_2)\neq(v_1,v_2); \\ \frac{1-\alpha}{n(nd_G+d_H)}, & x_1=x_2\in N_G(u); \\ 0, & \textit{otherwise}. \end{cases}$$

Through the analogous analysis in Theorem 4, we know the function D is a coupling between $m^\alpha_{(u,v_1)}$ and $m^\alpha_{(u,v_2)}$. Noting that $d((x,y_1),(x,y_2)) \leq d(y_1,y_2)$ and $d((x,y_1),(x,y_2)) \leq 2$ in $G[H]$, we obtain

$$\begin{aligned} &W(m^\alpha_{(u,v_1)},m^\alpha_{(u,v_2)}) \\ &\leq \sum_{\substack{(x_1,y_1)\in V(G[H])\\(x_2,y_2)\in V(G[H])}} D((x_1,y_1),(x_2,y_2))d((x_1,y_1),(x_2,y_2)) \\ &= \sum_{y_1,y_2\in V(H)}\frac{d_H}{nd_G+d_H}A(y_1,y_2)d((u,y_1),(u,y_2)) + \frac{\alpha nd_G}{nd_G+d_H} + \frac{1-\alpha}{n(nd_G+d_H)}\sum_{\substack{x_1=x_2\in N_G(u)\\y_1,y_2\in V(H)}} d((x_1,y_1),(x_2,y_2)) \\ &\leq \frac{d_H}{nd_G+d_H}W_H(m^\alpha_{v_1},m^\alpha_{v_2})+\frac{\alpha nd_G+2(1-\alpha)nd_G}{nd_G+d_H}. \end{aligned}$$

Hence for any $u \in V(G)$, $v_1 \sim v_2$ in H, we have

$$\begin{aligned} &k_\alpha^{G[H]}((u,v_1),(u,v_2)) \\ &= 1-W(m^\alpha_{(u,v_1)},m^\alpha_{(u,v_2)}) \\ &\geq 1-\frac{d_H}{nd_G+d_H}W_H(m^\alpha_{v_1},m^\alpha_{v_2})-\frac{\alpha nd_G+2(1-\alpha)nd_G}{nd_G+d_H} \\ &= \frac{d_H}{nd_G+d_H}(1-W_H(m^\alpha_{v_1},m^\alpha_{v_2}))-\frac{(1-\alpha)nd_G}{nd_G+d_H}. \end{aligned}$$

Thus

$$\begin{aligned} k^{G[H]}((u,v_1),(u,v_2)) &= \lim_{\alpha\to 1}\frac{k_\alpha^{G[H]}((u,v_1),(u,v_2))}{1-\alpha} \\ &\geq \frac{d_H}{nd_G+d_H}k^H(v_1,v_2)-\frac{nd_G}{nd_G+d_H}. \end{aligned}$$

■

Remark 2: We know the cycle C_n for $n \geq 6$ has a constant Ricci curvature 0. Combining with Theorem 5, we obtain the lower bound of the Ricci curvature for $G[C_n]$ is $-\frac{2nd_G}{nd_G+2}$, which is larger than -2.

IV. CONCLUSION

In this paper, we have proposed a precise definition of the weight function for the lexicographic product graph $G[H]$. This definition captures the intricate interplay between the vertices of the constituent graphs and provides a solid foundation for further analysis. Furthermore, we have derived an expression for the spectrum of the lexicographic product graph $G[H]$. By leveraging the spectrums of graphs G and H, with the additional assumption that H is a self-complementary graph, we have obtained a comprehensive understanding of the eigenvalues and associated properties of the lexicographic product graph. Additionally, our research has proved an essential result regarding the Ricci curvature for the lexicographic product of two regular graphs. We have established that the Ricci curvature for this operation exhibits a lower bound, shedding light on the geometric properties and constraints inherent in the lexicographic product. In the future, we will focus on investigating the expression for the spectrum of the lexicographic product graph $G[H]$, which will build upon the spectrums of graphs G and H assuming that H is not a self-complementary graph.

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