IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 31 May 2023, accepted 23 July 2023, date of publication 9 August 2023, date of current version 24 August 2023.

Digital Object Identifier 10.1109/ACCESS.2023.3298845

RESEARCH ARTICLE

Complex T-Spherical Fuzzy Frank Aggregation Operators and Their Application to Decision Making

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The authors extend their appreciation to the Deanship for Research & Innovation, Ministry of Education in Saudi Arabia for funding this research work through the project number: IFP22UQU4310396DSR035.

ABSTRACT Complex fuzzy (CF) sets (CFSs) play an important role in modeling two-dimensional information challenges. Researchers exploring decision-making systems have recently become interested in CFS extensions. The complex T-spherical fuzzy (CT-SF) set (CT-SFS) is a recent extension of the CFSs. The present study aims to devise the Frank operational laws for the CT-SF environment and to verify their required properties. Subsequently, some CT-SF Frank aggregation operators are explored, such as CT-SF Frank weighted averaging (CT-SFFWA) operator, CT-SF Frank weighted geometric (CT-SFFWG) operator, CT-SF Frank ordered weighted averaging (CT-SFFOWA) operator, and CT-SF Frank ordered weighted geometric (CT-SFFOWG) operator, CT-SF Frank hybrid averaging (CT-SFFHA) operator, CT-SF Frank hybrid geometric (CT-SFFHG) operator, and their peculiar cases are examined. Based on the devised operators, a novel multi-criteria group decision-making (MCGDM) methodology is investigated to tackle MCGDM problems under the CT-SF environment. Lastly, the practicality and effectiveness of the presented methodology are conducted by parameter analysis and comparative exploration.

INDEX TERMS Frank T-norms, complex T-spherical fuzzy set, aggregation operators, MCGDM.

I. INTRODUCTION

Multi-criteria group decision making (MCGDM) is a way to choose the best option or rank the options based on more than one factor, assessed by decision experts (DEs). This method has a wide variety of applications in many disciplines [1], [2], [3], [4], [5], [6], [7], [8]. Owing to the complexity and ambiguity of objective things and human cognition, the topic of MCGDM problems in uncertain contexts has received considerable interest. To handle problems in uncertain situations and find better solutions, Zadeh [9] developed fuzzy sets (FSs), which are defined by their membership grade μ. Since FS only gives a membership grade subject to a value within [0, 1] to support a fuzzy expression issue, but lacks a non-membership grade. In view of true and false membership grades, Atanassov [10] suggested intuitionistic fuzzy sets (IFSs) based on the FS theory, which consider both the membership grade $\mu(0 \leq \mu \leq 1)$ and non-membership grade $\nu(0 \leq \nu \leq 1)$, with the restriction that the sum of the two membership grades cannot exceed one, i.e., $\mu + \nu \leq 1$. Motivated by the idea of IFSs, to widen the space of the DEs's judgment regarding membership grade and non-membership grade, Yager [11], [12] originated two modified FSs called Pythagorean fuzzy sets (PyFSs) and q-rung orthopair fuzzy sets (q-ROFSs) one after another, which meet the requirements $\mu^2 + \nu^2 \leq 1$ and $\mu^q + \nu^q \leq 1 (q \geq 1)$, respectively. IFSs, PyFSs, and q-ROFSs are useful for tackling real MCGDM problems, and significant academic progress has been made [13], [14], [15], [16], [17].

However, there are a few scenarios in the real world where human opinions require more answers of types: yes, no, abstain, and rejection. Voting is an appropriate example of

The associate editor coordinating the review of this manuscript and approving it for publication was Amjad Ali.

such a circumstance, as human voters can be divided into four categories: vote for, abstain, vote against, and refuse to vote [18]. To address this type of situation, Cuong and Kreinovich [19] introduced the Picture FSs (PFSs), which are described by membership grade $\mu(0 \leq \mu \leq 1)$, non-membership grade $\nu(0 \leq \nu \leq 1)$, and neutral grade $\varsigma(0 \leq \varsigma \leq 1)$, with the constraint that the total of the three grades cannot cross one, i.e., $\mu + \nu + \varsigma \leq 1$. Soon after, motivated by the PyFSs and q-ROFSs, to widen the space of the DEs' judgment about membership grade, non-membership grade, and neutral grade, Mahmood et al. [20] presented the extended PFSs known as T-spherical fuzzy sets (T-SFSs), which meet the restriction $\mu^t + \nu^t + \varsigma^t \leq 1 (t \geq 1)$. Since their introduction, T-SFSs have gained increased interest from scholars. Ullah et al. [21] offered correlation coefficients for T-SFSs and put forward clustering and decision making methods based on the formulated correlation coefficients. The authors in [22] described several measures of similarity for T-SFSs, including the cosine similarity measure, the grey similarity measure, and the set-theoretic similarity measure. To aggregate T-spherical fuzzy information, numerous operators are investigated, such as T-spherical fuzzy power operators [23], T-spherical fuzzy Hamacher operators [24], T-spherical fuzzy Frank operators [25], and T-spherical fuzzy generalized Maclaurin symmetric mean operators [26], etc.

According to the prevalent studies cited above, such methodologies are constrained and unable to depict the partial ignorance of data and its variations over a specific period of time. To address this, Ramot et al. [27] developed complex FS (CFS). In addition, Alkouri and Salleh [28] introduced the doctrine of complex IFS (CIFS), which increased the range of the membership grade and the non-membership grade from real numbers to complex numbers with a unit disc, and which can express two-dimensional information. A CIFS is characterized by membership function ($\ddot{\mu} = \mu e^{i2\pi(\eth_\mu)}$) and non-membership function ($\ddot{\nu} = \nu e^{i2\pi(\eth_\nu)}$) such that $0 \leq \mu + \nu \leq 1$, $0 \leq \eth_\mu + \eth_\nu \leq 1$. In addition, Ullah et al. [29] reformed CIFS in order to investigate the complex PyFS (CPyFS) under the restriction that the sum of the squares of the real parts (including imaginary parts) of the membership grade and non-membership grade cannot exceed a unit interval. After the introduction of CPyFS, the complex q-ROFS (Cq-ROFS) [30], satisfying the conditions $0 \leq \mu^q + \nu^q \leq 1$, $0 \leq \eth_\mu^q + \eth_\nu^q \leq 1 (q \geq 1)$, was discovered to be an efficient tool for solving MCGDM problems. The theory of Cq-ROFSs has been implemented in various facets of daily life. However, these paradigms do not consider the issue when four aspects are used to describe an item, particularly when human opinion is involved. For example, with $0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51}$, Cq-ROFSs fail to tackle such situations because of the extra information, i.e., neutral part. To cope with such problems, Ali et al. [31] introduced the doctrine of complex T-spherical fuzzy sets (CT-SFSs) with more flexible conditions, i.e., $0 \leq \mu^t + \varsigma^t + \nu^t \leq 1$, $0 \leq \eth_\mu^t + \eth_\varsigma^t + \eth_\nu^t \leq 1 (t \geq 1)$. Karaaslan and Dawood [32] studied some Dombi aggregation operators under complex-T spherical fuzzy (CT-SF) environment and developed an MCGDM method. Nasir and his coauthors [33] studied the concept of CTSF relations and their desired properties in depth. A study by Zedam et al. [34] gave some CT-SF aggregation operators based on Hamacher operations and their application cleaner production evaluation in gold mines. Meanwhile, Ullah et al. [35] also familiarized some CT-SF Hamacher aggregation operators and employed them to decision making.

The Frank t-norm and t-conorm [36] are generalizations of the probabilistic, Lukasiewicz, algebraic, Einstein, and Hamacher t-conorm and t-conorm. These are more adaptable and appropriate for dealing with real-world decision making because they include a parameter that controls the power to which the argument values are raised. The Frank operators have attracted considerable attention from the scientific community in recent years, and it has produced numerous study outcomes on various FSs. Some operations and their relevant operators on numerous FSs have been propounded based on the Frank t-norm and t-conorm, such as single-valued neutrosophic Frank operations [37], probabilistic hesitant fuzzy Frank operations [38], Frank prioritized Bonferroni mean operations [39], fermatean fuzzy Frank operators [40], interval-valued probabilistic hesitant fuzzy aggregation operators [41], T-spherical fuzzy Frank operators [25], complex q-rung orthopair fuzzy Frank operators [42]. Several authors have also examined the mathematical properties of the Frank t-norms [43], [44], [45], [46]. To the best of our knowledge, the authors have conducted no research regarding Ct-SF Frank's operations. As previously noted, the CT-SFS covers the massive loss of information while we collect information from any real-life phenomenon to make decisions. It is especially effective to extract as much information as possible when human opinion is involved. Hence, utilization of CT-SFS in Frank operators has a high potential for improving MCGDM performance.

The following are the key motives and contributions of this article:

1). CFS and its generalizations play a crucial role in real-world decision making challenges using two-dimensional information. As a generalization of SFSs, a T-SFS comprehends voluminous amounts of information. However, it is insufficient for modeling a problem requiring two-dimensional data. In this regard, CT-SFS is of fundamental value. CT-SFS is the generalization of CFS, CIFS, CPFS, CPyFS, and CSFS theories. Frank operational rules are an important tool for modeling MCGDM problems, yet, no study on Frank operations of CT-SFS has been published in the literature. We define the Frank operations of CT-SFSs to fill this void in the literature.

2). Until now, the literature pertaining to aggregation operators of CT-SFS contains few references [32], [33], [34]. In light of the benefits of the Frank operators, we develop some novel aggregation operators CT-SFFWA, CT-SFFOWA, CT-SFFHA, CT-SFFWG, CT-SFFOWG, and CT-SFFHG based on the Frank t-norm and t-conorm to be employed in addressing a two-dimensional real-world problem.

3). To explore various aspects such as monotonicity, idempotency, boundedness, and some limiting instances of the formulated operators.

4). To build innovative MCGDM approach based on the designed operators.

5). To give the application of the outlined approach that validates the practicability and reliability of the suggested technique. In addition, by comparing the proposed technique to preexisting methods, we will confirm that the new method is superior to previous methods and demonstrate that the Frank t-conorm and t-norm aggregation operators enable the aggregation procedure more flexible.

The subsequent portions of this paper are structured as follows to meet the goals of our research: Section II offers a brief review of the important background material for this subject. Section III defines Frank operations of CT-SFNs and proves their essential properties. Section IV develops the theory of CT-SFFHWA and CT-SFFHWG operators and outlines their influential properties with numerous strong results. In Section V, we exploits the initiated operators to frame a decision-making algorithm for coping MCGDM problems utilizing CT-SFNs as characteristic values. In Section VI, the selection of water supply strategy problem is addressed to illustrate application, and a contrastive parameter analysis is conducted to highlight the stability of the propound method. Section VII comprises a discussion regarding the results achieved from the application of the presented method. Some concluding remarks and future outlooks are documented at the end.

II. SOME BASIC CONCEPTS

In the following, we present a concise overview of T-SFSs, CT-SFSs, Frank t-norm and Frank t-conorm to help readers comprehend the study.

T-SFS is proposed by Mahmood et al. [20] as a synthesis of SFS to provide a greater range of preferences for DEs and allow them to communicate their hesitation about an alternative. The following are some fundamental definitions of T-SFS and terms related to intended work.

Definition 1 [20]: Let X be a universal set. A T-spherical fuzzy set (T-SFS) $\mathcal{J}$ on X is characterized by

$$\mathcal{J} = \{\langle \hbar, \mu(\hbar), \varsigma(\hbar), \nu(\hbar)\rangle \,|\hbar \in X\}, \tag{1}$$

where $\mu(\hbar)$, $\varsigma(\hbar)$, $\nu(\hbar) \in [0,1]$ represent the membership, neutral and non-membership grades of $\hbar \in X$ to the set $\mathcal{J}$, respectively, with the condition that $0 \leq \mu^t(\hbar) + \varsigma^t(\hbar) + \nu^t(\hbar) \leq 1$. The degree of refusal is $\pi(\hbar) = \sqrt[t]{1-\mu^t(\hbar)-\varsigma^t(\hbar)-\nu^t(\hbar)}$. For convince, $\langle \mu(\hbar), \varsigma(\hbar), \nu(\hbar)\rangle$ is named a T-spherical fuzzy number (T-SFN), labeled by $\mathcal{J} = \langle \mu, \varsigma, \nu\rangle$.

Remark 1:

1) The Definition 1 deduced to SFS if we put $t = 2$.
2) The Definition 1 deduced to PFS if we put $t = 1$.
3) The Definition 1 deduced to q-OFS if we put $\varsigma = 0$.
4) The Definition 1 deduced to PyFS if we put $t = 2$ and $\varsigma = 0$.
5) The Definition 1 deduced to IFS if we put $t = 1$ and $\varsigma = 0$.

Definition 2 [47]: Let $\mathcal{J}_1 = \langle \mu_1, \varsigma_1, \nu_1\rangle$ and $\mathcal{J}_2 = \langle \mu_2, \varsigma_2, \nu_2\rangle$ be two T-SFNs and $\dagger > 0$, then

1) $\mathcal{J}_1 \oplus \mathcal{J}_2 = \left\langle \sqrt[t]{\mu_1^t+\mu_2^t-\mu_1^t\mu_2^t}, \varsigma_1\varsigma_2, \nu_1\nu_2\right\rangle$;
2) $\mathcal{J}_1 \otimes \mathcal{J}_2 = \left\langle \mu_1\mu_2, \sqrt[t]{\varsigma_1^t+\varsigma_2^t-\varsigma_1^t\varsigma_2^t}, \sqrt[t]{\nu_1^t+\nu_2^t-\nu_1^t\nu_2^t}\right\rangle$;
3) $\mathcal{J}_1^\dagger = \left\langle \mu_1^\dagger, \sqrt[t]{1-(1-\varsigma_1^t)^\dagger}, \sqrt[t]{1-(1-\nu_1^t)^\dagger}\right\rangle$;
4) $\dagger\mathcal{J}_1 = \left\langle \sqrt[t]{1-(1-\mu_1^t)^\dagger}, \varsigma_1^\dagger, \nu_1^\dagger\right\rangle$;
5) $\mathcal{J}_1^c = \langle \nu_1, \varsigma_1, \mu_1\rangle$.

Definition 3 [20], [48]: $\mathcal{J}_1 = \langle \mu_1, \varsigma_1, \nu_1\rangle$ and $\mathcal{J}_2 = \langle \mu_2, \varsigma_2, \nu_2\rangle$ be two T-SFNs, let $S(\mathcal{J}_1) = \mu_1^t - \varsigma_1^t - \nu_1^t + \left(\frac{\exp^{\mu_1^t-\varsigma_1^t-\nu_1^t}}{\exp^{\mu_1^t-\varsigma_1^t-\nu_1^t}+1} - \frac{1}{2}\right)\pi^t$ and $S(\mathcal{J}_2) = \mu_2^t - \varsigma_2^t - \nu_2^t + \left(\frac{\exp^{\mu_2^t-\varsigma_2^t-\nu_2^t}}{\exp^{\mu_2^t-\varsigma_2^t-\nu_2^t}+1} - \frac{1}{2}\right)\pi^t$ be the score values of $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively, and let $A(\mathcal{J}_1) = \mu_1^t + \varsigma_1^t + \nu_1^t$ and $A(\mathcal{J}_2) = \mu_2^t+\varsigma_2^t+\nu_2^t$ be the accuracy values of $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively. Then,

1) If $S(\mathcal{J}_1) < S(\mathcal{J}_1)$, then $\mathcal{J}_1 < \mathcal{J}_2$;
2) If $S(\mathcal{J}_1) = S(\mathcal{J}_1)$, then
   a. If $A(\mathcal{J}_1) < A(\mathcal{J}_1)$, then $\mathcal{J}_1 < \mathcal{J}_2$;
   b. If $S(\mathcal{J}_1) = S(\mathcal{J}_1)$, then $\mathcal{J}_1 = \mathcal{J}_2$.

Definition 4 [31]: A CT-SFS $\Im$ on a fixed set X is given by

$$\Im = \{(\hbar, \ddot{\mu}(\hbar), \ddot{\varsigma}(\hbar), \ddot{\nu}(\hbar))\,|\hbar \in X\}, \quad t \geq 1, \tag{2}$$

where $\ddot{\mu}(\hbar) = \mu(\hbar)e^{i2\pi(\eth_\mu)}$, $\ddot{\varsigma}(\hbar) = \varsigma(\hbar)e^{i2\pi(\eth_\varsigma)}$, $\ddot{\nu}(\hbar) = \nu(\hbar)e^{i2\pi(\eth_\nu)} \in [0,1]$ symbolizes the complex-valued membership, complex-valued neutral and complex-valued non-membership grades of $\hbar \in X$, respectively, accorded that $0 \leq (\mu(\hbar))^t + (\varsigma(\hbar))^t + (\nu(\hbar))^t \leq 1$, $0 \leq (\eth_\mu)^t + (\eth_\varsigma)^t + (\eth_\nu)^t \leq 1$, where $\mu,\varsigma\ \nu$, $\eth_\mu$, $\eth_\varsigma$, $\eth_\nu \in [0,1]$. The degree of hesitation is $\pi(\hbar)^t = \left(1-\left((\mu(\hbar))^t+(\varsigma(\hbar))^t+(\nu(\hbar))^t\right)\right)\left(1-\left((\eth_\mu)^t+(\eth_\varsigma)^t+(\eth_\nu)^t\right)\right)$. For convince, we termed $\Im = \left(\mu e^{i2\pi(\eth_\mu)}, \varsigma e^{i2\pi(\eth_\varsigma)}, \nu e^{i2\pi(\eth_\nu)}\right)$ a complex T-spherical fuzzy number (CT-SFN).

Definition 5 [32]: Let $\Im$ be a CT-SFN; then the score function is characterized by:

$$S(\Im) = \frac{1}{4}\cdot\left(2+\left(\mu^t-\varsigma^t-\nu^t\right)+\left(\eth_\mu^t-\eth_\varsigma^t-\eth_\nu^t\right)\right), \tag{3}$$

where $t \in [1,\infty)$, $S(\Im) \in [0,1]$. The larger the value of $S(\Im)$, the larger the CT-SFN $\mathcal{Q}$.

Definition 6 [32]: Let $\Im$ be a CT-SFN, then the degree of accuracy is given in the following manner:

$$A(\Im) = \frac{1}{4}\cdot\left(2+\left(\mu^t+\varsigma^t+\nu^t\right)+\left(\eth_\mu^t+\eth_\varsigma^t+\eth_\nu^t\right)\right), \tag{4}$$

where $A(\Im) \in [0, 1]$. When the computed score values are same, the larger the degree of accuracy $A(\Im)$, the larger the CT-SFN.

Definition 7 [31]: Let $\Im_1$ and $\Im_2$ be two CT-SFNs and $\dagger > 0$, then the basic rules of operation on them are listed as shown at the bottom of the page.

The T-spherical fuzzy aggregation operator has garnered a lot of attention as a useful tool in information fusion. Mahmood et al. [49] devised the T-spherical fuzzy weighted averaging (T-SFWA) operator and the T-spherical fuzzy weighted geometric (T-SFWG) operator as:

Definition 8 [49]: Given a set of T-SFNs $\mathcal{J}_{\dot{r}}$ $(\dot{r} = 1(1)n)$, then the T-spherical fuzzy weighted averaging (T-SFWA) operator is a mapping $\mathcal{J}^n \longrightarrow \mathcal{J}$ such that

$$T-SFWA(\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_n) = \natural_1\mathcal{J}_1 \oplus \natural_2\mathcal{J}_2 \oplus \ldots \oplus \natural_n\mathcal{J}_n = \left\langle \left(1 - \prod_{\dot{r}=1}^{n}\left(1 - \sigma_{\dot{r}}^t\right)^{\natural_{\dot{r}}}\right)^{1/t}, \prod_{\dot{r}=1}^{n}(\varsigma_{\dot{r}})^{\natural_{\dot{r}}}, \prod_{\dot{r}=1}^{n}(\varrho_{\dot{r}})^{\natural_{\dot{r}}} \right\rangle, \quad (5)$$

where $\natural = \{\natural_1, \natural_2, \ldots, \natural_n\}^T$ is the weight vector of $(\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_n)$ satisfying $0 \le \natural_{\dot{r}} \le 1$ and $\sum_{\dot{r}=1}^{n} \natural_{\dot{r}} = 1$.

Definition 9 [49]: Given a set of T-SFNs $\mathcal{J}_{\dot{r}}$ $(\dot{r} = 1(1)n)$, then the T-spherical fuzzy weighted geometric (T-SFWG) operator is a mapping $\mathcal{J}^n \longrightarrow \mathcal{J}$ such that

$$T-SFWG(\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_n) = \natural_1\mathcal{J}_1 \otimes \natural_2\mathcal{J}_2 \otimes \ldots \otimes \natural_n\mathcal{J}_n = \left\langle \prod_{\dot{r}=1}^{n}(\sigma_{\dot{r}})^{\natural_{\dot{r}}}, \left(1 - \prod_{\dot{r}=1}^{n}\left(1 - \varsigma_{\dot{r}}^t\right)^{\natural_{\dot{r}}}\right)^{1/t}, \times \left(1 - \prod_{\dot{r}=1}^{n}\left(1 - \varrho_{\dot{r}}^t\right)^{\natural_{\dot{r}}}\right)^{1/t} \right\rangle, \quad (6)$$

where $\natural = (\natural_1, \natural_2, \ldots, \natural_n)^T$ is the weight vector of $(\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_n)$ satisfying $0 \le \natural_{\dot{r}} \le 1$ and $\sum_{\dot{r}=1}^{n} \natural_{\dot{r}} = 1$.

Triangle norms have been extensively studied, beginning with Zadeh's presentation of the max and min operation as a pair of triangular norm (t-norm) and triangular conorm (t-conorm). Several t-norms and t-conorms, such as the product t-norm and probabilistic sum [50], can be mentioned. Einstein t-norm and t-conorm [51], Lukasiewicz t-norm and t-conorm [52], Hamacher t-norm and t-conorm [53], etc., are vehicles for FS operations. Frank's operations include Frank's product and Frank's sum, both of which are examples of triangular norms and conorms.

The Frank t-norm T_F and Frank t-conorm S_F are described as follows:

$$T_F(\hbar_1, \hbar_2) = \log_{£}\left(1 + \frac{\left(£^{\hbar_1} - 1\right)\left(£^{\hbar_2} - 1\right)}{£ - 1}\right) \times \forall \ (\hbar_1, \hbar_2) \in [0, 1]^2, \quad (7)$$

$$S_F(\hbar_1, \hbar_2) = 1 - \log_{£}\left(1 + \frac{\left(£^{1-\hbar_1} - 1\right)\left(£^{1-\hbar_2} - 1\right)}{£ - 1}\right) \times \forall \ (\hbar_1, \hbar_2) \in [0, 1]^2. \quad (8)$$

The following properties of the Frank t-norm and Frank t-conorm are mentioned [54].

$$T_F(\hbar_1, \hbar_2) + S_F(\hbar_1, \hbar_2) = \hbar_1 + \hbar_2, \quad (9)$$

$$\frac{\partial T_F(\hbar_1, \hbar_2)}{\partial \hbar_1} + \frac{\partial S_F(\hbar_1, \hbar_2)}{\partial \hbar_1} = 1. \quad (10)$$

The following desirable outcomes are easily verifiable using limit theory [54].

1). If $£ \longrightarrow 1$, then $T_F(\hbar_1, \hbar_2) \longrightarrow \hbar_1 + \hbar_2 - \hbar_1\hbar_2$, $S_F(\hbar_1, \hbar_2) \longrightarrow \hbar_1\hbar_2$, the Frank t-norm, and Frank t-conorm are reduced to probabilistic product and probabilistic sum.

2). If $£ \longrightarrow \infty$, then $T_F(\hbar_1, \hbar_2) \longrightarrow \min(\hbar_1 + \hbar_2, 1)$, $S_F(\hbar_1, \hbar_2) \longrightarrow \max(0, \hbar_1 + \hbar_2 - 1)$, the Frank t-norm and Frank t-conorm are deduced to the Lukasiewicz product and Lukasiewicz sum, respectively.

III. CT-SF FRANK OPERATIONAL LAWS

This section will present Frank's CT-SFN operations and investigate some of its noteworthy features.

1) $\Im_1 \oplus \Im_2 = \left(\left(\mu_1^t + \mu_2^t - \mu_1^t\mu_2^t\right)^{1/t} e^{i2\pi\left(\eth_{\mu 1}^t + \eth_{\mu 2}^t - \eth_{\mu 1}^t\eth_{\mu 2}^t\right)^{1/t}}, \varsigma_1\varsigma_2 e^{i2\pi\left(\eth_{\varsigma 1}\eth_{\varsigma 2}\right)}, \nu_1\nu_2 e^{i2\pi\left(\eth_{\nu 1}\eth_{\nu 2}\right)}\right);$

2) $\Im_1 \otimes \Im_2 = \left(\mu_1\mu_2 e^{i2\pi\left(\eth_{\mu 1}\eth_{\mu 2}\right)}, \left(\varsigma_1^t + \varsigma_2^t - \varsigma_1^t\varsigma_2^t\right)^{1/t} e^{i2\pi\left(\eth_{\varsigma 1}^t + \eth_{\varsigma 2}^t - \eth_{\varsigma 1}^t\eth_{\varsigma 2}^t\right)^{1/t}}, \left(\nu_1^t + \nu_2^t - \nu_1^t\nu_2^t\right)^{1/t} e^{i2\pi\left(\eth_{\nu 1}^t + \eth_{\nu 2}^t - \eth_{\nu 1}^t\eth_{\nu 2}^t\right)^{1/t}}\right);$

3) $\Im_1^{\dagger} = \left(\mu_1^{\dagger} e^{i2\pi\left(\eth_{\mu 1}^{\dagger}\right)}, \left(1 - \left(1 - \varsigma_1^t\right)^{\dagger}\right)^{1/t} e^{i2\pi\left(1 - \left(1 - \eth_{\varsigma 1}^t\right)^{\dagger}\right)^{1/t}}, \left(1 - \left(1 - \nu_1^t\right)^{\dagger}\right)^{1/t} e^{i2\pi\left(1 - \left(1 - \eth_{\nu 1}^t\right)^{\dagger}\right)^{1/t}}\right);$

4) $\dagger\Im_1 = \left(\left(1 - \left(1 - \mu_1^t\right)^{\dagger}\right)^{1/t} e^{i2\pi\left(1 - \left(1 - \eth_{\mu 1}^t\right)^{\dagger}\right)^{1/t}}, \varsigma_1^{\dagger} e^{i2\pi\left(\eth_{\varsigma 1}^{\dagger}\right)}, \nu_1^{\dagger} e^{i2\pi\left(\eth_{\nu 1}^{\dagger}\right)}\right);$

5) $\Im_1^c = \left(\nu_1 e^{i2\pi\eth_{\nu 1}}, \varsigma_1 e^{i2\pi\eth_{\varsigma 1}}, \mu_{Q_1} e^{i2\pi\eth_{\mu 1}}\right).$

Definition 10: Let $\Im_1 = \left(\sigma_1 e^{2i\pi \eth_{\sigma 1}}, \varsigma_1 e^{2i\pi \eth_{\varsigma 1}}, \varrho_1 e^{2i\pi \eth_{\varrho 1}}\right)$ and $\Im_2 = \left(\sigma_2 e^{2i\pi \eth_{\sigma 2}}, \varsigma_2 e^{2i\pi \eth_{\varsigma 2}}, \varrho_2 e^{2i\pi \eth_{\varrho 2}}\right)$ be two CT-SFNs and $\dagger > 0$, then, as 1)–3) and 4) are shown at the bottom of the page and the next page, respectively.

Remark 2:

1) The operational rules described in Definition 10 deduced to CSFNs if we put $t = 2$.
2) The operational rules described in Definition 10 deduced to CPFNS if we put $t = 1$.
3) The operational rules described in Definition 10 deduced to Cq-OFNS if we put $\varsigma = 0$.
4) The operational rules described in Definition 10 deduced to CPyFS if we put $t = 2$ and $\varsigma = 0$.
5) The operational rules described in Definition 10 deduced to CIFS if we put $t = 1$ and $\varsigma = 0$.

We investigate the following outcomes using the operating rules defined in Definition 10.

1) $$\Im_1 \oplus \Im_2 = \left(\begin{array}{c} \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\sigma_1^t}-1\right)\left(£^{1-\sigma_2^t}-1\right)}{£-1}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth_{\sigma 1}^t}-1\right)\left(£^{1-\eth_{\sigma 2}^t}-1\right)}{£-1}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varsigma_1^t}-1\right)\left(£^{\varsigma_2^t}-1\right)}{£-1}\right)}\, e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varsigma 1}^t}-1\right)\left(£^{\eth_{\varsigma 2}^t}-1\right)}{£-1}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varrho_1^t}-1\right)\left(£^{\varrho_2^t}-1\right)}{£-1}\right)}\, e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varrho 1}^t}-1\right)\left(£^{\eth_{\varrho 2}^t}-1\right)}{£-1}\right)}} \end{array}\right);$$

2) $$\Im_1 \otimes \Im_2 = \left(\begin{array}{c} \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\sigma_1^t}-1\right)\left(£^{\sigma_2^t}-1\right)}{£-1}\right)}\, e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\sigma 1}^t}-1\right)\left(£^{\eth_{\sigma 2}^t}-1\right)}{£-1}\right)}}, \\ \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\varsigma_1^t}-1\right)\left(£^{1-\varsigma_2^t}-1\right)}{£-1}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth_{\varsigma 1}^t}-1\right)\left(£^{1-\eth_{\varsigma 2}^t}-1\right)}{£-1}\right)}}, \\ \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\varrho_1^t}-1\right)\left(£^{1-\varrho_2^t}-1\right)}{£-1}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth_{\varrho 1}^t}-1\right)\left(£^{1-\eth_{\varrho 2}^t}-1\right)}{£-1}\right)}} \end{array}\right);$$

3) $$\Im_1^{\dagger} = \left(\begin{array}{c} \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\sigma_1^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}\, e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\sigma 1}^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}}, \\ \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\varsigma_1^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth_{\varsigma 1}^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}}, \\ \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\varrho_1^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth_{\varrho 1}^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}} \end{array}\right);$$

Theorem 1: Let $\Im_\ell = (\sigma_\ell e^{2i\pi ð_{\sigma\ell}}, \varsigma_\ell e^{2i\pi ð_{\varsigma\ell}}, \varrho_\ell e^{2i\pi ð_{\varrho\ell}})$ $(\ell = 1, 2)$ and $\Im = (\sigma e^{2i\pi ð_\sigma}, \varsigma e^{2i\pi ð_\varsigma}, \varrho e^{2i\pi ð_\varsigma})$ be three CT-SFNs, and $\dagger, \dagger_1, \dagger_2 > 0$, then

1) $\Im_1 \oplus \Im_2 = \Im_2 \oplus \Im_1$;
2) $\Im_1 \otimes \Im_2 = \Im_2 \otimes \Im_1$;
3) $\dagger(\Im_1 \oplus \Im_2) = \dagger\Im_1 \oplus \dagger\Im_2$;
4) $(\Im_1 \otimes \Im_2)^\dagger = \Im_1^\dagger \otimes \Im_2^\dagger$;
5) $\dagger_1\Im \oplus \dagger_2\Im = (\dagger_1 + \dagger_2)\Im$;
6) $\Im^{\dagger_1} \otimes \Im^{\dagger_2} = \Im^{\dagger_1+\dagger_2}$;
7) $(\dagger_1\dagger_2)\Im = \dagger_1(\dagger_2\Im)$.

Proof: We verify only parts 1, 3, 5 and 7 and analogously for others.

1. It is obvious.

3. As shown at the bottom of the page, by the Frank operational rule (4) in Definition 6, it results, as (11), shown at the bottom of the next page.

Now, (12), as shown at the bottom of page 8.

From Eqs. (11) and (12), we get $\dagger(\Im_1 \oplus \Im_2) = \dagger\Im_1 \oplus \dagger\Im_2$.

5. Equations (13) and (14), as shown at the bottom of pages 10 and 12, respectively.

Thus, from Eqs. (13) and (14), we get the desired result.

7. As shown in the equation at the bottom of page 12.

From this, we can further write the equation, as shown at the bottom of page 13. ■

Theorem 2: Let $\Im_1 = (\sigma_1 e^{2i\pi ð_{\sigma 1}}, \varsigma_1 e^{2i\pi ð_{\varsigma 1}}, \varrho_1 e^{2i\pi ð_{\varrho 1}})$ and $\Im_2 = (\sigma_2 e^{2i\pi ð_{\sigma 2}}, \varsigma_2 e^{2i\pi ð_{\varsigma 2}}, \varrho_2 e^{2i\pi ð_{\varrho 2}})$ be two CT-SFNs, then

1) $(\Im_1 \oplus \Im_2)^c = \Im_1^c \otimes \Im_2^c$;
2) $(\Im_1 \otimes \Im_2)^c = \Im_1^c \oplus \Im_2^c$.

Proof: The proof is trivial; therefore, it is omitted here. ■

IV. CT-SF FRANK AGGREGATION OPERATORS AND THEOREMS

In the following section, we suggest a set of weighted aggregation operators for CT-SFNs on the basis of the devised Frank operation rules.

A. CT-SF FRANK AVERAGING AGGREGATION OPERATORS

Definition 11: Let $\Im_{\dot r} = (\sigma_{\dot r} e^{2i\pi ð_{\sigma\dot r}}, \varsigma_{\dot r} e^{2i\pi ð_{\varsigma\dot r}}, \varrho_{\dot r} e^{2i\pi ð_{\varrho\dot r}})$ $(\dot r = 1(1)n)$ be a class of CT-SFNs, then the Complex T-spherical fuzzy Frank weighted averaging operator

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$$4)\ \dagger\Im_1 = \left(\begin{array}{c} \sqrt[t]{1-\log_£\left(1+\frac{\left(£^{1-\sigma_1^t}-1\right)^\dagger}{(£-1)^{\dagger-1}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_£\left(1+\frac{\left(£^{1-ð^t_{\sigma 1}}-1\right)^\dagger}{(£-1)^{\dagger-1}}\right)}}, \\ \sqrt[t]{\log_£\left(1+\frac{\left(£^{\varsigma_1^t}-1\right)^\dagger}{(£-1)^{\dagger-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_£\left(1+\frac{\left(£^{ð^t_{\varsigma 1}}-1\right)^\dagger}{(£-1)^{\dagger-1}}\right)}}, \\ \sqrt[t]{\log_£\left(1+\frac{\left(£^{\varrho_1^t}-1\right)^\dagger}{(£-1)^{\dagger-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_£\left(1+\frac{\left(£^{ð^t_{\varrho 1}}-1\right)^\dagger}{(£-1)^{\dagger-1}}\right)}} \end{array}\right);$$

$$3.\ \Im_1 \oplus \Im_2 = \left(\begin{array}{c} \sqrt[t]{1-\log_£\left(1+\frac{\left(£^{1-\sigma_1^t}-1\right)\left(£^{1-\sigma_2^t}-1\right)}{£-1}\right)}\, e^{2i\pi\sqrt[t]{1-\log_£\left(1+\frac{\left(£^{1-ð^t_{\sigma 1}}-1\right)\left(£^{1-ð^t_{\sigma 2}}-1\right)}{£-1}\right)}}, \\ \sqrt[t]{\log_£\left(1+\frac{\left(£^{\varsigma_1^t}-1\right)\left(£^{\varsigma_2^t}-1\right)}{£-1}\right)}\, e^{2i\pi\sqrt[t]{\log_£\left(1+\frac{\left(£^{ð^t_{\varsigma 1}}-1\right)\left(£^{ð^t_{\varsigma 2}}-1\right)}{£-1}\right)}}, \\ \sqrt[t]{\log_£\left(1+\frac{\left(£^{\varrho_1^t}-1\right)\left(£^{\varrho_2^t}-1\right)}{£-1}\right)}\, e^{2i\pi\sqrt[t]{\log_£\left(1+\frac{\left(£^{ð^t_{\varrho 1}}-1\right)\left(£^{ð^t_{\varrho 2}}-1\right)}{£-1}\right)}} \end{array}\right),$$

(CT-SFFWA) is:

$$CT-SFFWA\left(\Im_1,\Im_2,\ldots,\Im_n\right)=\oplus_{\dot{r}=1}^{n}\left(\perp_{\dot{r}}\Im_{\dot{r}}\right),z \quad (15)$$

where $\perp = (\perp_1, \perp_2, \ldots, \perp_n)^T$ is the weight vector of $\Im_{\dot{r}}$ $(\dot{r}=1(1)n)$ satisfying that $\perp_{\dot{r}} > 0$ and $\sum_{\dot{r}=1}^{n}\perp_{\dot{r}} = 1$.

$$\dagger\left(\Im_1\oplus\Im_2\right)=\left(\begin{array}{c}
\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\sigma_1^t}-1\right)\left(\pounds^{1-\sigma_2^t}-1\right)}{\pounds-1}\right)}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\eth_{\sigma 1}^t}-1\right)\left(\pounds^{1-\eth_{\sigma 2}^t}-1\right)}{\pounds-1}\right)}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}},\\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\varsigma_1^t}-1\right)\left(\pounds^{\varsigma_2^t}-1\right)}{\pounds-1}\right)}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varsigma 1}^t}-1\right)\left(\pounds^{\eth_{\varsigma 2}^t}-1\right)}{\pounds-1}\right)}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}},\\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\varrho_1^t}-1\right)\left(\pounds^{\varrho_2^t}-1\right)}{\pounds-1}\right)}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varrho 1}^t}-1\right)\left(\pounds^{\eth_{\varrho 2}^t}-1\right)}{\pounds-1}\right)}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}}
\end{array}\right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\sigma_1^t}-1\right)^{\dagger}\left(£^{1-\sigma_2^t}-1\right)^{\dagger}}{(£-1)^{2\dagger-1}}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth_{\sigma 1}^t}-1\right)^{\dagger}\left(£^{1-\eth_{\sigma 2}^t}-1\right)^{\dagger}}{(£-1)^{2\dagger-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varsigma_1^t}-1\right)^{\dagger}\left(£^{\varsigma_2^t}-1\right)^{\dagger}}{(£-1)^{2\dagger-1}}\right)}\, e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varsigma 1}^t}-1\right)^{\dagger}\left(£^{\eth_{\varsigma 2}^t}-1\right)^{\dagger}}{(£-1)^{2\dagger-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varrho_1^t}-1\right)^{\dagger}\left(£^{\varrho_2^t}-1\right)^{\dagger}}{(£-1)^{2\dagger-1}}\right)}\, e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varrho 1}^t}-1\right)^{\dagger}\left(£^{\eth_{\varrho 2}^t}-1\right)^{\dagger}}{(£-1)^{2\dagger-1}}\right)}} \end{array} \right). \qquad (11)$$

$$\dagger\Im_1 \oplus \dagger\Im_2 = \left(\begin{array}{c} \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\sigma_1^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth_{\sigma 1}^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varsigma_1^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}\, e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varsigma 1}^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varrho_1^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}\, e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varrho 1}^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}} \end{array} \right)$$

$$\oplus \left(\begin{array}{c} \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\sigma_2^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth_{\sigma 2}^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varsigma_2^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}\, e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varsigma 2}^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varrho_2^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}\, e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varrho 2}^t}-1\right)^{\dagger}}{(£-1)^{\dagger-1}}\right)}} \end{array} \right)$$

$$
= \left(\begin{array}{l}
\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\sigma_1^t}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}-1\right)\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\sigma_2^t}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}-1\right)}{\pounds-1}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\eth_{\sigma_1}^t}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}-1\right)\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\eth_{\sigma_2}^t}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}-1\right)}{\pounds-1}\right)}}, \\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\varsigma_1^t}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}-1\right)\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\varsigma_2^t}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}-1\right)}{\pounds-1}\right)}\, e^{2i\pi \sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varsigma_1}^t}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}-1\right)\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varsigma_2}^t}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}-1\right)}{\pounds-1}\right)}}, \\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\varrho_1^t}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}-1\right)\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\varrho_2^t}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}-1\right)}{\pounds-1}\right)}\, e^{2i\pi \sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varrho_1}^t}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}-1\right)\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varrho_2}^t}-1\right)^{\dagger}}{(\pounds-1)^{\dagger-1}}\right)}-1\right)}{\pounds-1}\right)}}
\end{array} \right)
$$

$$= \begin{pmatrix} \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\sigma_1^t}-1\right)^{\dagger}\left(£^{1-\sigma_2^t}-1\right)^{\dagger}}{(£-1)^{2\dagger-1}}\right)} e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth_{\sigma 1}^t}-1\right)^{\dagger}\left(£^{1-\eth_{\sigma 2}^t}-1\right)^{\dagger}}{(£-1)^{2\dagger-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varsigma_1^t}-1\right)^{\dagger}\left(£^{\varsigma_2^t}-1\right)^{\dagger}}{(£-1)^{2\dagger-1}}\right)} e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varsigma 1}^t}-1\right)^{\dagger}\left(£^{\eth_{\varsigma 2}^t}-1\right)^{\dagger}}{(£-1)^{2\dagger-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varrho_1^t}-1\right)^{\dagger}\left(£^{\varrho_2^t}-1\right)^{\dagger}}{(£-1)^{2\dagger-1}}\right)} e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varrho 1}^t}-1\right)^{\dagger}\left(£^{\eth_{\varrho 2}^t}-1\right)^{\dagger}}{(£-1)^{2\dagger-1}}\right)}} \end{pmatrix}. \tag{12}$$

Especially if $\perp = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the CT-SFFWA operator reduces to the Complex T-spherical fuzzy Frank averaging (CT-SFFA) operator of dimension n, which is described as below:

$$CT-SFFA\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \frac{1}{n} \oplus_{\dot{r}=1}^{n} \left(\Im_{\dot{r}}\right). \tag{16}$$

Theorem 3: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma \dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma \dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho \dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, then the result acquired by utilizing the CT-SFFWA operator is still a CT-SFN, and (17), as shown at the bottom of page 14.

Proof: We verify it using mathematical induction on n.

5.

$$\dagger_1\Im \oplus \dagger_2\Im = \begin{pmatrix} \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\sigma^t}-1\right)^{\dagger_1}}{(£-1)^{\dagger_1-1}}\right)} e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth_{\sigma}^t}-1\right)^{\dagger_1}}{(£-1)^{\dagger_1-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varsigma^t}-1\right)^{\dagger_1}}{(£-1)^{\dagger_1-1}}\right)} e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varsigma}^t}-1\right)^{\dagger_1}}{(£-1)^{\dagger_1-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varrho^t}-1\right)^{\dagger_1}}{(£-1)^{\dagger_1-1}}\right)} e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varrho}^t}-1\right)^{\dagger_1}}{(£-1)^{\dagger_1-1}}\right)}} \end{pmatrix} \oplus \begin{pmatrix} \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\sigma^t}-1\right)^{\dagger_2}}{(£-1)^{\dagger_2-1}}\right)} e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth_{\sigma}^t}-1\right)^{\dagger_2}}{(£-1)^{\dagger_2-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varsigma^t}-1\right)^{\dagger_2}}{(£-1)^{\dagger_2-1}}\right)} e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varsigma}^t}-1\right)^{\dagger_2}}{(£-1)^{\dagger_2-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\varrho^t}-1\right)^{\dagger_2}}{(£-1)^{\dagger_2-1}}\right)} e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varrho}^t}-1\right)^{\dagger_2}}{(£-1)^{\dagger_2-1}}\right)}} \end{pmatrix}$$

$$
=\left(\begin{array}{l}
\sqrt[t]{1-\log_£\left(1+\frac{\left(£^{\log_£\left(1+\frac{\left(£^{1-\sigma^t}-1\right)^{†_1}}{(£-1)^{†_1-1}}\right)}-1\right)\left(£^{\log_£\left(1+\frac{\left(£^{1-\sigma^t}-1\right)^{†_2}}{(£-1)^{†_2-1}}\right)}-1\right)}{£-1}\right)}\, e^{2i\pi\sqrt[t]{1-\log_£\left(1+\frac{\left(£^{\log_£\left(1+\frac{\left(£^{1-\eth_\sigma^t}-1\right)^{†_1}}{(£-1)^{†_1-1}}\right)}-1\right)\left(£^{\log_£\left(1+\frac{\left(£^{1-\eth_\sigma^t}-1\right)^{†_2}}{(£-1)^{†_2-1}}\right)}-1\right)}{£-1}\right)}},\\
\sqrt[t]{\log_£\left(1+\frac{\left(£^{\log_£\left(1+\frac{\left(£^{\varsigma^t}-1\right)^{†_1}}{(£-1)^{†_1-1}}\right)}-1\right)\left(£^{\log_£\left(1+\frac{\left(£^{\varsigma^t}-1\right)^{†_2}}{(£-1)^{†_2-1}}\right)}-1\right)}{£-1}\right)}\, e^{2i\pi\sqrt[t]{\log_£\left(1+\frac{\left(£^{\log_£\left(1+\frac{\left(£^{\eth_\varsigma^t}-1\right)^{†_1}}{(£-1)^{†_1-1}}\right)}-1\right)\left(£^{\log_£\left(1+\frac{\left(£^{\eth_\varsigma^t}-1\right)^{†_2}}{(£-1)^{†_2-1}}\right)}-1\right)}{£-1}\right)}},\\
\sqrt[t]{\log_£\left(1+\frac{\left(£^{\log_£\left(1+\frac{\left(£^{\varrho^t}-1\right)^{†_1}}{(£-1)^{†_1-1}}\right)}-1\right)\left(£^{\log_£\left(1+\frac{\left(£^{\varrho^t}-1\right)^{†_2}}{(£-1)^{†_2-1}}\right)}-1\right)}{£-1}\right)}\, e^{2i\pi\sqrt[t]{\log_£\left(1+\frac{\left(£^{\log_£\left(1+\frac{\left(£^{\eth_\varrho^t}-1\right)^{†_1}}{(£-1)^{†_1-1}}\right)}-1\right)\left(£^{\log_£\left(1+\frac{\left(£^{\eth_\varrho^t}-1\right)^{†_2}}{(£-1)^{†_2-1}}\right)}-1\right)}{£-1}\right)}}
\end{array}\right)
$$

$$
=\left(\begin{array}{c}
\sqrt[t]{1-\log_£\left(1+\frac{\left(£^{1-\sigma^t}-1\right)^{†_1+†_2}}{(£-1)^{†_1+†_2-1}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_£\left(1+\frac{\left(£^{1-\eth_\sigma^t}-1\right)^{†_1+†_2}}{(£-1)^{†_1+†_2-1}}\right)}},\\
\sqrt[t]{\log_£\left(1+\frac{\left(£^{\varsigma^t}-1\right)^{†_1+†_2}}{(£-1)^{†_1+†_2-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_£\left(1+\frac{\left(£^{\eth_\varsigma^t}-1\right)^{†_1+†_2}}{(£-1)^{†_1+†_2-1}}\right)}},\\
\sqrt[t]{\log_£\left(1+\frac{\left(£^{\varrho^t}-1\right)^{†_1+†_2}}{(£-1)^{†_1+†_2-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_£\left(1+\frac{\left(£^{\eth_\varrho^t}-1\right)^{†_1+†_2}}{(£-1)^{†_1+†_2-1}}\right)}}
\end{array}\right). \tag{13}
$$

For $n = 2$, we get the equation, as shown at the bottom of pages 14 and 15.

Thus, the result holds for $n = 2$.

If Eq. (17) holds for $n = k$, then for $n = k + 1$, we get the equation, as shown at the bottom of pages 16–18.

Thus, the result holds for $n = k + 1$, and hence, by the principle of mathematical induction, the result disclosed in Eq. (17) holds for all positive integer n. ■

Example 1: Let $\Im_1 = (0.4e^{2i\pi 0.3}, 0.3e^{2i\pi 0.4}, 0.5e^{2i\pi 0.5})$, $\Im_2 = (0.7e^{2i\pi 0.6}, 0.3e^{2i\pi 0.4}, 0.4e^{2i\pi 0.4})$, $\Im_3 = (0.6e^{2i\pi 0.5}, 0.7e^{2i\pi 0.8}, 0.8e^{2i\pi 0.8})$ be three CT-SFNs, and $\perp = (0.4, 0.3, 0.3)^T$ be the weight vector of $\Im_{\dot{r}} (\dot{r} = 1, 2, 3)$. Suppose $£ = 2$, then according to Definition 11 and Theorem 3, we can obtain (t=4), as shown in the equation at the bottom of page 18.

Theorem 4: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma \dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma \dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho \dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, and $£ > 1$.

As $£ \longrightarrow 1$, the CT-SFFWA operator approaches the following limit

$$\lim_{£ \longrightarrow 1} CT - SFFWA\,(\Im_1, \Im_2, \ldots, \Im_n)$$

$$= \begin{pmatrix} \sqrt[t]{1 - \prod_{\dot{r}=1}^{n} \left(1 - \sigma_{\dot{r}}^t\right)^{\perp_{\dot{r}}}}\, e^{2i\pi \sqrt[t]{1 - \prod_{\dot{r}=1}^{n} \left(1 - \eth_{\sigma_{\dot{r}}}^t\right)^{\perp_{\dot{r}}}}}, \\ \sqrt[t]{\prod_{\dot{r}=1}^{n} \left(\varsigma_{\dot{r}}^t\right)^{\perp_{\dot{r}}}}\, e^{2i\pi \sqrt[t]{\prod_{\dot{r}=1}^{n} \left(\eth_{\varsigma_{\dot{r}}}^t\right)^{\perp_{\dot{r}}}}}, \\ \sqrt[t]{\prod_{\dot{r}=1}^{n} \left(\varrho_{\dot{r}}^t\right)^{\perp_{\dot{r}}}}\, e^{2i\pi \sqrt[t]{\prod_{\dot{r}=1}^{n} \left(\eth_{\varrho_{\dot{r}}}^t\right)^{\perp_{\dot{r}}}}} \end{pmatrix}. \quad (18)$$

Proof: As $£ \longrightarrow 1$, then $\left(\prod_{\dot{r}=1}^{n} \left(£^{1-\sigma_{\dot{r}}^t} - 1\right)^{\perp_{\dot{r}}}, \prod_{\dot{r}=1}^{n} \left(£^{\varsigma_{\dot{r}}^t} - 1\right)^{\perp_{\dot{r}}}, \prod_{\dot{r}=1}^{n} \left(£^{\varrho_{\dot{r}}^t} - 1\right)^{\perp_{\dot{r}}}\right) \longrightarrow \langle 0, 0, 0\rangle$ by log property and the rule of infinitesimal changes, we obtain

$$\log_£ \left(1 + \prod_{\dot{r}=1}^{n} \left(£^{1-\sigma_{\dot{r}}^t} - 1\right)^{\perp_{\dot{r}}}\right)$$

$$= \frac{\ln \left(1 + \prod_{\dot{r}=1}^{n} \left(£^{1-\sigma_{\dot{r}}^t} - 1\right)^{\perp_{\dot{r}}}\right)}{\ln £} \longrightarrow \frac{\prod_{\dot{r}=1}^{n} \left(£^{1-\sigma_{\dot{r}}^t} - 1\right)^{\perp_{\dot{r}}}}{\ln £}$$

$$\log_£ \left(1 + \prod_{\dot{r}=1}^{n} \left(£^{\varsigma_{\dot{r}}^t} - 1\right)^{\perp_{\dot{r}}}\right)$$

$$(\dagger_1 + \dagger_2)\, \Im = \begin{pmatrix} \sqrt[t]{1 - \log_£ \left(1 + \frac{\left(£^{1-\sigma^t} - 1\right)^{\dagger_1 + \dagger_2}}{(£-1)^{\dagger_1 + \dagger_2 - 1}}\right)}\, e^{2i\pi \sqrt[t]{1 - \log_£ \left(1 + \frac{\left(£^{1-\eth_\sigma^t} - 1\right)^{\dagger_1 + \dagger_2}}{(£-1)^{\dagger_1 + \dagger_2 - 1}}\right)}}, \\ \sqrt[t]{\log_£ \left(1 + \frac{\left(£^{\varsigma^t} - 1\right)^{\dagger_1 + \dagger_2}}{(£-1)^{\dagger_1 + \dagger_2 - 1}}\right)}\, e^{2i\pi \sqrt[t]{\log_£ \left(1 + \frac{\left(£^{\eth_\varsigma^t} - 1\right)^{\dagger_1 + \dagger_2}}{(£-1)^{\dagger_1 + \dagger_2 - 1}}\right)}}, \\ \sqrt[t]{\log_£ \left(1 + \frac{\left(£^{\varrho^t} - 1\right)^{\dagger_1 + \dagger_2}}{(£-1)^{\dagger_1 + \dagger_2 - 1}}\right)}\, e^{2i\pi \sqrt[t]{\log_£ \left(1 + \frac{\left(£^{\eth_\varrho^t} - 1\right)^{\dagger_1 + \dagger_2}}{(£-1)^{\dagger_1 + \dagger_2 - 1}}\right)}} \end{pmatrix}. \quad (14)$$

$$\dagger_2 \Im = \begin{pmatrix} \sqrt[t]{1 - \log_£ \left(1 + \frac{\left(£^{1-\sigma^t} - 1\right)^{\dagger_2}}{(£-1)^{\dagger_2 - 1}}\right)}\, e^{2i\pi \sqrt[t]{1 - \log_£ \left(1 + \frac{\left(£^{1-\eth_\sigma^t} - 1\right)^{\dagger_2}}{(£-1)^{\dagger_2 - 1}}\right)}}, \\ \sqrt[t]{\log_£ \left(1 + \frac{\left(£^{\varsigma^t} - 1\right)^{\dagger_2}}{(£-1)^{\dagger_2 - 1}}\right)}\, e^{2i\pi \sqrt[t]{\log_£ \left(1 + \frac{\left(£^{\eth_\varsigma^t} - 1\right)^{\dagger_2}}{(£-1)^{\dagger_2 - 1}}\right)}}, \\ \sqrt[t]{\log_£ \left(1 + \frac{\left(£^{\varrho^t} - 1\right)^{\dagger_2}}{(£-1)^{\dagger_2 - 1}}\right)}\, e^{2i\pi \sqrt[t]{\log_£ \left(1 + \frac{\left(£^{\eth_\varrho^t} - 1\right)^{\dagger_2}}{(£-1)^{\dagger_2 - 1}}\right)}} \end{pmatrix}.$$

$$
\begin{aligned}
\dagger_1\left(\dagger_2\Im\right) &= \left(\begin{array}{c}
\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\sigma^t}-1\right)^{\dagger_2}}{(\pounds-1)^{\dagger_2-1}}\right)}-1\right)^{\dagger_1}}{(\pounds-1)^{\dagger_1-1}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\eth_{\sigma}^t}-1\right)^{\dagger_2}}{(\pounds-1)^{\dagger_2-1}}\right)}-1\right)^{\dagger_1}}{(\pounds-1)^{\dagger_1-1}}\right)}},\\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\varsigma^t}-1\right)^{\dagger_2}}{(\pounds-1)^{\dagger_2-1}}\right)}-1\right)^{\dagger_1}}{(\pounds-1)^{\dagger_1-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varsigma}^t}-1\right)^{\dagger_2}}{(\pounds-1)^{\dagger_2-1}}\right)}-1\right)^{\dagger_1}}{(\pounds-1)^{\dagger_1-1}}\right)}},\\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\varrho^t}-1\right)^{\dagger_2}}{(\pounds-1)^{\dagger_2-1}}\right)}-1\right)^{\dagger_1}}{(\pounds-1)^{\dagger_1-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varrho}^t}-1\right)^{\dagger_2}}{(\pounds-1)^{\dagger_2-1}}\right)}-1\right)^{\dagger_1}}{(\pounds-1)^{\dagger_1-1}}\right)}}
\end{array}\right)\\
&= \left(\begin{array}{c}
\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\sigma^t}-1\right)^{\dagger_1\dagger_2}}{(\pounds-1)^{\dagger_1\dagger_2-1}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\eth_{\sigma}^t}-1\right)^{\dagger_1\dagger_2}}{(\pounds-1)^{\dagger_1\dagger_2-1}}\right)}},\\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\varsigma^t}-1\right)^{\dagger_1\dagger_2}}{(\pounds-1)^{\dagger_1\dagger_2-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varsigma}^t}-1\right)^{\dagger_1\dagger_2}}{(\pounds-1)^{\dagger_1\dagger_2-1}}\right)}},\\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\varrho^t}-1\right)^{\dagger_1\dagger_2}}{(\pounds-1)^{\dagger_1\dagger_2-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varrho}^t}-1\right)^{\dagger_1\dagger_2}}{(\pounds-1)^{\dagger_1\dagger_2-1}}\right)}}
\end{array}\right) = \left(\dagger_1\dagger_2\right)\Im.
\end{aligned}
$$

$$= \frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\varsigma_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}}\right)}{\ln\pounds} \longrightarrow \frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{\varsigma_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}}}{\ln\pounds}$$

$$\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\varrho_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}}\right)$$

$$= \frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\varrho_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}}\right)}{\ln\pounds} \longrightarrow \frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{\varrho_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}}}{\ln\pounds}$$

Based upon Taylor's expansion rules, we get

$$\pounds^{1-\sigma_{\dot{r}}^t} = 1+\left(1-\sigma_{\dot{r}}^t\right)\ln\pounds+\frac{\left(\left(1-\sigma_{\dot{r}}^t\right)\ln\pounds\right)^2}{2!}+\ldots$$

$$\pounds^{\varsigma_{\dot{r}}^t} = 1+\left(\varsigma_{\dot{r}}^t\right)\ln\pounds+\frac{\left(\left(\varsigma_{\dot{r}}^t\right)\ln\pounds\right)^2}{2!}+\ldots$$

$$\pounds^{\varrho_{\dot{r}}^t} = 1+\left(\varrho_{\dot{r}}^t\right)\ln\pounds+\frac{\left(\left(\varrho_{\dot{r}}^t\right)\ln\pounds\right)^2}{2!}+\ldots$$

$$CT-SFFWA\left(\Im_1,\Im_2,\ldots,\Im_n\right) = \begin{pmatrix} \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\sigma_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth_{\sigma_{\dot{r}}}^t}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\varsigma_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\varsigma_{\dot{r}}}^t}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\varrho_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\varrho_{\dot{r}}}^t}-1\right)^{\perp_{\dot{r}}}\right)}} \end{pmatrix}. \tag{17}$$

$$CT-SFFWA\left(\Im_1,\Im_2\right)$$

$$=\perp_1\Im_1\oplus\perp_2\Im_2$$

$$= \begin{pmatrix} \sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\sigma_1^t}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}\right)\right)}-1\right)\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\sigma_2^t}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}\right)\right)}-1\right)}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\eth_{\sigma_1}^t}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}\right)\right)}-1\right)\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\eth_{\sigma_2}^t}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}\right)\right)}-1\right)}{\pounds-1}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\varsigma_1^t}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}\right)\right)}-1\right)\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\varsigma_2^t}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}\right)\right)}-1\right)}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varsigma_1}^t}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}\right)\right)}-1\right)\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varsigma_2}^t}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}\right)\right)}-1\right)}{\pounds-1}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\varrho_1^t}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}\right)\right)}-1\right)\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\varrho_2^t}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}\right)\right)}-1\right)}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varrho_1}^t}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}\right)\right)}-1\right)\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth_{\varrho_2}^t}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}\right)\right)}-1\right)}{\pounds-1}\right)}} \end{pmatrix}$$

$$
= \begin{pmatrix}
\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{1-\sigma_1^t}-1\right)^{\perp 1}}{(\pounds-1)^{\perp 1-1}}-1\right)\left(1+\frac{\left(\pounds^{1-\sigma_2^t}-1\right)^{\perp 2}}{(\pounds-1)^{\perp 2-1}}-1\right)}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{1-\eth^t_{\sigma_1}}-1\right)^{\perp 1}}{(\pounds-1)^{\perp 1-1}}-1\right)\left(1+\frac{\left(\pounds^{1-\eth^t_{\sigma_2}}-1\right)^{\perp 2}}{(\pounds-1)^{\perp 2-1}}-1\right)}{\pounds-1}\right)}}, \\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{\varsigma_1^t}-1\right)^{\perp 1}}{(\pounds-1)^{\perp 1-1}}-1\right)\left(1+\frac{\left(\pounds^{\varsigma_2^t}-1\right)^{\perp 2}}{(\pounds-1)^{\perp 2-1}}-1\right)}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{\eth^t_{\varsigma_1}}-1\right)^{\perp 1}}{(\pounds-1)^{\perp 1-1}}-1\right)\left(1+\frac{\left(\pounds^{\eth^t_{\varsigma_2}}-1\right)^{\perp 2}}{(\pounds-1)^{\perp 2-1}}-1\right)}{\pounds-1}\right)}}, \\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{\varrho_1^t}-1\right)^{\perp 1}}{(\pounds-1)^{\perp 1-1}}-1\right)\left(1+\frac{\left(\pounds^{\varrho_2^t}-1\right)^{\perp 2}}{(\pounds-1)^{\perp 2-1}}-1\right)}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{\eth^t_{\varrho_1}}-1\right)^{\perp 1}}{(\pounds-1)^{\perp 1-1}}-1\right)\left(1+\frac{\left(\pounds^{\eth^t_{\varrho_2}}-1\right)^{\perp 2}}{(\pounds-1)^{\perp 2-1}}-1\right)}{\pounds-1}\right)}}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{1-\sigma_1^t}-1\right)^{\perp 1}}{(\pounds-1)^{\perp 1-1}}-1\right)\left(1+\frac{\left(\pounds^{1-\sigma_2^t}-1\right)^{\perp 2}}{(\pounds-1)^{\perp 2-1}}-1\right)}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{1-\eth^t_{\sigma_1}}-1\right)^{\perp 1}}{(\pounds-1)^{\perp 1-1}}-1\right)\left(1+\frac{\left(\pounds^{1-\eth^t_{\sigma_2}}-1\right)^{\perp 2}}{(\pounds-1)^{\perp 2-1}}-1\right)}{\pounds-1}\right)}}, \\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{\varsigma_1^t}-1\right)^{\perp 1}}{(\pounds-1)^{\perp 1-1}}-1\right)\left(1+\frac{\left(\pounds^{\varsigma_2^t}-1\right)^{\perp 2}}{(\pounds-1)^{\perp 2-1}}-1\right)}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{\eth^t_{\varsigma_1}}-1\right)^{\perp 1}}{(\pounds-1)^{\perp 1-1}}-1\right)\left(1+\frac{\left(\pounds^{\eth^t_{\varsigma_2}}-1\right)^{\perp 2}}{(\pounds-1)^{\perp 2-1}}-1\right)}{\pounds-1}\right)}}, \\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{\varrho_1^t}-1\right)^{\perp 1}}{(\pounds-1)^{\perp 1-1}}-1\right)\left(1+\frac{\left(\pounds^{\varrho_2^t}-1\right)^{\perp 2}}{(\pounds-1)^{\perp 2-1}}-1\right)}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{\eth^t_{\varrho_1}}-1\right)^{\perp 1}}{(\pounds-1)^{\perp 1-1}}-1\right)\left(1+\frac{\left(\pounds^{\eth^t_{\varrho_2}}-1\right)^{\perp 2}}{(\pounds-1)^{\perp 2-1}}-1\right)}{\pounds-1}\right)}}
\end{pmatrix}
$$

$$= \left(\begin{array}{c} \sqrt[t]{1 - \log_{£}\left(1 + \left(\left(£^{1-\sigma_1^t} - 1\right)^{\perp_1}\right)\left(\left(£^{1-\sigma_2^t} - 1\right)^{\perp_2}\right)\right)} e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\left(\left(£^{1-\eth_{\sigma 1}^t}-1\right)^{\perp_1}\right)\left(\left(£^{1-\eth_{\sigma 2}^t}-1\right)^{\perp_2}\right)\right)}}, \\ \sqrt[t]{\log_{£}\left(1 + \left(\left(£^{\varsigma_1^t} - 1\right)^{\perp_1}\right)\left(\left(£^{\varsigma_2^t} - 1\right)^{\perp_2}\right)\right)} e^{2i\pi \sqrt[t]{\log_{£}\left(1+\left(\left(£^{\eth_{\varsigma 1}^t}-1\right)^{\perp_1}\right)\left(\left(£^{\eth_{\varsigma 2}^t}-1\right)^{\perp_2}\right)\right)}}, \\ \sqrt[t]{\log_{£}\left(1 + \left(\left(£^{\varrho_1^t} - 1\right)^{\perp_1}\right)\left(\left(£^{\varrho_2^t} - 1\right)^{\perp_2}\right)\right)} e^{2i\pi \sqrt[t]{\log_{£}\left(1+\left(\left(£^{\eth_{\varrho 1}^t}-1\right)^{\perp_1}\right)\left(\left(£^{\eth_{\varrho 2}^t}-1\right)^{\perp_2}\right)\right)}} \end{array} \right).$$

Also, since $£ > 1$, then $\ln £ > 0$, $£^{1-\sigma_{\dot r}^t} = 1 + \left(1-\sigma_{\dot r}^t\right)\ln £ + O(\ln £)$, $£^{\varsigma_{\dot r}^t} = 1 + \left(\varsigma_{\dot r}^t\right)\ln £ + O(\ln £)$, $£^{\varrho_{\dot r}^t} = 1 + \left(\varrho_{\dot r}^t\right)\ln £ + O(\ln £)$.

It follows that

$$\left(£^{1-\sigma_{\dot r}^t} - 1\right)^{\perp_{\dot r}} \longrightarrow \left(\left(1-\sigma_{\dot r}^t\right)\ln £\right)^{\perp_{\dot r}}$$

$$\prod_{\dot r=1}^{n}\left(£^{1-\sigma_{\dot r}^t} - 1\right)^{\perp_{\dot r}} \longrightarrow \prod_{\dot r=1}^{n}\left(1-\sigma_{\dot r}^t\right)^{\perp_{\dot r}}\prod_{\dot r=1}^{n}(\ln £)^{\perp_{\dot r}}$$

$$\prod_{\dot r=1}^{n}\left(£^{1-\sigma_{\dot r}^t} - 1\right)^{\perp_{\dot r}} \longrightarrow \prod_{\dot r=1}^{n}\left(1-\sigma_{\dot r}^t\right)^{\perp_{\dot r}}\ln(£)^{\sum_{\dot r=1}^{n}\perp_{\dot r}}$$

$$\frac{\prod_{\dot r=1}^{n}\left(£^{1-\sigma_{\dot r}^t} - 1\right)^{\perp_{\dot r}}}{\ln £} \longrightarrow \prod_{\dot r=1}^{n}\left(1-\sigma_{\dot r}^t\right).$$

$$\begin{aligned} &CT-SFFWA\left(\Im_1, \Im_2, \ldots, \Im_{k+1}\right) \\ &= CT-SFFWA\left(\Im_1, \Im_2, \ldots, \Im_k\right) \oplus \perp_{k+1}\Im_{k+1} \\ &= \left(\begin{array}{c} \sqrt[t]{1 - \log_{£}\left(1 + \prod_{\dot r=1}^{k}\left(£^{1-\sigma_{\dot r}^t} - 1\right)^{\perp_{\dot r}}\right)} e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot r=1}^{k}\left(£^{1-\eth_{\sigma \dot r}^t}-1\right)^{\perp_{\dot r}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1 + \prod_{\dot r=1}^{k}\left(£^{\varsigma_{\dot r}^t} - 1\right)^{\perp_{\dot r}}\right)} e^{2i\pi \sqrt[t]{\log_{£}\left(1+\prod_{\dot r=1}^{k}\left(£^{\eth_{\varsigma \dot r}^t}-1\right)^{\perp_{\dot r}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1 + \prod_{\dot r=1}^{k}\left(£^{\varrho_{\dot r}^t} - 1\right)^{\perp_{\dot r}}\right)} e^{2i\pi \sqrt[t]{\log_{£}\left(1+\prod_{\dot r=1}^{k}\left(£^{\eth_{\varrho \dot r}^t}-1\right)^{\perp_{\dot r}}\right)}} \end{array} \right) \\ &\oplus \left(\begin{array}{c} \sqrt[t]{1 - \log_{£}\left(1 + \frac{\left(£^{1-\sigma_{(k+1)}^t}-1\right)^{\perp(k+1)}}{(£-1)^{\perp(k+1)-1}}\right)} e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth_{\sigma(k+1)}^t}-1\right)^{\perp(k+1)}}{(£-1)^{\perp(k+1)-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1 + \frac{\left(£^{\varsigma_{(k+1)}^t}-1\right)^{\perp(k+1)}}{(£-1)^{\perp(k+1)-1}}\right)} e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varsigma(k+1)}^t}-1\right)^{\perp(k+1)}}{(£-1)^{\perp(k+1)-1}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1 + \frac{\left(£^{\varrho_{(k+1)}^t}-1\right)^{\perp(k+1)}}{(£-1)^{\perp(k+1)-1}}\right)} e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth_{\varrho(k+1)}^t}-1\right)^{\perp(k+1)}}{(£-1)^{\perp(k+1)-1}}\right)}} \end{array} \right) \end{aligned}$$

$$
= \left(\begin{array}{c}
\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{1-\sigma^t_{\dot r}}-1\right)^{\perp_{\dot r}}\frac{\left(\pounds^{1-\sigma^t_{(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\perp_{(k+1)}-1}}}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{1-\eth^t_{\sigma_{\dot r}}}-1\right)^{\perp_{\dot r}}\frac{\left(\pounds^{1-\eth^t_{\sigma(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\perp_{(k+1)}-1}}}{\pounds-1}\right)}}, \\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{\varsigma^t_{\dot r}}-1\right)^{\perp_{\dot r}}\frac{\left(\pounds^{\varsigma^t_{(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\perp_{(k+1)}-1}}}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{\eth^t_{\varsigma_{\dot r}}}-1\right)^{\perp_{\dot r}}\frac{\left(\pounds^{\eth^t_{\varsigma(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\perp_{(k+1)}-1}}}{\pounds-1}\right)}}, \\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{\varrho^t_{\dot r}}-1\right)^{\perp_{\dot r}}\frac{\left(\pounds^{\varrho^t_{(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\perp_{(k+1)}-1}}}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{\eth^t_{\varrho_{\dot r}}}-1\right)^{\perp_{\dot r}}\frac{\left(\pounds^{\eth^t_{\varrho(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\perp_{(k+1)}-1}}}{\pounds-1}\right)}}
\end{array} \right)
$$

$$
= \left(\begin{array}{c}
\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{1-\sigma^t_{\dot r}}-1\right)^{\perp_{\dot r}}\left(\pounds^{1-\sigma^t_{(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot r=1}^{k}\perp_{(\dot r)}-1}(\pounds-1)(\pounds-1)^{\perp_{(k+1)}-1}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{1-\eth^t_{\sigma_{\dot r}}}-1\right)^{\perp_{\dot r}}\left(\pounds^{1-\eth^t_{\sigma(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot r=1}^{k}\perp_{(\dot r)}-1}(\pounds-1)(\pounds-1)^{\perp_{(k+1)}-1}}\right)}}, \\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{\varsigma^t_{\dot r}}-1\right)^{\perp_{\dot r}}\left(\pounds^{\varsigma^t_{(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot r=1}^{k}\perp_{(\dot r)}-1}(\pounds-1)(\pounds-1)^{\perp_{(k+1)}-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{\eth^t_{\varsigma_{\dot r}}}-1\right)^{\perp_{\dot r}}\left(\pounds^{\eth^t_{\varsigma(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot r=1}^{k}\perp_{(\dot r)}-1}(\pounds-1)(\pounds-1)^{\perp_{(k+1)}-1}}\right)}}, \\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{\varrho^t_{\dot r}}-1\right)^{\perp_{\dot r}}\left(\pounds^{\varrho^t_{(k+1)}}-1\right)^{\perp_{(k+1)}}}{\sum_{\dot r=1}^{k}(\pounds-1)^{\perp_{(\dot r)}-1}(\pounds-1)(\pounds-1)^{\perp_{(k+1)}-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{\eth^t_{\varrho_{\dot r}}}-1\right)^{\perp_{\dot r}}\left(\pounds^{\eth^t_{\varrho(k+1)}}-1\right)^{\perp_{(k+1)}}}{\sum_{\dot r=1}^{k}(\pounds-1)^{\perp_{(\dot r)}-1}(\pounds-1)(\pounds-1)^{\perp_{(k+1)}-1}}\right)}}
\end{array} \right)
$$

$$
= \left(\begin{array}{c}
\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{1-\sigma^t_{\dot r}}-1\right)^{\perp_{\dot r}}\left(\pounds^{1-\sigma^t_{(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot r=1}^{k+1}\perp_{(\dot r)}-1}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{1-\eth^t_{\sigma_{\dot r}}}-1\right)^{\perp_{\dot r}}\left(\pounds^{1-\eth^t_{\sigma(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot r=1}^{k+1}\perp_{(\dot r)}-1}}\right)}}, \\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{\varsigma^t_{\dot r}}-1\right)^{\perp_{\dot r}}\left(\pounds^{\varsigma^t_{(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot r=1}^{k+1}\perp_{(\dot r)}-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{\eth^t_{\varsigma_{\dot r}}}-1\right)^{\perp_{\dot r}}\left(\pounds^{\eth^t_{\varsigma(k+1)}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot r=1}^{k+1}\perp_{(\dot r)}-1}}\right)}}, \\
\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{\varrho^t_{\dot r}}-1\right)^{\perp_{\dot r}}\left(\pounds^{\varrho^t_{(k+1)}}-1\right)^{\perp_{(k+1)}}}{\sum_{\dot r=1}^{k+1}(\pounds-1)^{\perp_{(\dot r)}-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot r=1}^{k}\left(\pounds^{\eth^t_{\varrho_{\dot r}}}-1\right)^{\perp_{\dot r}}\left(\pounds^{\eth^t_{\varrho(k+1)}}-1\right)^{\perp_{(k+1)}}}{\sum_{\dot r=1}^{k+1}(\pounds-1)^{\perp_{(\dot r)}-1}}\right)}}
\end{array} \right)
$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{k}\left(£^{1-\sigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(£^{1-\sigma_{(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}\right)} e^{2i\pi\sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{k}\left(£^{1-\eth_{\sigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(£^{1-\eth_{\sigma(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{k}\left(£^{\varsigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(£^{\varsigma_{(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}\right)} e^{2i\pi\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{k}\left(£^{\eth_{\varsigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(£^{\eth_{\varsigma(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{k}\left(£^{\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(£^{\varrho_{(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}\right)} e^{2i\pi\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{k}\left(£^{\eth_{\varrho\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(£^{\eth_{\varrho(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}\right)}} \end{array} \right).$$

Analogously, we can get

$$\frac{\prod_{\dot{r}=1}^{n}\left(£^{\varsigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{\ln £} \longrightarrow \prod_{\dot{r}=1}^{n}\left(\varsigma_{\dot{r}}^{t}\right)^{\perp_{\dot{r}}},$$

$$\frac{\prod_{\dot{r}=1}^{n}\left(£^{\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{\ln £} \longrightarrow \prod_{\dot{r}=1}^{n}\left(\varrho_{\dot{r}}^{t}\right)^{\perp_{\dot{r}}},$$

$$e^{2i\pi\frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\sigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{\ln £}} \longrightarrow e^{2i\pi\prod_{\dot{r}=1}^{n}\left(1-\eth_{\sigma_{\dot{r}}}^{t}\right)^{\perp_{\dot{r}}}},$$

$$e^{2i\pi\frac{\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\varsigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{\ln £}} \longrightarrow e^{2i\pi\prod_{\dot{r}=1}^{n}\left(\eth_{\varsigma_{\dot{r}}}^{t}\right)^{\perp_{\dot{r}}}},$$

and $e^{2i\pi\frac{\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\varrho\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{\ln £}} \longrightarrow e^{2i\pi\prod_{\dot{r}=1}^{n}\left(\eth_{\varrho_{\dot{r}}}^{t}\right)^{\perp_{\dot{r}}}}$.

$$CT-SFFWA\left(\Im_1,\Im_2,\Im_3\right)$$

$$= \left(\begin{array}{c} \sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\sigma_{\dot{r}}^{4}}-1\right)^{\perp_{\dot{r}}}\right)} e^{2i\pi\sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\eth_{\sigma\dot{r}}^{4}}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\varsigma_{\dot{r}}^{4}}-1\right)^{\perp_{\dot{r}}}\right)} e^{2i\pi\sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\eth_{\varsigma\dot{r}}^{4}}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\varrho_{\dot{r}}^{4}}-1\right)^{\perp_{\dot{r}}}\right)} e^{2i\pi\sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\eth_{\varrho\dot{r}}^{4}}-1\right)^{\perp_{\dot{r}}}\right)}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[4]{1-\log_2\left(1+\left(2^{1-0.4^4}-1\right)^{0.4}\left(2^{1-0.7^4}-1\right)^{0.3}\left(2^{1-0.6^4}-1\right)^{0.3}\right)} e^{2i\pi\sqrt[4]{1-\log_2\left(1+\left(2^{1-0.3^4}-1\right)^{0.4}\left(2^{1-0.6^4}-1\right)^{0.3}\left(2^{1-0.5^4}-1\right)^{0.3}\right)}}, \\ \sqrt[4]{\log_2\left(1+\left(2^{0.3^4}-1\right)^{0.4}\left(2^{0.3^4}-1\right)^{0.3}\left(2^{0.7^4}-1\right)^{0.3}\right)} e^{2i\pi\sqrt[4]{\log_2\left(1+\left(2^{0.4^4}-1\right)^{0.4}\left(2^{0.4^4}-1\right)^{0.3}\left(2^{0.8^4}-1\right)^{0.3}\right)}}, \\ \sqrt[4]{\log_2\left(1+\left(2^{0.5^4}-1\right)^{0.4}\left(2^{0.4^4}-1\right)^{0.3}\left(2^{0.8^4}-1\right)^{0.3}\right)} e^{2i\pi\sqrt[4]{\log_2\left(1+\left(2^{0.5^4}-1\right)^{0.4}\left(2^{0.4^4}-1\right)^{0.3}\left(2^{0.8^4}-1\right)^{0.3}\right)}} \end{array} \right)$$

$$= \left(0.5940e^{2i\pi 0.4988}, 0.3887e^{2i\pi 0.4964}, 0.5428e^{2i\pi 0.5428}\right).$$

Then, we have, as shown in the equation at page 20. which completes the proof. ■

Theorem 5: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, and £ > 1. As £ ⟶ ∞, the CT-SFFWA operator approaches the following limit

$$\lim_{£\longrightarrow\infty} CT-SFFWA\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \begin{pmatrix} \sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\sigma_{\dot{r}}^{t}\right)\right)}e^{2i\pi\sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth_{\sigma_{\dot{r}}}^{t}\right)\right)}}, \\ \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\varsigma_{\dot{r}}^{t}\right)\right)}e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth_{\varsigma_{\dot{r}}}^{t}\right)\right)}}, \\ \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\varrho_{\dot{r}}^{t}\right)\right)}e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth_{\varrho_{\dot{r}}}^{t}\right)\right)}} \end{pmatrix}. \tag{19}$$

Proof: According to Theorem 3, we have, as shown in the equation at the bottom of page 20.

Using limit rules, logarithmic transform, and L'Hospital's rule, it follows the equation, as shown at the bottom of pages 21–23, which completes the proof of Theorem 5. ■

Theorem 6 (Idempotency): Let $\Im_{\dot{r}} = (\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}})(\dot{r} = 1(1)n)$ be a class of CT-SFNs, if $\Im_{\dot{r}} = \Im_0 \ \forall \ \dot{r}$, then

$$CT-SFFWA\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \Im_0. \tag{20}$$

Proof: Since for all $\dot{r}$ $\Im_{\dot{r}} = \Im_0 = (\sigma_0e^{2i\pi\eth_{\sigma0}}, \varsigma_0e^{2i\pi\eth_{\varsigma0}}, \varrho_0e^{2i\pi\eth_{\varrho0}})$, and $\sum_{\dot{r}=1}^{n}\perp_{\dot{r}} = 1$ so according to Theorem 3, we get the equation, as shown at the bottom of page 23. ■

Thus, the proof is completed.

Theorem 7 (Monotonicity): Let $\Im_{\dot{r}} = (\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}})(\dot{r} = 1(1)n)$ and $\dot{\Im}_{\dot{r}} = (\dot{\sigma}_{\dot{r}}e^{2i\pi\eth_{\dot{\sigma}\dot{r}}}, \dot{\varsigma}_{\dot{r}}e^{2i\pi\eth_{\dot{\varsigma}1}}, \dot{\varrho}_{\dot{r}}e^{2i\pi\eth_{\dot{\varrho}\dot{r}}})(\dot{r} = 1(1)n)$ be two classes of CT-SFNs such that $\sigma_{\dot{r}} \geq \dot{\sigma}_{\dot{r}}$, $\varsigma_{\dot{r}} \leq \dot{\varsigma}_{\dot{r}}$, $\varrho_{\dot{r}} \leq \dot{\varrho}_{\dot{r}}$, $\eth_{\sigma_{\dot{r}}} \geq \eth_{\dot{\sigma}_{\dot{r}}}$, $\eth_{\varsigma_{\dot{r}}} \leq \eth_{\dot{\varsigma}_{\dot{r}}}$,and $\varrho_{\dot{r}} \leq \eth_{\dot{\varrho}_{\dot{r}}} \ \forall \ \dot{r}$, then

$$CT-SFFWA\left(\Im_1, \Im_2, \ldots, \Im_n\right) \geq CT-SFFWA\left(\dot{\Im}_1, \dot{\Im}_2, \ldots, \dot{\Im}_n\right). \tag{21}$$

Proof: Based on Definition 11, when $\sigma_{\dot{r}} \geq \dot{\sigma}_{\dot{r}}$, $\varsigma_{\dot{r}} \leq \dot{\varsigma}_{\dot{r}}$, $\varrho_{\dot{r}} \leq \dot{\varrho}_{\dot{r}}$, $\eth_{\sigma_{\dot{r}}} \geq \eth_{\dot{\sigma}_{\dot{r}}}$, $\eth_{\varsigma_{\dot{r}}} \leq \eth_{\dot{\varsigma}_{\dot{r}}}$, and $\varrho_{\dot{r}} \leq \eth_{\dot{\varrho}_{\dot{r}}} \ \forall \ \dot{r}$, then

$$\sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\sigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)} \geq \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\dot{\sigma}_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}$$

$$\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\varsigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)} \leq \sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\dot{\varsigma}_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}$$

$$\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)} \leq \sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\dot{\varrho}_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}$$

$$e^{2i\pi\sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\sigma_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}} \geq e^{2i\pi\sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\dot{\sigma}_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}}$$

$$e^{2i\pi\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\varsigma_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}} \leq e^{2i\pi\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\dot{\varsigma}_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}}$$

$$e^{2i\pi\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\varrho_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}} \leq e^{2i\pi\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\dot{\varrho}_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}}$$

Thus, $S\left(CT-SFFWA\left(\Im_1, \Im_2, \ldots, \Im_n\right)\right) \geq S(CT-SFFWA(\dot{\Im}_1, \dot{\Im}_2, \ldots, \dot{\Im}_n))$.

Hence, $CT-SFFWA\left(\Im_1, \Im_2, \ldots, \Im_n\right) \geq CT-SFFWA\left(\dot{\Im}_1, \dot{\Im}_2, \ldots, \dot{\Im}_n\right)$. ■

Theorem 8 (Boundedness): Let $\Im_{\dot{r}} = (\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma1}}, \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}})(\dot{r} = 1(1)n)$ be a class of CT-SFNs, and let

$$\Im^{-} = \left(\min_{1\leq\dot{r}\leq n}\sigma_{\dot{r}}e^{2i\pi\min_{1\leq\dot{r}\leq n}\eth_{\sigma_{\dot{r}}}}, \max_{1\leq\dot{r}\leq n}\varsigma_{\dot{r}}e^{2i\pi\max_{1\leq\dot{r}\leq n}\eth_{\varsigma_{\dot{r}}}}, \max_{1\leq\dot{r}\leq n}\varrho_{\dot{r}}e^{2i\pi\max_{1\leq\dot{r}\leq n}\eth_{\varrho_{\dot{r}}}}\right),$$

$$\Im^{+} = \left(\max_{1\leq\dot{r}\leq n}\sigma_{\dot{r}}e^{2i\pi\max_{1\leq\dot{r}\leq n}\eth_{\sigma_{\dot{r}}}}, \min_{1\leq\dot{r}\leq n}\varsigma_{\dot{r}}e^{2i\pi\min_{1\leq\dot{r}\leq n}\eth_{\varsigma_{\dot{r}}}}, \min_{1\leq\dot{r}\leq n}\varrho_{\dot{r}}e^{2i\pi\min_{1\leq\dot{r}\leq n}\eth_{\varrho_{\dot{r}}}}\right),$$

then

$$\Im^{-} \leq CT-SFFWA\left(\Im_1, \Im_2, \ldots, \Im_n\right) \leq \Im^{+}. \tag{22}$$

Proof: Since for all $\dot{r}$,

$$\min_{1\leq\dot{r}\leq n}\sigma_{\dot{r}}e^{2i\pi\min_{1\leq\dot{r}\leq n}\eth_{\sigma_{\dot{r}}}} \leq \sigma_{\dot{r}}e^{2i\pi\eth_{\sigma_{\dot{r}}}} \leq \max_{1\leq\dot{r}\leq n}\sigma_{\dot{r}}e^{2i\pi\max_{1\leq\dot{r}\leq n}\sigma_{\dot{r}}},$$

$$\min_{1\leq\dot{r}\leq n}\varsigma_{\dot{r}}e^{2i\pi\min_{1\leq\dot{r}\leq n}\varsigma_{\dot{r}}} \leq \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma_{\dot{r}}}} \leq \max_{1\leq\dot{r}\leq n}\varsigma_{\dot{r}}e^{2i\pi\max_{1\leq\dot{r}\leq n}\eth_{\varsigma_{\dot{r}}}}$$

and

$$\min_{1\leq\dot{r}\leq n}\varrho_{\dot{r}}e^{2i\pi\min_{1\leq\dot{r}\leq n}\eth_{\varrho_{\dot{r}}}} \leq \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho_{\dot{r}}}} \leq \max_{1\leq\dot{r}\leq n}\varrho_{\dot{r}}e^{2i\pi\max_{1\leq\dot{r}\leq n}\eth_{\varrho_{\dot{r}}}},$$

thusly on the basis of idempotency and monotonicity, we have $\Im^{-} \leq CT-SFFWA\left(\Im_1, \Im_2, \ldots, \Im_n\right) \leq \Im^{+}$. ■

$$\lim_{\pounds\longrightarrow 1} CT-SFFWA\left(\Im_1,\Im_2,\ldots,\Im_n\right)$$

$$= \lim_{\pounds\longrightarrow 1}\begin{pmatrix} \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\sigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth_{\sigma_{\dot{r}}}^{t}}-1\right)^{\perp\dot{r}}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\varsigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\varsigma_{\dot{r}}}^{t}}-1\right)^{\perp\dot{r}}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\varrho_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\varrho_{\dot{r}}}^{t}}-1\right)^{\perp\dot{r}}\right)}} \end{pmatrix}$$

$$= \lim_{\pounds\longrightarrow 1}\begin{pmatrix} \sqrt[t]{1-\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\sigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln\pounds}}e^{2i\pi\sqrt[t]{1-\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth_{\sigma_{\dot{r}}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln\pounds}}}, \\ \sqrt[t]{\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\varsigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln\pounds}}e^{2i\pi\sqrt[t]{\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\varsigma_{\dot{r}}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln\pounds}}}, \\ \sqrt[t]{\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\varrho_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln\pounds}}e^{2i\pi\sqrt[t]{\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\varrho_{\dot{r}}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln\pounds}}} \end{pmatrix}$$

$$= \lim_{\pounds\longrightarrow 1}\begin{pmatrix} \sqrt[t]{1-\frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\sigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}}{\ln\pounds}}e^{2i\pi\sqrt[t]{1-\frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth_{\sigma_{\dot{r}}}^{t}}-1\right)^{\perp\dot{r}}}{\ln\pounds}}}, \\ \sqrt[t]{\frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{\varsigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}}{\ln\pounds}}e^{2i\pi\sqrt[t]{\frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\varsigma_{\dot{r}}}^{t}}-1\right)^{\perp\dot{r}}}{\ln\pounds}}}, \\ \sqrt[t]{\frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{\varrho_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}}{\ln\pounds}}e^{2i\pi\sqrt[t]{\frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\varrho_{\dot{r}}}^{t}}-1\right)^{\perp\dot{r}}}{\ln\pounds}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\sigma_{\dot{r}}^{t}\right)^{\perp\dot{r}}}e^{2i\pi\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\eth_{\sigma_{\dot{r}}}^{t}\right)^{\perp\dot{r}}}}, \\ \sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\varsigma_{\dot{r}}^{t}\right)^{\perp\dot{r}}}e^{2i\pi\sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\eth_{\varsigma_{\dot{r}}}^{t}\right)^{\perp\dot{r}}}}, \sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\varrho_{\dot{r}}^{t}\right)^{\perp\dot{r}}}e^{2i\pi\sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\eth_{\varrho_{\dot{r}}}^{t}\right)^{\perp\dot{r}}}} \end{pmatrix},$$

$$\lim_{\pounds\longrightarrow \infty} CT-SFFWA\left(\Im_1,\Im_2,\ldots,\Im_n\right)$$

$$= \begin{pmatrix} \lim_{\pounds\longrightarrow\infty}\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\sigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}e^{2i\pi\lim_{\pounds\longrightarrow\infty}\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth_{\sigma_{\dot{r}}}^{t}}-1\right)^{\perp\dot{r}}\right)}}, \\ \lim_{\pounds\longrightarrow\infty}\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\varsigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}e^{2i\pi\lim_{\pounds\longrightarrow\infty}\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\varsigma_{\dot{r}}}^{t}}-1\right)^{\perp\dot{r}}\right)}}, \\ \lim_{\pounds\longrightarrow\infty}\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\varrho_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}e^{2i\pi\lim_{\pounds\longrightarrow\infty}\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\varrho_{\dot{r}}}^{t}}-1\right)^{\perp\dot{r}}\right)}} \end{pmatrix}.$$

Theorem 9 (Shift-Invariance): Let $\Im_{\dot{r}} = (\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho\dot{r}}})(\dot{r} = 1(1)n)$ be a class of CT-SFNs and $\dot{\Im} = (\dot{\sigma} e^{2i\pi \eth_{\dot{\sigma}}}, \dot{\varsigma} e^{2i\pi \eth_{\dot{\varsigma}1}}, \dot{\varrho} e^{2i\pi \eth_{\dot{\varrho}\dot{r}}})$ be any other CT-SFNs, then

$$CT - SFFWA \left(\Im_1 \oplus \dot{\Im}, \Im_2 \oplus \dot{\Im}, \ldots, \Im_n \oplus \dot{\Im}\right) = CT - SFFWA \left(\Im_1, \Im_2, \ldots, \Im_n\right) \oplus \dot{\Im}. \tag{23}$$

Theorem 10 (Homogeneity): Let $\Im_{\dot{r}} = (\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho\dot{r}}})(\dot{r} = 1(1)n)$ be a class of CT-SFNs and $\dagger > 0$ be any real number, then

$$CT - SFFWA \left(\dagger\Im_1, \dagger\Im_2, \ldots, \dagger\Im_n\right) = \dagger CT - SFFWA \left(\Im_1, \Im_2, \ldots, \Im_n\right). \tag{24}$$

To save space, the proof of the aforesaid two theorems can be simply deduced from the suggested Frank operational rules of CT-SFNs; consequently, it is skipped here.

Definition 12: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho\dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, then the Complex T-spherical fuzzy Frank ordered weighted averaging (T-SFFOWA) operator is:

$$T - SFFOWA \left(\Im_1, \Im_2, \ldots, \Im_n\right) = \oplus_{\dot{r}=1}^{n} \left(\natural_{\dot{r}} \Im_{\delta(\dot{r})}\right), \tag{25}$$

where $\natural = (\natural_1, \natural_2, \ldots, \natural_n)^T$ is the position weights of $\Im_{\dot{r}}$ $(\dot{r} = 1(1)n)$ such that $\natural_{\dot{r}} > 0$ and $\sum_{\dot{r}=1}^{n} \natural_{\dot{r}} = 1$. $(\delta(1), \delta(2), \ldots, \delta(n))$ is a permutation of $(1, 2, 3, \ldots, n)$ so that $\Im_{\delta(\dot{r}-1)} \geq \Im_{\delta(\dot{r})}$ for $\dot{r} = 2(1)n$.

Theorem 11: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma\dot{r}}}, \varrho_{j} e^{2i\pi \eth_{\varrho\dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, then the result acquired by utilizing the T-SFFOWA operator is still a CT-SFN, and (26), as shown at the top of page 24.

Proof: Here we omit the proof of this result because it is same to that of Theorem 3. ■

Example 2: Let $\Im_1 = (0.3e^{2i\pi 0.2}, 0.3e^{2i\pi 0.4}, 0.5e^{2i\pi 0.5})$, $\Im_2 = (0.7e^{2i\pi 0.6}, 0.4e^{2i\pi 0.5}, 0.5e^{2i\pi 0.5})$, $\Im_3 = (0.6e^{2i\pi 0.5}, 0.7e^{2i\pi 0.8}, 0.8e^{2i\pi 0.8})$ be three CT-SFNs, then according to Definition 3 we can get (t=4):

$$S\left(\Im_1\right) = 0.4628, \quad S\left(\Im_2\right) = 0.5392, \; S\left(\Im_3\right) = 0.1808.$$

Since $S\left(\Im_2\right) > S\left(\Im_1\right) > S\left(\Im_3\right)$, we have $\Im_{\delta(1)} = \left(0.7e^{2i\pi 0.6}, 0.4e^{2i\pi 0.5}, 0.5e^{2i\pi 0.5}\right)$, $\Im_{\delta(2)} = \left(0.3e^{2i\pi 0.2}, 0.3e^{2i\pi 0.4}, 0.5e^{2i\pi 0.5}\right)$, $\Im_{\delta(3)} = (0.6e^{2i\pi 0.5}, 0.7e^{2i\pi 0.8}, 0.8e^{2i\pi 0.8})$ and $\natural = (0.3, 0.4, 0.3)^T$ is the weight vector associated with the T-SFFOWA operator. Suppose $\pounds = 2$, then according to Definition 12 and Theorem 11, we have the equation, as shown at the middle of page 24.

Theorem 12: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho\dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, and $\pounds > 1$. As $\pounds \longrightarrow 1$, the T-SFFOWA operator approaches the limit (27), as shown at the bottom of page 24.

Theorem 13: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho\dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, and $\pounds > 1$. As $\pounds \longrightarrow \infty$, the T-SFFOWA operator approaches the limit (28), as shown at the top of page 25.

Likewise CT-SFFWA operator, the T-SFFOWA operator also adheres the boundedness, idempotency and monotonicity, shift-invariance, and homogeneity properties. Besides the aforesaid characteristics, the T-SFFOWA operator has the following noteworthy results.

Theorem 14: Let $\Im_{\dot{r}} = (\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho\dot{r}}})$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, then we have the following:

i). If $\natural = (1, 0, \ldots, 0)^T$ then $T - SFFOWA(\Im_1, \Im_2, \ldots, \Im_n) = \max\{\Im_1, \Im_2, \ldots, \Im_n\}$.

ii). If $\natural = (0, 0, \ldots, 1)^T$ then $T - SFFOWA(\Im_1, \Im_2, \ldots, \Im_n) = \min\{\Im_1, \Im_2, \ldots, \Im_n\}$.

iii). If $\natural_{\dot{r}} = 1$ and $\perp_i = 0 (i \neq \dot{r})$ then $T - SFFOWA(\Im_1, \Im_2, \ldots, \Im_n) = \Im_{\delta(\dot{r})}$ where $\Im_{\delta(\dot{r})}$ is the $\dot{r}$th largest of $\Im_{\dot{r}}$, $(\dot{r} = 1(1)n)$.

According to definition of CT-SFFWA and T-SFFOWA operators, we can notice that the CT-SFFWA operator can weights only the SFNs while T-SFFOWA operator weights only the ordered position of SFNs. In real-world practical situations, We should concentrate about both factors at the same time. Therefore, to circumvent this issue, we state the hybrid averaging operator based on Frank t-norm and t-conorm, which weight both the given CT-SFNs and their ordered positions.

$$\left(\begin{array}{c}
\sqrt[t]{1 - \lim\limits_{\pounds \longrightarrow \infty} \frac{\ln\left(1+\prod\limits_{\dot{r}=1}^{n}\left(\pounds^{1-\sigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln \pounds}}, e^{2i\pi\sqrt[t]{1-\lim\limits_{\pounds \longrightarrow \infty} \frac{\ln\left(1+\prod\limits_{\dot{r}=1}^{n}\left(\pounds^{1-\eth_{\sigma\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln \pounds}}} \\
\sqrt[t]{\lim\limits_{\pounds \longrightarrow \infty} \frac{\ln\left(1+\prod\limits_{\dot{r}=1}^{n}\left(\pounds^{\varsigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln \pounds}} e^{2i\pi\sqrt[t]{\lim\limits_{\pounds \longrightarrow \infty} \frac{\ln\left(1+\prod\limits_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\varsigma\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln \pounds}}}, \\
\sqrt[t]{\lim\limits_{\pounds \longrightarrow \infty} \frac{\ln\left(1+\prod\limits_{\dot{r}=1}^{n}\left(\pounds^{\varrho_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln \pounds}} e^{2i\pi\sqrt[t]{\lim\limits_{\pounds \longrightarrow \infty} \frac{\ln\left(1+\prod\limits_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\varrho\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln \pounds}}}
\end{array}\right)$$

$$
= \left(\begin{array}{c}
\sqrt[t]{1-\lim\limits_{£\longrightarrow\infty}\dfrac{\frac{\prod_{\dot r=1}^{n}\left(£^{1-\sigma_{\dot r}^{t}}-1\right)^{\perp_{\dot r}}}{1+\prod_{\dot r=1}^{n}\left(£^{1-\sigma_{\dot r}^{t}}-1\right)^{\perp_{\dot r}}}\left(\sum_{\dot r=1}^{n}\perp_{\dot r}\left(1-\sigma_{\dot r}^{t}\right)\frac{£^{-\sigma_{\dot r}^{t}}}{£^{1-\sigma_{\dot r}^{t}}-1}\right)}{\frac{1}{£}}}
\\
e^{2i\pi\sqrt[t]{1-\lim\limits_{£\longrightarrow\infty}\dfrac{\frac{\prod_{\dot r=1}^{n}\left(£^{1-\eth_{\sigma\dot r}^{t}}-1\right)^{\perp_{\dot r}}}{1+\prod_{\dot r=1}^{n}\left(£^{1-\eth_{\sigma\dot r}^{t}}-1\right)^{\perp_{\dot r}}}\left(\sum_{\dot r=1}^{n}\perp_{\dot r}\left(1-\eth_{\sigma\dot r}^{t}\right)\frac{£^{-\sigma_{\dot r}^{t}}}{£^{1-\eth_{\sigma\dot r}^{t}}-1}\right)}{\frac{1}{£}}}},
\\
\sqrt[t]{1-\lim\limits_{£\longrightarrow\infty}\dfrac{\frac{\prod_{\dot r=1}^{n}\left(£^{\varsigma_{\dot r}^{t}}-1\right)^{\perp_{\dot r}}}{1+\prod_{\dot r=1}^{n}\left(£^{\varsigma_{\dot r}^{t}}-1\right)^{\perp_{\dot r}}}\left(\sum_{\dot r=1}^{n}\perp_{\dot r}\left(\varsigma_{\dot r}^{t}\right)\frac{£^{\varsigma_{\dot r}^{t}-1}}{£^{\varsigma_{\dot r}^{t}}-1}\right)}{\frac{1}{£}}}
\\
e^{2i\pi\sqrt[t]{1-\lim\limits_{£\longrightarrow\infty}\dfrac{\frac{\prod_{\dot r=1}^{n}\left(£^{\eth_{\varsigma\dot r}^{t}}-1\right)^{\perp_{\dot r}}}{1+\prod_{\dot r=1}^{n}\left(£^{\eth_{\varsigma\dot r}^{t}}-1\right)^{\perp_{\dot r}}}\left(\sum_{\dot r=1}^{n}\perp_{\dot r}\left(\eth_{\varsigma\dot r}^{t}\right)\frac{£^{\eth_{\varsigma\dot r}^{t}-1}}{£^{\eth_{\varsigma\dot r}^{t}}-1}\right)}{\frac{1}{£}}}},
\\
\sqrt[t]{1-\lim\limits_{£\longrightarrow\infty}\dfrac{\frac{\prod_{\dot r=1}^{n}\left(£^{\varrho_{\dot r}^{t}}-1\right)^{\perp_{\dot r}}}{1+\prod_{\dot r=1}^{n}\left(£^{\varrho_{\dot r}^{t}}-1\right)^{\perp_{\dot r}}}\left(\sum_{\dot r=1}^{n}\perp_{\dot r}\left(\varrho_{\dot r}^{t}\right)\frac{£^{\varrho_{\dot r}^{t}-1}}{£^{\varrho_{\dot r}^{t}}-1}\right)}{\frac{1}{£}}}
\\
e^{2i\pi\sqrt[t]{1-\lim\limits_{£\longrightarrow\infty}\dfrac{\frac{\prod_{\dot r=1}^{n}\left(£^{\eth_{\varrho\dot r}^{t}}-1\right)^{\perp_{\dot r}}}{1+\prod_{\dot r=1}^{n}\left(£^{\eth_{\varrho\dot r}^{t}}-1\right)^{\perp_{\dot r}}}\left(\sum_{\dot r=1}^{n}\perp_{\dot r}\left(\eth_{\varrho\dot r}^{t}\right)\frac{£^{\eth_{\varrho\dot r}^{t}-1}}{£^{\eth_{\varrho\dot r}^{t}}-1}\right)}{\frac{1}{£}}}},
\end{array} \right)
$$

$$
= \left(\begin{array}{c}
\sqrt[t]{1-\lim\limits_{£\longrightarrow\infty}\dfrac{\prod_{\dot r=1}^{n}\left(£^{1-\sigma_{\dot r}^{t}}-1\right)^{\perp_{\dot r}}}{1+\prod_{\dot r=1}^{n}\left(£^{1-\sigma_{\dot r}^{t}}-1\right)^{\perp_{\dot r}}}\left(\sum_{\dot r=1}^{n}\perp_{\dot r}\left(1-\sigma_{\dot r}^{t}\right)\frac{£^{1-\sigma_{\dot r}^{t}}}{£^{1-\sigma_{\dot r}^{t}}-1}\right)}
\\
e^{2i\pi\sqrt[t]{1-\lim\limits_{£\longrightarrow\infty}\dfrac{\prod_{\dot r=1}^{n}\left(£^{1-\eth_{\sigma\dot r}^{t}}-1\right)^{\perp_{\dot r}}}{1+\prod_{\dot r=1}^{n}\left(£^{1-\eth_{\sigma\dot r}^{t}}-1\right)^{\perp_{\dot r}}}\left(\sum_{\dot r=1}^{n}\perp_{\dot r}\left(1-\eth_{\sigma\dot r}^{t}\right)\frac{£^{1-\eth_{\sigma\dot r}^{t}}}{£^{1-\eth_{\sigma\dot r}^{t}}-1}\right)}},
\\
\sqrt[t]{1-\lim\limits_{£\longrightarrow\infty}\dfrac{\prod_{\dot r=1}^{n}\left(£^{\varsigma_{\dot r}^{t}}-1\right)^{\perp_{\dot r}}}{1+\prod_{\dot r=1}^{n}\left(£^{\varsigma_{\dot r}^{t}}-1\right)^{\perp_{\dot r}}}\left(\sum_{\dot r=1}^{n}\perp_{\dot r}\left(\varsigma_{\dot r}^{t}\right)\frac{£^{\varsigma_{\dot r}^{t}}}{£^{\varsigma_{\dot r}^{t}}-1}\right)}
\\
e^{2i\pi\sqrt[t]{1-\lim\limits_{£\longrightarrow\infty}\dfrac{\prod_{\dot r=1}^{n}\left(£^{\eth_{\varsigma\dot r}^{t}}-1\right)^{\perp_{\dot r}}}{1+\prod_{\dot r=1}^{n}\left(£^{\eth_{\varsigma\dot r}^{t}}-1\right)^{\perp_{\dot r}}}\left(\sum_{\dot r=1}^{n}\perp_{\dot r}\left(\eth_{\varsigma\dot r}^{t}\right)\frac{£^{\eth_{\varsigma\dot r}^{t}}}{£^{\eth_{\varsigma\dot r}^{t}}-1}\right)}},
\\
\sqrt[t]{1-\lim\limits_{£\longrightarrow\infty}\dfrac{\prod_{\dot r=1}^{n}\left(£^{\varrho_{\dot r}^{t}}-1\right)^{\perp_{\dot r}}}{1+\prod_{\dot r=1}^{n}\left(£^{\varrho_{\dot r}^{t}}-1\right)^{\perp_{\dot r}}}\left(\sum_{\dot r=1}^{n}\perp_{\dot r}\left(\varrho_{\dot r}^{t}\right)\frac{£^{\varrho_{\dot r}^{t}}}{£^{\varrho_{\dot r}^{t}}-1}\right)}
\\
e^{2i\pi\sqrt[t]{1-\lim\limits_{£\longrightarrow\infty}\dfrac{\prod_{\dot r=1}^{n}\left(£^{\eth_{\varrho\dot r}^{t}}-1\right)^{\perp_{\dot r}}}{1+\prod_{\dot r=1}^{n}\left(£^{\eth_{\varrho\dot r}^{t}}-1\right)^{\perp_{\dot r}}}\left(\sum_{\dot r=1}^{n}\perp_{\dot r}\left(\eth_{\varrho\dot r}^{t}\right)\frac{£^{\eth_{\varrho\dot r}^{t}}}{£^{\eth_{\varrho\dot r}^{t}}-1}\right)}}
\end{array} \right)
$$

$$= \begin{pmatrix} \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\sigma_{\dot{r}}^{t}\right)\right)} e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\eth_{\sigma_{\dot{r}}}^{t}\right)\right)}}, \\ \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\varsigma_{\dot{r}}^{t}\right)\right)} e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth_{\varsigma_{\dot{r}}}^{t}\right)\right)}}, \\ \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\varrho_{\dot{r}}^{t}\right)\right)} e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth_{\varrho_{\dot{r}}}^{t}\right)\right)}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\sigma_{\dot{r}}^{t}\right)\right)} e^{2i\pi\sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth_{\sigma_{\dot{r}}}^{t}\right)\right)}}, \\ \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\varsigma_{\dot{r}}^{t}\right)\right)} e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth_{\varsigma_{\dot{r}}}^{t}\right)\right)}}, \\ \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\varrho_{\dot{r}}^{t}\right)\right)} e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth_{\varrho_{\dot{r}}}^{t}\right)\right)}} \end{pmatrix},$$

Definition 13: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}} e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}} e^{2i\pi\eth_{\varrho\dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, then the Complex T-spherical fuzzy Frank hybrid averaging (CT-SFFHA) operator is:

$$CT-SFFHA\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \oplus_{\dot{r}=1}^{n}\left(\natural_{\dot{r}}\hat{\Im}_{\delta(\dot{r})}\right), \quad (29)$$

where $\natural = (\natural_1, \natural_2, \ldots, \natural_n)^T$ is the weight vector associated with CT-SFFHA such that $\natural_{\dot{r}} > 0$ and $\sum_{\dot{r}=1}^{n}\natural_{\dot{r}} = 1$, $\perp = (\perp_1, \perp_2, \ldots, \perp_n)^T$ is the weight vector of $\Im_{\dot{r}}$ $(\dot{r} = 1(1)n)$ such that $\perp_{\dot{r}} > 0$ and $\sum_{\dot{r}=1}^{n}\perp_{\dot{r}} = 1$. $\hat{\Im}_{\delta(\dot{r})}$ is the $\dot{r}$th largest of the weighted CT-SFNs $\hat{\Im}_{\dot{r}}$ $\left(\hat{\Im}_{\dot{r}} = (n\perp_{\dot{r}})\Im_{\dot{r}}, \dot{r} = 1(1)n\right)$ and n is the balancing coefficient.

Theorem 15: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}} e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}} e^{2i\pi\eth_{\varrho\dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, then the result acquired by using the CT-SFFHA operator is still a CT-SFN, and (30), as shown at the bottom of page 25.

Proof: We skip the proof of this result since it is analogous to Theorem 3. ■

Example 3: Let $\Im_1 = (0.4e^{2i\pi 0.3}, 0.5e^{2i\pi 0.6}, 0.2e^{2i\pi 0.2})$, $\Im_2 = (0.6e^{2i\pi 0.5}, 0.7e^{2i\pi 0.8}, 0.8e^{2i\pi 0.8})$ $\Im_3 = (0.3e^{2i\pi 0.2}, 0.6e^{2i\pi 0.7}, 0.5e^{2i\pi 0.5})$, be three CT-SFNs (t=4), and $\perp = (0.4, 0.4, 0.2)^T$ is the weight vector of $\Im_{\dot{r}}(\dot{r} = 1, 2, 3)$. Suppose $£ = 2$, then according to Definition 10, we can obtain the weighted CT-SFNs, as shown in the equation at the bottom of pages 26 and 27.

Based on Definition 5, we can determine the score of $\hat{\Im}_{\dot{r}}$ $(\dot{r} = 1, 2, 3)$:

$$S\left(\hat{\Im}_1\right) = 0.4807, S\left(\hat{\Im}_2\right) = 0.2609, S\left(\hat{\Im}_3\right) = 0.3196.$$

$$CT-SFFWA\left(\Im_1, \Im_2, \ldots, \Im_n\right)$$

$$= \begin{pmatrix} \sqrt[t]{1-\log_£\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\sigma_0^t}-1\right)^{\perp_{\dot{r}}}\right)} e^{2i\pi\sqrt[t]{1-\log_£\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\sigma_0}^t}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \sqrt[t]{\log_£\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\varsigma_0^t}-1\right)^{\perp_{\dot{r}}}\right)} e^{2i\pi\sqrt[t]{\log_£\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\varsigma_0}^t}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \sqrt[t]{\log_£\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\varrho_0^t}-1\right)^{\perp_{\dot{r}}}\right)} e^{2i\pi\sqrt[t]{\log_£\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\varrho_0}^t}-1\right)^{\perp_{\dot{r}}}\right)}} \end{pmatrix}$$

$$= \left(\sqrt[t]{1-\log_£ £^{1-\sigma_0^t}} e^{2i\pi\sqrt[t]{1-\log_£ £^{1-\eth_{\sigma_0}^t}}}, \sqrt[t]{\log_£ £^{\varsigma_0^t}} e^{2i\pi\sqrt[t]{\log_£ £^{\eth_{\varsigma_0}^t}}}, \sqrt[t]{\log_£ £^{\varrho_0^t}} e^{2i\pi\sqrt[t]{\log_£ £^{\eth_{\varrho_0}^t}}}\right)$$

$$= \left(\sigma_0 e^{2i\pi\eth_{\sigma_0}}, \varsigma_0 e^{2i\pi\eth_{\varsigma_0}}, \varrho_0 e^{2i\pi\eth_{\varrho_0}}\right) = \Im_0.$$

Since $S\left(\hat{\Im}_1\right) > S\left(\hat{\Im}_3\right) > S\left(\hat{\Im}_2\right)$, we have $\hat{\Im}_{\delta(1)} = (0.4185e^{2i\pi 0.3114}, 0.4289e^{2i\pi 0.5363}, 0.1423e^{2i\pi 0.1423})$, $\hat{\Im}_{\delta(2)} = (0.2641e^{2i\pi 0.2166}, 0.7478e^{2i\pi 0.6468}, 0.6743e^{2i\pi 0.4301})$, $\hat{\Im}_{\delta(3)} = (0.6264e^{2i\pi 0.5230}, 0.6464e^{2i\pi 0.7619}, 0.7617e^{2i\pi 0.7619})$.

$$T - SFFOWA\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \begin{pmatrix} \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\sigma^t_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth^t_{\sigma_{\delta(\dot{r})}}}-1\right)^{\natural_{\dot{r}}}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\varsigma^t_{\delta(\dot{r})}}-1\right)^{\natural_j}\right)} e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth^t_{\varsigma_{\delta(\dot{r})}}}-1\right)^{\natural_{\dot{r}}}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\varrho^t_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth^t_{\varrho_{\delta(\dot{r})}}}-1\right)^{\natural_{\dot{r}}}\right)}} \end{pmatrix}. \tag{26}$$

$$\begin{aligned} &T - SFFOWA\left(\Im_{\delta(1)}, \Im_{\delta(2)}, \Im_{\delta(3)}\right) \\ &= \begin{pmatrix} \sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\sigma^4_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi\sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\eth^4_{\sigma_{\delta(j)}}}-1\right)^{\natural_j}\right)}}, \\ \sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\varsigma^4_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi\sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\eth^4_{\varsigma_{\delta(j)}}}-1\right)^{\natural_j}\right)}}, \\ \sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\varrho^4_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi\sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\eth^4_{\varrho_{\delta(j)}}}-1\right)^{\natural_j}\right)}} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt[4]{1-\log_2\left(1+\left(2^{1-0.7^4}-1\right)^{0.3}\left(2^{1-0.3^4}-1\right)^{0.4}\left(2^{1-0.6^4}-1\right)^{0.3}\right)} \\ e^{2i\pi\sqrt[4]{1-\log_2\left(1+\left(2^{1-0.6^4}-1\right)^{0.3}\left(2^{1-0.2^4}-1\right)^{0.4}\left(2^{1-0.5^4}-1\right)^{0.3}\right)}}, \\ \sqrt[4]{\log_2\left(1+\left(2^{0.4^4}-1\right)^{0.3}\left(2^{0.3^4}-1\right)^{0.4}\left(2^{0.7^4}-1\right)^{0.3}\right)} \\ e^{2i\pi\sqrt[4]{\log_2\left(1+\left(2^{0.5^4}-1\right)^{0.3}\left(2^{0.4^4}-1\right)^{0.4}\left(2^{0.8^4}-1\right)^{0.3}\right)}}, \\ \sqrt[4]{\log_2\left(1+\left(2^{0.5^4}-1\right)^{0.3}\left(2^{0.5^4}-1\right)^{0.4}\left(2^{0.8^4}-1\right)^{0.3}\right)} \\ e^{2i\pi\sqrt[4]{\log_2\left(1+\left(2^{0.5^4}-1\right)^{0.3}\left(2^{0.5^4}-1\right)^{0.4}\left(2^{0.8^4}-1\right)^{0.3}\right)}} \end{pmatrix} \\ &= \left(0.5861e^{2i\pi 0.4941}, 0.4236e^{2i\pi 0.5295}, 0.5790e^{2i\pi 0.5790}\right). \end{aligned}$$

$$\lim_{\pounds\longrightarrow 1} T - SFFOWA\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \begin{pmatrix} \sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\sigma^t_{\delta(\dot{r})}\right)^{\natural_{\dot{r}}}} e^{2i\pi\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\eth^t_{\sigma_{\delta(\dot{r})}}\right)^{\natural_{\dot{r}}}}}, \\ \sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\eth^t_{\varsigma_{\delta(\dot{r})}}\right)^{\natural_{\dot{r}}}} e^{2i\pi\sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\varsigma^t_{\delta(\dot{r})}\right)^{\natural_{\dot{r}}}}}, \\ \sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\varrho^t_{\delta(\dot{r})}\right)^{\natural_{\dot{r}}}} e^{2i\pi\sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\eth^t_{\varrho_{\delta(\dot{r})}}\right)^{\natural_{\dot{r}}}}} \end{pmatrix}. \tag{27}$$

Suppose $\natural = (0.3, 0.4, 0.3)^T$ is the weight vector associated with the CT-SFFHA operator. Then by Definition 13 and Theorem 15, we can obtain the equation, as shown at the bottom of pages 26 and 27.

Theorem 16: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma \dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma \dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho \dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, and $\pounds > 1$. As $\pounds \longrightarrow 1$, the CT-SFFHA operator approaches the limit (31), as shown at the bottom of page 27.

Theorem 17: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma \dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma \dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho \dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, and $\pounds > 1$. As $\pounds \longrightarrow \infty$, the CT-SFFHA operator approaches the limit (32), as shown at the bottom of page 27.

Likewise CT-SFFWA operator, the CT-SFFHA operator also adheres the boundedness, monotonicity and idempotency, shift-invariance, and homogeneity properties. Besides the aforesaid characteristics, the CT-SFFHA operator has the following particular cases.

Corollary 1: CT-SFFWA operator is a particular case of the CT-SFFHA operator.

Proof: Let $\natural = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then

$$\begin{aligned} &CT-SFFHA\left(\Im_1, \Im_2, \ldots, \Im_n\right) \\ &= \natural_1 \hat{\Im}_{\delta(1)} \oplus \natural_2 \hat{\Im}_{\delta(2)} \oplus \ldots \oplus \natural_n \hat{\Im}_{\delta(n)} \\ &= \frac{1}{n}\left(\hat{\Im}_{\delta(1)} \oplus \hat{\Im}_{\delta(2)} \oplus \ldots \oplus \hat{\Im}_{\delta(n)}\right) \\ &= \perp_1 \Im_1 \oplus \perp_2 \Im_2 \oplus \ldots \oplus \perp_n \Im_n \\ &= CT-SFFWA\left(\Im_1, \Im_2, \ldots, \Im_n\right). \end{aligned}$$

∎

Corollary 2: T-SFFOWA operator is a particular case of the CT-SFFHA operator.

Proof: Let $\perp = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then

$$\begin{aligned} &CT-SFFHA\left(\Im_1, \Im_2, \ldots, \Im_n\right) \\ &= \natural_1 \hat{\Im}_{\delta(1)} \oplus \natural_2 \hat{\Im}_{\delta(2)} \oplus \ldots \oplus \natural_n \hat{\Im}_{\delta(n)} \\ &= \natural_1 \Im_{\delta(1)} \oplus \natural_2 \Im_{\delta(2)} \oplus \ldots \oplus \natural_n \Im_{\delta(n)} \\ &= T-SFFOWA\left(\Im_1, \Im_2, \ldots, \Im_n\right). \end{aligned}$$

∎

B. CT-SF FRANK GEOMETRIC AGGREGATION OPERATORS

Based on devised Frank operations, this section provides a set of Complex T-spherical fuzzy Frank geometric aggregation operators. We will go through the CT-SFFWG, CT-SFFOWG, and T-SFFHWG, as well as the basic definitions, remarks, and results, corollary for these operators, which are formed on the Frank t-norm and t-conorm.

Definition 14: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma \dot{r}}}, \varsigma_j e^{2i\pi \eth_{\varsigma \dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho \dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, then the Complex T-spherical fuzzy Frank weighted geometric operator (CT-SFFWG) is:

$$CT-SFFWG\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \otimes_{\dot{r}=1}^{n} \left(\Im_{\dot{r}}\right)^{\perp_{\dot{r}}}, \tag{33}$$

where $\perp = (\perp_1, \perp_2, \ldots, \perp_n)^T$ is the weight vector of $\Im_{\dot{r}}$ $(\dot{r} = 1(1)n)$ such that $\perp_{\dot{r}} > 0$ and $\sum_{\dot{r}=1}^{n} \perp_{\dot{r}} = 1$. Especially if $\perp = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the CT-SFFWG operator reduces to the Complex T-spherical fuzzy Frank geometric (CT-SFFG) operator of dimension n, which is defined as given below:

$$CT-SFFA\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \otimes_{\dot{r}=1}^{n} \left(\Im_{\dot{r}}\right)^{\frac{1}{n}}. \tag{34}$$

$$\lim_{\pounds \longrightarrow \infty} T-SFFOWA\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \left(\begin{array}{c} \sqrt[t]{\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}} \left(\sigma_{\delta(\dot{r})}^{t}\right)\right)} e^{2i\pi \sqrt[t]{\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}} \left(\eth_{\sigma_{\delta(\dot{r})}^{t}}\right)\right)}}, \\ \sqrt[t]{1 - \left(\sum_{\dot{r}=1}^{n} w_{\dot{r}} \left(\vartheta_{\delta(\dot{r})}^{t}\right)\right)} e^{2i\pi \sqrt[t]{1 - \left(\sum_{\dot{r}=1}^{n} w_{\dot{r}} \left(\eth_{\vartheta_{\delta(\dot{r})}^{t}}\right)\right)}}, \\ \sqrt[t]{1 - \left(\sum_{\dot{r}=1}^{n} w_{\dot{r}} \left(\varrho_{\delta(\dot{r})}^{t}\right)\right)} e^{2i\pi \sqrt[t]{1 - \left(\sum_{\dot{r}=1}^{n} w_{\dot{r}} \left(\eth_{\varrho_{\delta(\dot{r})}}^{t}\right)\right)}} \end{array} \right). \tag{28}$$

$$CT-SFFHA\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \left(\begin{array}{c} \sqrt[t]{1 - \log_{\pounds}\left(1 + \prod_{\dot{r}=1}^{n} \left(\pounds^{1-\hat{\sigma}_{\delta(\dot{r})}^{t}} - 1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi \sqrt[t]{1 - \log_{\pounds}\left(1 + \prod_{\dot{r}=1}^{n} \left(\pounds^{1-\eth_{\hat{\sigma}\delta(\dot{r})}^{t}} - 1\right)^{\natural_{\dot{r}}}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1 + \prod_{\dot{r}=1}^{n} \left(\pounds^{\hat{\varsigma}_{\delta(\dot{r})}^{t}} - 1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi \sqrt[t]{\log_{\pounds}\left(1 + \prod_{\dot{r}=1}^{n} \left(\pounds^{\eth_{\hat{\varsigma}\delta(\dot{r})}^{t}} - 1\right)^{\natural_{\dot{r}}}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1 + \prod_{\dot{r}=1}^{n} \left(\pounds^{\hat{\varrho}_{\delta(\dot{r})}^{t}} - 1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi \sqrt[t]{\log_{\pounds}\left(1 + \prod_{\dot{r}=1}^{n} \left(\pounds^{\eth_{\hat{\varrho}\delta(\dot{r})}^{t}} - 1\right)^{\natural_{\dot{r}}}\right)}} \end{array} \right), \tag{30}$$

Theorem 18: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma \dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma \dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho \dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, then the result acquired by using the CT-SFFWG operator is still a CT-SFN, and (35), as shown at the bottom of page 28.

Proof: We verify it using mathematical induction on n.

For $n = 2$, we get the equation, as shown at the bottom of pages 28 and 29. Thus, the result holds for $n = 2$.

If Eq. (35) holds for $n = k$, then for $n = k + 1$, we get the equation, as shown at the bottom of pages 30–32.

Thus, result holds for $n = k+1$, and hence, by the principle of mathematical induction, the result disclosed in Eq. (35) holds for all positive integer n. ■

$$\hat{\Im}_1 = 3 \times 0.4 \times \Im_1 = \left(\begin{array}{c} \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.4^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)} e^{2i\pi \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.3^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}}, \\ \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.5^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)} e^{2i\pi \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.6^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}}, \\ \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.2^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)} e^{2i\pi \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.2^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}} \end{array} \right)$$

$$= \left(0.4185 e^{2i\pi 0.3114}, 0.4289 e^{2i\pi 0.5363}, 0.1423 e^{2i\pi 0.1423}\right);$$

$$\hat{\Im}_2 = 3 \times 0.4 \times \Im_2 = \left(\begin{array}{c} \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.6^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)} e^{2i\pi \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.5^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}}, \\ \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.7^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)} e^{2i\pi \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.8^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}}, \\ \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.8^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)} e^{2i\pi \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.8^4}-1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4-1}}\right)}} \end{array} \right)$$

$$= \left(0.6264 e^{2i\pi 0.5230}, 0.6464 e^{2i\pi 0.7619}, 0.7617 e^{2i\pi 0.7619}\right);$$

$$\hat{\Im}_3 = 3 \times 0.2 \times \Im_3 = \left(\begin{array}{c} \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.3^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)} e^{2i\pi \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.2^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)}}, \\ \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.6^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)} e^{2i\pi \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.7^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)}}, \\ \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.5^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)} e^{2i\pi \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.5^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)}} \end{array} \right)$$

$$= \left(0.2641 e^{2i\pi 0.2166}, 0.7478 e^{2i\pi 0.6468}, 0.6743 e^{2i\pi 0.4301}\right).$$

$$CT-SFFHA\left(\hat{\Im}_{\delta(1)},\hat{\Im}_{\delta(2)},\hat{\Im}_{\delta(3)}\right)=\left(\begin{array}{c}\sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\hat{\sigma}_{\delta(j)}^4}-1\right)^{\natural_j}\right)}e^{2i\pi\sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\eth_{\hat{\sigma}\delta(j)}^4}-1\right)^{\natural_j}\right)}},\\ \sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\hat{\varsigma}_{\delta(j)}^4}-1\right)^{\natural_j}\right)}e^{2i\pi\sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\eth_{\hat{\varsigma}\delta(\dot{r})}^4}-1\right)^{\natural_j}\right)}},\\ \sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\hat{\varrho}_{\delta(j)}^4}-1\right)^{\natural_j}\right)}e^{2i\pi\sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\eth_{\hat{\varrho}\delta(\dot{r})}^4}-1\right)^{\natural_j}\right)}}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt[4]{1-\log_2\left(1+\left(2^{1-0.4185^4}-1\right)^{0.3}\left(2^{1-0.2641^4}-1\right)^{0.4}\left(2^{1-0.6264^4}-1\right)^{0.3}\right)}\\ e^{2i\pi\sqrt[4]{1-\log_2\left(1+\left(2^{1-0.3114^4}-1\right)^{0.3}\left(2^{1-0.2166^4}-1\right)^{0.4}\left(2^{1-0.5230^4}-1\right)^{0.3}\right)}},\\ \sqrt[4]{\log_2\left(1+\left(2^{0.4289^4}-1\right)^{0.3}\left(2^{0.7478^4}-1\right)^{0.4}\left(2^{0.6464^4}-1\right)^{0.3}\right)}\\ e^{2i\pi\sqrt[4]{\log_2\left(1+\left(2^{0.5363^4}-1\right)^{0.3}\left(2^{0.6468^4}-1\right)^{0.4}\left(2^{0.7619^4}-1\right)^{0.3}\right)}},\\ \sqrt[4]{\log_2\left(1+\left(2^{0.1423^4}-1\right)^{0.3}\left(2^{0.6743^4}-1\right)^{0.4}\left(2^{0.7617^4}-1\right)^{0.3}\right)}\\ e^{2i\pi\sqrt[4]{\log_2\left(1+\left(2^{0.1423^4}-1\right)^{0.3}\left(2^{0.4301^4}-1\right)^{0.4}\left(2^{0.7619^4}-1\right)^{0.3}\right)}}\end{array}\right)$$

$$=\left(0.4926e^{2i\pi 0.4023},0.6089e^{2i\pi 0.6432},0.4428e^{2i\pi 0.3694}\right).$$

Example 4 (Continued from Example 1): According to Definition 14 and Theorem 18, we have the equation, as shown at the middle of page 33.

Theorem 19: Let $\Im_{\dot{r}}=\left(\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}},\varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}},\varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}}\right)$ $(\dot{r}=1(1)n)$ be a class of CT-SFNs, and £ >1. As £ $\longrightarrow 1$, the CT-SFFWG operator approaches the following

$$\lim_{£\longrightarrow 1}CT-SFFHA\left(\Im_1,\Im_2,\ldots,\Im_n\right)=\left(\begin{array}{c}\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\hat{\sigma}_{\delta(\dot{r})}^t\right)^{\natural_{\dot{r}}}}e^{2i\pi\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\eth_{\hat{\sigma}_{\delta(\dot{r})}}^t\right)^{\natural_{\dot{r}}}}},\\ \sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\hat{\varsigma}_{\delta(\dot{r})}^t\right)^{\natural_{\dot{r}}}}e^{2i\pi\sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\eth_{\hat{\varsigma}_{\delta(\dot{r})}}^t\right)^{\natural_{\dot{r}}}}},\\ \sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\hat{\varrho}_{\delta(\dot{r})}^t\right)^{\natural_{\dot{r}}}}e^{2i\pi\sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\eth_{\hat{\varrho}_{\delta(\dot{r})}}^t\right)^{\natural_{\dot{r}}}}}\end{array}\right). \quad (31)$$

$$\lim_{£\longrightarrow\infty}CT-SFFHA\left(\Im_1,\Im_2,\ldots,\Im_n\right)=\left(\begin{array}{c}\sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\natural_{\dot{r}}\left(\hat{\sigma}_{\delta(\dot{r})}^t\right)\right)}e^{2i\pi\sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\natural_{\dot{r}}\left(\eth_{\hat{\sigma}_{\delta(\dot{r})}}^t\right)\right)}},\\ \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\natural_{\dot{r}}\left(\hat{\varsigma}_{\delta(\dot{r})}^t\right)\right)}e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\natural_{\dot{r}}\left(\eth_{\hat{\varsigma}_{\delta(\dot{r})}}^t\right)\right)}},\\ \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\natural_{\dot{r}}\left(\hat{\varrho}_{\delta(\dot{r})}^t\right)\right)}e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\natural_{\dot{r}}\left(\eth_{\hat{\varrho}_{\delta(\dot{r})}}^t\right)\right)}}\end{array}\right). \quad (32)$$

limit

$$\lim_{\pounds\longrightarrow 1} CT-SFFWG\left(\Im_1,\Im_2,\ldots,\Im_n\right) = \left(\begin{array}{c} \sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\sigma_{\dot{r}}^{t}\right)^{\perp_{\dot{r}}}}\, e^{2i\pi\sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\eth^{t}_{\sigma_{\dot{r}}}\right)^{\perp_{\dot{r}}}}}, \\ \sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\varsigma_{\dot{r}}^{t}\right)^{\perp_{\dot{r}}}}\, e^{2i\pi\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\eth^{t}_{\varsigma_{\dot{r}}}\right)^{\perp_{\dot{r}}}}}, \\ \sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\varrho_{\dot{r}}^{t}\right)^{\perp_{\dot{r}}}}\, e^{2i\pi\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\eth^{t}_{\varrho_{\dot{r}}}\right)^{\perp_{\dot{r}}}}} \end{array}\right). \quad (36)$$

Proof: As $\pounds \longrightarrow 1$, then $\left(\prod_{\dot{r}=1}^{n}\left(\pounds^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}, \prod_{\dot{r}=1}^{n}\left(\pounds^{1-\varsigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}, \prod_{\dot{r}=1}^{n}\left(\pounds^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right) \longrightarrow \langle 0,0,0\rangle$ by log property and the rule of infinitesimal changes, we get the equation, as shown at the bottom of page 33.

According to Taylor's expansion formula, we get

$$\pounds^{\varsigma_{\dot{r}}^{t}} = 1+\left(\sigma_{\dot{r}}^{t}\right)\ln\pounds+\frac{\left(\left(\sigma_{\dot{r}}^{t}\right)\ln\pounds\right)^2}{2!}+\ldots$$

$$\pounds^{1-\sigma_{\dot{r}}^{t}} = 1+\left(1-\varsigma_{\dot{r}}^{t}\right)\ln\pounds+\frac{\left(\left(1-\varsigma_{\dot{r}}^{t}\right)\ln\pounds\right)^2}{2!}+\ldots$$

$$CT-SFFWG\left(\Im_1,\Im_2,\ldots,\Im_n\right) = \left(\begin{array}{c} \sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\sigma_{\dot{r}}}-1\right)^{\perp_{\dot{r}}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth^{t}_{\sigma_{\dot{r}}}}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\varsigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth^{t}_{\varsigma_{\dot{r}}}}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth^{t}_{\varrho_{\dot{r}}}}-1\right)^{\perp_{\dot{r}}}\right)}} \end{array}\right). \quad (35)$$

$$CT-SFFWG\left(\Im_1,\Im_2\right) = \Im_1^{\perp_1}\otimes\Im_2^{\perp_2}$$

$$= \left(\begin{array}{c} \sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\sigma_1^{t}}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}\right)\right)}-1\right)\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\sigma_2^{t}}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}\right)\right)}-1\right)}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth^{t}_{\sigma_1}}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}\right)\right)}-1\right)\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth^{t}_{\sigma_2}}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}\right)\right)}-1\right)}{\pounds-1}\right)}}, \\ \sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\varsigma_1^{t}}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}\right)\right)}-1\right)\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\varsigma_2^{t}}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}\right)\right)}-1\right)}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\eth^{t}_{\varsigma_1}}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}\right)\right)}-1\right)\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\eth^{t}_{\varsigma_2}}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}\right)\right)}-1\right)}{\pounds-1}\right)}}, \\ \sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\varrho_1^{t}}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}\right)\right)}-1\right)\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\varrho_2^{t}}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}\right)\right)}-1\right)}{\pounds-1}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\eth^{t}_{\varrho_1}}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}\right)\right)}-1\right)\left(\pounds^{1-\left(1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\eth^{t}_{\varrho_2}}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}\right)\right)}-1\right)}{\pounds-1}\right)}} \end{array}\right)$$

$$
= \left(\begin{array}{c}
\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{\sigma_1^t}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}-1\right)\left(1+\frac{\left(\pounds^{\sigma_2^t}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}-1\right)}{\pounds-1}\right)}\, e^{2i\pi \sqrt[t]{\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{\eth^t_{\sigma_1}}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}-1\right)\left(1+\frac{\left(\pounds^{\eth^t_{\sigma_2}}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}-1\right)}{\pounds-1}\right)}}, \\
\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{1-\varsigma_1^t}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}-1\right)\left(1+\frac{\left(\pounds^{1-\varsigma_2^t}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}-1\right)}{\pounds-1}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{1-\eth^t_{\varsigma_1}}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}-1\right)\left(1+\frac{\left(\pounds^{1-\eth^t_{\varsigma_2}}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}-1\right)}{\pounds-1}\right)}}, \\
\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{1-\varrho_1^t}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}-1\right)\left(1+\frac{\left(\pounds^{1-\varrho_2^t}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}-1\right)}{\pounds-1}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(1+\frac{\left(\pounds^{1-\eth^t_{\varrho_1}}-1\right)^{\perp_1}}{(\pounds-1)^{\perp_1-1}}-1\right)\left(1+\frac{\left(\pounds^{1-\eth^t_{\varrho_2}}-1\right)^{\perp_2}}{(\pounds-1)^{\perp_2-1}}-1\right)}{\pounds-1}\right)}}
\end{array} \right)
$$

$$
= \left(\begin{array}{c}
\sqrt[t]{\log_{\pounds}\left(1+\left(\left(\pounds^{\sigma_1^t}-1\right)^{\perp_1}\right)\left(\left(\pounds^{\sigma_2^t}-1\right)^{\perp_2}\right)\right)}\, e^{2i\pi \sqrt[t]{\log_{\pounds}\left(1+\left(\left(\pounds^{\eth^t_{\sigma_1}}-1\right)^{\perp_1}\right)\left(\left(\pounds^{\eth^t_{\sigma_2}}-1\right)^{\perp_2}\right)\right)}}, \\
\sqrt[t]{1-\log_{\pounds}\left(1+\left(\left(\pounds^{1-\varsigma_1^t}-1\right)^{\perp_1}\right)\left(\left(\pounds^{1-\varsigma_2^t}-1\right)^{\perp_2}\right)\right)}\, e^{2i\pi \sqrt[t]{1-\log_{\pounds}\left(1+\left(\left(\pounds^{1-\eth^t_{\varsigma_1}}-1\right)^{\perp_1}\right)\left(\left(\pounds^{1-\eth^t_{\varsigma_2}}-1\right)^{\perp_2}\right)\right)}}, \\
\sqrt[t]{1-\log_{\pounds}\left(1+\left(\left(\pounds^{1-\varrho_1^t}-1\right)^{\perp_1}\right)\left(\left(\pounds^{1-\varrho_2^t}-1\right)^{\perp_2}\right)\right)}\, e^{2i\pi \sqrt[t]{1-\log_{\pounds}\left(1+\left(\left(\pounds^{1-\eth^t_{\varrho_1}}-1\right)^{\perp_1}\right)\left(\left(\pounds^{1-\eth^t_{\varrho_2}}-1\right)^{\perp_2}\right)\right)}}
\end{array} \right).
$$

$$
\pounds^{1-\sigma_{\dot{r}}^t} = 1 + \left(1-\varrho_{\dot{r}}^t\right)\ln\pounds + \frac{\left(\left(1-\varrho_{\dot{r}}^t\right)\ln\pounds\right)^2}{2!} + \ldots
$$

Also, since $\pounds > 1$, then $\ln\pounds > 0$, $\pounds^{\sigma_{\dot{r}}^t} = 1 + \left(\sigma_{\dot{r}}^t\right)\ln\pounds + O\left(\ln\pounds\right)$, $\pounds^{1-\varsigma_{\dot{r}}^t} = 1 + \left(1-\varsigma_{\dot{r}}^t\right)\ln\pounds + O\left(\ln\pounds\right)$, $\pounds^{1-\varrho_{\dot{r}}^t} = 1 + \left(1-\varrho_{\dot{r}}^t\right)\ln\pounds + O\left(\ln\pounds\right)$.

As a result

$$
\left(\pounds^{\sigma_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}} \longrightarrow \left(\left(\sigma_{\dot{r}}^t\right)\ln\pounds\right)^{\perp_{\dot{r}}}
$$

$$
\prod_{\dot{r}=1}^{n}\left(\pounds^{\sigma_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}} \longrightarrow \prod_{\dot{r}=1}^{n}\left(\sigma_{\dot{r}}^t\right)^{\perp_{\dot{r}}}\prod_{\dot{r}=1}^{n}\left(\ln\pounds\right)^{\perp_{\dot{r}}}
$$

$$
\prod_{\dot{r}=1}^{n}\left(\pounds^{\sigma_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}} \longrightarrow \prod_{\dot{r}=1}^{n}\left(\sigma_{\dot{r}}^t\right)^{\perp_{\dot{r}}}\ln\left(\pounds\right)^{\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}}
$$

$$
\frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{\sigma_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}}}{\ln\pounds} \longrightarrow \prod_{\dot{r}=1}^{n}\left(\sigma_{\dot{r}}^t\right)^{\perp_{\dot{r}}}.
$$

Analogously, we can get

$$
\frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\varsigma_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}}}{\ln\pounds} \longrightarrow \prod_{\dot{r}=1}^{n}\left(1-\varsigma_{\dot{r}}^t\right)^{\perp_{\dot{r}}},
$$

$$
\frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\varrho_{\dot{r}}^t}-1\right)^{\perp_{\dot{r}}}}{\ln\pounds} \longrightarrow \prod_{\dot{r}=1}^{n}\left(1-\varrho_{\dot{r}}^t\right)^{\perp_{\dot{r}}},
$$

$$e^{2i\pi\frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth^{t}_{\sigma\dot{r}}}-1\right)^{\perp_{\dot{r}}}}{\ln\pounds}}\longrightarrow e^{2i\pi\prod_{\dot{r}=1}^{n}\left(\eth^{t}_{\sigma_{\dot{r}}}\right)^{\perp_{\dot{r}}}},$$

$$e^{2i\pi\frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth^{t}_{\varsigma\dot{r}}}-1\right)^{\perp_{\dot{r}}}}{\ln\pounds}}\longrightarrow e^{2i\pi\prod_{\dot{r}=1}^{n}\left(1-\eth^{t}_{\varsigma_{\dot{r}}}\right)^{\perp_{\dot{r}}}},$$

$$e^{2i\pi\frac{\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth^{t}_{\varrho\dot{r}}}-1\right)^{\perp_{\dot{r}}}}{\ln\pounds}}\longrightarrow e^{2i\pi\prod_{\dot{r}=1}^{n}\left(1-\eth^{t}_{\varrho_{\dot{r}}}\right)^{\perp_{\dot{r}}}}.$$

Then, we have the equation, as shown at the bottom of page 34, which completes the proof. ■

Theorem 20: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, and $\pounds > 1$. As $\pounds \longrightarrow \infty$, the CT-SFFWG operator approaches the following limit

$$\lim_{\pounds\longrightarrow\infty} CT-SFFWG\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \begin{pmatrix} \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\sigma^{t}_{\dot{r}}\right)\right)}e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth^{t}_{\sigma_{\dot{r}}}\right)\right)}}, \\ \sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\varsigma^{t}_{\dot{r}}\right)\right)}e^{2i\pi\sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth^{t}_{\varsigma_{\dot{r}}}\right)\right)}}, \\ \sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\varrho^{t}_{\dot{r}}\right)\right)}e^{2i\pi\sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth^{t}_{\varrho_{\dot{r}}}\right)\right)}} \end{pmatrix}. \quad (37)$$

$$CT-SFFWG\left(\Im_1, \Im_2, \ldots, \Im_{k+1}\right) = CT-SFFWG\left(\Im_1, \Im_2, \ldots, \Im_k\right) \otimes \Im_{k+1}^{\perp_{k+1}}$$

$$= \begin{pmatrix} \sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{k}\left(\pounds^{\sigma^{t}_{\dot{r}}}-1\right)^{\perp_{\dot{r}}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{k}\left(\pounds^{\eth^{t}_{\sigma\dot{r}}}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{k}\left(\pounds^{1-\varsigma^{t}_{\dot{r}}}-1\right)^{\perp_{\dot{r}}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{k}\left(\pounds^{1-\eth^{t}_{\varsigma\dot{r}}}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{k}\left(\pounds^{1-\varrho^{t}_{\dot{r}}}-1\right)^{\perp_{\dot{r}}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{k}\left(\pounds^{1-\eth^{t}_{\varrho\dot{r}}}-1\right)^{\perp_{\dot{r}}}\right)}} \end{pmatrix}$$

$$\otimes \begin{pmatrix} \sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\sigma^{t}_{(k+1)}}-1\right)^{\perp(k+1)}}{(\pounds-1)^{\perp(k+1)-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\left(\pounds^{\eth^{t}_{\sigma(k+1)}}-1\right)^{\perp(k+1)}}{(\pounds-1)^{\perp(k+1)-1}}\right)}}, \\ \sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\varsigma^{t}_{(k+1)}}-1\right)^{\perp(k+1)}}{(\pounds-1)^{\perp(k+1)-1}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\eth^{t}_{\varsigma(k+1)}}-1\right)^{\perp(k+1)}}{(\pounds-1)^{\perp(k+1)-1}}\right)}}, \\ \sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\varrho^{t}_{(k+1)}}-1\right)^{\perp(k+1)}}{(\pounds-1)^{\perp(k+1)-1}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\left(\pounds^{1-\eth^{t}_{\varrho(k+1)}}-1\right)^{\perp(k+1)}}{(\pounds-1)^{\perp(k+1)-1}}\right)}} \end{pmatrix}$$

$$
= \left(\begin{array}{c}
\sqrt[t]{\log_{£}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(£^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\frac{\left(£^{\sigma_{(k+1)}^{t}}-1\right)^{\perp(k+1)}}{(£-1)^{\perp(k+1)-1}}}{£-1}\right)}\, e^{2i\pi\sqrt[t]{\log_{£}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(£^{\eth_{\sigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\frac{\left(£^{\eth_{\sigma(k+1)}^{t}}-1\right)^{\perp(k+1)}}{(£-1)^{\perp(k+1)-1}}}{£-1}\right)}}, \\
\sqrt[t]{1-\log_{£}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(£^{1-\varsigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\frac{\left(£^{1-\varsigma_{(k+1)}^{t}}-1\right)^{\perp(k+1)}}{(£-1)^{\perp(k+1)-1}}}{£-1}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{£}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(£^{1-\eth_{\varsigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\frac{\left(£^{1-\eth_{\varsigma(k+1)}^{t}}-1\right)^{\perp(k+1)}}{(£-1)^{\perp(k+1)-1}}}{£-1}\right)}}, \\
\sqrt[t]{1-\log_{£}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(£^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\frac{\left(£^{1-\varrho_{(k+1)}^{t}}-1\right)^{\perp(k+1)}}{(£-1)^{\perp(k+1)-1}}}{£-1}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{£}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(£^{1-\eth_{\varrho\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\frac{\left(£^{1-\eth_{\varrho(k+1)}^{t}}-1\right)^{\perp(k+1)}}{(£-1)^{\perp(k+1)-1}}}{£-1}\right)}}
\end{array} \right)
$$

$$
= \left(\begin{array}{c}
\sqrt[t]{\log_{£}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(£^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(£^{\sigma_{(k+1)}^{t}}-1\right)^{\perp(k+1)}}{(£-1)^{\sum_{\dot{r}=1}^{k}\perp(\dot{r})-1}(£-1)(£-1)^{\perp(k+1)-1}}\right)}\, e^{2i\pi\sqrt[t]{\log_{£}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(£^{\eth_{\sigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(£^{\eth_{\sigma(k+1)}^{t}}-1\right)^{\perp(k+1)}}{(£-1)^{\sum_{\dot{r}=1}^{k}\perp(\dot{r})-1}(£-1)(£-1)^{\perp(k+1)-1}}\right)}}, \\
\sqrt[t]{1-\log_{£}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(£^{1-\varsigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(£^{1-\varsigma_{(k+1)}^{t}}-1\right)^{\perp(k+1)}}{(£-1)^{\sum_{\dot{r}=1}^{k}\perp(\dot{r})-1}(£-1)(£-1)^{\perp(k+1)-1}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{£}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(£^{1-\eth_{\varsigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(£^{1-\eth_{\varsigma(k+1)}^{t}}-1\right)^{\perp(k+1)}}{(£-1)^{\sum_{\dot{r}=1}^{k}\perp(\dot{r})-1}(£-1)(£-1)^{\perp(k+1)-1}}\right)}}, \\
\sqrt[t]{1-\log_{£}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(£^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(£^{1-\varrho_{(k+1)}^{t}}-1\right)^{\perp(k+1)}}{(£-1)^{\sum_{\dot{r}=1}^{k}\perp(\dot{r})-1}(£-1)(£-1)^{\perp(k+1)-1}}\right)}\, e^{2i\pi\sqrt[t]{1-\log_{£}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(£^{1-\eth_{\varrho\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(£^{1-\eth_{\varrho(k+1)}^{t}}-1\right)^{\perp(k+1)}}{(£-1)^{\sum_{\dot{r}=1}^{k}\perp(\dot{r})-1}(£-1)(£-1)^{\perp(k+1)-1}}\right)}}
\end{array} \right)
$$

$$
= \begin{pmatrix}
\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(\pounds^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(\pounds^{\sigma_{(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot{r}=1}^{k+1}\perp(\dot{r})-1}}\right)} e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(\pounds^{\eth_{\sigma_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(\pounds^{\eth_{\sigma(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot{r}=1}^{k+1}\perp(\dot{r})-1}}\right)}}, \\
\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(\pounds^{1-\varsigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(\pounds^{1-\varsigma_{(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot{r}=1}^{k+1}\perp(\dot{r})-1}}\right)} e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(\pounds^{1-\eth_{\varsigma_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(\pounds^{1-\eth_{\varsigma(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot{r}=1}^{k+1}\perp(\dot{r})-1}}\right)}}, \\
\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(\pounds^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(\pounds^{1-\varrho_{(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot{r}=1}^{k+1}\perp(\dot{r})-1}}\right)} e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\frac{\prod_{\dot{r}=1}^{k}\left(\pounds^{1-\eth_{\varrho_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(\pounds^{1-\eth_{\varrho(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}}{(\pounds-1)^{\sum_{\dot{r}=1}^{k+1}\perp(\dot{r})-1}}\right)}}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{k}\left(\pounds^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(\pounds^{\sigma_{(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}\right)} e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{k}\left(\pounds^{\eth_{\sigma_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(\pounds^{\eth_{\sigma(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}\right)}}, \\
\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{k}\left(\pounds^{1-\varsigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(\pounds^{1-\varsigma_{(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}\right)} e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{k}\left(\pounds^{1-\eth_{\varsigma_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(\pounds^{1-\eth_{\varsigma(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}\right)}}, \\
\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{k}\left(\pounds^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(\pounds^{1-\varrho_{(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}\right)} e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{k}\left(\pounds^{1-\eth_{\varrho_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\left(\pounds^{1-\eth_{\varrho(k+1)}^{t}}-1\right)^{\perp_{(k+1)}}\right)}}
\end{pmatrix}.
$$

Proof: Based on Theorem 18, we get the equation, as shown at the bottom of page 35.

Using limit rules, logarithmic transform, and L'Hospital's rule, it results, as shown in the equation at pages 36 and 37, which completes the proof of Theorem 20. ■

Theorem 21 (Idempotency): Let $\Im_{\dot{r}} = (\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho\dot{r}}})(\dot{r} = 1(1)n)$ be a class of CT-SFNs, if $\Im_{\dot{r}} = \Im_0 \ \forall \ \dot{r}$, then

$$
CT - SFFWG\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \Im_0. \tag{38}
$$

Proof: Since for all $\dot{r}$ $\Im_{\dot{r}} = \Im_0 = (\sigma_0 e^{2i\pi \eth_{\sigma_0}}, \varsigma_0 e^{2i\pi \eth_{\varsigma_0}}, \varrho_0 e^{2i\pi \eth_{\varrho_0}})$, and $\sum_{\dot{r}=1}^{n} \perp_{\dot{r}} = 1$ so by Theorem 18, we get the equation, as shown at the bottom of page 38.

Thus, the proof is completed. ■

Theorem 22 (Monotonicity): Let $\Im_{\dot{r}} = (\sigma_{\dot{r}} e^{2i\pi \eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}} e^{2i\pi \eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}} e^{2i\pi \eth_{\varrho\dot{r}}})(\dot{r} = 1(1)n)$ and $\dot{\Im}_{\dot{r}} = (\dot{\sigma}_{\dot{r}} e^{2i\pi \eth_{\dot{\sigma}1}}, \dot{\varsigma}_{\dot{r}} e^{2i\pi \eth_{\dot{\varsigma}\dot{r}}}, \dot{\varrho}_{\dot{r}} e^{2i\pi \eth_{\dot{\varrho}\dot{r}}})(\dot{r} = 1(1)n)$ be two families of CT-SFNs such that $\sigma_{\dot{r}} \geq \dot{\sigma}_{\dot{r}}$, $\varsigma_{\dot{r}} \leq \dot{\varsigma}_{\dot{r}}$, $\varrho_{\dot{r}} \leq \dot{\varrho}_{\dot{r}}$, $\eth_{\sigma_{\dot{r}}} \geq \eth_{\dot{\sigma}_{\dot{r}}}$, $\eth_{\varsigma_{\dot{r}}} \leq \eth_{\dot{\varsigma}_{\dot{r}}}$,and $\varrho_{\dot{r}} \leq \eth_{\dot{\varrho}_{\dot{r}}}$ $\forall$ $\dot{r}$, then

$$
\begin{aligned}
&CT - SFFWG\left(\Im_1, \Im_2, \ldots, \Im_n\right) \\
&\quad \geq CT - SFFWG\left(\dot{\Im}_1, \dot{\Im}_2, \ldots, \dot{\Im}_n\right).
\end{aligned} \tag{39}
$$

Proof: Based on Definition 14, when $\sigma_{\dot{r}} \geq \dot{\sigma}_{\dot{r}}$, $\varsigma_{\dot{r}} \leq \dot{\varsigma}_{\dot{r}}$, $\varrho_{\dot{r}} \leq \dot{\varrho}_{\dot{r}}$, $\eth_{\sigma_{\dot{r}}} \geq \eth_{\dot{\sigma}_{\dot{r}}}$, $\eth_{\varsigma_{\dot{r}}} \leq \eth_{\dot{\varsigma}_{\dot{r}}}$, and $\varrho_{\dot{r}} \leq \eth_{\dot{\varrho}_{\dot{r}}}$ $\forall$ $\dot{r}$, then

$$
\begin{aligned}
&\sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)} \\
&\quad \leq \sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\dot{\sigma}_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}
\end{aligned}
$$

$$\sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varsigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)} \geq \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\dot{\varsigma}_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}$$

$$\sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)} \geq \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\dot{\varrho}_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}$$

$$e^{2i\pi\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\sigma\dot{r}}^{t}}-1\right)^{\perp j}\right)}} \leq e^{2i\pi\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\dot{\sigma}\dot{r}}^{t}}-1\right)^{\perp j}\right)}}$$

$$e^{2i\pi\sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\varsigma\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}}$$

$$CT-SFFWG\left(\Im_1,\Im_2,\Im_3\right) = \begin{pmatrix} \sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\sigma_{\dot{r}}^4}-1\right)^{\perp j}\right)} e^{2i\pi\sqrt[4]{\log_2\left(1+\prod_{j=1}^{3}\left(2^{\eth_{\sigma\dot{r}}^4}-1\right)^{\perp\dot{r}}\right)}}, \\ \sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\varsigma_j^4}-1\right)^{\perp j}\right)} e^{2i\pi\sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\eth_{\varsigma j}^4}-1\right)^{\perp j}\right)}}, \\ \sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\varrho_j^4}-1\right)^{\perp j}\right)} e^{2i\pi\sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\eth_{\varrho\dot{r}}^4}-1\right)^{\perp j}\right)}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[4]{\log_2\left(1+\left(2^{0.4^4}-1\right)^{0.4}\left(2^{0.7^4}-1\right)^{0.3}\left(2^{0.6^4}-1\right)^{0.3}\right)} e^{2i\pi\sqrt[4]{\log_2\left(1+\left(2^{0.3^4}-1\right)^{0.4}\left(2^{0.6^4}-1\right)^{0.3}\left(2^{0.5^4}-1\right)^{0.3}\right)}}, \\ \sqrt[4]{1-\log_2\left(1+\left(2^{1-0.3^4}-1\right)^{0.4}\left(2^{1-0.3^4}-1\right)^{0.3}\left(2^{1-0.7^4}-1\right)^{0.3}\right)} e^{2i\pi\sqrt[4]{1-\log_2\left(1+\left(2^{1-0.4^4}-1\right)^{0.4}\left(2^{1-0.4^4}-1\right)^{0.3}\left(2^{1-0.8^4}-1\right)^{0.3}\right)}}, \\ \sqrt[4]{1-\log_2\left(1+\left(2^{1-0.5^4}-1\right)^{0.4}\left(2^{1-0.4^4}-1\right)^{0.3}\left(2^{1-0.8^4}-1\right)^{0.3}\right)} e^{2i\pi\sqrt[4]{1-\log_2\left(1+\left(2^{1-0.5^4}-1\right)^{0.4}\left(2^{1-0.4^4}-1\right)^{0.3}\left(2^{1-0.8^4}-1\right)^{0.3}\right)}} \end{pmatrix}$$

$$= \left(0.5363e^{2i\pi 0.4301}, 0.5358e^{2i\pi 0.6285}, 0.6418e^{2i\pi 0.6418}\right).$$

$$\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right) = \frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln £} \longrightarrow \frac{\prod_{\dot{r}=1}^{n}\left(£^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}}{\ln £}$$

$$\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varsigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right) = \frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varsigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln £} \longrightarrow \frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\varsigma_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}}{\ln £}$$

$$\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right) = \frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}\right)}{\ln £} \longrightarrow \frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp\dot{r}}}{\ln £}$$

$$\geq e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^{t}_{\dot{\varsigma}\dot{r}}}-1\right)^{\perp\dot{r}}\right)}}$$

$$e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^{t}_{\varrho\dot{r}}}-1\right)^{\perp\dot{r}}\right)}}$$

$$\geq e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^{t}_{\dot{\varrho}\dot{r}}}-1\right)^{\perp\dot{r}}\right)}}$$

Thus, $S(CT-SFFWG(\Im_1,\Im_2,\ldots,\Im_n)) \geq S(CT-SFFWG(\dot{\Im}_1,\dot{\Im}_2,\ldots,\dot{\Im}_n))$.

Hence, $CT-SFFWG(\Im_1,\Im_2,\ldots,\Im_n) \geq CT-SFFWG(\dot{\Im}_1,\dot{\Im}_2,\ldots,\dot{\Im}_n)$. ■

Theorem 23 (Boundedness): Let $\Im_{\dot{r}} = (\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}})(\dot{r}=1(1)n)$ be a class of CT-SFNs, and let

$$\Im^{-}=\left(\min_{1\leq\dot{r}\leq n}\sigma_{\dot{r}}e^{\min_{1\leq\dot{r}\leq n}\sigma_{\dot{r}}},\ \max_{1\leq\dot{r}\leq n}\varsigma_{\dot{r}}e^{\max_{1\leq\dot{r}\leq n}\varsigma_{\dot{r}}},\ \max_{1\leq\dot{r}\leq n}\varrho_{\dot{r}}e^{\max_{1\leq\dot{r}\leq n}\varrho_{\dot{r}}}\right),$$

$$\Im^{+}=\left(\max_{1\leq\dot{r}\leq n}\sigma_{\dot{r}}e^{\max_{1\leq\dot{r}\leq n}\sigma_{\dot{r}}},\ \min_{1\leq\dot{r}\leq n}\varsigma_{\dot{r}}e^{\min_{1\leq\dot{r}\leq n}\varsigma_{\dot{r}}},\ \min_{1\leq\dot{r}\leq n}\varrho_{\dot{r}}e^{\min_{1\leq\dot{r}\leq n}\varrho_{\dot{r}}}\right),$$

then

$$\Im^{-}\leq CT-SFFWG(\Im_1,\Im_2,\ldots,\Im_n)\leq\Im^{+}. \tag{40}$$

$$\lim_{£\longrightarrow 1} CT-SFFWG(\Im_1,\Im_2,\ldots,\Im_n)$$

$$=\lim_{£\longrightarrow 1}\left(\begin{array}{c}\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\sigma^{t}_{\dot{r}}}-1\right)^{\perp\dot{r}}\right)}, e^{2i\pi\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth^{t}_{\sigma\dot{r}}}-1\right)^{\perp j}\right)}}\\ \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varsigma^{t}_{\dot{r}}}-1\right)^{\perp\dot{r}}\right)}e^{2i\pi\sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^{t}_{\varsigma\dot{r}}}-1\right)^{\perp\dot{r}}\right)}},\\ \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varrho^{t}_{\dot{r}}}-1\right)^{\perp\dot{r}}\right)}e^{2i\pi\sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^{t}_{\varrho\dot{r}}}-1\right)^{\perp\dot{r}}\right)}}\end{array}\right)$$

$$=\lim_{£\longrightarrow 1}\left(\begin{array}{c}\sqrt[t]{\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\sigma^{t}_{\dot{r}}}-1\right)^{\perp\dot{r}}\right)}{\ln £}}e^{2i\pi\sqrt[t]{\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth^{t}_{\sigma\dot{r}}}-1\right)^{\perp\dot{r}}\right)}{\ln £}}},\\ \sqrt[t]{1-\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varsigma^{t}_{\dot{r}}}-1\right)^{\perp\dot{r}}\right)}{\ln £}}e^{2i\pi\sqrt[t]{1-\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^{t}_{\varsigma\dot{r}}}-1\right)^{\perp\dot{r}}\right)}{\ln £}}},\\ \sqrt[t]{1-\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varrho^{t}_{\dot{r}}}-1\right)^{\perp j}\right)}{\ln £}}e^{2i\pi\sqrt[t]{1-\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^{t}_{\varrho\dot{r}}}-1\right)^{\perp\dot{r}}\right)}{\ln £}}}\end{array}\right)$$

$$=\lim_{£\longrightarrow 1}\left(\begin{array}{c}\sqrt[t]{\frac{\prod_{\dot{r}=1}^{n}\left(£^{\sigma^{t}_{\dot{r}}}-1\right)^{\perp\dot{r}}}{\ln £}}e^{2i\pi\sqrt[t]{\frac{\prod_{\dot{r}=1}^{n}\left(£^{\eth^{t}_{\sigma\dot{r}}}-1\right)^{\perp\dot{r}}}{\ln £}}},\\ \sqrt[t]{1-\frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\varsigma^{t}_{\dot{r}}}-1\right)^{\perp\dot{r}}}{\ln £}}e^{2i\pi\sqrt[t]{1-\frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^{t}_{\varsigma\dot{r}}}-1\right)^{\perp\dot{r}}}{\ln £}}},\\ \sqrt[t]{1-\frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\varrho^{t}_{\dot{r}}}-1\right)^{\perp\dot{r}}}{\ln £}}e^{2i\pi\sqrt[t]{1-\frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^{t}_{\varrho\dot{r}}}-1\right)^{\perp\dot{r}}}{\ln £}}}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\sigma^{t}_{\dot{r}}\right)^{\perp\dot{r}}}e^{2i\pi\sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\eth^{t}_{\sigma_{\dot{r}}}\right)^{\perp\dot{r}}}},\\ \sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\varsigma^{t}_{\dot{r}}\right)^{\perp\dot{r}}}e^{2i\pi\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\eth^{t}_{\varsigma_{\dot{r}}}\right)^{\perp\dot{r}}}},\\ \sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\varrho^{t}_{\dot{r}}\right)^{\perp\dot{r}}}e^{2i\pi\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\eth^{t}_{\varrho_{\dot{r}}}\right)^{\perp\dot{r}}}}\end{array}\right),$$

Proof: Since for all $\dot{r}$, $\min\limits_{1\le\dot{r}\le n}\sigma_{\dot{r}}e^{\min\limits_{1\le\dot{r}\le n}\eth_{\sigma_{\dot{r}}}} \le \sigma_{\dot{r}}e^{\eth_{\sigma_{\dot{r}}}} \le \max\limits_{1\le\dot{r}\le n}\sigma_{\dot{r}}e^{\max\limits_{1\le\dot{r}\le n}\eth_{\sigma_{\dot{r}}}}$, $\min\limits_{1\le\dot{r}\le n}\vartheta_{\dot{r}}e^{\min\limits_{1\le\dot{r}\le n}\eth_{\vartheta_{\dot{r}}}} \le \vartheta_{\dot{r}}e^{\eth_{\vartheta_{\dot{r}}}} \le \max\limits_{1\le\dot{r}\le n}\vartheta_{\dot{r}}e^{\max\limits_{1\le\dot{r}\le n}\eth_{\vartheta_{\dot{r}}}}$ and $\min\limits_{1\le\dot{r}\le n}\varrho_{\dot{r}}e^{\min\limits_{1\le\dot{r}\le n}\eth_{\varrho_{\dot{r}}}} \le \varrho_{\dot{r}}e^{\eth_{\varrho_{\dot{r}}}} \le \max\limits_{1\le\dot{r}\le n}\varrho_{\dot{r}}e^{\max\limits_{1\le\dot{r}\le n}\eth_{\varrho_{\dot{r}}}}$, thereby on the basis of idempotency and monotonicity, we get

$$\Im^- \le CT - SFFWG\,(\Im_1, \Im_2, \ldots, \Im_n) \le \Im^+. \qquad \blacksquare$$

Theorem 24 (Shift-Invariance): Let $\Im_{\dot{r}} = (\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}})(\dot{r} = 1(1)n)$ be a class of CT-SFNs and $\dot{\Im} = \left(\dot{\sigma}e^{2i\pi\eth_{\dot{\sigma}1}}, \dot{\varsigma}e^{2i\pi\eth_{\dot{\varsigma}\dot{r}}}, \dot{\varrho}e^{2i\pi\eth_{\dot{\varrho}\dot{r}}}\right)$ be any other CT-SFNs, then

$$\begin{aligned} &CT - SFFWG\left(\Im_1 \otimes \dot{\Im}, \Im_2 \otimes \dot{\Im}, \ldots, \Im_n \otimes \dot{\Im}\right) \\ &\quad = CT - SFFWG\,(\Im_1, \Im_2, \ldots, \Im_n) \otimes \dot{\Im}. \end{aligned} \tag{41}$$

Theorem 25 (Homogeneity): Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs and $\dagger > 0$ be any real number, then

$$\begin{aligned} &CT - SFFWG\,(\dagger\Im_1, \dagger\Im_2, \ldots, \dagger\Im_n) \\ &\quad = \dagger CT - SFFWG\,(\Im_1, \Im_2, \ldots, \Im_n)\,. \end{aligned} \tag{42}$$

To save space, the proof of the aforesaid two theorems can be simply deduced from the suggested Frank operational rules of CT-SFNs; consequently, it is skipped here.

Definition 15: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, then the Complex T-spherical fuzzy Frank ordered weighted geometric (CT-SFFOWG) operator is:

$$CT - SFFOWG\,(\Im_1, \Im_2, \ldots, \Im_n) = \otimes_{\dot{r}=1}^{n}\left(\Im_{\delta(\dot{r})}^{\natural_{\dot{r}}}\right), \tag{43}$$

where $\natural = (\natural_1, \natural_2, \ldots, \natural_n)^T$ is the position weights of $\Im_{\dot{r}}\ (\dot{r} = 1(1)n)$ such that $\natural_{\dot{r}} > 0$ and $\sum\limits_{\dot{r}=1}^{n}\natural_{\dot{r}} = 1$. $(\delta(1), \delta(2), \ldots, \delta(n))$ is a permutation of $(1, 2, 3, \ldots, n)$ such that $\Im_{\delta(\dot{r}-1)} \ge \Im_{\delta(j)}$ for $\dot{r} = 1(1)n$.

Theorem 26: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, then the result acquired by using the CT-SFFOWG operator is still a CT-SFN, and (45), as shown at the bottom of page 38.

Proof: We omit the evidence of this result since it is identical to that of Theorem 18. ■

Example 5 (Continued From Example 2): Based on Definition 15 and Theorem 26, we can determine, as shown in the equation at the top of page 39.

Theorem 27: Let $\Im_{\dot{r}} = (\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}})$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, and $\pounds > 1$. As $\pounds \longrightarrow 1$, the CT-SFFOWG operator approaches the limit (46), as shown at page 39.

Theorem 28: Let $\Im_{\dot{r}} = \left(\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}}\right)$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, and $\pounds > 1$. As $\pounds \longrightarrow \infty$, the CT-SFFOWG operator approaches the limit (47), as shown at page 39.

Likewise CT-SFFWG operator, the CT-SFFOWG operator also adheres the boundedness, monotonicity and idempotency, shift-invariance, and homogeneity properties. Besides the aforesaid properties, the CT-SFFOWG operator has the following noteworthy results.

Theorem 29: Let $\Im_{\dot{r}} = (\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}}, \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}}, \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}})$ $(\dot{r} = 1(1)n)$ be a class of CT-SFNs, then we have the following:

i). If $\natural = (1, 0, \ldots, 0)^T$ then $CT - SFFOWG(\Im_1, \Im_2, \ldots, \Im_n) = \max\{\Im_1, \Im_2, \ldots, \Im_n\}$.

ii). If $\natural = (0, 0, \ldots, 1)^T$ then $CT - SFFOWG(\Im_1, \Im_2, \ldots, \Im_n) = \min\{\Im_1, \Im_2, \ldots, \Im_n\}$.

iii). If $\natural_{\dot{r}} = 1$ and $\perp_i = 0(i \ne \dot{r})$ then $CT - SFFOWG(\Im_1, \Im_2, \ldots, \Im_n) = \Im_{\delta(\dot{r})}$ where $\Im_{\delta(\dot{r})}$ is the $\dot{r}$th largest of $\Im_{\dot{r}}$, $(\dot{r} = 1(1)n)$.

According to the definition of CT-SFFWG and CT-SFFOWG operators, we can notice that the CT-SFFWG operator can weights only the SFNs while CT-SFFOWG operator weights only the ordered position of SFNs.

$$\lim_{\pounds\longrightarrow\infty} CT - SFFWG\,(\Im_1, \Im_2, \ldots, \Im_n)$$

$$= \left(\begin{array}{c} \lim\limits_{\pounds\longrightarrow\infty}\sqrt[t]{\log_{\pounds}\left(1 + \prod\limits_{\dot{r}=1}^{n}\left(\pounds^{\sigma_{\dot{r}}^{t}} - 1\right)^{\perp_{\dot{r}}}\right)}\, e^{2i\pi \lim\limits_{\pounds\longrightarrow\infty}\sqrt[t]{\log_{\pounds}\left(1+\prod\limits_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\sigma_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \lim\limits_{\pounds\longrightarrow\infty}\sqrt[t]{1 - \log_{\pounds}\left(1 + \prod\limits_{\dot{r}=1}^{n}\left(\pounds^{1-\varsigma_{\dot{r}}^{t}} - 1\right)^{\perp_{\dot{r}}}\right)}\, e^{2i\pi \lim\limits_{\pounds\longrightarrow\infty}\sqrt[t]{1-\log_{\pounds}\left(1+\prod\limits_{\dot{r}=1}^{n}\left(\pounds^{1-\eth_{\varsigma_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \lim\limits_{\pounds\longrightarrow\infty}\sqrt[t]{1 - \log_{\pounds}\left(1 + \prod\limits_{\dot{r}=1}^{n}\left(\pounds^{1-\varrho_{\dot{r}}^{t}} - 1\right)^{\perp_{\dot{r}}}\right)}\, e^{2i\pi \lim\limits_{\pounds\longrightarrow\infty}\sqrt[t]{1-\log_{\pounds}\left(1+\prod\limits_{\dot{r}=1}^{n}\left(\pounds^{1-\eth_{\varrho_{\dot{r}}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}} \end{array}\right).$$

$$\left(\begin{array}{c}\sqrt[t]{\lim_{£\longrightarrow\infty}\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}{\ln £}}\, e^{2i\pi\sqrt[t]{\lim_{£\longrightarrow\infty}\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\sigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}{\ln £}}},\\ \sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\vartheta_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}{\ln £}}\, e^{2i\pi\sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\vartheta\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}{\ln £}}},\\ \sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}{\ln £}}\, e^{2i\pi\sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\ln\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\varrho\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}\right)}{\ln £}}}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\frac{\prod_{\dot{r}=1}^{n}\left(£^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{1+\prod_{\dot{r}=1}^{n}\left(£^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\varsigma_{\dot{r}}^{t}\right)\frac{£^{\sigma_{\dot{r}}^{t}}-1}{£^{\sigma_{\dot{r}}^{t}}-1}\right)}{\frac{1}{£}}}\, e^{2i\pi\sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\frac{\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\sigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{1+\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\sigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\varsigma_{\dot{r}}^{t}\right)\frac{£^{\eth_{\sigma\dot{r}}^{t}-1}}{£^{\eth_{\sigma\dot{r}}^{t}}-1}\right)}{\frac{1}{£}}}},\\ \sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\varsigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varsigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\varsigma_{\dot{r}}^{t}\right)\frac{£^{-\varsigma_{\dot{r}}^{t}}}{£^{1-\varsigma_{\dot{r}}^{t}}-1}\right)}{\frac{1}{£}}}\, e^{2i\pi\sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\varsigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\varsigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\eth_{\varsigma\dot{r}}^{t}\right)\frac{£^{-\eth_{\varsigma\dot{r}}^{t}}}{£^{1-\eth_{\varsigma\dot{r}}^{t}}-1}\right)}{\frac{1}{£}}}},\\ \sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\varrho_{\dot{r}}^{t}\right)\frac{£^{-\varrho_{\dot{r}}^{t}}}{£^{1-\varrho_{\dot{r}}^{t}}-1}\right)}{\frac{1}{£}}}\, e^{2i\pi\sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\varrho\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\varrho\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\eth_{\varrho\dot{r}}^{t}\right)\frac{£^{-\eth_{\varrho\dot{r}}^{t}}}{£^{1-\eth_{\varrho\dot{r}}^{t}}-1}\right)}{\frac{1}{£}}}},\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\prod_{\dot{r}=1}^{n}\left(£^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{1+\prod_{\dot{r}=1}^{n}\left(£^{\sigma_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\sigma_{\dot{r}}^{t}\right)\frac{£^{\sigma_{\dot{r}}^{t}}}{£^{\sigma_{\dot{r}}^{t}}-1}\right)}\\ e^{2i\pi\sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\sigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{1+\prod_{\dot{r}=1}^{n}\left(£^{\eth_{\sigma\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth_{\sigma\dot{r}}^{t}\right)\frac{£^{\sigma_{\dot{r}}^{t}}}{£^{\eth_{\sigma\dot{r}}^{t}}-1}\right)}},\\ \sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\vartheta_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{1+\prod_{\dot{r}=1}^{n}\left(£^{1-\vartheta_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\vartheta_{\dot{r}}^{t}\right)\frac{£^{1-\vartheta_{\dot{r}}^{t}}}{£^{1-\vartheta_{\dot{r}}^{t}}-1}\right)}\\ e^{2i\pi\sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\vartheta\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\vartheta\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\eth_{\vartheta\dot{r}}^{t}\right)\frac{£^{1-\eth_{\vartheta\dot{r}}^{t}}}{£^{1-\eth_{\vartheta\dot{r}}^{t}}-1}\right)}},\\ \sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{1+\prod_{\dot{r}=1}^{n}\left(£^{1-\varrho_{\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\varrho_{\dot{r}}^{t}\right)\frac{£^{1-\varrho_{\dot{r}}^{t}}}{£^{1-\varrho_{\dot{r}}^{t}}-1}\right)}\\ e^{2i\pi\sqrt[t]{1-\lim_{£\longrightarrow\infty}\frac{\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\varrho\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}{1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth_{\varrho\dot{r}}^{t}}-1\right)^{\perp_{\dot{r}}}}\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\eth_{\varrho\dot{r}}^{t}\right)\frac{£^{1-\eth_{\varrho\dot{r}}^{t}}}{£^{1-\eth_{\varrho\dot{r}}^{t}}-1}\right)}}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\sigma_{\dot{r}}^{t}\right)\right)}e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth_{\sigma_{\dot{r}}}^{t}\right)\right)}},\\ \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\varsigma_{\dot{r}}^{t}\right)\right)}e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\eth_{\varsigma_{\dot{r}}}^{t}\right)\right)}},\\ \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\varrho_{\dot{r}}^{t}\right)\right)}e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(1-\eth_{\varrho_{\dot{r}}}^{t}\right)\right)}}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\sigma_{\dot{r}}^{t}\right)\right)}e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth_{\sigma_{\dot{r}}}^{t}\right)\right)}},\\ \sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\varsigma_{\dot{r}}^{t}\right)\right)}e^{2i\pi\sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth_{\varsigma_{\dot{r}}}^{t}\right)\right)}},\\ \sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\varrho_{\dot{r}}^{t}\right)\right)}e^{2i\pi\sqrt[t]{\left(\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}\left(\eth_{\varrho_{\dot{r}}}^{t}\right)\right)}}\end{array}\right),$$

In real-world practical situations, we should analyse both factors simultaneously. Thereby, to address this issue, we sate the hybrid geometric operator based on Frank t-norm and t-conorm, which weight both the given CT-SFNs and their ordered positions.

Definition 16: Let $\Im_{\dot{r}}=\left(\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}},\varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}},\varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}}\right)$ $(\dot{r}=1(1)n)$ be a class of CT-SFNs, then the Complex T-spherical fuzzy Frank hybrid geometric (CT-SFFHG) operator is:

$$CT-SFFHG\left(\Im_1,\Im_2,\ldots,\Im_n\right)=\otimes_{\dot{r}=1}^{n}\left(\hat{\Im}_{\delta(\dot{r})}\right)^{\natural_{\dot{r}}},\quad(44)$$

where $\natural=(\natural_1,\natural_2,\ldots,\natural_n)^T$ is the weight vector associated with CT-SFFHG so that $\natural_{\dot{r}}>0$ and $\sum_{\dot{r}=1}^{n}\natural_{\dot{r}}=1$, $\perp=(\perp_1,\perp_2,\ldots,\perp_n)^T$ is the weight vector of $\Im_{\dot{r}}$ $(\dot{r}=1(1)n)$ so that $\perp_{\dot{r}}>0$ and $\sum_{\dot{r}=1}^{n}\perp_{\dot{r}}=1$. $\hat{\Im}_{\delta(\dot{r})}$ is the $\dot{r}$th greatest of the weighted CT-SFNs $\hat{\Im}_{\dot{r}}\left(\hat{\Im}_{\dot{r}}=(\Im_{\dot{r}})^{n\perp_{\dot{r}}},\dot{r}=1(1)n\right)$ and n is the balancing coefficient.

Theorem 30: Let $\Im_{\dot{r}}=\left(\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}},\varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}},\varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}}\right)$ $(\dot{r}=1(1)n)$ be a class of CT-SFNs, then the result acquired by utilizing the CT-SFFHG operator is still a CT-SFN, and (48), as shown at the bottom of page 40.

Proof: We omit the verification of this result since it is identical to that of Theorem 18. ∎

Example 6 (Continued From Example 3): Based on Definition 10, we can determine the weighted CT-SFNs, as shown in the equation at the bottom of page 40.

According to Definition 5, we can get the score of $\hat{\Im}_{\dot{r}}$ $(\dot{r}=1,2,3)$:

$$S\left(\hat{\Im}_1\right)=0.4455,\quad S\left(\hat{\Im}_2\right)=0.1038,\ S\left(\hat{\Im}_3\right)=0.3972.$$

Since $S\left(\hat{\Im}_1\right)>S\left(\hat{\Im}_3\right)>S\left(\hat{\Im}_2\right)$, we have $\hat{\Im}_{\delta(1)}=(0.3275e^{2i\pi0.2317},\ 0.5227e^{2i\pi0.6260},\ 0.2093e^{2i\pi0.2166})$, $\hat{\Im}_{\delta(2)}=(0.5012e^{2i\pi0.1423},0.5307e^{2i\pi0.7292},0.4410e^{2i\pi0.5230})$, $\hat{\Im}_{\delta(3)}=(0.5353e^{2i\pi0.4301},0.7290e^{2i\pi0.8293},0.8292e^{2i\pi0.8293})$. Suppose $\natural=(0.3,0.4,0.3)^T$ is the weight vector associated with the CT-SFFHG operator. Then according to Definition 16 and Theorem 30, we can determine, as shown in the equation at page 41.

Theorem 31: Let $\Im_{\dot{r}}=\left(\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}},\varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}},\varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}}\right)$ $(\dot{r}=1(1)n)$ be a class of CT-SFNs, and £ > 1. As £ $\longrightarrow$ 1, the CT-SFFHG operator approaches the limit (49), as shown at the bottom of page 41.

Theorem 32: Let $\Im_{\dot{r}}=(\sigma_{\dot{r}}e^{2i\pi\eth_{\sigma\dot{r}}},\ \varsigma_{\dot{r}}e^{2i\pi\eth_{\varsigma\dot{r}}}e^{2i\pi},\ \varrho_{\dot{r}}e^{2i\pi\eth_{\varrho\dot{r}}})$ $(\dot{r}=1(1)n)$ be a class of CT-SFNs, and £ > 1. As £ $\longrightarrow\infty$, the CT-SFFHG operator approaches the limit (50), as shown at the bottom of page 42.

Similar to the CT-SFFWG operator, the CT-SFFHG operator also adheres the boundedness, idempotency and monotonicity, shift-invariance, and homogeneity properties. Besides the aforementioned properties, the CT-SFFHG operator has the following particular cases.

Corollary 3: CT-SFFWG operator is a particular case of the CT-SFFHG operator.

Proof: Let $\natural = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then

$$\begin{aligned} &CT - SFFHG\left(\Im_1, \Im_2, \ldots, \Im_n\right) \\ &= \hat{\Im}_{\delta(1)}^{\natural_1} \otimes \hat{\Im}_{\delta(2)}^{\natural_2} \otimes \ldots \otimes \hat{\Im}_{\delta(n)}^{\natural_n} \\ &= \left(\hat{\Im}_{\delta(1)} \otimes \hat{\Im}_{\delta(2)} \otimes \ldots \otimes \hat{\Im}_{\delta(n)}\right)^{\frac{1}{n}} \\ &= \Im_1^{\perp_1} \otimes \Im_2^{\perp_2} \otimes \ldots \otimes \Im_n^{\perp_n} \\ &= CT - SFFWG\left(\Im_1, \Im_2, \ldots, \Im_n\right). \end{aligned}$$

■

Corollary 4: CT-SFFOWG operator is a particular case of the CT-SFFHG operator.

Proof: Let $\perp = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then

$$\begin{aligned} &CT - SFFHG\left(\Im_1, \Im_2, \ldots, \Im_n\right) \\ &= \hat{\Im}_{\delta(1)}^{\natural_1} \otimes \hat{\Im}_{\delta(2)}^{\natural_2} \otimes \ldots \otimes \hat{\Im}_{\delta(n)}^{\natural_n} \\ &= \Im_{\delta(1)}^{\natural_1} \otimes \Im_{\delta(2)}^{\natural_2} \otimes \ldots \otimes \Im_{\delta(n)}^{\natural_n} \\ &= CT - SFFOWG\left(\Im_1, \Im_2, \ldots, \Im_n\right). \end{aligned}$$

■

V. MCGDM APPROACH

A. ENUMERATE

Let $\mathcal{O} = \{\mathcal{O}l_1, \mathcal{O}l_2, \ldots, \mathcal{O}l_m\}$ be a collection of m alternatives, $\mathcal{C} = \{\mathcal{C}r_1, \mathcal{C}r_2, \ldots, \mathcal{C}r_n\}$ be a collection of n criteria, and $\mathscr{D} = \{\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_l\}$ be the panel of l DEs which are authorized to evaluate the requirements of MCGDM problems, including the determination of the criteria and their weight, and the evaluation of the performance of the alternatives. Each DE $\mathscr{D}_x$ is assigned a weight $\kappa_x (x = 1(1)l)$ satisfying $\sum_{x=1}^{l} \kappa_x = 1$ to reflect his/her importance in the decision process. Our goal is to opt the optimal alternative from the set of m feasible alternatives regarding n criteria.

B. ALGORITHM

The step-wise mechanism of the framed approach is detailed as follows:

$$\begin{aligned} &CT - SFFWG\left(\Im_1, \Im_2, \ldots, \Im_n\right) \\ &= \left(\begin{array}{c} \sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\sigma_0^t}-1\right)^{\perp_{\dot{r}}}\right)} e^{2i\pi \sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\sigma_0}^t}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\varsigma_0^t}-1\right)^{\perp_{\dot{r}}}\right)} e^{2i\pi \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth_{\varsigma_0}^t}-1\right)^{\perp_{\dot{r}}}\right)}}, \\ \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\varrho_0^t}-1\right)^{\perp_{\dot{r}}}\right)} e^{2i\pi \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth_{\varrho_0}^t}-1\right)^{\perp_{\dot{r}}}\right)}} \end{array}\right) \\ &= \left(\sqrt[t]{\log_{\pounds} \pounds^{\sigma_0^t}} e^{2i\pi \sqrt[t]{\log_{\pounds} \pounds^{\eth_{\sigma_0}^t}}}, \sqrt[t]{1-\log_{\pounds} \pounds^{1-\varsigma_0^t}} e^{2i\pi \sqrt[t]{1-\log_{\pounds} \pounds^{1-\eth_{\varsigma_0}^t}}}, \sqrt[t]{1-\log_{\pounds} \pounds^{1-\varrho_0^t}} e^{2i\pi \sqrt[t]{1-\log_{\pounds} \pounds^{1-\eth_{\varrho_0}^t}}}\right) \\ &= \left(\sigma_0 e^{2i\pi \eth_{\sigma_0}}, \varsigma_0 e^{2i\pi \eth_{\varsigma_0}}, \varrho_0 e^{2i\pi \eth_{\varrho_0}}\right) = \Im_0. \end{aligned}$$

$$CT - SFFOWG\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \left(\begin{array}{c} \sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\sigma_{\delta(\dot{r})}^t}-1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi \sqrt[t]{\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{\eth_{\sigma_{\delta(\dot{r})}}^t}-1\right)^{\natural_{\dot{r}}}\right)}}, \\ \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\varsigma_{\delta(\dot{r})}^t}-1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth_{\varsigma_{\delta(\dot{r})}}^t}-1\right)^{\natural_{\dot{r}}}\right)}}, \\ \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\varrho_{\delta(j)}^t}-1\right)^{\natural_j}\right)} e^{2i\pi \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{\dot{r}=1}^{n}\left(\pounds^{1-\eth_{\varrho_{\delta(\dot{r})}}^t}-1\right)^{\natural_{\dot{r}}}\right)}} \end{array}\right). \quad (45)$$

Step 1: Creation of individual evaluation matrices: Each DE on the panel thoroughly examines the MCGDM problem and specifies the evaluation criteria for each alternative. Each DE evaluates the proficiencies and capabilities of alternatives in relation to particular criteria and assigns them lin-

$$CT-SFFOWG\left(\Im_{\delta(1)},\Im_{\delta(2)},\Im_{\delta(3)}\right) = \begin{pmatrix} \sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\sigma^4_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi\sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\eth^4_{\sigma_{\delta(\dot{r})}}}-1\right)^{\natural_{\dot{r}}}\right)}}, \\ \sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\varsigma^4_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi\sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\eth^4_{\varsigma_{\delta(\dot{r})}}}-1\right)^{\natural_{\dot{r}}}\right)}}, \\ \sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\varrho^4_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi\sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\eth^4_{\varrho_{\delta(\dot{r})}}}-1\right)^{\natural_{\dot{r}}}\right)}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[4]{\log_2\left(1+\left(2^{0.7^4}-1\right)^{0.3}\left(2^{0.3^4}-1\right)^{0.4}\left(2^{0.6^4}-1\right)^{0.3}\right)} \\ e^{2i\pi\sqrt[4]{\log_2\left(1+\left(2^{0.6^4}-1\right)^{0.3}\left(2^{0.2^4}-1\right)^{0.4}\left(2^{0.5^4}-1\right)^{0.3}\right)}}, \\ \sqrt[4]{1-\log_2\left(1+\left(2^{1-0.4^4}-1\right)^{0.3}\left(2^{1-0.3^4}-1\right)^{0.4}\left(2^{1-0.7^4}-1\right)^{0.3}\right)} \\ e^{2i\pi\sqrt[4]{1-\log_2\left(1+\left(2^{1-0.5^4}-1\right)^{0.3}\left(2^{1-0.4^4}-1\right)^{0.4}\left(2^{1-0.8^4}-1\right)^{0.3}\right)}}, \\ \sqrt[4]{1-\log_2\left(1+\left(2^{1-0.5^4}-1\right)^{0.3}\left(2^{1-0.5^4}-1\right)^{0.4}\left(2^{1-0.8^4}-1\right)^{0.3}\right)} \\ e^{2i\pi\sqrt[4]{1-\log_2\left(1+\left(2^{1-0.5^4}-1\right)^{0.3}\left(2^{1-0.5^4}-1\right)^{0.4}\left(2^{1-0.8^4}-1\right)^{0.3}\right)}} \end{pmatrix}$$

$$= \left(0.4785e^{2i\pi 0.3694}, 0.5442e^{2i\pi 0.6388}, 0.6509e^{2i\pi 0.6509}\right).$$

$$\lim_{\pounds\longrightarrow 1} CT-SFFOWG\left(\Im_1,\Im_2,\ldots,\Im_n\right) = \begin{pmatrix} \sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\sigma^t_{\delta(\dot{r})}\right)^{\natural_{\dot{r}}}} e^{2i\pi\sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\eth^t_{\sigma_{\delta(\dot{r})}}\right)^{\natural_{\dot{r}}}}}, \\ \sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\varsigma^t_{\delta(\dot{r})}\right)^{\natural_{\dot{r}}}} e^{2i\pi\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\eth^t_{\varsigma_{\delta(\dot{r})}}\right)^{\natural_{\dot{r}}}}}, \\ \sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\varrho^t_{\delta(\dot{r})}\right)^{\natural_{\dot{r}}}} e^{2i\pi\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\eth^t_{\varrho_{\delta(\dot{r})}}\right)^{\natural_{\dot{r}}}}} \end{pmatrix}. \tag{46}$$

$$\lim_{\pounds\longrightarrow\infty} CT-SFFOWG\left(\Im_1,\Im_2,\ldots,\Im_n\right) = \begin{pmatrix} \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}}\left(\sigma^t_{\delta(\dot{r})}\right)\right)} e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}}\left(\eth^t_{\sigma_{\delta(\dot{r})}}\right)\right)}}, \\ \sqrt[t]{\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}}\left(\vartheta^t_{\delta(\dot{r})}\right)\right)} e^{2i\pi\sqrt[t]{\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}}\left(\eth^t_{\vartheta_{\delta(\dot{r})}}\right)\right)}}, \\ \sqrt[t]{\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}}\left(\varrho^t_{\delta(\dot{r})}\right)\right)} e^{2i\pi\sqrt[t]{\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}}\left(\eth^t_{\varrho_{\delta(\dot{r})}}\right)\right)}} \end{pmatrix}. \tag{47}$$

$$CT-SFFHG\left(\Im_1,\Im_2,\ldots,\Im_n\right)=\left(\begin{array}{c}\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\hat{\sigma}^t_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)}e^{2i\pi\sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth^t_{\hat{\sigma}\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)}},\\ \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\hat{\varsigma}^t_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)}e^{2i\pi\sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^t_{\hat{\varsigma}\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)}},\\ \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\hat{\varrho}^t_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)}e^{2i\pi\sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^t_{\hat{\varrho}\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)}}\end{array}\right). \tag{48}$$

$$\hat{\Im}_1=\Im_1^{3\times0.4}=\left(\begin{array}{c}\sqrt[4]{\log_2\left(1+\frac{\left(2^{0.4^4}-1\right)^{3\times0.4}}{(2-1)^{3\times0.4-1}}\right)}e^{2i\pi\sqrt[4]{\log_2\left(1+\frac{\left(2^{0.3^4}-1\right)^{3\times0.4}}{(2-1)^{3\times0.4-1}}\right)}},\\ \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.5^4}-1\right)^{3\times0.4}}{(2-1)^{3\times0.4-1}}\right)}e^{2i\pi\sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.6^4}-1\right)^{3\times0.4}}{(2-1)^{3\times0.4-1}}\right)}},\\ \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.2^4}-1\right)^{3\times0.4}}{(2-1)^{3\times0.4-1}}\right)}e^{2i\pi\sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.2^4}-1\right)^{3\times0.4}}{(2-1)^{3\times0.4-1}}\right)}}\end{array}\right)=\left(0.3275e^{2i\pi0.2317},0.5227e^{2i\pi0.6260},0.2093e^{2i\pi0.2166}\right);$$

$$\hat{\Im}_2=\Im_2^{3\times0.4}=\left(\begin{array}{c}\sqrt[4]{\log_2\left(1+\frac{\left(2^{0.6^4}-1\right)^{3\times0.4}}{(2-1)^{3\times0.4-1}}\right)}e^{2i\pi\sqrt[4]{\log_2\left(1+\frac{\left(2^{0.5^4}-1\right)^{3\times0.4}}{(2-1)^{3\times0.4-1}}\right)}},\\ \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.7^4}-1\right)^{3\times0.4}}{(2-1)^{3\times0.4-1}}\right)}e^{2i\pi\sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.8^4}-1\right)^{3\times0.4}}{(2-1)^{3\times0.4-1}}\right)}},\\ \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.8^4}-1\right)^{3\times0.4}}{(2-1)^{3\times0.4-1}}\right)}e^{2i\pi\sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.8^4}-1\right)^{3\times0.4}}{(2-1)^{3\times0.4-1}}\right)}}\end{array}\right)=\left(0.5353e^{2i\pi0.4301},0.7290e^{2i\pi0.8293},0.8292e^{2i\pi0.8293}\right);$$

$$\hat{\Im}_3=\Im_3^{3\times0.2}=\left(\begin{array}{c}\sqrt[4]{\log_2\left(1+\frac{\left(2^{0.3^4}-1\right)^{3\times0.2}}{(2-1)^{3\times0.4-1}}\right)}e^{2i\pi\sqrt[4]{\log_2\left(1+\frac{\left(2^{0.2^4}-1\right)^{3\times0.2}}{(2-1)^{3\times0.4-1}}\right)}},\\ \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.6^4}-1\right)^{3\times0.2}}{(2-1)^{3\times0.4-1}}\right)}e^{2i\pi\sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.7^4}-1\right)^{3\times0.2}}{(2-1)^{3\times0.4-1}}\right)}},\\ \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.5^4}-1\right)^{3\times0.2}}{(2-1)^{3\times0.4-1}}\right)}e^{2i\pi\sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.5^4}-1\right)^{3\times0.2}}{(2-1)^{3\times0.4-1}}\right)}}\end{array}\right)=\left(0.5012e^{2i\pi0.1423},0.5307e^{2i\pi0.7292},0.4410e^{2i\pi0.5230}\right).$$

guistic terms based on his judgment. The CT-SFNs that correspond to linguistic terms assigned by the xth DE $\mathscr{D}_x$ are placed in a complex T-spherical fuzzy decision matrix (CT-SFDM) $M^{(x)}_{m\times n} = (\Im^{(x)}_{i\dot{r}})_{m\times n}$. In a similar manner, l DEs form l individual CT-SFDM $M^{1}_{m\times n}, M^{2}_{m\times n}, \ldots, M^{l}_{m\times n}$, as shown in the equation at page 42, where $x = 1(1)l$. Each entry $\Im^{(x)}_{i\dot{r}}$ of CT-SFDM of DE $\mathscr{D}_x$ has the form $\Im^{(x)}_{i\dot{r}} = \left(\ddot{\mu}^{(x)}_{i\dot{r}}, \ddot{\varsigma}^{(x)}_{i\dot{r}}, \ddot{\nu}^{(x)}_{i\dot{r}}\right) = \left(\mu^{(x)}_{i\dot{r}} e^{i2\pi\eth_{\mu^{(x)}_{i\dot{r}}}}, \varsigma^{(x)}_{i\dot{r}} e^{i2\pi\eth_{\varsigma^{(x)}_{i\dot{r}}}}, \nu^{(x)}_{i\dot{r}} e^{i2\pi\eth_{\nu^{(x)}_{i\dot{r}}}}\right)$.

Step 2: Normalization:
In this step, the individual CT-SFDMs must be transformed pursuant to certain benefit and cost criteria. The reaction of these two criteria is opposite; the bigger the value, the better the performance of the benefit criteria and the poorer the performance of the cost criteria. To verify that all criteria are compatible, we build the normalized CT-SFDM $\widetilde{M}^{(x)}_{m\times n} = (\widetilde{\Im}^{(x)}_{i\dot{r}})_{m\times n}$ according to Formula (51) to convert cost criteria to benefit criteria, where Ω_b is a collection of benefit criteria and Ω_c is a series of cost criteria.

$$\widetilde{\Im}^{(x)}_{i\dot{r}} = \begin{cases} \Im^{(x)}_{i\dot{r}}, & \text{for } \mathcal{C}r_{\dot{r}} \in \Omega_b \\ \left(\Im^{(x)}_{i\dot{r}}\right)^c, & \text{for } \mathcal{C}r_{\dot{r}} \in \Omega_c, \end{cases} \tag{51}$$

where $\left(\Im^{(x)}_{i\dot{r}}\right)^c$ represents the complement of $\Im^{(x)}_{i\dot{r}}$.

$$\begin{aligned}
&CT-SFFHG\left(\hat{\Im}_{\delta(1)}, \hat{\Im}_{\delta(2)}, \hat{\Im}_{\delta(3)}\right) \\
&= \begin{pmatrix}
\sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\hat{\sigma}^4_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi \sqrt[4]{\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{\hat{\eth}^4_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)}}, \\
\sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\hat{\varsigma}^4_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi \sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\hat{\varsigma}^4_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)}}, \\
\sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\hat{\varrho}^4_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)} e^{2i\pi \sqrt[4]{1-\log_2\left(1+\prod_{\dot{r}=1}^{3}\left(2^{1-\hat{\varrho}^4_{\delta(\dot{r})}}-1\right)^{\natural_{\dot{r}}}\right)}}
\end{pmatrix} \\
&= \begin{pmatrix}
\sqrt[4]{\log_2\left(1+\left(2^{0.3275^4}-1\right)^{0.3}\left(2^{0.5012^4}-1\right)^{0.4}\left(2^{0.5353^4}-1\right)^{0.3}\right)} \\
e^{2i\pi\sqrt[4]{\log_2\left(1+\left(2^{0.2317^4}-1\right)^{0.3}\left(2^{0.1423^4}-1\right)^{0.4}\left(2^{0.4301^4}-1\right)^{0.3}\right)}}, \\
\sqrt[4]{1-\log_2\left(1+\left(2^{1-0.5227^4}-1\right)^{0.3}\left(2^{1-0.5307^4}-1\right)^{0.4}\left(2^{1-0.7290^4}-1\right)^{0.3}\right)} \\
e^{2i\pi\sqrt[4]{1-\log_2\left(1+\left(2^{1-0.6260^4}-1\right)^{0.3}\left(2^{1-0.7292^4}-1\right)^{0.4}\left(2^{1-0.8293^4}-1\right)^{0.3}\right)}}, \\
\sqrt[4]{1-\log_2\left(1+\left(2^{1-0.2093^4}-1\right)^{0.3}\left(2^{1-0.4410^4}-1\right)^{0.4}\left(2^{1-0.8292^4}-1\right)^{0.3}\right)} \\
e^{2i\pi\sqrt[4]{1-\log_2\left(1+\left(2^{1-0.2166^4}-1\right)^{0.3}\left(2^{1-0.5230^4}-1\right)^{0.4}\left(2^{1-0.8293^4}-1\right)^{0.3}\right)}}
\end{pmatrix} \\
&= \left(0.4506e^{2i\pi 0.2317}, 0.6152e^{2i\pi 0.7464}, 0.6516e^{2i\pi 0.6632}\right).
\end{aligned}$$

$$\lim_{\pounds\longrightarrow 1} CT-SFFHG\left(\Im_1, \Im_2, \ldots, \Im_n\right) = \begin{pmatrix}
\sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\hat{\sigma}^t_{\delta(\dot{r})}\right)^{\natural_{\dot{r}}}} e^{2i\pi\sqrt[t]{\prod_{\dot{r}=1}^{n}\left(\hat{\eth}^t_{\delta(\dot{r})}\right)^{\natural_{\dot{r}}}}}, \\
\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\hat{\varsigma}^t_{\delta(\dot{r})}\right)^{\natural_{\dot{r}}}} e^{2i\pi\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\hat{\varsigma}^t_{\delta(\dot{r})}\right)^{\natural_{\dot{r}}}}}, \\
\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\hat{\varrho}^t_{\delta(\dot{r})}\right)^{\natural_{\dot{r}}}} e^{2i\pi\sqrt[t]{1-\prod_{\dot{r}=1}^{n}\left(1-\hat{\varrho}^t_{\delta(\dot{r})}\right)^{\natural_{\dot{r}}}}}
\end{pmatrix}. \tag{49}$$

Step 3: Determination of t:
Determine the smallest t for which each $\widetilde{\Im}_{i\dot{r}}^{(x)}$ satisfies the condition $0 \leq \left(\widetilde{\mu}_{i\dot{r}}^{(x)}\right)^t + \left(\widetilde{\varsigma}_{i\dot{r}}^{(x)}\right)^t + \left(\widetilde{\nu}_{i\dot{r}}^{(x)}\right)^t \leq 1$, $0 \leq \left(\eth_{\widetilde{\mu}_{i\dot{r}}^{(x)}}\right)^t + \left(\eth_{\widetilde{\varsigma}_{i\dot{r}}^{(x)}}\right)^t + \left(\eth_{\widetilde{\nu}_{i\dot{r}}^{(x)}}\right)^t \leq 1$.

Step 4: Creation of aggregated evaluation matrix:
Aggregate the normalized CT-SFDM $\widetilde{M}_{m\times n}^{(x)}(x = 1(1)l)$ into collective decision matrix $\widetilde{M}_{m\times n} = (\widetilde{\Im}_{i\dot{r}})_{m\times n}$ using CT-SFFWA operator (52), as shown at the bottom of the page.

$$\lim_{\pounds\longrightarrow\infty} CT-SFFHG(\Im_1, \Im_2, \ldots, \Im_n) = \begin{pmatrix} \sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}}\left(\hat{\sigma}_{\delta(\dot{r})}^{t}\right)\right)}e^{2i\pi\sqrt[t]{1-\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}}\left(\hat{\eth}_{\delta(\dot{r})}^{t}\right)\right)}}, \\ \sqrt[t]{\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}}\left(\hat{\vartheta}_{\delta(\dot{r})}^{t}\right)\right)}e^{2i\pi\sqrt[t]{\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}}\left(\hat{\vartheta}_{\delta(\dot{r})}^{t}\right)\right)}}, \\ \sqrt[t]{\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}}\left(\hat{\varrho}_{\delta(\dot{r})}^{t}\right)\right)}e^{2i\pi\sqrt[t]{\left(\sum_{\dot{r}=1}^{n} w_{\dot{r}}\left(\hat{\varrho}_{\delta(\dot{r})}^{t}\right)\right)}} \end{pmatrix}. \tag{50}$$

$$M_{m\times n}^{(x)} = \begin{matrix} & \mathcal{C}r_1 & \cdots & \mathcal{C}r_{\dot{r}} & \cdots & \mathcal{C}r_n \\ \mathcal{O}l_1 & \left(\ddot{\mu}_{11}^{(x)}, \ddot{\varsigma}_{11}^{(x)}, \ddot{\nu}_{11}^{(x)}\right) & \cdots & \left(\ddot{\mu}_{1\dot{r}}^{(x)}, \ddot{\varsigma}_{1\dot{r}}^{(x)}, \ddot{\nu}_{1\dot{r}}^{(x)}\right) & \cdots & \left(\ddot{\mu}_{1n}^{(x)}, \ddot{\varsigma}_{1n}^{(x)}, \ddot{\nu}_{1n}^{(x)}\right) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathcal{O}l_i & \left(\ddot{\mu}_{i1}^{(x)}, \sigma_{i1}^{(x)}, \ddot{\nu}_{i1}^{(x)}\right) & \cdots & \left(\ddot{\mu}_{i\dot{r}}^{(x)}, \ddot{\varsigma}_{i\dot{r}}^{(x)}, \ddot{\nu}_{i\dot{r}}^{(x)}\right) & \cdots & \left(\ddot{\mu}_{in}^{(x)}, \ddot{\varsigma}_{in}^{(x)}, \ddot{\nu}_{in}^{(x)}\right) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathcal{O}l_m & \left(\ddot{\mu}_{m1}^{(x)}, \ddot{\varsigma}_{m1}^{(x)}, \ddot{\nu}_{m1}^{(x)}\right) & \cdots & \left(\ddot{\mu}_{m\dot{r}}^{(x)}, \ddot{\varsigma}_{m\dot{r}}^{(x)}, \ddot{\nu}_{m\dot{r}}^{(x)}\right) & \cdots & \left(\ddot{\mu}_{mn}^{(x)}, \ddot{\varsigma}_{mn}^{(x)}, \ddot{\nu}_{mn}^{(x)}\right) \end{matrix},$$

$$\begin{aligned} \widetilde{\Im}_{i\dot{r}} &= CT-SFFWA\left(\widetilde{\Im}_{i\dot{r}}^{(1)}), \widetilde{\Im}_{i\dot{r}}^{(2)}), \ldots, \widetilde{\Im}_{i\dot{r}}^{(l)})\right) \\ &= \begin{pmatrix} \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{x=1}^{l}\left(\pounds^{1-\widetilde{\sigma}_{i\dot{r}}^{(x)t}}-1\right)^{\kappa_x}\right)}e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{x=1}^{l}\left(\pounds^{1-\eth^{t}_{\widetilde{\sigma}\,i\dot{r}}{}^{(x)}}-1\right)^{\kappa_x}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1+\prod_{x=1}^{l}\left(\pounds^{\widetilde{\varsigma}_{i\dot{r}}^{(x)t}}-1\right)^{\kappa_x}\right)}e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\prod_{x=1}^{l}\left(\pounds^{\eth^{t}_{\widetilde{\varsigma}\,i\dot{r}}{}^{(x)}}-1\right)^{\kappa_x}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1+\prod_{x=1}^{l}\left(\pounds^{\widetilde{\varrho}_{i\dot{r}}^{(x)t}}-1\right)^{\kappa_x}\right)}e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\prod_{x=1}^{l}\left(\pounds^{\eth^{t}_{\widetilde{\varrho}\,i\dot{r}}{}^{(x)}}-1\right)^{\kappa_x}\right)}} \end{pmatrix}. \end{aligned} \tag{52}$$

$$\begin{aligned} \perp_{\dot{r}} &= CT-SFFWA\left(\perp_{\dot{r}}^{(1)}, \perp_{\dot{r}}^{(2)}, \ldots, \perp_{\dot{r}}^{(l)}\right) \\ &= \begin{pmatrix} \sqrt[t]{1-\log_{\pounds}\left(1+\prod_{x=1}^{l}\left(\pounds^{1-\widetilde{\sigma}_{\dot{r}}^{(x)t}}-1\right)^{\kappa_x}\right)}e^{2i\pi\sqrt[t]{1-\log_{\pounds}\left(1+\prod_{x=1}^{l}\left(\pounds^{1-\eth^{t}_{\widetilde{\sigma}\,\dot{r}}{}^{(x)}}-1\right)^{\kappa_x}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1+\prod_{x=1}^{l}\left(\pounds^{\widetilde{\varsigma}_{\dot{r}}^{(x)t}}-1\right)^{\kappa_x}\right)}e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\prod_{x=1}^{l}\left(\pounds^{\eth^{t}_{\widetilde{\varsigma}\,\dot{r}}{}^{(x)}}-1\right)^{\kappa_x}\right)}}, \\ \sqrt[t]{\log_{\pounds}\left(1+\prod_{x=1}^{l}\left(\pounds^{\widetilde{\varrho}_{\dot{r}}^{(x)t}}-1\right)^{\kappa_x}\right)}e^{2i\pi\sqrt[t]{\log_{\pounds}\left(1+\prod_{x=1}^{l}\left(\pounds^{\eth^{t}_{\widetilde{\varrho}\,i\dot{r}}{}^{(x)}}-1\right)^{\kappa_x}\right)}} \end{pmatrix}, \end{aligned} \tag{53}$$

Step 5: Assigning criteria weights:
In an MCGDM problem, the criteria chosen by DEs may not be of equal value and importance. The panel of DEs evaluates the weight of criteria by giving linguistic terms to the criteria in accordance with their importance in the MCGDM problem. Let $\Im_{\dot{r}}^{(x)} = \left(\ddot{\mu}_{\dot{r}}^{(x)}, \ddot{\varsigma}_{\dot{r}}^{(x)}, \ddot{\nu}_{\dot{r}}^{(x)}\right) = \left(\mu_{\dot{r}}^{(x)} e^{i2\pi \eth_{\mu_{\dot{r}}}^{(x)}}, \varsigma_{\dot{r}}^{(x)} e^{i2\pi \eth_{\varsigma_{\dot{r}}}^{(x)}}, \nu_{\dot{r}}^{(x)} e^{i2\pi \eth_{\nu_{\dot{r}}}^{(x)}}\right)$ be the CT-SFN assigned to the criteria $\mathcal{C}r_{\dot{r}}$ by the expert $\mathscr{D}_x$. The CT-SFNs assigned to linguistic terms by DEs are gathered to yield the CT-SF weight vector weight vector $\perp = (\perp_1, \perp_2, \ldots, \perp_n)^T$ of criteria (53), as shown at the bottom of previous page, where $\dot{r} = 1, 2, \ldots, n$.

Step 6: Determination of weighted decision matrix:
Determine the weighted CT-SFDM (WCT-SFDM) by making usage of aggregated decision matrix $\widetilde{M}_{m\times n} = (\widetilde{\Im}_{i\dot{r}})_{m\times n}$ and the weight vector $\perp = (\perp_1, \perp_2, \ldots, \perp_n)^T$ of criteria. The entries of WCT-SFDM $\breve{M}_{m\times n} = (\breve{\Im}_{i\dot{r}})_{m\times n}$ can be ascertained by (54), as shown at the bottom of the next page.

Step 7: Determination of aggregated values:

a). According to the WCT-SFDM, the aggregated values $\dot{\Im}_i (i = 1(1)m)$ are computed using CT-SFFA operator (55), as shown at the bottom of the next page.

b). According to the WCT-SFDM, the aggregated values $\dot{\Im}_i = \left(\dot{\sigma}_i e^{2i\pi \eth_{\dot{\sigma} i}}, \dot{\varsigma}_1 e^{2i\pi \eth_{\dot{\varsigma} i}}, \dot{\varrho}_i e^{2i\pi \eth_{\dot{\varrho} i}}\right)$ $(i = 1(1)m)$ are computed using CT-SFFG operator (56), as shown at the bottom of the next page.

Step 8: Evaluation of score values:
Find the score value of the aggregated CT-SFNs $\dot{\Im}_i = \left(\dot{\sigma}_i e^{2i\pi \eth_{\dot{\sigma} i}}, \dot{\varsigma}_1 e^{2i\pi \eth_{\dot{\varsigma} i}}, \dot{\varrho}_i e^{2i\pi \eth_{\dot{\varrho} i}}\right)$ $(i = 1(1)m)$ based on the following equation:

$$S\left(\dot{\Im}_i\right) = \frac{1}{4} \cdot \left(2 + \left(\dot{\mu}^t - \dot{\varsigma}^t - \dot{\nu}^t\right) + \left(\eth_{\dot{\mu}}^t - \eth_{\dot{\varsigma}}^t - \eth_{\dot{\nu}}^t\right)\right). \quad (57)$$

Step 9: Ranking of alternatives:
After calculating the score values, the alternatives are ranked in descending score value order. The option with the greatest score represents the optimal solution for the MCGDM problem.

The graphical representation of the established MCGDM approach is depicted in Fig. 1.

VI. AN ILLUSTRATIVE EXAMPLE

In this section, we outline an MCGDM problem, namely "the selection of best strategy for water supply to Nohoor village in Iran" (taken from [55]), and solve it by employing the established approach in order to demonstrate the practical application of the proposed method.

A. BACKGROUND DESCRIPTION

Nohoor village is located in north-eastern Iran, in Khaf county, on the border between Khaf city and Qaen city, and shares a border with Afghanistan. The villagers of Nohoor make their living through animal rearing and contribute significantly to the province of Khorasan Razavi's dairy and meat requirements. Due to a shortage of water and a scorching climate, the fertile grounds of the town of Nohoor are unable to support animal husbandry. In the spring, the peasants migrate to higher ground for grazing, when plants and grass flourish thanks to the rain. Historically, Nohoor spring, located to the south of Nohoor village, was the only supply of water for the community. Currently, the Nohoor spring is drying because of the hot weather, lack of precipitation, and high rate of evaporation. The villagers of Nohoor village require water for drinking, agriculture, and livestock rearing. Due to the climate of the area, digging wells and constructing dams are ineffective solutions. This study aims to determine the optimal technique for supplying Nohoor hamlet with water from neighboring villages via subterranean pipelines. The following strategies are considered as alternatives for addressing this MCGDM issue:

$\mathcal{O}l_1$: In this strategy, water is transferred from subterranean supplies near the village of Chahpayab using pipes rather than water tankers. The primary characteristics of this technique are the slope of the terrain toward the settlement of Nohoor, the low pumping cost, the water pressure, and the relatively low likelihood of pipe damage. In contrast, the transmission distance for this technique is quite extensive, and any pipe failure may increase the risk of flooding.

$\mathcal{O}l_2$: This technique employs an expansion of a previously built system to feed Nohoor village with water from subterranean water resources near Mazhnabad village via pipelines from the distribution network of Chahzool village. Both Chahzool and Nooor villages may see a reduction in water pressure as a result of the addition of the pipeline to the prior system. The most significant restriction of this technique is the slope and steepness of the terrain, which may result in supply interruptions and pipeline damage. The primary benefit of this technique is its short transmission distance and straightforward deployment.

$\mathcal{O}l_3$: In this technique, water from the transmission line of Chahzool village to the primary source of Mazhnabad is kept in reservoirs and transported to Nohoor village via tanker before being stored in pools. The primary characteristic of this technique is its low cost relative to other strategies. The disadvantages of this technique include a lack of water security, a low capacity of water to meet villages' demands, and a shortage of water during unfavorable weather.

TABLE 1. Linguistic terms and their corresponding CT-SFNs.

Linguistic terms	CT-SFNs
Very high ($\mathbb{VH}$)/ Very important ($\mathbb{VI}$)	$\left(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28}\right)$
High ($\mathbb{H}$)/ Important ($\mathbb{I}$)	$\left(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36}\right)$
Medium ($\mathbb{M}$)/ Medium ($\mathbb{M}$)	$\left(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51}\right)$
Low ($\mathbb{L}$)/ Unimportant ($\mathbb{UI}$)	$\left(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83}\right)$
Very low ($\mathbb{VL}$)/ Very Unimportant ($\mathbb{VUI}$)	$\left(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91}\right)$

$\mathcal{O}l_4$: Pipelines are used in this technique to transport water from subsurface water resources found in the vicinity of Mazhnabad village via a well. The slope of the ground, excellent water security, and high water pressure are all advantages of this method.

$$\breve{\Im}_{i\dot{r}} = \widetilde{\Im}_{i\dot{r}} \otimes \perp_{\dot{r}} = \begin{pmatrix} \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\tilde{\sigma}^t}-1\right)\left(£^{\sigma^t}-1\right)}{£-1}\right)}\, e^{2i\pi \sqrt[t]{\log_{£}\left(1+\frac{\left(£^{\eth^t_{\tilde{\sigma}}}-1\right)\left(£^{\eth^t_{\sigma}}-1\right)}{£-1}\right)}}, \\ \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\tilde{\varsigma}^t}-1\right)\left(£^{1-\varsigma^t}-1\right)}{£-1}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth^t_{\tilde{\varsigma}}}-1\right)\left(£^{1-\eth^t_{\varsigma}}-1\right)}{£-1}\right)}}, \\ \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\tilde{\varrho}^t}-1\right)\left(£^{1-\varrho^t}-1\right)}{£-1}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\frac{\left(£^{1-\eth^t_{\tilde{\varrho}}}-1\right)\left(£^{1-\eth^t_{\varrho}}-1\right)}{£-1}\right)}} \end{pmatrix}. \tag{54}$$

$$\dot{\Im}_i = CT-SFFA\left(\breve{\Im}_{i1}, \breve{\Im}_{i2}, \ldots, \breve{\Im}_{in}\right) = \begin{pmatrix} \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\breve{\sigma}^t_{i\dot{r}}}-1\right)^{\frac{1}{n}}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^t_{\breve{\sigma}_{i\dot{r}}}}-1\right)^{\frac{1}{n}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\breve{\varsigma}^t_{i\dot{r}}}-1\right)^{\frac{1}{n}}\right)}\, e^{2i\pi \sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth^t_{\breve{\varsigma}_{i\dot{r}}}}-1\right)^{\frac{1}{n}}\right)}}, \\ \sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\breve{\varrho}^t_{i\dot{r}}}-1\right)^{\frac{1}{n}}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth^t_{\breve{\varrho}_{i\dot{r}}}}-1\right)^{\frac{1}{n}}\right)}} \end{pmatrix}. \tag{55}$$

$$\dot{\Im}_i = CT-SFFG\left(\breve{\Im}_{i1}, \breve{\Im}_{i2}, \ldots, \breve{\Im}_{in}\right) = \begin{pmatrix} \sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\breve{\sigma}^t_{i\dot{r}}}-1\right)^{\frac{1}{n}}\right)}\, e^{2i\pi \sqrt[t]{\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{\eth^t_{\breve{\sigma}_{i\dot{r}}}}-1\right)^{\frac{1}{n}}\right)}}, \\ \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\breve{\varsigma}^t_{i\dot{r}}}-1\right)^{\frac{1}{n}}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^t_{\breve{\varsigma}_{i\dot{r}}}}-1\right)^{\frac{1}{n}}\right)}}, \\ \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\breve{\varrho}^t_{i\dot{r}}}-1\right)^{\frac{1}{n}}\right)}\, e^{2i\pi \sqrt[t]{1-\log_{£}\left(1+\prod_{\dot{r}=1}^{n}\left(£^{1-\eth^t_{\breve{\varrho}_{i\dot{r}}}}-1\right)^{\frac{1}{n}}\right)}} \end{pmatrix}. \tag{56}$$

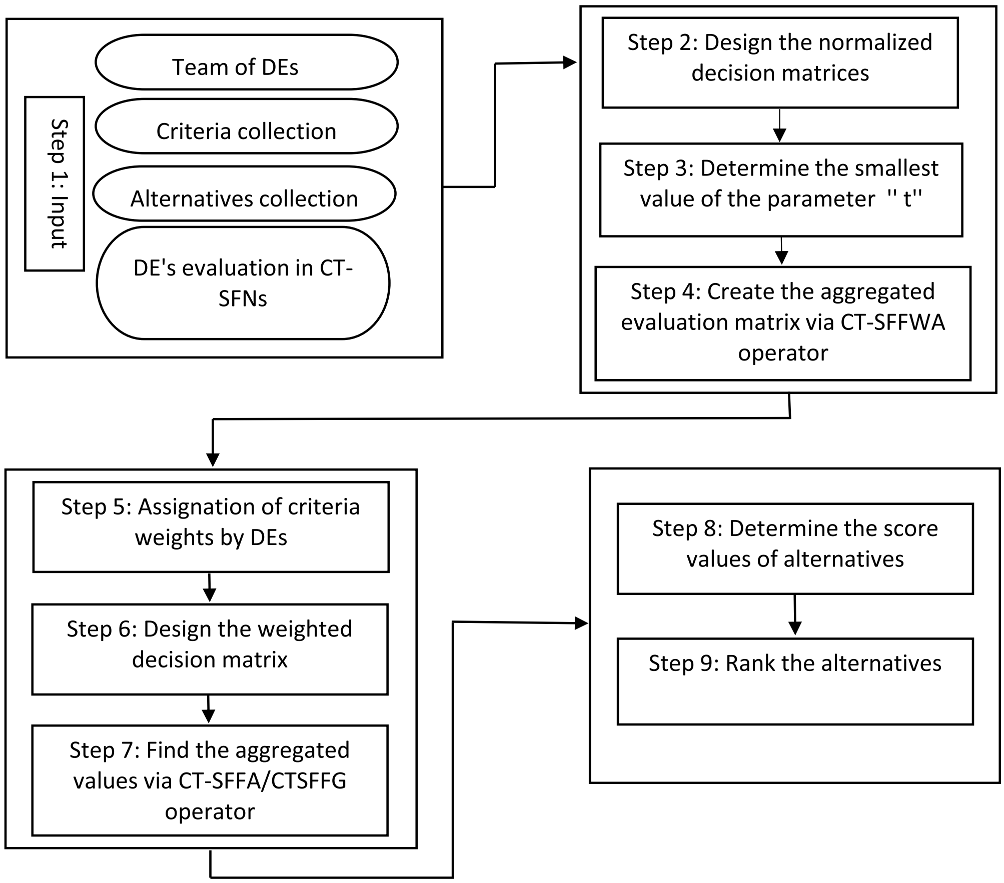

FIGURE 1. Graphical representation of the proposed method.

A panel comprising three DEs $\mathscr{D}_1$, $\mathscr{D}_2$, and $\mathscr{D}_3$ is formed to thoroughly investigate the critical demands and performance of various water supply options for this MCGDM problem. All of the experts agree on the following parameters as the decision criteria for this MCGDM problem:

$\mathcal{C}r_1$: Initial cost: This involves the costs associated with establishing the project, such as the cost of tankers, energy, pipes, pumping, and wages for labourers. The technique with the lowest starting expense is preferred.

$\mathcal{C}r_2$: Maintenance cost: This criterion includes the cost of repairing any potential water supply system damage or failure. The technique with the lowest cost of maintenance is preferable.

$\mathcal{C}r_3$: Water quality: The quality of drinking water is a crucial consideration. The technique that provides water of superior quality is desirable.

$\mathcal{C}r_4$: Environmental destruction: The degradation of pasturage and other natural water supplies has an impact on the villagers' cattle production. The distance between grazing lands and drinking water for cattle should be kept to a minimum. The plan with the fewest negative environmental consequences is preferred.

$\mathcal{C}r_5$: Water security and satisfaction of inhabitants: Nohoor village contributes significantly to the Iranian economy by producing meat and dairy products. As a result, water security and resident contentment are important factors in preventing village migration. The technique that results in greater villagers' contentment is desirable.

B. THE DECISION-MAKING PROCESS

The step-by-step solution to the aforesaid MCGDM problem using the framed technique is as follows:

Step 1: Table 1 is composed of the linguistic terms and their accompanying CT-SFNs to represent the level of satisfaction of DEs about potential strategies. Further, the assessments of the DEs for each strategy with respect to all criteria are represented in Table 2.

The individual CT-SFDMs of the DE $\mathscr{D}_1$, $\mathscr{D}_2$, and $\mathscr{D}_3$ are listed in Table 3.

Step 2: Since the criteria $\mathcal{C}r_1$, $\mathcal{C}r_2$ and $\mathcal{C}r_4$ are cost type. Thus by employing Eq. (51), the original CT-SFDM 3 are transformed into normalized CT-SFDM, which are shown in Table 4.

Step 3: As $0.28 + 0.30 + 0.92 = 1.50 \notin [0, 1]$, $0.28^2 + 0.30^2 + 0.92^2 = 1.0148 \notin [0, 1]$ and $0.28^3 + 0.30^3 + 0.92^3 =$

TABLE 2. DEs' assessment information corresponding to each criteria.

Criteria	Alternatives	$\mathscr{D}_1$	$\mathscr{D}_2$	$\mathscr{D}_3$
$\mathcal{C}r_1$	$\mathcal{O}l_1$	$\mathbb{VH}$	$\mathbb{H}$	$\mathbb{M}$
	$\mathcal{O}l_2$	$\mathbb{M}$	$\mathbb{L}$	$\mathbb{VH}$
	$\mathcal{O}l_3$	$\mathbb{VH}$	$\mathbb{VL}$	$\mathbb{L}$
	$\mathcal{O}l_4$	$\mathbb{VH}$	$\mathbb{M}$	$\mathbb{H}$
$\mathcal{C}r_2$	$\mathcal{O}l_1$	$\mathbb{M}$	$\mathbb{L}$	$\mathbb{L}$
	$\mathcal{O}l_2$	$\mathbb{L}$	$\mathbb{VL}$	$\mathbb{H}$
	$\mathcal{O}l_3$	$\mathbb{VH}$	$\mathbb{H}$	$\mathbb{M}$
	$\mathcal{O}l_4$	$\mathbb{M}$	$\mathbb{VL}$	$\mathbb{L}$
$\mathcal{C}r_3$	$\mathcal{O}l_1$	$\mathbb{VH}$	$\mathbb{M}$	$\mathbb{VL}$
	$\mathcal{O}l_2$	$\mathbb{L}$	$\mathbb{VL}$	$\mathbb{M}$
	$\mathcal{O}l_3$	$\mathbb{VH}$	$\mathbb{H}$	$\mathbb{VH}$
	$\mathcal{O}l_4$	$\mathbb{M}$	$\mathbb{L}$	$\mathbb{VL}$
$\mathcal{C}r_4$	$\mathcal{O}l_1$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{L}$
	$\mathcal{O}l_2$	$\mathbb{VH}$	$\mathbb{M}$	$\mathbb{H}$
	$\mathcal{O}l_3$	$\mathbb{M}$	$\mathbb{L}$	$\mathbb{VL}$
	$\mathcal{O}l_4$	$\mathbb{VH}$	$\mathbb{M}$	$\mathbb{H}$
$\mathcal{C}r_5$	$\mathcal{O}l_1$	$\mathbb{L}$	$\mathbb{VL}$	$\mathbb{M}$
	$\mathcal{O}l_2$	$\mathbb{VH}$	$\mathbb{H}$	$\mathbb{VH}$
	$\mathcal{O}l_3$	$\mathbb{VL}$	$\mathbb{M}$	$\mathbb{L}$
	$\mathcal{O}l_4$	$\mathbb{VH}$	$\mathbb{L}$	$\mathbb{M}$

0.8276 $\in$ [0, 1]. In a similar way, we found that all values in Table 4 belong to [0, 1] for $t = 3$.

Step 4: The normalized CT-SFDM 4 originally provided by DEs having the weight vector $\kappa = (0.3, 0.4, 0.3)^T$, are aggregated by employing the CT-SFFWA operator Eq. (52), and the outcomes are depicted in Table 5.

Step 5: The DEs assign linguistic terms to each criteria that represent the significance of that criteria in the MCGDM problem, as stated in Table 6. Further, the individual CT-SF weight of criteria is tabulated in Table 7. Deploying CT-SFWA operator Eq. (53), on Table 7 CT-SF weight of criteria is ascertained as follows:

$$\perp = \begin{pmatrix} \left(0.8396e^{2i\pi 0.6901}, 0.2970e^{2i\pi 0.4730}, 0.2832e^{2i\pi 0.4294}\right), \\ \left(0.5313e^{2i\pi 0.5339}, 0.5298e^{2i\pi 0.5326}, 0.4493e^{2i\pi 0.4210}\right), \\ \left(0.5786e^{2i\pi 0.8571}, 0.3737e^{2i\pi 0.3047}, 0.3737e^{2i\pi 0.4003}\right), \\ \left(0.6555e^{2i\pi 0.6605}, 0.4812e^{2i\pi 0.4701}, 0.4451e^{2i\pi 0.4451}\right), \\ \left(0.7048e^{2i\pi 0.6987}, 0.4451e^{2i\pi 0.4418}, 0.4148e^{2i\pi 0.4148}\right) \end{pmatrix}$$

Step 6: According to Eq. (54), the WCT-SFDM is obtained as shown in Table 8.

Step 7: Based on Eq. (55), the aggregated values $\dot{\Im}_i(i = 1(1)m)$ are obtained as given below.

$$\dot{\Im}_1 = \left(0.2900e^{2i\pi 0.3857}, 0.5038e^{2i\pi 0.5038}, 0.6681e^{2i\pi 0.6848}\right),$$
$$\dot{\Im}_2 = \left(0.4152e^{2i\pi 0.4213}, 0.4941e^{2i\pi 0.4983}, 0.5914e^{2i\pi 0.6087}\right),$$
$$\dot{\Im}_3 = \left(0.4348e^{2i\pi 0.5199}, 0.4820e^{2i\pi 0.4949}, 0.5554e^{2i\pi 0.5845}\right),$$
$$\dot{\Im}_4 = \left(0.3214e^{2i\pi 0.3490}, 0.5093e^{2i\pi 0.5093}, 0.5903e^{2i\pi 0.6674}\right).$$

Based on Eq. (56), the aggregated values $\dot{\Im}_i(i = 1(1)m)$ are obtained as given below.

$$\dot{\Im}_1 = \left(0.2711e^{2i\pi 0.2970}, 0.5148e^{2i\pi 0.5164}, 0.7003e^{2i\pi 0.7221}\right),$$
$$\dot{\Im}_2 = \left(0.3340e^{2i\pi 0.3704}, 0.5030e^{2i\pi 0.5108}, 0.6281e^{2i\pi 0.6551}\right),$$
$$\dot{\Im}_3 = \left(0.3477e^{2i\pi 0.3904}, 0.5019e^{2i\pi 0.5061}, 0.6122e^{2i\pi 0.6354}\right),$$
$$\dot{\Im}_4 = \left(0.2861e^{2i\pi 0.3189}, 0.5161e^{2i\pi 0.5210}, 0.6203e^{2i\pi 0.7008}\right).$$

Step 8: The score value of each aggregated value obtained by Eq. (55) is computed by employing Eq. (57) as follows:

$$S\left(\dot{\Im}_1\right) = 0.7392, S\left(\dot{\Im}_2\right) = 0.7058,$$
$$S\left(\dot{\Im}_3\right) = 0.7067, S\left(\dot{\Im}_4\right) = 0.7107.$$

The score value of each aggregated value obtained by Eq. (56) is computed by employing Eq. (57) as follows:

$$S\left(\dot{\Im}_1\right) = 0.7601, S\left(\dot{\Im}_2\right) = 0.7194,$$
$$S\left(\dot{\Im}_3\right) = 0.7109, S\left(\dot{\Im}_4\right) = 0.7294.$$

Step 9: We get the following ranking of water supply strategies based on the derived score values.

For CT-SFFA operator:

$$\dot{\Im}_1 > \dot{\Im}_4 > \dot{\Im}_3 > \dot{\Im}_2.$$

For CT-SFFG operator:

$$\dot{\Im}_1 > \dot{\Im}_4 > \dot{\Im}_2 > \dot{\Im}_3.$$

C. IMPACT OF PARAMETER t ON DECISION-MAKING RESULTS

This section addresses the influence of parameter t on the alternative ranking results. We varied the value of the parameter t using Eqs. (55) and (56) in order to examine the effect of different parametric values on the final ranking results. To accomplish this, we assign various values of $t = 3, 5, 7, 9, 11, 13$, and 15 and repeat the analytical calculations as tabulated in Tables 9 and 10.

In Tables 9 and 10, the parameters have an effect on the score values since the score values decrease gradually as parameter t increases. However, the overall results of the ranking remain constant.

Figures 2 and 3 depict a geometrical depiction of the intended work delineated in Tables 9 and 10.

According to Figures 2 and 3, the DE has the option of selecting different aggregation operators by altering the adjustment parameter t. However, it should be emphasized that the ranking results remain unaltered. Thereby, it demonstrates that the CT-SF Frank operators initiated in this research are feasible and highly stable.

D. IMPACT OF PARAMETER £ ON DECISION-MAKING RESULTS

Theorems 3 and 18 reveal that the CT-SFFWA operator and CT-SFFWG operator provide a large class of CT-SF aggregation operators via parameters £. To thoroughly comprehend the execution of aggregation, we experiment with parameter values ranging from 3 to 150 for the aforementioned strategies selection problem. The score values acquired by the CT-SFFA operator and CT-SFFG operator are represented graphically in Figures 4 and 5 and summarized in Tables 11 and 12, respectively. From Table 11, we may deduce that the score values generated by the CT-SFFA operator increase gradually as the parameter £ increases, whereas

TABLE 3. CT-SF information corresponding to each criteria.

Criteria	Alternatives	$\mathscr{D}_1$	$\mathscr{D}_2$	$\mathscr{D}_3$
$\mathcal{C}r_1$	$\mathcal{O}l_1$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$
	$\mathcal{O}l_2$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$
	$\mathcal{O}l_3$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$
	$\mathcal{O}l_4$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$
$\mathcal{C}r_2$	$\mathcal{O}l_1$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$
	$\mathcal{O}l_2$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$
	$\mathcal{O}l_3$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$
	$\mathcal{O}l_4$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$
$\mathcal{C}r_3$	$\mathcal{O}l_1$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$
	$\mathcal{O}l_2$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$
	$\mathcal{O}l_3$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$
	$\mathcal{O}l_4$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$
$\mathcal{C}r_4$	$\mathcal{O}l_1$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$
	$\mathcal{O}l_2$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$
	$\mathcal{O}l_3$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$
	$\mathcal{O}l_4$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$
$\mathcal{C}r_5$	$\mathcal{O}l_1$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$
	$\mathcal{O}l_2$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$
	$\mathcal{O}l_3$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$
	$\mathcal{O}l_4$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$

TABLE 4. Normalized CT-SF information corresponding to each criteria.

Criteria	Alternatives	$\mathscr{D}_1$	$\mathscr{D}_2$	$\mathscr{D}_3$
$\mathcal{C}r_1$	$\mathcal{O}l_1$	$(0.24e^{2i\pi 0.28}, 0.25e^{2i\pi 0.30}, 0.87e^{2i\pi 0.92})$	$(0.36e^{2i\pi 0.36}, 0.38e^{2i\pi 0.39}, 0.78e^{2i\pi 0.76})$	$(0.51e^{2i\pi 0.51}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.56})$
	$\mathcal{O}l_2$	$(0.51e^{2i\pi 0.51}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.56})$	$(0.79e^{2i\pi 0.83}, 0.36e^{2i\pi 0.32}, 0.32e^{2i\pi 0.28})$	$(0.24e^{2i\pi 0.28}, 0.25e^{2i\pi 0.30}, 0.87e^{2i\pi 0.92})$
	$\mathcal{O}l_3$	$(0.24e^{2i\pi 0.28}, 0.25e^{2i\pi 0.30}, 0.87e^{2i\pi 0.92})$	$(0.84e^{2i\pi 0.91}, 0.25e^{2i\pi 0.17}, 0.22e^{2i\pi 0.19})$	$(0.79e^{2i\pi 0.83}, 0.36e^{2i\pi 0.32}, 0.32e^{2i\pi 0.28})$
	$\mathcal{O}l_4$	$(0.24e^{2i\pi 0.28}, 0.25e^{2i\pi 0.30}, 0.87e^{2i\pi 0.92})$	$(0.51e^{2i\pi 0.51}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.56})$	$(0.36e^{2i\pi 0.36}, 0.38e^{2i\pi 0.39}, 0.78e^{2i\pi 0.76})$
$\mathcal{C}r_2$	$\mathcal{O}l_1$	$(0.51e^{2i\pi 0.51}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.56})$	$(0.79e^{2i\pi 0.83}, 0.36e^{2i\pi 0.32}, 0.32e^{2i\pi 0.28})$	$(0.79e^{2i\pi 0.83}, 0.36e^{2i\pi 0.32}, 0.32e^{2i\pi 0.28})$
	$\mathcal{O}l_2$	$(0.79e^{2i\pi 0.83}, 0.36e^{2i\pi 0.32}, 0.32e^{2i\pi 0.28})$	$(0.84e^{2i\pi 0.91}, 0.25e^{2i\pi 0.17}, 0.22e^{2i\pi 0.19})$	$(0.36e^{2i\pi 0.36}, 0.38e^{2i\pi 0.39}, 0.78e^{2i\pi 0.76})$
	$\mathcal{O}l_3$	$(0.24e^{2i\pi 0.28}, 0.25e^{2i\pi 0.30}, 0.87e^{2i\pi 0.92})$	$(0.36e^{2i\pi 0.36}, 0.38e^{2i\pi 0.39}, 0.78e^{2i\pi 0.76})$	$(0.51e^{2i\pi 0.51}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.56})$
	$\mathcal{O}l_4$	$(0.51e^{2i\pi 0.51}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.56})$	$(0.84e^{2i\pi 0.91}, 0.25e^{2i\pi 0.17}, 0.22e^{2i\pi 0.19})$	$(0.79e^{2i\pi 0.83}, 0.36e^{2i\pi 0.32}, 0.32e^{2i\pi 0.28})$
$\mathcal{C}r_3$	$\mathcal{O}l_1$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$
	$\mathcal{O}l_2$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$
	$\mathcal{O}l_3$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$
	$\mathcal{O}l_4$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$
$\mathcal{C}r_4$	$\mathcal{O}l_1$	$(0.36e^{2i\pi 0.36}, 0.38e^{2i\pi 0.39}, 0.78e^{2i\pi 0.76})$	$(0.36e^{2i\pi 0.36}, 0.38e^{2i\pi 0.39}, 0.78e^{2i\pi 0.76})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$
	$\mathcal{O}l_2$	$(0.24e^{2i\pi 0.28}, 0.25e^{2i\pi 0.30}, 0.87e^{2i\pi 0.92})$	$(0.51e^{2i\pi 0.51}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.56})$	$(0.36e^{2i\pi 0.36}, 0.38e^{2i\pi 0.39}, 0.78e^{2i\pi 0.76})$
	$\mathcal{O}l_3$	$(0.51e^{2i\pi 0.51}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.56})$	$(0.79e^{2i\pi 0.83}, 0.36e^{2i\pi 0.32}, 0.32e^{2i\pi 0.28})$	$(0.84e^{2i\pi 0.91}, 0.25e^{2i\pi 0.17}, 0.22e^{2i\pi 0.19})$
	$\mathcal{O}l_4$	$(0.24e^{2i\pi 0.28}, 0.25e^{2i\pi 0.30}, 0.87e^{2i\pi 0.92})$	$(0.51e^{2i\pi 0.51}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.56})$	$(0.36e^{2i\pi 0.36}, 0.38e^{2i\pi 0.39}, 0.78e^{2i\pi 0.76})$
$\mathcal{C}r_5$	$\mathcal{O}l_1$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$
	$\mathcal{O}l_2$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$
	$\mathcal{O}l_3$	$(0.22e^{2i\pi 0.19}, 0.25e^{2i\pi 0.17}, 0.84e^{2i\pi 0.91})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$
	$\mathcal{O}l_4$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$

TABLE 5. Aggregated CT-SF information matrix.

	$\mathcal{C}r_1$	$\mathcal{C}r_2$	$\mathcal{C}r_3$
$\mathcal{O}l_1$	$(0.3985e^{2i\pi 0.4035}, 0.3784e^{2i\pi 0.3961}, 0.7201e^{2i\pi 0.7448})$	$(0.7351e^{2i\pi 0.7723}, 0.4122e^{2i\pi 0.3737}, 0.3688e^{2i\pi 0.3477})$	$(0.6637e^{2i\pi 0.7208}, 0.3477e^{2i\pi 0.3212}, 0.4873e^{2i\pi 0.5256})$
$\mathcal{O}l_2$	$(0.6377e^{2i\pi 0.6719}, 0.3704e^{2i\pi 0.3671}, 0.5116e^{2i\pi 0.5131})$	$(0.7517e^{2i\pi 0.8151}, 0.3167e^{2i\pi 0.2646}, 0.3688e^{2i\pi 0.3319})$	$(0.3796e^{2i\pi 0.3990}, 0.3568e^{2i\pi 0.2917}, 0.7201e^{2i\pi 0.7596})$
$\mathcal{O}l_3$	$(0.7473e^{2i\pi 0.8128}, 0.2803e^{2i\pi 0.2426}, 0.3860e^{2i\pi 0.3603})$	$(0.3985e^{2i\pi 0.4035}, 0.3784e^{2i\pi 0.3961}, 0.7201e^{2i\pi 0.7448})$	$(0.8396e^{2i\pi 0.8756}, 0.2970e^{2i\pi 0.3340}, 0.2832e^{2i\pi 0.3096})$
$\mathcal{O}l_4$	$(0.4160e^{2i\pi 0.4202}, 0.3947e^{2i\pi 0.4097}, 0.6917e^{2i\pi 0.7236})$	$(0.7629e^{2i\pi 0.8234}, 0.3568e^{2i\pi 0.2917}, 0.3189e^{2i\pi 0.2970})$	$(0.3844e^{2i\pi 0.4023}, 0.3704e^{2i\pi 0.3096}, 0.3704e^{2i\pi 0.7517})$
	$\mathcal{C}r_4$	$\mathcal{C}r_5$	
$\mathcal{O}l_1$	$(0.3490e^{2i\pi 0.3400}, 0.3737e^{2i\pi 0.3671}, 0.7829e^{2i\pi 0.7809})$	$(0.3796e^{2i\pi 0.3990}, 0.3568e^{2i\pi 0.2917}, 0.7201e^{2i\pi 0.7596})$	
$\mathcal{O}l_2$	$(0.4160e^{2i\pi 0.4202}, 0.3947e^{2i\pi 0.4097}, 0.6917e^{2i\pi 0.7236})$	$(0.8396e^{2i\pi 0.8756}, 0.2970e^{2i\pi 0.3340}, 0.2832e^{2i\pi 0.3096})$	
$\mathcal{O}l_3$	$(0.7565e^{2i\pi 0.8119}, 0.3704e^{2i\pi 0.3096}, 0.3298e^{2i\pi 0.3096})$	$(0.4058e^{2i\pi 0.4320}, 0.3875e^{2i\pi 0.3277}, 0.6858e^{2i\pi 0.7182})$	
$\mathcal{O}l_4$	$(0.4160e^{2i\pi 0.4202}, 0.3947e^{2i\pi 0.4097}, 0.6917e^{2i\pi 0.7236})$	$(0.6613e^{2i\pi 0.7150}, 0.3704e^{2i\pi 0.3671}, 0.4991e^{2i\pi 0.5340})$	

TABLE 6. Importance of criteria relative to the panel of DEs.

	$\mathcal{C}r_1$	$\mathcal{C}r_2$	$\mathcal{C}r_3$	$\mathcal{C}r_4$	$\mathcal{C}r_5$
$\mathscr{D}_1$	$\mathbb{VI}$	$\mathbb{M}$	$\mathbb{UI}$	$\mathbb{M}$	$\mathbb{I}$
$\mathscr{D}_2$	$\mathbb{I}$	$\mathbb{M}$	$\mathbb{VI}$	$\mathbb{I}$	$\mathbb{M}$
$\mathscr{D}_3$	$\mathbb{VI}$	$\mathbb{UI}$	$\mathbb{VI}$	$\mathbb{M}$	$\mathbb{I}$

the score function of the CT-SFFG aggregation operator drops gradually as the parameter £ increases. This implies that DEs can utilize their preferences to choose the most preferred values depending on realistic decision circumstances.

By virtue of score values, we are able to obtain the alternatives' absolute ranking results. The preferred ranking in Table 11 is always $\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$. The preferred ranking in Table 12 is always $\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$. Thus, the CT-SFFA operator and CT-SFFG operator provide only optimal results. In addition, based on Figure 4, we understand that the ranking outcomes of alternatives are the same when the parameter values differ in the example, and the consistency of the suggested CT-SFFA operators is indicated by the uniform ranking outcomes. Figure 5 also demonstrates that the ranking outcomes of the alternatives remain the same

TABLE 7. Importance information of criteria in terms of CT-SFNs.

	$\mathcal{C}r_1$	$\mathcal{C}r_2$	$\mathcal{C}r_3$
$\mathscr{D}_1$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$
$\mathscr{D}_2$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$
$\mathscr{D}_3$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$	$(0.32e^{2i\pi 0.28}, 0.36e^{2i\pi 0.32}, 0.79e^{2i\pi 0.83})$	$(0.87e^{2i\pi 0.92}, 0.25e^{2i\pi 0.30}, 0.24e^{2i\pi 0.28})$
	$\mathcal{C}r_4$	$\mathcal{C}r_5$	
$\mathscr{D}_1$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$	
$\mathscr{D}_2$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	
$\mathscr{D}_3$	$(0.51e^{2i\pi 0.56}, 0.56e^{2i\pi 0.53}, 0.51e^{2i\pi 0.51})$	$(0.78e^{2i\pi 0.76}, 0.38e^{2i\pi 0.39}, 0.36e^{2i\pi 0.36})$	

TABLE 8. WCT-SF information matrix.

	$\mathcal{C}r_1$	$\mathcal{C}r_2$	$\mathcal{C}r_3$
$\mathcal{O}l_1$	$(0.3022e^{2i\pi 0.2343}, 0.4302e^{2i\pi 0.5480}, 0.7313e^{2i\pi 0.7780})$	$(0.3420e^{2i\pi 0.3655}, 0.5973e^{2i\pi 0.5840}, 0.5174e^{2i\pi 0.4864})$	$(0.3298e^{2i\pi 0.5829}, 0.4534e^{2i\pi 0.3934}, 0.5483e^{2i\pi 0.5891})$
$\mathcal{O}l_2$	$(0.4949e^{2i\pi 0.4110}, 0.4243e^{2i\pi 0.5345}, 0.5373e^{2i\pi 0.5930})$	$(0.3514e^{2i\pi 0.3933}, 0.5626e^{2i\pi 0.5520}, 0.5174e^{2i\pi 0.4789})$	$(0.1791e^{2i\pi 0.3120}, 0.4587e^{2i\pi 0.3749}, 0.7449e^{2i\pi 0.7851})$
$\mathcal{O}l_3$	$(0.5914e^{2i\pi 0.5184}, 0.3632e^{2i\pi 0.4924}, 0.4302e^{2i\pi 0.4988})$	$(0.1714e^{2i\pi 0.1714}, 0.5835e^{2i\pi 0.5927}, 0.7620e^{2i\pi 0.7763})$	$(0.4461e^{2i\pi 0.7298}, 0.4269e^{2i\pi 0.4023}, 0.4206e^{2i\pi 0.4527})$
$\mathcal{O}l_4$	$(0.3143e^{2i\pi 0.2466}, 0.4429e^{2i\pi 0.5547}, 0.7042e^{2i\pi 0.7601})$	$(0.3586e^{2i\pi 0.3975}, 0.5755e^{2i\pi 0.5582}, 0.4956e^{2i\pi 0.4638})$	$(0.1791e^{2i\pi 0.3143}, 0.4668e^{2i\pi 0.3862}, 0.4668e^{2i\pi 0.7780})$
	$\mathcal{C}r_4$	$\mathcal{C}r_5$	
$\mathcal{O}l_1$	$(0.1929e^{2i\pi 0.1863}, 0.5434e^{2i\pi 0.5322}, 0.8138e^{2i\pi 0.8123})$	$(0.2254e^{2i\pi 0.2386}, 0.5086e^{2i\pi 0.4791}, 0.7536e^{2i\pi 0.7878})$	
$\mathcal{O}l_2$	$(0.2300e^{2i\pi 0.2343}, 0.5533e^{2i\pi 0.5528}, 0.7375e^{2i\pi 0.7639})$	$(0.5535e^{2i\pi 0.5797}, 0.4837e^{2i\pi 0.4958}, 0.4534e^{2i\pi 0.4642})$	
$\mathcal{O}l_3$	$(0.4461e^{2i\pi 0.4924}, 0.5422e^{2i\pi 0.5092}, 0.4968e^{2i\pi 0.4885})$	$(0.2426e^{2i\pi 0.2577}, 0.5239e^{2i\pi 0.4932}, 0.7243e^{2i\pi 0.7521})$	
$\mathcal{O}l_4$	$(0.2300e^{2i\pi 0.2343}, 0.5533e^{2i\pi 0.5528}, 0.7375e^{2i\pi 0.7639})$	$(0.4135e^{2i\pi 0.4483}, 0.5153e^{2i\pi 0.5111}, 0.5760e^{2i\pi 0.6016})$	

TABLE 9. Ranking results for different values of t using CT-SFFA operator.

t	$S(\dot{\Im}_1)$	$S(\dot{\Im}_2)$	$S(\dot{\Im}_3)$	$S(\dot{\Im}_4)$	Ranking
$t=3$	0.7392	0.7058	0.7067	0.7107	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
$t=5$	0.5908	0.5625	0.5632	0.5700	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
$t=7$	0.5366	0.5195	0.5206	0.5250	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
$t=9$	0.5155	0.5065	0.5073	0.5095	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
$t=11$	0.5067	0.5022	0.5030	0.5038	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
$t=13$	0.5030	0.5007	0.5013	0.5016	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
$t=15$	0.5013	0.5003	0.5005	0.5006	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$

TABLE 10. Ranking results for different values of t using CT-SFFG operator.

t	$S(\dot{\Im}_1)$	$S(\dot{\Im}_2)$	$S(\dot{\Im}_3)$	$S(\dot{\Im}_4)$	Ranking
$t=3$	0.7601	0.7194	0.7109	0.7294	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
$t=5$	0.6198	0.5837	0.5786	0.5927	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
$t=7$	0.5637	0.5382	0.5358	0.5433	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
$t=9$	0.5365	0.5195	0.5182	0.5222	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
$t=11$	0.5219	0.5106	0.5098	0.5119	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
$t=13$	0.5134	0.5059	0.5056	0.5067	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
$t=15$	0.5083	0.5035	0.5032	0.5038	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$

regardless of the values of £ in the example, which reflects the consistency of the suggested CT-SFFG operators.

VII. COMPARATIVE ANALYSIS

In this section, a comparison is made between the proposed technique and existing approaches to demonstrate the suggested technique's validity and precision. In addition, this part has a comprehensive discussion highlighting the supremacy of the framed approach.

For the purpose of this comparison, the following different types of aggregation operators have been chosen: CT-SF averaging (CT-SFA) operator [31], CT-SF geometric (CT-SFG) operator [31], T-spherical fuzzy Frank averaging (T-SFFA) operator [25], T-spherical fuzzy Frank geometric (T-SFFG) operator [25], complex q-rung orthopair fuzzy Frank averaging operator (Cq-OFFA) [42], complex q-rung orthopair fuzzy Frank geometric operator (Cq-OFFG) [42], CT-SF Dombi averaging (CT-FDA) operator [32], CT-SF Dombi geometric (CT-SFDG) operator [32], complex

TABLE 11. Ranking results by CT-SFFA with various £.

£	$S(\dot{\Im}_1)$	$S(\dot{\Im}_2)$	$S(\dot{\Im}_3)$	$S(\dot{\Im}_4)$	Ranking
£ = 3	0.7385	0.7052	0.7063	0.7101	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
£ = 5	0.7401	0.7063	0.7072	0.7113	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
£ = 7	0.7410	0.7070	0.7077	0.7119	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
£ = 9	0.7419	0.7076	0.7082	0.7125	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
£ = 50	0.7465	0.7112	0.7118	0.7161	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
£ = 75	0.7474	0.7120	0.7126	0.7169	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
£ = 100	0.7482	0.7126	0.7132	0.7173	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
£ = 150	0.7490	0.7134	0.7140	0.7180	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$

TABLE 12. Ranking results by CT-SFFG with various £.

£	$S(\dot{\Im}_1)$	$S(\dot{\Im}_2)$	$S(\dot{\Im}_3)$	$S(\dot{\Im}_4)$	Ranking
£ = 3	0.7607	0.7199	0.7113	0.7299	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
£ = 5	0.7596	0.7192	0.7105	0.7291	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
£ = 7	0.7589	0.7185	0.7101	0.7286	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
£ = 9	0.7583	0.7181	0.7098	0.7283	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
£ = 50	0.7561	0.7168	0.7090	0.7269	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
£ = 75	0.7558	0.7166	0.7091	0.7266	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
£ = 100	0.7556	0.7165	0.7091	0.7265	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
£ = 150	0.7553	0.7164	0.7092	0.7264	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$

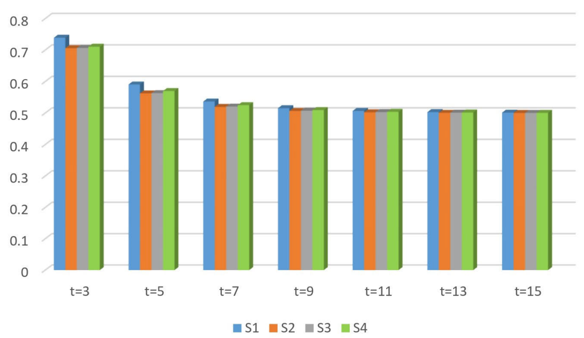

FIGURE 2. Ranking of alternatives by CT-SFFA operator for different values of *t*.

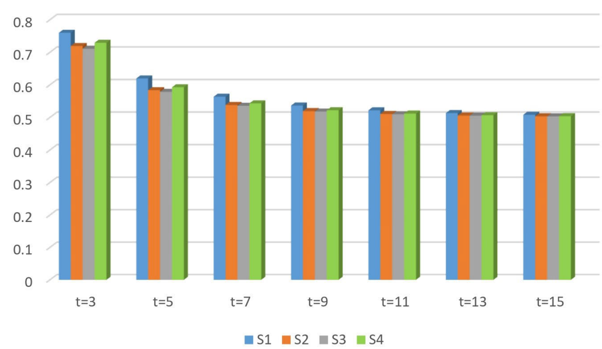

FIGURE 3. Ranking of alternatives by CT-SFFG operator for different values of *t*.

spherical fuzzy prioritized averaging (CSFPA) operator [56], complex spherical fuzzy prioritized geometric (CSFPG) operator [56] and T-spherical fuzzy fairly weighted averaging operator(T-SFFFWA) operator [58]. As most of these existing approaches are unable for group decision issues with CT-SF criteria weight information. We implement these operators on Table 8 to make comparisons possible and more effective. The score values and ranking order of alternatives deploying preexisting and devised operators are displayed in Table 13.

On the basis of Table 13 's ranking, a Spearman correlation analysis is performed to produce Figure 6.

From Table 13, it can be seen that the ranking results provided by CT-SFA and CT-SFG operators [31] are identical to those produced by our developed CT-SFFG operator. According to Theorems 4 and 19, if £ $\longrightarrow$ 1, the operators CT-SFFA and CT-SFFG are reduced to CT-SFA and CT-SFG operators, respectively. Consequently, CT-SFA and CT-SFG operators are the special cases of the described Frank operators. Ali et al. [31] approach is based on a score function that has multiple flaws, as pointed out by [32], rendering this approach ineffective. Secondly, we can notice from Table 13 that Mahnaz et al. [25] and Du et al. [42] operators yield the Ranking $\mathcal{O}l_3 > \mathcal{O}l_2 > \mathcal{O}l_4 > \mathcal{O}l_1$, which is quite different from the propound approach and rest of the existing operators ranking. According to Mahnaz et al. [25], Du et al. [42] and Farid et al. [58] operators $\mathcal{O}l_1$ is the worst alternative but the other existing operators, and our presented operators have

TABLE 13. Ranking results derived by different aggregation operators.

Method	$S(\mathcal{O}l_1)$	$S(\mathcal{O}l_2)$	$S(\mathcal{O}l_3)$	$S(\mathcal{O}l_4)$	Ranking
CT-SFA [31]	0.3860	0.2570	0.1798	0.3381	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
CT-SFG [31]	0.4824	0.3588	0.3311	0.4088	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
T-SFFA [25]	-0.4562	-0.2941	-0.2330	-0.3521	$\mathcal{O}l_3 > \mathcal{O}l_2 > \mathcal{O}l_4 > \mathcal{O}l_1$
T-SFFG [25]	-0.5164	-0.3869	-0.3608	-0.4051	$\mathcal{O}l_3 > \mathcal{O}l_2 > \mathcal{O}l_4 > \mathcal{O}l_1$
Cq-OFFA [42]	-0.2688	-0.09990	-0.07414	-0.2136	$\mathcal{O}l_3 > \mathcal{O}l_2 > \mathcal{O}l_4 > \mathcal{O}l_1$
Cq-OFFG [42]	-0.3369	-0.2204	-0.1922	-0.2635	$\mathcal{O}l_3 > \mathcal{O}l_2 > \mathcal{O}l_4 > \mathcal{O}l_1$
CT-SFDA [32]	0.7503	0.6735	0.7049	0.6458	$\mathcal{O}l_1 > \mathcal{O}l_3 > \mathcal{O}l_2 > \mathcal{O}l_4$
CT-SFDG [32]	0.7836	0.7420	0.7330	0.7426	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$
CSFPA [56]	-	-	-	-	Not applicable
CSFPG [56]	-	-	-	-	Not applicable
T-SFFFWA [58]	-0.1310	0.1151	0.1914	0.01823	$\mathcal{O}l_3 > \mathcal{O}l_2 > \mathcal{O}l_4 > \mathcal{O}l_1$
CT-SFFA	0.7392	0.7058	0.7067	0.7107	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_3 > \mathcal{O}l_2$
CT-SFFG	0.7601	0.7194	0.7109	0.7294	$\mathcal{O}l_1 > \mathcal{O}l_4 > \mathcal{O}l_2 > \mathcal{O}l_3$

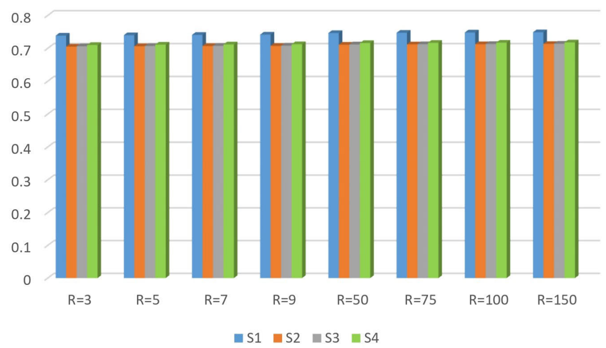

FIGURE 4. Ranking of alternatives by CT-SFFA operator for different values of £.

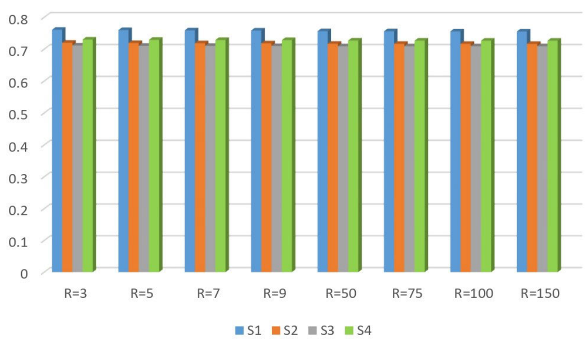

FIGURE 5. Ranking of alternatives by CT-SFFG operator for different values of £.

ranked $\mathcal{O}l_1$ as the best alternative. This discrepancy results from the fact that the previous operators [25], [58] can only handle one dimension. During their execution, we eliminated the complex portion of the data. Although the Du et al. [42] operators process complex data, but they disregard the neutral portion. The operators' disregard for the neutral portion in the case of Du et al. and their failure to deal with complex data in the case of Mahnaz et al. lead to a significant loss of information and erroneous decision outcomes.

The preexistent operators CT-SFDA and CT-SFDG [32] demonstrate (from Table 13) that the best alternative is $\mathcal{O}l_1$, which is the same as avowed by the devised operators. Further, we can notice from the comparison Table 13 that according to CT-SFDA operator [32], $\mathcal{O}l_3$ is the second best alternative, whereas it is the worst alternative if we deploy CT-SFDA operator [32]. Therefore, the Ranking results of the two Dombi operators do not coincide. The primary cause of this mismatch may be the use of the identical formulation of the neutral component in both the addition and multiplication operational rules (see Section III of Ref. [32]). Though, there is also a small bit of dispute in the ranking of our initiated operators,i.e., as per the ranking order of CT-SFFA operator $\mathcal{O}l_2$ is the worst alternative but in line with CT-SFFG operator, it is the second worst alternative. This occurs due to the less difference between the score values of alternatives $\mathcal{O}l_2$ and $\mathcal{O}l_3$. Despite Dombi operators, if we consider the complex spherical fuzzy operators [56], then it is evident from the comparison Table 13 that these operators failed to classify the objects. Albeit CSFPA and CSFPG operators [56] are quite efficient in dealing with data having a prioritization relationship but due to their restricted condition, i.e., $0 \leq (\mu(\hbar))^2 + (\varsigma(\hbar))^2 + (\nu(\hbar))^2 \leq 1$, $0 \leq \left(\eth_\mu\right)^2 + \left(\eth_\varsigma\right)^2 + (\eth_\nu)^2 \leq 1$, these operators are unable to satisfy the CT-SFS requirement for the considered data.

Based on the above analysis, some key merits of the formulated CT-SF frank operators are enlisted as follows:

1). The initiated operators are based on CT-SFS, which generalize the existing operators [25], [31], [42], [57]: i). If £ $\longrightarrow$ 1 these operators reduced to [31], ii). If we consider the complex portion zero, the proposed operators are reduced to [25], iii). If we set the neutral portion to zero, the devised operators reduced to [42], iv). If we fix the neutral portion zero and £ $=$ 2, the formulated operators reduced to [57]. Consequently, the introduced CT-SF Frank aggregation operators can be used to cope with more uncertain and complex data in real-world decision making issues.

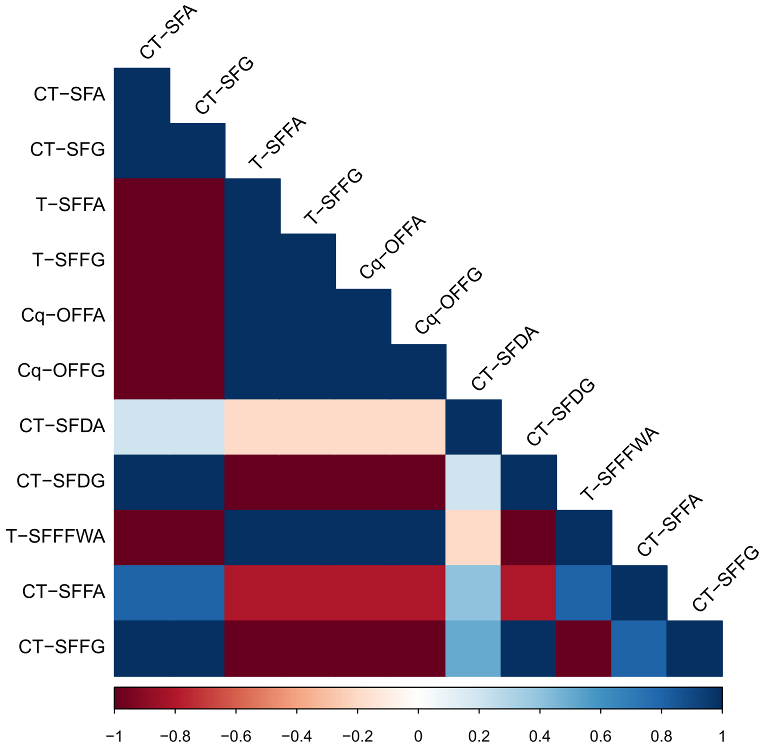

FIGURE 6. **Spearman correlation plot.**

2). The designed method is capable of deriving the criteria weight from the CT-SF information provided by the DEs, which is then used to determine the final ranking of decision alternatives. In contrast, previous methods, except [25], only execute with a known weight vector.
3). The most notable feature of the suggested method is the involvement of parameters t and £. DEs can modify their values according to their own preferences. Since Frank t-norm and t-conorm are a generalization of algorithms such as Algebraic (£ $\longrightarrow$ 1), Einstein (£ $=$ 2), and Hamacher's (£ $=$ 3) t-norm and t-conorm, it is more general in coping with MCGDM problems.
4). None of the existing approaches consider the DEs' weights, whereas the framed approach has the capability of discriminating among DEs by using their importance values.

VIII. CONCLUSION AND FUTURE WORK PLAN

In this study, we extended Frank operations to the CT-SF environment based on the definitions of CT-SFS and Frank t-norm and t-conorm. To begin with, several Frank operational rules of CT-SFS were designed. Meanwhile, we introduced a series of CT-SF Frank operators, such as CT-SFFWA, CT-SFFOWA, CT-SFFHA, CT-SFFWG, CT-SFFOWG, and CT-SFFHG operators. Furthermore, we developed certain features for the aforementioned operators and supplied empirical evidence to support the principles and ideas underlying the operators we have created. Aside from that, we implemented the formulated operators to build an approach to the MCGDM problems with CT-SF data, and thus we added a new direction for addressing MCGDM problems. With the use of a practical decision-making problem involving the selection of the best strategy for water supply, we were able to illustrate both potency and practicability. We analyzed the impact of the involved parameters on decision-making results and found the method quite stable. Finally, we compared the newly devised operators to the previous operators to demonstrate their utility and suitability.

Despite having some advantages over modern methodologies, the created model is not without limitations. Consequently, its structure is incapable of handling problems with completely unknown weight information. In addition, our approach for resolving the MCGDM problem may be computationally intensive because they need laborious and complex calculations. Future research will concentrate on the creation of more complex MCGDM strategies, such as the CT-SF-PROMETHE method, the CT-SF-VIKOR method, the CT-SF-AHP method, and the CT-SF -ELECTRE method. Our intention is to investigate the potential application scope of the CT-SF model in other domains.

ACRONYMS

CF: complex fuzzy. CFS: complex fuzzy set. CT-SF: complex T-spherical fuzzy. CT-SFS: complex T-spherical fuzzy set. CT-SFFWA: CT-SF Frank weighted averaging. CT-SFFWG: CT-SF Frank weighted geometric. CT-SFFOWA: CT-SF Frank ordered weighted averaging. CT-SFFOWG: CT-SF

Frank ordered weighted geometric. CT-SFFHA: CT-SF Frank hybrid averaging. CT-SFFHG: CT-SF Frank hybrid geometric. MCGDM: multi-criteria group decision making. DEs: decision experts. FSs: fuzzy sets. IFSs: intuitionistic fuzzy sets. PyFSs: Pythagorean fuzzy sets. q-ROFSs: q-rung orthopair fuzzy sets. PFSs: Picture FSs. T-SFSs: T-spherical fuzzy sets. CFS: complex FS. CIFS: complex IFS. CPyFS: complex PyFS. Cq-ROFS: complex q-ROFS. CT-SFSs: complex T-spherical fuzzy sets. CT-SF: complex-T spherical fuzzy. CT-SFDM: CT-SF decision matrix. WCT-SFDM: weighted CT-SFDM. CT-SFA: CT-SF averaging. CT-SFG: CT-SF geometric. T-SFFA: T-spherical fuzzy Frank averaging. T-SFFG: T-spherical fuzzy Frank geometric. Cq-OFFA: complex q-rung orthopair fuzzy Frank averaging. Cq-OFFG: complex q-rung orthopair fuzzy Frank geometric. CT-FDA: CT-SF Dombi averaging. CT-FDG: CT-SF Dombi geometric. CSFPA: complex spherical fuzzy prioritized averaging. CSFPG: complex spherical fuzzy prioritized geometric.

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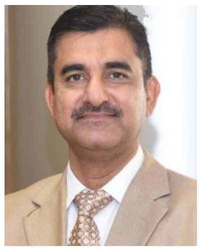

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